Queueing Theory with Applications . . . : Solution Manual

QUEUEING THEORY WITH APPLICATIONS TO PACKET TELECOMMUNICATION: SOLUTION MANUAL

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Chapter 1

TERMINOLOGY AND EXAMPLES

EXERCISE 1.1 Assume values of $\tilde{x}$ and $\tilde{t}$ are drawn from truncated geometric distributions. In particular, let $P\{\tilde{x} = n\} = 0$ and $P\{\tilde{t} = n\} = 0$ except for $1 \leq n \leq 10$, and let $P\{\tilde{x} = n\} = \alpha p_x(1 - p_x)^n$ and $P\{\tilde{t} = n\} = \beta p_t(1 - p_t)^n$ for $1 \leq n \leq 10$ with $p_x = 0.292578$ and $p_t = 0.14358$.

1. Using your favorite programming language or a spreadsheet, generate a sequence of 100 random variates each for $\tilde{x}$ and $\tilde{t}$.
2. Plot $\tilde{u}(t)$ as a function of t, compute $\tilde{z}_n$ from (1.3) and $\tilde{w}_n$ from (1.2) for $1 \leq n \leq 20$.
3. Compute $\tilde{\tau}_n$ for $1 \leq n \leq 20$ and verify that $\tilde{w}_n$ can be obtained from $\tilde{z}(t)$.
4. Compute $\tilde{z}_n$ from (1.3) and $\tilde{w}_n$ from (1.2) for $1 \leq n \leq 100$ and compute the average waiting times for the 100 customers.

Solution Part 1. We must first determine the constants α and β. Since the probabilities must sum to unity, we find

$$\sum_{n=1}^{10} P\left\{\tilde{x} = n\right\} = \sum_{n=1}^{10} P\left\{\tilde{t} = n\right\} = 1.$$

Thus, using the formula for the partial sum of a geometric series, we find

$$\alpha\left[(1 - p_x) - (1 - p_x)^{11}\right] = \beta\left[(1 - p_t) - (1 - p_t)^{11}\right] = 1,$$

from which we may solve to find $\alpha = 1.45939316$ and $\beta = 1.48524892$.

Now, in general,

$$\begin{aligned} P\{\tilde{x} \leq n\} &= \sum_{i=1}^{n} P\{\tilde{x} = i\} = \sum_{i=1}^{n} \alpha p_x (1-p_x)^i \\ &= \alpha \left[(1-p_x) - (1-p_x)^{n+1}\right]. \end{aligned}$$

Solving $P\{\tilde{x} = n\} = \alpha p_x (1-p_x)^n$ for n, we find

$$n = \frac{\ln\left[(1-p_x) - (1/\alpha)P\{\tilde{x} \leq n\}\right]}{\ln(1-p_x)} - 1.$$

We may therefore select random variates from a uniform distribution and use these numbers in place of the above formula in place of $P\{\tilde{x} \leq n\}$ and then round up to obtain the variates for $\tilde{x}$. Similarly, by solving $P\{\tilde{t} = n\} = \beta p_t (1-p_t)^n$ for n we may find the variates for $\tilde{t}$ from

$$n = \frac{\ln\left[(1-p_t) - (1/\beta)P\{\tilde{t} \leq n\}\right]}{\ln(1-p_t)} - 1.$$

By drawing samples from a uniform distribution and using those values for $P\{\tilde{x} \leq n\}$ and $P\{\tilde{t} \leq n\}$ in the above formulas, one can obtain the variates for x_n and t_n for $n = 1, 2, \cdots, 100$, and then apply the formulas to obtain the following table:

n	x_n	t_n	z_n	w_n	τ_n
1	3	6	2	0	6
2	5	1	-1	2	7
3	1	6	-4	1	13
4	1	5	-1	0	18
5	7	2	4	0	20
6	2	3	-1	4	23
7	2	3	-3	3	26
8	4	5	3	0	31
9	2	1	1	3	32
10	1	1	0	4	33
11	2	1	-1	4	34
12	1	3	-3	3	37
13	3	4	-3	0	41
14	4	6	2	0	47
15	7	2	3	2	49
16	2	4	-2	5	53
17	5	4	1	3	57
18	1	4	-6	4	61
19	1	7	-6	0	68
20	4	7	-1	0	75

n	x_n	t_n	z_n	w_n	τ_n
21	1	5	-6	0	80
22	5	7	-1	0	87
23	5	6	2	0	93
24	2	3	-8	2	96
25	4	10	-4	0	106
26	1	8	-1	0	114
27	1	2	-2	0	116
28	4	3	2	0	119
29	3	2	-3	2	121
30	1	6	0	0	127
31	2	1	-2	0	128
32	2	4	-6	0	132
33	1	8	-2	0	140
34	9	3	2	0	143
35	3	7	-6	2	150
36	2	9	-3	0	159
37	5	5	3	0	164
38	3	2	2	3	166
39	4	1	3	5	167
40	2	1	-3	8	168
41	6	5	3	5	173
42	3	3	0	8	176
43	2	3	-2	8	179
44	1	4	-9	6	183
45	5	10	-2	0	193
46	2	7	-8	0	200
47	1	10	-1	0	210
48	1	2	0	0	212
49	2	1	-1	0	213
50	1	3	-4	0	216
51	10	5	8	0	221
52	2	2	-2	8	223
53	3	4	1	6	227
54	1	2	-1	7	229
55	5	2	2	6	231
56	2	3	1	8	234
57	2	1	-1	9	235
58	1	3	-1	8	238
59	1	2	-1	7	240
60	5	2	1	6	242
61	2	4	0	7	246
62	3	2	1	7	248
63	1	2	-1	8	250
64	1	2	-4	7	252
65	5	5	-5	3	257
66	3	10	-2	0	267
67	2	5	0	0	272
68	2	2	-1	0	274
69	1	3	-2	0	277
70	4	3	-2	0	280

n	x_n	t_n	z_n	w_n	τ_n
71	1	6	-2	0	286
72	4	3	-1	0	289
73	5	5	-5	0	294
74	1	10	-3	0	304
75	1	4	-9	0	308
76	1	10	0	0	318
77	4	1	-4	0	319
78	4	8	-4	0	327
79	5	8	-1	0	335
80	4	6	3	0	341
81	6	1	5	3	342
82	2	1	-1	8	343
83	3	3	2	7	346
84	1	1	-3	9	347
85	6	4	0	6	351
86	9	6	2	6	357
87	3	7	-3	8	364
88	7	6	3	5	370
89	3	4	2	8	374
90	4	1	2	10	375
91	2	2	-1	12	377
92	3	3	2	11	380
93	5	1	1	13	381
94	3	4	2	14	385
95	3	1	-4	16	386
96	8	7	4	12	393
97	2	4	1	16	397
98	1	1	0	17	398
99	4	1	-4	17	399
100	2	8	0	13	407
Averages	3.05	4.07	-0.98	3.95	

Solution Part 2. Figure 2.1 shows the unfinished work as a function of time. The values of z_n and w_n are shown in the above Table.

Solution Part 3. The values of τ_n and w_n are shown in the above Table.From Figure 2.1, it is readily verified that

$$w_n = u(\tau_n^-).$$

For example, $\tau_3 = 13$, and $u(13^-) = 1$. Therefore, $w_3 = 1$. That is, if an arrival occurs at time t, then an amount of time $u(t^-)$ would pass before service on that customer's job began. That is, the server would first complete the work present in the system at time t^-, and then would begin the service on the customer that arrived at time t.

Solution Part 4. The average waiting time is shown at the bottom of the above table. It is found by simply summing the waiting times and dividing by the number of customers.

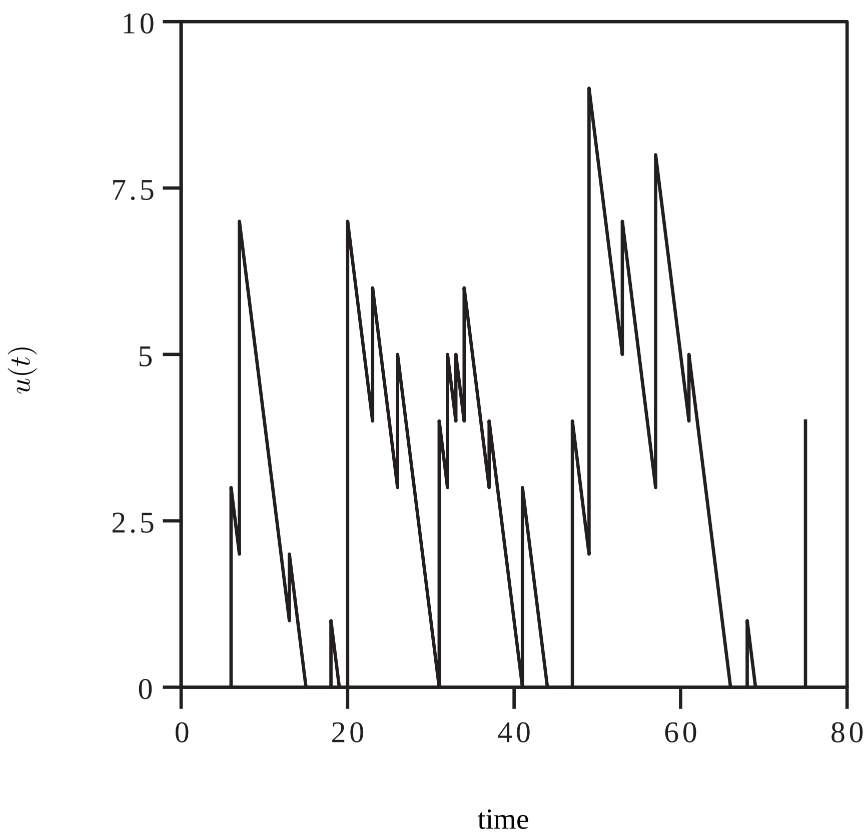

Figure 2.1 Unfinished work as a function of time.

EXERCISE 1.2 Assume values of $\tilde{x}$ and $\tilde{t}$ are drawn from truncated geometric distributions as given in Exercise 1.1.

1. Using the data obtained in Exercise 1.1, determine the lengths of all busy periods that occur during the interval.
2. Determine the average length of the busy period.
3. Compare the average length of the busy period obtained in the previous step to the average waiting time computed in Exercise 1.1. Based on the results of this comparison, speculate about whether or not the average length of the busy period and the average waiting time are related.

Solution Part 1. A busy period starts whenever a customer waits zero time. The length of the busy period is the total service time accumulated up to the point in time at which the next busy period starts, that is, whenever some later arriving customer has a waiting time of zero. From the table of values given in the previous problem's solution, we see that a busy period is in progress at the time of arrival of C_100. Therefore we must decide whether or not to count that busy period. We arbitrarily decide not to count that busy period. The lengths of the busy periods are given in the following table:

1	9
2	1
3	11
4	10
5	3
6	19
7	1
8	4
9	1
10	5
11	7
12	4
13	1
14	1
15	7
16	1
17	2
18	2
19	1
20	12
21	2
22	26
23	5
24	2
25	1

26	1
27	2
28	1
29	44
30	3
31	2
32	2
33	1
34	4
35	1
36	4
37	5
38	1
39	1
40	1
41	4
42	4
43	5
Average	5.209302326

Solution Part 2. The average length of the busy period is determined by summing the lengths of the busy periods and dividing by the number of busy periods. The result is shown at the end of the above table.

Solution Part 3. The computed averages are $E[\tilde{w}] = 3.95$ and $E[\tilde{y}] = 5.21$. We see that based on this very small simulation, the results are not all that different. It is sometimes possible to establish a definitive relationship between the two quantities. We will see later on that under certain sets of assumptions, the busy period and the waiting time have the same value.

EXERCISE 1.3 For the general case where the packet arrival process has run lengths, it is seen that the survivor functions decrease with decreasing run lengths. Determine whether or not there exists an average run length at which the packet arrival process becomes a sequence of independent Bernoulli trials. If such a choice is possible, find the value of run length at which independence occurs. Discuss the result of reducing the run length below that point. [Hint: A packet arrival occurs whenever the system transitions into the on state. Thus, if the probability of transitioning into the on state is independent of the current state, the arrival process becomes a sequence of independent Bernoulli trials.]

Solution. The arrival process will be Bernoulli if the probability of having a packet in a slot is independent of the current state. In general, there will be a packet arrival in a slot if the system is in the on state. Hence we want to set transition probabilities p_{01} and p_{11} equal. Since we want to have a 90% traffic intensity, we choose $p_{01} = p_{11} = 0.9/N$. Since the run lengths have the geometric distribution, the average run length will be $1/p_{10}$. Thus, the average run length will be

$$\frac{N}{N-0.9.}$$

Similarly the average off time will be

$$\frac{N}{0.9.}$$

Chapter 2

REVIEW OF RANDOM PROCESSES

EXERCISE 2.1 For the experiment described in Example 2.1, specify all of the event probabilities.

Solution. For the experiment described in Example 2.1, the events are as follows: $\emptyset$, $\{N\}$, $\{S\}$, $\{C\}$, $\{T\}$, $\{S, N\}$, $\{C, N\}$, $\{T, N\}$, $\{C, S\}$, $\{T, S\}$, $\{T, C\}$, $\{C, S, N\}$, $\{T, S, N\}$, $\{T, C, N\}$, $\{T, C, S\}$, $\{T, C, S, N\}$. Since each of the events is a union of elementary events, which are mutually exclusive, the probability of each event is the sum of the probabilities of its constituent events. For example,

$$P\{T, C, S, N\} = P\{T\}+P\{C\}+P\{S\}+P\{N\} = 0.11+0.66+0.19+0.04 = 1.0.$$

By proceeding in this manner, we find the results given in the following table:

Event	Probability	Event	Probability
$\emptyset$	0.00	$\{C, S\}$	0.85
$\{N\}$	0.04	$\{T, S\}$	0.30
$\{S\}$	0.19	$\{T, C\}$	0.77
$\{C\}$	0.66	$\{C, S, N\}$	0.89
$\{T\}$	0.11	$\{T, S, N\}$	0.34
$\{S, N\}$	0.23	$\{T, C, N\}$	0.81
$\{C, N\}$	0.70	$\{T, C, S\}$	0.96
$\{T, N\}$	0.15	$\{T, C, S, N\}$	1.00

EXERCISE 2.2 For the experiment described in Example 2.1, there are a total of four possible outcomes and the number of events is 16. Show that it is always true that $\mathrm{card}\ (\Omega) = 2^{\mathrm{card}\ (\mathcal{S})}$, where $\mathrm{card}\ (\mathcal{A})$ denotes the cardinality of the set $\mathcal{A}$, which, in turn, is the number of elements of $\mathcal{A}$. [Hint: The events can be put in one-to-one correspondence with the $\mathrm{card}\ (\mathcal{S})$-bit binary numbers.]

Solution. Every event in the event space consists of a union of the elementary events of the experiment. Thus, every event is characterized by the presence or absence of elementary events. Suppose an experiment has $\mathrm{card}\ (\mathcal{S})$ elementary events, then these events can be ordered and put in one-to-one correspondence with the digits of an $\mathrm{card}\ (\mathcal{S})$-digit binary number, say,

$$d_{\mathrm{card}\ (\mathcal{S})-1} d_{\mathrm{card}\ (\mathcal{S})-2} \cdots d_0.$$

If elementary event i is present in some event, the $d_i = 1$, else $d_i = 0$. Since there are a total of 2^N N-digit binary numbers, there are exactly $2^{\mathrm{card}\ (\mathcal{S})}$ events in any experiment. For example, in the experiment of Eample 2.1, we have four events. Let T, C, S, and N be represented by the binary digits d_3, d_2, d_1, and d_0, respectively. Then, 1010 represents the event $\{T, S\}$. The total number of events is $2^4 = 16$.

EXERCISE 2.3 Repeat the computations of Example 2.2 by constructing restricted experiments based on $\mathcal{S}_A$ and $\mathcal{S}_B$. [Hint: The probabilities of the elementary events in the restricted experiments must be normalized by dividing the probabilities of the individual elementary events of the restricted experiment by the sum of the probabilities of the constituent elementary events of the restricted experiment.]

Solution. The idea is to construct two different experiments, A and B, and compute $P\{\omega_1\}$ in each of the experiments. The set of all possible outcomes in experiment A is $\mathcal{S}_A = \{s_1, s_2, s_4, s_6, s_8\}$.

For experiment A, we compute the probabilities of each elementary event to be $P\{s_i\} = k_A \frac{i}{36}$, where k_A is a normalizing parameter such that

$$\sum_{s_i \in \mathcal{S}_A} P\{s_i\} = 1.$$

We find $K_A = \frac{36}{21}$ so that $P\{s_i\} = \frac{i}{21}$ for $s_i \in \mathcal{S}_A$ and $P\{s_i = 0\}$ otherwise. We then find for experiment A,

$$P\{\omega_1\} = P\{s_1\} + P\{s_2\} + P\{s_3\} = \frac{1}{21} + \frac{2}{21} + 0 = \frac{1}{7}.$$

Similarly, for experiment B, we find $P\{s_i\} = \frac{i}{15}$ for $s_i \in \mathcal{S}_B$ and $s_i = 0$ otherwise. Then, for experiment B, we have

$$P\{\omega_1\} = P\{s_1\} + P\{s_2\} + P\{s_3\} = \frac{0}{15} + \frac{0}{15} + \frac{3}{15} = \frac{1}{5}.$$

We recognize these probabilities to be the same as the conditional probabilities previously computed.

EXERCISE 2.4 Suppose $\omega_1, \omega_2 \in \Omega$ with $P\{\omega_1\} \neq 0$ and $P\{\omega_2\} \neq 0$. Show that if ω_1 is statistically independent of ω_2, then necessarily, ω_2 is statistically independent of ω_1. In other words, show that statistical independence between events is a mutual property.

Solution. From Bayes' rule, we have

$$P\{\omega_2|\omega_1\} = \frac{P\{\omega_1|\omega_2\} P\{\omega_2\}}{P\{\omega_1\}}.$$

If ω_1 is statistically independent of ω_2, then $P\{\omega_1|\omega_2\} = P\{\omega_1\}$. Thus,

$$P\{\omega_2|\omega_1\} = \frac{P\{\omega_1\} P\{\omega_2\}}{P\{\omega_1\}} = P\{\omega_2\}.$$

Since

$$P\{\omega_2|\omega_1\} = P\{\omega_2\}$$

is the definition of ω_2 being statistically independent of ω_1, the result follows.

EXERCISE 2.5 Suppose $\omega_1, \omega_2 \in \Omega$ but $P\{\omega_1\} = 0$ or $P\{\omega_2\} = 0$. Discuss the concept of independence between ω_1 and ω_2.

Solution. We assume ω_1 and ω_2 are two different events. Suppose $P\{\omega_1\} = 0$ but $P\{\omega_2\} \neq 0$. Then

$$P\{\omega_1|\omega_2\} = \frac{P\{\omega_1\omega_2\}}{P\{\omega_2\}}.$$

But, $P\{\omega_1\omega_2\} \leq P\{\omega_1\}$ and $P\{\omega_1\omega_2\} \geq 0$. Thus, $P\{\omega_1\omega_2\} = 0$, which implies $P\{\omega_1|\omega_2\} = 0$, and thus, ω_1 is independent of ω_2. If $P\{\omega_1\} = 0$, then ω_1 is equivalent to the null event. Therefore, the probability that any event has occurred given that the null event has actually occurred is zero; that is, all subsets of the null event are the null event. Hence no event other than the null event can be statistically independent of the null event. In words, if we know the null event has occurred, then we are certain no other event has occurred. Since the other event is not the null event, its unconditional probability of occurrence is greater than zero.

EXERCISE 2.6 Develop the expression for $dF_{\tilde{c}}(x)$ for the random variable $\tilde{c}$ defined in Example 2.4.

Solution. The random variable $\tilde{c}$ defined in Example 2.4 is discrete, and its support set is

$$\mathcal{C} = \{56.48, 70.07, 75.29, 84.35, 96.75, 99.99, 104.54, 132.22\}.$$

We then find,

$$\begin{aligned} dF_{\tilde{c}}(x) &= \sum_{c\in\mathcal{C}} P\{\tilde{c}=c\}\,\delta(x-c)dx \\ &= \sum_{c\in\{56.48,70.07,75.29,84.35,96.75,99.99,104.54,132.22\}} P\{\tilde{c}=c\}\,\delta(x-c)dx. \end{aligned} \tag{2.1}$$

Thus,

$$\begin{aligned} dF_{\tilde{c}}(x) &= \frac{8}{36}\delta(x-56.48)dx+\frac{7}{36}\delta(x70.07)dx+\frac{6}{36}\delta(x-75.29)dx+ \\ &\quad \frac{5}{36}\delta(x-84.35)dx+\frac{4}{36}\delta(x-96.75)dx+\frac{3}{36}\delta(x-99.99)dx+ \\ &\quad \frac{2}{36}\delta(x-104.54)dx+\frac{1}{36}\delta(x-132.22)dx \end{aligned}$$

EXERCISE 2.7 Find $E[\tilde{c}]$ and $\mathrm{var}\,(\tilde{c})$ for the random variable $\tilde{c}$ defined in Example 2.4.

Solution. By definition,

$$E\,[\tilde{c}] = \sum_{c\in\mathcal{C}} cP\,\{\tilde{c}=c\}\,,$$

$$\mathrm{var}\,(\tilde{c}) = E\left[\tilde{c}^2\right] - E^2[\tilde{c}],$$

and

$$E\left[\tilde{c}^2\right] = \sum_{c\in\mathcal{C}} c^2 P\,\{\tilde{c}=c\}\,.$$

From Example 2.4 and the previous problem, we have the following table:

c	$P\{\tilde{c}=c\}$
56.48	0.222222222
70.07	0.194444444
75.29	0.166666667
84.35	0.138888889
96.75	0.111111111
99.99	0.083333333
104.54	0.055555556
132.22	0.027777778

Upon performing the calculations, we find $E\,[\tilde{c}] = 79.0025$, $E\left[\tilde{c}^2\right] = 6562.507681$, and $\mathrm{var}\,(\tilde{c}) = 321.1126743$.

EXERCISE 2.8 Let $\mathcal{X}_1$ and $\mathcal{X}_2$ denote the support sets for $\tilde{x}_1$ and $\tilde{x}_2$, respectively. Specialize (2.12) where $\mathcal{X}_1 = \{5, 6, \ldots, 14\}$ and $\mathcal{X}_2 = \{11, 12, \ldots, 22\}$.

Solution. Equation (2.12) is as follows:

$$P\{\tilde{x}_1 + \tilde{x}_2 = n\} = \sum_{i=0}^{n} P\{\tilde{x}_2 = n - i \mid \tilde{x}_1 = i\} P\{\tilde{x}_1 = i\}.$$

First we note that the smallest possible value of $\tilde{x}_1 + \tilde{x}_2$ is $5 + 11 = 16$, and the largest possible value $\tilde{x}_1 + \tilde{x}_2$ can take on is $14 + 22 = 36$. We also see that $\tilde{x}_1 + \tilde{x}_2$ can take on all values between 16 and 36. Thus, we find immediately that the support set for $\tilde{x}_1 + \tilde{x}_2$ is $\{16, 17, \ldots, 36\}$. We also see that since $\tilde{x}_1$ ranges only over the integers from 5 to 14, $P\{\tilde{x}_1 = i\} > 0$ only if i is over only the integers from 5 to 14.

Similarly, $prob\tilde{x}_2 = n - i > 0$ only if $n - i \geq 11$ and $n - i \leq 22$, so that $prob\tilde{x}_2 = n - i > 0$ only if $n - 11 \geq i$ and $n - 22 \leq i$ or $i \leq n - 11$ and $i \geq n - 22$. Hence, for $P\{\tilde{x}_1 = i\} > 0$, we need $i \geq \max\{5, n - 22\}$ and $i \leq \min set 14, n - 11$. We, therefore have the following result. For $n \neq \{5, 6, \ldots, 36\}$, $P\{\tilde{x}_1 + \tilde{x}_2 = n\} = 0$, and for $n \in \{5, 6, \ldots, 36\}$,

$$P\{\tilde{x}_1 + \tilde{x}_2 = n\} = \sum_{i=\max\{5, n-22\}}^{\min\{14, n-11\}} P\{\tilde{x}_2 = n - i \mid \tilde{x}_1 = i\} P\{\tilde{x}_1 = i\}.$$

For example,

$$\begin{aligned} P\{\tilde{x}_1 + \tilde{x}_2 = 18\} &= \sum_{i=\max\{5, 18-22\}}^{\min\{14, 18-11\}} P\{\tilde{x}_2 = n - i \mid \tilde{x}_1 = i\} P\{\tilde{x}_1 = i\} \\ &= \sum_{i=5}^{7} P\{\tilde{x}_2 = n - i \mid \tilde{x}_1 = i\}. \end{aligned}$$

EXERCISE 2.9 Derive (2.14) as indicated in the previous paragraph. .

Solution. We wish to find an expression for $f_{\tilde{x}_1+\tilde{x}_2}(x)$ given the densities $f_{\tilde{x}_1}(x)$ and $f_{\tilde{x}_2}(x)$. We first note that $f_{\tilde{x}_1+\tilde{x}_2}(x) = \frac{d}{dx} F_{\tilde{x}_1+\tilde{x}_2}(x)(x)$. But, $F_{\tilde{x}_1+\tilde{x}_2}(x)(x) = P\{\tilde{x}_1 + \tilde{x}_2 \leq x\}$. Now, suppose we partition the interval $(0, x]$ into N equal-length intervals an assume that the random variable $\tilde{x}_1$ has masses at the points $i\Delta X$, where $\Delta X = \frac{x}{N}$. Then,

$$P\{\tilde{x}_1 + \tilde{x}_2 \leq x\} = \sum_{i=1}^{N} P\{\tilde{x}_1 + \tilde{x}_2 \leq x | \tilde{x}_1 = i\Delta x\} P\{\tilde{x}_1 = i\Delta X\}.$$

But

$$P\{\tilde{x}_1 = i\Delta X\} \approx f_{\tilde{x}_1}(i\Delta X)\Delta X,$$

so,

$$\begin{aligned} P\{\tilde{x}_1 + \tilde{x}_2 \leq x\} &= \sum_{i=1}^{N} P\{\tilde{x}_1 + \tilde{x}_2 \leq x | \tilde{x}_1 = i\Delta x\} f_{\tilde{x}_1}\Delta X \\ &= \sum_{i=1}^{N} P\{\tilde{x}_2 \leq x - i\Delta X | \tilde{x}_1 = i\Delta x\} f_{\tilde{x}_1}(i\Delta X)\Delta X. \end{aligned}$$

In the limit, we can pass $i\Delta X$ to the continuous variable y and ΔX to dy. The sum then passes to the integral over $(0, x]$, which yields the result

$$P\{\tilde{x}_1 + \tilde{x}_2 \leq x\} = \int_0^x P\{\tilde{x}_2 \leq x - y | \tilde{x}_1 = y\} f_{\tilde{x}_1}(y)dy,$$

or

$$F_{\tilde{x}_1+\tilde{x}_2}(x) = \int_0^x F_{\tilde{x}_2|\tilde{x}_1}(x - y|y) f_{\tilde{x}_1}(y)dy.$$

Differentiating both sides with respect to x then yields

$$f_{\tilde{x}_1+\tilde{x}_2}(x) = \int_0^x f_{\tilde{x}_2|\tilde{x}_1}(x - y|y) f_{\tilde{x}_1}(y)dy.$$

EXERCISE 2.10 Prove Lemma 2.1.

Solution. Recall the definition of memoryless: A random variable $\tilde{x}$ is said to be memoryless if, and only if, for every $\alpha, \beta \geq 0$,

$$P\{\tilde{x} > \alpha + \beta | \tilde{x} > \beta\} = P\{\tilde{x} > \alpha\}.$$

In order to prove the lemma, we simply show that the exponential distribution satisfies the definition of memoryless; that is, we compute $P\{\tilde{x} > \alpha + \beta | \tilde{x} > \beta\}$ and see whether this quantity is equal to $P\{\tilde{x} > \alpha\}$. From the definition of conditional probability, we find

$$P\{\tilde{x} > \alpha + \beta | \tilde{x} > \beta\} = \frac{P\{\tilde{x} > \alpha + \beta, \tilde{x} > \beta\}}{P\{\tilde{x} > \beta\}}.$$

But the joint probability of the numerator is just $P\{\tilde{x} > \alpha + \beta\}$, and for the exponential distribution is

$$P\{\tilde{x} > \alpha + \beta\} = e^{-\lambda(\alpha+\beta)}.$$

Also,

$$P\{\tilde{x} > \beta\} = e^{-\lambda\beta}.$$

Thus,

$$P\{\tilde{x} > \alpha + \beta | \tilde{x} > \beta\} = \frac{e^{-\lambda(\alpha+\beta)}}{e^{-\lambda\beta}} = e^{-\lambda\alpha} P\{\tilde{x} > \alpha\}.$$

Thus, the definition of memoryless is satisfied and the conclusion follows.

EXERCISE 2.11 Prove Lemma 2.2. [*Hint*: Start with rational arguments. Extend to the real line using a continuity argument. The proof is given in Feller, [1968] pp. 458 - 460, but it is strongly recommended that the exercise be attempted without going to the reference.]

Solution. We first prove the proposition that $g(n/m) = g(1/m)^n$ for an arbitrary integer m. Let T denote the truth set for the proposition. Trivially, $n \in T$, so assume $(n-1) \in T$. Then

$$g(n/m) = g\left(\sum_{j=1}^{n} \frac{1}{m}\right) = g\left(\sum_{j=1}^{n-1} \frac{1}{m} + \frac{1}{m}\right).$$

But, by hypothesis of the Lemma 2.2, $g(t+s) = g(t)g(s)\ \forall s, t > 0$. Thus,

$$g(n/m) = g\left(\sum_{j=1}^{n-1} \frac{1}{m}\right) g\left(\frac{1}{m}\right),$$

and by using the inductive hypothesis, we have

$$g(n/m) = g\left(\frac{1}{m}\right)^{n-1} g\left(\frac{1}{m}\right) = g\left(\frac{1}{m}\right)^{n} \quad \text{for all integer } m, n.$$

This completes the proof of the proposition. Now, with $m = 1$, the result of the proposition gives $g(n) = g(1)^n$, and with $n = m$, the same result gives $g(1) = g(1/m)^m$ from which it follows that $g(1/m) = g(1)^{1/m}$. Substitution of the latter result into the conclusion of the proposition yields

$$g(n/m) = g\left(\frac{1}{m}\right)^{n-1} g\left(\frac{1}{m}\right) = g(1)^{n/m} \quad \text{for all integer } m, n.$$

If we now take $\lambda = -\ln[g(1)]$, we have

$$g(n/m) = e^{-\lambda(n/m)} \quad \text{for all integer } m, n.$$

Since the rationals are dense in the reals, each real number is the limit of a sequence of rational numbers. Further, g is a right continuous function so that by analytic continuity,

$$g(t) = e^{-\lambda t} \quad \text{for all integer } t > 0.$$

EXERCISE 2.12 Repeat Example 2.5, assuming all students have a deterministic holding time of one unit. How do the results compare? Would an exponential assumption on service-time give an adequate explanation of system performance if the service-time is really deterministic?

Solution. $P\{\text{Alice before Charlie}\}$ must be zero. We see this as follows: Alice, Bob and Charlie will use the phone exactly one time unit each. Even if Bob finishes his call at precisely the moment Alice walks up, Charlie has already begun his. So, Charlie's time remaining is strictly less than one, while Alice's time to completion is exactly one in the best case. Thus, it is impossible for Alice to finish her call before Charlie. Since assuming exponential holding times results in $P\{\text{Alice before Charlie}\} = 1/4$, exponentiality is not an appropriate assumption in this case.

EXERCISE 2.13 Prove Theorem 2.2. Theorem2.2. Let $\tilde{x}$ be a non-negative random variable with distribution $F_{\tilde{x}}(x)$, and let $F^*_{\tilde{x}}(s)$ be the Laplace-Stieltjes transform of $\tilde{x}$. Then

$$E[\tilde{x}^2] = (-1)^n \frac{d^n}{ds^n} F^*_{\tilde{x}}(s)\Big|_{s=0}.$$

Solution. We first prove the proposition that

$$\frac{d^n}{ds^n} F^*_{\tilde{x}}(s) = (-1)^n \int_0^\infty x^n e^{-sx} dF_{\tilde{x}}(x).$$

Let T be the truth set for the proposition. By Definition 2.8, $F^*_{\tilde{x}}(s) = E[e^{-sx}]$, so that

$$\frac{d}{ds} F^*_{\tilde{x}}(s) = \int_0^\infty (-x) e^{-sx} dF_{\tilde{x}}(x),$$

and so $1 \in T$. Now assume that $n \in T$. Then

$$\begin{aligned}
\frac{d^{n+1}}{ds^{n+1}} F^*_{\tilde{x}}(s) &= \frac{d}{ds}\left\{\frac{d^n}{ds^n} F^*_{\tilde{x}}(s)\right\} \\
&= \frac{d}{ds}\left\{(-1)^n \int_0^\infty x^n e^{-sx} dF_{\tilde{x}}(x)\right\} \\
&= (-1)^{n+1} \int_0^\infty x^{n+1} e^{-sx} dF_{\tilde{x}}(x).
\end{aligned} \tag{2.2}$$

This proves the proposition. We may now show

$$(-1)^n \frac{d^n}{ds^n} F^*_{\tilde{x}}(s)\Big|_{s=0} \quad = \textstyle\int_0^\infty x^n e^{-sx} dF_{\tilde{x}}(x)\Big|_{s=0}$$

$$= \int_0^\infty x^n dF_{\tilde{x}}(s)$$
$$= E[\tilde{x}^n]. \tag{2.3}$$

EXERCISE 2.14 Prove Theorem 2.3. Let $\tilde{x}$ and $\tilde{y}$ be independent, nonnegative, random variables having Laplace-Stieltjes transforms $F^*_{\tilde{x}}(s)$ and $F^*_{\tilde{y}}(s)$, respectively. Then, the Laplace-Stieltjes transform for the random variable $\tilde{z} = \tilde{x} + \tilde{y}$ is given by the product of $F^*_{\tilde{x}}(s)$ and $F^*_{\tilde{y}}(s)$.

Solution. By the definition of Laplace-Stieltjes transform,

$$\begin{aligned} F^*_{\tilde{z}}(s) &= E[e^{-s\tilde{z}}] = E[e^{-s(\tilde{x}+\tilde{y})}] \\ &= E[e^{-s\tilde{x}}e^{-s\tilde{y}}]. \end{aligned}$$

Now, because $\tilde{x}$ and $\tilde{y}$ are independent, $\tilde{h}_1(\tilde{x})$ and $\tilde{h}_2(\tilde{y})$ are also independent for arbitrary functions $\tilde{h}_1$ and $\tilde{h}_2$. In particular, $e^{-s\tilde{x}}$ and $e^{-s\tilde{y}}$ are independent. Since the expectation of the product of independent random variables is equal to the product of the expectations,

$$E[e^{-s\tilde{x}}e^{-s\tilde{y}}] = E[e^{-s\tilde{x}}]E[e^{-s\tilde{y}}].$$

That is,

$$F^*_{\tilde{z}}(s) = F^*_{\tilde{x}}(s)F^*_{\tilde{y}}(s).$$

EXERCISE 2.15 Let $\tilde{x}$ be an exponentially distributed random variable with parameter λ. Find $F^*_{\tilde{x}}(s)$.

Solution. By definition,

$$F^*_{\tilde{x}}(s) = E\left[e^{-s\tilde{x}}\right] = \int_0^\infty e^{-sx} dF_{\tilde{x}}(x).$$

For $\tilde{x}$ exponential, $dF_{\tilde{x}}(x) = \lambda e^{-\lambda x}dx$, so

$$\begin{aligned} F^*_{\tilde{x}}(s) &= \int_0^\infty e^{-sx}\lambda e^{-\lambda x}dx \\ &= \int_0^\infty \lambda e^{-(s+\lambda)x}dx \\ &= \frac{\lambda}{s+\lambda}\int_0^\infty (s+\lambda)e^{-(s+\lambda)x}dx \\ &= \frac{\lambda}{s+\lambda} \end{aligned}$$

EXERCISE 2.16 Let $\tilde{x}$ be an exponentially distributed random variable with parameter λ. Derive expressions for $E[\tilde{x}]$, $E[\tilde{x}^2]$, and $Var(\tilde{x})$. [*Hint*: Use Laplace transforms.]

Solution. By Theorem 2.2,

$$E[\tilde{x}^n] = (-1)^n \frac{d^n}{ds^n} F^*_{\tilde{x}}(s)\Big|_{s=0},$$

where by Exercise 2.5, $F^*_{\tilde{x}}(s) = \frac{\lambda}{\lambda+s}$. Then

$$\begin{aligned} E[\tilde{x}] &= (-1)\frac{d}{ds}\left(\frac{\lambda}{\lambda+s}\right)\Big|_{s=0} \\ &= (-1)\left[\lambda\left(\frac{1}{\lambda+s}\right)^{-2}\right]\Big|_{s=0} = \frac{1}{\lambda}. \\ E[\tilde{x}^2] &= (-1)^2 \tfrac{d^2}{ds^2}\left(\tfrac{\lambda}{\lambda+s}\right)\Big|_{s=0} \\ &= \left[2\lambda\left(\frac{1}{\lambda+s}\right)^{-3}\right]\Big|_{s=0} = \frac{2}{\lambda^2}. \end{aligned}$$

We may now compute $\text{Var}(\tilde{x})$ as follows:

$$\begin{aligned} \text{Var}(\tilde{x}) &= E[\tilde{x}^2] - E^2[\tilde{x}] \\ &= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \\ &= \tfrac{1}{\lambda}. \end{aligned}$$

Note that for the exponential random variable, the mean and variance are equal.

EXERCISE 2.17 Let $\tilde{x}$ and $\tilde{y}$ be independent exponentially distributed random variables with parameters α and β, respectively.

1. Find the distribution of $\tilde{z} = min\{\tilde{x}, \tilde{y}\}$. [*Hint*: Note that $\tilde{z} = min\{\tilde{x}, \tilde{y}\}$ and $\tilde{z} > z$ means $\tilde{x} > z$ and $\tilde{y} > z$.]
2. Find $P\{\tilde{x} < \tilde{y}\}$.
3. Show that the conditional distribution $F_{\tilde{z}|\tilde{x}<\tilde{y}}(z) = F_{\tilde{z}}(z)$.

Solution.

1. First, note that $P\{\tilde{z} \le z\} = 1 - P\{\tilde{z} > z\}$. As noted in the hint, $\tilde{z} = \min\{\tilde{x},\ \tilde{y}\}$ and $\tilde{z} > z$ means $\tilde{x} > z$ and $\tilde{y} > z$. Therefore,

$$P\{\tilde{z} > z\} = P\{\tilde{x} > z, \tilde{y} > z\}.$$

Since $\tilde{x}$ and $\tilde{y}$ are independent,

$$P\{\tilde{x} > z, \tilde{y} > z\} \quad = P\{\tilde{x} > z\}\, P\{\tilde{y} > z\}$$

$$= e^{-\alpha x} e^{-\beta x} = e^{-(\alpha+\beta)x}.$$

Thus,

$$P\left\{\tilde{z} \le z\right\} = 1 - e^{-(\alpha+\beta)x},$$

so that $\tilde{z}$ is exponentially distributed with parameter $\lambda + \alpha$.

2. We compute the required probability by conditioning on the value of $\tilde{x}$ as follows:

$$\begin{aligned} P\left\{\tilde{x} < \tilde{y}\right\} &= \int_0^\infty P\left\{\tilde{x} < \tilde{y} | \tilde{x} = x\right\} dF_{\tilde{x}}(x) \\ &= \int_0^\infty P\left\{x < \tilde{y}\right\} dF_{\tilde{x}}(x) \\ &= \int_0^\infty e^{-\beta x} \alpha e^{-\alpha x} dx \\ &= \frac{\alpha}{\alpha+\beta} \int_0^\infty (\alpha+\beta) e^{-(\alpha+\beta)x} dx \\ &= \frac{\alpha}{\alpha+\beta}. \end{aligned}$$

3. We begin with the definition of $F_{\tilde{z}|\tilde{x}<\tilde{y}}(z)$. We have

$$F_{\tilde{z}|\tilde{x}<\tilde{y}}(z) = P\left\{\tilde{z} \le z | \tilde{x} < \tilde{y}\right\}.$$

Thus,

$$\begin{aligned} F_{\tilde{z}|\tilde{x}<\tilde{y}}(z) &= 1 - P\left\{\tilde{z} > z | \tilde{x} < \tilde{y}\right\} \\ &= 1 - \frac{P\{\tilde{z} > z, \tilde{x} < \tilde{y}\}}{P\{\tilde{x} < \tilde{y}\}}. \end{aligned}$$

But, since $\tilde{z} = \min\{\tilde{x}, \tilde{y}\}$

$$\begin{aligned} P\left\{\tilde{z} > z, \tilde{x} < \tilde{y}\right\} &= P\left\{\tilde{x} > z, \tilde{x} < \tilde{y}\right\} \\ &= \int_z^\infty P\left\{\tilde{x} < \tilde{y} | \tilde{x} = x\right\} dF_{\tilde{x}}(x) \\ &= \int_z^\infty P\left\{x < \tilde{y}\right\} dF_{\tilde{x}}(x) \\ &= \int_z^\infty e^{-\beta x} \alpha e^{-\alpha x} dx \\ &= \frac{\alpha}{\alpha+\beta} \int_z^\infty (\alpha+\beta) e^{-(\alpha+\beta)x} dx \\ &= \frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)z}. \end{aligned}$$

Thus, upon using th eresult of part (ii), we find

$$P\left\{\tilde{z} > z | \tilde{x} < \tilde{y}\right\} = e^{-(\alpha+\beta)z}$$

so that

$$F_{\tilde{z}|\tilde{x}<\tilde{y}}(z) = 1 - e^{-(\alpha+\beta)z} = F_{\tilde{z}}(z),$$

and the proof is complete.

EXERCISE 2.18 Suppose Albert and Betsy run a race repeatedly. The time required for Albert to complete the race, $\tilde{a}$, is exponentially distributed with parameter α and the time required for Betsy to complete, $\tilde{b}$, is exponentially distributed with parameter β. Let $\tilde{n}_b$ denote the number of times Betsy wins before Albert wins his first race. Find $P\{\tilde{n}_b = n\}$ for $n \ge 0$.

Solution. First consider $P\{\tilde{n}_b = 0\}$, the probability that Albert wins the very first race. Since Albert's time must be less than Betsy's for him to win, this probability is $P\left\{\tilde{a} < \tilde{b}\right\}$. By Exercise 2.7 (iii), $P\left\{\tilde{a} < \tilde{b}\right\} = \frac{\alpha}{\alpha+\beta}$. Now consider $P\{\tilde{n}_b = 1\}$, the probability that Betsy wins one race and then Albert wins the second. Both $\tilde{a}$ and $\tilde{b}$ are exponential random variables; hence they are memoryless. That is, during the second race they don't 'remember' what happened during the first race. Thus the times for each race for each runner are independent of each other. Let $\tilde{a}_i$, $\tilde{b}_i$ denote the races times of Albert and Betsy, respectively, on the i^{th} race. Then

$$\begin{aligned} P\{\tilde{n}_b = 1\} &= P\left\{\tilde{a}_1 > \tilde{b}_1, \tilde{a}_2 < \tilde{b}_2\right\} \\ &= P\left\{\tilde{a}_1 > \tilde{b}_1\right\} P\left\{\tilde{a}_2 < \tilde{b}_2\right\} \\ &= \tfrac{\beta}{\alpha+\beta} \cdot \tfrac{\alpha}{\alpha+\beta}. \end{aligned}$$

Repeating this argument for arbitrary n, we see that

$$\begin{aligned} P\{\tilde{n}_b = n\} &= P\left\{\tilde{a}_1 > \tilde{b}_1, \tilde{a}_2 > \tilde{b}_2, \cdots, \tilde{a}_n > \tilde{b}_n, \tilde{a}_{n+1} < \tilde{b}_{n+1}\right\} \\ &= P\left\{\tilde{a}_1 > \tilde{b}_1\right\} P\left\{\tilde{a}_2 > \tilde{b}_2\right\} \cdots P\left\{\tilde{a}_n > \tilde{b}_n\right\} P\left\{\tilde{a}_{n+1} < \tilde{b}_{n+1}\right\} \\ &= \left(\tfrac{\beta}{\alpha+\beta}\right)^n \tfrac{\alpha}{\alpha+\beta}. \end{aligned}$$

That is, $\tilde{n}_b$ is geometric with parameter $\frac{\alpha}{\alpha+\beta}$.

EXERCISE 2.19 Let $\{\tilde{x}_i, i = 1, 2, \ldots\}$ be a sequence of exponentially distributed random variables and let $\tilde{n}$ be a geometrically distributed random variable with parameter p, independent of $\{\tilde{x}_i, i = 1, 2, \ldots\}$. Let

$$\tilde{y} = \sum_{i=1}^{\tilde{n}} \tilde{x}_i.$$

Show that $\tilde{y}$ has the exponential distribution with parameter $p\alpha$.

Solution. First we prove a lemma. *Lemma*: If $\tilde{x}_i, i = 1, 2, \cdots$ are as in the statement of the exercise and n is a $fixed$ integer, then $\tilde{y} = \sum_{i=1}^{n} \tilde{x}_i$ has the gamma distribution with parameter (n, α).

Proof: If $n = 1$, the result holds trivially. So suppose the result holds for $n = N - 1$, $N \geq 2$. If $f_{\sum_{i=1}^{N-1} \tilde{x}_i}(y)$ is the probability density function of $\tilde{y}$, then

$$f_{\sum_{i=1}^{N-1} \tilde{x}_i}(y) = \alpha e^{-\alpha y} \frac{(\alpha y)^{N-2}}{(N-2)!}.$$

Using the fact that $\sum_{i=1}^{N} \tilde{x}_i = \sum_{i=1}^{N-1} + \tilde{x}_N$ leads to

$$f_{\sum_{i=1}^{N} \tilde{x}_i}(y) \quad = \int_0^\infty f_{\tilde{x}_N}(y - t) f_{\sum_{i=1}^{N-1} \tilde{x}_i}(t) dt$$

$$= \int_0^\infty \alpha e^{-\alpha(y-t)} \alpha e^{-\alpha t} \frac{(\alpha t)^{N-2}}{(N-2)!} dt$$
$$= \alpha e^{-\alpha y} \frac{(\alpha y)^{N-1}}{(N-1)!}.$$

This proves the Lemma. To complete the exercise, we condition the probability density function of $\tilde{y}$ on the value of $\tilde{n}$. Recall that here $\tilde{n}$ is a random variable and is independent of $\tilde{x}_i, i = 1, 2, \cdots$. Thus,

$$\begin{aligned} f_{\tilde{y}}(y) &= f_{\sum_{i=1}^{\tilde{n}} \tilde{x}_i}(y) \\ &= \sum_{n=1}^{\infty} f_{\sum_{i=1}^{n} \tilde{x}_i}(y) P\{\tilde{n} = n\} \\ &= \sum_{n=1}^{\infty} \alpha e^{-\alpha y} \frac{(\alpha y)^{N-1}}{(N-1)!} p(1-p)^{n-1} \\ &= p\alpha e^{-\alpha y} \sum_{n=0}^{\infty} \frac{(\alpha y(1-p))^n}{n!} \\ &= p\alpha e^{-\alpha y} e^{\alpha y(1-p)} = p\alpha e^{-p\alpha y}. \end{aligned}$$

EXERCISE 2.20 This exercise is intended to reinforce the meaning of Property 2 of exponential random variables. Let $\tilde{x}$ and $\tilde{y}$ denote the two independent exponential random variables with rates 1 and 2, respectively, and define $\tilde{z} = min\{\tilde{x}, \tilde{y}\}$. Using a spreadsheet (or a computer programming language), generate a sequence of 100 variables for each of the random variables. Denote the ith variate for $\tilde{x}$ and $\tilde{y}$ by x_i and y_i, respectively, and set $\tilde{z} = min\{\tilde{x}, \tilde{y}\}$ for $i = 1, 2, \ldots, 100$.

Let n denote the number of values of i such that $x_i < y_i$, let i_j denote the $j$the such value and define $w_j = z_i$, for $j = 1, 2, \ldots, n$. Compute the sample averages for the variates; that is compute $\bar{x} = (1/100)\sum_{i=1}^{100} x_i$, $\bar{y} = (1/100)\sum_{i=1}^{100} y_i$, $\bar{z} = (1/100)\sum_{i=1}^{100} z_i$, and $\bar{w} = (1/100)\sum_{j=1}^{100} w_j$. Compare the results. Is $\bar{w}$ closer to $\bar{x}$ or $\bar{z}$?

Now give an intuitive explanation for the statement, “It is tempting to conclude that if one knows the state change was caused by the event having its interevent time drawn from the distribution $F_{\tilde{x}}(x)$, then the time to state change is exponentially distributed with parameter α, but this is false.”

Solution. The following sequences for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, and $\tilde{w}$ were generated using a spreadsheet package:

n	x_n	y_n	z_n	w_n
1	1.84397193	0.16714174	0.16714174	
2	1.24667042	0.18449179	0.18449179	
3	2.2392642	0.07468417	0.07468417	
4	0.7575555	0.08376639	0.08376639	
5	1.42970352	0.39268839	0.39268839	
6	4.01878152	1.34361333	1.34361333	
7	3.43923471	0.08348682	0.08348682	
8	1.1702069	0.64094628	0.64094628	
9	2.46251518	1.1929263	1.1929263	
10	2.89695466	0.74196329	0.74196329	
11	0.65195416	0.11265824	0.11265824	
12	0.38827964	0.77950928	0.38827964	0.38827964
13	0.13928522	0.01412904	0.01412904	
14	0.56052792	0.31830474	0.31830474	
15	1.98337703	1.1304179	1.1304179	
16	0.58175933	0.61897506	0.58175933	0.58175933
17	0.4729621	0.32353766	0.32353766	
18	1.18397153	0.1701992	0.1701992	
19	0.27687641	0.15029115	0.15029115	
20	0.04403085	0.21006062	0.04403085	0.04403085
21	0.1339849	0.25982677	0.1339849	0.1339849
22	0.93645887	0.2989296	0.2989296	
23	0.58498602	0.2435585	0.2435585	
24	1.20431873	0.05653863	0.05653863	
25	1.28216752	2.32710226	1.28216752	1.28216752
26	0.36686726	0.08247806	0.08247806	
27	0.73515498	1.20688382	0.73515498	0.73515498
28	0.29300628	0.21725415	0.21725415	
29	1.41245136	0.06286597	0.06286597	
30	1.43283182	1.14791652	1.14791652	
31	0.45688209	0.07873111	0.07873111	
32	3.43212736	0.7438511	0.7438511	
33	6.08640858	0.53356533	0.53356533	
34	2.75348525	0.63894337	0.63894337	
35	0.63137432	0.19140255	0.19140255	
36	0.17315955	0.64144138	0.17315955	0.17315955
37	0.46050301	0.53641231	0.46050301	0.46050301
38	0.54172938	0.27631063	0.27631063	
39	1.81441406	0.55841417	0.55841417	
40	0.95500193	0.83148476	0.83148476	

n	x_n	y_n	z_n	w_n
41	0.8008909	0.02778907	0.02778907	
42	0.02385716	0.67811659	0.02385716	0.02385716
43	0.4176623	0.06398327	0.06398327	
44	0.44562071	1.30356611	0.44562071	0.44562071
45	0.93021164	0.58957361	0.58957361	
46	0.28880168	0.13168134	0.13168134	
47	2.15051176	0.42603754	0.42603754	
48	0.38888728	0.33542262	0.33542262	
49	0.04647351	0.13146297	0.04647351	0.04647351
50	0.24678196	1.08658913	0.24678196	0.24678196
51	1.29128385	0.12054774	0.12054774	
52	1.19616038	0.58372847	0.58372847	
53	0.70889554	0.50033913	0.50033913	
54	0.03148097	0.19271322	0.03148097	0.03148097
55	2.00060148	0.73581018	0.73581018	
56	2.37414692	0.05800158	0.05800158	
57	0.5614103	1.02321168	0.5614103	0.5614103
58	0.28880168	0.48428777	0.28880168	0.28880168
59	2.10829625	0.55708733	0.55708733	
60	0.00647539	0.28160211	0.00647539	0.00647539
61	1.21198004	0.42532257	0.42532257	
62	0.25354251	0.16509961	0.16509961	
63	0.25010753	0.31332586	0.25010753	0.25010753
64	0.42281911	1.21254329	0.42281911	0.42281911
65	0.43420203	0.08457892	0.08457892	
66	2.10942768	0.48084293	0.48084293	
67	0.57224943	0.73587666	0.57224943	0.57224943
68	1.4642691	0.36727522	0.36727522	
69	0.92824082	0.34664989	0.34664989	
70	0.36374508	0.83064034	0.36374508	0.36374508
71	0.26106201	0.31306894	0.26106201	0.26106201
72	1.86907458	0.1030878	0.1030878	
73	0.56825514	0.27826342	0.27826342	
74	1.15881452	0.23169248	0.23169248	
75	0.01464839	0.00071004	0.00071004	
76	0.52810458	0.05496071	0.05496071	
77	0.87417563	0.43087249	0.43087249	
78	0.03337237	0.20835641	0.03337237	0.03337237
79	1.75585187	0.85560022	0.85560022	
80	0.79283399	1.17958999	0.79283399	0.79283399

n	x_n	y_n	z_n	w_n
81	0.07335002	2.02886531	0.07335002	0.07335002
82	2.25514443	0.58648222	0.58648222	
83	1.81272977	0.14403435	0.14403435	
84	0.08817172	0.78172674	0.08817172	0.08817172
85	0.66163598	0.30661797	0.30661797	
86	0.74250208	0.40806585	0.40806585	
87	0.44464432	0.08091491	0.08091491	
88	2.93930994	0.28810271	0.28810271	
89	0.06949872	0.08515753	0.06949872	0.06949872
90	0.62340183	0.10592748	0.10592748	
91	0.9035322	0.9430666	0.9035322	0.9035322
92	0.13090162	0.06881894	0.06881894	
93	0.27739983	0.18747991	0.18747991	
94	0.68718355	0.23571032	0.23571032	
95	0.37903079	0.38083248	0.37903079	0.37903079
96	0.25523489	0.05314163	0.05314163	
97	0.26009176	0.68434451	0.26009176	0.26009176
98	0.41854311	0.90496445	0.41854311	0.41854311
99	0.00704379	0.21641837	0.00704379	0.00704379
100	0.14198992	0.01055528	0.01055528	
	98.8205643	46.0988392	33.8636937	10.3453931
	Mean of $x_i's$	Mean of $y_i's$	Mean of $z_i's$	Mean of $w_i's$
	0.98820564	0.46098839	0.33863694	0.33372236

Solution. Observe that the sampled means of $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are close to their expected values: 0.98820564 as compared to the expected value of 1.0 for $\tilde{x}$; 0.46098839 as compared to the expected value of 0.5 for $\tilde{y}$; and 0.33863694 for $\tilde{z}$. (Recall that if α and β are the rates for $\tilde{x}$ and $\tilde{y}$, respectively, then $E[\tilde{z}] = \frac{1}{\alpha+\beta}$. In this exercise $\alpha = 1$ and $\beta = 2$; hence $E[\tilde{z}] = 0.33333333$.) Note that the sampled mean of $\tilde{w}$ is very close to the sampled mean of $\tilde{z}$.

Now, the w_i are selected from the samples of $\tilde{z}$. Once the samples of $\tilde{z}$ are chosen, it does not matter whether the original variates came from sampling the distribution of $\tilde{x}$ or of $\tilde{y}$. Hence the same result would hold if $\tilde{w}$ represented those y_i that were less than their corresponding x_i values. The distribution of $\tilde{w}$, and therefore of $\tilde{z}$, does not depend on whether $\alpha < \beta$ or $\beta < \alpha$. Thus $\tilde{z}$ is not a representation of $\tilde{x}$, and so will not be exponential with rate α.

EXERCISE 2.21 Define counting processes which you think have the following properties:

1. independent but not stationary increments,
2. stationary but not independent increments,
3. neither stationary nor independent increments, and
4. both stationary and independent increments.

What would you think would be the properties of the process which counts the number of passengers which arrive to an airport by June 30 of a given year if time zero is defined to be midnight, December 31 of the previous year?

Solution.

1. Independent, not stationary: Count the number of customers who enter a store over an hour-long period. Assume the store has infinte capacity. The number of customers that enter during one 1-hour time interval likely will be independent of the number that enter during a non-overlapping 1-hour period. However, because of "peak" hours and slow times, this counting process is not likely to have stationary increments.

2. Stationary, not independent: Count the number of failures of a light bulb in any small time interval. Let the failure rate be exponential, so it is memoryless. Hence the probability it fails in one fixed-length interval is the same as in any other. This process does not have independent increments, however. If the light burns out in one increment, then there is a 100% probability it will not burn out in the next, since it can have only one failure.

3. Neither stationary nor independent: Count the Earth's population. Certainly this is not stationary; it is generally agreed that the birth rate increases with time. Nor is this process independent. The birth of every individual alive today was dependent upon the births of his or her parents. And their births were dependent upon the births of their parents, and so on.

4. Both stationary and independent: Count the number of phone calls through an operator's switchboard between 3:00 am and 5:00 am in Watseka, Illinois (a small farming community). Then the process can be modeled using stationary and independent increments.

For the final part of this exercise, assume the airport has an infinite airplane capacity. Then this process would likely have independent increments. Passengers usually will not be interested in how many people arrived before them;

they are concerned in their own, current, flight. Furthermore, this counting process is not likely to have stationary increments since there will be peak and slow times for flying. (For example: “red eye” flights are during slow times.)

EXERCISE 2.22 Show that $E[\tilde{n}(t+s) - \tilde{n}(s)] = \lambda t$ if $\{\tilde{n}(t), t > 0\}$ is a Poisson process with rate λ.

Solution.

$$\begin{aligned}
E\left[\tilde{n}(t+s) - \tilde{n}(s)\right] &= \sum_{n=0}^{\infty} nP\left\{\tilde{n}(t+s) - \tilde{n}(s) = n\right\} \\
&= \sum_{n=0}^{\infty} n\left[\frac{(\lambda t)^n e^{-\lambda t}}{n!}\right] \\
&= \sum_{n=1}^{\infty} n\left[\frac{(\lambda t)^n e^{-\lambda t}}{n!}\right] \\
&= \sum_{n=1}^{\infty} n\left[\frac{(\lambda t)^n e^{-\lambda t}}{n-1!}\right] \\
&= (\lambda t)\, e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{n-1!} \\
&= (\lambda t)\, e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} \\
&= (\lambda t)\, e^{-\lambda t} e^{\lambda t} \\
&= \lambda t
\end{aligned}$$

EXERCISE 2.23 For each of the following functions, determine whether the function is $o(h)$ or not. Your determination should be in the form of a formal proof.

1. $f(t) = t$,
2. $f(t) = t^2$,
3. $f(t) = t^{\frac{1}{2}}$,
4. $f(t) = e^{-at}$ for $a, t > 0$
5. $f(t) = te^{-at}$ for $a, t > 0$

Solution.

1. $\lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$. Therefore $f(t) = t$ is not $o(h)$.
2. $\lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$. Therefore $f(t) = t^2$ is $o(h)$.
3. $\lim_{h\to 0} \frac{h^{\frac{1}{2}}}{h} = \lim_{h\to 0} h^{-\frac{1}{2}} = \infty$. Therefore $f(t) = t^{\frac{1}{2}}$ is not $o(h)$.

4. $\lim_{h\to 0} \frac{e^{-ah}}{h} = \lim_{h\to 0} \frac{e^{-ah}}{h} = \infty$, since $e^{-ah} \to 1$ as $h \to 0$. Therefore $f(t) = e^{-at}$ for $a, t > 0$ is not $o(h)$.

5. $\lim_{h\to 0} \frac{he^{-ah}}{h} = \lim_{h\to 0} e^{-ah} = e^0 = 1$, Therefore $f(t) = te^{-at}$ for $a, t > 0$ is not $o(h)$.

