Supplementary Material of "Multi-Objective Generalized Linear Bandits"

A Proof of Theorem 1

We follow standard techniques for analyzing the confidence sets [Zhang *et al.*, 2016; Jun *et al.*, 2017] and start with the following lemma.

Lemma 1 *For any $t \geq 1$ and $i \in [m]$,*

$$\ell_{t,i}(u) - \ell_{t,i}(v) \geq \nabla \ell_{t,i}(v)^\top (u - v) + \frac{\kappa}{2}(u^\top x_t - v^\top x_t)^2, \forall u, v \in \mathcal{B}_D.$$

Proof. Define $\tilde{\ell}_{t,i}(z) = -y_t^i z + g_i(z), z \in [-D, D]$. By Assumption 3 we have

$$\tilde{\ell}''_{t,i}(z) = \mu'_i(z) \geq \kappa > 0, \ \forall z \in (-D, D)$$

which implies that $\tilde{\ell}_{t,i}(z)$ is κ-strongly convex on $[-D, D]$, i.e.,

$$\tilde{\ell}_{t,i}(z) - \tilde{\ell}_{t,i}(z') \geq \tilde{\ell}'_{t,i}(z')(z - z') + \frac{\kappa}{2}(z - z')^2, \forall z, z' \in [-D, D]. \tag{17}$$

Note that

$$\ell_{t,i}(u) = \tilde{\ell}_{t,i}(u^\top x_t), \quad \ell_{t,i}(v) = \tilde{\ell}_{t,i}(v^\top x_t), \quad \nabla \ell_{t,i}(v)^\top (u - v) = \tilde{\ell}'_{t,i}(v^\top x_t)(u - v)^\top x_t.$$

We complete the proof by substituting $z = u^\top x_t$ and $z' = v^\top x_t$ into (17). □

Let $f_{t,i}(\theta) = \mathbb{E}_{y_t^i}[\ell_{t,i}(\theta)]$ be the conditional expectation over y_t^i. The following lemma shows that θ_i is the minimum point of $f_{t,i}(\theta)$ on the bounded domain.

Lemma 2 *For any $t \geq 1$ and $i \in [m]$, we have $f_{t,i}(\theta) - f_{t,i}(\theta_i) \geq 0, \ \forall \theta \in \mathcal{B}_D$.*

Proof. Fix t and i. For any $\theta \in \mathcal{B}_D$, we have

$$\begin{aligned}
& f_{t,i}(\theta) - f_{t,i}(\theta_i) \\
=& \mathbb{E}_{y_t^i}[\ell_{t,i}(\theta) - \ell_{t,i}(\theta_i)] \\
=& g_i(\theta^\top x_t) - g_i(\theta_i^\top x_t) + \mu_i(\theta_i^\top x_t)(\theta_i^\top x_t - \theta^\top x_t) \\
\geq& g'_i(\theta_i^\top x_t)(\theta^\top x_t - \theta_i^\top x_t) + \mu_i(\theta_i^\top x_t)(\theta_i^\top x_t - \theta^\top x_t) \\
=& 0.
\end{aligned}$$

where the second and the last equality follow from properties of GLM, and the inequality holds due to the convexity of g_i. □

To exploit the property of the estimation method in (1), we introduce the following lemma from Zhang *et al.* [2016].

Lemma 3 *For any $t \geq 1$ and $i \in [m]$,*

$$\nabla \ell_{t,i}(\hat{\theta}_{t,i})^\top (\hat{\theta}_{t,i} - \theta_i) - \frac{1}{2}\|\nabla \ell_{t,i}(\hat{\theta}_{t,i})\|^2_{Z_{t+1}^{-1}} \leq \frac{1}{2}\left(\|\hat{\theta}_{t,i} - \theta_i\|^2_{Z_{t+1}} - \|\hat{\theta}_{t+1,i} - \theta_i\|^2_{Z_{t+1}}\right). \tag{18}$$

To proceed, we bound the norm of $\nabla \ell_{t,i}(\hat{\theta}_{t,i})$ as follows.

$$\|\nabla \ell_{t,i}(\hat{\theta}_{t,i})\|^2_{Z_{t+1}^{-1}} = \| - y_t^i x_t + \mu_i(\hat{\theta}_{t,i}^\top x_t) x_t\|^2_{Z_{t+1}^{-1}} \leq (R + U)^2 \|x_t\|^2_{Z_{t+1}^{-1}} \tag{19}$$

where the inequality holds because of Assumptions 3 and 4. We are now ready to prove Theorem 1. By Lemma 1, we have

$$\ell_{t,i}(\hat{\theta}_{t,i}) - \ell_{t,i}(\theta_i) \leq \nabla \ell_{t,i}(\hat{\theta}_{t,i})^\top (\hat{\theta}_{t,i} - \theta_i) - \frac{\kappa}{2}(\theta_i^\top x_t - \hat{\theta}_{t,i}^\top x_t)^2.$$