EXERCISE 2.24 Suppose that $f(t)$ and $g(t)$ are both $o(h)$. Determine whether each of the following functions is $o(h)$.

1. $s(t) = f(t) + g(t)$
2. $d(t) = f(t) - g(t)$
3. $p(t) = f(t)g(t)$
4. $q(t) = f(t)/g(t)$
5. $i(t) = \int_0^t f(x)\,dx$

Solution.

1. $s(t) = f(t) + g(t)$ is $o(h)$ by the additive property of limits.

2. $d(t) = f(t) - g(t)$ is $o(h)$ by the additive and scalar multiplication properties of limits. (Here, the scalar is -1.)

   We now prove an itermediate result. *Lemma*: If $f(t)$ is $o(h)$, then $f(0) = 0$.

   proof: Clearly, by the definition of $o(h)$, $f(t)$ is continuous. So suppose that $f(0)$ is not equal to 0, say $f(0) = a, a > 0$. Then

$$\lim_{h\to 0} \frac{f(h)}{h} = \infty$$

   since $a > 0$. But $f(t)$ is $o(h)$, so that the required limit is zero. Hence it must be that $a = 0$. So if $f(t)$ is $o(h)$ then $f(0) = 0$.

3. By the Lemma, $\lim_{h\to 0} g(h) = \lim_{h\to 0} f(h) = 0$. So

$$\lim_{h\to 0} \frac{f(h)g(h)}{h} = \left(\lim_{h\to 0} \frac{f(h)}{h}\right)\left(\lim_{h\to 0} g(h)\right) = 0 \cdot 0 = 0$$

   So $p(t) = f(t)g(t)$ is $o(h)$.

4. Since $lim_{h\to 0} f(h)$ and $lim_{h\to 0} g(h)$ exist independently,

$$\lim_{h\to 0} \frac{q(h)}{h} = \frac{\lim_{h\to 0} f(h)}{\lim_{h\to 0} g(h)h} = \frac{0}{0}$$

The limit is then indeterminant, and whether or not $q(h)$ is $o(h)$ depends upon the specific form of $f(h)$ and $g(h)$.That is, we now need to know what $f(t)$, $g(t)$ are to proceed further. For example, if $f(t) = t^4, g(t) = t^2$, then

$$\lim_{h\to 0} \frac{q(h)}{h} = \lim_{h\to 0} \frac{h^4}{h^2 h} = \lim_{h\to 0} h = 0$$

making $q(t)$ an $o(h)$ function. If, on the other hand, $f(t) = t^2$, $g(t) = t^4$ then

$$\lim_{h\to 0} \frac{q(h)}{h} = \lim_{h\to 0} \frac{h^2}{h^4 h} = \lim_{h\to 0} \frac{1}{h^3} = \infty$$

and here $q(t)$ is not $o(h)$.

5. Since $f(t)$ is continuous, there exists some n in $[0, h]$ such that

$$\int_0^h f(x)\, dx = h \cdot f(n)$$

Therefore

$$\begin{aligned} lim_{h\to 0}\frac{i(h)}{h} &= \lim_{h\to 0} \tfrac{hf(n)}{h} \\ &= \lim_{h\to 0} f(n), \quad n \text{ in } [0,\text{h}] \\ &= 0 \end{aligned}$$

by the Lemma above. Hence $i(t) = \int_0^t f(x)\, dx$ is $o(h)$.

EXERCISE 2.25 Show that definition 1 of the Poisson process implies definition 2 of the Poisson process.

Solution. Denote Property (j) of Poisson process Definition n by $(j)_n$.

1 $(i)_2$: Immediate.

2 $(ii)_2$: Property $(ii)_1$ gives us independent increments; it remains to show they are also stationary. By $(iii)_1$, we see that $P\{\tilde{n}(t + s) - \tilde{n}(s) = n\}$ is independent of s. This defines stationary increments.

3 $(iii)_2$: By $(iii)_1$,

$$\begin{aligned} P\left\{\tilde{n}(h) = 1\right\} &= \tfrac{\lambda h^1 e^{-\lambda h}}{1!} \\ &= \lambda h e^{-\lambda h} \\ &= \lambda h e^{-\lambda h} \\ &= \lambda h e^{-\lambda h} + \lambda h - \lambda h \\ &= \lambda h \left(e^{-\lambda h} - 1\right) + \lambda h \\ &= g(h) + \lambda h \end{aligned}$$

Note that

$$\begin{aligned}\lim_{h\to 0}\frac{g(h)}{h} &= \lim_{h\to 0}\tfrac{\lambda h\left(e^{-\lambda h}-1\right)}{h}\\ &= \lambda\lim_{h\to 0}\left(e^{-\lambda h}-1\right)\\ &= \lambda\left(1-1\right)\\ &= 0\end{aligned}$$

by Exercise 2.15 (v). So $g(h)$ is $o(h)$. Thus

$$\begin{aligned}P\left\{\tilde{n}(h)=1\right\} &= \lambda h + g(h)\\ &= \lambda h + o(h)\end{aligned}$$

And so $(iii)_2$ holds.

4 $(iv)_2$: By $(iii)_1$,

$$P\left\{\tilde{n}(h)\geq 2\right\} = \sum_{n=2}^{\infty}\frac{(\lambda h)^n\, e^{-\lambda h}}{n!}$$

Let $n \geq 2$ be arbitrary. Then

$$\begin{aligned}\lim_{h\to 0}\frac{(\lambda h)^n\, e^{-\lambda h}}{n!} &= \lim_{h\to 0}\lambda^n e^{-\lambda h}h^{n-1}\\ &= \lambda^n\lim_{h\to 0}e^{-\lambda h}h^{n-1}\\ &= \lambda^n\cdot 0\\ &= 0\end{aligned}$$

since $\lim_{h\to 0}e^{\lambda h} = 1$ and $\lim_{h\to 0}h^{n-1} = 0, \quad n \geq 2$. And by Exercise 2.16 (i), sums of $o(h)$ functions are $o(h)$. So

$$\sum_{n=2}^{\infty}\frac{(\lambda h)^n\, e^{\lambda h}}{n!}$$

is $o(h)$. This shows $(iv)_2$.

EXERCISE 2.26 Show that Definition 2 of the Poisson process implies Definition 1 of the Poisson process. [*Hint*: After satisfying the first two properties of definition 1, establish that $P_0(t) = \exp\{-\lambda t\}$ where $P_n(t) = P\{\tilde{n}(t) = n\}$ and then prove the validity of Property 3 of Definition 1 by induction.]

Solution. Denote propery (j) of Poisson process definition n by $(j)_n$.

1 $(i)_1$: Immediate.

2 $(ii)_1$: Immediate by the first half of $(ii)_2$.

3 $(iii)_1$: Define

$$P_n(t) = P\{\tilde{n}(t) = n\}, \qquad n \geq 0$$

with

$$P_n(t) = 0, \qquad n < 0.$$

Then $P_n(t+h) = P\left\{\tilde{n}(t+h) = n\right\}$. Conditioning upon the value of $\tilde{n}(t)$, we find

$$\begin{aligned} P_n(t+h) &= \textstyle\sum_{k=0}^{n} P\left\{\tilde{n}(t+h) = n | \tilde{n}(t) = k\right\} P\left\{\tilde{n}(t) = k\right\} \\ &= \textstyle\sum_{k=0}^{n} P\left\{\tilde{n}(t+h) - \tilde{n}(t) = n-k\right\} P_k(t) \\ &= P\left\{\tilde{n}(t+h) - \tilde{n}(t) = 0\right\} P_n(t) \\ &\quad + P\left\{\tilde{n}(t+h) - \tilde{n}(t) = 1\right\} P_{n-1}(t) \\ &\quad + P\left\{\tilde{n}(t+h) - \tilde{n}(t) = 2\right\} P_{n-2}(t) \\ &\quad + \cdots \\ &\quad + P\left\{\tilde{n}(t+h) - \tilde{n}(t) = n\right\} P_0(t) \end{aligned}$$

But by $(iii)_2$, $(iv)_2$,

$$\begin{aligned} P\left\{\tilde{n}(t+h) - \tilde{n}(t) = 1\right\} &= \lambda h + o(h), \quad \text{and} \\ P\left\{\tilde{n}(t+h) - \tilde{n}(t) \geq 2\right\} &= o(h). \end{aligned}$$

Thus,

$$\begin{aligned} P\left\{\tilde{n}(t+h) - \tilde{n}(t) = 0\right\} &= P\left\{\tilde{n}(h) = 0\right\} \\ &= 1 - \left[P\left\{\tilde{n}(h) = 1\right\} + P\left\{\tilde{n}(h) \geq 2\right\}\right] \\ &= 1 - \left[(\lambda h + o(h)) + o(h)\right] \\ &= 1 - \lambda h + o(h). \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{k=2}^{n} P\left\{\tilde{n}(t+h) - \tilde{n}(t) = k\right\} P_{n-k}(t) &= \textstyle\sum_{k=2}^{n} o(h) P_{n-k}(t) \leq o(h) \cdot 1 \\ &= o(h) \end{aligned}$$

since $P_{n-k}(t)$ is a scalar for all $k \leq n$, and any scalar multiplied by a $o(h)$ function is still $o(h)$. (Think of the limit definition of $o(h)$ to see why this is true.) Thus

$$P_n(t+h) = P_n(t)\left[1 - \lambda h + o(h)\right] + P_{n-1}(t)\left[\lambda h + o(h)\right] + o(h).$$

Rearranging the terms,

$$P_n(t+h) - P_n(t) = -\lambda h P_n(t) + \lambda h P_{n-1}(t) + o(h).$$

We then divide by h and take the limit as $h \to 0$

$$\lim_{h\to 0}\frac{P_n(t+h)-P_n(t)}{h}=\lim_{h\to 0}\frac{-\lambda h P_n(t)}{h}+\lim_{h\to 0}\frac{\lambda h P_{n-1}(t)}{h}+\lim_{h\to 0}\frac{o(h)}{h},$$

so that

$$\begin{aligned}P_n^{'}(t) &= -\lambda P_n(t)+\lambda P_{n-1}(t)+0\\ &= -\lambda P_n(t)+\lambda P_{n-1}(t), \qquad n\geq 1.\end{aligned} \tag{2.26.1}$$

For $n=0$,

$$P_0^{'}(t)=-\lambda P_0(t).$$

i.e.,

$$P_0^{'}(t)+\lambda P_0(t)=0.$$

Observe that

$$\frac{d}{dt}\left[e^{\lambda t}P_0(t)\right]=0,$$

leads to

$$e^{\lambda t}P_0^{'}(t)+\lambda e^{\lambda t}P_0(t)=0$$

and

$$P_0(t)=Ke^{-\lambda t}. \tag{2.26.2}$$

But by $(i)_2$, $P_0(0)=1$. Using this result in (2.26.2) we find $K=1$, so that

$$P_0(t)=e^{\lambda t}.$$

Let T denote the truth set for the Proposition that

$$P_n(t)=\frac{(\lambda t)^n e^{\lambda t}}{n!}.$$

Then $0\in T$. Now suppose $n-1\in T$. That is,

$$P_{n-1}(t)=\left[\frac{(\lambda h)^{n-1}e^{\lambda h}}{(n-1)!}\right]. \tag{2.26.3}$$

By (2.26.1) from above,

$$P_n^{'}(t)=-\lambda P_n(t)+\lambda P_{n-1}(t)$$

Multiplying by $e^{-\lambda t}$ and rearranging terms:

$$e^{-\lambda t}\left[P_n^{'}(t)+\lambda P_n(t)\right]=\lambda e^{\lambda t}P_{n-1}(t).$$

Hence,

$$\frac{d}{dt}e^{\lambda t}P_n(t) = \lambda e^{\lambda t}P_{n-1}(t)$$

Since $n-1 \in T$, we have by (2.26.3),

$$\begin{aligned} &= \lambda e^{\lambda t}\left[\tfrac{(\lambda t)^{n-1}e^{-\lambda t}}{(n-1)!}\right] \\ &= \tfrac{\lambda(\lambda t)^{n-1}}{(n-1)!} \\ &= \tfrac{\lambda^n t^{(n-1)}}{(n-1)!} \end{aligned}$$

Integrating both sides with respect to t:

$$\begin{aligned} e^{-\lambda t}P_n(t) &= \tfrac{\lambda^n t^n}{n(n-1)!} + c \\ &= \tfrac{(\lambda t)^n}{n!} + c. \end{aligned}$$

Thus,

$$P_n(t) = e^{-\lambda t}\left[\frac{(\lambda t)^n}{n!} + c\right]$$

But by $(i)_2$, $\tilde{n}(0) = 0$. So for $n > 0$, $P_n(0) = 0$. Therefore,

$$\begin{aligned} P_n(0) &= e^{-\lambda(0)}\left[\tfrac{(\lambda\cdot 0)^n}{n!} + c\right] \\ &= 1\,(c) = 0. \end{aligned}$$

That is,

$$P_n(t) = \frac{(\lambda t)^n e^{\lambda t}}{n!}.$$

Thus, $n \in T$, the Proposition is proved, and all properties of Definition 1 are satisfied.

EXERCISE 2.27 Show that the sequence of interarrival times for a Poisson process with rate λ forms a set of mutually iid exponential random variables with parameter λ.

Solution. Recall that if $\{\tilde{n}(t), t \geq 0\}$ is a Poisson process with rate λ, then for all s,

$$P\{\tilde{n}(t+s) - \tilde{n}(s) = 0\} = P\{\tilde{n}(t) = 0\} = e^{-\lambda t}$$

Now, in order for no events to have occurred by time t, the time of the first event must be greater than t. That is,

$$P\{\tilde{t}_1 > t\} = P\{\tilde{n}(t) = 0\} = e^{-\lambda t}$$

This is equivalent to

$$P\{\tilde{t}_1 \leq t\} = 1 - e^{-\lambda t}$$

Observe that this is the cumulative distribution function for an exponential random variable with parameter λ. Since the c.d.f. is unique, $\tilde{t}_1$ is exponentially distributed with rate λ.

Note that the second interarrival time begins at the end of the first interarrival time. Furthermore, the process has stationary and independent increments so that $\tilde{t}_2$ has the same distribution as $\tilde{t}_1$ and is independent of $\tilde{t}_1$.

Repeating these arguments for $n \geq 3$, we see that $\{\tilde{t}_1, \tilde{t}_2, \tilde{t}_3, \cdots\}$ are independent exponential variables with parameter λ.

EXERCISE 2.28 Show that

$$\frac{d}{dt}P\{\tilde{s}_n \leq t\} = \frac{\lambda(\lambda t)^{n-1}e^{-\lambda t}}{(n-1)!}.$$

[*Hint*: Start by noting $\tilde{s}_n \leq t \Longleftrightarrow \tilde{n}(t) \geq n$].

Solution. First note that $\tilde{s}_n \leq t$ if and only if $\tilde{n}(t) \geq n$. Hence $\tilde{n}(t)$ a Poisson process that

$$P\left\{\tilde{s}_n \leq t\right\} = P\left\{\tilde{n}(t) \geq n\right\} = \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!},$$

since $\{\tilde{n}(t), t \geq 0\}$ is a Poisson process with rate λ. But this implies

$$\begin{aligned}\frac{d}{dt}P\left\{\tilde{s}_n \leq t\right\} &= \tfrac{d}{dt}\textstyle\sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\\ &= \textstyle\sum_{k=n}^{\infty} \frac{d}{dt}\left[\frac{(\lambda t)^k e^{-\lambda t}}{k!}\right]\\ &= \textstyle\sum_{k=n}^{\infty} \left[\frac{k\lambda^k t^{k-1} e^{-\lambda t}}{k!} - \frac{\lambda(\lambda t)^k e^{-\lambda t}}{k!}\right]\\ &= \textstyle\sum_{k=n}^{\infty} \left[\frac{k\lambda^k t^{k-1} e^{-\lambda t}}{k!}\right] - \sum_{k=n}^{\infty} \left[\frac{\lambda(\lambda t)^k e^{-\lambda t}}{k!}\right]\\ &= \textstyle\sum_{k=n}^{\infty} \frac{\lambda(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!} - \sum_{k=n}^{\infty} \frac{\lambda(\lambda t)^k e^{-\lambda t}}{k!}.\end{aligned}$$

Observe that we may separate the first summation as

$$\begin{aligned}\sum_{k=n}^{\infty}\left[\frac{k\lambda^k t^{k-1} e^{-\lambda t}}{k!}\right] &= \tfrac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}\\ &\quad + \textstyle\sum_{k=n+1}^{\infty} \frac{\lambda(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!} - \sum_{k=n}^{\infty} \frac{\lambda(\lambda t)^k e^{-\lambda t}}{k!}.\end{aligned}$$

Thus, upon reindexing the first summation,

$$\begin{aligned}\frac{d}{dt}P\left\{\tilde{s}_n \leq t\right\} &= \tfrac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!} + \textstyle\sum_{k=n}^{\infty} \frac{\lambda(\lambda t)^k e^{-\lambda t}}{k!} - \sum_{k=n}^{\infty} \frac{\lambda(\lambda t)^k e^{-\lambda t}}{k!}\\ &= \tfrac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}\end{aligned}$$

EXERCISE 2.29 Show that Definition 3 of the Poisson process implies Definition 1 of the Poisson process.

Solution.

(i) Clearly, since $\tilde{n}(0) = \max\{n : \tilde{s}_n \leq 0\}$, $\tilde{n}(0) = 0$.

(ii) This follows immediately from the independence of $\{\tilde{t}_i, i \geq 1\}$. Furthermore, because these are exponential random variables and thus are memoryless, $\tilde{n}$ has stationary increments.

(iii) Using Exercise 2.20, note that

$$\begin{aligned} P\{\tilde{n}(t) = n\} &= P\{\tilde{n}(t) \geq n\} + P\{\tilde{n}(t) \geq n+1\} \\ &= P\{\tilde{s}_n \leq t\} - P\{\tilde{s}_{n+1} \leq t\}. \\ \frac{d}{dt}P\{\tilde{n}(t) = n\} &= \tfrac{\lambda(\lambda t)^{n-1}e^{-\lambda t}}{(n-1)!} - \tfrac{\lambda(\lambda t)^n e^{-\lambda t}}{n!} \end{aligned}$$

But the right-hand side is:

$$\frac{d}{dt}\left[\frac{(\lambda t)^n e^{-\lambda t}}{n!}\right].$$

Therefore

$$P\{\tilde{n}(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

and the proof is complete.

EXERCISE 2.30 Let $\tilde{n}_1$ and $\tilde{n}_2$ be independent Poisson random variables with rates α and β, respectively. Define $\tilde{n} = \tilde{n}_1 + \tilde{n}_2$. Show that $\tilde{n}$ has the Poisson distribution with rate $\alpha + \beta$. Using this result, prove Property 1 of the Poisson process.

Solution. From the definition of the Poisson distribution,

$$P\{\tilde{n}_1 = n\} = \frac{\alpha^n e^{-n}}{n!}$$

and

$$P\{\tilde{n}_2 = n\} = \frac{\beta^n e^{-n}}{n!}$$

Now, condition on the value of $\tilde{n}_2$ to find:

$$\begin{aligned} P\{\tilde{n} = n\} &= \textstyle\sum_{k=0}^{n} P\left[\tilde{n}_1 + \tilde{n}_2 = n, \middle| \tilde{n}_2 = k\right] P\{\tilde{n}_2 = k\} \\ &= \textstyle\sum_{k=0}^{n} P\{\tilde{n}_1 + k = n\} P\{\tilde{n}_2 = k\} \\ &= \textstyle\sum_{k=0}^{n} P\{\tilde{n}_1 = n-k\} P\{\tilde{n}_2 = k\} \\ &= \textstyle\sum_{k=0}^{n} \left[\tfrac{(\alpha)^{n-k}e^{-\alpha}}{(n-k)!}\right]\left[\tfrac{(\beta)^k e^{-\beta}}{k!}\right] \end{aligned}$$

since both $\tilde{n}_1, \tilde{n}_2$ are Poisson.

$$\begin{aligned} &= \frac{e^{-(\alpha+\beta)}}{n!}\sum_{k=0}^{n}\frac{n!(\alpha)^{n-k}(\beta)^k}{(n-k)!k!} \\ &= \frac{e^{-(\alpha+\beta)}}{n!}\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k \\ &= \frac{e^{-(\alpha+\beta)}}{n!}(\alpha+\beta)^n \\ &= \frac{[(\alpha+\beta)\cdot t]^n e^{-(\alpha+\beta)}}{n!} \end{aligned}$$

This shows $\tilde{n} = \tilde{n}_1 + \tilde{n}_2$ is Poisson with rate $\alpha + \beta$.

To prove Property 1, define $\{\tilde{n}_1(t), t \geq 0\}$ and $\{\tilde{n}_2(t), t \geq 0\}$ to be Poisson processes with rates α and β, respectively, and $\tilde{n}(t) = \tilde{n}_1(t) + \tilde{n}_2(t)$. Then, from Definition 1 of the Poisson process, $\tilde{n}_1(t)$ and $\tilde{n}_2(t)$ are Poisson distributed with rates αt and βt, respectively. Thus, by the result just shown, $\tilde{n}(t)$ is Poisson distributed with parameter $(\alpha + \beta)t$; this settles property (iii) of Definition 1 of the Poisson process.

Since, by property (i) of Definition 1 of the Poisson process , $\tilde{n}_1(0) = \tilde{n}_2(0) = 0$, and $\tilde{n}(0) = \tilde{n}_1(0) + \tilde{n}_2(0)$, we find $\tilde{n}(0) = 0$; this settles property (i) of Definition 1 of the Poisson process.

It remains to show that $\{\tilde{n}(t), t \geq 0\}$ has independent increments. Consider two non-overlapping intervals of time, say (t_0, t_1) and (t_2, t_3). Now,

$$\tilde{n}(t_1) - \tilde{n}(t_0) = [\tilde{n}_1(t_1) - \tilde{n}_1(t_0)] + [\tilde{n}_2(t_1) - \tilde{n}_2(t_0)],$$

and

$$\tilde{n}(t_3) - \tilde{n}(t_2) = [\tilde{n}_1(t_3) - \tilde{n}_1(t_2)] + [\tilde{n}_2(t_3) - \tilde{n}_2(t_2)].$$

Since the random variables $[\tilde{n}_1(t_1) - \tilde{n}_1(t_0)]$ and $[\tilde{n}_2(t_1) - \tilde{n}_2(t_0)]$ are independent, and $[\tilde{n}_1(t_3) - \tilde{n}_1(t_2)]$ and $[\tilde{n}_2(t_3) - \tilde{n}_2(t_2)]$ are independent, it follows that $[\tilde{n}(t_1) - \tilde{n}(t_0)]$ and $[\tilde{n}(t_3) - \tilde{n}(t_2)]$ are independent. Thus, $\{\tilde{n}(t), t \geq 0\}$ has independent increments, and the proof of Property 1 of Poisson processes is complete.

> EXERCISE 2.31 Suppose an urn contains $\tilde{n}$ balls, where $\tilde{n}$ is a Poisson random variable with parameter λ. Suppose the balls are either red or green, the proportion of red balls being p. Show that the distribution of the number of red balls, $\tilde{n}_r$, in the urn is Poisson with parameter $p\lambda$, the distribution of green balls, $\tilde{n}_g$ is Poisson with parameter $(1-p)\lambda$, and that $\tilde{n}_r$ and $\tilde{n}_g$ are independent random variables. Use this result to prove Property 2 of the Poisson process. [*Hint*: Condition on the total number of balls in the urn and use the fact that the number of successes in a sequence of n repeated Bernoulli trials has the binomial distribution with parameters n and p.]

Solution. Condition on the value of $\tilde{n}$, the total number of balls in the urn:

$$P\{\tilde{n}_r = r, \tilde{n}_g = g\} = \sum_{k=0}^{\infty} P\{\tilde{n}_r = r, \tilde{n}_g = g | \tilde{n} = k\} P\{\tilde{n} = k\}$$

Note that the only way for $\tilde{n}_r = r$ and $\tilde{n}_g = g$, given that $\tilde{n} = k$, is if their sum is $r + g$; that is, if $k = r + g$. So for $k \neq r + g$,

$$P\{\tilde{n}_r = r, \tilde{n}_g = g | \tilde{n} = k\} = 0.$$

Hence

$$P\{\tilde{n}_r = r, \tilde{n}_g = g\} = P\{\tilde{n}_r = r, \tilde{n}_g = g | \tilde{n} = r + g\} P\{\tilde{n} = k\}$$

Consider a red ball to be a success and a green ball to be a failure. Then we may think of all of the balls in the urn as being a series of $\tilde{n}$ Bernoulli experiments. Hence $P\{\tilde{n}_r = r, \tilde{n}_g = g | \tilde{n} = r + g\}$ is Binomial having parameters $(r + g), p$. Then, since $\tilde{n}$ is Poisson with rate λ,

$$P\{\tilde{n}_r = r, \tilde{n}_g = g | \tilde{n} = r + g\} P\{\tilde{n} = k\}$$

is

$$\binom{r+g}{r} p^r (1-p)^{(r+g)-r} \left[\frac{e^{-\lambda t} (\lambda t)^{r+g}}{(r+g)!} \right] = \frac{e^{-\lambda t} (\lambda t)^{r+g}}{(r+g)!}$$

After some rearranging of terms:

$$= \left[\frac{e^{-p\lambda t} (p\lambda t)^r}{r!} \right] \left[\frac{e^{-(1-p)\lambda t} \{(1-p)\lambda t\}^g}{g!} \right]$$

Summing over all possible values of $\tilde{n}_g$, we find

$$\begin{aligned}
P\{\tilde{n}_r = r\} &= \sum_{g=0}^{\infty} P\{\tilde{n}_r = r, \tilde{n}_g = g\} \\
&= \left[\frac{(p\lambda t)^r e^{-p\lambda t}}{r!} \right] e^{-(1-p)\lambda t} \sum_{g=0}^{\infty} \left[\frac{\{(1-p)\lambda t\}^g}{g!} \right] \\
&= \left[\frac{(p\lambda t)^r e^{-p\lambda t}}{r!} \right] e^{-(1-p)\lambda t} e^{(1-p)\lambda t} \\
&= \frac{(p\lambda t)^r e^{-p\lambda t}}{r!}
\end{aligned}$$

Therefore $\tilde{n}_r$ is Poisson with rate $p\lambda$. Similarly, by summing over all possible values of $\tilde{n}_r$, $P\{\tilde{n}_g = g\}$ is shown to be Poisson with rate $(1-p)\lambda$. Furthermore, since

$$P\{\tilde{n}_r = r, \tilde{n}_g = g\} = P\{\tilde{n}_r = r\} P\{\tilde{n}_g = g\}$$

$\tilde{n}_r$ and $\tilde{n}_g$ are independent.

Now show $\tilde{n}_r(t)$ and $\tilde{n}_g(t)$ are Poisson processes. Let $\{\tilde{n}(t), t \geq 0\}$ be a Poisson process with rate λ. By Definition 1 of the Poisson process $\tilde{n}(t)$

is Poisson distributed with parameter λt. Let each event be recorded with probability p. Define $\tilde{n}_r(t)$ to be the number of events recorded by time t, and $\tilde{n}_g(t)$ to be the number of events not recorded by time t. Then by the result just shown,$\tilde{n}(t) = \tilde{n}_r(t) + \tilde{n}_g(t)$, where $\tilde{n}_r(t)$ and $\tilde{n}_g(t)$ are Poisson distributed with rates $p\lambda t$ and $(1-p)\lambda t$, respectively. This proves property (iii) of Definition 1 of the Poisson process.

By property (iii) of Definition 1 of the Poisson process, $\tilde{n}(0) = 0$. Since $\tilde{n}_r(t)$ and $\tilde{n}_g(t)$ are non-negative for all $t \geq 0$, $\tilde{n}_r(0) = \tilde{n}_g(0) = 0$. Thus property (i) of Definition 1 of the Poisson process holds.

It remains to show property (ii) of Definition 1 of the Poisson process. Consider two non-overlapping intervals of time, say (t_0, t_1) and (t_2, t_3). Then, since $\tilde{n}(t) = \tilde{n}_r(t) + \tilde{n}_g(t)$,

$$\tilde{n}(t_1) - \tilde{n}(t_0) = [\tilde{n}_r(t_1) - \tilde{n}_r(t_0)] + [\tilde{n}_g(t_1) - \tilde{n}_g(t_0)],$$

and

$$\tilde{n}(t_3) - \tilde{n}(t_2) = [\tilde{n}_r(t_3) - \tilde{n}_r(t_2)] + [\tilde{n}_g(t_3) - \tilde{n}_g(t_2)].$$

By the result shown above, $\tilde{n}_r(t)$ and $\tilde{n}_g(t)$ are independent random variables. That is, $[\tilde{n}_r(t_1) - \tilde{n}_r(t_0)]$ and $[\tilde{n}_g(t_1) - \tilde{n}_g(t_0)]$ are independent, and $[\tilde{n}_r(t_3) - \tilde{n}_r(t_2)]$ and $[\tilde{n}_g(t_3) - \tilde{n}_g(t_2)]$ are independent. Furthermore, $\{\tilde{n}(t), t \geq 0\}$ is a Poisson process so it has independent increments: $\tilde{n}(t_1) - \tilde{n}(t_0)$ and $\tilde{n}(t_3) - \tilde{n}(t_2)$ are independent. Since $\tilde{n}_r(t)$ and $\tilde{n}_g(t)$ are independent of each other across the sums, which are then in turn independent of each other, $\tilde{n}_r(t)$ and $\tilde{n}_g(t)$ are independent across the intervals. i.e., $[\tilde{n}_r(t_1) - \tilde{n}_r(t_0)]$ and $[\tilde{n}_r(t_3) - \tilde{n}_r(t_2)]$ and independent, and $[\tilde{n}_g(t_1) - \tilde{n}_g(t_0)]$ and $[\tilde{n}_g(t_3) - \tilde{n}_g(t_2)]$ and independent. This proves property (ii) of Definition 1 of the Poisson process.

Since all three properties hold, $\{\tilde{n}_r(t), t \geq 0\}$ and $\{\tilde{n}_g(t), t \geq 0\}$ are Poisson processes.

> EXERCISE 2.32 Events occur at a Poisson rate λ. Suppose all odd numbered events and no even numbered events are recorded. Let $\tilde{n}_1(t)$ be the number of events recorded by time t and $\tilde{n}_2(t)$ be the number of events not recorded by time t. Do the processes $\{\tilde{n}_1(t), t \geq 0\}$ and $\{\tilde{n}_2(t), t \geq 0\}$ each have independent increments? Do they have stationary increments? Are they Poisson processes?

Solution. Since only odd numbered events are recorded, the time between recorded events is the sum of two exponentially distributed random variables with parameter λ. Hence, the processes $\{\tilde{n}_1(t), t \geq 0\}$ and $\{\tilde{n}_2(t), t \geq 0\}$ are clearly not Poisson. Now, suppose an event is recorded at time t_0. Then, the probability that an event will be recorded in $(t_0, t_0 + h)$ is $o(h)$ because this would require two events from the original Poisson process in a period of length h. Therefore, the increments are not independent. On the other hand, the event probabilities are independent of t_0 because the original process is Poisson, so the increments are stationary.

EXERCISE 2.33 Determine the one-step *transition probability matrix* for the Markov chain $\{q_k, k = 0, 1, \ldots\}$ of Section 1.2.2 for the case where $\{\tilde{v}_k, k = 1, 2, \ldots\}$ is assumed to be a sequence of independent, identically distributed binomial random variables with parameters N and p.

Solution. The dynamical equations are given as follows:

$$\tilde{q}_{k+1} = (\tilde{q}_k - 1)^+ + \tilde{v}_{k+1}.$$

By definition, the one-step transition probability is

$$P\left\{\tilde{q}_{k+1} = j | \tilde{q}_k = i\right\} \quad \text{for all integers} \quad i, j$$

But,

$$P\left\{\tilde{q}_{k+1} = j | \tilde{q}_k = i\right\} = P\left\{(\tilde{q}_k - 1)^+ + \tilde{v}k + 1 = j | \tilde{q}_k = i\right\}.$$

In turn,

$$\begin{aligned} P\left\{(\tilde{q}_k - 1)^+ + \tilde{v}k + 1 = j | \tilde{q}_k = i\right\} &= P\left\{(i - 1)^+ + \tilde{v}k + 1 = j | \tilde{q}_k = i\right\} \\ &= P\left\{\tilde{v}k + 1 = j - (i - 1)^+ | \tilde{q}_k = i\right\} \end{aligned}$$

But, $\tilde{v}_{k+1}$ is independent of $\tilde{q}_k = i$. It follows that

$$P\left\{\tilde{v}k + 1 = j - (i - 1)^+ | \tilde{q}_k = i\right\} = P\left\{\tilde{v}k + 1 = j - (i - 1)^+\right\}.$$

For $i = 0$ or $i = 1$,

$$P\left\{\tilde{v}k + 1 = j - (i - 1)^+\right\} = P\left\{\tilde{v}k + 1 = j\right\}.$$

For $i \geq 2$,

$$P\left\{\tilde{v}k + 1 = j - (i - 1)^+\right\} = P\left\{\tilde{v}k + 1 = j + 1 - i\right\},$$

where we note that the previous probability is zero whenever $j < i - 1$. Define $a_j = P\left\{\tilde{v}k + 1 = j\right\}$ for $i = 0, 1, \ldots, N$. Then, we have shown that

$$\mathcal{P} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & \cdots & a_N & 0 & \cdots \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & \cdots & a_N & 0 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \cdots & a_{N-1} & \ddots & \ddots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \cdots & a_{N-2} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \end{bmatrix}.$$

EXERCISE 2.34 Staring with (2.18), show that the rows of $\mathcal{P}^\infty$ must be identical. [Hint: First calculate $\mathcal{P}^\infty$ under the assumption $\beta_0 = [1 \quad 0 \quad 0 \quad \dots]$. Next, calculate $\mathcal{P}^\infty$ under the assumption $\beta_0 = [0 \quad 1 \quad 0 \quad \dots]$. Continue along these lines.]

Solution. The general idea is that π is independent of β_0; that is, the value of π is the same no matter how we choose the distribution of the starting state. Suppose we choose $\beta_0 = [1 \quad 0 \quad 0 \quad \dots]$. Then, $\beta_0\mathcal{P}^\infty$ is equal to the first row of $\mathcal{P}^\infty$. Since $\pi = \beta_0\mathcal{P}^\infty$, we then have the first row of $\mathcal{P}^\infty$ must be π. if we choose $\beta_0 = [0 \quad 1 \quad 0 \quad \dots]$, then $\beta_0\mathcal{P}^\infty$ is equal to the second row of $\mathcal{P}^\infty$. Again, since $\pi = \beta_0\mathcal{P}^\infty$, we then have the first row of $\mathcal{P}^\infty$ must be π. In general, if we choose the starting state as i with probability 1, then we conclude that the i-th row of $\mathcal{P}^\infty$ is equal to π. thus, all rows of $\mathcal{P}^\infty$ must be equal to π.

EXERCISE 2.35 Suppose $\{\tilde{x}_k, k = 0, 1, \dots\}$ is a Markov chain such that

$$\mathcal{P} = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}.$$

Determine the stationary vector of $\{\tilde{x}_k, k = 0, 1, \dots\}$.

Solution. We have $\pi = \pi\mathcal{P}$ and $\pi\mathbf{e} = 1$. Thus,

$$[\,\pi_1 \quad \pi_2\,] = [\,\pi_1 \quad \pi_2\,] \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix},$$

or

$$[\,\pi_1 \quad \pi_2\,] \begin{bmatrix} 0.4 & -0.4 \\ -0.5 & 0.5 \end{bmatrix} = [\,0 \quad 0\,].$$

Also $\pi\mathbf{e} = 1$ means

$$[\,\pi_1 \quad \pi_2\,] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = .$$

Upon substitution of the last equation into the second equation of the matrix equation, we have

$$[\,\pi_1 \quad \pi_2\,] \begin{bmatrix} 0.4 & 1 \\ -0.5 & 1 \end{bmatrix} = [\,0 \quad 1\,].$$

Adding $\frac{1}{2}$ of the second equation to the first yields

$$[\,\pi_1 \quad \pi_2\,] \begin{bmatrix} 0.9 & 1 \\ 0 & 1 \end{bmatrix} = [\,0.5 \quad 1\,].$$

Then dividing the first equation by 0.9 yields

$$[\,\pi_1 \quad \pi_2\,] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = [\,\tfrac{5}{9} \quad 1\,].$$

And, finally subtracting the first equation from the second yields

$$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & \frac{4}{9} \end{bmatrix}.$$

or

$$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & \frac{4}{9} \end{bmatrix}.$$

EXERCISE 2.36 Develop the one-step state probability transition matrix for the special case $\{\tilde{v}_k, k = 0, 1, \ldots\}$ for the special case of $N = 5$.

Solution. We wish to find $P\{\tilde{v}_{k+1} = j|\tilde{v}_k = i\}$ for $i, j \in \{0, 1, 2, 3, 4, 5\}$. Now, $\tilde{v}_k$ is the number of sources in the on state during interval k. Thus, if $\tilde{v}_k = i$, then i sources are in the on state during period k and $5 - i$ sources are in the off state during that period. Let $\tilde{a}_i$ denote the number of sources that are in the on state in period $k + 1$ if i sources are in the on state in period k and $\tilde{b}_i$ denote the number of sources that are in the on state in period $k+1$ if i sources are in the off state in period k. HTen, the number of on sources during period $k + 1$ given i on sources in period k is equal to $\tilde{a}_i + \tilde{b}_{5-i}$. Thus,

$$P\{\tilde{v}_{k+1} = j|\tilde{v}_k = i\} = P\left\{\tilde{a}_i + \tilde{b}_{5-i} = j\right\}.$$

But,

$$\begin{aligned} P\left\{\tilde{a}_i + \tilde{b}_{5-i} = j\right\} &= \sum_{\ell=0}^{i} P\left\{\tilde{a}_i + \tilde{b}_{5-i} = j|\tilde{a}_i = \ell\right\} P\{\tilde{a}_i = \ell\} \\ &= \sum_{\ell=0}^{i} P\left\{\tilde{b}_{5-i} = j - \ell\right\} P\{\tilde{a}_i = \ell\}, \end{aligned}$$

the latter step following by independence among the behavior of the sources. In the summation, if $\ell > j$, then $P\left\{\tilde{b}_{5-i} = j - \ell\right\} = 0$, so we may terminate the summation at $\min\{i, j\}$. In addition, $j - \ell > 5 - i$ or $\ell < j + i - 5$, then $P\left\{\tilde{b}_{5-i} = j - \ell\right\} = 0$. Thus, we may begin the summation at $\max\{0, j + i - 5\}$. Thus

$$P\{\tilde{v}_{k+1} = j|\tilde{v}_k = i\} = \sum_{\ell=\max\{0,j+i-5\}}^{\min\{i,j\}} P\left\{\tilde{b}_{5-i} = j - \ell\right\} P\{\tilde{a}_i = \ell\}.$$

Since each of the off source turns on with probability p_{01}.

$$P\left\{\tilde{b}_{5-i} = j - \ell\right\} = \binom{5-i}{j-\ell} p_{01}^{j-\ell} p_{00}^{(5-i)-(j-\ell)}.$$

Similarly, since each of the on source remains on with probability p_{11}.

$$P\left\{\tilde{a}_i = \ell\right\} = \binom{i}{\ell} p_{11}^{\ell} p_{10}^{i-\ell}.$$

The transition probabilities can now be obtained by performing the indicate summations. Some examples are

$$P\left\{\tilde{v}_{k+1} = 0 | \tilde{v}_k = 0\right\} = p_{00}^5,$$

$$P\left\{\tilde{v}_{k+1} = 1 | \tilde{v}_k = 0\right\} = 5 p_{01} p_{00}^4,$$

and

$$\begin{aligned}
P\left\{\tilde{v}_{k+1} = 3 | \tilde{v}_k = 4\right\} &= \sum_{\ell=2}^{3} P\left\{\tilde{b}_1 = 3 - \ell\right\} P\left\{\tilde{a}_3 = \ell\right\} \\
&= P\left\{\tilde{b}_1 = 1\right\} P\left\{\tilde{a}_4 = 2\right\} + P\left\{\tilde{b}_1 = 0\right\} P\left\{\tilde{a}_4 = 3\right\} \\
&= p_{01} \binom{4}{2} p_{11}^2 p_{10}^2 + p_{00} \binom{4}{3} p_{11}^3 p_{10} \\
&= 6 p_{01} p_{11}^2 p_{10}^2 + 4 p_{00} p_{11}^3 p_{10}
\end{aligned}$$

EXERCISE 2.37 For the example discussed in Section 1.2.2 in the case where arrivals occur according to an on off process, determine whether or not $\{\tilde{q}_k, k = 0, 1, \ldots\}$ is a DPMC. Defend your conclusion mathematically; that is show whether or not $\{\tilde{q}_k, k = 0, 1, \ldots\}$ satisfies the definition of a Markov chain.

Solution. We are given

$$\tilde{q}_{k+1} = (\tilde{q}_k - 1)^+ + \tilde{v}_{k+1},$$

and we want to know whether or not

$$P\left\{\tilde{q}_{k+1} = j | \tilde{q}_0 = i_0, \tilde{q}_1 = i_1, \ldots, \tilde{q}_k = i\right\} = P\left\{\tilde{q}_{k+1} = j | \tilde{q}_k = i\right\}.$$

For the case where the arrival process is an independent sequence of random variables, the answer is “Yes.” The sequence of values of $\tilde{q}_k$ does reveal the number of arrivals in each interval, but the probability mass function for the queue length at epoch $k+1$ can be computed solely form the value of information is not needed to compute the new value of $\tilde{q}_k$. For the case where the arrival process is on-off, the sequence of queue lengths reveals the sequence of arrivals. For example, if $\tilde{q}_k = 7$ and $\tilde{q}_{k-1} = 3$, then we know that there were 5 arrivals during interval k, which means that there were 5 sources in the on state during interval k. Thus, the distribution of the number of new arrivals during interval $k+1$, and therefore the distribution of $\tilde{q}_{k+1}$, is dependent upon the values of $\tilde{q}_k$ and $\tilde{q}_{k-1}$. Thus, if we know only that $\tilde{q}_k = i$, we cannot compute the distribution of $\tilde{q}_{k+1}$. Hence for the general on-off arrival process, $\{\tilde{q}_k, k = 0, 1, \ldots\}$ is not a DPMC.

EXERCISE 2.38 Suppose $\{\tilde{x}(t), t \geq 0\}$ is a time-homogeneous CTMC having infinitesimal generator $\mathcal{Q}$ defined as follows:

$$Q_{ij} = \begin{cases} -\lambda, & \text{if } j = i, \\ \lambda, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\{\tilde{x}_1(t), t \geq 0\}$ is a Poisson process. [*Hint*: Simply solve the infinite matrix differential equation term by term starting with $\mathcal{P}_{00}(t)$ and completing each column in turn.]

Solution. Solve the infinite matrix differential equation:

$$\frac{d}{dt}P(t) = P(t)\mathcal{Q}.$$

Due to independent increments, we need only compute the first row of the matrix $P(t)$. Begin with $P_{0,0}(t)$:

$$\frac{d}{dt}P_{0,0}(t) = -\lambda P_{0,0}(t).$$

We see immediately that

$$P_{0,0}(t) = e^{-\lambda t}.$$

Now solve for the second column of $P(t)$:

$$\begin{aligned} \frac{d}{dt}P_{0,1}(t) &= \lambda P_{0,0}(t) - \lambda P_{0,1}(t) \\ &= \lambda e^{-\lambda t} - \lambda P_{0,1}(t). \end{aligned}$$

This solves to

$$P_{0,1}(t) = \lambda t e^{-\lambda t}$$

Repeating this process for each column of $P(t)$, we find the solution of $P(t)$ to be

$$P_{0,n}(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Observe that this probability is that of a Poisson random variable with parameter λt. Hence the time between events is exponential. By Definition 2.16, $\{\tilde{x}(t), t \geq 0\}$ is a Poisson process. Furthermore, this is strictly a birth process; that is, the system cannot lose customers once they enter the system. This should be apparent from the definition of $\mathcal{Q}$.

EXERCISE 2.39 Let $\{\tilde{x}(t), t \geq 0\}$ be a CTMC such that

$$\mathcal{Q} = \begin{bmatrix} -2.00 & 1.25 & 0.75 \\ 0.50 & -1.25 & 0.75 \\ 1.00 & 2.00 & -3.00 \end{bmatrix}.$$

1. Solve for $P(\infty)$ directly by solving $P(\infty)\mathcal{Q} = 0$ $P_(\infty)\mathbf{e} = 1$.
2. Solve for π for the DPMC embedded at points of state transition using (2.24).
3. Find $\mathcal{P}$ for for the DPMC embedded at points of state transition.
4. Show that the value of π found in part 2 of this problem satisfies $\pi = \pi\mathcal{P}$ for the $\mathcal{P}$ found in part 3 of this problem.

Solution.

1.

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,] \begin{bmatrix} -2.00 & 1.25 & 0.75 \\ 0.50 & -1.25 & 0.75 \\ 1.00 & 2.00 & -3.00 \end{bmatrix} = [\,0 \quad 0 \quad 0\,],$$

and

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1$$

yield

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,] \begin{bmatrix} -2.00 & 1.25 & 1 \\ 0.50 & -1.25 & 1 \\ 1.00 & 2.00 & 1 \end{bmatrix} = [\,0 \quad 0 \quad 1\,].$$

Then dividing the first equation by -2 and the second equation by -1.25 yields

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,] \begin{bmatrix} 1 & -1 & 1 \\ -\frac{1}{4} & 1 & 1 \\ -\frac{1}{2} & -\frac{8}{5} & 1 \end{bmatrix} = [\,0 \quad 0 \quad 1\,].$$

Then adding the first equation to the second and subtracting the first equation from the third yields:

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,] \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{3}{4} & \frac{5}{4} \\ -\frac{1}{2} & -\frac{21}{10} & \frac{3}{2} \end{bmatrix} = [\,0 \quad 0 \quad 1\,].$$

Dividing the second equation by 3/4 yields

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,]\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & \frac{5}{4} \\ -\frac{1}{2} & -\frac{14}{5} & \frac{3}{2} \end{bmatrix} = [\,0 \quad 0 \quad 1\,].$$

Subtracting 5/4 of the second equation from the third yields

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,]\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{2} & -\frac{14}{5} & 5 \end{bmatrix} = [\,0 \quad 0 \quad 1\,].$$

Dividing the third equation by 5 yields

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,]\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{2} & -\frac{14}{5} & 1 \end{bmatrix} = [\,0 \quad 0 \quad 0.2\,].$$

Then adding 14/5 times the third equation to the second yields

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,]\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} = [\,0 \quad \tfrac{14}{25} \quad 0.2\,].$$

Finally adding 1/4 of the second equation and one half of the third equation to the first equation yields

$$[\,P(\infty)_0 \quad P(\infty)_1 \quad P(\infty)_2\,] = [\,\tfrac{6}{25} \quad \tfrac{14}{25} \quad \tfrac{5}{25}\,].$$

2. We have from (2.24)

$$\pi_j = \frac{\sum_{i\neq j} P_i(\infty)\mathcal{Q}_{ij}}{\sum_\ell \sum_{i\neq \ell} P_i(\infty)\mathcal{Q}_{i\ell}}.$$

Plugging in numbers, we find

$$\sum_{i\neq 0} P_i(\infty)\mathcal{Q}_{i0} = \frac{14}{25}\frac{1}{2} + \frac{5}{25} = \frac{12}{25},$$

$$\sum_{i\neq 1} P_i(\infty)\mathcal{Q}_{i1} = \frac{6}{25}\frac{5}{4} + \frac{5}{25}2 = \frac{70}{100},$$

and

$$\sum_{i\neq 2} P_i(\infty)\mathcal{Q}_{i1} = \frac{6}{25}\frac{3}{4} + \frac{14}{25}\frac{3}{4} = \frac{60}{100},$$

Thus,

$$\pi_0 = \frac{48}{48+70+60} = \frac{48}{178}. \quad \pi_1 = \frac{70}{178}, \quad \text{and} \quad \pi_2 = \frac{60}{178}.$$

3. From

$$P\{\tilde{x}_{k+1} = j | \tilde{x}_k = i\} = \frac{\mathcal{Q}_{ij}}{-\mathcal{Q}_{ii}},$$

for $i \neq j$ and $P\{\tilde{x}_{k+1} = j | \tilde{x}_k = j\} = 0$, we have immediately

$$\mathcal{P} = \begin{bmatrix} 0 & \frac{5}{8} & \frac{3}{8} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}.$$

4. We wish to show that the value of π computed in Part 2 satisfies $\pi = \pi\mathcal{P}$. We have

$$\begin{aligned} \left[\, \tfrac{48}{178} \quad \tfrac{70}{178} \quad \tfrac{60}{178} \,\right] \begin{bmatrix} 0 & \frac{5}{8} & \frac{3}{8} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} &= \left[\, \tfrac{70}{178}\tfrac{2}{5} + \tfrac{60}{178}\tfrac{1}{3} \quad \tfrac{48}{178}\tfrac{5}{8} + \tfrac{60}{178}\tfrac{2}{3} \quad \tfrac{48}{178}\tfrac{3}{8} + \tfrac{70}{178}\tfrac{3}{5} \,\right] \\ &= \left[\, \tfrac{48}{178} \quad \tfrac{70}{178} \quad \tfrac{60}{178} \,\right]. \end{aligned}$$

Chapter 3

ELEMENTARY CONTINUOUS-TIME MARKOV CHAIN-BASED QUEUEING MODELS

EXERCISE 3.1 Carefully pursue the analogy between the random walk and the occupancy of the M/M/1 queueing system. Determine the probability of an increase in the queue length, and show that this probability is less than 0.5 if and only if $\lambda < \mu$.

Solution. Consider the occupancy of the M/M/1 queueing system as the position of a random walker on the nonnegative integers, where a wall is erected to the left of position zero. When there is an increase in the occupancy of the queue this is like the walker taking a step to the right; a decrease is a step to the left. However, if the queue is empty (the walker is at position zero), then a 'decrease' in the occupancy is analogous to the walker attempting a step to the left – but he hits the wall and so remains at position zero (queue empty).

Let λ be the rate of arriving customers and μ be the rate of departing customers. Denote the probability of an arrival by p^+ and a departure by p^-. Then p^+ is simply the probability that an arrival occurs before a departure. From Chapter 2, this probability is $\frac{\lambda}{\lambda+\mu}$. It follows that $p^+ < \frac{1}{2}$ if and only if $\lambda < \mu$.

EXERCISE 3.2 Prove Theorem 3.1 and its continuos analog

$$E[\tilde{x}] = \int_0^\infty P\{\tilde{x} > x\}dx.$$

Solution.

$$\begin{aligned}
E\left[\tilde{n}\right] &= \sum_{n=0}^{\infty} nP\left\{\tilde{n} = n\right\} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} 1\right) P\left\{\tilde{n} = n\right\}
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} P\{\tilde{n}=n\}\right)$$

Changing the orders of summation

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} P\{\tilde{n}=n\}\right) &= \sum_{n=0}^{\infty}\left(\sum_{k=n+1}^{\infty} P\{\tilde{n}=k\}\right)\\ &= \sum_{n=0}^{\infty} P\{\tilde{n}>n\}\end{aligned}$$

For the continuous analog, we have

$$E[\tilde{x}] = \int_0^\infty x f_{\tilde{x}}(x)dx.$$

But,

$$x = \int_0^x dy,$$

so

$$E[\tilde{x}] = \int_0^\infty \int_0^x dy f_{\tilde{x}}(x)dx.$$

Changing order of integration yields

$$E[\tilde{x}] = \int_0^\infty \int_y^\infty f_{\tilde{x}}(x)dxdy = \int_0^\infty P\{\tilde{x}>y\}dy = \int_0^\infty P\{\tilde{x}>x\}dx.$$

EXERCISE 3.3 Prove Theorem 3.2.

Solution. The theorem is as follows: Suppose $\tilde{x}$ and $\tilde{y}$ are any two nonnegative random variables. Then $E[\min\{\tilde{x},\tilde{y}\}] \leq \min\{E[\tilde{x}], E[\tilde{y}]\}$.

First we consider the continuous case. Define $\tilde{z} = \min\{\tilde{x},\tilde{y}\}$. Then, $\tilde{z} > z$ if, and only if, $\{\tilde{x} > z, \tilde{y} > z\}$. But,

$$E[\tilde{z}] = \int_0^\infty P\{\tilde{z}>z\}dz = \int_0^\infty P\{\tilde{x}>z, \tilde{y}.z\}.$$

But, $P\{\tilde{x}>z, \tilde{y}.z\} \leq P\{\tilde{x}>z\}$ and $P\{\tilde{x}>z, \tilde{y}.z\} \leq P\{\tilde{y}>z\}$ because the event $\{\tilde{x}>z, \tilde{y}>z\} \subseteq \{\tilde{x}>z\}$ and $\{\tilde{x}>z, \tilde{y}>z\} \subseteq \{\tilde{y}>z\}$. Therefore,

$$E[\tilde{z}] \leq \int_0^\infty P\{\tilde{x}>z\} \quad \text{and} \quad E[\tilde{z}] \leq \int_0^\infty P\{\tilde{y}>z\}.$$

Equivalently,

$$E[\tilde{z}] \leq E[\tilde{x}] \quad \text{and} \quad E[\tilde{z}] \leq E[\tilde{y}].$$

The previous statement is equivalent to

$$E\left[\min\left\{\tilde{x}, \tilde{y}\right\}\right] \leq \min\left\{E\tilde{x}, E\left[\tilde{y}\right]\right\}.$$

The discrete and mixed cases are proved in the same way.

EXERCISE 3.4 Suppose customers arrive to a system at the end of every even-numbered second and each customer requires exactly one second of service. Compute the stochastic equilibrium occupancy distribution, that is, the time-averaged distribution of the number of customers found in the system. Compute the occupancy distribution as seen by arriving customers. Compare the two distributions. Are they the same?

Solution. Since a customer arrives every two seconds and leaves after one second, there will either be one customer in the system or no customers in the system. Each of these events is equally likely since the time intervals that they occur in is the same (1 second). Hence

$$\begin{aligned}
E\left[\tilde{s}\right] &= E\left[\tilde{s}|\tilde{n}=0\right]P\left\{\tilde{n}=0\right\} + E\left[\tilde{s}|\tilde{n}=1\right]P\left\{\tilde{n}=1\right\} \\
&= 0\cdot\frac{1}{2} + 1\cdot\frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

On the other hand, because a customer finishes service at the end of every odd-numbered second, she has left the system by the time the next cusomer arrives. Hence an arriving customer sees the system occupancy as empty; that is, as zero. Obviously this is not the same as $E\left[\tilde{s}\right]$ and we conclude that the distributions as seen by an arbitrary arriving customer and that of an arbitrary observer cannot be the same.

EXERCISE 3.5 For the ordinary M/M/1 queueing system, determine the limiting distribution of the system occupancy

1. as seen by departing customers, [*Hint*: Form the system of equations $\pi_d = \pi_d \mathcal{P}_d$, and then solve the system as was done to obtain $P\{\tilde{n} = n\}$.]

2. as seen by arriving customers, and [*Hint*: First form the system of equations $\pi_a = \pi_a \mathcal{P}_a$, and then try the solution $\pi_a = \pi_d$.]

3. at instants of time at which the occupancy changes. That is, embed a Markov chain at the instants at which the occupancy changes, defining the state to be the number of customers in the system immediately following the state change. Define $\pi = [\,\pi_0 \quad \pi_1 \quad \cdots]$ to be the stationary probability vector and P to be the one-step transition probability matrix for this embedded Markov chain. Determine π, and then compute the stochastic equilibrium distribution for the process $\{\tilde{n}(t), t \geq 0\}$ according to the following well known result from the theory of Markov chains as discussed in Chapter 2:

$$P_i = \frac{\pi_i E[\tilde{s}_i]}{\sum\limits_{i=0}^{\infty} \pi_i E[\tilde{s}_i]},$$

   where $\tilde{s}_i$ denotes the time the systems spends in state i on each visit.

Observe that the results of parts 1, 2, and 3 are identical, and that these are all equal to the stochastic equilibrium occupancy probabilities determined previously.

Solution.

1. Let $\tilde{n}_d(k)$ denote the number of customers left by the $k-th$ departing customer. Then the $(i,j)-th$ element of $\mathcal{P}_d$ is given by

$$P\{\tilde{n}_d(k+1) = j|\tilde{n}_d(n) = i\}.$$

   This probability is given by the probability that exactly $j-(i-1)^+$ customers arrive during the $k-th$ service time. For $j-(i-1)^+ < 0$,

$$P\{\tilde{n}_d(n+1) = j|\tilde{n}_d(n) = i\} = 0,$$

   since you can't have a negative number of arrivals. Otherwise, letting $\mathcal{A}$ denote the event of an arrival before the $k-th$ service completion,

$$P\{j-(i-1)^+ \quad \mathcal{A}\} = \left(\frac{\lambda}{\lambda+\mu}\right)^{j-(i-1)^+} \frac{\mu}{\lambda+\mu}.$$

This results in

$$\mathcal{P}_d = \begin{pmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu}\frac{\mu}{\lambda+\mu} & \left(\frac{\lambda}{\lambda+\mu}\right)^2\frac{\mu}{\lambda+\mu} & \cdots \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu}\frac{\mu}{\lambda+\mu} & \left(\frac{\lambda}{\lambda+\mu}\right)^2\frac{\mu}{\lambda+\mu} & \cdots \\ 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu}\frac{\mu}{\lambda+\mu} & \cdots \\ 0 & 0 & \frac{\mu}{\lambda+\mu} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Using this matrix, form the system of equations $\pi_d = \pi_d \mathcal{P}_d$. Then, for $i = 0$,

$$\begin{aligned} \pi_{d_0} &= \sum_{i=0}^{\infty} \pi_{d_i}\mathcal{P}_{d_{i0}} \\ &= \pi_{d_0}\mathcal{P}_{d_{00}} + \pi_{d_1}\mathcal{P}_{d_{10}} + \sum_{i=2}^{\infty}\pi_{d_i}\mathcal{P}_{d_{i0}} \\ &= \pi_{d_0}\left(\frac{\mu}{\lambda+\mu}\right) + \pi_{d_1}\left(\frac{\lambda}{\lambda+\mu}\right)\left(\frac{\mu}{\lambda+\mu}\right) + 0. \end{aligned}$$

Hence $\pi_{d_1} = \frac{\lambda}{\mu}\pi_{d_0}$. Substitute this into the equation for π_{d_1}:

$$\begin{aligned} \pi_{d_1} &= \textstyle\sum_{i=0}^{\infty} \pi_{d_i}\mathcal{P}_{d_{i1}} \\ &= \pi_{d_1}\mathcal{P}_{d_{01}} + \pi_{d_1}\mathcal{P}_{d_{11}} + \pi_{d_2}\mathcal{P}_{d_{21}}\sum_{i=3}^{\infty}\pi_{d_i}\mathcal{P}_{d_{i1}} \\ &= \pi_{d_0}\left(\frac{\mu}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right) + \frac{\lambda}{\mu}\pi_{d_0}\left(\frac{\lambda}{\lambda+\mu}\right)\left(\frac{\mu}{\lambda+\mu}\right) \\ &\quad +\pi_{d_2}\left(\frac{\mu}{\lambda+\mu}\right) + 0 \end{aligned}$$

which gives the solution $\pi_{d_2} = \left(\frac{\lambda}{\mu}\right)^2 \pi_{d_0}$. Repeating this process we see that $\pi_{d_j} = \left(\frac{\lambda}{\mu}\right)^j \pi_{d_0}$, $j = 0, 1, 2, \ldots$.

Now use the normalizing constraint$\sum_{j=0}^{\infty}\pi_{d_j} = 1$. Then for $\lambda < \mu$,

$$\pi_{d_0}\sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^j = \pi_{d_0}\left(\frac{1}{1-\frac{\lambda}{\mu}}\right) = 1.$$

Thus $\pi_{d_0} = 1 - \frac{\lambda}{\mu}$, and $\pi_{d_j} = \left(\frac{\lambda}{\mu}\right)^j\left(1-\frac{\lambda}{\mu}\right)$.

2. Let $\tilde{n}_a(k)$ denote the number of customers as seen by the $k-th$ arriving customer. Then the $(i,j)-th$ element of $\mathcal{P}_a$ is given by

$$P\left\{\tilde{n}_a(k+1) = j | \tilde{n}_d(n) = i\right\}.$$

For $j = 0, 1, \ldots i+1$ this probability is given by the probability that exactly $i + 1 - j$ customers arrive during the $k - th$ service time:

$$\begin{aligned} P\left\{\tilde{n}_a(k+1) = j | \tilde{n}_d(n) = 0\right\} &= \left(\frac{\mu}{\lambda+\mu}\right)^{1-j}, \\ P\left\{\tilde{n}_a(k+1) = j | \tilde{n}_d(n) = i\right\} &= \frac{\lambda}{\lambda+\mu}\left(\frac{\mu}{\lambda+\mu}\right)^{i+1-j}, \quad i > 0 \end{aligned}$$

For $j > i + 1$, this probability is equal to zero since only departures may occur between arrivals. This results in

$$\mathcal{P}_a = \begin{pmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 & 0 & 0 & \ldots \\ \left(\frac{\mu}{\lambda+\mu}\right)^2 & \frac{\lambda}{\lambda+\mu}\frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \ldots \\ \left(\frac{\mu}{\lambda+\mu}\right)^3 & \frac{\lambda}{\lambda+\mu}\left(\frac{\mu}{\lambda+\mu}\right)^2 & \frac{\lambda}{\lambda+\mu}\frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Form the system of equations $\pi_a = \pi_a \mathcal{P}_a$. Then, for $j = 0$,

$$\begin{aligned} \pi_{a_0} &= \sum_{i=0}^{\infty} \pi_{a_i} \mathcal{P}_{a_{i0}} \\ &= \frac{\mu}{\lambda+\mu} \sum_{i=0}^{\infty} \pi_{a_i} \left(\frac{\mu}{\lambda+\mu}\right)^i. \end{aligned}$$

Now, to show that $\pi_a = \pi_d$, it suffices to show that π_d is a solution to $\pi_a = \pi_a \mathcal{P}_a$. Substitute π_{d_0} into the equation just found for π_{a_0}:

$$\begin{aligned} \pi_{a_0} &= \frac{\mu}{\lambda+\mu} \sum_{i=0}^{\infty} \frac{\mu-\lambda}{\mu} \left(\frac{\lambda}{\mu}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)^i \\ &= \frac{\mu-\lambda}{\lambda+\mu} \sum_{i=0}^{\infty} \left(\frac{\lambda}{\lambda+\mu}\right)^i \\ &= \frac{\mu-\lambda}{\lambda+\mu} \cdot \frac{\lambda+\mu}{\mu} \\ &= \frac{\mu-\lambda}{\mu} \end{aligned}$$

which is the same result found for π_{d_0} in part (a).

Now generalize to π_{a_j}, $j > 0$, and substitute π_{d_i} for π_{a_i}:

$$\pi_{a_j} = \frac{\lambda}{\lambda+\mu} \sum_{i=0}^{\infty} \left(\frac{\mu}{\lambda+\mu}\right)^i \pi_{a_{j-1+i}}$$

$$\begin{aligned}
&= \frac{\lambda}{\lambda+\mu}\left(\frac{\mu-\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{j-1}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\lambda+\mu}\right)^{i} \\
&= \frac{\lambda}{\lambda+\mu}\left(\frac{\mu-\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{j-1}\frac{\lambda+\mu}{\mu} \\
&= \frac{\mu-\lambda}{\mu}\left(\frac{\lambda}{\mu}\right)^{j}.
\end{aligned}$$

Thus $\pi_a = \pi_d$.

3. Determine $\mathcal{P}$, the one-step transition probability matrix. If the system is in state 0 at time k, then with probability 1 it will be in state 1 at time $(k+1)$. The probability of it being in any other state at time $(k+1)$ is 0. If the system is in state i, ($i \neq 0$) at time k, then the next event will either be a departure or an arrival. If it's a departure, the system wil be in state $(i-1)$ at time $(k+1)$. This happens with probability $\frac{\mu}{\lambda+\mu}$. If the next event is an arrival, then the system will be in state $(i+1)$ at time $(k+1)$. This will occur with probability $\frac{\lambda}{\lambda+\mu}$. Thus,

$$\mathcal{P} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
\frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \cdots \\
0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \cdots \\
0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Form the system of equations $\pi_j = \sum_{i=0}^{\infty}\pi_j\mathcal{P}_{ij}$. Then for $j=0$:

$$\pi_0 = 0 + \pi_1\left(\frac{\mu}{\lambda+\mu}\right) + \sum_{i=2}^{\infty}\pi_j \cdot 0.$$

That is, $\pi_1 = \pi_0\frac{\lambda+\mu}{\mu}$. Substituting this into $\pi_1 = \sum_{i=0}^{\infty}\pi_j\mathcal{P}_{1j}$ gives the solution $\pi_2 = \pi_0\left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)$. Repeating this process we see that for $j \geq 1$,

$$\pi_j = \pi_0\frac{\lambda+\mu}{\mu}\left(\frac{\lambda}{\mu}\right)^{j-1}.$$

Now use the normalizing constraint to find π_0:

$$\begin{aligned}
1 &= \pi_0 + \sum_{j=0}^{\infty}\pi_0\left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{j-1} \\
&= \pi_0\left[1+\left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{1}{1-\frac{\lambda}{\mu}}\right)\right].
\end{aligned}$$

i.e., $\pi_0 = (\mu - \lambda)/2\mu$. And so for $j \geq 1$,

$$\pi_j = \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{j-1}\frac{\mu-\lambda}{2\mu}.$$

To compute P_i, first determine $E[\tilde{s}_i]$, the expected amount of time the system spends in state i on each visit. Note that $\tilde{s}_i$ is an exponential random variable, since the time spent in state i is determined by when the next arrival or departure may occur. The rate for $\tilde{s}_0$ is λ, since only an arrival may occur. And the rates for all other $\tilde{s}_i$, $i > 0$, is $\lambda + \mu$, since either an arrival or a departure may occur. Thus by the properties of the exponential distribution,

$$\begin{aligned} E[\tilde{s}_0] &= \frac{1}{\lambda} \\ E[\tilde{s}_i] &= \frac{1}{\lambda+\mu}, \qquad i > 0. \end{aligned}$$

Then

$$\begin{aligned} \sum_{i=0}^{\infty}\pi_i E[\tilde{s}_i] &= \pi_0 E[\tilde{s}_0] + \sum_{i=1}^{\infty}\left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{i-1}\left(\frac{\mu-\lambda}{2\mu}\right)\left(\frac{1}{\lambda+\mu}\right) \\ &= \frac{\mu-\lambda}{2\mu}\cdot\frac{1}{\lambda} + \frac{\mu-\lambda}{2\mu^2}\left[\frac{1}{1-\frac{\lambda}{\mu}}\right] \\ &= \frac{1}{2\lambda}. \end{aligned}$$

Using this result and the $\pi_j's$ obtained above in

$$P_i = \frac{\pi_i E[\tilde{s}_i]}{\sum\limits_{i=0}^{\infty}\pi_i E[\tilde{s}_i]}$$

we get the solutions

$$\begin{aligned} P_0 &= \frac{\frac{\mu-\lambda}{2\mu}\cdot\frac{1}{\lambda}}{\frac{1}{2\lambda}} = 1 - \frac{\lambda}{\mu} \\ P_i &= \frac{\left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{i-1}\left(\frac{\mu-\lambda}{2\mu}\right)\left(\frac{1}{\lambda+\mu}\right)}{\frac{1}{2\lambda}} \\ &= \left(\frac{\lambda}{\mu}\right)^j\left(1-\frac{\lambda}{\mu}\right). \end{aligned}$$

This is precisely the results of parts (a) and (b).

EXERCISE 3.6 Using Little's result, show that the probability that the server is busy at an arbitrary point in time is equal to the quantity (λ/μ).

Solution. Let $\mathcal{B}$ denote the event that the server is busy. Define the system to be the server only (i.e., no queue). Then

$$P\{\mathcal{B}\} = P\{\tilde{n} = 1\}$$

where $\tilde{n}_s$ is the number of customers in the system (i.e., $\tilde{n}_s$ is the number in service). Hence

$$\begin{aligned} E[\tilde{n}] &= 0 \cdot P\{\tilde{n} = 0\} + 1 \cdot P\{\tilde{n} = 1\} \\ &= P\{\tilde{n} = 1\} \\ &= P\{\mathcal{B}\}. \end{aligned}$$

Furthermore, the average waiting time in the system is just the average service time. That is, $E[\tilde{w}] = \frac{1}{\mu}$. By Little's Result, $E[\tilde{n}] = \lambda E[\tilde{w}]$. Thus

$$P\{\mathcal{B}\} = E[\tilde{n}] = \frac{\lambda}{\mu}.$$

EXERCISE 3.7 Let $\tilde{w}$ and $\tilde{s}$ denote the length of time an arbitrary customer spends in the queue and in the system, respectively, in stochastic equilibrium. Let $F_{\tilde{s}}(x) \equiv P\{\tilde{s} \leq x\}$ and $F_{\tilde{w}}(x) \equiv P\{\tilde{w} \leq x\}$. Show that

$$F_{\tilde{s}}(x) = 1 - e^{-\mu(1-\rho)x}, \qquad \text{for } x \geq 0,$$

and

$$F_{\tilde{w}}(x) = 1 - \rho e^{-\mu(1-\rho)x}, \qquad \text{for } x \geq 0,$$

without resorting to the use of Laplace-Stieltjes transform techniques.

Solution. To compute $F_{\tilde{s}}(x)$,condition on the value of $\tilde{n}$, the number of customers in the system:

$$F_{\tilde{s}}(x) = \sum_{n=0}^{\infty} P\{\tilde{s} \leq x | \tilde{n} = n\} P\{\tilde{n} = n\}, \tag{3.6.1}$$

where $P\{\tilde{n} = n\} = \rho^n(1-\rho)$.
If a customer arrives at time t and finds n customers in the system, then the probability that she departs the system by time $t + x$ equals the probability that there will be $n+1$ service completions in the interval $(t, t+x]$: the original n customers plus herself. This probability is $P\{\tilde{s}_{n+1} \leq x\}$. Recall Exercise 2.20:

$$\frac{d}{dx} P\{\tilde{s}_{n+1} \leq x\} = \frac{\mu(\mu x)^n e^{-\mu x}}{n!}.$$

By differentiating and then integrating both sides of $(3.6.1)$,

$$\begin{aligned}
F_{\tilde{s}}(x) &= \int_0^x \sum_{n=0}^{\infty} \frac{\mu(\mu\alpha)^n e^{-\mu\alpha}}{n!}\rho^n(1-\rho)d\alpha \\
&= \int_0^x \mu(1-\rho)e^{-\mu\alpha}\sum_{n=0}^{\infty}\frac{(\mu\rho\alpha)^n}{n!}d\alpha \\
&= \int_0^x \mu(1-\rho)e^{-\mu(1-\rho)\alpha}d\alpha \\
&= 1-e^{-\mu(1-\rho)x}, \qquad x \geq 0.
\end{aligned}$$

Now find $F_{\tilde{w}}(x)$:

$$\begin{aligned}
F_{\tilde{w}}(x) &= P\{\tilde{w} \leq 0\} + P\{\tilde{w} < x\}, \qquad x > 0 \\
&= P\{\tilde{w} \leq 0\} + int_0^x f_{\tilde{w}}(\alpha)d\alpha, \qquad x > 0.
\end{aligned}$$

But $P\{\tilde{w} \leq 0\} = P\{\tilde{w} = 0\} = P\{\tilde{n} = 0\} = 1-\rho$, so

$$F_{\tilde{w}}(x) = (1-\rho) + \int_0^x f_{\tilde{w}}(\alpha)d\alpha, \qquad x > 0.$$

Condition $f_{\tilde{w}}$ on the value of $\tilde{n}$, the number of customers in the system and use the fact that the system is in stochastic equilibrium:

$$\begin{aligned}
F_{\tilde{w}}(x) &= (1-\rho) + \int_0^x \sum_{n=1}^{\infty}\frac{\mu(\mu\alpha)^{n-1}e^{-\mu\alpha}}{(n-1)!}\rho^n(1-\rho)d\alpha \\
&= (1-\rho) + \int_0^x \mu\rho(1-\rho)e^{-\mu\alpha}\sum_{n=0}^{\infty}\frac{(\mu\rho\alpha)^n}{n!}d\alpha \quad \text{(reindexing)} \\
&= (1-\rho) + \int_0^x \mu\rho(1-\rho)e^{-\mu(1-\rho)\alpha}d\alpha \\
&= 1-\rho e^{-\mu(1-\rho)x}, \qquad x \geq 0.
\end{aligned}$$

EXERCISE 3.8 **M/M/1 Departure Process.** Show that the distribution of an arbitrary interdeparture time for the M/M/1 system in stochastic equilibrium is exponential with the same parameter as the interarrival-time distribution. Argue that the interdeparture times are independent so that the departure process for this system is Poisson with the same rate as the arrival process (Burke [1956]). [*Hint*: Use the fact that the Poisson arrival sees the system in stochastic equilibrium. Then condition on whether or not the ith departing customer leaves the system empty.]

Solution. Observe that the Poisson arrivals will see the system in stochastic equilibrium. Using this fact, condition on whether or not the $i-th$ departing customer leaves the system empty. Now, if the system is not left empty, the time to the next departure is simply the time it takes for the $(i+1)-st$ customer to complete service. This is Poisson with parameter μ. If the system is left empty, however, the time to the next departure is the time to the next arrival plus the time for that arrival to complete service. Denoting the event that the system is left empty by $\mathcal{A}$,

$$P\left\{\tilde{d} \le d|\mathcal{A}\right\} = P\left\{\tilde{a} + \tilde{x} \le d\right\}.$$

Recall that in stochastic equilibrium the probability that the system is empty is $(1-\rho)$. Then if $\mathcal{B}$ denotes the event that the system is not left empty,

$$\begin{aligned} P\left\{\tilde{d} \le d\right\} &= P\left\{\tilde{d} \le d|\mathcal{B}\right\} P\left\{\mathcal{B}\right\} + P\left\{\tilde{d} \le d|\mathcal{A}\right\} P\left\{\mathcal{A}\right\} \\ &= \rho P\left\{\tilde{x} \le d\right\} + (1-\rho) P\left\{\tilde{a} + \tilde{x} \le d\right\}. \end{aligned}$$

Sum over all possible values of $\tilde{a}$ to get $P\left\{\tilde{a} + \tilde{x} \le d\right\}$, noting that $P\left\{\tilde{a} + \tilde{x} \le d\right\} = 0$ if $\tilde{a} > d$. Hence

$$P\left\{\tilde{d} \le d\right\} = \rho\left(1 - e^{-\mu d}\right) + (1-\rho)\int_0^d P\left\{\tilde{a} + \tilde{x} \le d|\tilde{a} = a\right\} dF_{\tilde{a}}.$$

Since $\tilde{a}$ and $\tilde{x}$ are independent of each other, this is

$$\begin{aligned} P\left\{\tilde{d} \le d\right\} &= \rho\left(1 - e^{-\mu d}\right) + (1-\rho)\int_0^d P\left\{a + \tilde{x} \le d\right\} \lambda e^{-\lambda a} da \\ &= \rho\left(1 - e^{-\mu d}\right) + (1-\rho)\int_0^d \left(1 - e^{-\mu(d-a)}\right) \lambda e^{-\lambda a} da \\ &= 1 - e^{-\lambda d}. \end{aligned}$$

This shows the departure process $\tilde{d}$ is Poisson with parameter λ. That is, the departure process occurs at the same rate as the arrival process, which we know to be independent. Because of this 'rate in equals rate out' characteristic, this implies that the interdeparture times are also independent. If the interdeparture times were not independent, then its distribution could not be that of the interarrival times.

EXERCISE 3.9 M/M/1 with Instantaneous Feedback. A queueing system has exogenous Poisson arrivals with rate λ and exponential service with rate μ. At the instant of service completion, each potentially departing customer rejoins the service queue, independent of system state, with probability p.

1. Determine the distribution of the total amount of service time the server renders to an arbitrary customer.

2. Compute the distribution of the number of customers in the system in stochastic equilibrium. How does your solution compare to that of the M/M/1 queueing system? What explains this behavior? [*Hint*: Consider the remaining service time required for each customer in the queue. Suppose customers that required additional increments of service returned immediately to service rather than joining the tail of the queue. What would be the effect on the queue occupancy?]

3. Argue that the departure process from the system is a Poisson process with rate λ.

4. Compute the average sojourn time for this system and comment on computation of the distribution of the sojourn time.

Solution.

1. Let $\tilde{t}$ be the total amount of service time rendered to an arbitrary customer. Condition on $\tilde{n}$, the number of times the customer joins the queue.

$$F_{\tilde{t}}(x) = \sum_{n=1}^{\infty} P\left\{\tilde{t} \leq x | \tilde{n} = n\right\} P\left\{\tilde{n} = n\right\}. \tag{3.8.1}$$

Now, the probability of the customer entering the queue n times is simply $p^{n-1}(1-p)$ since he returns to the queue $(n-1)$ times with probability p^{n-1} and leaves after the n-th visit with probability $(1-p)$. Furthermore, if the customer was in the queue n times, then $P\left\{\tilde{t} \leq x | \tilde{n} = n\right\}$ is the probability that the sum of his n service times will be less than x. That is,

$$\begin{aligned} P\left\{\tilde{t} \leq x | \tilde{n} = n\right\} &= P\left\{\sum_{i=1}^{\infty} \tilde{x}_i \leq x\right\} \\ &= P\left\{\tilde{s}_n \leq x\right\}, \qquad \tilde{x}_i \geq 0. \end{aligned}$$

Recall Exercise 2.20:

$$\frac{d}{dx} P\{\tilde{s}_{n+1} \leq x\} = \frac{\mu(\mu x)^n e^{-\mu x}}{n!}.$$

Then, upon differentiating and then integrating both sides of (3.8.1),

$$\begin{aligned}
F_{\tilde{t}}(x) &= \int_0^x \sum_{n=1}^{\infty} \frac{\mu(\mu\alpha)^{n-1}e^{-\mu\alpha}}{(n-1)!} p^{n-1}(1-p)d\alpha \\
&= \int_0^x \mu e^{-\mu\alpha}(1-p)\sum_{n=1}^{\infty}\frac{(\mu p\alpha)^n}{n!}d\alpha \\
&= \int_0^x \mu e^{-\mu\alpha}(1-p)e^{\mu p\alpha}d\alpha \\
&= 1-e^{-\mu(1-p)x}, \qquad x \geq 0.
\end{aligned}$$

2. Consider those customers who, on their first pass through the server, still have increments of service time remaining. Suppose that instead of joining the end of the queue, they immediately reenter service. In effect, they 'use up' all of their service time on the first pass. This will simply rearrange the order of service. Since the queue occupancy does not depend on order of service, reentering these customers immediately will have no effect on the queue occupancy. Observe that the arrival process is now a Poisson process: since customers complete service now in one pass and don't join the end of the queue as they did before, they arrive to the queue only once. Hence, arrivals are independent. With this is mind, we see that the system can now be modeled as an ordinary M/M/1 system with no feedback. From part (a), the total amount of service time for each customer is exponential with parameter $(1-p)\mu$. The distribution of the number of customers in the system in stochastic equilibrium is the same as that of an ordinary M/M/1 system, with this new service rate: $P_n = (1-\rho)\rho^n$, where $\rho = \frac{\lambda}{(1-p)\mu}$.

3. As argued in part(b), we can model this feedback system as one in which there is no feedback and whose service rate is $(1-p)\mu$. Since the departure process does not depend on the actual rate of the server, it will remain a Poisson process with the same rate as that of the arrival process.

4. Using the results of part(b), model this system as an ordinary M/M/1 queueing system whose service is exponential with rate $(1-\rho)\mu$. Then $E[\tilde{s}] = \frac{1}{(1-p)\mu}\frac{1}{1-\rho}$, where $\rho = \frac{\lambda}{(1-\rho)\mu}$.

EXERCISE 3.10 For the M/M/1 queueing system,

1. find $E[\tilde{h}]$, the expected number of customers served in busy period, and
2. find $E[e^{-s\tilde{y}}]$, the Laplace-Stieltjes transform of the distribution of the length of a busy period. Show that $(d/dy)F_{\tilde{y}}(y) = 1/(y\sqrt{\rho})e^{-(\lambda+\mu)y}I_1(2y\sqrt{\lambda\mu})$. A Laplace transform pair,

$$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}} \Longleftrightarrow \frac{1}{t}e^{-at}I_1(at),$$

taken from *Mathematical Tables from the Handbook of Physics and Chemistry*, will be useful in accomplishing this exercise.

Solution.

1. Let $\mathcal{D}$ denote the event that the first customer completes service before the first arrival after the busy period has begun, and let $\mathcal{A}$ denote the complement of $\mathcal{D}$. Then

$$E[\tilde{h}] = E[\tilde{h}|\mathcal{D}]P\{\mathcal{D}\} + E[\tilde{h}|\mathcal{A}]P\{\mathcal{A}\}.$$

If $\mathcal{D}$ occurs then clearly only the initial customer will complete service during that busy period; that is, $E[\tilde{h}|\mathcal{D}] = 1$. On the other hand, if $\mathcal{D}$ occurs, then the length of the busy period has been shown to be the length of the interarrival time plus twice the length of $\tilde{y}$, a generic busy period. Now, no service will be completed during the interarrival period, but $\tilde{h}$ service completions will take place during each busy period length $\tilde{y}$. Thus,

$$\begin{aligned} E[\tilde{h}] &= 1\cdot P\{\mathcal{D}\} + E[\tilde{h}+\tilde{h}]P\{\mathcal{A}\} \\ &= \frac{\mu}{\lambda+\mu} + 2E[\tilde{h}]\frac{\lambda}{\lambda+\mu}. \end{aligned}$$

Then, if $\rho = \frac{\lambda}{\mu}$,

$$E[\tilde{h}] = \frac{\mu}{\mu-\lambda} = \frac{1}{1-\rho}.$$

2. Condition $E[e^{-s\tilde{y}}]$ on whether or not the first customer completes service before the first arrival after the busy period has begun. Let $\mathcal{D}$, $\mathcal{D}$ be the same as in part (i). Then

$$E[e^{-s\tilde{y}}] = E[e^{-s\tilde{y}}|\mathcal{D}]P\{\mathcal{D}\} + E[e^{-s\tilde{y}}|\mathcal{A}]P\{\mathcal{A}\}.$$

Now,

$$E[e^{-s\tilde{y}}|\mathcal{D}] = E[e^{-s\tilde{z}_1}],$$

and

$$\begin{aligned} E[e^{-s\tilde{y}}|\mathcal{A}] &= E[e^{-s(\tilde{z}_1+\tilde{y}+\tilde{y})}] \\ &= E[e^{-s\tilde{z}_1}]\left(E[e^{-s\tilde{y}}]\right)^2. \end{aligned}$$

Thus,

$$E[e^{-s\tilde{y}}] = P\{\mathcal{D}\}E[e^{-s\tilde{z}_1}] + P\{\mathcal{A}\}E[e^{-s\tilde{z}_1}]\left(E[e^{-s\tilde{y}}]\right)^2.$$

This implies that

$$0 = P\{\mathcal{A}\}E[e^{-s\tilde{z}_1}]\left(E[e^{-s\tilde{y}}]\right)^2 - E[e^{-s\tilde{y}}] + P\{\mathcal{D}\}E[e^{-s\tilde{z}_1}].$$

Substituting the expressions for $E[e^{-s\tilde{z}_1}]$, $P\{\mathcal{A}\}$, and $P\{\mathcal{D}\}$, and using the quadratic formula,

$$E[e^{-s\tilde{y}}] = \frac{(s+\lambda+\mu) \pm \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\lambda}.$$

We now must decide which sign gives the proper function. By the definition of the Laplace-Stieltjes transform, $E[e^{-s\tilde{y}}]|_{s=0}$ is simply the cumulative distribution function of the random variable evaluated at infinity. This value is known to be 1. Then

$$\begin{aligned} E[e^{-0\cdot\tilde{y}}] &= 1 \\ &= \frac{(\lambda+\mu) \pm \sqrt{(\lambda+\mu)^2 - 4\lambda\mu}}{2\lambda} \\ &= \frac{(\lambda+\mu) \pm \sqrt{(\lambda-\mu)^2}}{2\lambda} \\ &= \frac{(\lambda+\mu) \pm \sqrt{(\mu-\lambda)^2}}{2\lambda} \\ &= \frac{(\lambda+\mu) \pm (\mu-\lambda)}{2\lambda}. \end{aligned}$$

This of course implies that the sign should be negative. i.e.,

$$E[e^{-s\tilde{y}}] = \frac{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\lambda}.$$

It remains to show that

$$\frac{d}{dy}F_{\tilde{y}}(y) = \frac{1}{y\sqrt{\rho}}e^{-(\lambda+\mu)y}I_1(2y\sqrt{\lambda\mu}).$$

We first prove a lemma. *Lemma*: Let $g(t) = e^{-at}f(t)$. Then $G(s) =$

$F(s+a)$.

proof:

Recall that if $F(s)$ is the Laplace transform of $f(t)$, then

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Observe that if s is replaced by $(s+a)$ we get

$$\begin{aligned} F(s+a) &= \int_0^\infty f(t)e^{-(s+a)t}dt \\ &= \int_0^\infty f(t)e^{-at}e^{-st}dt \\ &= \int_0^\infty g(t)e^{-st}dt \\ &= G(s) \end{aligned}$$

by the definition of a Laplace transform. This proves the Lemma. Using this Lemma, we wish to invert

$$\alpha(s) = \frac{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\lambda}$$

given

$$\frac{\sqrt{s+2a} - \sqrt{s}}{\sqrt{s+2a} + \sqrt{s}} \Longleftrightarrow \frac{1}{t}e^{-at}I_1(at).$$

First note

$$\begin{aligned} \beta(s) &= \frac{\sqrt{s+2a} - \sqrt{s}}{\sqrt{s+2a} + \sqrt{s}} \\ &= \frac{\left(\sqrt{s+2a} - \sqrt{s}\right)}{\left(\sqrt{s+2a} + \sqrt{s}\right)} \frac{\left(\sqrt{s+2a} - \sqrt{s}\right)}{\left(\sqrt{s+2a} - \sqrt{s}\right)} \\ &= \frac{(s+a) - \sqrt{s(s+2a)}}{a} \end{aligned}$$

where

$$\beta(s) \Longleftrightarrow \frac{1}{t}e^{-at}I_1(at).$$

To apply the Lemma to invert $\alpha(s)$, we first need to manipulate $\alpha(s)$ so it is in the form of $\beta(s)$. Define

$$\gamma(s) \quad = \quad \beta(s+b) = \frac{(s+a+b) - \sqrt{(s+b)(s+2a+b)}}{a}$$

$$= \frac{(s+a+b) - \sqrt{(s+2a+b)^2 - a^2}}{a}.$$

Let $(a+b) = (\lambda + \mu)$ and $a^2 = 4\lambda\mu$. Then $b = (\lambda+\mu) - 2\sqrt{\lambda\mu}$.

$$\begin{aligned}
\gamma(s) &= \beta(s + (\lambda+\mu) - 2\sqrt{\lambda\mu}) \\
&= \frac{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\sqrt{\lambda\mu}} \\
&= \frac{1}{\rho}\left[\frac{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\lambda}\right].
\end{aligned}$$

Note that

$$\gamma(s) = \alpha(s)\sqrt{\rho} \Longrightarrow \alpha(s) = \frac{1}{\sqrt{\rho}}\gamma(s),$$

Then by applying the Lemma,

$$\begin{aligned}
L^{-1}[\gamma(s)] &= e^{-bt}L^{-1}[\beta(s)] \\
&= e^{-bt}\frac{1}{t}e^{-at}I_1(at) \\
&= \frac{1}{t}e^{-(a+b)t}I_1(at).
\end{aligned}$$

By linearity, $\quad af(t) \Longleftrightarrow aL[f(t)]$.

$$\begin{aligned}
L^{-1}[\alpha(s)] &= \frac{1}{\sqrt{\rho}}L^{-1}[\gamma(s)] \\
&= \frac{1}{\sqrt{\rho}}\frac{1}{t}e^{-(a+b)t}I_1(at) \\
&= \frac{1}{\sqrt{\rho}}\frac{1}{t}e^{-(\lambda+\mu)t}I_1(2t\sqrt{\lambda\mu}).
\end{aligned}$$

Or, replacing t with y,

$$\frac{d}{dy}F_{\tilde{y}}(y) = \frac{1}{\sqrt{\rho}}\frac{1}{y}e^{-(\lambda+\mu)y}I_1(2y\sqrt{\lambda\mu}).$$

This is the desired result.

EXERCISE 3.11 For the M/M/1 queueing system, argue that $\tilde{h}$ is a stopping time for the sequence $\{\tilde{x}_i, i = 1, 2, \ldots\}$ illustrated in Figure 3.5. Find $E[\tilde{h}]$ by using the results given above for $E[\tilde{y}]$ in combination with Wald's equation.

Solution. Recall that for $\tilde{h}$ to be a stopping time for a sequence of random variables, $\tilde{h}$ must be independent of $\tilde{x}_{\tilde{h}+1}$. Now, $\tilde{h}$ describes the number of customers served in a busy period and so $\tilde{x}_{\tilde{h}}$ is the last customer served during that particular busy period; all subsequent customers $\tilde{x}_{\tilde{h}+j}$, $j \geq 1$, are served in another busy period. Hence $\tilde{h}$ is independent of those customers. This defines $\tilde{h}$ to be a stopping time for the sequence $\{\tilde{x}_i, i = 1, 2, \cdots\}$. We can thus apply Wald's equation

$$E[\tilde{y}] = E\left[\sum_{i=1}^{\tilde{h}} \tilde{x}_i\right] = E[\tilde{h}]E[\tilde{x}].$$

Hence,

$$\begin{aligned} E[\tilde{h}] &= \frac{E[\tilde{y}]}{E[\tilde{x}]} \\ &= \left[\frac{1/\mu}{1-\rho}\right] \Big/ \frac{1}{\mu} \\ &= \frac{1}{1-\rho}. \end{aligned}$$

EXERCISE 3.12 For the M/M/1 queueing system, argue that $E[\tilde{s}]$, the expected amount of time a customer spends in the system, and the expected length of a busy period are equal. [*Hint*: Consider the expected waiting time of an arbitrary customer in the M/M/1 queueing system under a nonpreemptive LCFS and then use Little's result.]

Solution. Let $\mathcal{B}$ be the event that an arbitrary customer finds the system empty upon arrival, and let I be the event that the system is found to be idle. Then

$$E[\tilde{s}] = E[\tilde{s}|\mathcal{B}]P\left\{\mathcal{B}\right\} + E[\tilde{s}|I]P\left\{I\right\}.$$

Now, if the system is idle with probability $(1-\rho)$, then the customer's sojourn time will just be her service time. On the other hand, if the system is busy upon arrival, then the customer has to wait until the system is empty again to receive service. That is, $\tilde{w}$ in this case is equivalent to $\tilde{y}$, and so

$$\begin{aligned} E[\tilde{s}|\mathcal{B}] &= E[\tilde{w}] + E\tilde{x}] \\ &= E[\tilde{y}] + \frac{1}{\mu} \\ &= \frac{\frac{1}{\mu}}{1-\rho} + \frac{1}{\mu} \\ &= \frac{1}{\mu}\left[\frac{1}{1-\rho} + 1\right]. \end{aligned}$$

Combining the two conditional probabilities, we see that

$$E[\tilde{s}] \quad = \quad \frac{\rho}{\mu}\left[\frac{1}{1-\rho} + 1\right] + \frac{1-\rho}{\mu}$$

$$= \frac{1}{\mu}\left[\frac{1}{1-\rho}\right]$$
$$= E[\tilde{y}].$$

EXERCISE 3.13 Let $\tilde{s}_{\mathrm{LCFS}}$ denote the total amount of time an arbitrary customer spends in the M/M/1 queueing system under a nonpreemptive discipline. Determine the Laplace-Stieltjes transform for the distribution of $\tilde{s}_{\mathrm{LCFS}}$.

Solution. Condition the Laplace-Stieltjes transform for the distribution of $\tilde{s}_{\mathrm{LCFS}}$ on whether or not the service is busy when an arbitrary customer arrives. If the server is idle, the customer will immediately enter service and so the Laplace-Stieltjes transform of $\tilde{s}_{\mathrm{LCFS}}$ is that of $\tilde{x}$, the service time. If the server is busy, however, the customer's total time in the system will be waiting time in the queue plus service time. It has already been shown in Exercise 3.11 that in a LCFS discipline the waiting time in queue has the same distribution as $\tilde{y}$, an arbitrary busy period. Let $\mathcal{B}$ denote the event that the customer finds the server is busy, and let $\mathcal{B}^c$ denote the event that the server is idle.

$$\begin{aligned}
E[e^{-s\tilde{s}_{\mathrm{LCFS}}}] &= E[e^{-s\tilde{s}_{\mathrm{LCFS}}}|\mathcal{B}^c]P\left\{\mathcal{B}^c\right\} + E[e^{-s\tilde{s}_{\mathrm{LCFS}}}|\mathcal{B}]P\left\{\mathcal{B}\right\} \\
&= (1-\rho)E[e^{-s\tilde{x}}] + \rho\{E[e^{-s\tilde{y}}] + E[e^{-s\tilde{x}}]\} \\
&= E[e^{-s\tilde{x}}] + \rho E[e^{-s\tilde{y}}] \\
&= \frac{\mu}{s+\mu} + \rho\cdot\frac{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\lambda}.
\end{aligned}$$

EXERCISE 3.14 Determine the Laplace-Stieltjes transform for the length of the busy period for the M/M/2 queueing system, the system having Poisson arrivals, exponential service, two parallel servers, and an infinite waiting room capacity. [*Hint*: Condition on whether or not an arrival occurs prior to the completion of the first service of a busy period. Then note that there is a very close relationship between the time required to reduce the occupancy from two customers to one customer in the M/M/2 and the length of the busy period in the ordinary M/M/1 system.]

Solution. Let $\tilde{y}_2$ denote the length of a generic busy period in the M/M/1 queueing system and suppose that a customer arrives, beginning the busy period. Condition $E[e^{-s\tilde{y}_2}]$ on whether or not the customer finishes service before the next customer arrives. Let $\mathcal{A}$ denote the event of an arrival before completion of service, and let $\mathcal{D}$ denote the event the customer completes service before the next arrival:

$$E[e^{-s\tilde{y}_2}] = E[e^{-s\tilde{y}_2}|\mathcal{D}]P\left\{\mathcal{D}\right\} + E[e^{-s\tilde{y}_2}|\mathcal{A}]P\left\{\mathcal{A}\right\}.$$

If the original customer departs before this arrival, then the length of the busy period is the minimum of the service time and the interarrival time. Hence we see that

$$E[e^{-s\tilde{y}_2}|\mathcal{D}] = E[e^{-s\tilde{z}_1}],$$

which we know to be $\frac{\mu+\lambda}{s+\mu+\lambda}$. If the next arrival occurs before the original customer finishes service, then there will be two customers in the system. Call this state of the system 'state 2'. The second server will be activated and the overall service rate of the system will be 2μ. It will continue to be 2μ until there is only one customer in the system again. Call this state of the system 'state 1'. Consider $\tilde{f}_{2,1}$, the time it takes to return to state 1 from state 2. (This is called the 'first passage time from state 2 to state 1.') Think of the time spent in state 2 as an ordinary M/M/1 busy period, one in which the service rate is 2μ. Then

$$E[e^{-s\tilde{f}_{2,1}}] = \frac{(s+\lambda+2\mu) - \sqrt{(s+\lambda+2\mu)^2 - 8\lambda\mu}}{2\lambda}.$$

This follows directly from the definition of $E[e^{-s\tilde{y}}]$, the Laplace-Stieltjes transformof a generic busy period in the M/M/1 system, with 2μ substituted in for the service rate. Once the system returns to state 1 again, note that this is exactly the state it was in originally. Because of the Markovian properties, the expected length of the remaining time in the busy period should be the same as it was originally: $E[e^{-s\tilde{y}_2}]$. With these observations, we see that

$$E[e^{-s\tilde{y}_2}|\mathcal{A}] = E[e^{-s(\tilde{z}_1+\tilde{f}_{2,1}+\tilde{y}_2)}],$$

where $\tilde{z}_1$ represents the interarrival time between the original customer and the first arrival after the busy period begins. Hence,

$$\begin{aligned} E[e^{-s\tilde{y}_2}] &= P\{\mathcal{D}\}\, E[e^{-s\tilde{y}_2}|\mathcal{D}] + P\{\mathcal{A}\}\, E[e^{-s\tilde{y}_2}|\mathcal{A}] \\ &= P\{\mathcal{D}\}\, E[e^{-s\tilde{z}_1}] + P\{\mathcal{A}\}\, E[e^{-s\tilde{z}_1}]E[e^{-s\tilde{f}_{2,1}}]E[e^{-s\tilde{y}_2)}]. \end{aligned}$$

Substituting the expressions for $E[e^{-s\tilde{z}_1}]$, $E[e^{-s\tilde{f}_{2,1}}]$, $P\{\mathcal{A}\}$, and $P\{\mathcal{D}\}$, and rearranging terms,

$$E[e^{-s\tilde{y}_2}] = \frac{2\mu}{(s+\lambda) + \sqrt{(s+\lambda+2\mu)^2 - 8\lambda\mu}}.$$

EXERCISE 3.15 We have shown that the number of arrivals from a Poisson process with parameter λ, that occur during an exponentially distributed service time with parameter μ, is geometrically distributed with parameter $\mu/(\mu+\lambda)$; that is, the probability of n arrivals during a service time is given by $[\lambda/(\lambda+\mu)]^n[\mu/(\lambda+\mu)]$. Determine the mean length of the busy period by conditioning on the number of arrivals that occur during the first service time of the busy period. For example, let $\tilde{n}_1$ denote the number of arrivals that occur during the first service time, and start your solution with the statement

$$E[\tilde{y}] = \sum_{n=0}^{\infty} E[\tilde{y}|\tilde{n}_1 = n]P\{\tilde{n}_1 = n\}.$$

[*Hint*: The arrivals segment the service period into a sequence of intervals.]

Solution. Condition $E[\tilde{y}]$ on the number of customers who arrive during the service time of the original customer. Now, if no customers arrive, then the busy period ends with the completion of the original customer's service time. It has already been shown that $\tilde{y}|\{\tilde{x}_1 < \tilde{t}_1\} = \tilde{z}_1$. Thus,

$$E[\tilde{y}|\tilde{n} = 0] = E[\tilde{z}_1] = \frac{1}{\mu+\lambda}.$$

Now suppose that exactly one new customer arrives during the original customer's service period. Then due to the memoryless property of the exponential distribution, this service time starts over. So the remaining time in this busy period is equal to the length of a busy period in which there are initially two customers present, one of whom's busy period ends service with no new arrivals. That is,

$$\tilde{y}|\{\tilde{x}_1 > \tilde{t}_1\} = \tilde{z}_1 + \tilde{y} + \left(\tilde{y}|\{\tilde{x}_1 < \tilde{t}_1\}\right),$$

where the first term, $\tilde{z}_1$, represents the interarrival time between the original customer and the first arrival. We've already show above that the last term of this sum is equivalent to $\tilde{z}_1$. Hence,

$$E[\tilde{y}|\tilde{n} = 1] = 2E[\tilde{z}_1] + E[\tilde{y}].$$

Repeating this process, we see that if there are n arrivals during the service period of the initial customer, then this is equivalent to the length of a busy period that has $(n+1)$ initial customers present: n customers we know nothing about (and so have generic busy periods), and 1 customer whose service period ends with no new arrivals. i.e.

$$\begin{aligned} E[\tilde{y}|\tilde{n} = n] &= nE[\tilde{z}_1] + nE[\tilde{y}] + E[\tilde{z}_1] \\ &= (n+1)E[\tilde{z}_1] + nE[\tilde{y}]. \end{aligned}$$

Then, using the geometric properties of $P\{\tilde{n}_1 = n\}$,

$$\begin{aligned}
E[\tilde{y}] &= \sum_{n=0}^{\infty} E[y|\tilde{n}_1 = n]P\{\tilde{n}_1 = n\} \\
&= \sum_{n=0}^{\infty} [(n+1)E[\tilde{z}_1] + nE[\tilde{y}]] \left(\frac{\lambda}{\lambda+\mu}\right)^n \frac{\mu}{\lambda+\mu} \\
&= \frac{\mu}{\lambda+\mu}\left[E[\tilde{z}_1] + \frac{\lambda}{\lambda+\mu}E[\tilde{y}]\right] \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\lambda+\mu}\right)^{n-1} \\
&= \frac{\mu}{\lambda+\mu}\left[E[\tilde{z}_1] + \frac{\lambda}{\lambda+\mu}E[\tilde{y}]\right] \left(\frac{\lambda+\mu}{\mu}\right)^2 \\
&= \frac{\mu}{\lambda+\mu}\left[\frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}E[\tilde{y}]\right] \left(\frac{\lambda+\mu}{\mu}\right)^2 \\
&= \frac{1}{\mu} + \frac{\lambda}{\mu}E[\tilde{y}],
\end{aligned}$$

which implies

$$E[\tilde{y}] = \frac{\frac{1}{\mu}}{1-\rho}$$

EXERCISE 3.16 Suppose that the arrival and service time distributions are memoryless, but that their rates depend upon the number of customers in the system. Let the arrival rate when there are k customers in the system be λ_k, and let the service rate when there are k customers in the system be μ_k. Show that the dynamical equations are as follows:

$$P_n'(t) = \begin{cases} -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) & \\ +\mu_{n+1}P_{n+1}(t), & \text{for } n > 0; \\ \lambda_0 P_0(t) + \mu_1 P_1(t) & \text{for } n = 0. \end{cases}$$

Solution. Let $P_n(t)$ denote the probability of having n customers in the system at time t. Then

$$P_n(t+h) = \sum_{k=0}^{\infty} P\{\tilde{n}(t+h) = n|\tilde{n}(t) = k\}P_k(t) \qquad (3.15.1)$$

where $\tilde{n}(t+h) = n|\tilde{n}(t) = k$ signifies the event of having $n-k$ arrivals or $k-n$ departures in the time interval $(t, t+h]$. But these arrivals and departures are Poisson, so we may apply the properties of Definition 2 of the Poisson process (modified to take into account the possibility of deaths. In this case we may think of Definition 2 as governing changes in state and not simply the event of births.) Let $\mathcal{D}_i$, $\mathcal{A}_i$ denote the events of i departures and i arrivals, respectively, in the time interval $(t, t+h]$. And note that for $i \geq 2$, $P\{\mathcal{D}_i\} = P\{\mathcal{A}_i\} = o(h)$. Hence, using general λ and μ,

$$P\{\mathcal{D}_2\} = P\{\mathcal{A}_2\} = o(h)$$

$$\begin{aligned} P\left\{\mathcal{D}_1\right\} &= \mu h + o(h). \\ P\left\{\mathcal{A}_1\right\} &= \lambda h + o(h). \\ P\left\{\mathcal{D}_0\right\} &= 1 - \mu h + o(h). \\ P\left\{\mathcal{A}_0\right\} &= 1 - \lambda h + o(h). \end{aligned}$$

Then, substituting these in (3.15.1) and using state-dependent arrival and departure rates,

$$\begin{aligned} P_n(t+h) &= P\left\{\mathcal{A}_0\right\} P\left\{\mathcal{D}_0\right\} P_n(t) \\ &= \quad +P\left\{\mathcal{A}_1\right\} P_{n-1}(t) + P\left\{\mathcal{D}_1\right\} P_{n+1}(t) + o(h) \\ &= [1 - \lambda_n h + o(h)][1 - \mu_n h + o(h)]P_n(t) \\ &= \quad +[\lambda_{n-1}h + o(h)]P_{n-1}(t) + [\mu_{n+1}h + o(h)]P_{n+1}(t) + o(h) \\ &= [1 - (\lambda_n + \mu_n)h]P_n(t) + \lambda_{n-1}hP_{n-1}(t) \\ &= \qquad +\mu_{n+1}hP_{n+1}(t) + o(h). \end{aligned}$$

Rearranging terms and dividing by h:

$$\frac{P_n(t+h) + P_n(t)}{h} = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t) + \frac{o(h)}{h}.$$

Now let $h \to 0$:

$$P_n'(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t).$$

Finally, note that for $n = 0$, $P_{n-1}(t) = 0 = \mu_0$, so that

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t).$$

EXERCISE 3.17 Prove Theorem 3.5.

Solution. Let $\mathcal{Q}$ be a $(K+1)$-dimensional square matrix having distinct eigenvalues $\sigma_0, \sigma_1, \cdots, \sigma_K$, and define

$$W(\sigma) = (\sigma I - \mathcal{Q}).$$

Then, for det$W(\sigma) \neq 0$, (i.e., for $\sigma \neq \sigma_i$, $i = 1, 2, \cdots, K$), we find

$$W(\sigma)W^{-1}(\sigma) = I.$$

Since $W^{-1}(\sigma) = \mathrm{adj}\ W(\sigma)/\det W(\sigma)$,

$$\mathrm{adj}\ W(\sigma)W(\sigma) = \det W(\sigma)I.$$

Now, $W(\sigma)$, $\mathrm{adj}\ W(\sigma)$, and $\det\ W(\sigma)$ are all continuous functions of σ. Therefore,

$$\lim_{\sigma \to \sigma_i} \mathrm{adj}\ W(\sigma)W(\sigma) = \lim_{\sigma \to \sigma_i} \det W(\sigma)I,$$

so that, since $\lim_{\sigma \to \sigma_i} \det W(\sigma) = 0$,

$$\text{adj } W(\sigma_i) W(\sigma_i) = O_{(K+1)\times(K+1)},$$

where $O_{n\times m}$ denotes an $n \times m$ matrix of zeroes. Thus, we have

$$\text{adj } (\sigma_i I - \mathcal{Q})\,(\sigma_i I - \mathcal{Q}) = O_{(K+1)\times(K+1)}.$$

Now, by definition, X is a left eigenvector of $\mathcal{Q}$ corresponding to σ_i if $\sigma_i X = X\mathcal{Q}$ or $X(\sigma_i I - \mathcal{Q}) = 0$. Therefore, every row of $\text{adj } (\sigma_i I - \mathcal{Q})$ is a left eigenvector of $\mathcal{Q}$ corresponding to σ_i. Hence $\mathcal{M}_i$ is proportional to the rows of $\text{adj } (\sigma_i I - \mathcal{Q})$.

EXERCISE 3.18 Prove Theorem 3.6.

Solution. By Exercise 3.16, since the rows of $\text{adj } (\sigma_i I - \mathcal{Q})$ are proportional to $\mathcal{M}_i$, the left eigenvector of $\mathcal{Q}$ corresponding to σ_i, they must be proportional to each other. Similary, we may use the same technique of Exercise 3.16 to show that the columns of $\text{adj } (\sigma_i I - \mathcal{Q})$ are proportional to each other. Since $W(\sigma)W^{-1}(\sigma) = I$, and $W^{-1}(\sigma) = \text{adj } W(\sigma)/\det W(\sigma)$, we find that

$$W(\sigma)\text{adj } W(\sigma) = \det W(\sigma) I.$$

By the continuity of $W(\sigma)$, $\text{adj } W(\sigma)$, and $\det W(\sigma)$,

$$\lim_{\sigma \to \sigma_i} W(\sigma)\text{adj } W(\sigma) = \lim_{\sigma \to \sigma_i} \det W(\sigma) I.$$

This implies, since $\lim_{\sigma \to \sigma_i} \det W(\sigma) = 0$, that

$$\text{adj } W(\sigma_i) W(\sigma_i) = O_{(K+1)\times(K+1)},$$

where $O_{n\times m}$ denotes an $n \times m$ matrix of zeroes. Thus, we have

$$(\sigma_i I - \mathcal{Q})\,\text{adj } (\sigma_i I - \mathcal{Q}) = O_{(K+1)\times(K+1)}.$$

Now, by definition, X is an eigenvector of $\mathcal{Q}$ corresponding to σ_i if $\sigma_i X = \mathcal{Q}X$ or $(\sigma_i I - \mathcal{Q})X = 0$. Therefore, every column of $\text{adj } (\sigma_i I - \mathcal{Q})$ is an eigenvector of $\mathcal{Q}$ corresponding to σ_i. Hence the rows of $\text{adj } (\sigma_i I - \mathcal{Q})$ are proportional to each other.

EXERCISE 3.19 Let $K = 1$. Use Definition 2 of the Poisson process to write an equation of the form

$$\frac{d}{dt}\,[\,P_0(t) \quad P_1(t)\,] = [\,P_0(t) \quad P_1(t)\,]\,Q.$$

Show that the eigenvalues of the matrix Q are real and nonnegative. Solve the equation for $P_0(t), P_1(t)$ and show that they converge to the solution given in Example 3.2 regardless of the values $P_0(0), P_1(0)$. [*Hint*: First, do a similarity transformation on the matrix Q, which converts the matrix to a symmetric matrix $\hat{Q}$. Then show that the matrix $\hat{Q}$ is negative semi-definite.]

Solution. As seen in the proof of Exercise 3.15, we can use Definition 2 of the Poisson process to write the dynamical equations as:

$$\begin{array}{rcl} P_0'(t) & = & -\lambda_0 P_0(t) + \mu_1 P_1(t) \\ P_1'(t) & = & -\mu_n P_1(t) + \lambda_0 P_0(t). \end{array}$$

Then, using the definition

$$\hat{q}_{i,i+1} = \sqrt{q_{i,i+1} q_{i+1,i}} \quad \text{for} \quad i = 0, 1, \cdots, K-1,$$

we see that $\hat{Q}$ is as follows:

$$\begin{pmatrix} -\lambda & \sqrt{\mu\lambda} \\ \sqrt{\mu\lambda} & -\mu \end{pmatrix}$$

It is easily seen that the eigenvalues of $\hat{Q}$ are $\sigma_0 = 0$ and $\sigma_1 = -(\lambda + \mu)$. Then, using the equation

$$P(t) = P(0)\,\mathcal{M}\ \mathrm{diag}\ \left(e^{\sigma_0 t}, e^{\sigma_1 t}, \cdots, e^{\sigma_K t}\right)\mathcal{M}^{-1},$$

and arbitrary $P_0(0)$ and $P_1(0)$, we see that

$$P(t) = P(0)\begin{bmatrix} 1 & -\lambda \\ 1 & \mu \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-(\lambda+\mu)t} \end{bmatrix}\begin{bmatrix} \frac{\mu}{\mu+\lambda} & \frac{\lambda}{\mu+\lambda} \\ \frac{-1}{\mu+\lambda} & \frac{1}{\mu+\lambda} \end{bmatrix}.$$

This implies that

$$\begin{array}{rcl} P_0(t) & = & \dfrac{\mu}{\lambda+\mu}[P_0(0) + P_1(0)] + \dfrac{e^{-(\lambda+\mu)t}}{\lambda+\mu}[\lambda P_0(0) - \mu P_1(0)] \quad , \\ P_1(t) & = & \dfrac{\lambda}{\lambda+\mu}[P_0(0) + P_1(0)] + \dfrac{e^{-(\lambda+\mu)t}}{\lambda+\mu}[-\lambda P_0(0) + \mu P_1(0)]. \end{array}$$

Note that $P_0(0) + P_1(0) = 1$, and then let $t \to \infty$:

$$P_0(t) \quad = \quad \frac{\mu}{\lambda+\mu} \quad ,$$

$$P_1(t) \quad = \quad \frac{\mu}{\lambda + \mu}.$$

Since $P_0(0)$ and $P_1(0)$ were arbitrary, we see that P_0, P_1 converge to the solution given in the example regardless of the values $P_0(0), P_1(0)$.

EXERCISE 3.20 For the specific example given here show that the equilibrium probabilities for the embedded Markov chain are the same as those for the continuous-time Markov chain.

Solution. As already shown, the probability matrix for the discrete-time embedded Markov chain is

$$\mathcal{P}_e = \begin{pmatrix} 1-\rho & \rho \\ 1 & 0 \end{pmatrix}.$$

Thus, using the equation $\pi_e = \pi_e \mathcal{P}_e$,

$$\begin{aligned} \pi_{e_0} &= \frac{1}{1+\rho} \\ \pi_{e_1} &= \frac{\rho}{1+\rho} \end{aligned}$$

Recall that in the original system, the continuous-time Markov chain was

$$\mathcal{P}_O = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, however, the system can transition from one state back into itself. The one-step transition probability matrix is then

$$\mathcal{P}_C = \begin{pmatrix} 1-\rho & 1 \\ 1 & 0 \end{pmatrix}.$$

Since this is precisely the probability matrix for the discrete-time embedded Markov chain, we see that

$$\begin{aligned} \pi_{c_0} &= \pi_{e_0} = \frac{1}{1+\rho} \\ \pi_{c_1} &= \pi_{e_1} = \frac{\rho}{1+\rho}. \end{aligned}$$

Furthermore, by the very definition of being uniformized, the mean occupancy times for each state on each entry are equal. That is, $E[\tilde{s}_0] = E[\tilde{s}_1] = 1/\nu$. This implies

$$P_{C_i} \quad = \quad \frac{\pi_{c_i} E[\tilde{s}_i]}{\sum_{i=0}^{1} \pi_{c_i} E[\tilde{s}_i]}$$

$$
\begin{aligned}
&= \frac{\pi_{c_i} \cdot \frac{1}{\nu}}{\pi_{c_0} \cdot \frac{1}{\nu} + \pi_{c_1} \cdot \frac{1}{\nu}} \\
&= \frac{\pi_{c_i}}{\pi_{c_0} + \pi_{c_1}} \\
&= \pi_{c_i} \\
&= \pi_{e_i}.
\end{aligned}
$$

Thus we see that the equilibrium probabilities for the embedded discrete-time Markov chain are equal to the equilibrium probabilities for the continuous-time Markov chain when we randomize the system.

EXERCISE 3.21 Show that the equilibrium probabilities for the embedded Markov chain underlying the continuous-time Markov chain are equal to the equilibrium probabilities for the continuous-time Markov chain.

Solution. Let the capacity of the finite M/M/1 queueing system be K, K>1. In this general case, we must take ν to be at least as large as $\lambda + \mu$, the rate leaving states $1, 2, \cdots, K-1$. Then

$$
\mathcal{P}_e = \begin{pmatrix}
\frac{\nu-\lambda}{\nu} & \frac{\lambda}{\nu} & 0 & 0 & \dots & 0 & 0 \\
\frac{\mu}{\nu} & \frac{\nu-(\lambda+\mu)}{\nu} & \frac{\lambda}{\nu} & 0 & \dots & 0 & 0 \\
0 & \frac{\mu}{\nu} & \frac{\nu-(\lambda+\mu)}{\nu} & \frac{\lambda}{\nu} & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & \frac{\mu}{\nu} & \frac{\nu-\mu}{\nu}
\end{pmatrix}.
$$

This produces the solution

$$
\begin{aligned}
\pi_{e_0} &= \frac{1-\rho}{1-\rho^{K+1}} \\
\pi_{e_i} &= \rho^i \pi_{e_0}, \qquad \rho = \frac{\lambda}{\mu}
\end{aligned}
$$

By the same reasoning as in the proof of Exercise 3.19, the one-step transition probability matrix $\mathcal{P}_C$ is equal to $\mathcal{P}_e$. Hence for $i = 0, 1, \cdots, K$, $\pi_{e_i} = \pi_{c_i}$. And, since the system has been uniformized, the expected amount of time spent in each state before transition is unity. Hence,

$$
\begin{aligned}
P_{C_i} &= \frac{\pi_{c_i} E[\tilde{s}_i]}{\sum_{i=0}^{\infty} \pi_{c_i} E[\tilde{s}_i]} \\
&= \frac{\pi_{c_i} \cdot 1}{\sum_{i=0}^{\infty} \pi_{c_i} \cdot 1} \\
&= \pi_{c_i} \\
&= \pi_{e_i}.
\end{aligned}
$$

EXERCISE 3.22 For the special case of the finite capacity M/M/1 queueing system with $K = 2$, $\lambda_0 = \lambda_1 = 0.8$, and $\mu_1 = \mu_2 = 1$, determine the time-dependent state probabilities by first solving the differential equation (3.32) directly and then using uniformization for $t = 0.0, 0.2, 0.4, \ldots, 1.8, 2.0$ with $P_0(0) = 1$, plotting the results for $P_0(t)$, $P_1(t)$, and $P_2(t)$. Compare the quality of the results and the relative difficulty of obtaining the numbers.

Solution. For $K = 2$, the matrix $\mathcal{Q}$ is:

$$\begin{aligned}\mathcal{Q} &= \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & \mu & -\mu \end{bmatrix} \\ &= \begin{bmatrix} -0.8 & 0.8 & 0 \\ 1.0 & -1.8 & 0.8 \\ 0 & 1.0 & -1.0 \end{bmatrix}.\end{aligned}$$

The eigenvalues of $\mathcal{Q}$ are found to be 0, $-.9055728$, and -2.6944272, and their corresponding eigenvectors are proportional to $[\,1 \quad 1 \quad 1\,]^T$, $[\,-7.5777088 \quad 1 \quad 10.5901699\,]^T$, and $[\,-0.42229124 \quad 1 \quad -0.5901699\,]^T$, respectively. Thus, we find

$$P(t) = [\,P_0(0) \quad P_1(0) \quad P_2(0)\,]\,\mathcal{M}e^{\Lambda t}\mathcal{M}^{-1},$$

where

$$\begin{aligned}\mathcal{M} &= \begin{bmatrix} 1 & -7.5777088 & -0.42229124 \\ 1 & 1 & 1 \\ 1 & 10.5901699 & -0.5901699 \end{bmatrix} \\ e^{\Lambda t} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-0.9055728t} & 0 \\ 0 & 0 & e^{-2.6944272t} \end{bmatrix} \\ \mathcal{M}^{-1} &= \begin{bmatrix} 0.4098361 & 0.3278688 & 0.2622951 \\ -0.0582906 & 0.0061539 & 0.0521367 \\ -0.3515454 & 0.6659772 & -0.3144318 \end{bmatrix}.\end{aligned}$$

The time dependent state probabilities are then

$$\begin{aligned}P_0(t) &= 0.4098361 + 0.4417094e^{-0.9055728t} + 0.1484545e^{-2.6944272t} \\ P_1(t) &= 0.3278688 + 0.0466325e^{-0.9055728t} - 0.2812364e^{-2.6944272t} \\ P_2(t) &= 0.2622951 - 0.3950769e^{-0.9055728t} + 0.1327818e^{-2.6944272t}.\end{aligned}$$

To complete the exercise, Equation 3.48 was programmed on a computer for $t = 0.0, 0.2, 0.4, \cdots, 2.0$. The resulting values of $P_0(t)$, $P_1(t)$, and $P_2(t)$ are shown in Figure 3.1.