Taking expectation in both side and using Lemma 2, we obtain

$$\begin{aligned}
0 &\leq f_{t,i}(\hat{\theta}_{t,i}) - f_{t,i}(\theta_i) \\
&\leq \nabla f_{t,i}(\hat{\theta}_{t,i})^\top (\hat{\theta}_{t,i} - \theta_i) - \frac{\kappa}{2}(\theta_i^\top x_t - \hat{\theta}_{t,i}^\top x_t)^2 \\
&= \underbrace{(\nabla f_{t,i}(\hat{\theta}_{t,i}) - \nabla \ell_{t,i}(\hat{\theta}_{t,i}))^\top (\hat{\theta}_{t,i} - \theta_i)}_{a_{t,i}} - \frac{\kappa}{2}\underbrace{(\theta_i^\top x_t - \hat{\theta}_{t,i}^\top x_t)^2}_{b_{t,i}} + \nabla \ell_{t,i}(\hat{\theta}_{t,i})^\top (\hat{\theta}_{t,i} - \theta_i) \\
&\overset{(18,19)}{\leq} a_{t,i} - \frac{\kappa}{2} b_{t,i} + \frac{(R+U)^2}{2}\|x_t\|^2_{Z_{t+1}^{-1}} + \frac{1}{2}\left(\|\hat{\theta}_{t,i} - \theta_i\|^2_{Z_{t+1}} - \|\hat{\theta}_{t+1,i} - \theta_i\|^2_{Z_{t+1}}\right) \\
&= a_{t,i} - \frac{\kappa}{4} b_{t,i} + \frac{(R+U)^2}{2}\|x_t\|^2_{Z_{t+1}^{-1}} + \frac{1}{2}\left(\|\hat{\theta}_{t,i} - \theta_i\|^2_{Z_t} - \|\hat{\theta}_{t+1,i} - \theta_i\|^2_{Z_{t+1}}\right)
\end{aligned}$$

where the last equality is due to $Z_{t+1} = Z_t + \frac{\kappa}{2} x_t x_t^\top$. Summing the above inequality over 1 to t and rearranging, we have

$$\|\hat{\theta}_{t+1,i} - \theta_i\|_{Z_{t+1}}^2 \leq \lambda D^2 + 2\sum_{s=1}^{t} a_{s,i} - \frac{\kappa}{2}\sum_{s=1}^{t} b_{s,i} + (R+U)^2 \sum_{s=1}^{t} \|x_s\|_{Z_{s+1}^{-1}}^2. \tag{20}$$

We propose the following lemma to bound $\sum_{s=1}^t a_{s,i}$.

Lemma 4 *With probability at least* $1-\delta$*, for any* $i \in [m]$ *and* $t \geq 1$*,*

$$\sum_{s=1}^{t} a_{s,i} \leq \frac{\kappa}{4}\sum_{s=1}^{t} b_{s,i} + \frac{\kappa}{4} + \frac{8(R+U)^2}{\kappa} \log\left(\frac{m}{\delta}\sqrt{1+4D^2 t}\right). \tag{21}$$

Proof. Let $\eta_{s,i} = y_s^i - \mu_i(\theta_i^\top x_s)$, which is a martingale difference sequence. By Assumptions 3 and 4, $|\eta_{s,i}| \leq (R+U)$ holds almost surely, which implies that $\eta_{s,i}$ is $(R+U)$-sub-Gaussian. Thus, we can apply the self-normalized bound for martingales (Abbasi-Yadkori *et al.* 2012, Corollary 8) and use the union bound to obtain that with probability at least $1-\delta$,

$$\sum_{s=1}^{t} a_{s,i} \leq (R+U)\cdot\sqrt{\left(2+2\sum_{s=1}^{t} b_{s,i}\right)}\cdot\sqrt{\log\left(\frac{m}{\delta}\sqrt{1+\sum_{s=1}^{t} b_{s,i}}\right)},\ \forall i \in [m], \forall t \geq 1.$$

We conclude the proof by noticing that $b_{s,i} \leq 4D^2$ and using the well-known inequality $\sqrt{pq} \leq \frac{p}{c} + cq$, where $c = \frac{8(R+U)}{\kappa}$. □

It remains to bound $\sum_{s=1}^t \|x_s\|_{Z_{s+1}^{-1}}^2$. To this end, we employ Lemma 12 in Hazan *et al.* [2007] and obtain

$$\sum_{s=1}^{t} \|x_s\|_{Z_{s+1}^{-1}}^2 \leq \frac{2}{\kappa}\sum_{s=1}^{t} \log\frac{\det(Z_{s+1})}{\det(Z_s)} \leq \frac{2}{\kappa}\log\frac{\det(Z_{t+1})}{\det(Z_1)}. \tag{22}$$

We complete the proof by combining (20), (21), and (22). □