For $K = 2$, it was easier to solve for the equilibrium probabilities by hand. For larger values of K, however, this would become increasingly difficult. The

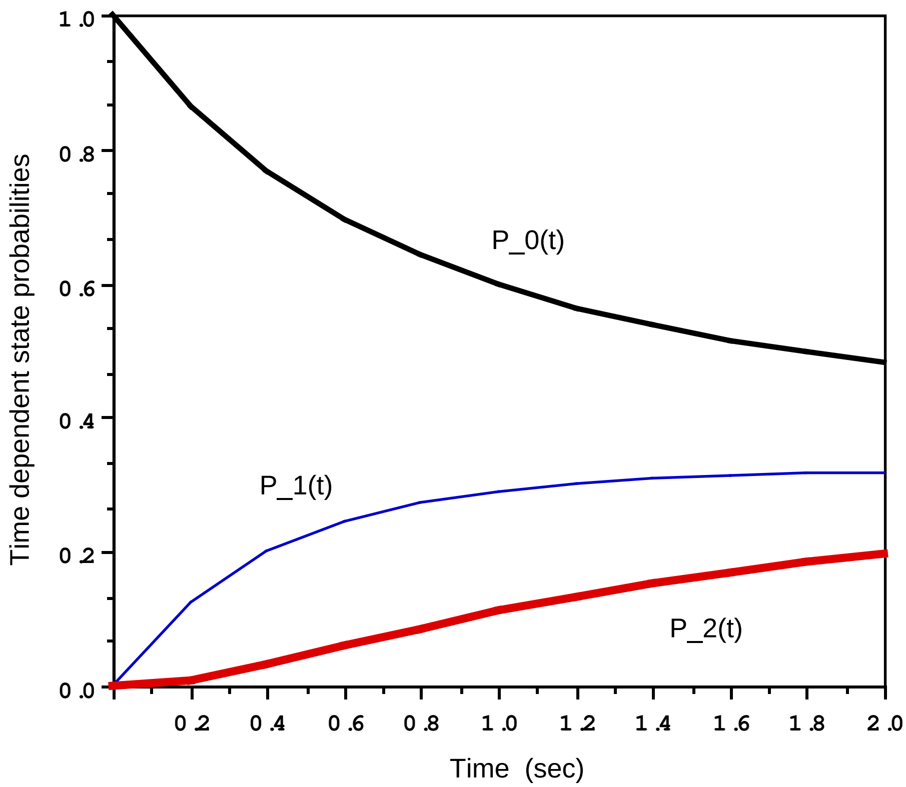

Figure 3.1. Time dependent state probabilities for Exercise 3.22

quality of the computer results differed with the values of t and ϵ, where ϵ determines when the summation in Equation 3.48 may be stopped. As t, depedent on ϵ, increased to threshold value, the equilibrium probabilities improved. And as ϵ decreased, the values of t were allowed to get larger and larger, bringing the equilibrium probabilities closer to those determined by hand.

EXERCISE 3.23 For the special case of the finite-capacity M/M/1 system, show that for $K = 1, 2, \ldots,$

$$P_B(K) = \frac{\rho P_B(K-1)}{1 + \rho P_B(K-1)},$$

where $P_B(0) = 1$.

Solution. By Equation (3.60), if the finite capacity is K,

$$P_B(K) = \left[\frac{1-\rho}{1-\rho^{K+1}}\right]\rho^K.$$

It follows that if the finite capacity is $K-1$,

$$P_B(K-1) = \left[\frac{1-\rho}{1-\rho^K}\right]\rho^{K-1}.$$

That is,

$$(1-\rho)\rho^{K-1} = P_B(K-1)(1-\rho^K).$$

So

$$\begin{aligned} P_B(K) &= \left[\frac{1-\rho}{1-\rho^{K+1}}\right]\rho\rho^{K-1} \\ &= \frac{\rho P_B(K-1)(1-\rho^K)}{1-\rho^{K+1}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{1-\rho^K}{1-\rho^{K+1}} &= \frac{1-\rho^K}{1-\rho^K+\rho^K-\rho^{K+1}} \\ &= \frac{1-\rho^K}{1-\rho^K+\rho^K(1-\rho)} \\ &= \frac{1}{1+\frac{\rho\rho^{K-1}(1-\rho)}{1-\rho^K}} \\ &= \frac{1}{1+\rho P_B(K-1)}. \end{aligned}$$

Hence,

$$P_B(K) = \frac{\rho P_B(K-1)}{1+\rho P_B(K-1)}.$$

EXERCISE 3.24 For the finite-state general birth-death process, show that for $K = 1, 2, \dots,$

$$P_B(K) = \frac{(\lambda_K/\mu_K)P_B(K-1)}{1+(\lambda_K/\mu_K)P_B(K-1)},$$

where $P_B(0) = 1$.

Solution. If the finite capacity of the M/M/1 queueing system is K, then by combining Equations (3.56) and (3.58),

$$P_K = \frac{1}{1+\sum_{n=1}^{K}\left[\frac{\prod_{i=0}^{n-1}\lambda_i}{\prod_{i=1}^{n}\mu_i}\right]}\left[\frac{\prod_{i=0}^{K-1}\lambda_i}{\prod_{i=1}^{K}\mu_i}\right].$$

Similarly if the finite capacity is $K-1$,

$$P_{K-1} = \frac{1}{1 + \sum_{n=1}^{K-1} \left[\frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \right]} \left[\frac{\prod_{i=0}^{K-2} \lambda_i}{\prod_{i=1}^{K-1} \mu_i} \right].$$

Now, note that the numerator of P_K is equal to

$$\left(\frac{\lambda_{K-1}}{\mu_K}\right) \frac{\prod_{i=0}^{K-2} \lambda_i}{\prod_{i=1}^{K-1} \mu_i},$$

and the denominator of P_K is

$$1 + \sum_{n=1}^{K-1} \left[\frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \right] + \left(\frac{\lambda_{K-1}}{\mu_K}\right) \frac{\prod_{i=0}^{K-2} \lambda_i}{\prod_{i=1}^{K-1} \mu_i}.$$

Dividing both numerator and denominator by

$$1 + \sum_{n=1}^{K-1} \left[\frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \right],$$

we see that the numerator becomes

$$\frac{\left(\frac{\lambda_{K-1}}{\mu_K}\right) \frac{\prod_{i=0}^{K-2} \lambda_i}{\prod_{i=1}^{K-1} \mu_i}}{1 + \sum_{n=1}^{K-1} \left[\frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \right]},$$

which is $(\lambda_{K-1}/\mu_K)P_{K-1}$. And the denominator becomes

$$1 + (\frac{\lambda_{K-1}}{\mu_K})\frac{\left[\frac{\prod_{i=0}^{K-2}\lambda_i}{\prod_{i=1}^{K-1}\mu_i}\right]}{1+\sum_{n=1}^{K-1}\left[\frac{\prod_{i=0}^{n-1}\lambda_i}{\prod_{i=1}^{n}\mu_i}\right]},$$

which is $1 + (\lambda_{K-1}/\mu_K)P_{K-1}$. Thus,

$$P_B(K) = \frac{(\lambda_{K-1}/\mu_K)P_B(K-1)}{1 + (\lambda_{K-1}/\mu_K)P_B(K-1)},$$

where $P_B(0) = 1$.

EXERCISE 3.25 Let K be arbitrary. Use the balance equation approach to write an equation of the form

$$\frac{d}{dt}P(t) = P(t)Q$$

where $P(t) = [\,P_0(t) \quad P_1(t) \quad \cdots \quad P_K(t)\,]$. Show that the eigenvalues of the matrix Q are real and nonpositive.

Solution. Observe that $\frac{d}{dt}P(t)$ is $\{$the rate of arrivals to the system at time $t\}$ minus $\{$the rate of departures from the system at time $t\}$. Therefore, using the table in Section 3.4 of the text, we may write

$$\frac{d}{dt}P(t) = \begin{bmatrix} \mu_1 P_1(t) - \lambda_0 P_0(t) \\ \lambda_0 P_0(t) + \mu_2 P_2(t) - (\lambda_1 + \mu_1)\, P_1(t) \\ \vdots \\ \lambda_{K-2}P_{K-2}(t) + \mu_K P_K(t) - (\lambda_{K-1} + \mu_{K-1})\, P_{K-1}(t) \\ \lambda_{K-1}P_{K-1}(t) - \mu_K P_K(t) \end{bmatrix}$$

We may rewrite this as $\frac{d}{dt}P(t) = P(t)\mathcal{Q}$, where

$$\mathcal{Q} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & & \ddots & & \\ & & \mu_{K-1} & -(\lambda_{K-1}+\mu_{K-1}) & \lambda_{K-1} \\ & & 0 & \mu_K & -\mu_K \end{bmatrix}.$$

Now, $\mathcal{Q}$ is a diagonal, real matrix, whose main diagonal is negative and whose off-diagonal elements are all of the same sign (positive). Hence, its eigenvalues are all real and nonpositive.

EXERCISE 3.26 Using the concept of local balance, write and solve the balance equations for the general birth-death process shown in Figure 3.12.

Solution. In Figure 3.12, the initial boundary is between the nodes representing state 1 and state 2. We see that the rate going from the left side of the boundary into the right side (that is, from state 1 into state 2) is λ_1. The rate going from the right side of the boundary into the left side (from state 2 into state 1) is μ_2. Since the 'rate in' must equal the 'rate out', we must be have

$$\begin{array}{rrcl} & \lambda_1 P_1 & = & \mu_2 P_2 \\ i.e. & P_2 & = & \dfrac{\lambda_1}{\mu_2} P_1. \end{array}$$

Moving the boundary between state 0 and state 1, the local balance is

$$P_1 = \frac{\lambda_0}{\mu_1} P_0.$$

Substituting this into the earlier result for P_2, we see that

$$P_2 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0.$$

Repeating this process by moving the boundary between every pair of states, we get the general result

$$P_n = \frac{\sum_{i=0}^{n-1} \lambda_i}{\sum_{i=1}^{n} \mu_i} P_0.$$

EXERCISE 3.27 Prove (3.66).

Solution.

$$E[\tilde{x}(\tilde{x}-1)\cdots(\tilde{x}-n+1)] = \frac{d^n}{dz^n} \mathcal{F}_{\tilde{x}}(z)\bigg|_{z=1} \tag{3.66}$$

Since $\mathcal{F}_{\tilde{x}}(z) = E[z^{\tilde{x}}]$, we may differentiate each side with respect to z:

$$\begin{aligned} \frac{d}{dz}\mathcal{F}_{\tilde{x}}(z) &= \frac{d}{dz} E[z^{\tilde{x}}] \\ &= \frac{d}{dz} \int_x z^x dF_{\tilde{x}}(x) \\ &= \int_x x z^{x-1} dF_{\tilde{x}}(x) \\ &= E[\tilde{x} z^{\tilde{x}-1}]. \end{aligned}$$

Hence,

$$\lim_{z\to 1} \frac{d}{dz}\mathcal{F}_{\tilde{x}}(z) = \lim_{z\to 1} E[\tilde{x} z^{\tilde{x}-1}]$$

$$= \quad E[\tilde{x}].$$

Now use induction on n, assuming the result holds for $n-1$. That is, assume that

$$\frac{d^{n-1}}{dz^{n-1}}\mathcal{F}_{\tilde{x}}(z) = E\left[\tilde{x}(\tilde{x}-1)\cdots(\tilde{x}-n+2)z^{\tilde{x}-n+1}\right].$$

Then, differentiating this expression with respect to z,

$$\begin{aligned}
\frac{d^n}{dz^n}\mathcal{F}_{\tilde{x}}(z) &= \frac{d}{dz}\int_x x(x-1)\cdots(x-n+2)z^{x-n+1}dF_{\tilde{x}}(x) \\
&= \int_x x(x-1)\cdots(x-n+2)(x-n+1)z^{x-n}dF_{\tilde{x}}(x) \\
&= E\left[\tilde{x}(\tilde{x}-1)\cdots(\tilde{x}-n+1)z^{\tilde{x}-n}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{z\to 1}\frac{d^n}{dz^n}\mathcal{F}_{\tilde{x}}(z) &= \lim_{z\to 1} E\left[\tilde{x}(\tilde{x}-1)\cdots(\tilde{x}-n+1)z^{\tilde{x}-n}\right] \\
&= E[\tilde{x}(\tilde{x}-1)\cdots(\tilde{x}-n+1)].
\end{aligned}$$

EXERCISE 3.28 Prove (3.67).

Solution.

$$P\{\tilde{x}=n\} = \frac{1}{n!}\frac{d^n}{dz^n}\mathcal{F}_{\tilde{x}}(z)\bigg|_{z=0}. \qquad (3.67)$$

By definition of expected value,

$$\begin{aligned}
\mathcal{F}_{\tilde{x}}(z) &= E[z^{\tilde{x}}] \\
&= \sum_{i=0}^{\infty} z^n P\{\tilde{x}=i\}.
\end{aligned}$$

On the other hand, the Maclaurin series expansion for $\mathcal{F}_{\tilde{x}}(z)$ is as follows:

$$\mathcal{F}_{\tilde{x}}(z) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n}{dz^n}\mathcal{F}_{\tilde{x}}(z)\bigg|_{z=0} z^n.$$

By the uniqueness of power series, we may match coefficients of z^n. The result follows.

EXERCISE 3.29 Use (3.66) to find $E[\tilde{n}]$ and $E[\tilde{n}^2]$.

Solution. Use equation (3.66) with $n=1$:

$$E[\tilde{n}(\tilde{n}-1)\cdots(\tilde{n}-1+1)] \quad = \quad E[\tilde{n}] = \frac{d}{dz}\mathcal{F}_{\tilde{n}}(z)\bigg|_{z=1}$$

$$\begin{aligned} &= \frac{d}{dz}\frac{P_0}{(1-\rho z)}\bigg|_{z=1} \\ &= \frac{\rho}{1-\rho}. \end{aligned}$$

To solve for $E[\tilde{n}^2]$, let $n = 2$:

$$\begin{aligned} E[\tilde{n}(\tilde{n}-1)\cdots(\tilde{n}-2+1)] &= E[\tilde{n}^2 - \tilde{n}] = \frac{d^2}{dz^2}\mathcal{F}_{\tilde{x}}(z)\bigg|_{z=1} \\ &= \frac{2\rho^2}{(1-\rho)^2}. \end{aligned}$$

But $E[\tilde{n}^2 - \tilde{n}] = E[\tilde{n}^2] - E[\tilde{n}]$, so after a little arithmetic we find that

$$E[\tilde{n}^2] = \frac{\rho^2 + \rho}{(1-\rho)^2}.$$

3.1 Supplementary Problems

3-1 Messages arrive to a statistical multiplexing system according to a Poisson process having rate λ. Message lengths, denoted by $\tilde{m}$, are specified in octets, groups of 8 bits, and are drawn from an exponential distribution having mean $1/\mu$. Messages are multiplexed onto a single trunk having a transmission capacity of C bits per second according to a FCFS discipline.

(a) Let $\tilde{x}$ denote the time required for transmission of a message over the trunk. Show that $\tilde{x}$ has the exponential distribution with parameter $\mu C/8$.

(b) Let $E[\tilde{m}] = 128$ octets and $C = 56$ kilobits per second (kb/s). Determine $\lambda_{\max}$, the maximum message-carrying capacity of the trunk.

(c) Let $\tilde{n}$ denote the number of messages in the system in stochastic equilibrium. Under the conditions of (b), determine $P\{\tilde{n} > n\}$ as a function of λ. Determine the maximum value of λ such that $P\{\tilde{n} > 50\} < 10^{-2}$.

(d) For the value of λ determined in part (c), determine the minimum value of s such that $P\{\tilde{s} > s\} < 10^{-2}$, where $\tilde{s}$ is the total amount of time a message spends in the system.

(e) Using the value of λ obtained in part (c), determine the maximum value of K, the system capacity, such that $P_B(K) < 10^{-2}$.

Solution:

(a) Since trunk capacity is C, the time to transmit a message of length m bytes is $x = 8m/C$. Therefore, $\tilde{x} = 8\tilde{m}/C$, or $\tilde{m} = \tilde{x}C/8$. Thus,

$$\begin{aligned} P\left\{\tilde{x} \leq x\right\} &= P\left\{\frac{8\tilde{m}}{C} \leq x\right\} \\ &= P\left\{\tilde{m} \leq \frac{xC}{8}\right\} \\ &= 1 - e^{\frac{-\mu C}{8}x}, \end{aligned}$$

where the last equation follows from the fact that $\tilde{x}$ is exponential with parameter μ. Therefore, $\tilde{m}$ is exponential with parameter $\mu C/8$.

(b) Since $E[\tilde{m}] = 128$ octets, $\mu = \frac{1}{128}$. Therefore, $\tilde{x}$ is exponentially distributed with parameter $\frac{\mu C}{8} = \frac{7}{128}$ Kbps, or $\frac{7}{128} \times 10^3$ bps. Then $E[\tilde{x}] = \frac{128}{7} \times 10^{-3}$ sec. Since $\lambda_{max}E[\tilde{x}] = \rho < 1$,

$$\lambda < \frac{1}{E[\tilde{x}]}$$

$$\begin{aligned} &= \frac{7}{128} \times 10^3 \\ &= 54.6875 \quad \text{msg/sec.} \end{aligned}$$

(c) We know from (3.10) that $P\{\tilde{n} > n\} = \rho^{n+1}$. Thus, $P\{\tilde{n} > 50\} = \rho^{51}$. Now, $\rho^{51} < 10^{-2}$implies

$$51 \log_{10} \rho < -2.$$

i.e.,

$$\log_{10} \rho < -\frac{2}{51}.$$

Hence, $\rho = \lambda E[\tilde{x}] < 10^{-\frac{2}{51}} = 0.91366$, so that

$$\lambda < 49.966 \quad \text{msg/sec.}$$

(d) Now, $\tilde{s}$ is exponential with parameter $\mu(1-\rho)$, so $P\{\tilde{s} > x\} = e^{-\mu(1-\rho)x}$. Then for $P\{\tilde{s} > s\} < 10^{-2}$, we must have

$$e^{-\mu(1-\rho)s} < 10^{-2}$$

or

$$-\mu\,[1-\rho]\,s < -2\,(\ln 10)$$

i.e.,

$$\begin{aligned} s &> \frac{2\,(\ln 10)}{\mu\,(1-\rho)} \\ &= \frac{2\,(\ln 10)}{\frac{7}{128} \times 10^3\,(1-0.91366)} \\ &= \frac{4.60517}{4.7217} \\ &= 0.9753 \text{ sec} \\ &= 975.3 \text{ ms} \end{aligned}$$

(e) Recall the equation for $P_B(K)$:

$$P_B(K) = \left(\frac{1-\rho}{1-\rho^{K+1}}\right)\rho^K$$

We wish to find K such that $P_B(K) < 10^{-2}$. First, we set $P_B(K) = 10^{-2}$ and solve for K.

$$\left(\frac{1-\rho}{1-\rho^{K+1}}\right)\rho^K = 10^{-2}$$

$$\begin{aligned}
(1-\rho)\,\rho^K &= 10^{-2}\left(1-\rho^{K+1}\right) \\
&= 10^{-2} - 10^{-2}\rho^{K+1} \\
(1-\rho)\,\rho^K + 10^{-2}\rho^{K+1} &= 10^{-2} \\
\left(1-\rho+10^{-2}\rho\right)\rho^K &= 10^{-2} \\
\rho^K &= \frac{10^{-2}}{1-0.99\rho}
\end{aligned}$$

Therefore,

$$\begin{aligned}
K &= \frac{-2\,(\ln 10) - \ln(1-0.99\rho)}{\ln\rho} \\
&= \frac{-[2\,(\ln 10) + \ln(1-0.99\rho)]}{\ln\rho}
\end{aligned}$$

For $\rho = 0.91366$,

$$\begin{aligned}
K &= \frac{-[4.60517 - 2.348874]}{-0.09029676} \\
&= \frac{2.256296}{0.09029676} \\
&= 25.799
\end{aligned}$$

Therefore, for a blocking probability less than 10^{-2}, we need $K \geq 26$. From part (c), note that for the non-blocking system, the value of K such that $P\{\tilde{n} > K\} < 10^{-2}$ is 50 for this particular value of λ. Thus, it is seen that the buffer size required to achieve a given blocking probability cannot be obtained directly from the survivor function. In this case, approximating buffer requirements from the survivor function would have resulted in $K = 50$, but in reality, only 26 storage spots are needed.

3-2 A finite population, K, of users attached to a statistical multiplexing system operate in a continuous cycle of *think, wait, service*. During the think phase, the length of which is denoted by $\tilde{t}$, the user generates a message. The message then waits in a queue behind any other messages, if any, that may be awaiting transmission. Upon reaching the head of the queue, the user receives service and the corresponding message is transmitted over a communication channel. Message service times, $\tilde{x}$, and think times, $\tilde{t}$, are drawn from exponential distributions with rates μ and λ, respectively. Let the state of the system be defined as the total number of users waiting and in service and be denoted by $\tilde{n}$.

(a) The first passage time from state i to state $i-1$ is the total amount of time the systems spends in all states from the time it first enters state i until it makes its first transition to the state $i-1$. Let $\tilde{s}_i$ denote the total cumulative time the system spends in state i during the first passage time from state i to state $i-1$. Determine the distribution of $\tilde{s}_i$.

(b) Determine the distribution of the number of visits from state i to state $i+1$ during the first passage time from state i to $i-1$.

(c) Show that $E[\tilde{y}_K]$, the expected length of a busy period, is given by the following recursion:

$$E[\tilde{y}_K] = \frac{1}{\mu}\left(1 + \lambda(K-1)E\left[\tilde{y}_{K-1}\right]\right) \quad \text{with} \quad E\left[\tilde{y}_0\right] = 0.$$

[*Hint*: Use the distribution found in part (b) in combination with the result of part (a) as part of the proof.]

(d) Let $P_0(K)$ denote the stochastic equilibrium probability that the communication channel is idle. Determine $P_0(K)$ using ordinary birth-death process analysis.

(e) Let $E[\tilde{\imath}_K]$ denote the expected length of the idle period for the communication channel. Verify that $P_0(K)$ is given by the ratio of the expected length of the idle period to the sum of the expected lengths of the idle and busy periods; that is,

$$P_0(K) = \frac{E[\tilde{\imath}_K]}{E[\tilde{\imath}_K] + E[\tilde{y}_K]}$$

which can be determined iteratively by

$$P_0(K) = \frac{1}{1 + [(K\lambda)/\mu]\ \{1 + (K-1)\lambda\ E[\tilde{y}_{K-1}]\}}.$$

That is, show that $P_0(K)$ computed by the formula just stated is identical to that obtained in part (d).

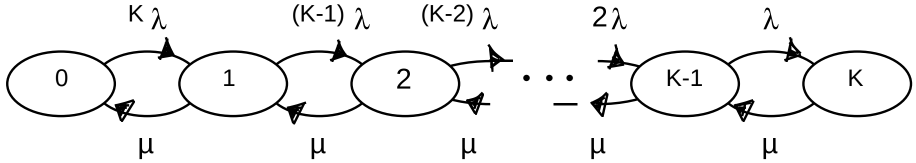

Figure 3.2. State diagram for Supplemental Exercise 3.2

Solution:

(a) The state diagram is as in Figure 3.2. For a typical state, say i, we wish to know the total amount of time spent in state i before the system transitions from state i to $i-1$ for the first time, $i = 1, 2, \cdots, K$. Now, on each visit to state i, the time spent in state i is the minimum of two exponential random variables: the time to first arrival, which has parameter $(K-i)\lambda$, and the time to first departure, which has parameter μ. Thus, the time spent in state i on each visit is exponentially distributed with parameter $(K-i)\lambda + \mu$. Now, the time spent in state i on the j^{-th} visit is independent of everything, and in particular on the number of visits to state i before transitioning to state $i-1$. Let $\tilde{s}_{ij}$ denote the time spent in state i on the j^{-th} visit and $\tilde{v}_i$ denote the number of visits to state i during the first passage from i to $i-1$. Then $\tilde{s}_i = \sum_{j=1}^{\tilde{v}_i} \tilde{s}_{ij}$. Now, the number of visits to state i is geometrically distributed (see proof of exponential random variables, page 35 of the text) with parameter $\mu / [(K-i)\lambda + \mu]$ because the probability that a given visit is the last visit is the same as the probability of a service completion before arrival (see Proposition 3 of exponential random variables, page 35 of the text) and this probability is independent of the number of visits that have occurred up to this point. Thus, $\tilde{s}_i$ is the geometric sum of exponential random variables and is therefore exponentially distributed with parameter $[(K-i)\lambda + \mu] \frac{\mu}{(K-i)\lambda+\mu} = \mu$, which follows directly from Property 5 of exponential random variables, given on page 35 of the text. In summary, the total amount of time spent in any state before transitioning to the next lower state is exponentially distributed with parameter μ.

(b) From arguments similar to those of part (a), the number of visits to state $i+1$ from state i is also geometric. In this case,

$$P\{\tilde{v}_i = k\} = \frac{\mu}{(K-i)\lambda + \mu} \left[\frac{(K-i)\lambda}{(K-i)\lambda + \mu}\right]^k$$

so that the mean number of visits is

$$\frac{(K-i)\lambda + \mu}{\mu} - 1 = \frac{(K-i)\lambda}{\mu} \quad \text{for} \quad i = 1, 2, \cdots, K-1.$$

For $i = K$, the number of visits is exactly 1.

(c) First note that $\tilde{y}_{K-i}$, the busy period for a system having $K-i$ customers, has the same distribution as the first passage time from state $i+1$ to i. From part (b), the expected minimum number of visits to state $i+1$ from state i is $(K-i)\frac{\lambda}{\mu}$. Thus, the total time spent in states $i+1$ and above during a first passage time from state i to $i-1$ is

$(K-i)\frac{\lambda}{\mu}E[\tilde{y}_{K-i}]$. Also, the total time spent in state i is $1/\mu$, so that the first passage time from state i to $i-1$ is given by

$$\begin{aligned} E[\tilde{y}_{K-(i-1)}] &= \frac{1}{\mu} + (K-i)\frac{\lambda}{\mu}E[\tilde{y}_{K-i}] \\ &= \frac{1}{\mu}\left(1 + (K-i)\lambda E[\tilde{y}_{K-i}]\right). \end{aligned}$$

With $i = 1$, we have

$$E[\tilde{y}_K] = \frac{1}{\mu}\left(1 + (K-1)\lambda E[\tilde{y}_{K-1}]\right).$$

We then have

$$\begin{aligned} E[\tilde{y}_1] &= \frac{1}{\mu} \\ E[\tilde{y}_2] &= \frac{1}{\mu}\left(1 + 1\cdot\lambda E[\tilde{y}_1]\right) \\ &= \frac{1}{\mu}\left(1 + \frac{\lambda}{\mu}\right) \\ E[\tilde{y}_3] &= \frac{1}{\mu}\left(1 + 2\lambda E[\tilde{y}_2]\right) \\ &= \frac{1}{\mu}\left(1 + 2\frac{\lambda}{\mu}\left[1 + \frac{\lambda}{\mu}\right]\right) \end{aligned}$$

so that we may compute $E[\tilde{y}_K]$ for any value of K by using the recursion, starting with $n = 1$,

$$E[\tilde{y}_n] = \frac{1}{\mu}\left(1 + (K-1)\lambda E[\tilde{y}_{n-1}]\right),$$

with $E[\tilde{y}_0] = 0$, and iterating to K.

(d) Using ordinary birth-death analysis, we find from (3.58) that

$$\begin{aligned} P_0 &= \frac{1}{1 + \sum_{n=1}^{\infty}\left[\prod_{i=0}^{n-1}\lambda_i \Big/ \prod_{i=1}^{n}\mu_i\right]} \\ &= \frac{1}{1 + \sum_{n=1}^{K}\left[\prod_{i=0}^{n-1}\lambda_i \Big/ \prod_{i=1}^{n}\mu_i\right]}. \end{aligned}$$

Now, $\mu_i = \mu$ and $\lambda_i = (K-i)\lambda$ for this particular case. Thus,

$$P_0 = \frac{1}{1 + \sum_{n=1}^{K}\left[\prod_{i=0}^{n-1}(K-i)\lambda \Big/ \prod_{i=1}^{n}\mu\right]}$$

$$= \quad \frac{1}{1+\sum_{n=1}^{K}\left(\frac{\lambda}{\mu}\right)^n \prod_{i=0}^{n-1}(K-i)}.$$

But $\prod_{i=0}^{n-1}(K-i) = K(K-1)\cdots K-(n-1) = \frac{K!}{(K-n)!}$. So

$$P_0 = \frac{1}{1+\sum_{n=1}^{K}\left(\frac{\lambda}{\mu}\right)^n \frac{K!}{(K-n)!}}.$$

(e) The expression given for $P_0(K)$ is readily converted to the final form given as follows:

$$\begin{aligned} P_0(K) &= \frac{\frac{1}{K\lambda}}{\frac{1}{K\lambda}+E[\tilde{y}_K]} \\ &= \frac{\frac{1}{K\lambda}}{\frac{1}{K\lambda}+\frac{1}{\mu}\left(1+\lambda(K-1)E[\tilde{y}_(K-1)]\right)} \\ &= \frac{1}{1+\frac{K\lambda}{\mu}\left(1+\lambda(K-1)E[\tilde{y}_(K-1)]\right)} \end{aligned}$$

We now prove the proposition

$$E[\tilde{y}_K] = \frac{1}{\mu}\left[1+\sum_{n=1}^{K-1}\frac{(K-1)!}{(K-1-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]$$

proof: Let T denote the truth set for the proposition. With $K=1$, we find $E[\tilde{y}_1] = \frac{1}{\mu}$, which is clearly correct. Now, suppose $(K-1) \in T$. Then, by hypothesis,

$$E[\tilde{y}_{K-1}] = \frac{1}{\mu}\left[1+\sum_{n=1}^{K-2}\frac{(K-2)!}{(K-2-n)!}\left(\frac{\lambda}{\mu}\right)^n\right] \qquad (*)$$

and, from part (c),

$$E[\tilde{y}_K] = \frac{1}{\mu}\left[1+(K-1)\lambda E[\tilde{y}_{K-1}]\right]$$

which is always true. Substituting $(*)$ into this last expression gives

$$\begin{aligned} E[\tilde{y}_K] &= \frac{1}{\mu}\left[1+\lambda(K-1)\left(\frac{1}{\mu}\left[1+\sum_{n=1}^{K-2}\frac{(K-2)!}{(K-2-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]\right)\right] \\ &= \frac{1}{\mu}\left[1+\lambda\frac{(K-1)}{\mu}+\sum_{n=2}^{K-1}\frac{(K-1)(K-2)!}{(K-1-n)!}\left(\frac{\lambda}{\mu}\right)^n\right] \end{aligned}$$

$$= \frac{1}{\mu}\left[1 + \sum_{n=1}^{K-1} \frac{(K-1)!}{(K-1-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]$$

This proves the proposition.

We now substitute the expression for $E[\tilde{y}_{K-1}]$ into the given expression

$$\begin{aligned} P_0(K) &= \left[1 + \lambda\frac{K}{\mu}\left(1 + (K-1)\lambda E[\tilde{y}_{K-1}]\right)\right]^{-1} \\ &= \left[1 + \lambda\frac{K}{\mu}\left(1 + \sum_{n=1}^{K-1} \frac{(K-1)!}{(K-1-n)!}\left(\frac{\lambda}{\mu}\right)^n\right)\right]^{-1} \\ &= \left[1 + \sum_{n=1}^{K} \frac{K!}{(K-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}, \end{aligned}$$

which is identical to that obtained in part (d).

3-3 *Traffic engineering with finite population.* Ten students in a certain graduate program share an office that has four telephones. The students are always busy doing one of two activities: *doing queueing homework* (work state) or *using the telephone* (service state); no other activities are allowed - ever. Each student operates continuously as follows: the student is initially in the work state for an exponential, rate β, period of time. The student then attempts to use one of the telephones. If all telephones are busy, then the student is blocked and returns immediately to the work state. If a telephone is available, the student uses the telephone for a length of time drawn from an exponential distribution with rate μ and then returns to the work state.

(a) Define an appropriate state space for this service system.

(b) Draw a state diagram for this system showing all transition rates.

(c) Write the balance equations for the system.

(d) Specify a method of computing the ergodic blocking probability for the system - that is the proportion of attempts to join the service system that will be blocked - in terms of the system parameters and the ergodic state probabilities.

(e) Specify a formula to compute the average call generation rate.

(f) Let $\mu = 1/3$ calls per minute; that is, call holding times have a mean of three minutes. Compute the call blocking probability as a function of β for $\beta \in (0, 30)$.

(g) Compare the results of part (f) to those of the Erlang loss system having 4 servers and total offered traffic equal to that of part (f). That is, for each value of β, there is a total offered traffic rate for the system specified in this problem. Use this total offered traffic to obtain a value of λ, and then obtain the blocking probability that would result in the Erlang loss system, and plot this result on the same graph as the results obtained in (f). Then compare the results.

Solution.

(a) Define the state of the system as the number of students using the telephones. This can have values $0, 1, 2, 3, 4$.

(b) See Figure 3.3.

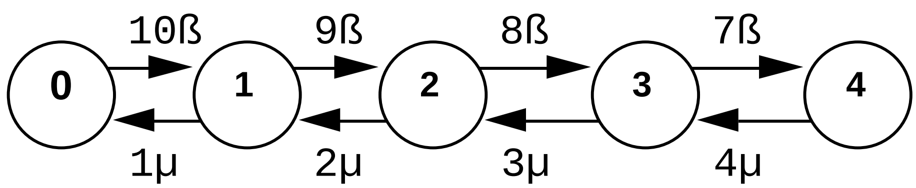

Figure 3.3. State diagram for Supplemental Exercise 3.3

(c) The balance equations can be either local or global; we choose local.

$$\begin{aligned} 10\beta P_0 &= \mu P_1 \\ 9\beta P_1 &= 2\mu P_2 \\ 8\beta P_1 &= 3\mu P_3 \\ 7\beta P_3 &= 4\mu P_4. \end{aligned}$$

(d) All arrivals that occur while the system is in state 4 are blocked. The arrival rate while in state 4 is 6β. Over a long period of time T, the number of blocked attempts is $6\beta P_4 T$ while the total number of arrivals is $(10\beta P_0 + 9\beta P_1 + 8\beta P_2 + 7\beta P_3 + 6\beta P_4)\,T$. Taking ratios, we find that the proportion of blocked calls is

$$\frac{6P_4}{10P_0 + 9P_1 + 8P_2 + 7P_3 + 6P_4}.$$

(e) The average call generation rate is $10P_0\beta + 9P_1\beta + 8P_2\beta + 7P_3\beta + 6P_4\beta$.

(f) First we solve the balance equations.

$$\begin{aligned} P_1 &= \frac{10\beta}{\mu}P_0 = \binom{10}{1}\frac{\beta}{\mu}P_0 \\ P_2 &= \frac{9\beta}{2\mu}P_1 = \binom{10}{2}\left(\frac{\beta}{\mu}\right)^2 P_0 \\ P_3 &= \frac{8\beta}{3\mu}P_2 = \binom{10}{3}\left(\frac{\beta}{\mu}\right)^3 P_0 \\ P_4 &= \frac{7\beta}{4\mu}P_3 = \binom{10}{4}\left(\frac{\beta}{\mu}\right)^4 P_0 \\ . \end{aligned}$$

So,

$$1 = P_0 \sum_{i=0}^{4} \binom{10}{i}\left(\frac{\beta}{\mu}\right)^i.$$

i.e.,

$$P_0 = \frac{1}{\sum_{i=0}^4 \binom{10}{i}\left(\frac{\beta}{\mu}\right)^i}$$

Therefore,

$$\begin{aligned} P_1 &= \frac{10\frac{\beta}{\mu}}{\sum_{i=0}^4 \binom{10}{i}\left(\frac{\beta}{\mu}\right)^i} \\ P_i &= \frac{\binom{10}{i}\left(\frac{\beta}{\mu}\right)^i}{\sum_{j=0}^4 \binom{10}{j}\left(\frac{\beta}{\mu}\right)^j} \qquad \text{for } i = 0, 1, 2, 3, 4 \end{aligned}$$

The call generation rate is then

$$\begin{aligned} \lambda &= \sum_{i=0}^{4} (10-i)\beta P_i \\ &= \frac{\beta \sum_{i=0}^4 (10-i)\binom{10}{i}\left(\frac{\beta}{\mu}\right)^i}{\sum_{j=0}^4 \binom{10}{j}\left(\frac{\beta}{\mu}\right)^j}. \end{aligned}$$

And the call blocking probability is $6P_4\frac{\beta}{\lambda}$. Hence,

$$P_B = \frac{6\beta\binom{10}{4}\left(\frac{\beta}{\mu}\right)^4}{\beta\sum_{i=0}^4 (10-i)\binom{10}{i}\left(\frac{\beta}{\mu}\right)^i}$$

$$= \frac{6\binom{10}{4}}{\sum_{i=0}^{4}(10-i)\binom{10}{i}\left(\frac{\mu}{\beta}\right)^{4-i}}.$$

With $\mu = \frac{1}{3}$, we have

$$\lambda(\beta) = \frac{\beta\sum_{i=0}^{4}(10-i)\binom{10}{i}(3\beta)^i}{\sum_{j=0}^{4}\binom{10}{j}(3\beta)^j}$$

and

$$P_B(\beta) = \frac{6\binom{10}{4}(3\beta)^4}{\sum_{j=0}^{4}(10-j)\binom{10}{j}(3\beta)^j}.$$

(g) From (3.64) and (3.65), we have

$$a = \frac{\lambda}{\mu} = 3\lambda$$

and

$$\begin{aligned} B(4,a) &= \frac{\left(\frac{a^4}{4!}\right)}{\sum_{i=0}^{4}\left(\frac{a^i}{i!}\right)} \\ &= \frac{\left(\frac{(3\lambda)^4}{4!}\right)}{\sum_{i=0}^{4}\left(\frac{(3\lambda)^i}{i!}\right)} \\ &= \frac{(3\lambda)^4}{4!\left[1 + 3\lambda + \frac{(3\lambda)^2}{2!} + \frac{(3\lambda)^3}{3!} + \frac{(3\lambda)^4}{4!}\right]} \\ &= \frac{(3\lambda)^4}{24 + (24 + [12 + (4 + 3\lambda)3\lambda]3\lambda)3\lambda} \end{aligned}$$

The idea is to compute λ first and then compute $\beta(4,\lambda)$ using this formula. It can be seen that for the Erlang blocking formula, a smooth traffic assumption would result in a substantial underestimation of the actual blocking probability. This will always be true for relatively small numbers of users. As an example, if $\beta = 1$, then the actual blocking probability is approximately 0.1, but the Erland blocking formula would yield 0.00025, which is off by a factor of 400.

3-4 A company has six employees who use a leased line to access a database. Each employee has a *think* time which is exponentially distributed with parameter λ. Upon completion of the think time, the employee needs the database and joins a queue along with other employees who may be waiting for the leased line to access the database. Holding times are exponentially distributed with parameter μ. When the number of waiting employees reaches a level 2, use of an auxiliary line is authorized. The time required for the employee to obtain the authorization is exponentially distributed with rate τ. If the authorization is completed when there are less than three employees waiting or if the number of employees waiting drops below two at any time while the extra line is in use, the extra line is immediately disconnected.

(a) Argue that the set $\{0, 1, 2, 3, 3r, 3a, 4r, 4r, 5r, 5a, 6r, 6a\}$, where the numbers indicate the number of employees waiting and in service, the letter *r* indicates that authorization has been requested, and the letter *a* indicates that the auxiliary line is actually available for service, is a suitable state space for this process.

(b) The situation in state 4r is that there are employees waiting and in service and an authorization has been requested. With the process in state 4r at time t_0, list the events that would cause a change in the state of the process.

(c) Compute the probability that each of the possible events listed in part (b) would actually cause the change of state, and specify the new state of the process following the event.

(d) What is the distribution of the amount of time the system spends in state 4r on each visit? Explain.

(e) Draw the state transition rate diagram.

(f) Write the balance equations for the system.

Solution.

(a) Two waiting employees means that there are three employees waiting and using the system. At a point in time, if an employee requests a line, then there is a waiting period before the line is authorized. The idea is that the system behaves differently when the number of waiting persons is increasing than when the number of waiting persons is decreasing. When the third person arrives, the system is immediately in the request mode $3a$. But, if no other customers arrive before the line is authorized, the line is dropped, leaving 3 in the system, but only one line. If a person arrives while the system is in state 3, the line is again requested, putting the system in state $4r$. If the line authorization is complete while the system is in $4r$, $5r$, or $6r$, then the system goes into state $4a$, $5a$, or $6a$, respectively.

(b) While in state $4r$, the possible next events are customer arrival, customer service, line authorization complete, with rates 2λ, μ, and τ, respectively.

(c) Let $\mathcal{A}$, S, and AC denote the events customer arrival, customer service, line authorization complete, respectively. Then

$$
\begin{aligned}
P\{\mathcal{A}\} &= \frac{2\lambda}{2\lambda+\mu+\tau} \\
P\{S\} &= \frac{\mu}{2\lambda+\mu+\tau} \\
P\{AC\} &= \frac{\tau}{2\lambda+\mu+\tau}.
\end{aligned}
$$

(d) The distribution of the amount of time spent in state $4r$ is exponentially distributed with parameter $2\lambda+\mu+\tau$ because this time is the minimum of 3 exponential random variables with parameter 2λ, μ, and τ, respectively.

(e) See Figure 3.4 for the state transition rate diagram.

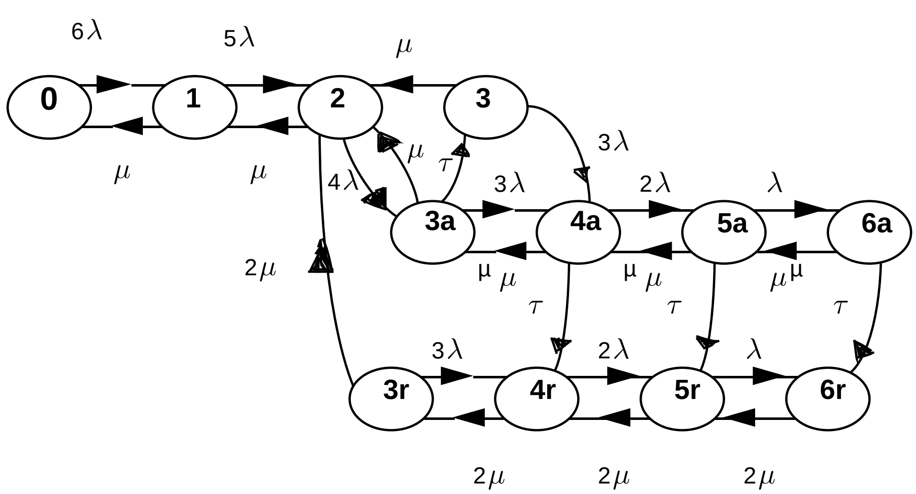

Figure 3.4. State diagram for Supplemental Exercise 3.4

(f) The balance equations are as follows:

state	rate leaves		rate enters
0	$6\lambda P_0$	$=$	μP_1
1	$(5\lambda + \mu)P_1$	$=$	$6\lambda P_0 + \mu P_2$
2	$(4\lambda + \mu)P_2$	$=$	$5\lambda P_1 + \mu P_3 + \mu P_{3a} + 2\mu P_{3r}$
3	$(3\lambda + \mu)P_3$	$=$	τP_{3a}
$3a$	$(3\lambda + \tau + \mu)P_{3a}$	$=$	$4\lambda P_2 + \mu P_{4a}$
$3r$	$(3\lambda + 2\mu)P_{3r}$	$=$	$2\mu P_{4r}$
$4a$	$(2\lambda + \tau + \mu)P_{4a}$	$=$	$3\lambda P_3 + 3\lambda P_{3a} + \mu P_{5a}$
$4r$	$(2\lambda + 2\mu)P_{4r}$	$=$	$3\lambda P_{3r} + \tau P_{4a} + 2\mu P_{5r}$
$5a$	$(\lambda + \tau + \mu)P_{5a}$	$=$	$2\lambda P_{4a} + \mu P_{6a}$
$5r$	$(\lambda + 2\mu)P_{5r}$	$=$	$2\lambda P_{4r} + 2\mu P_{6r} + \tau P_{5a}$
$6a$	$(\mu + \tau)P_{6a}$	$=$	λP_{5a}
$6r$	$2\mu P_{6r}$	$=$	$\lambda P_{5r} + \tau P_{6a}$

3-5 Messages arrive to a statistical multiplexer at a Poisson rate λ for transmission over a communication line having a capacity of C in octets per second. Message lengths, specified in octets, are exponentially distributed with parameter μ. When the waiting messages reach a level 3, the capacity of the transmission line is increased to C_e by adding a dial-up line. The time required to set up the dial-up line to increase the capacity is exponentially distributed with rate τ. If the connection is completed when there are less than three messages waiting or if the number of messages waiting drops below two at any time while the additional capacity is in use, the extra line is immediately disconnected.

(a) Define a suitable state space for this queueing system.

(b) Draw the state transition-rate diagram.

(c) Organize the state vector for this system according to level, where the level corresponds to the number of messages waiting and in service, and write the vector balance equations for the system.

(d) Determine the infinitesimal generator for the underlying Markov chain for this system and comment on its structure relative to matrix geometric solutions.

Solution.

(a) Figure 3.5 details two state diagram for the system. Note that the first passage time from state 2 to state 1 in the first diagram is the same as the length of the busy period in the second diagram. i.e., the M/M/1 with service rate 2μ. Thus, $\tilde{f}_{21} = E[\tilde{y}_{M/M/1}] = \frac{1}{2\mu}/1 - \frac{\lambda}{2\mu}$.

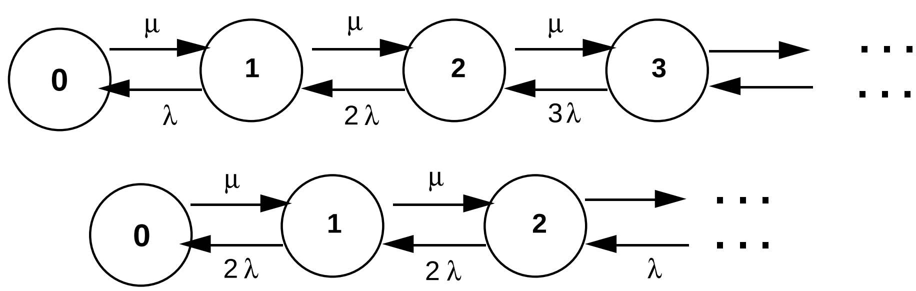

Figure 3.5. State diagram for Supplemental Exercise 3.5

(b) Let $\mathcal{A}$ denote the event of an arrival before the first departure from a busy period, and let $\mathcal{A}^c$ denote its complement. Then,

$$\begin{aligned} E[\tilde{y}] &= E[\tilde{y}|\mathcal{A}]P\{\mathcal{A}\} + E[\tilde{y}|\mathcal{A}^c]P\{\mathcal{A}^c\} \\ &= E[\tilde{z} + \tilde{y}_2]P\{\mathcal{A}\} + E[\tilde{z}]P\{\mathcal{A}^c\} \\ &= E[\tilde{z} + \tilde{f}_{21} + \tilde{y}]\frac{\lambda}{\lambda+\mu} + E[\tilde{z}]\frac{\mu}{\lambda+\mu} \end{aligned}$$

.

Thus,

$$E[\tilde{y}]\frac{\mu}{\lambda+\mu} = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\left(\frac{\frac{1}{2\mu}}{1-\frac{\lambda}{2\mu}}\right)$$

i.e.,

$$E[\tilde{y}] = \frac{1}{\mu} + \frac{\lambda}{\mu}\left(\frac{\frac{1}{2\mu}}{1-\frac{\lambda}{2\mu}}\right).$$

(c) $\tilde{c} = \tilde{i} + \tilde{y}$ implies that $E[\tilde{c}] = \frac{1}{\lambda} + \frac{1}{\mu} + \frac{\lambda}{\mu}\left(\frac{\frac{1}{2\mu}}{1-\frac{\lambda}{2\mu}}\right)$. Therefore,

$$\begin{aligned} P\{\text{system is idle}\} &= \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{1}{\mu} + \frac{\lambda}{\mu}\left(\frac{\frac{1}{2\mu}}{1-\frac{\lambda}{2\mu}}\right)} \\ &= \frac{1}{1 + \frac{\lambda}{\mu}\left(1 + \frac{\frac{\lambda}{2\mu}}{1-\frac{\lambda}{2\mu}}\right)} \end{aligned}$$

$$= \frac{1}{1+\frac{\lambda}{\mu}\left(\frac{1}{1-\frac{\lambda}{2\mu}}\right)}.$$

(d) From an earlier exercise, we know that the total amount of time the system spends i state 1 before entering state 0 is exponential with parameter μ because there are a geometric number of visits, and for each visit the amount of time spent is exponential with parameter $(\lambda+\mu)$. Thus,

$$\begin{aligned} P\{\tilde{n}=1\} &= \frac{\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}+\frac{\lambda}{\mu}\left(\frac{\frac{1}{2\mu}}{1-\frac{\lambda}{2\mu}}\right)} \\ &= \frac{\frac{\lambda}{\mu}}{1+\frac{\lambda}{\mu}\left(\frac{1}{1-\frac{\lambda}{2\mu}}\right)}. \end{aligned}$$

3-6 Consider the M/M/2 queueing system, the system having Poisson arrivals, exponential service, 2 parallel servers, and an infinite waiting room capacity.

(a) Determine the expected first passage time from state 2 to state 1. [*Hint*: How does this period of time compare to the length of the busy period for an ordinary M/M/1 queueing system?]

(b) Determine the expected length of the busy period for the ordinary M/M/2 queueing system by conditioning on whether or not an arrival occurs before the first service completion of the busy period and by using the result from part (a).

(c) Define $\tilde{c}$ as the length of time between successive entries into busy periods, that is, as the length of one busy/idle cycle. Determine the probability that the system is idle at an arbitrary point in time by taking the ratio of the expected length of an idle period to the expected length of a cycle.

(d) Determine the total expected amount of time the system spends in state 1 during a busy period. Determine the probability that there is exactly one customer in the system by taking the ratio of the expected amount of time that there is exactly one customer in the system during a busy period to the expected length of a cycle.

(e) Check the results of (c) and (d) using classical birth-death analysis.

(f) Determine the expected sojourn time, $E[\tilde{s}]$, for an arbitrary customer by conditioning on whether an arbitrary customer finds either zero, one, or two or more customers present. Consider the nonpreemptive last-come-first-serve discipline together with Little's result and the fact that the distribution of the number of customers in the system is not affected by order of service.

Solution.

(a) If the occupancy is n, then there are $(n-1)$ customers waiting. So, if there are 3 customers waiting, occupancy is 4. Less than 3 messages waiting means occupancy is less than 4. We assume that if the number waiting drops below 3 while attempting to set up the extra capacity, then the attempt is aborted. In this case, the required states are $0, 1, 2, 3, 3e, 4, 4e, \cdots$, where e indicates extra capacity.

(b) See Figure 3.6.

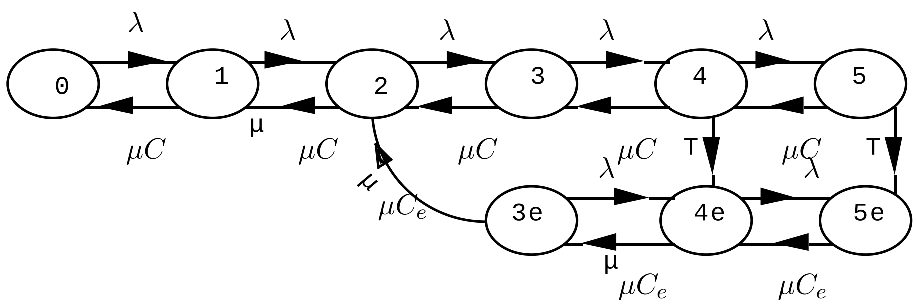

Figure 3.6. State diagram for Supplemental Exercise 3.6

(c) Organizing the state vector for this system according to level, we see that

$$
\begin{array}{lcl}
\text{level } 0 & : & P_0 \\
\text{level } 1 & : & P_1 \\
\text{level } 2 & : & P_2 \\
\text{level } n & : & P_n = [\,P_{n0} \quad P_{n1}\,] \quad n > 2
\end{array}
$$

And the balance equations for the system are

$$
\begin{aligned}
\lambda P_0 &= \mu C P_1 \\
(\lambda + \mu C)\, P_1 &= \mu P_2 + \lambda P_0 \\
(\lambda + \mu C)\, P_2 &= \lambda P_1 + \lambda P_3 \begin{bmatrix} \mu C \\ \mu C_e \end{bmatrix}
\end{aligned}
$$

$$P_3 \begin{bmatrix} \lambda + \mu C & 0 \\ 0 & \lambda + \mu C_e \end{bmatrix} = P_2 \begin{bmatrix} \lambda & 0 \end{bmatrix} + P_4 \begin{bmatrix} \mu C & 0 \\ 0 & \mu C_e \end{bmatrix}$$

For all remaining n, $n \geq 4$,

$$P_n \begin{bmatrix} \lambda + \mu C + \tau & 0 \\ 0 & \lambda + \mu C_e \end{bmatrix} = P_{n-1} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + P_n \begin{bmatrix} 0 & \tau \\ 0 & 0 \end{bmatrix} + P_{n+1} \begin{bmatrix} \mu C & 0 \\ 0 & \mu C_e \end{bmatrix}$$

(d) The infinitesimal generator is readily written by inspection of the balance equations by simply taking all terms to the right-hand side. If we let $\mathcal{R} = (\lambda + \mu C)$ and $\mathcal{R}_e = (\lambda + \mu C_e)$, the resulting generator is

$$\tilde{\mathcal{Q}} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & \dots \\ \mu C & -\mathcal{R} & \lambda & 0 & 0 & 0 & 0 & \dots \\ 0 & \mu C & -\mathcal{R} & \lambda & 0 & 0 & 0 & \dots \\ 0 & 0 & \mu C & -\mathcal{R} & 0 & \lambda & 0 & \dots \\ 0 & 0 & \mu C_e & 0 & -\mathcal{R}_e & 0 & \lambda & \dots \\ 0 & 0 & 0 & \mu C & 0 & -\mathcal{R}_e - \tau & \tau & \dots \\ 0 & 0 & 0 & 0 & \mu C & 0 & -\mathcal{R}_e & \dots \\ 0 & 0 & 0 & 0 & 0 & \mu C & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu C_e & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

We may also write $\tilde{\mathcal{Q}}$ as

$$\tilde{\mathcal{Q}} = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & 0 & \dots \\ B_{10} & B_{11} & B_{12} & B_{13} & 0 & 0 & 0 & \dots \\ 0 & B_{21} & B_{22} & B_{23} & 0 & 0 & 0 & \dots \\ 0 & 0 & B_{32} & B_{33} & A_0 & 0 & 0 & \dots \\ 0 & 0 & 0 & B_{43} & 0 & A_1 & A_0 & \dots \\ 0 & 0 & 0 & 0 & A_2 & A_1 & A_0 & \dots \\ 0 & 0 & 0 & 0 & 0 & A_2 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

We see that this latter form is the same as that given on page 140 of the text, except that the boundary conditions are very complex. The text by Neuts treats problems of this form under the heading of queues with complex boundary conditions.

Chapter 4

ADVANCED CONTINUOUS-TIME MARKOV CHAIN-BASED QUEUEING MODELS

EXERCISE 4.1 Using Little's result, determine the average time spent in the system for an arbitrary customer when the system is in stochastic equilibrium.

Solution. Let $\tilde{s}$ denote the time spent in the system for an arbitrary customer, and let γ be the net arrival rate into the system, $\gamma = \sum_{i=1}^{M} \gamma_i$. By Little's result,

$$E[\tilde{s}] = \frac{E[\tilde{n}]}{\gamma}$$

where $\tilde{n}$ is the total number of customers in the system in stochastic equilibrium. Observe that the total number of customers in the system is the sum of the customers at each node. i.e., $\tilde{n} = \sum_{i=1}^{M} \tilde{n}_i$. Therefore,

$$\begin{aligned} E[\tilde{s}] &= \frac{1}{\gamma} E\left[\sum_{i=1}^{M} \tilde{n}_i\right] \\ &= \frac{1}{\gamma} \sum_{i=1}^{M} E[\tilde{n}_i] \\ &= \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\rho_i}{1-\rho_i}, \end{aligned}$$

the latter following from (4.11) and (3.11).

EXERCISE 4.2 Argue that the matrix R is stochastic, and that, therefore, the vector λ is proportional to the equilibrium probabilities of the Markov chain for which R is the one-step transition probability matrix.

Solution. The element r_{ij} represents the probability that a unit currently at node i will proceed next to node j. Since the system is closed, a unit must go to some node within the network. Therefore,

$$\sum_{j=1}^{M} r_{ij} = 1 \forall i.$$

Hence R is stochastic. From $\lambda[I-R]=0$, we have

$$\lambda R = \lambda.$$

Therefore, the vector λ is the left eigenvector of R corresponding to its eigenvalue at 1, and all solutions λ are proportional to this eigenvector. But, this equation has exactly the form $\pi R = \pi$, where R is a one-step transition matrix and π is the stationary probability vector of the Markov chain for which R is the one-step transition probability matrix. Thus, λ is proportional to π.

EXERCISE 4.3 Let x denote any integer. Show that

$$\frac{1}{j2\pi}\oint_C \phi^x d\phi = \begin{cases} 1, & \text{for } x=-1 \\ 0, & \text{otherwise.} \end{cases}$$

by direct integration.

Solution. Suppose $x \neq -1$. Then

$$\frac{1}{j2\pi}\oint_C \phi^x d\phi = \frac{1}{j2\pi}\cdot\frac{\phi^{x+1}}{x+1}|_C.$$

Note that the closed path C may be traversed by beginning at the point $re^{j\theta}$ and ending at $re^{j(\theta+2\pi)}$. The integral is then

$$\begin{aligned}
\frac{1}{j2\pi}\cdot\frac{\phi^{x+1}}{x+1}|_C &= \frac{1}{j2\pi}\left[re^{j(\theta+2\pi)}|^{x+1} - re^{j\theta}|^{x+1}\right] \\
&= \frac{1}{j2\pi}\left[r^{x+1}e^{j(\theta+2\pi)(x+1)} - r^{x+1}e^{j\theta(x+1)}\right] \\
&= \frac{1}{j2\pi}\left[r^{x+1}e^{j\theta(x+1)}\cdot 1 - r^{x+1}e^{j\theta(x+1)}\right] \\
&= 0.
\end{aligned}$$

Now suppose $x=-1$.

$$\begin{aligned}
\frac{1}{j2\pi}\oint_C \phi^{-1} d\phi &= \frac{1}{j2\pi}\oint_C \frac{d\phi}{\phi} \\
&= \frac{1}{j2\pi}\ln\phi\Big|_C \\
&= \frac{1}{j2\pi}\ln\phi\Big|_{re^{j\theta}}^{re^{j(\theta+2\pi)}}
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{j2\pi}\left[\ln r + j(\theta + 2\pi) - \ln r - j\theta\right] \\ &= \frac{1}{j2\pi} j2\pi = 1. \end{aligned}$$

EXERCISE 4.4 Suppose that the expression for $g(N, M, \phi)$ can be written as

$$g(N, M, \phi) = \prod_{i=1}^{r} \frac{1}{(1 - \sigma_i \phi)^{\nu_i}}, \tag{4.1}$$

where $\sum_{i=1}^{r} \nu_i = M$. That is, there are exactly r distinct singular values of $g(N, M, \phi)$- these are called σ_1,σ_2,. . .,σ_r - and the multiplicity of σ_i is ν_i. We may rewrite (4.35) as

$$g(N, M, \phi) = \sum_{i=1}^{r} \sum_{j=1}^{\nu_j} \frac{c_{ij}}{(1 - \sigma_i \phi)^j}. \tag{4.2}$$

Show that

$$\begin{aligned} c_{ij} &= \frac{1}{(\nu_i - j)!}\left(-\frac{1}{\sigma_i}\right)^{(\nu_i - j)} \\ &\quad \frac{d^{(\nu_i - j)}}{d\phi^{(\nu_i - j)}}\left[(1 - \sigma_i \phi)^{\nu_i} g(N, M, \phi)\right]\Big|_{\phi = 1/\sigma_i}. \end{aligned}$$

Solution. Let m be a fixed but otherwise arbitrary index, $1 \leq m \leq r$. Multiply both sides of $g(N, M, \phi)$ by $(1 - \sigma_m \phi)^{\nu_m}$:

$$(1 - \sigma_m \phi)^{\nu_m} g(N, M, \phi) = \sum_{j=1}^{\nu_m} c_{mj}\left(1 - \sigma_m \phi\right)^{\nu_m - j} + (1 - \sigma_m \phi)^{\nu_m} F(\phi),$$

where $F(\phi)$ is the remainder of the terms in the sum not involving σ_m. Now fix n, $1 \leq n \leq \nu_m$. Differentiate both sides of this expression and evaluate at $\phi = \frac{1}{\sigma_m}$. Thus,

$$\frac{d^{(\nu_m - n)}}{d\phi^{(\nu_m - n)}}(1 - \sigma_m \phi)^{\nu_m} g(N, M, \phi)\Big|_{\phi = \frac{1}{\sigma_i}} = (-\sigma_m)^{(\nu_m - n)} c_{mn} (\nu_m - n)!$$

i.e.,

$$\begin{aligned} c_{mn} &= \frac{1}{(\nu_m - n)!}\left(\frac{-1}{\sigma_m}\right)^{(\nu_m - n)} \frac{d^{(\nu_m - n)}}{d\phi^{(\nu_m - n)}}\left[(1 - \sigma_m \phi)^{\nu_m} g(N, M, \phi)\right] \\ &= \frac{1}{(\nu_i - n)!}\left(\frac{-1}{\sigma_i}\right)^{(\nu_i - n)} \frac{d^{(\nu_i - n)}}{d\phi^{(\nu_i - n)}}\left[(1 - \sigma_i \phi)^{\nu_i} g(N, M, \phi)\right]\Big|_{\phi = \frac{1}{\sigma_m}}. \end{aligned}$$

EXERCISE 4.5 Define b_{nN} to be the coefficient of ϕ^N in the expansion of $(1-\sigma_i\phi)^{-n}$. Show that

$$b_{nN} = \binom{N+n-1}{N} \sigma_i^N. \tag{4.3}$$

Solution. Define $h(\phi) = (1-\sigma_i\phi)^{-n}$. Then by the Maclaurin series expansion for h,

$$h(\phi) = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)(-\sigma_i\phi)^k}{k!}.$$

In particular, the N^{-th} term of this sum is

$$\frac{h^{(N)}(0)(-\sigma_i\phi)^N}{N!}.$$

But $h^{(N)}(0) = (-n)(-n-1)\dots(-n-M+1)(1)^N$, so that the coefficient of ϕ^N is

$$\begin{aligned} b_{nN} &= \frac{(-1)^N(N+n-1)!(-\sigma_i)^N}{N!(n-1)!} \\ &= \binom{N+n-1}{N}\sigma_i^N. \end{aligned}$$

EXERCISE 4.6 Verify that the probabilities as specified by (4.39) sum to unity.

Solution. In this exercise, we must first determine the set

$$\mathcal{S}_{4,5} = \left\{ (n_1, n_2, \cdots, n_5) | \sum_{i=1}^{5} n_i = 4 \right\}.$$

Then for each member $(n_1, n_2, n_3, n_4, n_5)$ of this set, we'll substitute n_3, n_4, and n_5 into $2^{n_3+n_4}4^{n_5}$, producing an intermediate result, which we'll call I. Note that I is equal to $P\{n_1, n_2, n_3, n_4, n_5\}$ multiplied by 1497. It suffices, then, to show that the sum of these intermediate results is equal to 1497. The following table gives all possible members of the set $\mathcal{S}_{4,5}$, with the last column of the table being the intermediate result I. These members were obtained by finding all possible combinations of $n_1, n_2, \cdots, n_5$ with $n_5 = 4$; then finding all combinations with $n_5 = 3$, etc.

$\underline{i}$	$\underline{n}_1$	$\underline{n}_2$	$\underline{n}_3$	$\underline{n}_4$	$\underline{n}_5$	$\underline{I}$
1	0	0	0	0	4	256
2	0	0	0	1	3	128
3	0	0	1	3	3	128
4	0	1	0	0	3	64
5	1	0	0	0	3	64
6	0	0	0	2	2	64
7	0	0	2	0	2	64
8	0	0	1	1	2	64
9	0	2	0	0	2	16
10	2	0	0	0	2	16
11	1	1	0	0	2	16
12	0	1	0	1	2	32
13	1	0	0	1	2	32
14	0	1	1	0	2	32
15	1	0	1	0	2	32
16	0	0	3	0	1	32
17	0	0	0	3	1	32
18	0	0	1	2	1	32
19	0	0	2	1	1	32
20	0	1	2	0	1	16
21	1	0	2	0	1	16
22	1	0	0	2	1	16
23	0	1	0	2	1	16
24	0	1	1	1	1	16
25	1	0	1	1	1	16
26	0	2	1	0	1	8
27	2	0	1	0	1	8
28	0	2	0	1	1	8
29	2	0	0	1	1	8
30	1	1	1	0	1	8
31	1	1	0	1	1	8
32	0	3	0	0	1	4
33	3	0	0	0	1	4
34	1	2	0	0	1	4
35	2	1	0	0	1	4
36	0	0	4	0	0	16
37	0	0	0	4	0	16
38	0	0	3	1	0	16
39	0	0	1	3	0	16

$\underline{i}$	$\underline{n}_1$	$\underline{n}_2$	$\underline{n}_3$	$\underline{n}_4$	$\underline{n}_5$	$\underline{I}$
40	0	0	2	2	0	16
41	1	0	3	0	0	8
42	0	1	3	0	0	8
43	1	0	0	3	0	8
44	0	1	0	3	0	8
45	1	0	2	1	0	8
46	0	1	2	1	0	8
47	1	0	1	2	0	8
48	0	1	1	2	0	8
49	0	2	2	0	0	4
50	2	0	2	0	0	4
51	0	2	0	2	0	4
52	2	0	0	2	0	4
53	1	1	2	0	0	4
54	1	1	0	2	0	4
55	1	1	1	1	0	4
56	0	2	1	1	0	4
57	2	0	1	1	0	4
58	0	3	1	0	0	2
59	3	0	1	0	0	2
60	0	3	0	1	0	2
61	3	0	0	1	0	2
62	1	2	0	1	0	2
63	2	1	0	1	0	2
64	1	2	1	0	0	2
65	2	1	1	0	0	2
66	0	4	0	0	0	1
67	4	0	0	0	0	1
68	1	3	0	0	0	1
69	3	1	0	0	0	1
70	2	2	0	0	0	1

It is easy to verify that the last column does indeed sum to 1497. This shows that the probabilities given by (3.105) sum to unity.

EXERCISE 4.7 Carefully develop the argument leading from (4.43) to (4.44).

Solution. In the development of (4.43), N was never assigned a specific number. Rather, it is a variable and so it simply represents some integer. Hence (4.43) is true for all integers. Since $N - n$ also represents some integer, we see immediately that (4.44) follows from (4.43).

EXERCISE 4.8 Using the recursion of (4.43) together with the initial conditions, verify the expression for $g(N,5)$ for the special case $N=6$ numerically for the example presented in this section.

Solution. Letting $\rho=4$ as in the example in the chapter, we find that

$$\begin{aligned}\rho_1 &= \rho_2 = \frac{\rho}{4} = 1,\\ \rho_3 &= \rho_4 = \frac{\rho}{2} = 2,\\ \rho_5 &= \rho = 4.\end{aligned}$$

Then, using (3.109) along with the initial conditions we may compute $g(1,1)$:

$$\begin{aligned}g(1,1) &= g(1,0) + \rho_1 g(0,1)\\ &= 0 + 1\cdot 1 = 1.\end{aligned}$$

We then substitute this value into the expression for $g(1,2)$:

$$\begin{aligned}g(1,2) &= g(1,1) + \rho_2 g(0,2)\\ &= +1\cdot 1 = 2.\end{aligned}$$

Repeating this for each value of N and M until $g(6,5)$ is reached, we may form a table of normalizing constants:

				M		
	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
N 0 :	0	1	1	1	1	1
1 :	0	1	2	4	6	10
2 :	0	1	3	11	23	63
3 :	0	1	4	26	72	324
4 :	0	1	5	57	201	1497
5 :	0	1	6	120	522	6510
6 :	0	1	7	247	1291	27331

where $g(6,5)$ is in the lower right-hand corner of the table.

EXERCISE 4.9 Develop an expression for throughput at node M using Little's result and (4.47).

Solution. Define $\tilde{n}_{sM}$ and $\tilde{s}_{sM}$ to be the number in service and time in service, respectively, at node M. Each node in the system may have only one customer in service at a time, so that by (4.47),

$$\begin{aligned}E[\tilde{n}_M] &= P\left\{\tilde{n} > 0\right\}\\ &= \rho_M \frac{g(N-1,M)}{g(N,M)}\end{aligned}$$

$$= \quad L,$$

and

$$\begin{aligned} E[\tilde{s}_M] &= \frac{1}{\mu} \\ &= W. \end{aligned}$$

The throughput may now be found by applying Little's result to L and W,

$$\begin{aligned} \gamma &= \frac{L}{W} \\ &= \lambda_M \frac{g(N-1, M)}{g(N, M)}. \end{aligned}$$

EXERCISE 4.10 Show that $\det \mathcal{A}(z)$ is a polynomial, the order of which is not greater than $2(K+1)$.

Solution. The proof is by induction. First consider the case where $\mathcal{A}(z)$ is a 1×1 matrix. Clearly $\det \mathcal{A}(z)$ is a polynomial whose order is at most 2. Now assume that for an $n \times n$ matrix, its determinant is at most $2(n+1)$. Let $\mathcal{A}(z)$ be an $(n+1) \times (n+1)$ matrix. We wish to show $\det \mathcal{A}(z)$ is at most $2(n+2)$. Now, the determinant of $\mathcal{A}(z)$ will be

$$\det \mathcal{A}(z) = \sum_{j=0}^{K} a_{0,j} \mathcal{A}_{0,j},$$

where $a_{0,j}$ is the element in the jth column of the first row of $\mathcal{A}(z)$ and $\mathcal{A}_{0,j}$ represents its respective cofactor. Note that for all j, $j = 0, 1, \cdots, n$, $a_{0,j}$ is of at most order 2. Furthermore, each $\mathcal{A}_{0,j}$ is the determinant of an $n \times n$ matrix. By assumption, $\mathcal{A}_{0,j}$ is of order at most $2(n+1)$. Hence the product $a_{0,j}\mathcal{A}_{0,j}$ is of at most order $2(n+2)$. Thus the result holds for all $K \geq 0$.

EXERCISE 4.11 Repeat the above numerical example for the parameter values $\beta = \delta = 2$, $\lambda = \mu = 1$. That is, the proportion of time spent in each phase is the same and the transition rate between the phases is faster than in the original example. Compare the results to those of the example by plotting curves of the respective complementary distributions. Compute the overall traffic intensity and compare.

Solution. Substituting $\beta = \delta = 2$, $\lambda = \mu = 1$, into the example in Chapter 3, we find

$$\mathcal{Q} = \begin{bmatrix} -\beta & \beta \\ \delta & -\delta \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix},$$

$$\mathcal{M} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the matrices $\mathcal{M}$ and Λ remain unchanged from the example in the chapter. Then, upon substitution of these definitions into (3.137), we find

$$\mathcal{A}(z) = \begin{bmatrix} 1-3z & 2z \\ 2z & z^2-4z+1 \end{bmatrix},$$

$$\text{adj } \mathcal{A}(z) = \begin{bmatrix} z^2-4z+1 & -2z \\ -2z & 1-3z \end{bmatrix},$$

and

$$\begin{aligned} \det \mathcal{A}(z) &= -3z^3 + 9z^2 - 7z + 1 \\ &= 3(1-z)(1+\frac{\sqrt{6}}{3}-z)(1-\frac{\sqrt{6}}{3}-z). \end{aligned}$$

Thus, we find that

$$\text{adj } \mathcal{A}(z)\Big|_{z=1-\frac{\sqrt{6}}{3}} = \begin{bmatrix} 0.2996598 & -0.3670068 \\ -0.3670068 & 0.4494897 \end{bmatrix},$$

$$\text{adj } \mathcal{A}(z)\Big|_{z=1} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix},$$

and

$$\frac{\det \mathcal{A}(z)}{(1-z)}\Big|_{z=1} = -2.$$

Substituting these numbers into (3.140) and (3.141), we find that

$$[\,0 \quad 1\,] = P_0 \begin{bmatrix} -0.0673470 & 2 \\ +0.0824829 & 2 \end{bmatrix}.$$

Thus, we find that

$$P_0 = [\,0.2752551 \quad 0.2247449\,].$$

Upon substitution of this result into (3.138), we find after some algebra that

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix}^T = \frac{1}{1-0.5505103z} \begin{bmatrix} 0.2752551 - 0.0505103z \\ 0.2247449 \end{bmatrix}^T.$$

After some additional algebra, this result reduces to

$$[\,G_0(z) \quad G_1(z)\,] = \quad [\,0.2752551 \quad 0.2247449\,]$$

$$+ \quad [\, 0.1835034 \quad 0.2247449 \,] \sum_{n=1}^{\infty} (0.5505103)^n z^n.$$

Thus, for $n \geq 1$, we find

$$[\, P_{n0} \quad P_{n1} \,] = [\, 0.1835034 \quad 0.2247449 \,]\, (0.5505103)^n.$$

The probability generating function for the occupancy distribution can now be computed from (3.139) or by simply summing $G_0(z)$ and $G_1(z)$. We find

$$\mathcal{F}_{\tilde{n}}(z) = 0.5 + 0.4082483 \sum_{n=1}^{\infty} (0.5505103)^n z^n.$$

From this probability generating function, we find $P_0 = 0.5$ and

$$P_n \mathbf{e} = 0.4082483 \times (0.5505103)^n \qquad \text{for } n \geq 1.$$

To compute the traffic intensity we determine the probability that the system will be busy. This is simply the complement of the probability that the system is not busy: $1 - P_0$. Since P_0 did not change in this exercise from the example in the chapter, the traffic intensity for both problems will be the same:

$$1 - 0.5 = 0.5$$

Figure 4.1 shows the complementary distributions of both the original example and its modification $\beta = \delta = 2$,$\lambda = \mu = 1$.

EXERCISE 4.12 Suppose we want to use the packet switch and transmission line of the above numerical example to serve a group of users who collectively generate packets at a Poisson rate γ, independent of the computer's activity, in addition to the computer. This is a simple example of integrated traffic. Assuming that the user packets also require exponential service with rate μ, show the impact the user traffic has on the occupancy distribution of the packet switch by plotting curves for the cases of $\gamma = 0$ and $\gamma = 0.1$.

Solution. Since the group of users generate packets independent of the computer's activity, the overall arrival rate to the transmission line is simple the sum of the arrivals from the computer and from the group of users. That is,

$$\Lambda = \begin{bmatrix} \gamma & 0 \\ 0 & \lambda + \gamma \end{bmatrix}.$$

The matrices $\mathcal{Q}$ and $\mathcal{M}$ remain unchanged from the example. Now, in the first case where $\gamma = 0$, it should be clear that the matrix Λ will also remain

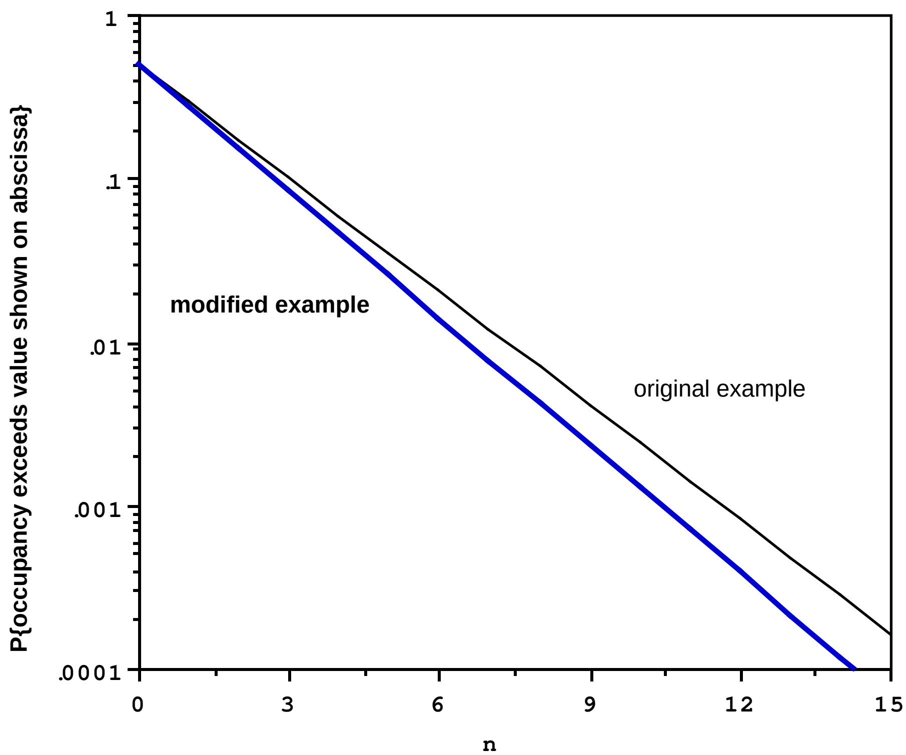

Figure 4.1. Complementary distributions

unchanged. i.e., the problem is the same as that in the example. In the second case, where $\gamma = 0.1$, the matrix Λ will be:

$$\Lambda = \begin{bmatrix} 0.1 & 0 \\ 0 & 1.1 \end{bmatrix}.$$

We repeat the definitions of $\mathcal{Q}$ and $\mathcal{M}$:

$$\mathcal{Q} = \begin{bmatrix} -\beta & \beta \\ \delta & -\delta \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$\mathcal{M} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We proceed in the same manner as in Exercise 3.39 and in the example in the chapter. Substituting these definitions into (3.137) yields:

$$\mathcal{A}(z) = \begin{bmatrix} .1z^2 - 2.1z + 1 & z \\ z & 1.1z^2 - 3.1z + 1 \end{bmatrix},$$

$$\text{adj } \mathcal{A}(z) = \begin{bmatrix} 1.1z^2 - 3.1z + 1 & -z \\ -z & .1z^2 - 2.1z + 1 \end{bmatrix},$$

and

$$\begin{aligned} \det \mathcal{A}(z) &= .11z^4 - 2.62z^3 + 6.71z^2 - 5.2z + 1 \\ &= 0.11(1-z)(1.5091141 - z)(21.0225177 - z)(0.2865501 - z). \end{aligned}$$

Thus, we find that

$$\text{adj } \mathcal{A}(z)\Big|_{z=0.2865501} = \begin{bmatrix} 0.2020168 & -0.2865501 \\ -0.2865501 & 0.4064559 \end{bmatrix},$$

$$\text{adj } \mathcal{A}(z)\Big|_{z=1} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

and

$$\frac{\det \mathcal{A}(z)}{(1-z)}\Big|_{z=1} = -0.8.$$

Substituting these numbers into (3.140) and (3.141), we find that

$$[\,0 \quad 1\,] = P_0 \begin{bmatrix} -0.0845332 & 2.5 \\ +0.1199059 & 2.5 \end{bmatrix}.$$

Thus, we find that

$$P_0 = [\,0.2346046 \quad 0.1653954\,].$$

Upon substitution of this result into (3.138), we find after some algebra that the matrix $[G_0(z) G_1(z)]$ is

$$\frac{1}{(1 - 0.6626404z)(1 - 0.0475680z)} \begin{bmatrix} 0.2346046 - 0.0739485z \\ 0.1653954 - 0.0047394 \end{bmatrix}^T.$$

After some additional algebra, this result reduces to

$$\begin{aligned} [\,G_0(z) \quad G_1(z)\,] = \quad & [\,0.2346046 \quad 0.1653954\,] \\ & + \quad [\,0.1325209 \quad 0.1704812\,] \sum_{n=1}^{\infty} (0.6626404)^n z^n \\ & + \quad [\,0.1020837 \quad -0.0050858\,] \sum_{n=1}^{\infty} (0.0475680)^n z^n. \end{aligned}$$

Thus, for $n \geq 1$, we find

$$[\,P_{n0} \quad P_{n1}\,] \quad = \quad [\,0.1325209 \quad 0.1704812\,]\,(0.6626404)^n$$

$$+ \quad [\,0.1020837 \quad -0.0050858\,]\,(0.0475680)^n.$$

The probability generating function for the occupancy distribution can now be computed from (3.139) or by simply summing $G_0(z)$ and $G_1(z)$. We find

$$\mathcal{F}_{\tilde{n}}(z) = \quad 0.4 + 0.3030021 \sum_{n=1}^{\infty} (0.6626404)^n z^n$$
$$+ \quad 0.0969979 \sum_{n=1}^{\infty} (0.0475680)^n z^n.$$

From this probability generating function, we find $P_0 = 0.4$ and

$$P_n\mathbf{e} = 0.3030021 \times (0.6626404)^n + 0.0969979 \times (0.0475680)^n \qquad \text{for } n \geq 1.$$

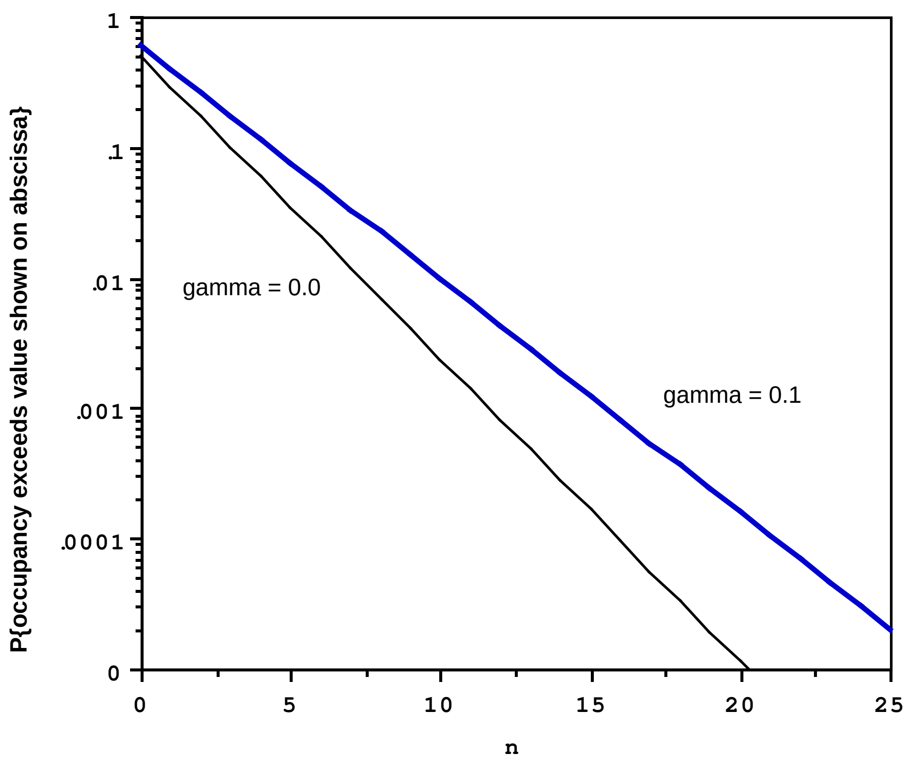

Figure 4.2. Complementary distributions

Figure 4.2 shows the occupancy distributions for the cases $\gamma = 0$, $\gamma = 0.1$.

EXERCISE 4.13 Let $\mathcal{Q}$ denote the infinitesimal generator for a finite, discrete-valued, continuous-time Markov chain. Show that the rows of $\mathrm{adj}\ \mathcal{Q}$ are equal.

Solution. We first note that the rows of $\mathrm{adj}\ \mathcal{Q}$ are all proportional to each other and, in addition, these are proportional to the stationary vector for the CTMC. Let the stationary vector be denoted by ϕ. Then, with k_i, $0 \le i \le K$, a scalar,

$$\mathrm{adj}\ \mathcal{Q} = \begin{bmatrix} k_0\phi \\ k_1\phi \\ \vdots \\ k_K\phi \end{bmatrix} = \begin{bmatrix} k_0\phi_0 & k_0\phi_1 & \dots & k_0\phi_K \\ k_1\phi_0 & k_1\phi_1 & \dots & k_1\phi_K \\ \vdots & \vdots & & \vdots \\ k_K\phi_0 & k_K\phi_1 & \dots & k_K\phi_K \end{bmatrix}$$

In addition, the columns of $\mathrm{adj}\ \mathcal{Q}$ are all proportional to each other and these are all proportional to the right eigenvector of $\mathcal{Q}$ corresponding to its zero eigenvalue. Since $\mathcal{Q}\mathbf{e} = 0$, this right eigenvector is proportional to $\mathbf{e}$. Therefore,

$$\mathrm{adj}\ \mathcal{Q} = [\, j_0\mathbf{e} \quad j_1\mathbf{e} \quad \dots \quad j_K\mathbf{e}\,] = \begin{bmatrix} j_0 & j_1 & \cdots & j_K \\ j_0 & j_1 & \cdots & j_K \\ \vdots & \vdots & & \vdots \\ j_0 & j_1 & \dots & j_K \end{bmatrix},$$

where j_i, $0 \le j \le K$, is a scalar. Comparing the two expressions for $\mathrm{adj}\ \mathcal{Q}$, we see that $k_i\phi_l = j_l$ for all i. So that if $\phi_l \ne 0$, then $k_i = j_l/\phi_l$ for all i. Since ϕ_l cannot be zero for all i unless all states are transient, which is impossible for a finite state CTMC, it follows that the k_i are identical so that the rows of $\mathrm{adj}\ \mathcal{Q}$ are identical and the proof is complete.

EXERCISE 4.14 Obtain (4.73) by starting out with (4.64), differentiating both sides with respect to z, postmultiplying both sides by $\mathbf{e}$, and then taking limits as $z \to 1$.

Solution. Recall Equation (4.64):

$$\mathcal{G}(z)\mathcal{A}(z) = (1 - z)P_0\,\mathcal{M},$$

where

$$\mathcal{A}(z) = \Lambda z^2 - (\Lambda - \mathcal{Q} + \mathcal{M})z + \mathcal{M}.$$

Now, differentiating both sides of (3.137) with respect to z, we see that

$$\mathcal{G}'(z)\mathcal{A}(z) + \mathcal{G}(z)\,[2\Lambda z - (\Lambda - \mathcal{Q} + \mathcal{M})] = -zP_0\mathcal{M}.$$

Taking limits as $z \to 1$,

$$\mathcal{G}'(1)\mathcal{Q} + \mathcal{G}(1)\,[\Lambda + \mathcal{Q} - \mathcal{M}] = -P_0\mathcal{M}.$$

Then postmultiply both sides by $\mathbf{e}$, and noting that $\mathcal{Q}\mathbf{e} = 0$ yields the desired result:

$$\mathcal{G}(1)\,[\mathcal{M} - \Lambda] = P_0\mathcal{M}.$$

EXERCISE 4.15 Show that the sum of the columns of $\mathcal{A}(z)$ is equal to the column vector $(1-z)(\mathcal{M}-z\Lambda)\mathbf{e}$ so that $\det \mathcal{A}(z)$ has a (1-z) factor.

Solution. To find the sum of the column vectors of $\mathcal{A}(z)$, postmultiply the matrix by the column vector $\mathbf{e}$:

$$\begin{aligned}\mathcal{A}(z)\mathbf{e} &= \Lambda z^2\mathbf{e} - (\Lambda - \mathcal{Q} + \mathcal{M})\,\mathbf{e} + \mathcal{M}\mathbf{e} \\ &= -z(1-z)\Lambda\mathbf{e} + (1-z)\mathcal{M}\mathbf{e} \\ &= (1-z)\,(\mathcal{M} - z\Lambda)\,\mathbf{e}.\end{aligned}$$

EXERCISE 4.16 Show that the zeros of the determinant of the λ-matrix $\mathcal{A}lA(z)$ are all real and nonnegative. [*Hint*: First, do a similarity transformation, transforming $\mathcal{A}(z)$ into a symmetric matrix, $\hat{\mathcal{A}}(z)$. Then, form the inner product $< X_z, \hat{\mathcal{A}}(z)X_z >$, where X_z is the null vector of $\hat{\mathcal{A}}(z)$ corresponding to z. Finally, examine the zeros of the resulting quadratic equation.]

Solution. First we show that the null values are all positive by transforming $\mathcal{A}(z)$ into a symmetric matrix, $\hat{\mathcal{A}}(z)$, and then forming the quadratic form of $\hat{\mathcal{A}}(z)$. This quadratic form results in a scalar quadratic equation whose roots are positive and one of which is a null value of $\mathcal{A}(z)$. Recall the definition of $\mathcal{A}(z)$:

$$\mathcal{A}(z) = \Lambda z^2 - (\Lambda - \mathcal{Q} + \mathcal{M})z + \mathcal{M},$$

where

$$\begin{aligned}\Lambda &= \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_K), \\ \mathcal{M} &= \mathrm{diag}(\mu_0, \mu_1, \cdots, \mu_K),\end{aligned}$$

and

$$\mathcal{Q} = \begin{pmatrix} -\beta_0 & \beta_0 & 0 & & & & \\ \delta_1 & -(\beta_1+\delta_1) & \beta_1 & & & & \\ & & & \ddots & & & \\ & & & & \delta_{K-1} & -(\beta_{K-1}+\delta_{K-1}) & \beta_{K-1} \\ & & & & 0 & \delta_K & -\delta_K \end{pmatrix}.$$

Now, since $\mathcal{Q}$ is tridiagonal and the off-diagonal terms have the same sign, it is possible to transform $\mathcal{Q}$ to a symmetric matrix using a diagonal similarity transformation. Define

$$W_{ij} = \begin{cases} \prod_{j=0}^{i} l_j & \text{for } i = j = 0, 1, \cdots, K \\ 0, & \text{else} \end{cases}$$

where

$$l_0 = 1,$$

and

$$l_j = \sqrt{\frac{\delta_j}{\beta_{j-1}}} \quad \text{for} \quad 1 \le j \le K.$$

Then

$$\hat{\mathcal{Q}} = W^{-1}\mathcal{Q}W,$$

where $\hat{\mathcal{Q}}$ is the following tridiagonal symmetric matrix:

$$\hat{\mathcal{Q}} = \begin{pmatrix} -\beta_0 & \sqrt{\beta_0\delta_1} & 0 & & & \\ \sqrt{\beta_0\delta_1} & -(\beta_1+\delta_1) & \sqrt{\beta_1\delta_2} & & & \\ & & & \ddots & & \\ & & & 0 & \sqrt{\beta_{K-1}\delta_K} & -\delta_K \end{pmatrix}.$$

It is then straightforward to show that $\hat{\mathcal{Q}}$ is negative semi-definite. Also, since the determinant of a product of matrices is equal to the product of the individual determinants, and W is non-singular, it follows that

$$\det\hat{\mathcal{A}}(z) = \det W^{-1}\mathcal{A}(z)W.$$

Furthermore, since Λ, $\mathcal{M}$, and W are all diagonal, it follows that

$$\det\hat{\mathcal{A}}(z) = \det\left[\Lambda z^2 - (\Lambda - \hat{\mathcal{Q}} + \mathcal{M})z + \mathcal{M}\right].$$

Now, suppose that $\det\mathcal{A}(z) = 0$. Then $\det\hat{\mathcal{A}}(z) = 0$, and there exists some nontrivial X_σ, such that

$$\hat{\mathcal{A}}(z)X_\sigma = 0.$$

Therefore,

$$< X_\sigma, \hat{\mathcal{A}}(z)X_\sigma >= 0,$$

where $< \cdot, \cdot >$ denotes the inner product. Thus,

$$< X_\sigma, \hat{\mathcal{A}}(z)X_\sigma >= l\sigma^2 - (l-p+m)\sigma + m = 0, \tag{3.44.1}$$

where

$$\begin{aligned} l &= < X_\sigma, \Lambda X_\sigma >, \\ m &= < X_\sigma, \mathcal{M}X_\sigma >, \\ p &= < X_\sigma, \hat{\mathcal{Q}}X_\sigma > . \end{aligned}$$

But, Λ and $\mathcal{M}$ are positive semi-definite, so l and m are nonnegative. Also, p is negative semi-definite, so $p \le 0$. The discriminant of the quadratic equation for (3.44.1) is given by

$$(l-p+m)^2 - 4lm = (l-m)^2 - 2p(l+m) + p^2.$$

We see that the right-hand side of this equation is the sum of nonnegative terms, and so the value of σ must be real. Hence all null values of $\mathcal{A}(z)$ are real.
We now show that these null values are positive. We see from above that either $l = 0$, or $l > 0$. In the first case, we find from (3.44.1) that

$$\sigma = \frac{m}{m-p} > 0.$$

In the second case we find from the quadratic equation that

$$\sigma = \frac{1}{2l}\left[(l-p+m) \pm \sqrt{(l-p+m)^2 - 4lm}\right].$$

Clearly, $(l-p+m) > \sqrt{(l-p+m)^2 - 4lm} > 0$. Therefore, both roots of (3.44.1) are positive so that all of the null values of $\mathcal{A}(z)$ are positive.

EXERCISE 4.17 The traffic intensity for the system is defined as the probability that the server is busy at an arbitrary point in time.

1. Express the traffic intensity in terms of the system parameters and P_0.
2. Determine the average amount of time a customer spends in service using the results of part 1 and Little's result.
3. Check the result obtained in part 2 for the special case $\mathcal{M} = \mu I$.

Solution.

1. Recall (4.74):
$$\mathcal{G}(1)\Lambda\mathbf{e} = [\mathcal{G}(1) - P_0]\,\mathcal{M}\mathbf{e}.$$
The left-hand side expresses the average rate at which units enter the service the system, and the right-hand side expresses the average rate at which units leave the system. Hence, analogous to the special case of the M/M/1 system of $\lambda = (1-P_0)\mu$ where $\rho = \frac{\lambda}{\mu} = (1-P_0)$, we may write the traffic intensity using (4.74) as
$$\rho = [\mathcal{G}(1) - P_0]\,\mathbf{e}.$$

2. Let T represent time. Then, if $[\mathcal{G}(1) - P_0]\,\mathbf{e}$ is the traffic intensity, $[\mathcal{G}(1) - P_0]\,T\mathbf{e}$ is the total time spent serving. In addition, the number of customers completing service over time is $[\mathcal{G}(1) - P_0]\,\mathcal{M}T\mathbf{e}$. This is due to the fact that the right-hand side of (4.74) represents the average rate at which customers leave the system. Therefore, by Little's result, the average service time is then
$$\frac{[\mathcal{G}(1) - P_0]\,T\mathbf{e}}{[\mathcal{G}(1) - P_0]\,\mathcal{M}T\mathbf{e}} = \frac{[\mathcal{G}(1) - P_0]\,\mathbf{e}}{[\mathcal{G}(1) - P_0]\,\mathcal{M}\mathbf{e}}.$$

3. If $\mathcal{M} = \mu I$, then the average service time is
$$\frac{E[\tilde{x}] = [\mathcal{G}(1) - P_0]\,\mathbf{e}}{[\mathcal{G}(1) - P_0]\,\mu\mathbf{e}} = \frac{1}{\mu},$$
which is a familiar result.

EXERCISE 4.18 Show that adj $\mathcal{A}(z_i)\mathbf{e}$ is proportional to the null vector of $\mathcal{A}(z)$ corresponding to z_i.

Solution. Since $\mathcal{A}^{-1}(z_i) = \text{adj}\ \mathcal{A}(z_i)/\det\ \mathcal{A}(z_i)$, and $\mathcal{A}(z_i)\mathcal{A}^{-1}(z_i) = I$, we find that
$$\mathcal{A}(z_i)\text{adj}\ \mathcal{A}(z_i) = \det\ \mathcal{A}(z_i)I.$$
By the continuity of $\mathcal{A}(z_i)$, adj $\mathcal{A}(z_i)$, and $\det\ \mathcal{A}(z_i)$, and the fact that z_i is a null value of $\mathcal{A}(z)$,
$$\lim_{\sigma\to\sigma_i} \mathcal{A}(z_i)\text{adj}\ \mathcal{A}(z_i) = \lim_{\sigma\to\sigma_i} \det\mathcal{A}(z_i)I.$$
This implies, since $\lim\limits_{\sigma\to\sigma_i} \det\mathcal{A}(z_i) = 0$, that
$$\mathcal{A}(z_i)\text{adj}\ \mathcal{A}(z_i) = O_{(K+1)\times(K+1)},$$
where $O_{n\times m}$ denotes an $n \times m$ matrix of zeroes. Thus, we have
$$\mathcal{A}(z_i)\text{adj}\ \mathcal{A}(z_i)\mathbf{e} = \mathbf{0}.$$
This defines adj $\mathcal{A}(z_i)$ to be a null vector of $\mathcal{A}(z)$ corresponding to z_i.

EXERCISE 4.19 Show that if the system described by (4.57) and (4.60) is ergodic, then there are exactly K zeros of $\det\ \mathcal{A}lA(z)$ in the interval $(0, 1)$. [*Hint*: First show that this is the case if $\delta_i = 0\ \forall\ i$. Then show that it is not possible for $\det\ \mathcal{A}(z)/(1 - z)$ to be zero for any choice of δ_is unless $P_0 = 0$, which implies no equilibrium solution exists. The result is that the number of zeros in $(0, 1)$ does not change when the δ_i change.]

Solution. First note that det $\mathcal{A}(z)$ is equal to the determinant of the matrix obtained from $\mathcal{A}(z)$ by replacing its last column by $\mathcal{A}(z)\mathbf{e}$. Now, it is easy to show that
$$\mathcal{A}(z)\mathbf{e} = \begin{bmatrix} \lambda_0 z^2 - (\lambda_0 + \mu_0)z + \mu_0 \\ \lambda_1 z^2 - (\lambda_1 + \mu_1)z + \mu_1 \\ \\ \cdots \\ \lambda_K z^2 - (\lambda_K + \mu_K)z + \mu_K \end{bmatrix} = (1 - z)(\mathcal{M} - \Lambda z)\mathbf{e}.$$
Therefore, det $\mathcal{A}(z) = (1 - z)$det $\mathcal{B}(z)$, where $\mathcal{B}(z)$ is the matrix obtained by replacing the last column of $\mathcal{A}(z)$ by the column vector $(\mathcal{M} - \Lambda z)\mathbf{e}$.

Now, we may compute the determinant of $\mathcal{B}(1)$ by expanding about its last column. In doing this, we find that the vector of cofactors corresponding to the last column in proportional to μ_K, the left eigenvector of $\mathcal{Q}$ corresponding to its zero eigenvalue. That is,

$$\mathcal{G}(1) = \frac{\phi_K}{\phi_K \mathbf{e}}.$$

Thus,

$$\det \mathcal{B}(1) = \alpha \mathcal{G}(1) \left[\mathcal{M} - \Lambda \right] \mathbf{e}, \tag{4.57.1}$$

for some $\alpha = \phi_K \mathbf{e} \neq 0$. But from equation (4.74), $\mathcal{G}(1) \left[\mathcal{M} - \Lambda \right] \mathbf{e} = \pi_0 \mathcal{M} \mathbf{e}$. The left-hand side of (4.57.1) is zero only if $\pi_0 = 0$. But $\pi_0 > 0$ if an equilibrium solution exists, so that if an equilibrium solution exists, then $\mathcal{B}(1)$ is nonsingular. Equivalently, if an equilibrium solution exists, then $\mathcal{B}(z)$ has no null value at $z = 1$.

We note that the null values of a λ-matrix are continuous functions of the parameters of the matrix. Thus, we may choose to examine the behavior of the null values of $\mathcal{B}(z)$ as a function of the death rates of the phase process while keeping the birth rates positive. Toward this end, we define

$$\hat{\mathcal{B}}(z, \delta_1, \delta_2, \cdots, \delta_K).$$

First, consider det $\hat{\mathcal{B}}(z, 0, 0, \cdots, 0)$. Since the lower subdiagonal of this λ-matrix is zero, the determinant of the matrix is equal to the product of the diagonal elements. These elements are as follows:

$$\lambda_i z^2 - (\lambda_i + \beta_i + \mu_i) z + \mu_i, \quad 0 \leq i < K,$$

$$\mu_K - \lambda_K z.$$

The null values of $\hat{\mathcal{B}}(z, 0, 0, \cdots, 0)$ may therefore be found by setting each of the above terms equal to zero. Thus, the null values of $\hat{\mathcal{B}}(z, 0, 0, \cdots, 0)$ may be obtained by solving the following equations for z:

$$\lambda_i z^2 - (\lambda_i + \beta_i + \mu_i) z + \mu_i = 0, \quad 0 \leq i < K, \tag{4.57.2}$$

and

$$\mu_K - \lambda_K = 0.$$

When solving (4.57.2), two possibilities must be considered: $\lambda_i = 0$, and $\lambda_i > 0$. In the first case,

$$z = \frac{\mu_i}{\beta_i + \mu_i},$$

so that for each i, the corresponding null value is between zero and one. In the latter case, we find that

$$\lim_{z \to 0} \lambda_i z^2 - (\lambda_i + \beta_i + \mu_i) z + \mu_i \quad = \quad \mu_i > 0;$$

$$\lim_{z\to 1} \lambda_i z^2 - (\lambda_i + \beta_i + \mu_i)z + \mu_i \quad = \quad -\beta_i < 0;$$

and that the expression

$$\lim_{z\to\infty} \lambda_i z^2 - (\lambda_i + \beta_i + \mu_i)z + \mu_i$$

tends to $+\infty$. Therefore, each quadratic equation of (4.57.2) has one root in $(0,1)$ and one root in $(1,\infty)$ if $\lambda_i > 0$. In summary, the equations (4.57.2) yield exactly K null values of $\hat{\mathcal{B}}(z,0,0,\cdots,0)$ in $(0,1)$, and between 0 and K null values of $\hat{\mathcal{B}}(z,0,0,\cdots,0)$ in $(1,\infty)$, the latter quantity being equal to the number of positive values of λ_i.

Now, due to (4.57.1), there exists no $(\delta_1,\delta_2,\cdots,\delta_K)$ such that 0 = det $\hat{\mathcal{B}}(1,\delta_1,\delta_2,\cdots,\delta_K)$. In addition, all of the null values of $\hat{\mathcal{B}}(z,\delta_1,\delta_2,\cdots,\delta_K)$ are real if $\mathcal{Q}$ is a generator for an ergodic birth and death process. Therefore, it is impossible for any of the null values of $\hat{\mathcal{B}}(z,\delta_1,\delta_2,\cdots,\delta_K)$ in the interval $(0,1)$ to assume that value 0 for some value of $(\delta_1,\delta_2,\cdots,\delta_K)$. Hence, the number of null values of $\hat{\mathcal{B}}(1,\delta_1,\delta_2,\cdots,\delta_K)$ in $(0,1)$ is independent of the death rates. Therefore, $\mathcal{B}(z)$, and consequently $\mathcal{A}(z)$, has exactly K null values in the interval $(0,1)$.

EXERCISE 4.20 Beginning with (4.86) through (4.88), develop expressions for the joint and marginal complementary ergodic occupancy distributions.

Solution. From (4.86),

$$P_n = \sum_{z_{K+1}\in Z^{(1,\infty)}} \mathcal{A}_i \, z_{K+i}^{-n}.$$

Therefore,

$$\begin{aligned}
P\{\tilde{n} > n\} &= \sum_{j=n+1}^{\infty} \sum_{z_{K+1}\in Z^{(1,\infty)}} \mathcal{A}_i \, z_{K+i}^{-j} \\
&= \sum_{j=n+1}^{\infty} \sum_{z_{K+1}\in Z^{(1,\infty)}} \mathcal{A}_i \left(\frac{1}{z_{K+i}}\right)^j \\
&= \sum_{z_{K+1}\in Z^{(1,\infty)}} \mathcal{A}_i \sum_{j=n+1}^{\infty} \left(\frac{1}{z_{K+i}}\right)^j \\
&= \sum_{z_{K+1}\in Z^{(1,\infty)}} \mathcal{A}_i \frac{\left(\frac{1}{z_{K+i}}\right)^{n+1}}{1-\frac{1}{z_{K+i}}} \\
&= \sum_{z_{K+1}\in Z^{(1,\infty)}} \mathcal{A}_i \left[\frac{z_{K+i}^{(n+1)}}{1-z_{K+i}^{-1}}\right].
\end{aligned}$$

Similarly,

$$P\{\tilde{n} > n\}\,\mathbf{e} = \sum_{z_{K+1} \in Z^{(1,\infty)}} \mathcal{A}_i \mathbf{e} \left[\frac{z_{K+i}^{(n+1)}}{1 - z_{K+i}^{-1}} \right].$$

EXERCISE 4.21 Develop an expression for $\mathrm{adj}\ \mathcal{A}(z_i)$ in terms of the outer products of two vectors using LU decomposition. [*Hint*: The term in the lower right-hand corner, and consequently the last row, of the upper triangular matrix will be zero. What then is true of its adjoint?]

Solution. We let $\mathcal{A}(z) = L(z)U(z)$, where $L(z)$ is lower triangular and $U(z)$ is upper triangular. We follow the usual convention that the elements of the major diagonal of $L(z)$ are all unity. From matrix theory, we know that for $\mathcal{A}(z)$ nonsingular,

$$\begin{aligned}
\mathrm{adj}\ \mathcal{A} &= \det \mathcal{A}(z)\ \mathcal{A}(z)^{-1} \\
&= \det \mathcal{A}(z) U^{-1}(z) L^{-1}(z) \\
&= \det \mathcal{A}(z) \frac{1}{\det U(z)} \mathrm{adj}\ U(z) \frac{1}{\det L(z)} \mathrm{adj}\ L(z) \\
&= \mathrm{adj}\ U(z)\ \mathrm{adj}\ L(z).
\end{aligned}$$

Since $\mathrm{adj}\ \mathcal{A}(z)$ is a continuous function of z for an arbitrary matrix $\mathcal{G}(z)$, we then have

$$\mathrm{adj}\ \mathcal{A}(z_i) = \mathrm{adj}\ U(z_i) \mathrm{adj}\ L(z_i).$$

Now, if z_i is a simple null value of $U(z)$, the LU decomposition can be arranged such that the last diagonal element of $U(z)$ is zero. This means the entire last row of $U(z)$ is zero so that all elements of $\mathrm{adj}\ U(z)$ are zero except for its last column. This means that $\mathrm{adj}\ \mathcal{A}(z_i)$ will be given by the product of the last column of $\mathrm{adj}\ U(z_i)$ with the last row of $\mathrm{adj}\ L(z_i)$. Let $y(z_i)$ denote the last column of $\mathrm{adj}\ U(z_i)$ and $x(z_i)$ denote the last row of $\mathrm{adj}\ L(z_i)$. Since $\det L(z_i) = 1$, we may solve for the last row of $\mathrm{adj}\ L(z_i)$ by solving

$$x(z_i) L(z_i) = f_K,$$

where f_i is the row vector of all zeros except for a 1 in position i. Since the last column of $\mathrm{adj}\ U(z)$ is unaffected by the elements of the last row of $U(z)$, we may determine the last column of $\mathrm{adj}\ U(z_i)$ in the following way. First, we replace the (K, K) element of $U(z_i)$ by $1/\prod_{j=0}^{K-1} U_{jj}(z_i)$. Call this matrix $\hat{U}(z_i)$ and the note that this matrix is nonsingular and that its determinant is 1. Now solve the linear system

$$\hat{U}(z_i) y(z_i) = \mathbf{e}_K.$$

We then have

$$y(z_i) = \frac{1}{\det \hat{U}(z_i)} \mathrm{adj}\ \hat{U}(z_i) \mathbf{e}_K.$$

But $\det \hat{U}(z_i) = 1$ and, in addition, the last column of $\text{adj } \hat{U}(z_i)$ is the same as the last column of $U(z_i)$. Consequently $y(z_i)$ is identically the last column of $\text{adj } U(z_i)$.

In summary, we solve $\hat{U}(z_i)y(z_i) = \mathbf{e}_K$, $x(z_i)L(z_i) = f_K$ and then we have

$$\text{adj } \mathcal{A}(z_i) = y(z_i)x(z_i).$$

We note in passing that solving the linear systems is trivial because of the form of $\hat{U}(z_i)$ and $L(z_i)$. We illustrate the above method with an example. Let

$$\mathcal{A} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 8 & 11 \\ 6 & 13 & 18 \end{bmatrix}.$$

We may then decompose $\mathcal{A}$ as described above:

$$\begin{aligned} L &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} \\ U &= \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Now, det L is 1, so that $\text{adj } L = L^{-1}$. That is, if $\hat{x}$ is the last row of $\text{adj } L$, then

$$\hat{x}L = [0 \quad 0 \quad 1]$$

Thus,

$$\hat{x} = [1 \quad -2 \quad 1]$$

We now replace the $(3, 3)$ element of U of $\frac{1}{2+2} = \frac{1}{4}$, so that

$$\hat{U} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Since the determinant of this matrix is 1, $\text{adj } \hat{U} =$ det $\hat{U}^{-1}$. We may compute the last column $\hat{y}$ of $\text{adj } \hat{U}$ (and, hence, of $\text{adj } U$) by solving the system

$$\hat{U}\hat{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

It is easy to show that $\hat{y} = [1 \quad -6 \quad 4]^T$. The product is then

$$\hat{x}\hat{y} = \begin{bmatrix} 1 & -2 & 1 \\ -6 & 12 & -6 \\ 4 & -8 & 4 \end{bmatrix}$$

By computing $\text{adj } \mathcal{A}$ directly by forming the minors of $\mathcal{A}$, we see that the product $\hat{x}\hat{y}$ is, indeed, equal to $\text{adj } \mathcal{A}$.

EXERCISE 4.22 Solve for the equilibrium state probabilities for Example 4.5 using the matrix geometric approach. Specify the results in terms of the matrix $\mathcal{A}lR$. Determine numerically the range of values of τ for which (4.98) converges. Also, verify numerically that the results are the same as those obtained above.

Solution. A program was written to compute the different iterations of Equation (3.172). $\mathcal{R}_0$ was taken to be diag$(0.5, 0.5)$. Using differing values of τ and the parameters from Example 3.7 into (3.172), we find

$$\mathcal{R}_i = \tau^{-1}\mathcal{R}_{i-1}^2 + \mathcal{R}_{i-1}\left[I - \tau^{-1}\begin{pmatrix} 2.1 & -1.0 \\ -1 & 3.1 \end{pmatrix}\right] + \tau^{-1}\begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 1.1 \end{bmatrix}.$$

The matrix $\mathcal{R}$ was determined to be

$$\mathcal{R} = \begin{bmatrix} 0.070500 & 0.029500 \\ 0.460291 & 0.639709 \end{bmatrix}$$

for values of τ between 1.7215 and 2795. Since the vector $\mathcal{G}(1)$ is proportional to the left eigenvector of $\mathcal{Q}$ corresponding to the eigenvalue 0, we find

$$\mathcal{G}(1) = [\,0.5 \quad 0.5\,].$$

Substituting these result into (3.167) yields

$$\begin{aligned} P_0 &= [\,0.5 \quad 0.5\,]\left[I - \begin{pmatrix} 0.070500 & 0.029500 \\ 0.460291 & 0.639709 \end{pmatrix}\right] \\ &= [\,0.234604 \quad 0.165395\,]. \end{aligned}$$

The solution expressed in matrix geometric form as in (3.163) is then

$$\begin{aligned} P_n &= P_0\mathcal{R}^n \\ &= [\,0.234604 \quad 0.165395\,]\begin{bmatrix} 0.070500 & 0.029500 \\ 0.460291 & 0.639709 \end{bmatrix}^n. \end{aligned}$$

These are the same results as those found in Example 3.7. In particular, we see that $P_0 = 0.4$.
The following are sample data from the program, where $\epsilon = 1 \times 10^{-11}$, and n is the number of iterations it took for the program to converge. Note that the resultant matrix $\mathcal{R}$ was the same for each of the sampled τ.

$$\mathcal{R} = \begin{bmatrix} 0.070500 & 0.029500 \\ 0.460291 & 0.639709 \end{bmatrix},$$

$\underline{\tau}$	$\underline{n}$
2	300
50	3200
200	11800
500	28000
2795	10928

EXERCISE 4.23 Solve Exercise 3.22 using the matrix geometric approach. Evaluate the relative difficulty of using the matrix geometric approach to that of using the probability generating function approach.

Solution. Substituting different values of τ and the parameters from Exercise 3.39 into the computer program from Exercise 3.50, we find

$$\mathcal{R}_i = \tau^{-1}\mathcal{R}_{i-1}^2 + \mathcal{R}_{i-1}\left[I - \tau^{-1}\begin{pmatrix} 3.0 & -2.0 \\ -2 & 4.0 \end{pmatrix}\right] + \tau^{-1}\begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}.$$

The matrix $\mathcal{R}$ was determined to be

$$\mathcal{R} = \begin{bmatrix} 0.0 & 0.0 \\ 0.449490 & 0.550510 \end{bmatrix}$$

. Since the vector $\mathcal{G}(1)$ is proportional to the left eigenvector of $\mathcal{Q}$ corresponding to the eigenvalue 0, we find

$$\mathcal{G}(1) = [\,0.5 \quad 0.5\,].$$

Substituting these result into (3.167) yields

$$\begin{aligned} P_0 &= [\,0.5 \quad 0.5\,]\left[I - \begin{pmatrix} 0.0 & 0.0 \\ 0.449490 & 0.550510 \end{pmatrix}\right] \\ &= [\,0.275255 \quad 0.224745\,]. \end{aligned}$$

The solution expressed in matrix geometric form as in (3.163) is then

$$\begin{aligned} P_n &= P_0\mathcal{R}^n \\ &= [\,0.275255 \quad 0.224745\,]\begin{bmatrix} 0.0 & 0.0 \\ 0.449490 & 0.550510 \end{bmatrix}^n. \end{aligned}$$

These are the same results as those found in Exercise 3.39. In particular, we see that $P_0 = 0.5$. Note that $P_{0,0}$ does not play a role in determining P_n since the first row of $\mathcal{R}$ is zero.

The following are sample data from the program, where $\epsilon = 1 \times 10^{-11}$, and n is the number of iterations it took for the program to converge. As in Exercise 3.50, the resultant matrix $\mathcal{R}$ was the same for each of the sampled τ.

$$\mathcal{R} = \begin{bmatrix} 0.0 & 0.0 \\ 0.449490 & 0.550510 \end{bmatrix},$$

τ	n
3	300
10	600
50	2400
100	4500
500	20400

For this exercise, $K = 1$. Since the matrices involved were 2×2, using the probability generating function approach was fairly straight forward and did not involve a great deal of computation. However, as the dimension of these matrices increases, the difficulty involved with using this method also increases. On the other hand, solving the problem using the matrix-geometric approach involved developing a computer program. Although this is not always an easy task, this particular program was relatively simple. Furthermore, the program should be adaptable to different parameters: matrix dimension, τ, λ_i, β_i, etc. So although there is initial overhead, it is reusable. Plus, as the dimension of the matrices increases even slightly, the matrix geometric method is much more practical than the probability generating function approach.

EXERCISE 4.24 Prove the result given by (4.112) for the n-th factorial moment of $\tilde{n}$.

Solution. First we prove a lemma:

$$\frac{d^n}{dz^n}\mathcal{G}(z) = n!\mathcal{G}(z)\left[I - z\mathcal{R}\right]^{-n}\mathcal{R}^n, \qquad n \geq 1.$$

Suppose $n = 1$. Then using the identity $\mathcal{G}(z) = P_0\left[I - z\mathcal{R}\right]^{-1}$,

$$\begin{aligned} \frac{d}{dz}\mathcal{G}(z) &= P_0\left[I - z\mathcal{R}\right]^{-2}\mathcal{R} \\ &= P_0\left[I - z\mathcal{R}\right]^{-1}\left[I - z\mathcal{R}\right]^{-1}\mathcal{R} \\ &= \mathcal{G}(z)\left[I - z\mathcal{R}\right]^{-1}\mathcal{R}. \end{aligned}$$

Now assume the result holds true for n; we wish to show it holds true for $n+1$.

$$\begin{aligned}\frac{d^{(n+1)}}{dz^{(n+1)}}\mathcal{G}(z) &= \frac{d}{dz}\left[\frac{d^n}{dz^n}\mathcal{G}(z)\right] \\ &= \frac{d}{dz}\left(n!\mathcal{G}(z)\left[I - z\mathcal{R}\right]^{-n}\mathcal{R}^n\right),\end{aligned}$$

by assumption. Thus,

$$\begin{aligned}\frac{d^{(n+1)}}{dz^{(n+1)}}\mathcal{G}(z) &= -(n+1)n!\left[I - z\mathcal{R}\right]^{-(n+1)}\mathcal{R}^n\left(-\mathcal{R}\right) \\ &= (n+1)!\mathcal{G}(z)\left[I - z\mathcal{R}\right]^{-(n+1)}\mathcal{R}^{n+1}.\end{aligned}$$

This proves the lemma. Using this result, we evaluate the n^{-th} derivative of $\mathcal{G}(z)$ at $z = 1$, and recall that $\mathcal{R}^n = \mathcal{V}^{-1}\mathcal{N}^n\mathcal{V}$,

$$\begin{aligned}\frac{d^n}{dz^n}\mathcal{G}(z)\bigg|_{z=1}\mathbf{e} &= n!\mathcal{G}(z)\left[I - z\mathcal{R}\right]^{-n}\mathcal{R}^n\bigg|_{z=1}\mathbf{e} \\ &= n!\mathcal{G}(1)\left[I - \mathcal{R}\right]^{-n}\mathcal{R}^n\mathbf{e} \\ &= n!\mathcal{G}(1)\prod_{j=1}^{n}\left[\sum_{i=0}^{\infty}\mathcal{R}^i\right]\mathcal{R}^n\mathbf{e} \\ &= n!\mathcal{G}(1)\prod_{j=1}^{n}\left[\sum_{i=0}^{\infty}\mathcal{V}^{-1}\mathcal{N}^i\mathcal{V}\right]\mathcal{V}^{-1}\mathcal{N}^n\mathcal{V}\mathbf{e} \\ &= n!\mathcal{G}(1)\mathcal{V}^{-1}\prod_{j=1}^{n}\left[\sum_{i=0}^{\infty}\mathcal{N}^i\right]\mathcal{N}^n\mathcal{V}\mathbf{e} \\ &= n!\mathcal{G}(1)\mathcal{V}^{-1}\left[I - \mathcal{N}\right]^{-n}\mathcal{N}^n\mathcal{V}\mathbf{e}.\end{aligned}$$

Now, $\mathcal{N}$ is a diagonal matrix, and so $[I - \mathcal{N}]^{-1}$ is the matrix

$$\mathrm{diag}\left[\ \tfrac{1}{1-\nu_0} \quad \cdots \quad \tfrac{1}{1-\nu_K}\ \right].$$

Furthermore, multiplication of diagonal matrices is again a diagonal matrix, whose elements are the products of the diagonal elements of the individual matrices. In particular, powers of diagonal matrices are the diagonal matrix of powers of the diagonal elements of the original matrix. Thus,

$$\frac{d^n}{dz^n}\mathcal{G}(z)\bigg|_{z=1}\mathbf{e} = n!\mathcal{G}(1)\nu^{-1}\left[\ \left(\tfrac{\nu_0}{1-\nu_0}\right)^n, \quad \cdots \quad \left(\tfrac{\nu_K}{1-\nu_K}\right)^n\ \right]\mathcal{V}\mathbf{e}.$$

EXERCISE 4.25 Beginning with (4.113), show that

$$\sum_{i=0}^{\infty} P_i = \pi.$$

[Hint: First, sum the elements of the right hand side of (4.113) from $i = 1$ to $i = \infty$. This will yield $0 = [\sum_{i=0}^{\infty} P_i]\,[A_0 + A_1 + A_2] + \theta(P_0, P_1)$. Next, use the fact that, because $\tilde{Q}$ is stochastic, the sum of the elements of each of the rows of $\tilde{Q}$ must be a zero matrix to show that $\theta(P_0, P_1) = 0$. Then complete the proof in the obvious way.]

Solution. We emphasize that we are dealing here with the specific case that the arrival and service rates are dependent upon the phase of an auxiliary process. The arrival, service, and phase processes are independent of the level except that at level zero, there is no service. For a system in which the previous condition does not hold, the result is in general not true.

The implication of the fact that $\tilde{Q}$ is stochastic is that the sum of the block matrices at each level is stochastic; that is $[A_0 + A_1 + A_2]$, $[B_1 + A_1 + A_0]$, and $[B_0 + A_0]$ are all stochastic, and each of these stochastic matrices is the infinitesimal generator of the phase process at its level. Thus, in order for the phase process to be independent of the level, the three stochastic matrices must be the same. This can be found by comparing the first two to see that $B_1 = A_2$ and the first and third to see that $B_0 = A_1 + A_2$.

Upon summing the elements of the right hand side of (4.113) from $i = 1$ to $i = \infty$, we find

$$0 = \left[\sum_{i=0}^{\infty} P_i\right] [A_0 + A_1 + A_2] - P_0\,[A_1 + A_2] - P_1 A_2.$$

But, for our special case, $A_1 + A_2 = B_0$ and $A_2 = B_1$. Thus the previous summation yields

$$0 = \left[\sum_{i=0}^{\infty} P_i\right] [A_0 + A_1 + A_2] - P_0 B_0 - P_1 B_1.$$

But, from the boundary condition of (4.113), we have $P_0 B_0 - P_1 B_1 = 0$. Therefore, we have

$$0 = \left[\sum_{i=0}^{\infty} P_i\right] [A_0 + A_1 + A_2]$$

from which we see that

$$\sum_{i=0}^{\infty} P_i = \pi$$

because both are the unique left eigenvector of $[A_0 + A_1 + A_2]$ whose elements sum to unity.

In a very general case, the arrival and services while in a given phase could be accompanied by a phase transition. For xample, we would denote by λ_{ij} the arrival rate when the process that starts in phase i and results in an increase in level by 1 and a transition to phase j simultaneously, and define μ_{ij} in a parallel way. At the same time, we could have an independently operating phase process, whose infinitesimal generator is $\mathcal{Q}$.

Let $\hat{\Lambda}$ and $\hat{\mathcal{M}}$ denote the more general arrival and service matrices, respectively, and let Λ and $\mathcal{M}$ denote the diagonal matrices whose i-th elements denote the total arrival and service rates while the process is in phase i. It is easy to show that we then have the following dynamical equations:

$$0 = P_0(\mathcal{Q} - \Lambda) + P_1\hat{\mathcal{M}} \quad \text{and} \quad 0 = P_{i-1}\hat{\Lambda} + P_i(\mathcal{Q} - \Lambda - \mathcal{M}) + P_{i+1}\hat{\mathcal{M}}.$$

It is then easy to see that the phase process is independent of the level if and only if the $\mathcal{M} = \hat{\mathcal{M}}$. That is, the service process must not result in a phase shift. Alternatively, the phase process at level zero could be modified to incorporate the phase changes resulting from the service process at all other levels, but then this would contaminate the definition of the independent phase process at each level.

EXERCISE 4.26 With $\tilde{Q}$ defined as in (4.138) and b any vector of probability masses such that $b\mathbf{e} = 1$, show that the matrix $\mathcal{S} + \mathcal{S}^0 b$ is always an infinitesimal generator.

Solution. We first note that since $\tilde{Q}$ is stochastic, it follows that $\mathcal{S}\mathbf{e} + \mathcal{S}^0 = 0$. Since $b\mathbf{e} = 1$, it follows immediately that $\mathcal{S}^0 b\mathbf{e} = \mathcal{S}^0$. Thus, $\mathcal{S}\mathbf{e} + \mathcal{S}^0 b\mathbf{e} = 0$. But, $\mathcal{S}^0 b$ is a matrix whose dimensions are the same as those of $\mathcal{S}$ so that $\mathcal{S}$ and $\mathcal{S}^0 b$ may be summed term-by-term. Hence we have $(\mathcal{S} + \mathcal{S}^0 b)\mathbf{e} = 0$. It remains to show that the elements of the diagonal of $\mathcal{S} + \mathcal{S}^0 b$ are nonpositive and the off diagonal elements are nonnegative. To show this, first note that the matrix $\mathcal{S}$ has these properties. Since $\tilde{Q}$ is stochastic, the magnitude of the diagonal elements of $\mathcal{S}$ must be at least as large as the corresponding row elements of $\mathcal{S}^0$, so that the sum of the diagonal elements of $\mathcal{S}$ and the corresponding row elements of $\mathcal{S}^0$ cannot exceed 0. The diagonal elements of $\mathcal{S}^0 b$ cannot exceed the corresponding row elements of $\mathcal{S}^0$ because b is a vector of probability masses, and hence each element cannot exceed 1. Since the off-diagonal elements of $\mathcal{S}$ and the elements of $\mathcal{S}^0 b$ are all nonnegative, the elements of the result follows.

EXERCISE 4.27 Consider a single-server queueing system having Poisson arrivals. Suppose upon entering service, each customer initially receives a type 1 service increment. Each time a customer receives a type 1 service increment, the customer leaves the system with probability $(1-p)$ or else receives a type 2 service increment followed by an additional type 1 service increment. Suppose type 1 and type 2 service increment times are each drawn independently from exponential distributions with parameters μ_1 and μ_2, respectively. Define the phase of the system to be 1 if a customer in service is receiving a type 2 service increment. Otherwise, the system is in phase 0. Define the state of the system to be the 0 when the system is empty and by the pair (i, j) where $i > 0$ is the system occupancy and $j = 0, 1$ is the phase of the service process. Define $P_i = [\, P_{i0} \quad P_{i1}\,]$ for $i > 0$ and P_0, a scalar. Draw the state diagram, and determine the matrix $\mathcal{Q}$, the infinitesimal generator for the continuous-time Markov chain defining the occupancy process for this system.

Solution. The state diagram can be determined by first defining the states in the diagram which have the from (i, j), where i is the system occupancy and j is the phase, except for $i = 0$. It is then a matter of determining transition rates. Since all phase 0 service completions result in lowering the occupancy by 1 with probability p, the rate from state $(i, 0)$ to $(i-1, 0)$ is $\mu_1(1-p)$. Similarly, the rate from $(i, 0)$ to $(i, 1)$ is $\mu_1 p$. This is a direct result of Property 2 of the Poisson Process given on page 44. Since service completions at phase 1 always result in an additional phase 0 service, the transition rate from $(i, 1)$ to $(i, 0)$ is μ_2. Increases in level occur due to arrivals only, so for every state, the rate of transition to the next level is λ. The state diagram is detailed in Figure 4.3.

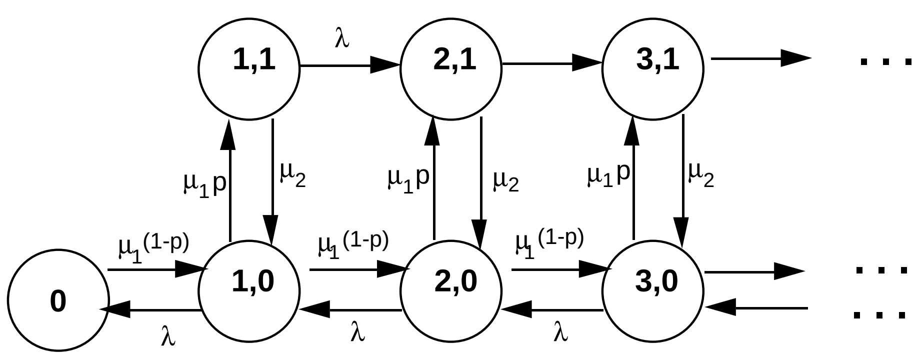

Figure 4.3. State Diagram for Problem 4.27

By inspection of the state diagram, we find

$$\mathcal{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu_1(1-p) & -\mu_1-\lambda & \mu_1 p & \lambda & 0 & \cdots \\ 0 & \mu_2 & -\lambda-\mu_2 & 0 & \lambda & \cdots \\ 0 & \mu_1(1-p) & 0 & -\mu_1-\lambda & \mu_1 p & \cdots \\ 0 & 0 & 0 & \mu_2 & -\mu_2-\lambda \ldots & \\ 0 & 0 & 0 & \mu_1(1-p) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This compares to the infinitessimal generator for a queueing system with phase dependent arrivals and service as follows:

$$\begin{aligned} \Lambda &= \lambda I, \\ P &= \begin{bmatrix} -\mu_1 p & \mu_1 p \\ \mu_2 & -\mu_2 \end{bmatrix}, \\ \mathcal{M} &= \begin{bmatrix} \mu_1(1-p) & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

It is different in the first level, but it is exactly an M/MP/1 system because the service always starts in phase 0.

EXERCISE 4.28 Suppose

$$\mathcal{S} = \begin{bmatrix} -\mu & \mu \\ 0 & -\mu \end{bmatrix} \quad \mathcal{S}^0 = \begin{bmatrix} 0 \\ \mu \end{bmatrix}, \quad \text{and} \quad P_t(0) = [\,1 \quad 0\,].$$

Find $F_{\tilde{x}}(t) = P\{\tilde{x} \le t\}$ and $f_{\tilde{x}}(t)$, and identify the form of $f_{\tilde{x}}(t)$. [*Hint*: First solve for $P_0(t)$, then for $P_1(t)$, and then for $P_2(t) = P_a(t)$. There is never a need to do matrix exponentiation.]

Solution. We have

$$\frac{d}{dt}P(t) = P(t)\tilde{\mathcal{Q}},$$

or

$$\frac{d}{dt}[\,P_0(t) \quad P_1(t) \quad P_2(t)\,] = [\,P_0(t) \quad P_1(t) \quad P_2(t)\,] \begin{bmatrix} -\mu & \mu & 0 \\ 0 & -\mu & \mu \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\frac{d}{dt}P_0(t) = \mu P_0(t)$, so that $P_0(t) = ke^{-\mu t}$. But $P_0(0) = 1$, so $k = 1$. Therefore $P_0(t) = e^{-\mu t}$.

$$\frac{d}{dt}P_1(t) = \mu P_0(t) - \mu P_1(t).$$

or

$$\frac{d}{dt}P_1(t) + \mu P_1(t) = \mu P_0(t).$$

Multiplying both sides by $e^{\mu t}$,

$$e^{\mu t}\frac{d}{dt}P_1(t) + \mu e^{\mu t}P_1(t) = \mu e^{\mu t}P_0(t).$$

Equivalently,

$$\frac{d}{dt}\left[e^{\mu t}P_1(t)\right] = \mu e^{\mu t}e^{-\mu t} = \mu.$$

Therefore, $e^{\mu t}P_1(t) = \mu t + k$, or $P_1(t) = \mu t e^{-\mu t} + k e^{-\mu t}$. Now, $P_1(0) = 0$, so $k = 0$ and we have $P_1(t) = \mu t e^{-\mu t}$.

$$\begin{aligned}\frac{d}{dt}P_2(t) &= \mu P_1(t) \\ &= \mu^2 t e^{-\mu t}.\end{aligned}$$

Since the probabilities sum to 1, it must be the case that

$$\begin{aligned}P_2(t) &= 1 - P_0(t) - P_1(t) \\ &= 1 - e^{-\mu t}\mu t e^{-\mu t}.\end{aligned}$$

We may check this by evaluating $P_2(t)$ directly.

$$\begin{aligned}\frac{d}{dt}P_2(t) &= \frac{d}{dt}\left[1 - e^{-\mu t}\mu t e^{-\mu t}\right] \\ &= 0 + \mu e^{-\mu t} + \mu^2 t e^{-\mu t} - \mu e^{-\mu t} \\ &= \mu(\mu t)e^{-\mu t}.\end{aligned}$$

Now,

$$\begin{aligned}P_2(t) &= P\left\{\text{in state 2 at time t}\right\} \\ &= P\left\{\tilde{x} \le t\right\}.\end{aligned}$$

That is,

$$F_{\tilde{x}}(t) = 1 - e^{-\mu t} - \mu t e^{-\mu t},$$

and

$$f_{\tilde{x}}(t) = \mu(\mu t)e^{-\mu t}.$$

EXERCISE 4.29 Starting with (4.139) as given, prove the validity of (4.140).

Solution. First, find the Laplace-Stieltjes transform of $\tilde{x}$.

$$\begin{aligned}F^*_{\tilde{x}}(s) &= \int_0^\infty e^{-sx}dF_{\tilde{x}}(x) \\ &= \int_0^\infty e^{-sx}\left[-bSe^{Sx}\mathbf{e}\right]dx\end{aligned}$$

$$\begin{aligned}
&= -bS\int_0^\infty e^{-[sI-S]x}dx\mathbf{e} \\
&= bS\left[sI-S\right]^{-1}\int_0^\infty \left(-\left[sI-S\right]\right)e^{-[sI-S]x}dx\mathbf{e} \\
&= -bS\left[sI-S\right]^{-1}\mathbf{e}.
\end{aligned}$$

Now, $E[\tilde{x}^n] = (-1)^n \frac{d^n}{ds^n} F^*_{\tilde{x}}(s)|_{s=0}$, so

$$\begin{aligned}
E[\tilde{x}] &= (-1)\frac{d}{ds}F^*_{\tilde{x}}(s)\Big|_{s=0} \\
&= (-1)\frac{d^n}{ds^n}\left[-bS\left[sI-S\right]^{-1}\mathbf{e}\right]\Big|_{s=0} \\
&= (-1)\left[bS\left[sI-S\right]^{-2}\mathbf{e}\right]\Big|_{s=0} \\
&= -bS^{-1}\mathbf{e}.
\end{aligned}$$

We now prove that proposition that

$$\frac{d^n}{ds^n}F^*_{\tilde{x}}(s) = -n!(-1)^n bS\left[sI-S\right]^{-(n+1)}\mathbf{e}.$$

For n=1,

$$\begin{aligned}
\frac{d}{ds}F^*_{\tilde{x}}(s) &= bS\left[sI-S\right]^{-2}\mathbf{e} \\
&= 1!(-1)(-1)bS\left[sI-S\right]^{-2}\mathbf{e}.
\end{aligned}$$

So $1 \in T$, the truth set for the proposition. If $(n-1) \in T$, then

$$\frac{d^{n-1}}{ds^{n-1}}F^*_{\tilde{x}}(s) = -(n-1)!(-1)^{(n-1)}bS\left[sI-S\right]^{-n}\mathbf{e},$$

so that

$$\begin{aligned}
\frac{d^n}{ds^n}F^*_{\tilde{x}}(s) &= -(n-1)!(-1)^{(n-1)}bS(-n)\left[sI-S\right]^{-(n+1)}\mathbf{e} \\
&= -n!(-1)^n bS\left[sI-S\right]^{-(n+1)}\mathbf{e}.
\end{aligned}$$

and the proof of the proposition is complete. Therefore, continuing with the proof of the exercise,

$$\begin{aligned}
E[\tilde{x}^n] &= (-1)^n\left[-n!(-1)^n bS\left[sI-S\right]^{-(n+1)}\mathbf{e}\right]\Big|_{s=0} \\
&= -n!bS(-1)^{(n+1)}\mathbf{e} \\
&= n!(-1)^n bS^{-n}\mathbf{e}.
\end{aligned}$$

4.1 Supplementary Problems

4-1 Let $\mathcal{Q}$ be an $(m+1)$-square matrix representing the infinitesimal generator for a continuous-time Markov chain with state space $\{0, 1, \ldots, m\}$. Let

$$\tilde{\mathcal{Q}} = \begin{bmatrix} \mathcal{T} & \mathcal{T}^0 \\ 0 & 0 \end{bmatrix},$$

where $\mathcal{T}$ is an m-square matrix, $\mathcal{T}^0$ is an $m \times 1$ column vector, and the remaining terms are chosen to conform, be a matrix obtained by replacing any row of $\mathcal{Q}$ by a row of zeros and then exchanging rows so that the final row is a vector of zeros. Let $P(t) = [\, P_t(t) \quad P_a(t)\,]$ denote the state probability vector for the Markov chain for which $\tilde{\mathcal{Q}}$ is the infinitesimal generator, with $P_t(t)$ a row vector of dimension m and $P_a(t)$ a scalar.

(a) Argue that if $\mathcal{Q}$ is the infinitesimal generator for an irreducible Markov chain, then the states $0, 1, \ldots, m-1$ of the modified chain are all transient, and state m is an absorbing state.

(b) Prove that if $\mathcal{Q}$ is the infinitesimal generator for an irreducible Markov chain, then the matrix $\mathcal{T}$ must be nonsingular. [*Hint*: Solve for $P_t(t)$, then prove by contradiction. Make use of the fact that if $\mathcal{T}$ is singular, then $\mathcal{T}$ has a zero eigenvalue.]

(c) Show that $P_a(t) = 1 - P_t(0)\exp\{\mathcal{T}t\}\mathbf{e}$. [*Hint*: Use the fact that $P_t(t)\mathbf{e}$ is the probability that the state of the modified Markov chain is in the set $\{0, \ldots, m-1\}$ at time t.]

(d) Let $\tilde{x}$ be the time required for the modified Markov chain to reach state m given an initial probability vector $P(0) = [\, P_t(0) \quad 0\,]$, that is, with $P_t(0)\mathbf{e} = 1$. Argue that $P\{\tilde{x} \le t\} = P_a(t)$, that is,

$$P\{\tilde{x} \le t\} = 1 - P_t(0)\exp\{\mathcal{T}t\}\mathbf{e}.$$

(e) Argue that if $\mathcal{Q}$ is the infinitesimal generator for an irreducible Markov chain, then the matrix $\tilde{\mathcal{T}} = \mathcal{T} + \mathcal{T}^0 P_t(0)$ is the infinitesimal generator for an irreducible Markov chain with state space $\{0, 1, \ldots, m-1\}$.

Solution:

(a) Since the Markov chain is irreducible, all states can be reached from all other states; therefore, state m is reachable from all other states in the original chain. Since the dynamics of the system have not been changed for states $\{0, 1, \cdots, m-1\}$ in the modified chain, state m is still reachable from all states. But, once in state m, the modified system will never leave state m because its departure rate to all other states is 0. Therefore, all states other than state m are transient states in the modified chain and state m is an absorbing state.

(b) Observe that

$$\frac{d}{dt}\left[\, P_t(t) \quad P_a(t) \,\right] = P_t(t)\mathcal{T}$$

so that

$$\frac{d}{dt}P_t(t) = P_t(0)e^{\mathcal{T}t}$$

Now, we know that

$$\lim_{t\to 0} P_t(t) = 0$$

is independent of $P_t(0)$. This means that all eigenvalues of $\mathcal{T}$ must have strictly negative real parts. But, if all eigenvalues have strictly negative real parts, the matrix $\mathcal{T}$ cannot be singluar.

(c) From the matrix above, $d/dtP_a(t) = P_t(t)\mathcal{T}^0$. Therefore,

$$\frac{d}{dt}P_a(t) = P_{t_0}e^{\mathcal{T}t}\mathcal{T}^0$$

Now, $P_a(t)$ is the probability that the system is in an absorbing state at time t, which is equal to the complement of the probability that the system is in some other state at time t. Hence,

$$\begin{aligned} P_a(t) &= 1 - P_t(t)\mathbf{e} \\ &= 1 - P_t(0)e^{\mathcal{T}t}\mathbf{e} \end{aligned}$$

Furthermore, from the differential equation,

$$P_a(t) = P_t(0)e^{\mathcal{T}t}\mathcal{T}^{-1}\mathcal{T}^0 + K$$

But $\mathcal{T}\mathbf{e} + \mathcal{T}^0 = 0$, so that $\mathcal{T}^{-1}\mathcal{T}^0 = -\mathbf{e}$. Thus,

$$P_a(t) = K - P_t(0)\mathbf{e}e^{\mathcal{T}t}\mathbf{e},$$

and

$$\lim_{t\to 0} P_a(t) = 1 = 1 - 0$$

implies that $K = 1$, and $P_a(t) = 1 - P_t(0)e^{\mathcal{T}t}\mathbf{e}$.

(d) The argument is straight-forward. If the chain is in the absorbing state at time t, then it necessarily has left the transient states by time t. Thus,

$$\begin{aligned} P\left\{\tilde{x} \le t\right\} &= P_a(t) \\ &= 1 - P_t(0)e^{\mathcal{T}t}\mathbf{e} \end{aligned}$$

(e) The vector $\mathcal{T}^0$ represents the rates at which the system leaves the transient states to enter the absorbing state. The matrix $\mathcal{T}^0 P_t(0)$ is then a nonnegative $m \times m$ matrix. Since $[\mathcal{T}\ \mathcal{T}^0]\mathbf{e} = 0$, it follows that $\mathcal{T} + \mathcal{T}_0 P_t(0)$ is a matrix whose diagonal terms are nonpositive. This is because $\mathcal{T}_{ii} + \mathcal{T}_i^0 \leq 0$ so that $\mathcal{T}_{ii} + \mathcal{T}_i^0 P_{ti}(0) \leq 0$. This is due to the fact that $P_{ti}(0) \leq 1$. All nondiagonal terms are nonnegative since all off-diagonal terms of $\mathcal{T}$ are nonnegative and $\mathcal{T}^0$ is nonnegative. It remains only to determine if

$$\tilde{\mathcal{T}}_e = [\,\mathcal{T} + \mathcal{T}^0 P_t(0)\,]\,\mathbf{e} = 0$$

Now, $P_t(0)\mathbf{e} = 1$, so $[\mathcal{T} + \mathcal{T}^0 P_t(0)]\mathbf{e} = \mathcal{T}\mathbf{e} + \mathcal{T}^0$, which we already know is equal to a zero vector. Thus, $\mathcal{T} + \mathcal{T}^0 P_t(0)$ is the infinitesimal generator for an irreducible Markov chain with state space $\{0, 1, \ldots, m-1\}$.

4-2 Consider an m-server queueing system having Poisson arrivals. Suppose upon entering service, each customer initially receives a type 1 service increment. Each time a customer receives a type 1 service increment, the customer leaves the system with probability $(1-p)$ or else receives a type 2 service increment followed by an additional type 1 service increment. Suppose type 1 and type 2 service increment times are each drawn independently from exponential distributions with parameters μ_1 and μ_2, respectively. With the service process defined as in Problem 4-1, suppose there are m servers. Define the phase of the system to be j if there are j customers receiving a type 2 service increment, $j = 0, 1, \cdots, m$. Define the state of the system to be 0 when the system is empty and by the pair (i, j) where $i \geq 0$ is the system occupancy and $j = 0, \cdots, i$ is the phase of the service process. Define $P_i = [\,P_{i0} \quad P_{i1} \quad \cdots \quad P_{i,\min\{i,m\}}\,]$ for $i > 0$ and P_0, a scalar.

(a) Draw the state transition diagram for the special case of $m = 3$.

(b) Write the matrix balance equations for the special case of $m = 3$.

(c) Write the matrix balance equations for the case of general values of m.

(d) Determine the matrix $\mathcal{Q}$, the infinitesimal generator for the continuous-time Markov chain defining the occupancy process for this system.

(e) Comment on the structure of the matrix $\mathcal{Q}$ relative to that for the phase-dependent arrival and service rate queueing system and to the M/PH/1 system. What modifications in the solution procedure would have to be made to solve this problem? [*Hint*: See Neuts [1981a], pp. 24-26.]

Solution:

(a) For this exercise, we may think of a customer as staying with the same server thoughout its service; the server simply does a different task. See Figure 4.4 for the state diagram of the system.

(b) We first introduce the following notation, where I_k is the $k\times k$ identity matrix:

$$
\begin{aligned}
\Lambda_0 &= \lambda, \hat{\Lambda}_0 = [\,\lambda \quad 0\,] \\
\Lambda_1 &= \lambda I_2, \quad \hat{\Lambda}_1 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{bmatrix} \\
\Lambda_2 &= \lambda I_3, \quad \hat{\Lambda}_2 = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{bmatrix} \\
\Lambda_3 &= \lambda I_4 \\
\mathcal{M}_1 &= \begin{bmatrix} \mu_1(1-\rho) & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\mathcal{M}}_1 = \begin{bmatrix} \mu_1(1-\rho) \\ 0 \end{bmatrix}, \\
\mathcal{M}_2 &= \begin{bmatrix} 2\mu_1(1-\rho) & 0 & 0 \\ 0 & \mu_1(1-\rho) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\hat{\mathcal{M}}_2 &= \begin{bmatrix} 2\mu_1(1-\rho) & 0 \\ 0 & \mu_1(1-\rho) \\ 0 & 0 \end{bmatrix}, \\
\mathcal{M}_3 &= \begin{bmatrix} 3\mu_1(1-\rho) & 0 & 0 & 0 \\ 0 & 2\mu_1(1-\rho) & 0 & 0 \\ 0 & 0 & \mu_1(1-\rho) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\hat{\mathcal{M}}_3 &= \begin{bmatrix} 3\mu_1(1-\rho) & 0 & 0 \\ 0 & 2\mu_1(1-\rho) & 0 \\ 0 & 0 & \mu_1(1-\rho) \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathcal{P}_1 &= \begin{bmatrix} 0 & -\mu_1\rho \\ -\mu_2 & 0 \end{bmatrix}, \mathcal{P}_2 = \begin{bmatrix} 0 & -2\mu_1 & 0 \\ -\mu_2 & 0 & -\mu_1\rho \\ 0 & -2\mu_2 & 0 \end{bmatrix}, \\
\mathcal{P}_3 &= \begin{bmatrix} 0 & -3\mu_1 & 0 & 0 \\ -\mu_2 & 0 & -2\mu_1 & 0 \\ 0 & 0 & -3\mu_2 & 0 \end{bmatrix}.
\end{aligned}
$$

With this notation, we may compactly write the balance equations as follows:

$$
\begin{aligned}
P_0\Lambda_0 &= P_1\hat{\mathcal{M}}_1 \\
P_1\,[\,\Lambda_1 + \mathcal{P}_1 + \mathcal{M}_1\,] &= P_0\hat{\Lambda}_0 + P_2\hat{\mathcal{M}}_2 \\
P_2\,[\,\Lambda_2 + \mathcal{P}_2 + \mathcal{M}_2\,] &= P_1\hat{\Lambda}_0 + P_3\hat{\mathcal{M}}_3
\end{aligned}
$$

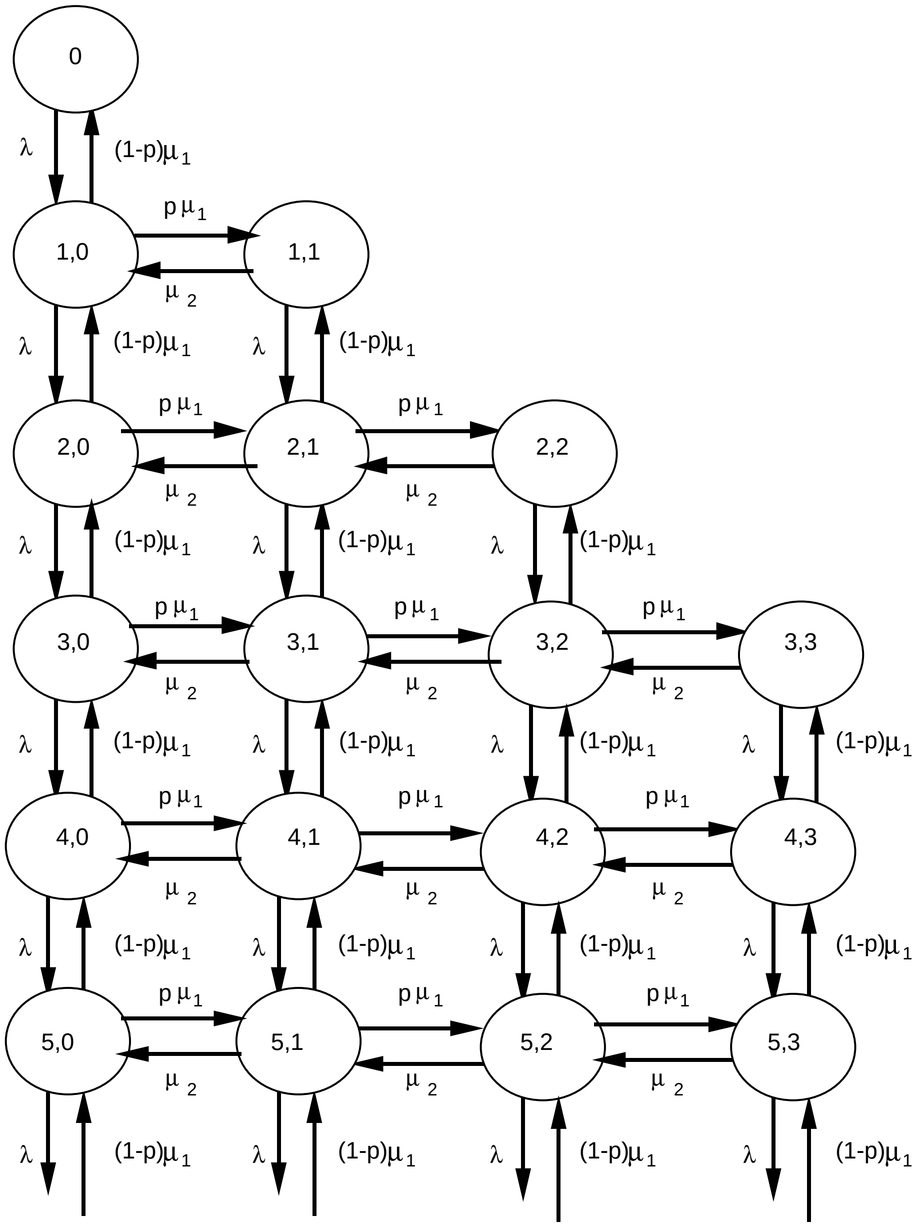

Figure 4.4. State Diagram for Problem 2a

$$P_3\left[\,\Lambda_3 + \mathcal{P}_3 + \mathcal{M}_3\,\right] \;=\; P_2\hat{\Lambda}_2 + P_4\mathcal{M}_3,$$

and for $n \geq 4$,

$$P_n\left[\,\Lambda_3 + \mathcal{P}_3 + \mathcal{M}_3\,\right] = P_{n-1}\hat{\Lambda}_3 + P_{n+1}\mathcal{M}_3$$

(c) To obtain the balance equations for general m, we simply define Λ_i, $i = 1, 2, \cdots, m$; $\hat{\Lambda}_i$, $i = 0, 1, \cdots, m-1$; and $\mathcal{P}_i$, $\mathcal{M}_i$, $\hat{\mathcal{M}}_i$, $i = 0, 1, \cdots, m$ as in part (b). The equations will then follow the same pattern as in part (b). i.e.,

$$\begin{aligned} P_0\Lambda_0 &= P_1\hat{\mathcal{M}}_1, \\ P_n\left[\,\Lambda_n + \mathcal{P}_n + \mathcal{M}_n\,\right] &= P_{n-1}\hat{\Lambda}_{n-1} + P_{n+1}\mathcal{M}_{n+1}, \end{aligned}$$

where $n = 1, 2, \cdots, m$. For $n > m$,

$$P_n\left[\,\Lambda_m + \mathcal{P}_m + \mathcal{M}_m\,\right] = P_{n-1}\hat{\Lambda}_m + P_{n+1}\mathcal{M}_m$$

(d) Define

$$\begin{aligned} \Delta &= \mathrm{diag}[-\Lambda_0, -\left[\Lambda_1 + \mathcal{P}_1 + \mathcal{M}_1\right), -\left(\Lambda_2 + \mathcal{P}_2 + \mathcal{M}_2\right), \\ &= \qquad -\left(\Lambda_3 + \mathcal{P}_3 + \mathcal{M}_3\right), -\left(\Lambda_3 + \mathcal{P}_3 + \mathcal{M}_3\right), \cdots] \\ \Delta^u &= \mathrm{superdiag}\left[\hat{\Lambda}_0, \hat{\Lambda}_1, \hat{\Lambda}_2, \Lambda_3, \Lambda_3, \Lambda_3, \cdots\right] \\ \Delta_l &= \mathrm{subdiag}\left[\hat{\mathcal{M}}_0, \hat{\mathcal{M}}_1, \hat{\mathcal{M}}_2, \hat{\mathcal{M}}_3, \mathcal{M}_3, \mathcal{M}_3, \cdots\right] \end{aligned}$$

Then, by inspection of the balance equations in part (c),

$$\hat{\mathcal{Q}} = \Delta + \Delta^u + \Delta_l$$

(e) This exercise will be harder to solve than the M/PH/1 system; this is due to the fact that there is memory from service to service. That is, a new customer starts service at exactly the phase of the previously completed customer as long as there are customers waiting. In the M/PH/1 system, the starting phase is selected at random. Since there is more than one possible starting phase, this problem will be more difficult to solve. This matrix $\mathcal{Q}$ has the form

$$\begin{bmatrix} B_{00} & B_{01} & B_{02} & B_{03} & 0 & \dots & & \\ B_{10} & B_{11} & B_{12} & B_{13} & 0 & \dots & & \\ B_{20} & B_{21} & B_{22} & B_{23} & 0 & \dots & & \\ B_{30} & B_{31} & B_{32} & B_{33} & A_0 & \dots & & \\ 0 & 0 & 0 & B_{43} & A_1 & A_0 & \dots & \\ 0 & 0 & 0 & 0 & A_2 & A_1 & A_0 & \dots \\ 0 & 0 & 0 & 0 & 0 & A_2 & A_1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & A_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which is similar to the form of the matrix of the G/M/1 type, but with more complex boundary conditions.

(f) We know that $P_4 = P_3\mathcal{R}$, but we do not know P_0, P_1, P_2, P_3. On the other hand, we know $P\mathcal{Q} = 0$. Now consider only those elements of P that are multiplied by the B submatrices. That is, P_0, P_1, P_2, P_3, P_4. Then,

$$0 = [\,P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4\,] \begin{bmatrix} B_{00} & B_{01} & B_{02} & B_{03} \\ B_{10} & B_{11} & B_{12} & B_{13} \\ B_{20} & B_{21} & B_{22} & B_{23} \\ B_{30} & B_{31} & B_{32} & B_{33} \\ 0 & 0 & 0 & B_{43} \end{bmatrix}$$

But $P_4 = P_3\mathcal{R}$, so that

$$0 = [\,P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_3\mathcal{R}\,] \begin{bmatrix} B_{00} & B_{01} & B_{02} & B_{03} \\ B_{10} & B_{11} & B_{12} & B_{13} \\ B_{20} & B_{21} & B_{22} & B_{23} \\ B_{30} & B_{31} & B_{32} & B_{33} \\ 0 & 0 & 0 & B_{43} \end{bmatrix}$$

Then combining the columns involving P_3, we have

$$\begin{aligned} 0 &= [\,P_0 \quad P_1 \quad P_2 \quad P_3\,] \begin{bmatrix} B_{00} & B_{01} & B_{02} & B_{03} \\ B_{10} & B_{11} & B_{12} & B_{13} \\ B_{20} & B_{21} & B_{22} & B_{23} \\ B_{30} & B_{31} & B_{32} & B_{33} + \mathcal{R}B_{43} \end{bmatrix} \\ &= [\,P_0 \quad P_1 \quad P_2 \quad P_3\,]\, B(\mathcal{R}) \end{aligned}$$

Thus, we see that the vector $[\,P_0 \quad P_1 \quad P_2 \quad P_3\,]$ is proportional to the left eigenvector of $B(\mathcal{R})$ corresponding to its zero eigenvalue. We may then write the zero eigenvalue of $B(\mathcal{R})$ as x and $[\,P_0 \quad P_1 \quad P_2 \quad P_3\,] = k\,[\,x_0 \quad x_1\,]$, where k is a normalizing constant, and x_1 is proportional to P_3. Thus,

$$kx_0\mathbf{e} + \sum_{i=1}^{\infty} kx_1\mathcal{R}^i\mathbf{e} = i,$$

or

$$kx_0\mathbf{e} + kx_1\,[I - \mathcal{R}]^{-1}\,\mathbf{e} = 1,$$

so that

$$k = \frac{1}{x_0\mathbf{e} + kx_1\,[I - \mathcal{R}]^{-1}\,\mathbf{e}}$$

Once k is know, we can compute

$$[\,P_0 \quad P_1 \quad P_2\,] \;=\; kx_0$$

$$P_3 = kx_1$$

and

$$P_{n+3} = P_3 \mathcal{R}^n, \qquad n \geq 1$$

Chapter 5

THE BASIC M/G/1 QUEUEING SYSTEM

EXERCISE 5.1 With $\tilde{x}$, $\{\tilde{n}(t), t \geq 0\}$, and $\tilde{y}$ defined as in Theorem 5.2, show that $E[\tilde{y}(\tilde{y}-1)\cdots(\tilde{y}-n+1)] = \lambda^n E[\tilde{x}^n]$.

Solution. Recall that $\mathcal{F}_{\tilde{y}}(z) = \sum_{y=0}^{\infty} z^y P\{\tilde{y} = y\}$. Then

$$\frac{d^n}{dz^n}\mathcal{F}_{\tilde{y}}(z) = \sum_{\tilde{y}=0}^{\infty} [y(y-1)\cdots(y-n+1)]\, z^{y-n} P\{\tilde{y} = y\},$$

so that

$$\left.\frac{d^n}{dz^n}\mathcal{F}_{\tilde{y}}(z)\right|_{z=1} = E\left[\tilde{y}(\tilde{y}-1)\cdots(\tilde{y}-n+1)\right].$$

On the other hand,

$$\begin{aligned}\frac{d^n}{dz^n}F^*_{\tilde{x}}(\lambda[1-z]) &= \frac{d^n}{dz^n}\int_0^{\infty} e^{-\lambda[1-z]}dF_{\tilde{x}}(x) \\ &= \int_0^{\infty} (\lambda x)^n e^{-\lambda[1-z]}dF_{\tilde{x}}(x),\end{aligned}$$

which implies

$$\left.\frac{d^n}{dz^n}F^*_{\tilde{x}}(\lambda[1-z])\right|_{z=1} = \lambda^n E[\tilde{x}^n].$$

Since $\mathcal{F}_{\tilde{y}}(z) = F^*_{\tilde{x}}(\lambda[1-z])$, it follows that

$$E\left[\tilde{y}(\tilde{y}-1)\cdots(\tilde{y}-n+1)\right] = \lambda^n E[\tilde{x}^n]$$

Alternatively,

$$\mathcal{F}_{\tilde{y}}(z) = F^*_{\tilde{x}}\left(s(z)\right),$$

where $s(z) = \lambda[1-z]$ implies

$$\begin{aligned}
\frac{d^n}{dz^n}\mathcal{F}_{\tilde{y}}(z) &= \frac{d^n}{ds^n}F^*_{\tilde{x}}\left(s(z)\right)\left(\frac{d}{dz}s(z)\right)^n \\
&= \frac{d^n}{ds^n}F^*_{\tilde{x}}\left(-\lambda\right)^n \\
&= (-\lambda)^n\left[(-1)^n E[\tilde{x}^n]\right] \\
&= \lambda^n E[\tilde{x}^n].
\end{aligned}$$

EXERCISE 5.2 Prove Theorem 5.3.

Solution. The proof follows directly from the definition of $\mathcal{F}_{\tilde{x}}(z)$.

$$\begin{aligned}
\mathcal{F}_{(\tilde{x}-1)^+}(z) &= \sum_{i=0}^{\infty} z^i P\left\{(\tilde{x}-1)^+ = i\right\} \\
&= z^0 P\left\{(\tilde{x}-1)^+ = 0\right\} + \sum_{i=1}^{\infty} z^i P\left\{(\tilde{x}-1)^+ = i\right\}.
\end{aligned}$$

But $(\tilde{x}-1)^+ = 0$ when either $\tilde{x}=1$ or $\tilde{x}=0$, so that

$$\begin{aligned}
\mathcal{F}_{(\tilde{x}-1)^+}(z) &= P\left\{\tilde{x}=0\right\} + P\left\{\tilde{x}=1\right\} + \frac{1}{z}\sum_{i=1}^{\infty} z^{(i+1)} P\left\{\tilde{x}=i+1\right\} \\
&= P\left\{\tilde{x}=0\right\} + P\left\{\tilde{x}=1\right\} \\
&= \qquad + \frac{1}{z}\left[\mathcal{F}_{\tilde{x}}(z) - P\left\{\tilde{x}=0\right\} - zP\left\{\tilde{x}=1\right\}\right] \\
&= \left(1-\frac{1}{z}\right)P\left\{\tilde{x}=0\right\} - \frac{1}{z}\mathcal{F}_{\tilde{x}}(z).
\end{aligned}$$

EXERCISE 5.3 Starting with (5.5), use the properties of Laplace transforms and probability generating functions to establish (5.6).

Solution. Observe that if we take the limit of the right-hand side of (5.5) as $z \to 1$, both numerator and denominator go to zero. Thus we can apply L'Hôpital's rule, and then take the limit as $z \to 1$ of the resultant expression, using the properties of Laplace transforms and probability generating functions. Let $\alpha(z)$ denote the numerator and $\beta(z)$ denote the denominator of the right-hand side of (5.5). Then the derivatives are

$$\begin{aligned}
\frac{d}{dz}\alpha(z) &= -P\left\{\tilde{q}=0\right\}\left[F^*_{\tilde{x}}(\lambda[1-z]) - (1-z)\frac{d}{dz}F^*_{\tilde{x}}(\lambda[1-z])\right] \\
\frac{d}{dz}\beta(z) &= \frac{d}{dz}F^*_{\tilde{x}}(\lambda[1-z]) - 1.
\end{aligned}$$

Now take the limits of both sides as $z \to 1$. It is easy to verify using Theorem 2.2 that $\lim_{z\to 1}\frac{d}{dz}F^*_{\tilde{x}}(\lambda[1-z]) = \lambda E[\tilde{x}] = \rho$. Thus, after some simple algebra,

$$-\frac{P\{\tilde{q}=0\}}{\rho-1}=1.$$

i.e.,

$$P\{\tilde{q}=0\}=1-\rho.$$

EXERCISE 5.4 Establish (5.6) directly by using Little's result.

Solution. Define the system to be the server only. Then there is either one in the system (i.e., in service) or zero, so that

$$\begin{aligned} E[\tilde{n}] &= 0\cdot P\{\tilde{n}=0\}+1\cdot P\{\tilde{n}=1\} \\ &= P\{\tilde{n}=1\}. \end{aligned}$$

Note that this is the probability that the server is busy. On the other hand, we have by Little's result that

$$\begin{aligned} E[\tilde{n}] &= \lambda E[\tilde{s}] \\ &= \lambda E[\tilde{x}] \\ &= \rho. \end{aligned}$$

i.e., the probability that the server is busy is ρ, or, equivalently, that the probability that no one is in the system is $(1-\rho)$. Now, it has already been pointed out that the Poisson arrival's view of the system is the same as that of a random observer. Thus, $P\{\tilde{n}=n\} = P\{\tilde{q}=n\}$. In particular, we see that $P\{\tilde{q}=0\}=1-\rho$.

EXERCISE 5.5 **Batch Arrivals.** Suppose arrivals to the system occur in batches of size $\tilde{b}$, and the batches occur according to a Poisson process at rate λ. Develop an expression equivalent to (5.5) for this case. Be sure to define each of the variables carefully.

Solution. It should be easy to see that the basic relationship

$$\tilde{q}_{n+1}=(\tilde{q}_n-1)^{+}+\tilde{v}_{n+1}$$

will remain valid. i.e., the number of customers left in the system by the $(n+1)$-st departing customer will be the sum of the number of customers left by the n-th customer plus the number of customers who arrive during the $(n+1)$-st customer's service. It should be clear, however, that the number of customers who arrive during the $(n+1)$-st customer's service will vary according to the batch size, and so we need to re-specify $\tilde{v}_{n+1}$. Now,

$$\begin{aligned} \mathcal{F}_{\tilde{q}_{n+1}}(z) &= \mathcal{F}_{(\tilde{q}_n-1)}(z)\mathcal{F}_{\tilde{v}_{n+1}}(z) \\ &= \left[\left(1-\frac{1}{z}\right)P\{\tilde{q}_n=0\}+\frac{1}{z}\mathcal{F}_{\tilde{q}_n}(z)\right]\mathcal{F}_{\tilde{v}_{n+1}}(z). \end{aligned}$$

Taking the limit of this equation as $n \to \infty$,

$$\begin{aligned}
\mathcal{F}_{\tilde{q}}(z) &= \left[\left(1-\frac{1}{z}\right)P\left\{\tilde{q}=0\right\}+\frac{1}{z}\mathcal{F}_{\tilde{q}}(z)\right]\mathcal{F}_{\tilde{v}}(z) \\
&= \frac{\left(1-\frac{1}{z}\right)P\left\{\tilde{q}=0\right\}\mathcal{F}_{\tilde{v}}(z)}{1-\frac{1}{z}\mathcal{F}_{\tilde{q}}(z)\mathcal{F}_{\tilde{v}}(z)}.
\end{aligned}$$

But, conditioning upon the batch size and the length of the service, we find that

$$\begin{aligned}
\mathcal{F}_{\tilde{v}}(z) &= E[z^{\tilde{v}}] \\
&= \int_0^\infty \sum_{n=0}^{\infty} E\left[z^{\tilde{v}}\middle|\tilde{n}=n,\tilde{x}=x\right]P\left\{\tilde{n}=n|\tilde{x}=x\right\}dF_{\tilde{x}}(x) \\
&= \int_0^\infty \sum_{n=0}^{\infty} E\left[z^{\sum_{i=1}^{n} b_i}\right]\frac{(\lambda x)^n}{n!}e^{-\lambda x}dF_{\tilde{x}}(x) \\
&= \int_0^\infty \left\{\sum_{n=0}^{\infty} E\left[z^{\tilde{b}}\right]\frac{(\lambda x)^n}{n!}\right\}e^{-\lambda x}dF_{\tilde{x}}(x) \\
&= \int_0^\infty e^{\lambda x\mathcal{F}_{\tilde{b}}(z)-\lambda x}dF_{\tilde{x}}(x) \\
&= F_{\tilde{x}}^{*}\left(\lambda\left[1-\mathcal{F}_{\tilde{b}}(z)\right]\right).
\end{aligned}$$

EXERCISE 5.6 Using the properties of the probability generating function, show that

$$\begin{aligned} E[\tilde{n}] &= \rho + \frac{\lambda\rho}{1-\rho}\frac{E[\tilde{x}^2]}{2E[\tilde{x}]} \\ &= \rho\left(1 + \frac{\rho}{1-\rho}\frac{C_{\tilde{x}}{}^2+1}{2}\right). \end{aligned} \tag{5.1}$$

[*Hint*: The algebra will be greatly simplified if (5.8) is first rewritten as

$$\mathcal{F}_{\tilde{n}}(z) = \alpha(z)/\beta(z),$$

where

$$\alpha(z) = (1-\rho)F_{\tilde{x}}^*(\lambda[1-z]),$$

and

$$\beta(z) = 1 - \frac{1-F_{\tilde{x}}^*(\lambda[1-z])}{1-z}.$$

Then, in order to find

$$\lim_{z\to 1}\frac{d}{dz}\mathcal{F}_{\tilde{n}}(z),$$

first find the limits as $z \to 1$ of $\alpha(z)$, $\beta(z)$, $d\alpha(z)/dz$, and $d\beta(z)/dz$, and then substitute these limits into the formula for the derivative of a ratio. Alternatively, multiply both sides of (5.8) to clear fractions and then differentiate and take limits.]

Solution. Following the hint given in the book, we first rewrite $\mathcal{F}_{\tilde{n}}(z)$ as $\frac{\alpha(z)}{\beta(z)}$, where

$$\alpha(z) = (1-\rho)F_{\tilde{x}}^*(\lambda[1-z]),$$

and

$$\beta(z) = 1 - \frac{1-F_{\tilde{x}}^*(\lambda[1-z])}{1-z}.$$

It is straight forward to find the limit as $z \to 1$ of $\frac{d}{dz}\mathcal{F}_{\tilde{n}}$, $\alpha(z)$ and $\frac{d}{dz}\alpha(z)$:

$$\begin{aligned} \lim_{z\to 1}\alpha(z) &= 1-\rho \\ \lim_{z\to 1}\frac{d}{dz}\alpha(z) &= (1-\rho)\rho \\ \lim_{z\to 1}\frac{d}{dz}\mathcal{F}_{\tilde{n}}(z) &= E[\tilde{n}]. \end{aligned}$$

The limit as $z \to 1$ of $\beta(z)$ and its derivative are more complicated. Note that if we take the limit of $\beta(z)$ that both the numerator and the denominator of the fraction go to zero. We may apply L'Hôpital's rule in this case.

$$\begin{aligned}
\lim_{z\to 1}\beta(z) &= 1-\lim_{z\to 1}\left[\frac{1-F_{\tilde{x}}^{*}\left(\lambda[1-z]\right)}{1-z}\right] \\
&= 1-\lim_{z\to 1}\left[\frac{\lambda\frac{d}{dz}F_{\tilde{x}}^{*}\left(\lambda[1-z]\right)}{-1}\right] \\
&= 1-\lambda E[\tilde{x}] \\
&= 1-\rho.
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
\frac{d}{dz}\beta(z) &= -\frac{d}{dz}\left[\frac{1-F_{\tilde{x}}^{*}\left(\lambda[1-z]\right)}{1-z}\right] \\
&= -\frac{d}{dz}\left[\frac{1-F_{\tilde{x}}^{*}\left(s(z)\right)}{1-z}\right] \\
&= \frac{-(1-z)\lambda\frac{d}{ds}F_{\tilde{x}}^{*}\left(s(z)\right)-\left[1-F_{\tilde{x}}^{*}\left(s(z)\right)\right]}{(1-z)^2}.
\end{aligned}$$

Upon taking the limit of this ratio and applying L'Hôpital's rule twice, we see that

$$\begin{aligned}
\lim_{z\to 1}\frac{d}{dz}\beta(z) &= \lim_{z\to 1}\left[\frac{(1-z)\lambda^2\frac{d^2}{ds^2}F_{\tilde{x}}^{*}\left(s(z)\right)}{2(z-1)}\right] \\
&= \lim_{z\to 1}\left[\frac{-(1-z)\lambda^3\frac{d^3}{ds^3}F_{\tilde{x}}^{*}\left(s(z)\right)-\lambda^2\frac{d^2}{ds^2}F_{\tilde{x}}^{*}\left(s(z)\right)}{2}\right] \\
&= \frac{-\lambda^2}{2}E[\tilde{x}^2].
\end{aligned}$$

To find $E[\tilde{n}]$, we substitute these limits into the formula for the derivative of $\mathcal{F}_{\tilde{n}}(z)$,

$$\frac{d}{dz}\mathcal{F}_{\tilde{n}}(z)=\frac{\beta(z)\frac{d}{dz}\alpha(z)-\alpha(z)\frac{d}{dz}\beta(z)}{\beta^2(z).}$$

Thus,

$$\begin{aligned}
E[\tilde{n}] &= \frac{(1-\rho)^2\rho+(1-\rho)\frac{\lambda^2}{2}E[\tilde{x}^2]}{(1-\rho)^2} \\
&= \rho+\frac{\lambda^2}{1-\rho}\frac{E[\tilde{x}^2]}{2} \\
&= \rho+\frac{\lambda^2}{1-\rho}\frac{\left(E[\tilde{x}^2]-E^2[\tilde{x}]+E^2[\tilde{x}]\right)}{2}
\end{aligned}$$

$$\begin{aligned}
&= \rho + \frac{\lambda^2}{1-\rho} E^2[\tilde{x}] \left(\frac{E[\tilde{x}^2] - E^2[\tilde{x}] + E^2[\tilde{x}]}{2E^2[\tilde{x}]} \right) \\
&= \rho + \frac{\rho^2}{1-\rho} \left(\frac{Var(\tilde{x}) + 1}{2E^2[\tilde{x}]} \right) \\
&= \rho + \frac{\rho^2}{1-\rho} \left(\frac{C_{\tilde{x}}{}^2 + 1}{2} \right) \\
&= \rho \left(1 + \frac{\rho}{1-\rho} \frac{C_{\tilde{x}}{}^2 + 1}{2} \right),
\end{aligned}$$

EXERCISE 5.7 Let $\delta_n = 1$ if $\tilde{q}_n = 0$ and let $\delta_n = 0$ if $\tilde{q}_n > 0$ so that $\tilde{q}_{n+1} = \tilde{q}_n - 1 + \delta_n + \tilde{v}_{n+1}$. Starting with this equation, find $E[\delta_\infty]$ and $E[\tilde{n}]$. Interpret $E[\delta_\infty]$. [*Hint*: To find $E[\tilde{n}]$, start off by squaring both sides of the equation for $\tilde{q}_{n+1}$.]

Solution. First take expectations of the equation for $\tilde{q}_{n+1}$.

$$E[\tilde{q}] = E[\tilde{q}] - 1 + E[\delta_\infty] + E[\tilde{v}].$$

By Little's result, we know that $E[\tilde{v}] = \lambda E[\tilde{x}] = \rho$. Substitute this value into the above equation to find that $E[\delta_\infty] = 1 - \rho$.
To find $E[\tilde{n}]$, we begin by squaring the equation given in the problem statement.

$$\begin{aligned}
E[\tilde{q}^2] &= E[\tilde{q}^2] + E[\tilde{v}^2] + E[\delta_\infty^2] + 1 + 2E[\tilde{q}\tilde{v}] + 2E[\tilde{q}\delta] + 2E[\tilde{v}\delta_\infty] \\
&\quad -2E[\tilde{q}] - 2E[\tilde{v}] - 2E[\delta_\infty].
\end{aligned}$$

But $E[\tilde{q}] = E[\tilde{n}]$, and $E[\delta_\infty^2] = E[\delta_\infty]$ since δ_n can take on values of 0 or 1 only. In addition, $\tilde{q}_n$ and $\tilde{v}_{n+1}$ are independent, as are δ_n and $\tilde{v}_{n+1}$. Clearly their limits are also independent. However, δ_n and $\tilde{q}_n$ are not independent. Furthermore, their product will always be zero; hence their expected value will also be zero. Using these observations and solving for $E[\tilde{n}]$,

$$E[\tilde{n}]\,(2 - 2E[\tilde{v}]) = E[\delta_\infty^2] + E[\tilde{v}^2] + 1 + 2E[\tilde{v}]E[\delta_\infty] - 2E[\tilde{v}] - 2E[\delta_\infty].$$

That is,

$$E[\tilde{n}] = \frac{\rho(1-2\rho) + E[\tilde{v}^2]}{2(1-\rho)}.$$

But by Exercise 5.1, $E[\tilde{v}^2] - E[\tilde{v}] = \lambda^2 E[\tilde{x}^2]$, so that $E[\tilde{v}^2] = \rho^2 E[\tilde{x}^2] + \rho$. Substituting this into the expression for $E[\tilde{n}]$,

$$E[\tilde{n}] = \frac{\rho}{1-\rho}.$$

EXERCISE 5.8 Using (5.14) and the properties of the Laplace transform, show that

$$\begin{aligned} E[\tilde{s}] &= \frac{\rho}{1-\rho}\frac{E[\tilde{x}^2]}{2E[\tilde{x}]} + E[\tilde{x}] \\ &= \left(\frac{\rho}{1-\rho}\frac{C_{\tilde{x}}{}^2+1}{2}+1\right)E[\tilde{x}]. \end{aligned} \tag{5.2}$$

Combine this result with that of Exercise 5.6 to verify the validity of Little's result applied to the M/G/1 queueing system. [*Hint*: Use (5.15) rather than (5.14) as a starting point, and use the hint for Exercise 5.6]

Solution. Begin with (5.15) and rewrite $F^*_{\tilde{s}}(s)$ as $\alpha(s)/\beta(s)$, where

$$\beta(s) = 1 - \rho\left[\frac{1-F^*_{\tilde{x}}(s)}{sE[\tilde{x}]}\right].$$

Simple calculations give us the limits of $\frac{d}{ds}F^*_{\tilde{s}}(s)$, $\alpha(s)$, and $\frac{d}{ds}\alpha(s)$ as $s \to 0$.

$$\begin{aligned} \lim_{s\to 0}\alpha(s) &= 1-\rho \\ \lim_{s\to 0}\frac{d}{ds}\alpha(s) &= -(1-\rho)E[\tilde{x}] \\ \lim_{s\to 0}\frac{d}{ds}\mathcal{F}_{\tilde{s}}(s) &= -E[\tilde{s}]. \end{aligned}$$

Taking the limit of the $\beta(s)$as $s \to 0$ results in both a zero numerator and denominator in the second term. We may apply L'Hôpital's rule in this case to find

$$\begin{aligned} \lim_{s\to 0}\beta(s) &= 1-\rho\lim_{s\to 0}\left[\frac{-\frac{d}{ds}F^*_{\tilde{x}}(s)}{E[\tilde{x}]}\right] \\ &= 1-\rho\frac{E[\tilde{x}]}{E[\tilde{x}]} \\ &= 1-\rho. \end{aligned}$$

We now find the limit of $\beta^{'}(s)$, where

$$\frac{d}{ds}\beta(s) = \frac{\rho}{E[\tilde{x}]}\left[\frac{sE[\tilde{x}]\frac{d}{ds}F^*_{\tilde{x}}(s) + \left[1-F^*_{\tilde{x}}(s)\right]E[\tilde{x}]}{s^2}\right].$$

Then, taking the limit of this expression as $s \to 0$ and twice applying L'Hôpital's rule,

$$\lim_{s\to 0}\frac{d}{ds}\beta(s) = \frac{\rho}{E[\tilde{x}]}\lim_{s\to 0}\left[\frac{\frac{d}{ds}F^*_{\tilde{x}}(s) + s\frac{d^2}{ds^2}F^*_{\tilde{x}}(s) - \frac{d}{ds}F^*_{\tilde{x}}(s)}{2s}\right]$$

$$
\begin{aligned}
&= \frac{\rho}{E[\tilde{x}]} \lim_{s \to 0} \left[\frac{\frac{d^2}{ds^2} F^*_{\tilde{x}}(s) + s \frac{d^3}{ds^3} F^*_{\tilde{x}}(s)}{2} \right] \\
&= \rho \frac{E[\tilde{x}^2]}{2E[\tilde{x}]}.
\end{aligned}
$$

Substituting these expressions into the formula for the derivative of $F^*_{\tilde{s}}(s)$, we find that

$$
\begin{aligned}
-E[\tilde{s}] &= \frac{-(1-\rho)^2 E[\tilde{x}] - (1-\rho)\frac{\lambda}{2} E[\tilde{x}^2]}{(1-\rho)^2} \\
&= -E[\tilde{x}] - \frac{\rho}{1-\rho} \frac{E[\tilde{x}^2]}{2E[\tilde{x}]} \\
&= -E[\tilde{x}] - \frac{\rho}{1-\rho} \frac{(E[\tilde{x}^2] - E^2[\tilde{x}] + E^2[\tilde{x}])}{2E[\tilde{x}]} \\
&= -E[\tilde{x}] - \frac{\rho}{1-\rho} E[\tilde{x}] \left(\frac{E[\tilde{x}^2] - E^2[\tilde{x}] + E^2[\tilde{x}]}{2E^2[\tilde{x}]} \right) \\
&= -E[\tilde{x}] - \frac{\rho}{1-\rho} E[\tilde{x}] \left(\frac{Var(\tilde{x}) + E^2[\tilde{x}]}{2E^2[\tilde{x}]} \right) \\
&= -E[\tilde{x}] - \frac{\rho}{1-\rho} E[\tilde{x}] \left(\frac{{C_{\tilde{x}}}^2 + 1}{2} \right) \\
&= -E[\tilde{x}] \left(\frac{\rho}{1-\rho} \frac{{C_{\tilde{x}}}^2 + 1}{2} + 1 \right).
\end{aligned}
$$

i.e.,

$$
E[\tilde{s}] = E[\tilde{x}] \left(\frac{\rho}{1-\rho} \frac{{C_{\tilde{x}}}^2 + 1}{2} + 1 \right).
$$

To complete the exercise, combine this result with that of Exercise 5.6.

$$
\begin{aligned}
\frac{E[\tilde{n}]}{E[\tilde{s}]} &= \frac{\rho \left(1 + \frac{\rho}{1-\rho} \frac{{C_{\tilde{x}}}^2+1}{2} \right)}{E[\tilde{x}] \left(\frac{\rho}{1-\rho} \frac{{C_{\tilde{x}}}^2+1}{2} + 1 \right)} \\
&= \frac{\rho}{E[\tilde{x}]} \\
&= \lambda.
\end{aligned}
$$

This verifies Little's result applied to the M/G/1 queueing system.

EXERCISE 5.9 Using (5.17) and the properties of the Laplace transform, show that

$$E[\tilde{w}] = \frac{\rho}{1-\rho}\frac{{C_{\tilde{x}}}^2+1}{2}E[\tilde{x}]. \tag{5.3}$$

Combine this result with the result of Exercise 5.6 to verify the validity of Little's result when applied to the waiting line for the M/G/1 queueing system.

Solution. Recall equation (5.17):

$$F_{\tilde{w}}^*(s) = \frac{(1-\rho)}{1-\rho\left[\frac{1-F_{\tilde{x}}^*(s)}{sE[\tilde{x}]}\right]}.$$

As in Exercise 5.8, we rewrite $F_{\tilde{w}}^*(s)$ as $(1-\rho)\beta^{-1}(s)$, where

$$\beta(s) = 1-\rho\left[\frac{1-F_{\tilde{x}}^*(s)}{sE[\tilde{x}]}\right].$$

To compute $E[\tilde{w}]$, we need to find

$$\frac{d}{ds}F_{\tilde{w}}^*(s) = -(1-\rho)\beta^{-2}(s)\frac{d}{ds}\beta(s),$$

and its limit as $s \to 0$. We first calculate $\beta(s)$ at $s=0$. Clearly, however, this would result in both a zero numerator and denominator. Therefore, applying L'Hôpital's rule,

$$\begin{aligned}
\lim_{s\to 0}\beta(s) &= 1-\rho\lim_{s\to 0}\left[\frac{-\frac{d}{ds}F_{\tilde{x}}^*(s)}{E[\tilde{x}]}\right] \\
&= 1-\rho.
\end{aligned}$$

We now calculate the limit of $\frac{d}{ds}\beta(s)$ by applying L'Hôpital's rule twice. This results in

$$\begin{aligned}
\lim_{s\to 0}\frac{d}{ds}\beta(s) &= \lim_{s\to 0}-\frac{\rho}{E[\tilde{x}]}\left[\frac{-s\frac{d}{ds}F_{\tilde{x}}^*(s)-[1-F_{\tilde{x}}^*(s)]}{s^2}\right] \\
&= \lim_{s\to 0}\frac{\rho}{E[\tilde{x}]}\left[\frac{\frac{d}{ds}F_{\tilde{x}}^*(s)+s\frac{d^2}{ds^2}F_{\tilde{x}}^*(s)-\frac{d}{ds}F_{\tilde{x}}^*(s)}{2s}\right] \\
&= \lim_{s\to 0}\frac{\rho}{E[\tilde{x}]}\left[\frac{\frac{d^2}{ds^2}F_{\tilde{x}}^*(s)+s\frac{d^3}{ds^3}F_{\tilde{x}}^*(s)}{2}\right] \\
&= \rho\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}.
\end{aligned}$$

Noting that $E[\tilde{w}]$ is $-d/dsF^*_{\tilde{w}}(s)$ at $s=0$, we may now specify $E[\tilde{w}]$ as

$$\begin{aligned}
E[\tilde{w}] &= -\lim_{s\to 0}\frac{d}{ds}F^*_{\tilde{w}}(s)\\
&= (1-\rho)(1-\rho)^{-2}\rho\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}\\
&= \left[\frac{\rho}{1-\rho}\right]\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}\\
&= \frac{\rho}{1-\rho}\left[\frac{E^2[\tilde{x}]-E^2[\tilde{x}]+E^2[\tilde{x}]}{2E[\tilde{x}]}\right]\\
&= \frac{\rho}{1-\rho}\frac{C^2_{\tilde{x}}+1}{2}E[\tilde{x}].
\end{aligned}$$

Combine this result with that of Exercise 5.6 to complete the exercise. That is, show that

$$\frac{E[\tilde{n}]-\lambda E[\tilde{x}]}{\lambda} = E[\tilde{w}],$$

where $E[\tilde{n}]-\lambda E[\tilde{x}] = E[\tilde{n}_q]$, the expected number of customers in queue.

$$\begin{aligned}
\frac{E[\tilde{n}]-\lambda E[\tilde{x}]}{\lambda} &= E[\tilde{x}]\left[1+\frac{\rho}{1-\rho}\frac{{C_{\tilde{x}}}^2+1}{2}\right]-E[\tilde{x}]\\
&= E[\tilde{x}]\left[1+\frac{\rho}{1-\rho}\frac{{C_{\tilde{x}}}^2+1}{2}-1\right]\\
&= E[\tilde{x}]\frac{\rho}{1-\rho}\frac{{C_{\tilde{x}}}^2+1}{2}\\
&= E[\tilde{w}].
\end{aligned}$$

EXERCISE 5.10 Using properties of the Laplace transform, show that

$$E[\tilde{y}] = \frac{E[\tilde{x}]}{1-\rho}. \qquad (5.4)$$

Solution. Recall equation (5.21)

$$F^*_{\tilde{y}}(s) = F^*_{\tilde{x}}[s+\lambda-\lambda F^*_{\tilde{y}}(s)].$$

We see that $F^*_{\tilde{y}}(s)$ has the form $f(s)=F(g(s))$, so that

$$\frac{d}{ds}f(s) = \frac{d}{dg}F(g(s))\cdot\frac{d}{ds}g(s).$$

This results in the expression

$$\begin{aligned}\frac{d}{ds}F^*_{\tilde{y}}(s) &= \frac{d}{ds}F^*_{\tilde{x}}(s)\frac{d}{ds}\left(s+\lambda-\lambda F^*_{\tilde{y}}(s)\right)\\ &= \frac{d}{ds}F^*_{\tilde{x}}(s)\left[1-\lambda\frac{d}{ds}F^*_{\tilde{y}}(s)\right].\end{aligned}$$

Evaluate both sides at $s=0$, and recall that $\frac{d}{ds}F^*_{\tilde{y}}(s)\Big|_{s=0} = -E[\tilde{y}]$, and $\frac{d}{ds}F^*_{\tilde{x}}(s)\Big|_{s=0} = -E[\tilde{x}]$. Thus,

$$-E[\tilde{y}] = -E[\tilde{x}]\left(1+\lambda E[\tilde{y}]\right).$$

Solving for $E[\tilde{y}]$, this equation becomes

$$E[\tilde{y}] = \frac{E[\tilde{x}]}{1-\rho}.$$

EXERCISE 5.11 For the ordinary M/G/1 queueing system determine $E[\tilde{y}]$ without first solving for $F_{\tilde{y}}(s)$. [*Hint*: Condition on the length of the first customer's service and the number of customers that arrive during that period of time.]

Solution. To determine $E[\tilde{y}]$, condition on the length of the first customer's service.

$$E[\tilde{y}]\int_0^\infty E[\tilde{y}|\tilde{x}=x]dF_{\tilde{x}}(x). \qquad (5.11.1)$$

Next, condition this expected value on the number of customers who arrive during the first customer's service. i.e.,

$$E[\tilde{y}|\tilde{x}=x] = \sum_{v=0}^{\infty} E[\tilde{y}|\tilde{x}=x,\tilde{v}=v]\frac{(\lambda x)^v}{v!}e^{-\lambda x}. \qquad (5.11.2)$$

Now, given that the length of the first customer's service is x and the number of customers who arrive during this time is v, it is easy to see that the length of the busy period will be the length of the first customer's service plus the length of the sub busy periods generated by those v customers. That is,

$$E[\tilde{y}|\tilde{x}=x,\tilde{v}=v] = x + \sum_{i=0}^{v} E[\tilde{y}_i]$$

where $\tilde{y}_0 = 0$ with probability 1. Thus,

$$E[\tilde{y}|\tilde{x}=x,\tilde{v}=v] \quad = \quad x + \sum_{i=1}^{v} E[\tilde{y}]$$

$$= \quad x + vE[\tilde{y}],$$

since each of the sub busy periods $\tilde{y}_i$ are drawn from the same distribution. If we substitute this expression into (5.11.2),

$$\begin{aligned} E[\tilde{y}|\tilde{x} = x] &= \sum_{v=0}^{\infty} (x + vE[\tilde{y}]) \frac{(\lambda x)^v}{v!} e^{-\lambda x} \\ &= x + E[\tilde{v}]E[\tilde{y}] \\ &= x + \rho E[\tilde{y}]. \end{aligned}$$

To complete the derivation, substitute this result into (5.11.1) and solve for $E[\tilde{y}]$. Thus,

$$\begin{aligned} E[\tilde{y}] &= \int_0^{\infty} (x + \rho E[\tilde{y}])\, dF_{\tilde{x}}(x) \\ &= \int_0^{\infty} x dF_{\tilde{x}}(x) + \rho E[\tilde{y}] \\ &= E[\tilde{x}] + \rho E[\tilde{y}]. \end{aligned}$$

i.e.,

$$E[\tilde{y}] = \frac{E[\tilde{x}]}{1 - \rho}.$$

EXERCISE 5.12 **M/G/1 with Exceptional First Service**. A queueing system has Poisson arrivals with rate λ. The service time for the first customer in each busy period is drawn from the service-time distribution $F_{\tilde{x}_e}(x)$, and the remaining service times in each busy period are drawn from the general distribution $F_{\tilde{x}}(x)$. Let $\tilde{y}_e$ denote the length of the busy period for this system. Show that

$$E[\tilde{y}_e] = \frac{E[\tilde{x}_e]}{1 - \rho},$$

where $\rho = \lambda E[\tilde{x}]$.

Solution. Let $\tilde{v}_e$ denote the number of customers who arrive during the first customer's service. Then it is easy to see that

$$E[\tilde{y}_e] = E[\tilde{x}_e] + E[\tilde{v}_e]E[\tilde{y}],$$

where $\tilde{y}$ denotes the length of the busy period for each of the subsequent $\tilde{v}_e$ customers. By Little's result applied to $E[\tilde{v}_e]$, $E[\tilde{v}_e] = \lambda E[\tilde{x}_e]$. Then, using the result from Exercise 5.11,

$$E[\tilde{y}_e] \quad = \quad E[\tilde{x}_e] + \frac{\rho}{1 - \rho} E[\tilde{x}_e]$$

$$= \frac{E[\tilde{x}_e]}{1-\rho}.$$

EXERCISE 5.13 For the M/G/1 queueing system with exceptional first service as defined in the previous exercise, show that $F^*_{\tilde{y}_e}(s) = F^*_{\tilde{x}_e}(s + \lambda - \lambda F^*_{\tilde{y}}(s))$.

Solution. Recall that we may write $F^*_{\tilde{y}}(s)$ as $E[e^{-s\tilde{y}_e}]$. As in the proof of Exercise 5.11, we shall first condition this expected value on the length of the first customer's service, and then condition it again on the number of customers who arrive during the first customer's service. That is,

$$E[e^{-s\tilde{y}_e}] = \int_0^\infty E[e^{-s\tilde{y}_e}|\tilde{x}_e = x]dF_{\tilde{x}}(x), \qquad (5.13.1)$$

where

$$E[e^{-s\tilde{y}_e}|\tilde{x}_e = x] = \sum_{v=0}^{\infty} E[e^{-s\tilde{y}_e}|\tilde{x}_e = x, \tilde{v} = v]P\left\{\tilde{v} = v\right\} dF_{\tilde{x}}(x). \qquad (5.13.2)$$

Here, $\tilde{x}_e$ represents the length of the first customer's service and $\tilde{v}$ is the number of arrivals during that time. Now, the length of the busy period will be the length of the first customer's service plus the length of the sub busy periods generated by those customers who arrive during the first customer's service. Hence,

$$E[e^{-s\tilde{y}_e}|\tilde{x}_e = x, \tilde{v} = v] = E\left[e^{-s(x+\sum_{i=0}^{\infty}\tilde{y}_i)}\right],$$

where $\tilde{y}_0 = 0$ with probability 1. Furthermore, x and v are constants at this stage, and the $\tilde{y}_i$ are independent identically distributed random variables, so that

$$\begin{aligned} E[e^{-s\tilde{y}_e}|\tilde{x}_e = x, \tilde{v} = v] &= e^{-sx}\prod_{i=1}^{v} E[e^{-s\tilde{y}_i}] \\ &= e^{-sx}E^v[e^{-s\tilde{y}}]. \end{aligned}$$

Substituting this into $(5.13.2)$,

$$\begin{aligned} E[e^{-s\tilde{y}_e}|\tilde{x}_e = x] &= e^{-sx}e^{-\lambda x}\sum_0^\infty \frac{\left(E[e^{-s\tilde{y}}]\lambda x\right)^v}{v!} \\ &= e^{-sx}e^{-\lambda x}e^{\left(-\lambda x E[e^{-s\tilde{y}}]\right)}. \end{aligned}$$

To complete the proof, substitute this into $(5.13.1)$, and then combine the exponents of e.

$$E[e^{-s\tilde{y}_e}] = \int_0^\infty e^{-x\left(s+\lambda-\lambda E[e^{-s\tilde{y}}]\right)}dF_{\tilde{x}}(x)$$

$$= F^*_{\tilde{y}_e}\left(s + \lambda - \lambda E[e^{-s\tilde{y}}]\right)$$
$$= F^*_{\tilde{y}_e}\left(s + \lambda - \lambda F^*_{\tilde{y}}(s)\right).$$

EXERCISE 5.14 Comparison of the formulas for the expected waiting time for the M/G/1 system and the expected length of a busy period for the M/G/1 system with the formula for exceptional first service reveals that they both have the same form; that is, the expected waiting time in an ordinary M/G/1 system is the same as the length of the busy period of an M/G/1 system in which the expected length of the first service is given by

$$E[\tilde{x}_e] = \rho \frac{E[\tilde{x}^2]}{2E[\tilde{x}]}.$$

Explain why these formulas have this relationship. What random variable must $\tilde{x}_e$ represent in this form? [*Hint*: Consider the operation of the M/G/1 queueing system under a nonpreemptive, LCFS, service discipline and apply Little's result, taking into account that an arriving customer may find the system empty.]

Solution. Consider the waiting time under LCFS. We know from Little's result that $E[\tilde{w}_{LCFS}]$ is the same as $E[\tilde{w}_{FCFS}]$ because $E[\tilde{n}_{FCFS}] = E[\tilde{n}_{LCFS}]$. But $E[\tilde{w}_{LCFS}|\text{no customers present}] = 0$ and $E[\tilde{w}_{LCFS}|\text{at least 1 customer present}] = E[\text{length of busy period started by customer in service}] = E[\tilde{x}_e]/(1-\rho)$. But the probability of at least one customer present is ρ, so that

$$E[\tilde{w}] = \frac{\rho}{1-\rho} E[\tilde{x}_e].$$

On the other hand, we know that

$$E[\tilde{w}] = \frac{\rho}{1-\rho} \frac{E[\tilde{x}^2]}{2E[\tilde{x}]}.$$

Therefore

$$E[\tilde{x}_e] = \frac{E[\tilde{x}^2]}{2E[\tilde{x}]}$$

represents the expected remaining service time of the customer in service, if any, at the arrival time of an arbitrary customer.

EXERCISE 5.15 Show that the final summation on the right-hand side of (5.28) is $K + 1$ if $\ell = (K + 1)m + n$ and zero otherwise.

Solution. First, suppose that $\ell = (K + 1)m + n$. Then $\frac{\ell - n}{K+1} = m$, and the final summation is

$$\sum_{k=0}^{K} e^{-j2\pi km} = \sum_{k=0}^{K} 1 = K + 1$$

Now suppose that ℓ is not of the above form; that is, $\ell = (K+1)m + n + r$, $0 < r < (K+1)$. Then

$$\frac{2\pi k(\ell - n)}{K+1} = \frac{2\pi k\,[m(K+1)+r]}{K+1} = 2\pi km + \frac{2\pi kr}{K+1},$$

and the final summation becomes

$$\sum_{k=0}^{K} e^{-j\frac{2\pi k[m(K+1)+r]}{K+1}} = \sum_{k=0}^{K} e^{-j\frac{2\pi kr}{K+1}},$$

since $e^{j2\pi km} = 1$ for all integers k, m. Now, note that $0 < r < K+1$ implies that this summation may be summed as a geometric series. That is,

$$\begin{aligned} \sum_{k=0}^{K} e^{-j\frac{2\pi kr}{K+1}} &= \frac{1 - \left(e^{-j\frac{2\pi r}{K+1}}\right)^{K+1}}{1 - e^{-j\frac{2\pi r}{K+1}}} \\ &= \frac{1 - e^{-j2\pi r}}{1 - e^{-j\frac{2\pi r}{K+1}}} = 0. \end{aligned}$$

| EXERCISE 5.16 Argue the validity of the expression for b_1 in (5.31).

Solution. First, multiply both sides of (5.30) by $(z - z_0)$ to obtain

$$(z - z_0)\mathcal{F}_{\tilde{n}}(z) = (z - z_0)\frac{(1-\rho)(z-1)}{z - F^*_{\tilde{x}}(\lambda[1-z])}F^*_{\tilde{x}}(\lambda[1-z]).$$

Now, taking the limit as $z \to z_0$, the denominator of the right-hand side goes to zero by definition of z_0. Hence we may apply L'Hôpital's rule to the right-hand side:

$$\frac{(z - z_0)\left[\frac{d}{dz}(1-\rho)(z-1)F^*_{\tilde{x}}(\lambda[1-z])\right] + (1-\rho)(z-1)F^*_{\tilde{x}}(\lambda[1-z])}{1 - \frac{d}{dz}F^*_{\tilde{x}}(\lambda[1-z])}.$$

If we evaluate this expression at $z = z_0$, the first term in the numerator goes to zero, and so we have that

$$\begin{aligned} b_1 &= \lim_{z \to z_0} (z - z_0)\mathcal{F}_{\tilde{n}}(z) \\ &= \left[\frac{(1-\rho)(z-1)F^*_{\tilde{x}}(\lambda[1-z])}{1 - \frac{d}{dz}F^*_{\tilde{x}}(\lambda[1-z])}\right]_{z=z_0}. \end{aligned}$$

EXERCISE 5.17 Show that the denominator of the right-hand side of (5.30) for the probability generating of the occupancy distribution has only one zero for $z > 1$. [*Hint*: From Theorem 5.2, we know that $F^*_{\tilde{x}}(\lambda[1-z])$ is the probability generating function for the number of arrivals during a service time. Therefore, $F^*_{\tilde{x}}(\lambda[1-z])$ can be expressed as a power series in which the coefficients are probabilities and therefore nonnegative. The function and all of its derivatives are therefore nonnegative for nonnegative z, and so on. Now compare the functions $f_1(z) = z$ and $f_2(z) = F^*_{\tilde{x}}(\lambda[1-z])$, noting that the expression $\mathcal{F}_{\tilde{n}}(z)$ can have poles neither inside nor on the unit circle (Why?).]

Solution. From Theorem 5.2, we know that $F^*_{\tilde{x}}(\lambda[1-z]$ is the probability generating function for the number of arrivals during a service time. Therefore, $F^*_{\tilde{x}}(\lambda[1-z])$ can be expressed as

$$F^*_{\tilde{x}}(\lambda[1-z]) = \sum_{n=0}^{\infty} P\left\{\tilde{n}_a = n\right\} z^n,$$

where each of the $P\left\{\tilde{n}_a = n\right\}$ is, of course, nonnegative, and whose sum is equal to one. Now $\mathcal{F}_{\tilde{n}}(z)$ is also a probability generating function and so is bounded within the unit circle for $z > 0$. Hence it can have no poles, and its denominator no zeros, in that region. If $\mathcal{F}_{\tilde{n}}(z)$ does have a zero at $z = 0$, then $z = F^*_{\tilde{x}}(z) = 0$ implies that $P\left\{\tilde{n}_a\right\} = 0$. That is, the probability that the queue is ever empty is zero, and so stability is never achieved. Since this is not possible we conclude that the denominator has no zero at $z = 0$. This means that $P\left\{\tilde{n}_a = 0\right\} > 0$.

Consider the graphs of the functions $f_1(z) = z$ and $f_2(z) = F^*_{\tilde{x}}(z)$. The graph of $f_1(z)$ is a line which starts at 0 and whose slope is 1. On the other hand, the graph of $f_2(z)$ starts above the origin and slowly increases towards 1 as $z \to 1$, with $f_1(z) = f_2(z)$ at $z = 1$. At $z = 1$, $\frac{d}{dz}f_2(z) = \rho < 1$, and $\frac{d}{dz}f_1(z) = 1$. But the slope of $f_2(z)$ is an increasing function, so that as z increases, $\frac{d}{dz}f_2(z)$ also increases. This means that eventually, $f_2(z)$ will intersect $f_1(z)$. This will happen at the point $z = z_0$. However, since $\frac{d}{dz}f_2(z)$ is increasing, while $\frac{d}{dz}f_1(z)$ is constant, $f_2(z)$ and $f_1(z)$ will intersect only once for $z > 1$.

EXERCISE 5.18 Starting with (5.29) and (5.32), establish the validity of (5.33) through (5.36).

Solution. If $i = K + 1$ in (5.29), then $p_{K+1+n} \approx p_{K+1}r_0^n$. In addition, if $i = K$ and $n = 1$, then $p_{K+1} \approx p_K r_0$. Combining these results,

$$p_{K+1+n} \approx p_{K+1}r_0^n \approx (p_K r_0)r_0^n = p_K r_0^{n+1}. \qquad (5.5.18.1)$$

To show the validity of (5.34), let $n = 0$ in (5.29) to get

$$c_{0,K} = p_0 + \sum_{m=1}^{\infty} p_{m(K+1)},$$

which implies

$$c_{0,K} - p_0 = \sum_{m=1}^{\infty} p_{m(K+1)}. \qquad (5.18.2)$$

Now,

$$\begin{aligned}
\sum_{m=1}^{\infty} p_{m(K+1)} &= p_{K+1} + \sum_{m=1}^{\infty} p_{m(K+1)} - p_{K+1} \\
&= p_{K+1} + \sum_{m=1}^{\infty} p_{(m+1)(K+1)} \\
&= p_{K+1} + \sum_{m=1}^{\infty} p_{m(K+1)+K+1} \\
&\approx p_K r_0 + \sum_{m=1}^{\infty} r_0 p_{m(K+1)+K} \\
&= r_0 \left[p_K + \sum_{m=1}^{\infty} p_{m(K+1)+K} \right] \\
&= r_0 c_{K,K},
\end{aligned}$$

where the last equality follows from (5.29) with $n = K$. Substituting this result into (5.18.2) and solving for r_0,

$$r_0 \approx \frac{c_{0,K} - p_0}{c_{K,K}}. \qquad (5.34)$$

To show the validity of (5.35), note that

$$\sum_{m=1}^{\infty} p_{m(K+1)+n} \approx r_0^n \sum_{m=1}^{\infty} p_{m(K+1)}.$$

Furthermore, by the definition of $c_{n,K}$,

$$c_{0,K} = \sum_{m=0}^{\infty} p_{m(K+1)},$$

or

$$c_{0,K} - p_0 = \sum_{m=1}^{\infty} p_{m(K+1)}.$$

Substituting this into (5.29), we find

$$\begin{aligned} c_{n,K} &= p_n + \sum_{m=1}^{\infty} p_{m(K+1)+n} \\ &= \approx p_n + \sum_{m=1}^{\infty} p_{m(K+1)} r_0^n \\ &= p_n + (c_{0,K} - p_0)\, r_0^n . \end{aligned}$$

Thus, solving for p_n,

$$p_n \approx c_{n,K} - (c_{0,K} - p_0) r_0^n . \qquad (5.35)$$

Finally, by the definition of probability generating function,

$$\mathcal{F}_{\tilde{n}}(0) = \sum_{n=0}^{\infty} z^n p_n \Big|_{z=0} = p_0 . \qquad (5.36)$$

EXERCISE 5.19 Approximate the distribution of the service time for the previous example by an exponential distribution with an appropriate mean. Plot the survivor function for the corresponding M/M/1 system at 95% utilization. Compare the result to those shown in Figure 5.2.

Solution. Figure 5.19.1 illustrates the survivor function for the M/M/1 system, with an exponential service distribution and $\rho = .95$. Observe that this function closely approximates that of the truncated geometric distribution in Figure 5.2 *of the text*, with $a = 1$, $b = 5000$, but that the survivor function is significantly different for the rest of the curves. Thus, the exponential distribution is a good approximation only in the special case where the truncated distribution closely resembles the exponential but not otherwise.

EXERCISE 5.20 Starting with (5.39), demonstrate the validity of (5.47).

Solution. Beginning with (5.39),

$$\begin{aligned} F^*_{\tilde{x}_r}(s) &= \frac{1 - F^*_{\tilde{x}}(s)}{sE[\tilde{x}]} \\ &= \frac{1 - F^*_{\tilde{x}}(s)}{\lambda[1-z]\frac{1}{\mu}} \\ &= \left[\frac{1 - F^*_{\tilde{x}}(s)}{\rho}\right] \sum_{i=0}^{\infty} z^i \end{aligned}$$

By (5.43), $F^*_{\tilde{x}}(s) = \sum_{i=0}^{\infty} z^i P_{\tilde{x},i}$, so that

$$F^*_{\tilde{x}_r}(s) = \frac{1}{\rho}\left[1 - \sum_{i=0}^{\infty} z^i P_{\tilde{x},i}\right] \sum_{i=0}^{\infty} z^i$$

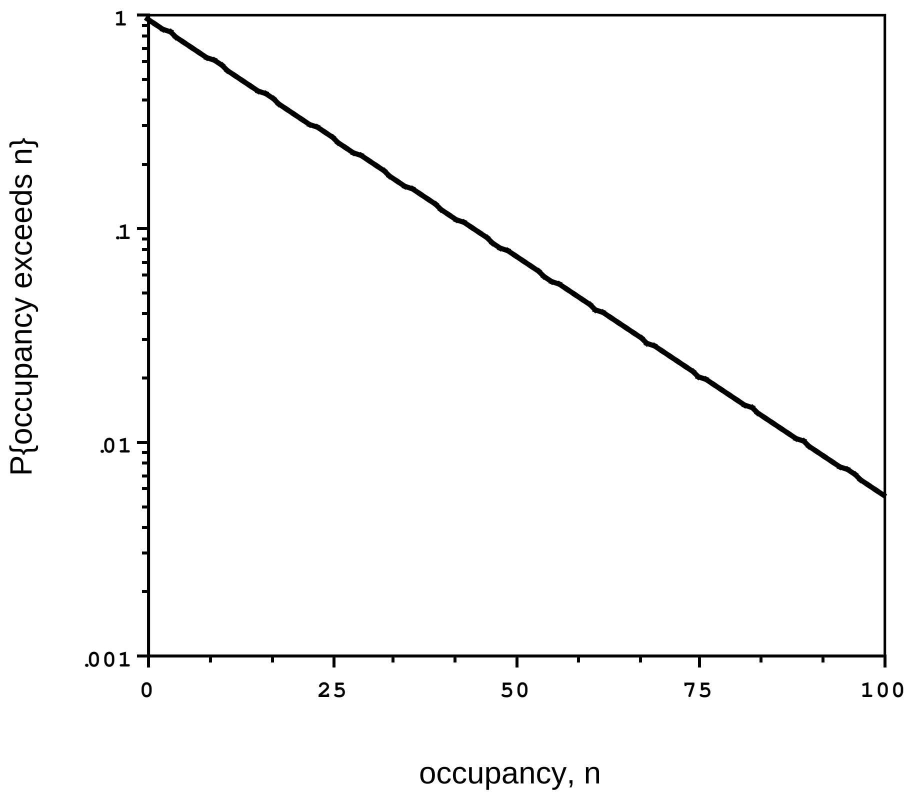

Figure 5.19.1 Survivor function for Exercise 5.19.

$$= \frac{1}{\rho}\sum_{i=0}^{\infty} z^i \left[1 - \sum_{n=0}^{i} P_{\tilde{x},n}\right].$$

But, by applying (5.43) to $F^*_{\tilde{x}_r}(s)$, we see that

$$\begin{aligned} F^*_{\tilde{x}_r}(s) &= \sum_{i=0}^{\infty} z^i P_{\tilde{x}_r,i} \\ &= \frac{1}{\rho}\sum_{i=0}^{\infty} z^i \left[1 - \sum_{n=0}^{i} P_{\tilde{x},n}\right]. \end{aligned}$$

Matching coefficients of z_i, we obtain

$$P_{\tilde{x}_r,i} = \frac{1}{\rho}\left[1 - \sum_{n=0}^{i} P_{\tilde{x},n}\right].$$

EXERCISE 5.21 Evaluate $P_{\tilde{x},i}$ for the special case in which $\tilde{x}$ has the exponential distribution with mean $1/\mu$. Starting with (5.50), show that the ergodic occupancy distribution for the M/M/1 system is given by $P_i = (1-\rho)\rho^i$, where $\rho = \lambda/\mu$.

Solution. Since the interarrival times of the Poisson process are exponential, rate λ, we have by Proposition 4, page 35, that

$$\begin{aligned} P_{\tilde{x},i} &= \left(\frac{\lambda}{\lambda+\mu}\right)^i \frac{\mu}{\lambda+\mu} \\ &= \left(\frac{\rho}{1+\rho}\right)^i \frac{1}{1+\rho}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=0}^{i} P_{\tilde{x},n} &= \frac{1}{1+\rho} \sum_{n=0}^{i} \left(\frac{\rho}{1+\rho}\right)^n \\ &= \frac{1}{1+\rho} \left[\frac{1-\left(\frac{\rho}{1+\rho}\right)^{i+1}}{1-\frac{\rho}{1+\rho}} \right], \\ &= \frac{1}{1+\rho} \left[1-\left(\frac{\rho}{1+\rho}\right)^{i+1} \right]. \end{aligned}$$

We now show that $P_i = (1-\rho)\rho^i$ where $\rho = \lambda/\mu$. Let T denote the truth set for this proposition. Then $0 \in T$ since from (5.94),

$$P_0 = \frac{(1-\rho)P_{\tilde{x},0}}{P_{\tilde{x},0}} = 1-\rho.$$

Now assume that $i \in T$. We wish to show this implies $(i+1) \in T$. By hypothesis,

$$P_n = (1-\rho)\rho^n, \qquad n < i,$$

so that

$$P_{i-n} = (1-\rho)\rho^{i-n} \qquad n > 0.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{i} \left(1 - \sum_{m=0}^{n} P_{\tilde{x},n}\right) P_{i-n} &= \sum_{n=1}^{i} \left(\frac{\rho}{1+\rho}\right)^{n+1} (1-\rho)\rho^{i-n} \\ &= (1-\rho)\rho^{i+1} \sum_{n=1}^{i} \left(\frac{1}{1+\rho}\right)^{n+1} \end{aligned}$$

$$
\begin{aligned}
&= (1-\rho)\rho^{i+1}\left[\frac{\left(\frac{1}{1+\rho}\right)^2 - \left(\frac{1}{1+\rho}\right)^{i+1}}{1-\frac{1}{1+\rho}}\right] \\
&= \frac{(1-\rho)\rho^i}{1+\rho}\left[1-\left(\frac{1}{1+\rho}\right)^{i+1}\right].
\end{aligned}
$$

Thus, using (5.50) and the hypothesis,

$$
\begin{aligned}
P_i &= \left\{(1-\rho)\frac{\rho^i}{(1+\rho)^{i+1}} + \frac{(1-\rho)\rho^i}{1+\rho}\left[1-\frac{1}{(1+\rho)^i}\right]\right\}\Big/\frac{1}{1+\rho} \\
&= (1-\rho)\rho^i,
\end{aligned}
$$

and the proof is complete.

EXERCISE 5.22 Evaluate $P_{\tilde{x},i}$ for the special case in which $P\{\tilde{x}=1\} = 1$. Use (5.50) to calculate the occupancy distribution. Compare the complementary occupancy distribution ($P\{N > i\}$) for this system to that of the M/M/1 system with $\mu = 1$.

Solution. If $\tilde{x} = 1$ with probability 1, then it should be clear that the distribution describing the number of Poisson arrivals is simply a Poisson distribution with $t = 1$ (Definition 1, part iii). i.e.,

$$
P_{\tilde{x},i} = \frac{e^{-\lambda}\lambda^i}{i!}.
$$

To calculate the complementary occupancy probabilities, a computer program was written using the recursive formula (5.50) using different values of λ. For $\lambda = 0.1$, the distributions were very similar, with the probabilities of the deterministic and exponenitial summing to 1 after $n = 6$ and $n = 9$, respectively. (Accuracy was to within 10^{-8}.) However, as n increased, the two clearly diverged, with the deterministic system summing to 1 faster than that of the exponential. The graphs of the complementary probabilities vs. those of the M/M/1 system for $\lambda = 0.5$ and $\lambda = 0.9$ is shown in Figure 5.22.1.

EXERCISE 5.23 Evaluate $P_{\tilde{x},i}$ for the special case in which $P\{\tilde{x} = \frac{1}{2}\} = P\{\tilde{x} = \frac{3}{2}\} = \frac{1}{2}$. Use (5.50) to calculate the occupancy distribution. Compare the complementary occupancy distribution ($P\{N > i\}$) for this system to that of the M/M/1 system with $\mu = 1$.

Solution. As in Exercise 5.5.22, if $\tilde{x}$ is deterministic, then $P_{\tilde{x},i}$ is a weighted Poisson distribution, with

$$
P_{\tilde{x},i} = \left(\frac{1}{2}\right)\frac{e^{-\frac{1}{2}\lambda}\left(\frac{1}{2}\lambda\right)^i}{i!} + \left(\frac{1}{2}\right)\frac{e^{-\frac{3}{2}\lambda}\left(\frac{3}{2}\lambda\right)^i}{i!}.
$$

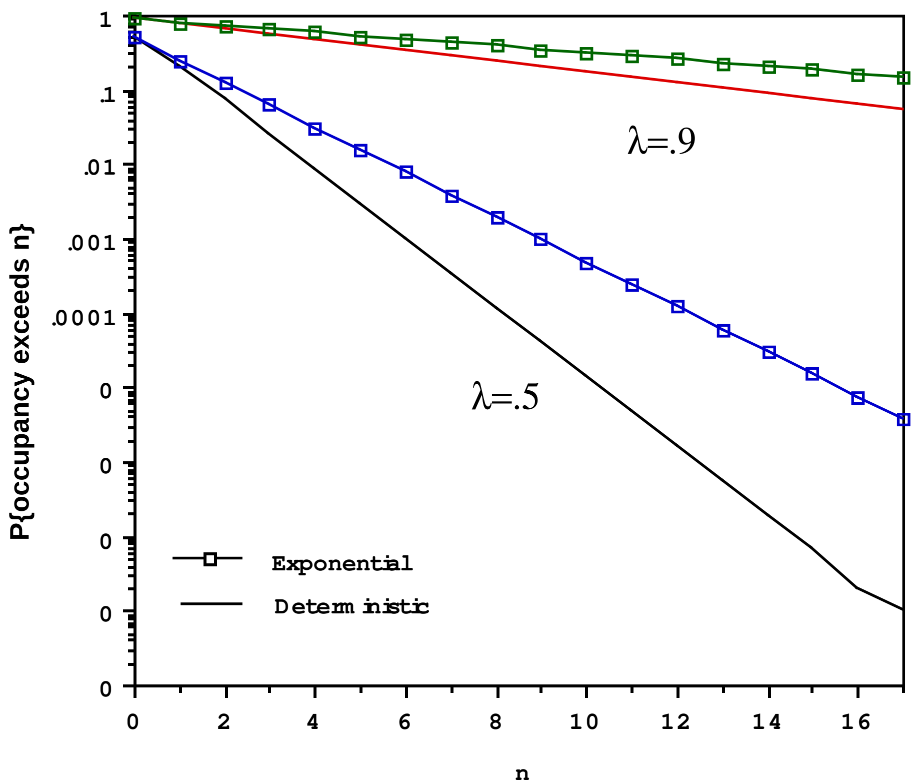

Figure 5.22.1 Survivor function for Exercise 5.22.

The computer program used in Exercise 22 was modified and the new occupancy probabilites calculated. As in Exercise 22, the graphs of both distributions were very similar for $\lambda = 0.1$, with the probabilities of both summing to unity within $n = 9$. As λ increased, however, the differences between the two systems became apparent, with the deterministic complementary distribution approaching zero before that of the exponential system. The graphs comparing the resultant complementary probabilites to those of the M/M/1 system for $\lambda = 0.5$ and $\lambda = 0.9$ is shown in Figure 5.23. 1.

EXERCISE 5.24 Beginning with (5.59), suppose $F_{\tilde{x}_e}(x) = F_{\tilde{x}}(x)$. Show that (5.59) reduces to the standard Pollaczek-Khintchine transform equation for the queue length distribution in an ordinary M/G/1 system.

Solution. Equation 5.59 is as follows:

$$\mathcal{F}_{\tilde{q}}(z)\left[z - \mathcal{F}_{\tilde{a}}(z)\right] = \pi_0 \left[z\,\mathcal{F}_{\tilde{b}}(z) - \mathcal{F}_{\tilde{a}}(z)\right]. \qquad (5.59)$$

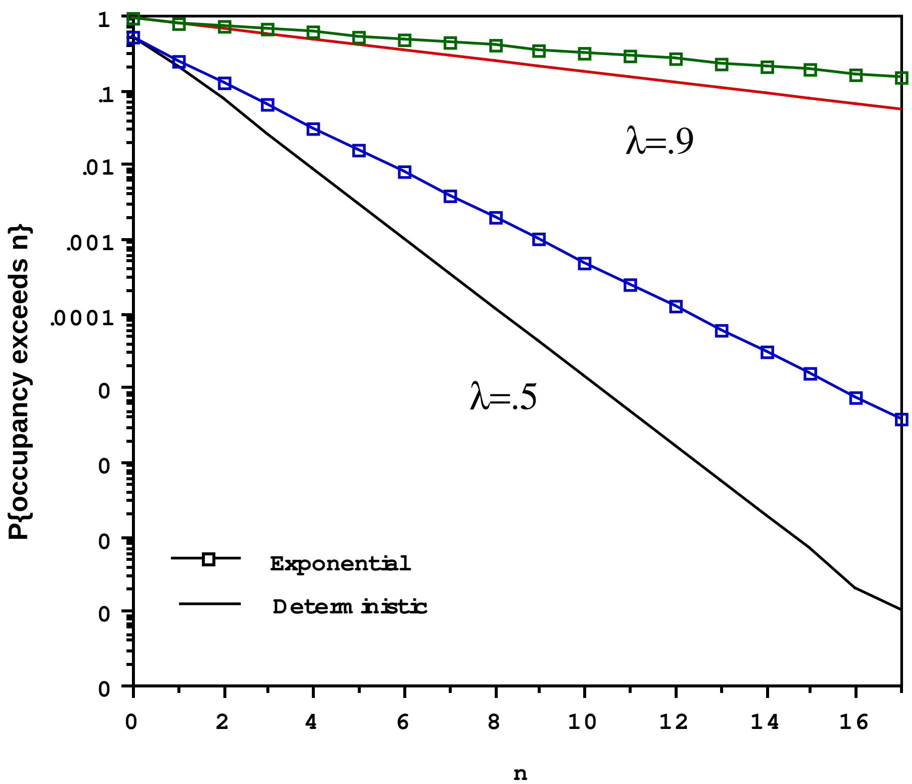

Figure 5.23.1 Survivor function for Exercise 5.23.

Since $F_{\tilde{x}_e}(x) = F_{\tilde{x}}(x)$, we have $\mathcal{F}_{\tilde{b}}(z) = \mathcal{F}_{\tilde{a}}(z)$. Upon substitution of this result in (5.59), we have

$$\mathcal{F}_{\tilde{q}}(z)\left[z - \mathcal{F}_{\tilde{a}}(z)\right] = \pi_0(z-1)\mathcal{F}_{\tilde{a}}(z).$$

Putting $\mathcal{F}_{\tilde{a}}(z) = F^*_{\tilde{x}}(\lambda[1-z])$ leads to

$$\mathcal{F}_{\tilde{q}}(z)\left[z - F^*_{\tilde{x}}(\lambda[1-z])\right] = \pi_0(z-1)F^*_{\tilde{x}}(\lambda[1-z]).$$

Solving the previous equation, we find

$$\mathcal{F}_{\tilde{q}}(z) = \frac{(1-z)\pi_0 F^*_{\tilde{x}}(\lambda[1-z])}{F^*_{\tilde{x}}(\lambda[1-z]) - z},$$

which is the same as (5.5) with $\pi = 1 - P\left\{\tilde{q} = 0\right\}$.

EXERCISE 5.25 Beginning with (5.59), use the fact that $\lim_{z\to 1}\mathcal{F}_{\tilde{q}}(z) = 1$ to show that

$$\pi_0 = \frac{1-\mathcal{F}'_{\tilde{a}}(1)}{1-\mathcal{F}'_{\tilde{a}}(1)+\mathcal{F}'_{\tilde{b}}(1)}. \tag{5.5}$$

Solution. Equation 5.59 is as follows:

$$\mathcal{F}_{\tilde{q}}(z)\left[z-\mathcal{F}_{\tilde{a}}(z)\right] = \pi_0\left[z\,\mathcal{F}_{\tilde{b}}(z)-\mathcal{F}_{\tilde{a}}(z)\right]. \tag{5.59}$$

First take the derivative of both sides of (5.59) with respect to z to find

$$\frac{d}{dz}\mathcal{F}_{\tilde{q}}(z)\left[z-\mathcal{F}_{\tilde{a}}(z)\right]+\mathcal{F}_{\tilde{q}}(z)\left[1-\frac{d}{dz}\mathcal{F}_{\tilde{a}}(z)\right] = \pi_0\left[z\,\frac{d}{dz}\mathcal{F}_{\tilde{b}}(z)+\mathcal{F}_{\tilde{b}}(z)-\frac{d}{dz}\mathcal{F}_{\tilde{a}}(z)\right].$$

Now, take the limit as $z\to 1$ and recognize that $\mathcal{F}_{\tilde{q}}(1)=1$, $\mathcal{F}_{\tilde{a}}(1)=1$, and $\mathcal{F}_{\tilde{b}}(1)=1$. We then have

$$\left[1-\frac{d}{dz}\mathcal{F}_{\tilde{a}}(1)\right] = \pi_0\left[\frac{d}{dz}\mathcal{F}_{\tilde{b}}(1)+1-\frac{d}{dz}\mathcal{F}_{\tilde{a}}(1)\right]$$

from which the result follows directly.

EXERCISE 5.26 Argue rigorously that in order for the M/G/1 queue to be stable, we must have $a_0 > 0$.

Solution. Since a_0 represents the probability that there are no arrivals during an ordinary service time, there only two choices, $a_0 = 0$ or $a_0 > 0$. The latter choice indicates that there are at least one arrival during an ordinary service time and hence the expected number of arrivals during a service time is at least unity. Suppose the queueing system ever enters level one. Then the dynamics of returning to level zero are identical to that of an ordinary M/G/1 queueing system. Therefore, we may consider the ordinary M/G/1 queueing system. In that case the expected number of arrivals during a service time is the traffic intensity. Thus, if $a_0 = 0$, the traffic intensity is at least unity and the system is unstable. Therefore, we know that for stability of the present queueing system, we ned $a_0 > 0$.

EXERCISE 5.27 Verify (5.62).

Solution. The restatement of (5.62) is as follows:

$$z_0 D = \pi_0 N$$

where $z_0 = [\,\pi_1 \quad \pi_2 \quad \dots \quad \pi_{m+1}\,]$, $N = [\,b_0 - 1 \quad b_1 \quad \dots \quad b_m\,]$ and

$$D = \begin{bmatrix} -a_0 & 1-a_1 & -a_2 & \cdots & -a_m \\ 0 & -a_0 & 1-a_1 & \ddots & -a_{m-1} \\ 0 & 0 & -a_0 & \ddots & -a_{m-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_0 \end{bmatrix}.$$

This is derived directly from (5.61), which is as follows:

$$[\,\pi_0 \quad \pi_1 \quad \ldots \quad \pi_m\,] = \pi_0\,[\,b_0 \quad b_1 \quad \ldots \quad b_m\,] +$$
$$[\,\pi_1 \quad \pi_2 \quad \ldots \quad \pi_{m+1}\,] \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_m \\ 0 & a_0 & a_1 & \cdots & a_{m-1} \\ 0 & 0 & a_0 & \cdots & a_{m-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}.$$

We may write the left side of the previous equation as

$$\pi_0\,[\,1 \quad 0 \quad \ldots \quad 0\,] + [\,\pi_1 \quad \pi_2 \quad \ldots \quad \pi_{m+1}\,] \begin{bmatrix} 0 & I_m \\ 0 & 0 \end{bmatrix},$$

where $[\,1 \quad 0 \quad \ldots \quad 0\,]$ is an m-vector and I_m is an m-square identity matrix. We thus have for (5.59),

$$\pi_0\,[\,1 \quad 0 \quad \ldots \quad 0\,] + z_0 \begin{bmatrix} 0 & I_m \\ 0 & 0 \end{bmatrix} = \pi_0\,[\,b_0 \quad b_1 \quad \ldots \quad b_m\,] +$$
$$z_0 \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_m \\ 0 & a_0 & a_1 & \cdots & a_{m-1} \\ 0 & 0 & a_0 & \cdots & a_{m-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}.$$

The result follows by noting that $N = [\,b_0 \quad b_1 \quad \ldots \quad b_m\,] - [\,1 \quad 0 \quad \ldots \quad 0\,]$ and

$$D = \begin{bmatrix} 0 & I_m \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_m \\ 0 & a_0 & a_1 & \cdots & a_{m-1} \\ 0 & 0 & a_0 & \cdots & a_{m-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}.$$

EXERCISE 5.28 For $i = 1, 2, \ldots$, show that

$$\mathcal{F}^{(i)}_{\tilde{q}}(z) = \sum_{j=0}^{\infty} \pi_{i+j} z^j.$$

Solution. Since $\mathcal{F}_{\tilde{q}}(z) = \sum_{j=0}^{\infty} \pi_j z^j$, we see immediately that $\mathcal{F}_{\tilde{q}}(z) - \pi_0 = \sum_{j=1}^{\infty} \pi_j z^j$. Therefore,

$$\frac{1}{z}\left[\mathcal{F}_{\tilde{q}}(z) - \pi_0\right] = \sum_{j=1}^{\infty} \pi_j z^{j-1} = \sum_{j=0}^{\infty} \pi_{j+1} z^j.$$

Thus, the result holds for $i = 1$. Assume the result hold for $i - 1$, then

$$\mathcal{F}_{\tilde{q}}^{(i-1)}(z) = \sum_{j=0}^{\infty} \pi_{i-1+j} z^j.$$

Now,

$$\mathcal{F}_{\tilde{q}}^{(i)}(z) = \frac{1}{z}\left[\mathcal{F}_{\tilde{q}}^{(i-1)}(z) - \pi_{i-1}\right],$$

we have

$$\mathcal{F}_{\tilde{q}}^{(i)}(z) = \frac{1}{z}\left[\sum_{j=0}^{\infty} \pi_{i-1+j} z^j - \pi_{i-1}\right] = \frac{1}{z}\left[\sum_{j=1}^{\infty} \pi_{i-1+j} z^j\right].$$

Hence

$$\mathcal{F}_{\tilde{q}}^{(i)}(z) = \sum_{j=0}^{\infty} \pi_{i+j} z^j,$$

and the proof is complete.

EXERCISE 5.29 Define

$$\mathcal{F}_{\tilde{q}}^{(1)}(z) = \frac{1}{z}\left[\mathcal{F}_{\tilde{q}}(z) - \pi_0\right] \text{ and } \mathcal{F}_{\tilde{q}}^{(i+1)}(z) = \frac{1}{z}\left[\mathcal{F}_{\tilde{q}}^{(i)}(z) - \pi_i\right], i \geq 1.$$

Starting with (5.68), substitute a function of $\mathcal{F}_{\tilde{q}}^{(1)}(z)$ for $\mathcal{F}_{\tilde{q}}(z)$, then a function of $\mathcal{F}_{\tilde{q}}^{(2)}(z)$ for $\mathcal{F}_{\tilde{q}}^{(1)}(z)$, and continue step by step until a function of $\mathcal{F}_{\tilde{q}}^{(C)}(z)$ is substituted for $\mathcal{F}_{\tilde{q}}^{(C-1)}(z)$. Show that at each step, one element of

$$\sum_{j=0}^{C-1} \pi_j z^j \mathcal{F}_{\tilde{a}}(z)$$

is eliminated, resulting in (5.69).

Solution. First we solve

$$\mathcal{F}_{\tilde{q}}^{(1)}(z) = \frac{1}{z}\left[\mathcal{F}_{\tilde{q}}(z) - \pi_0\right]$$

for $\mathcal{F}_{\tilde{q}}(z)$ to find

$$\mathcal{F}_{\tilde{q}}(z) = z\mathcal{F}_{\tilde{q}}^{(1)}(z) + \pi_0$$

and

$$\mathcal{F}_{\tilde{q}}^{(i+1)}(z) = \frac{1}{z}\left[\mathcal{F}_{\tilde{q}}^{(i)}(z) - \pi_i\right], i \geq 1$$

for $\mathcal{F}_{\tilde{q}}^{(i)}(z)$ to find

$$\mathcal{F}_{\tilde{q}}^{(i)}(z) = z\mathcal{F}_{\tilde{q}}^{(i+1)}(z) + \pi_i.$$

Now, (5.68) is

$$\mathcal{F}_{\tilde{q}}(z)\left[z^C - \mathcal{F}_{\tilde{a}}(z)\right] = \sum_{j=0}^{C-1} \pi_j \left[z^C \mathcal{F}_{\tilde{b},j}(z) - z^j \mathcal{F}_{\tilde{a}}(z)\right]. \tag{5.68}$$

To begin, we substitute $\mathcal{F}_{\tilde{q}}(z) = z\mathcal{F}_{\tilde{q}}^{(1)}(z) + \pi_0$ into (5.68) to obtain

$$z\mathcal{F}_{\tilde{q}}(z)^{(1)}\left[z^C - \mathcal{F}_{\tilde{a}}(z)\right] = \sum_{j=0}^{C-1} \pi_j z^C \mathcal{F}_{\tilde{b},j}(z) - \sum_{j=0}^{C-1} z^j \mathcal{F}_{\tilde{a}}(z) + z^C \pi_0 + \pi_0 \mathcal{F}_{\tilde{a}}(z).$$

Simplifying, we find

$$z\mathcal{F}_{\tilde{q}}(z)^{(1)}\left[z^C - \mathcal{F}_{\tilde{a}}(z)\right] = \sum_{j=0}^{C-1} \pi_j z^C \mathcal{F}_{\tilde{b},j}(z) - \sum_{j=1}^{C-1} z^j \mathcal{F}_{\tilde{a}}(z) + z^C \pi_0. \tag{5.68.1}$$

Similarly, we substitute $z\mathcal{F}_{\tilde{q}}^{(2)}(z) + \pi_i$ for $\mathcal{F}_{\tilde{q}}^{(1)}(z)$ in (5.68.1) to find

$$\begin{aligned} z^2\mathcal{F}_{\tilde{q}}(z)^{(2)}\left[z^C - \mathcal{F}_{\tilde{a}}(z)\right] &= \sum_{j=0}^{C-1} \pi_j z^C \mathcal{F}_{\tilde{b},j}(z) - \sum_{j=1}^{C-1} z^j \mathcal{F}_{\tilde{a}}(z) \\ &\quad - z^C \pi_0 - z^{C+1}\pi_1 + \pi_1 z \mathcal{F}_{\tilde{a}}(z), \end{aligned}$$

and simplifying leads to

$$z^2\mathcal{F}_{\tilde{q}}(z)^{(1)}\left[z^C - \mathcal{F}_{\tilde{a}}(z)\right] = \sum_{j=0}^{C-1} \pi_j z^C \mathcal{F}_{\tilde{b},j}(z) - \sum_{j=2}^{C-1} z^j \mathcal{F}_{\tilde{a}}(z) - z^C \pi_0 - z^{C+1}\pi_1. \tag{5.68.2}$$

Continuing in this way leads to

$$\begin{aligned} z^C\mathcal{F}_{\tilde{q}}(z)^{(C)}\left[z^C - \mathcal{F}_{\tilde{a}}(z)\right] &= \sum_{j=0}^{C-1} \pi_j z^C \mathcal{F}_{\tilde{b},j}(z) - \sum_{j=C-1}^{C-1} z^j \mathcal{F}_{\tilde{a}}(z) \\ &\quad + \sum_{j=0}^{C-1} z^{C+j}\pi_j + \pi_{C-1} z^{C-1} \mathcal{F}_{\tilde{a}}(z), \end{aligned}$$

and simplifying leads to

$$z^C\mathcal{F}_{\tilde{q}}(z)^{(C)}\left[z^C - \mathcal{F}_{\tilde{a}}(z)\right] = \sum_{j=0}^{C-1} \pi_j z^C \mathcal{F}_{\tilde{b},j}(z) - \sum_{j=0}^{C-1} z^C + j\pi_j,$$

or

$$\mathcal{F}_{\tilde{q}}(z)^{(C)}\left[z^C - \mathcal{F}_{\tilde{a}}(z)\right] = \sum_{j=0}^{C-1} \pi_j \left[\mathcal{F}_{\tilde{b},j}(z) - z^j\right]. \tag{5.69}$$

EXERCISE 5.30 Suppose $\mathcal{F}_{\tilde{a}}(z)$ and $\mathcal{F}_{\tilde{b}}(z)$ are each polynomials of degree m as discussed in the first part of this section. Define $w(z) = 1$. Find ν, D, N, A, and E using (5.72), (5.73), and (5.78). Compare the results to those presented in (5.62) and (5.66).

Solution. We are considering the case where $C = 1$. Recall that $\mathcal{F}_{\tilde{a}}(z) = u(z)/w(z)$. With $w(z) = 1$, $u(z) = \mathcal{F}_{\tilde{a}}(z)$, Similarly, $v(z) = \mathcal{F}_{\tilde{b}}(z)$. Thus, $d(z) = z - \mathcal{F}_{\tilde{a}}(z)$ and $n_0(z) = n(z) = \mathcal{F}_{\tilde{b}}(z) - 1$ for this special case. Now, $m \geq 1$ because otherwise, there would never be any arrivals. Recall that ν_d and ν_n are the degrees of $d(z)$ and $n(z)$, respectively. Since $m \geq 1$, $\nu_d = \nu_n = m$. Thus,

$$\nu = \max\{nu_d, \nu_n + 1\} = \max\{m, m+1\} = m+1.$$

Using (5.73), we then have

$$d_1 = 1 - a_1,\ d_i = -a_i, i \neq 1, 0 \leq i \leq m \text{ and } n_0 = b_0 - 1, n_i = b_i, 1 \leq i \leq m.$$

We then have from (5.76)

$$D = \begin{bmatrix} -a_0 & 1-a_1 & -a_2 & \cdots & -a_m \\ 0 & -a_0 & 1-a_1 & \ddots & -a_{m-1} \\ 0 & 0 & -a_0 & \ddots & -a_{m-2} \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & -a_0 \end{bmatrix},$$

and $N = [\,b_0 - 1 \quad b_1 \quad b_2 \quad \cdots \quad b_m\,]$. From (5.78), where E and A are defined as follows where $E = \text{diag }(1, 1, \ldots, 1, d_0)$, and

$$A = \begin{bmatrix} 0 & 0 & \cdots & \cdots & d_{\nu-1} \\ 1 & 0 & \ddots & \ddots & d_{\nu-2} \\ 0 & 1 & \ddots & \ddots & -d_{\nu-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & d_1 \end{bmatrix},$$

we find $E = \text{diag }(1, 1, \ldots, 1, -a_0)$ and

$$A = \begin{bmatrix} 0 & 0 & \cdots & \cdots & -a_m \\ 1 & 0 & \ddots & \ddots & -a_{m-1} \\ 0 & 1 & \ddots & \ddots & -a_{m-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & (1-a_1) \end{bmatrix}.$$

These are the same results as presented in (5.62) and (5.63).

Table 5.1. Occupancy values as a function of the number of units served during a service period for the system analyzed in Example 5.8.

Occupancy	$C = 1$	$C = 2$	$C = 4$	$C = 8$
-	1.00000	1.00000	1.00000	1.00000
0	0.9000	0.9493	0.9861	0.9985
1	0.7633	0.8507	0.9453	0.9920
2	0.6350	0.7347	0.8772	0.9762
3	0.5261	0.6233	0.7914	0.9481
4	0.4356	0.5254	0.6997	0.9071
5	0.3606	0.4419	0.6107	0.8547
6	0.2985	0.3715	0.5292	0.7942
7	0.2472	0.3123	0.4569	0.7292

EXERCISE 5.31 Table 5.5 gives numerical values for the survivor function of the occupancy distributions shown in Figure 5.5. From this table, determine the probability masses for the first few elements of the distributions and then compute the mean number of units served during a service time for $C = 1, 2, 4,$ and 8. Analyze the results of your calculations.

Solution. The main idea here is that if there are at least C units present in the system, then the number of units served will be C. Otherwise, there will be the same as the number of units present, or equivalently, the servers will serve all units present at the beginning of the service interval. Hence, the expected number of units served will be

$$\sum_{i=1}^{C-1} iP\{\tilde{q} = i\} + CP\{\tilde{q} \geq C\}.$$

To obtain the probability masses, we simply subtract successive values of the survivor functions. For example, for the case $C = 2$, we have $P\{\tilde{q} = 0\} = 1 - 0.9493 = 0.0507$, $P\{\tilde{q} = 1\} = 0.9493 - 0.8507 = 0.0986$, $P\{\tilde{q} \geq 2\} = P\{\tilde{q} > 1\} = 0.8507$. Hence the expected number of services during a service interval is $1 \times 0.0986 + 2 \times 0.8507 = 1.8$. From this, we see that the system utilization is 0.9. If we repeat this for $C = 4$, we will find that the expected number of services that occur during a service time is 3.6 so that again the system utilization is 0.9.

EXERCISE 5.32 Use a busy period argument to establish the validity of (5.90). [*Hint*: Consider the M/G/1 system under the nonpreemptive LCFS service discipline.]

Solution. Let $\mathcal{B}$ be the event that an arbitrary customer finds the system busy upon arrival, and let I represent its complement. Then

$$\begin{aligned} E[\tilde{w}] &= E[\tilde{w}|\mathcal{B}]P\{\mathcal{B}\} + E[\tilde{w}|I]P\{I\} \\ &= \rho E[\tilde{w}|\mathcal{B}] + E[\tilde{w}|I](1-\rho). \end{aligned}$$

But $E[\tilde{w}|I] = 0$ since the customer enters service immediately if the system is idle upon arrival. Therefore,

$$E[\tilde{w}] = \rho E[\tilde{w}|\mathcal{B}].$$

Now, the total amount of waiting time for the arbitrary customer in a LCFS discipline will be the length of the busy period started by the customer in service and generated by the newly arrived customers. Hence,

$$E[\tilde{w}|\mathcal{B}] = \frac{E[\tilde{x}_e]}{1-\rho]}.$$

Multiplying this expression by ρ, the probability that the system is busy upon arrival, we find that

$$E[\tilde{w}] = \rho E[\tilde{w}|\mathcal{B}] = \frac{\rho}{1-\rho}E[\tilde{x}_e],$$

which is the desired result.

EXERCISE 5.33 Show that the Laplace-Stieltjes transform for the distribution of the residual life for the renewal process having renewal intervals of length $\tilde{z}$ is given by

$$F^*_{\tilde{z}_r}(s) = \frac{1 - F^*_{\tilde{z}}(s)}{s\, E[\tilde{z}]}. \tag{5.98}$$

Solution. By Definition 2.8,

$$\begin{aligned} F^*_{\tilde{z}_r}(s) &= \int_0^\infty e^{-sz} dF_{\tilde{z}_r}(z) \\ &= \int_0^\infty e^{-sz} \left[\frac{1 - F_{\tilde{z}}(z)}{E[\tilde{z}]}\right] dz \\ &= \frac{1}{E[\tilde{z}]} \int_0^\infty e^{-sz} \int_x^\infty dF_{\tilde{z}}(y) dz \\ &= \frac{1}{E[\tilde{z}]} \int_0^\infty \left\{ \int_0^y e^{-sz} dz \right\} dF_{\tilde{z}}(y) \\ &= \frac{1}{E[\tilde{z}]} \int_0^\infty \frac{1}{s} \left[1 - e^{-sy}\right] dF_{\tilde{z}}(y) \\ &= \frac{1}{sE[\tilde{z}]} \left[1 - F_{\tilde{z}}(s)\right]. \end{aligned}$$

EXERCISE 5.34 For an arbitrary nonnegative random variable, $\tilde{x}$, show that

$$E\left[\tilde{x}_r^n\right] = \frac{E\left[\tilde{x}^{n+1}\right]}{(n+1)E[\tilde{x}]}. \tag{5.102}$$

Solution. Recall that

$$f_{\tilde{x}_r}(z) = \frac{[1 - F_{\tilde{x}}(z)]}{E[\tilde{x}]}.$$

We may then write

$$E[\tilde{x}_r] = \int_0^\infty x^n f_{\tilde{x}_r}(x)dx,$$

so that

$$\begin{aligned} E[\tilde{x}_r] &= \frac{1}{E[\tilde{x}]}\int_0^\infty x^n\left[1 - F_{\tilde{x}}(z)\right]dx \\ &= \frac{1}{E[\tilde{x}]}\int_0^\infty x^n\left[1 - \int_x^\infty f_{\tilde{x}}(z)dz\right]dx. \end{aligned}$$

Changing the order of integration,

$$\begin{aligned} E[\tilde{x}_r] &= \frac{1}{E[\tilde{x}]}\int_0^\infty \left[\int_0^z x^n dx\right] f_{\tilde{x}}(z)dz \\ &= \frac{1}{E[\tilde{x}]}\int_0^\infty \left.\frac{x^{n+1}}{n+1}\right|_0^z f_{\tilde{x}}(z)dz \\ &= \frac{E[\tilde{x}^{n+1}]}{(n+1)E[\tilde{x}]}. \end{aligned}$$

EXERCISE 5.35 For the M/G/1 system, suppose that $\tilde{x} = \tilde{x}_1 + \tilde{x}_2$ where $\tilde{x}_1$ and $\tilde{x}_2$ are independent, exponentially distributed random variables with parameters μ_1 and μ_2, respectively. Show that $C_{\tilde{x}}^2 \leq 1$ for all μ_1, μ_2 such that $E[\tilde{x}] = 1$.

Solution. First, by the fact that $E[\tilde{x}] = 1$,

$$1 = E^2[\tilde{x}] = E^2[\tilde{x}_1 + \tilde{x}_2]$$

$$\begin{aligned}
&= E^2[\tilde{x}_1] + 2E[\tilde{x}_1]E[\tilde{x}_2] + E^2[\tilde{x}_2] \\
&= \frac{1}{\mu_1^2} + \frac{2}{\mu_1\mu_2} + \frac{1}{\mu_2^2},
\end{aligned}$$

which implies that $1/\mu_1^2 + 1/\mu_2^2 \le 1$ since $2/\mu_1\mu_2 \ge 0$. Therefore,

$$\begin{aligned}
C_{\tilde{x}}^2 &= \frac{\mathrm{Var}(\tilde{x})}{E^2[\tilde{x}]} \\
&= \frac{\mathrm{Var}(\tilde{x}_1 + \tilde{x}_2)}{1} \\
&= \mathrm{Var}(\tilde{x}_1) + \mathrm{Var}(\tilde{x}_2) \\
&= \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} \le 1.
\end{aligned}$$

EXERCISE 5.36 Compute the expected waiting time for the M/G/1 system with unit mean deterministic service times and for the M/G/1 system with service times drawn from the unit mean Erlang-2 distribution. Plot on the same graph $E[\tilde{w}]$ as a function of ρ for these two distributions and for the M/M/1 queueing system with $\mu = 1$ on the same graph. Compare the results.

Solution. To compute the expected waiting time, recall equation (5.101):

$$E[\tilde{w}] = \frac{\rho E[\tilde{z}]}{1-\rho}\left[\frac{1 + C_{\tilde{z}}^2}{2}\right],$$

where

$$C_{\tilde{z}}^2 = \frac{\mathrm{Var}(\tilde{z})}{E^2[\tilde{z}]}.$$

For all three systems, note that $E[\tilde{z}] = 1$. Thus, we need only find the variance (which will be equal to $C_{\tilde{z}}^2$ in this case) of each to compute its expected waiting time. For the deterministic system, the variance is zero, so that

$$E[\tilde{w}_d] = \frac{\rho}{1-\rho}\left[\frac{1+0}{2}\right] = \frac{\frac{\rho}{2}}{1-\rho}.$$

For the Erlang-2 system,

$$\begin{aligned}
\mathrm{Var}(z_e) &= \mathrm{Var}(z_1) + \mathrm{Var}(z_2) \\
&= \frac{1}{4\mu^2} + \frac{1}{4\mu^2} \\
&= \frac{1}{2},
\end{aligned}$$

and the mean waiting time for this system is

$$E[\tilde{w}_e] = \frac{\rho}{1-\rho}\left[\frac{1+\frac{1}{2}}{2}\right] = \frac{\frac{3}{4}\rho}{1-\rho}.$$

For the M/M/1 system, $\mathrm{Var}(\tilde{z}_m) = 1/\mu^2 = 1$, so that

$$E[\tilde{w}_m] = \frac{\rho}{1-\rho}\left[\frac{1+1}{2}\right] = \frac{\rho}{1-\rho}.$$

From Figure 5.1, we see that as the coefficient of variance increases, the performance of the system deteriorates. i.e., the coefficient of variance and the mean waiting time are directly proportional. In addition, we see from the figure that the mean waiting time is also directly proportional to ρ. This should be clear since if $\rho = \frac{\lambda}{\mu}$ increases, then the arrival rate also increases, putting a greater load on the system.

EXERCISE 5.37 For the M/G/1 system, suppose that $\tilde{x}$ is drawn from the distribution $F_{\tilde{x}_1}(x)$ with probability p and from $F_{\tilde{x}_2}(x)$ otherwise, where $\tilde{x}_1$ and $\tilde{x}_2$ are independent, exponentially distributed random variables with parameters μ_1 and μ_2, respectively. Let $E[\tilde{x}] = 1$. Show that $C_{\tilde{x}}^2 \geq 1$ for all $p \in [0, 1]$.

Solution. By the definition of $C_{\tilde{x}}^2$,

$$C_{\tilde{x}}^2 = \frac{\mathrm{Var}(\tilde{x})}{E^2[\tilde{x}]} = \frac{E[\tilde{x}^2] - E^2[\tilde{x}]}{E^2[\tilde{x}]}.$$

That is, proving $C_{\tilde{x}}^2 \geq 1$ is equivalent to showing $E[\tilde{x}^2] \geq 2$. Let $\mathcal{D}_i$ denote the event that distribution i is selected, $i = 1, 2$. Then

$$\begin{aligned} P\{\tilde{x} \leq x\} &= P\{\tilde{x} \leq x|\mathcal{D}_1\} P\{\mathcal{D}_1\} + P\{\tilde{x} \leq x|\mathcal{D}_1\} P\{\mathcal{D}_1\} \\ &= pF_{\tilde{x}_1}(x) + (1-p)F_{\tilde{x}_2}(x), \end{aligned}$$

so that

$$dF_{\tilde{x}}(x) = pdF_{\tilde{x}_1}(x) + (1-p)dF_{\tilde{x}_2}(x).$$

Therefore,

$$E[\tilde{x}^n] = pE[\tilde{x}_1^n] + (1-p)E[\tilde{x}_2^n].$$

For the exponential distribution with rate μ, $E[\tilde{x}^2] = 2/\mu^2$, and so

$$E[\tilde{x}^2] = p\frac{2}{\mu_1^2} + (1-\rho)\frac{2}{\mu_2^2} = 2\left[\frac{p}{\mu_1^2} + \frac{1-p}{\mu_2^2}\right].$$

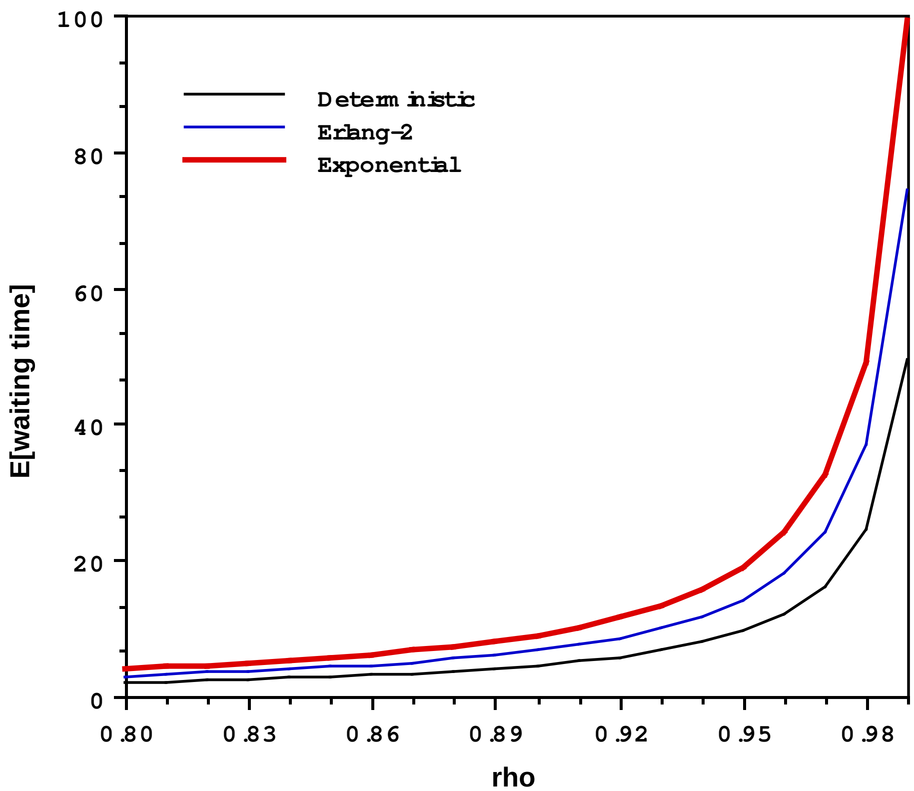

Figure 5.36.1 Survivor function for Exercise 5.36.

Therefore, showing $E[\tilde{x}^2] \geq 2$ is equivalent to showing

$$\frac{p}{\mu_1^2} + \frac{(1-p)}{\mu_2^2} \geq 1.$$

i.e.,

$$\frac{p}{\mu_1^2} + \frac{1-p}{\mu_2^2} - 1 \geq 0.$$

Now,

$$E[\tilde{x}] = \frac{p}{\mu_1} + \frac{1-p}{\mu_2} = 1,$$

so that

$$E^2[\tilde{x}] = \frac{p^2}{\mu_1^2} + \frac{2p(1-p)}{\mu_1\mu_2} + \frac{(1-p)^2}{\mu_2^2} = 1.$$

Thus,

$$\begin{aligned}
\frac{p}{\mu_1^2}+\frac{1-p}{\mu_2^2}-1 &= \frac{p}{\mu_1^2}+\frac{1-p}{\mu_2^2}-\frac{p^2}{\mu_1^2}-\frac{2p(1-p)}{\mu_1\mu_2}-\frac{(1-p)^2}{\mu_2^2} \\
&= \frac{p(1-p)}{\mu_1^2}+\frac{p(1-p)}{\mu_2^2}-\frac{2p(1-p)}{\mu_1\mu_2} \\
&= p(1-p)\left[\frac{1}{\mu_1^2}-\frac{2}{\mu_1\mu_2}+\frac{1}{\mu_2^2}\right] \\
&= p(1-p)\left[\frac{1}{\mu_1}-\frac{1}{\mu_2}\right]^2 \geq 0,
\end{aligned}$$

and the proof is complete.

EXERCISE 5.38 With $\tilde{x}$ and p defined as in Exercise 5.37, let $p=\frac{1}{2}$. Find μ_1 and μ_2 such that $C_{\tilde{x}}^2=2$. Would it be possible to determine p, μ_1, and μ_2 uniquely for a given value of $C_{\tilde{x}}^2$? Explain.

Solution. First, we will compute the general case for *any* p, and then we will substitute for the specific value of $p=\frac{1}{2}$. Now, $1=E[\tilde{x}]$, and $C_{\tilde{x}}^2=E[\tilde{x}^2]-1$, so that we have the following two equations

$$\frac{p}{\mu_1}+\frac{(1-p)}{\mu_2} = 1 \qquad (5.38.1)$$

$$\frac{2p}{\mu_1^2}+\frac{2(1-p)}{\mu_2^2} = C_{\tilde{x}}^2+1 \qquad (5.38.2)$$

Rearranging (5.38.2),

$$\frac{1}{\mu_1^2}+\frac{(1-p)/p}{\mu_2^2} = \frac{C_{\tilde{x}}^2+1}{2p}. \qquad (5.38.3)$$

Solving equation (5.38.1) for $\frac{1}{\mu_1}$, and squaring both sides

$$\frac{1}{\mu_1^2}=\frac{1}{p^2}\left[1-\frac{2(1-p)}{\mu_2}+\frac{(1-p)^2}{\mu_2^2}\right].$$

Substitute this expression into (5.38.3) to yield

$$\frac{1}{\mu_2^2}-\frac{2}{\mu_2}+\frac{1-\frac{p}{2}\left(C_{\tilde{x}}^2+1\right)}{1-p}=0.$$

And solving this equation for $\frac{1}{\mu_2}$, we obtain

$$\frac{1}{\mu_2} = 1\pm\sqrt{\left(\frac{p}{1-p}\right)\left(\frac{1}{2}\right)\left(C_{\tilde{x}}^2-1\right)}. \qquad (5.38.4)$$

Substitute this result into (1) to yield

$$\frac{1}{\mu_1} \quad = \quad 1 \pm \sqrt{\left(\frac{1-p}{2p}\right)(C^2_{\tilde{x}} - 1)}. \qquad (5.38.5)$$

This shows that one can find μ_1 and μ_2 given p and $C^2_{\tilde{x}}$ if $C^2_{\tilde{x}} \geq 1$. If we take $1/\mu_1 = 1 + \sqrt{1/2}$, then

$$\frac{1}{\mu_2} = 1 - \sqrt{\frac{1}{2}}.$$

From (5.38.4) and (5.38.5), we see that it is necessary to specify p in addition to $C^2_{\tilde{x}}$ to obtain unique values of μ_1 and μ_2. This should be clear from equations (5.38.1) and (5.38.2), which is a system of two equations in three unknowns.

EXERCISE 5.39 (Ross[1989]) Consider an ordinary renewal process with renewal interval $\tilde{z}$. Choose a real number c arbitrarily. Now suppose the renewal process is observed at a random point in time, t_0. If the age of the observed interval is less than c, define the system to be in an x-period, else define the system to be in a y-period. Thus, the expected cycle length is $E[\tilde{z}]$, and the expected length of the it x-period is $E[\min\{c, \tilde{z}\}]$. Show that

$$E[\min\{c, \tilde{z}\}] = \int_0^c [1 - F_{\tilde{z}}(z)]dz$$

so that

$$\frac{d}{dz} F_{\tilde{z}_a}(z) = \frac{1 - F_{\tilde{z}}(z)}{E[\tilde{z}]},$$

as was shown in the previous subsection.

Solution.

EXERCISE 5.40 Formalize the informal discussion of the previous paragraph.

Solution. Let $Y(t)$ denote the total length of time the server has spent serving up to time t, so that $Y(t)/t$ represents the proportion of time the server has spent serving up to time t. Now let $N(t)$ denote the total number of busy periods completed up to time t. So long as there has been at least one busy period up to this time, we may write

$$\frac{Y(t)}{t} = \frac{Y(t)}{t}\frac{N(t)}{N(t)} = \frac{Y(t)}{N(t)}\frac{N(t)}{t}.$$

For a fixed t, $Y(t)$ is a random variable. Thus, the long-term average proportion of time the server is busy is

$$\lim_{t\to\infty}\frac{Y(t)}{t} = \lim_{t\to\infty}\frac{Y(t)}{t}\frac{N(t)}{N(t)}.$$

It remains to show

$$\lim_{t\to\infty}\frac{Y(t)}{t}\frac{N(t)}{N(t)} = \lim_{t\to\infty}\frac{Y(t)}{N(t)}\lim_{t\to\infty}\frac{N(t)}{t}.$$

That is, that these limits exist separately.

We consider the limit of $t/N(t)$, as $t \to \infty$. Let $c_n = x_n + y_n$ denote the length of the n-th cycle. Then

$$\lim_{t\to\infty}\frac{t}{N(t)} = \lim_{t\to\infty}\frac{\sum_{n=0}^{N(t)} c_n}{N(t)} \le \frac{t}{N(t)} \le \frac{\sum_{n=0}^{N(t)+1} c_n}{N(t)},$$

so that

$$\lim_{t\to\infty}\frac{t}{N(t)} = E[\tilde{c}] = E[\tilde{x}] + E[\tilde{y}].$$

Therefore,

$$\lim_{t\to\infty}\frac{N(t)}{t} = \frac{1}{E[\tilde{x}] + E[\tilde{y}]}.$$

Since the limits exist separately,

$$\begin{aligned}\lim_{t\to\infty}\frac{Y(t)}{t} &= \lim_{t\to\infty}\frac{Y(t)}{N(t)}\lim_{t\to\infty}\frac{N(t)}{t} \\ &= \frac{E[\tilde{y}]}{E[\tilde{x}] + E[\tilde{y}]}.\end{aligned}$$

5-1 Consider a communication system in which messages are transmitted over a communication line having a capacity of C octets/sec. Suppose the messages have length $\tilde{m}$ (in octets), and the lengths are drawn from a geometric distribution having a mean of $E[\tilde{m}]$ octets, but truncated at a and b characters on the lower and upper ends of the distribution, respectively. That is, message lengths are drawn from a distribution characterized as follows:

$$P\{\tilde{m} = m\} = k\theta(1-\theta)^{m-1} \quad \text{for} \quad a \le m \le b,$$

where $\tilde{m}$ is the number of characters in a message and k is a normalizing constant.

(a) Given that

$$P\{\tilde{m} = m\} = k\theta(1-\theta)^{m-1} \quad \text{for} \quad a \le m \le b,$$

show that

$$k = \left[(1-\theta)^{a-1} - (1-\theta)^{b}\right]^{-1},$$

$$E[z^{\tilde{m}}] = z^{(a-1)} \frac{\theta z}{1-(1-\theta)z} \frac{1-[(1-\theta)z]^{(b-[a-1])}}{1-(1-\theta)^{(b-[a-1])}},$$

and

$$E[\tilde{m}] = a - 1 + \frac{1}{\theta} - \frac{(b-[a-1])(1-\theta)^{(b-[a-1])}}{1-(1-\theta)^{(b-[a-1])}}.$$

(b) Rearrange the expression for $E[\tilde{m}]$ given above by solving for θ^{-1} to obtain an equation of the form

$$\frac{1}{\theta} = f(E[\tilde{m}], a, b, \theta),$$

and use this expression to obtain a recursive expression for θ of the form

$$\frac{1}{\theta_{i+1}} = f(E[\tilde{m}], a, b, \theta_i).$$

(c) Write a simple program to implement the recursive relationship defined in part (b) to solve for θ in the special case of $a = 10$, $b = 80$, and $E[\tilde{m}] = 30$. Use $\theta_0 = E^{-1}[\tilde{m}]$ as the starting value for the recursion.

(d) Argue that $F^*_{\tilde{x}}(s) = F^*_{\tilde{m}}(s/C)$, where C is the transmission capacity in octets/sec.

(e) Using the computer program given in the Appendix, obtain the complementary occupancy distribution for the transmission system under its actual message length distribution at a traffic utilization of 95%, assuming a transmission capacity of 30 characters/sec.

(f) Compare this complementary distribution to one obtained under the assumption that the message lengths are drawn from an ordinary geometric distribution. Comment on the suitability of making the geometric assumption.

Solution:

(a) From the laws of total probability,

$$\sum_{m=a}^{b} P\{\tilde{m} = m\} = 1.$$

And from (1.1),

$$\begin{aligned}
\sum_{m=a}^{b} P\{\tilde{m}=m\} &= k\sum_{m=a}^{b}\theta(1-\theta)^{m-1} \\
&= k\theta\frac{(1-\theta)^{a-1}-(1-\theta)^{b}}{1-(1-\theta)} \\
&= k\left[(1-\theta)^{a-1}-(1-\theta)^{b}\right]=1.
\end{aligned}$$

Therefore,

$$k=\left[(1-\theta)^{a-1}-(1-\theta)^{b}\right]^{-1}.$$

To find $E[\tilde{m}]$, observe that

$$\begin{aligned}
E[z^{\tilde{m}}] &= \sum_{m=a}^{b} z^{m}P\{\tilde{m}=m\} \\
&= \sum_{m=a}^{b} z^{m}k\theta(1-\theta)^{m-1} \\
&= k\theta z\sum_{m=a}^{b}[z(1-\theta)]^{m-1} \\
&= k\theta z\frac{[z(1-\theta)]^{a-1}-[z(1-\theta)]^{b}}{1-z(1-\theta)} \\
&= z^{a-1}\frac{\theta z\left[(1-\theta)^{a-1}-z^{(b-(a-1))}(1-\theta)^{b}\right]}{1-z(1-\theta)} \\
&\qquad \cdot\frac{1}{(1-\theta)^{a-1}-(1-\theta)^{b}} \\
&= z^{a-1}\frac{\theta z}{1-z(1-\theta)}\,\frac{1-[(1-\theta)z]^{b-(a-1)}}{1-(1-\theta)^{b-(a-1)}}.
\end{aligned}$$

To compute $E[\tilde{m}]$, first note the following probabilities:

$$\begin{aligned}
P\{\tilde{m}>x\} &= \begin{cases} 1, & x<a, \\ \sum_{m=x+1}^{b} P\{\tilde{m}=m\}, & a\le x<b, \\ 0, & x\ge b \end{cases}, \\
&= \begin{cases} 1, & x<a, \\ k\left[(1-\theta)^{x}-(1-\theta)^{b}\right], & a\le x<b, \\ 0, & x\ge b \end{cases},
\end{aligned}$$

so that

$$E[\tilde{m}] = \sum_{x=0}^{\infty} P\{\tilde{m}>x\}$$

$$
\begin{aligned}
&= k\left[\sum_{x=a}^{b-1}(1-\theta)^x-(b-a)(1-\theta)^b\right]+a\\
&= k\left[\frac{(1-\theta)^a-(1-\theta)^b}{1-(1-\theta)}-(b-a)(1-\theta)^b\right]+a\\
&= \frac{k}{\theta}\left[(1-\theta)^a-(1-\theta)^b-\theta(b-a)(1-\theta)^b\right]+a\\
&= \frac{1}{\theta}\left[\frac{(1-\theta)^a-(1-\theta)^b-\theta(b-a)(1-\theta)^b}{(1-\theta)^{a-1}-(1-\theta)^b}\right]+a\\
&= \frac{1}{\theta}\left[\frac{(1-\theta)-(1-\theta)^{b-(a-1)}-\theta(b-a)(1-\theta)^{b-(a-1)}}{1-(1-\theta)^{b-(a-1)}}\right]\\
&\qquad +a\\
&= \frac{1}{\theta}-\frac{1+(b-a)(1-\theta)^{b-(a-1)}}{1-(1-\theta)^{b-(a-1)}}+a\\
&= \frac{1}{\theta}-\frac{1-(1-\theta)^{b-(a-1)}+[b-(a-1)](1-\theta)^{b-(a-1)}}{1-(1-\theta)^{b-(a-1)}}\\
&\qquad +a\\
&= (a-1)+\frac{1}{\theta}-\frac{[b-(a-1)](1-\theta)^{b-(a-1)}}{1-(1-\theta)^{b-(a-1)}}.
\end{aligned}
$$

We may also compute the mean of $\tilde{m}$ by considering the following. Define $\tilde{m}_1$ to be geometric random variable with parameter θ. Then $E[\tilde{m}] = a-1+E[\tilde{m}_1|\tilde{m}_1 \le c]$, where the latter expression is equal to the mean of geometric random variable truncated at the point $c = b-(a-1)$. Now,

$$
\begin{aligned}
E[\tilde{m}_1] &= E[\tilde{m}_1|\tilde{m}_1 \le c]P\{\tilde{m} \le c\}\\
&\quad +E[\tilde{m}_1|\tilde{m}_1 \le c]P\{\tilde{m} > c\}. \qquad (5.1.1)
\end{aligned}
$$

But

$$
\begin{aligned}
P\{\tilde{m} > c\} &= \sum_{i=c}^{\infty}\theta(1-\theta)^{i-1}\\
&= \frac{\theta(1-\theta)^c}{1-(1-\theta)}\\
&= (1-\theta)^c,
\end{aligned}
$$

and

$$E[\tilde{m}_1|\tilde{m} > c] = c+\frac{1}{\theta}.$$

Thus,

$$P\{\tilde{m} \le c\} = 1-(1-\theta)^c,$$

and from (S5.1.1), we then have

$$
\begin{aligned}
E[\tilde{m}_1 | \tilde{m}_1 \leq c] &= \frac{\frac{1}{\theta} - \left(c + \frac{1}{\theta}\right)(1-\theta)^c}{1-(1-\theta)^c} \\
&= \frac{1}{\theta} - \frac{c(1-\theta)^c}{1-(1-\theta)^c}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E[\tilde{m}] &= a - 1 + \frac{1}{\theta} - \frac{c(1-\theta)^c}{1-(1-\theta)^c} \\
&= a - 1 + \frac{1}{\theta} - \frac{[b-(a-1)](1-\theta)^{b-(a-1)}}{1-(1-\theta)^{b-(a-1)}}.
\end{aligned}
$$

(b) From part (a),

$$
\frac{1}{\theta} = E[\tilde{m}] = (a-1) + \frac{[b-(a-1)](1-\theta)^{b-(a-1)}}{1-(1-\theta)^{b-(a-1)}},
$$

or

$$
\frac{1}{\theta_{i+1}} = E[\tilde{m}] - (a-1) + \frac{[b-(a-1)](1-\theta_i)^{b-(a-1)}}{1-(1-\theta_i)^{b-(a-1)}}.
$$

(c) The computer program resulted in $\theta = 0.039531$.

(d) Since $\tilde{x} = \frac{\tilde{m}}{C}$, $P\{\tilde{x} \leq x\} = P\left\{\frac{\tilde{m}}{C} \leq x\right\} = P\{\tilde{m} \leq Cx\}$, so that

$$
F^*_{\tilde{x}}(x) = \int_0^\infty e^{-sx} dF_{\tilde{x}}(x) = \int_0^\infty e^{-sx} dF_{\tilde{m}}(Cx).
$$

Let $y = Cx$; then $x = y/C$, and

$$
\begin{aligned}
F^*_{\tilde{x}}(s) = \int_0^\infty e^{-s\frac{y}{C}} dF_{\tilde{m}}(y) & \\
&= F^*_{\tilde{m}}\left(\frac{s}{C}\right).
\end{aligned}
$$

(e) See the graph in Figure 1.

(f) The comparison is roughly that shown in Figure 5.1 between the $(10, 80)$ curve and the $(1, 5000)$ curve. For n small, the two distributions can be said to approximate one another. However, as n increases, the truncated geometric diverges from that of the ordinary geometric and so the geometric assumption is not valid in this case.

medskip

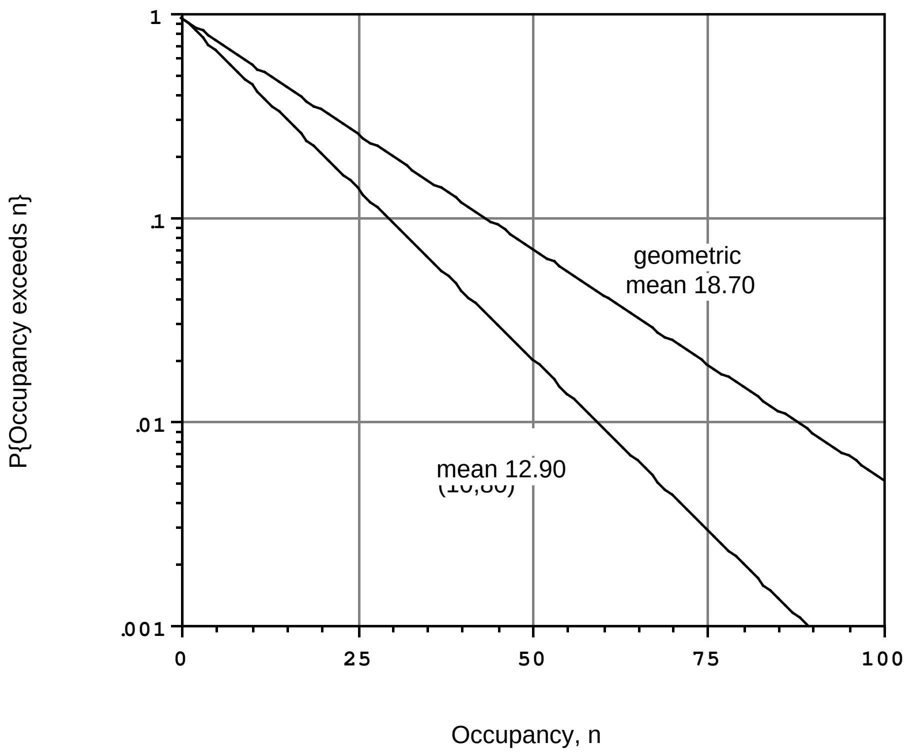

Figure P1.1 Survivor function for Supplementary Problem 1.

5-2 Using the properties of the probability generating function, determine a formula for $E[\tilde{n}^2]$, the second moment of the occupancy distribution for the ordinary M/G/1 system, in terms of the first three moments of $F_{\tilde{x}}(x)$, the service time distribution. Verify the formula for $E[\tilde{n}]$ along the way. [*Hint*: The algebra will be greatly simplified if **(??)** is first rewritten as

$$\mathcal{F}_{\tilde{n}}(z) = \alpha(z)/\beta(z),$$

where

$$\alpha(z) = (1-\rho)F^*_{\tilde{x}}(\lambda[1-z]),$$

$$\beta(z) = 1 - \rho F^*_{\tilde{x}_r}(\lambda[1-z]),$$

and $F_{\tilde{x}_r}(x)$ is the distribution for the forward recurrence time of the service time. Then, in order to find

$$\lim_{z \to 1} \frac{d^2}{dz^2}\mathcal{F}_{\tilde{n}}(z),$$

first find the limits as $z \to 1$ of $\alpha(z)$, $\beta(z)$, $d\alpha(z)/dz$, $d\beta(z)/dz$, $d^2\alpha(z)/dz^2$, $d^2\beta(z)/dz^2$, and then substitute these limits into the formula for the second derivative of the ratio.]

Solution: First rewrite (5.8) as

$$\mathcal{F}_{\tilde{n}}(z) = \frac{\alpha(z)}{\beta(z)},$$

where

$$\alpha(z) = (1-\rho)F^*_{\tilde{x}}(\lambda[1-z]),$$

and

$$\beta(z) = 1 - \rho F^*_{\tilde{x}_r}(\lambda[1-z]),$$

where $F_{\tilde{x}_r}(x)$ is the distribution for the forward recurrence time of the service time. Then,

$$\frac{d}{dz}\mathcal{F}_{\tilde{n}}(z) = \frac{\alpha^{'}(z)}{\beta(z)} - \frac{\alpha(z)\beta^{'}(z)}{\beta^2(z)},$$

and

$$\begin{aligned}
\frac{d^2}{dz^2}\mathcal{F}_{\tilde{n}}(z) &= \frac{\alpha^{''}(z)}{\beta(z)} - \frac{\alpha^{'}(z)\beta^{'}(z)}{\beta^2(z)} \\
&\qquad - \left(\frac{\alpha^{'}(z)\beta^{'}(z)}{\beta^2(z)} + \frac{\alpha(z)\beta^{''}(z)}{\beta^2(z)} - \frac{2\alpha(z)[\beta^{'}(z)]^2}{\beta^3(z)}\right) \\
&= \frac{\alpha^{''}(z)}{\beta(z)} - \frac{2\alpha^{'}(z)\beta^{'}(z)}{\beta^2(z)} - \frac{\alpha(z)\beta^{''}(z)}{\beta^2(z)} + \frac{2\alpha(z)[\beta^{'}(z)]^2}{\beta^3(z)}.
\end{aligned}$$

Now, $\alpha(1) = 1-\rho$, and $\beta(1) = 1-\rho$, so that

$$\alpha^{'}(z) = (1-\rho)(-\lambda)\frac{d}{ds}F^*_{\tilde{x}}(x)$$

implies

$$\alpha^{'}(1) = (1-\rho)(-\lambda)\frac{d}{ds}F^*_{\tilde{x}}(0) = \rho(1-\rho),$$

and

$$\beta^{'}(z) = -\rho(-\lambda)\frac{d}{ds}F^*_{\tilde{x}_r}(x)$$

implies

$$\beta^{'}(1) = -\rho(-\lambda)(-E[\tilde{x}_r]) = -\lambda\rho E[\tilde{x}_r].$$

Therefore

$$E[\tilde{n}] = \frac{d}{ds}\mathcal{F}_{\tilde{n}}(1)$$

$$
\begin{aligned}
&= \frac{\alpha^{'}(1)}{\beta(1)} - \frac{\alpha(1)\beta^{'}(1)}{\beta^2(1)} \\
&= \frac{(1-\rho)\rho}{1-\rho} + \frac{(1-\rho)\lambda\rho E[\tilde{x}_r]}{(1-\rho)^2} \\
&= \rho + \frac{\lambda\rho}{1-\rho}\frac{E[\tilde{x}^2]}{2E[\tilde{x}]} \\
&= \rho\left[1 + \frac{\rho}{2(1-\rho)}\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}\right]
\end{aligned}
$$

.

In terms of $C^2_{\tilde{x}}$,

$$E[\tilde{n}] = \rho\left[1 + \frac{\rho}{1-\rho}\frac{1+C^2_{\tilde{x}}}{2}\right].$$

Taking the second derivatives of $\alpha(z)$ and $\beta(z)$ evaluated at $z=1$,

$$\alpha^{''}(z)\Big|_{z=1} = (1-\rho)(-\lambda)^2\frac{d^2}{ds^2}F^*_{\tilde{x}}(s)\Big|_{s=0} = (1-\rho)\lambda^2E[\tilde{x}^2],$$

and

$$\beta^{''}(z)\Big|_{z=1} = (-\rho)(-\lambda)^2\frac{d^2}{ds^2}F^*_{\tilde{x}}(s)\Big|_{s=0} = -\rho\lambda^2E[\tilde{x}^2_r].$$

Now, $F^*_{\tilde{x}_r}(x) = \frac{1-F^*_{\tilde{x}}(s)}{sE[\tilde{x}]}$. Thus,

$$
\begin{aligned}
\frac{d}{ds}F^*_{\tilde{x}_r}(s) &= \frac{1}{E[\tilde{x}]}\left[-\left(1-F^*_{\tilde{x}}(s)\right)\frac{1}{s^2} - \frac{1}{s}\frac{d}{ds}F^*_{\tilde{x}}(s)\right] \\
&= -\frac{1}{E[\tilde{x}]}\left[\frac{1-F^*_{\tilde{x}}(s) + s\frac{d}{ds}F^*_{\tilde{x}}(s)}{s^2}\right].
\end{aligned}
$$

Applying L'Hôpital's rule twice,

$$\lim_{s\to 0}\frac{d}{ds}F^*_{\tilde{x}_r}(s) = -\frac{1}{E[\tilde{x}]}\lim_{s\to 0}\left[\frac{s\frac{d^3}{ds^3}F^*_{\tilde{x}}(s) + \frac{d^2}{ds^2}F^*_{\tilde{x}}(s)}{2}\right],$$

so that

$$E[\tilde{x}_r] = \frac{E[\tilde{x}^2]}{2E[\tilde{x}]}.$$

Next we obtain

$$\frac{d^2}{ds^2}F^*_{\tilde{x}}(s) = \frac{1}{E[\tilde{x}]}\left[\frac{2[1-F^*_{\tilde{x}}(s) + s\frac{d}{ds}F^*_{\tilde{x}}(s)] + s^2\frac{d^2}{ds^2}F^*_{\tilde{x}}(s)}{s^3}\right].$$

By applying L'Hôpital's rule four times, we find

$$\lim_{s\to 0}\frac{d^2}{ds^2}F^*_{\tilde{x}_r}(s) = \frac{E[\tilde{x}^3]}{3E[\tilde{x}]}.$$

Therefore,

$$E[\tilde{x}_r^2] = \frac{E[\tilde{x}^3]}{3E[\tilde{x}]}.$$

Consequently,

$$lim_{z\to 1}\beta^{''}(z) = \frac{-\rho\lambda^2}{3}\frac{E[\tilde{x}^3]}{E[\tilde{x}]}.$$

We may now specify

$$\begin{aligned}
\frac{d^2}{dz^2}\mathcal{F}_{\tilde{n}}(z)\bigg|_{z=1} &= E[\tilde{n}(\tilde{n}-1)] = E[\tilde{n}^2] - E[\tilde{n}] \\
&= \lambda^2 E[\tilde{x}^2] - \frac{2\rho(1-\rho)[-\lambda\rho]\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}}{(1-\rho)^2} \\
&\quad + \frac{(1-\rho)\rho\lambda^2}{(1-\rho)^2}\frac{E[\tilde{x}^3]}{3E[\tilde{x}]} + \frac{2(1-\rho)\lambda^2\rho^2}{(1-\rho)^3}\left[\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}\right]^2 \\
&= \lambda^2 E[\tilde{x}^2] + \frac{\rho^3}{(1-\rho)}\frac{E[\tilde{x}^2]}{2E^2[\tilde{x}]} + \frac{\rho^3}{3(1-\rho)}\frac{E[\tilde{x}^3]}{E^3[\tilde{x}]} \\
&\quad + \frac{2\rho^4}{4(1-\rho)^2}\left[\frac{E[\tilde{x}^2]}{E[\tilde{x}]}\right]^2 \\
&= \rho^2\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]} + \frac{\rho^3}{1-\rho}\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]} + \frac{\rho^3}{3(1-\rho)}\frac{E[\tilde{x}^3]}{E^3[\tilde{x}]} \\
&\quad + \frac{\rho^4}{2(1-\rho)^2}\left[\frac{E[\tilde{x}^2]}{E[\tilde{x}]}\right]^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[\tilde{n}^2] &= \rho^2\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]} + \frac{\rho^3}{1-\rho}\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]} + \frac{\rho^3}{3(1-\rho)}\frac{E[\tilde{x}^3]}{E^3[\tilde{x}]} + \\
&\quad \frac{\rho^4}{2(1-\rho)^2}\left\{\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]}\right\}^2 + \rho + \frac{\rho^2}{2(1-\rho)}\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]} \\
&= \frac{\rho^2}{2(1-\rho)}\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]}\left\{3 + \frac{\rho^2}{1-\rho}\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]}\right\} + \frac{\rho^3}{3(1-\rho)}\frac{E[\tilde{x}^3]}{E^3[\tilde{x}]} + \rho.
\end{aligned}$$

Alternatively, we may determine a formula for $E[\tilde{n}^2]$ using an operator argument as follows. Rewrite $\tilde{n} = \tilde{n}_1 + \tilde{n}_2$, where

$$\mathcal{F}_{\tilde{n}_1}(z) = (1-\rho)\left[1 - \rho F^*_{\tilde{x}_r}(\lambda[1-z])\right]^{-1},$$

and

$$\mathcal{F}_{\tilde{n}_2}(z) = F^*_{\tilde{x}}(\lambda[1-z]).$$

Then

$$\begin{aligned} E[\tilde{n}^2] &= E[(\tilde{n}_1 + \tilde{n}_2)^2] \\ &= E[\tilde{n}_1^2] + 2E[\tilde{n}_1]E[\tilde{n}_2] + E[\tilde{n}_2^2]. \end{aligned} \qquad (S5.2.1)$$

Now,

$$E[\tilde{n}_j] = \frac{d}{dz}\mathcal{F}_{\tilde{n}_j}(z)\Big|_{z=1}$$

and

$$E[\tilde{n}_j(\tilde{n}_j - 1)] = \frac{d^2}{ds^2}\mathcal{F}_{\tilde{n}_j}(z)\Bigg|_{z=1},$$

or, rearranging terms,

$$\begin{aligned} E[\tilde{n}_j^2] &= \frac{d^2}{dz^2}\mathcal{F}_{\tilde{n}_j}(z)\Big|_{z=1} + \frac{d}{dz}\mathcal{F}_{\tilde{n}_j}(z)\Big|_{z=1} \\ &= \frac{d^2}{dz^2}\mathcal{F}_{\tilde{n}_j}(z)\Big|_{z=1} + E[\tilde{n}_j]. \end{aligned}$$

Then, for $\tilde{n}_1$ and $\tilde{n}_2$ as defined above, and for $s(z) = \lambda[1-z]$,

$$\begin{aligned} E[\tilde{n}_1] &= \left\{-(1-\rho)\left[1 - \rho F^*_{\tilde{x}_r}(\lambda[1-z])\right]^{-2}\left(\rho\lambda\frac{d}{ds}F^*_{\tilde{x}_r}(s(z))\right)\right\}\Bigg|_{z=1} \\ &= \frac{1-\rho}{(1-\rho)^2}(\rho\lambda)E[\tilde{x}_r], \end{aligned}$$

and

$$E[\tilde{n}_2] = \left\{-\lambda\frac{d}{ds}F^*_{\tilde{x}}(s)\right\}\Bigg|_{z=1} = \rho.$$

The second moments of $\tilde{n}_1$ and $\tilde{n}_2$ are thus

$$\begin{aligned} E[\tilde{n}_1^2] = &\left\{2(1-\rho)\left[1 - \rho F^*_{\tilde{x}_r}(\lambda[1-z])\right]^{-3}\left[\rho\lambda\frac{d}{ds}F^*_{\tilde{x}_r}(s)\right]^2\right. \\ &+(1-\rho)\left[1 - \rho F^*_{\tilde{x}_r}(\lambda[1-z])\right]^{-2} \\ &\left.\cdot\left[\rho\lambda\frac{d^2}{ds^2}F^*_{\tilde{x}_r}(s(z))\right]\right\}\Bigg|_{z=1} + E[\tilde{n}_1] \end{aligned}$$

$$= \frac{2(1-\rho)}{(1-\rho)^3}(\rho\lambda)^2 E^2[\tilde{x}_r] + \frac{(1-\rho)}{(1-\rho)^2}\rho\lambda^2 E[\tilde{x}_r^2] + \frac{\rho\lambda}{1-\rho}E[\tilde{x}_r],$$

and

$$\begin{aligned} E[\tilde{n}_2^2] &= \left\{ (-\lambda)^2 \frac{d^2}{ds^2} F^*_{\tilde{x}}(s(z)) \right\}\Big|_{z=1} + E[\tilde{n}_1] \\ &= \lambda^2 E[\tilde{x}^2] + \rho. \end{aligned}$$

We may now substitute these results into (S5.2.1) to obtain

$$\begin{aligned} E[\tilde{n}^2] &= \frac{2}{(1-\rho)^2}(\rho\lambda)^2 E^2[\tilde{x}_r] + \frac{\rho}{1-\rho}\lambda^2 E[\tilde{x}_r^2] + \frac{\rho\lambda}{1-\rho}E[\tilde{x}_r] \\ &\qquad + 2\frac{\rho^2\lambda}{1-\rho}E[\tilde{x}_r] + \lambda^2 E[\tilde{x}_r] + \rho \\ &= \frac{2\rho^2\lambda^2}{(1-\rho)^2}\left(\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}\right)^2 + \frac{\rho\lambda}{1-\rho}\frac{E[\tilde{x}^2]}{2E[\tilde{x}]} + 2\frac{\rho^2\lambda}{1-\rho}\frac{E[\tilde{x}^2]}{2E[\tilde{x}]} \\ &\qquad + \lambda^2 E[\tilde{x}^2] + \rho + \frac{\rho\lambda^2}{1-\rho}E[\tilde{x}_r^2] \\ &= \frac{\rho^2}{(1-\rho)^2}\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}\left\{ \frac{2\rho^2 E[\tilde{x}^2]}{2E^2[\tilde{x}]} + 3(1-\rho) \right\} + \rho + \frac{\rho\lambda^2}{1-\rho}\frac{E[\tilde{x}^3]}{3E[\tilde{x}]} \\ &= \frac{\rho^2}{2(1-\rho)}\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]}\left\{ \frac{\rho^2}{1-\rho}\frac{E[\tilde{x}^2]}{E^2[\tilde{x}]} + 3 \right\} + \rho + \frac{\rho\lambda^2}{3(1-\rho)}\frac{E[\tilde{x}^3]}{E[\tilde{x}]}. \end{aligned}$$

This is the same expression as was found using the first method.

5-3 Jobs arrive to a single server system at a Poisson rate λ. Each job consists of a random number of tasks, $\tilde{m}$, drawn from a general distribution $F_{\tilde{m}}(m)$, independent of everything. Each task requires a service time drawn from a common distribution, $F_{\tilde{x}_t}$, independent of everything.

(a) Determine the Laplace-Stieltjes transform of the job service-time distribution.

(b) Determine the mean forward recurrence time of the service-time distribution using the result of part (a) and transform properties.

(c) Determine the stochastic equilibrium mean sojourn time for jobs in this system.

(d) Determine the mean number of tasks remaining for a job in service at an arbitrary point in time, if any.

Solution:

(a) Since the time between interruptions is exponentially distributed with parameter β, the process that counts the interruptions is a Poisson process. Hence

$$P\left\{\tilde{n}=n|\tilde{x}=x\right\}=\frac{(\beta x)^n e^{-\beta x}}{n!}.$$

Therefore,

$$P\left\{\tilde{n}=n\right\}=\int_0^\infty \frac{(\beta x)^n e^{-\beta x}}{n!}dF_{\tilde{x}}(x),$$

and

$$\begin{aligned}
\mathcal{F}_{\tilde{n}}(z) &= \sum_{n=0}^{\infty} z^n P\left\{\tilde{n}=n\right\} \\
&= \sum_{n=0}^{\infty} z^n \infty \frac{(\beta x)^n e^{-\beta x}}{n!} dF_{\tilde{x}}(x) \\
&= \int_0^\infty \sum_{n=0}^{\infty} \frac{(z\beta x)^n}{n!} e^{-\beta x} F_{\tilde{x}}(x) \\
&= \int_0^\infty e^{z\beta x} e^{-\beta x} F_{\tilde{x}}(x).
\end{aligned}$$

Therefore

$$\mathcal{F}_{\tilde{n}}(z)=F^*_{\tilde{x}}(\beta[1-z]),$$

which, of course, is just the pgf for the distribution of the number of arrivals from a Poisson process over a period $\tilde{x}$.

(b) Let $\tilde{n}$ represent the total number of interruptions. Then $\tilde{c}=\tilde{x}+\sum_{i=0}^{\tilde{n}} \tilde{x}_{s_i}$. Therefore,

$$\begin{aligned}
E[e^{-s\tilde{c}}] &= E\left[e^{-s\left(\tilde{x}+\sum_{i=0}^{\tilde{n}} \tilde{x}_{s_i}\right)}\right] \\
&= \int_0^\infty \sum_{n=0}^{\infty} \left\{ E\left[e^{-s\left(\tilde{x}+\sum\limits_{i=0}^{\tilde{n}} \tilde{x}_{s_i}\right)} \Big|_{\tilde{n}=n,\tilde{x}=x} \right] \right. \\
&\qquad\qquad \left. P\left\{\tilde{n}=n|\tilde{x}=x\right\} \right\} dF_{\tilde{x}}(x) \\
&= \int_0^\infty \sum_{n=0}^{\infty} e^{-sx}\left[F^*_{\tilde{x}_s}(s)\right]^n \frac{(\beta x)^n}{n!} e^{-\beta x} dF_{\tilde{x}}(x) \\
&= \int_0^\infty e^{-sx} e^{\beta x F^*_{\tilde{x}_s}(s)} e^{-\beta x} dF_{\tilde{x}}(x).
\end{aligned}$$

Thus,

$$F_{\tilde{c}}(s) = F^*_{\tilde{x}}\left(s + \beta - \beta F^*_{\tilde{x}_s}(s)\right).$$

(c) The formula for the M/G/1 busy period is as follows:

$$F^*_{\tilde{y}}(s) = F^*_{\tilde{x}}\Big(s + \lambda - \lambda F^*_{\tilde{y}}(s)\Big).$$

The relationship between the two formulas is that if the service time of the special customer has the same distribution as the length of a busy period, then the completion time has the same distribution as the length of an ordinary busy period. The relationship is explained by simply allowing the first customer of the busy period to be interrupted by any other arriving customer. Then each interruption of the first customer has the same distribution as the ordinary busy period. When service of the first customer is complete, the busy period is over. Thus, the busy period relationship is simply a special case of the completion time.

(d) Since $\tilde{c} = \tilde{x} + \sum_{i=0}^{\tilde{n}} \tilde{x}_{s_i}$,

$$E[\tilde{c}|x] = x + \beta x E[\tilde{x}_s],$$

which implies

$$E[\tilde{c}] = E[\tilde{x}] + \beta E[\tilde{x}]E[\tilde{x}_s].$$

Therefore

$$E[\tilde{c}] = E[\tilde{x}](1 + \rho_s),$$

where $\rho_s = \beta E[\tilde{x}_s]$. As a check,

$$\frac{d}{ds}F^*_{\tilde{c}}(s)\Big|_{s=0} = \frac{d}{ds}F^*_{\tilde{x}}(s)\Big|_{s=0}\left[1 - \beta\frac{d}{ds}F_{\tilde{x}_s}(s)\right]\Big|_{s=0}.$$

It follows that

$$E[\tilde{c}] = E[\tilde{x}]\left(1 + \beta E[\tilde{x}_s]\right).$$

Similarly,

$$\frac{d^2}{ds^2}F_{\tilde{c}}(s) = \frac{d^2}{ds^2}F^*_{\tilde{x}}(s)\left[1 - \beta\frac{d}{ds}F^*_{\tilde{x}}(s)\right]^2 + \frac{d}{ds}F^*_{\tilde{x}}(s)\left[-\beta\frac{d^2}{ds^2}F^*_{\tilde{x}}(s)\right].$$

Taking the limit as $s \to 0$,

$$E[\tilde{c}^2] = E[\tilde{x}^2]\left(1 + \beta E[\tilde{x}_s]\right)^2 + E[\tilde{x}]\beta E[\tilde{x}_s^2].$$

(e) Note that the probability that the server is busy is just the expected value of the number of customer is service. That is, if $\mathcal{B}$ represents the event the the server is busy,

$$P\{\mathcal{B}\} \quad = \quad \lambda E[\tilde{c}] = \lambda E[\tilde{x}](1 + \rho_s)$$

$$= \quad \rho(1+\rho_s).$$

The stability condition is then $\rho(1+\rho_s) < 1$. Rewriting ρ as $\lambda E[\tilde{x}]$ the stability condition is then

$$\lambda < \frac{1}{E[\tilde{x}]} \frac{1}{(1+\rho_s)}.$$

5-4 Consider a queueing system in which ordinary customers have service times drawn from a general distribution with mean $1/\mu$. There is a special customer who receives immediate service whenever she enters the system, her service time being drawn, independently on each entry, from a general distribution, $F_{\tilde{x}_s}(x)$, which has mean $1/\alpha$. Upon completion of service, the special customer departs the system and then returns after an exponential, rate β, length of time. Let $\tilde{x}_{si}$ denote the length of the ith interruption of an ordinary customer by the special customer, and let $\tilde{n}$ denote the number of interruptions. Also, let $\tilde{c}$ denote the time that elapses from the instant an ordinary customer enters service until the instant the ordinary customer departs.

(a) Suppose that service time for the ordinary customer is chosen once. Following an interruption, the ordinary customer's service resumes from the point of interruption. Determine $P\left\{\tilde{n}=n|\tilde{x}=x\right\}$, the conditional probability that the number of interruptions is n, and $\mathcal{F}_{\tilde{n}}(z)$, the probability generating function for the number of interruptions suffered by the ordinary customer.

(b) Determine $F_{\tilde{c}}^*(s)$, the Laplace-Stieltjes transform for $\tilde{c}$ under this policy. [*Hint*: Condition on the the length of the service time of the ordinary customer and the number of service interruptions that occur.]

(c) Compare the results of part (b) with the Laplace-Stieltjes transform for the length of the M/G/1 busy period. Explain the relationship between these two results.

(d) Determine $E[\tilde{c}]$ and $E[\tilde{c}^2]$.

(e) Determine the probability that the server will be busy at an arbitrary point in time in stochastic equilibrium, and the stability condition for this system.

Solution:

(a) Since the time between interruptions is exponentially distributed with parameter β, the process that counts the interruptions is a Poisson process. Hence

$$P\left\{\tilde{n}=n|\tilde{x}=x\right\}=\frac{(\beta x)^n e^{-\beta x}}{n!}.$$

Therefore,

$$P\left\{\tilde{n}=n\right\}=\int_0^\infty \frac{(\beta x)^n e^{-\beta x}}{n!}dF_{\tilde{x}}(x),$$

and

$$\begin{aligned}\mathcal{F}_{\tilde{n}}(z) &= \sum_{n=0}^{\infty} z^n P\left\{\tilde{n}=n\right\}\\ &= \sum_{n=0}^{\infty} z^n \infty \frac{(\beta x)^n e^{-\beta x}}{n!}dF_{\tilde{x}}(x)\\ &= \int_0^\infty \sum_{n=0}^{\infty} \frac{(z\beta x)^n}{n!}e^{-\beta x}F_{\tilde{x}}(x)\\ &= \int_0^\infty e^{z\beta x}e^{-\beta x}F_{\tilde{x}}(x).\end{aligned}$$

Therefore

$$\mathcal{F}_{\tilde{n}}(z)=F^*_{\tilde{x}}(\beta[1-z]),$$

which, of course, is just the pgf for the distribution of the number of arrivals from a Poisson process over a period $\tilde{x}$.

(b) Let $\tilde{n}$ represent the total number of interruptions. Then $\tilde{c}=\tilde{x}+\sum_{i=0}^{\tilde{n}}\tilde{x}_{s_i}$. Therefore,

$$\begin{aligned}E[e^{-s\tilde{c}}] &= E\left[e^{-s\left(\tilde{x}+\sum_{i=0}^{\tilde{n}}\tilde{x}_{s_i}\right)}\right]\\ &= \int_0^\infty \sum_{n=0}^{\infty}\left\{E\left[e^{-s\left(\tilde{x}+\sum_{i=0}^{\tilde{n}}\tilde{x}_{s_i}\right)}\Big|_{\tilde{n}=n,\tilde{x}=x}\right]\right.\\ &\qquad\left. P\left\{\tilde{n}=n|\tilde{x}=x\right\}\right\}dF_{\tilde{x}}(x)\\ &= \int_0^\infty \sum_{n=0}^{\infty} e^{-sx}\left[F^*_{\tilde{x}_s}(s)\right]^n \frac{(\beta x)^n}{n!}e^{-\beta x}dF_{\tilde{x}}(x)\\ &= \int_0^\infty e^{-sx}e^{\beta x F^*_{\tilde{x}_s}(s)}e^{-\beta x}dF_{\tilde{x}}(x).\end{aligned}$$

Thus,

$$F_{\tilde{c}}(s) = F^*_{\tilde{x}}\left(s + \beta - \beta F^*_{\tilde{x}_s}(s)\right).$$

(c) The formula for the M/G/1 busy period is as follows:

$$F^*_{\tilde{y}}(s) = F^*_{\tilde{x}}\left(s + \lambda - \lambda F^*_{\tilde{y}}(s)\right).$$

The relationship between the two formulas is that if the service time of the special customer has the same distribution as the length of a busy period, then the completion time has the same distribution as the length of an ordinary busy period. The relationship is explained by simply allowing the first customer of the busy period to be interrupted by any other arriving customer. Then each interruption of the first customer has the same distribution as the ordinary busy period. When service of the first customer is complete, the busy period is over. Thus, the busy period relationship is simply a special case of the completion time.

(d) Since $\tilde{c} = \tilde{x} + \sum_{i=0}^{\tilde{n}} \tilde{x}_{s_i}$,

$$E[\tilde{c}|x] = x + \beta x E[\tilde{x}_s],$$

which implies

$$E[\tilde{c}] = E[\tilde{x}] + \beta E[\tilde{x}]E[\tilde{x}_s].$$

Therefore

$$E[\tilde{c}] = E[\tilde{x}](1 + \rho_s),$$

where $\rho_s = \beta E[\tilde{x}_s]$. As a check,

$$\frac{d}{ds}F^*_{\tilde{c}}(s)\bigg|_{s=0} = \frac{d}{ds}F^*_{\tilde{x}}(s)\bigg|_{s=0}\left[1 - \beta\frac{d}{ds}F_{\tilde{x}_s}(s)\right]\bigg|_{s=0}.$$

It follows that

$$E[\tilde{c}] = E[\tilde{x}]\left(1 + \beta E[\tilde{x}_s]\right).$$

Similarly,

$$\frac{d^2}{ds^2}F_{\tilde{c}}(s) = \frac{d^2}{ds^2}F^*_{\tilde{x}}(s)\left[1 - \beta\frac{d}{ds}F^*_{\tilde{x}}(s)\right]^2 + \frac{d}{ds}F^*_{\tilde{x}}(s)\left[-\beta\frac{d^2}{ds^2}F^*_{\tilde{x}}(s)\right].$$

Taking the limit as $s \to 0$,

$$E[\tilde{c}^2] = E[\tilde{x}^2]\left(1 + \beta E[\tilde{x}_s]\right)^2 + E[\tilde{x}]\beta E[\tilde{x}_s^2].$$

(e) Note that the probability that the server is busy is just the expected value of the number of customer is service. That is, if $\mathcal{B}$ represents the event the the server is busy,

$$P\{\mathcal{B}\} = \lambda E[\tilde{c}] = \lambda E[\tilde{x}](1 + \rho_s)$$

$$= \quad \rho(1+\rho_s).$$

The stability condition is then $\rho(1+\rho_s) < 1$. Rewriting ρ as $\lambda E[\tilde{x}]$ the stability condition is then

$$\lambda < \frac{1}{E[\tilde{x}]}\frac{1}{(1+\rho_s)}.$$

5-5 Consider a queueing system that services customers from a finite population of K identical customers. Each customer, while not being served or waiting, *thinks* for an exponentially distributed length of time with parameter λ and then joins a FCFS queue to wait for service. Service times are drawn independently from a general service time distribution $F_{\tilde{x}}(x)$.

(a) Given the expected length of the busy period for this system, describe a procedure through which you could obtain the expected waiting time. [*Hint*: Use alternating renewal theory.]

(b) Given the expected length of the busy period with $K = 2$, describe a procedure for obtaining the expected length of the busy period for the case of $K = 3$.

Solution:

(a) First note if $\mathcal{B}$ denotes the event that the system is busy,

$$P\{\mathcal{B}\} = \frac{E[\tilde{y}]}{E[\tilde{y}] + E[\tilde{i}]}.$$

Then, by Little's result,

$$P\{\mathcal{B}\} = \lambda_{eff} E[\tilde{x}],$$

where λ_{eff} is the average arrival rate of jobs to the server. Since the customer are statistically identical, $\lambda_c = \frac{\lambda_{eff}}{K}$ is the job generation rate per customer. Now, a customer generates one job per cycle; hence λ_c is equal to the inverse of the expected cycle length. i.e.,

$$\lambda_c = \frac{1}{E[\tilde{t}] + E[\tilde{w}] + E[\tilde{x}]}.$$

Therefore, by Little's result,

$$P\{\mathcal{B}\} = K\frac{E[\tilde{x}]}{E[\tilde{t}] + E[\tilde{w}] + E[\tilde{x}]} = \frac{E[\tilde{y}]}{E[\tilde{y}] + E[\tilde{c}]}.$$

We may then solve for $E[\tilde{w}]$ to find

$$\begin{aligned}
E[\tilde{w}] &= \frac{KE[\tilde{x}]}{P\left\{\mathcal{B}\right\}} - E[\tilde{x}] - E[\tilde{t}] \\
&= \frac{E[\tilde{x}]}{P\left\{\mathcal{B}\right\}}\left[K - P\left\{\mathcal{B}\right\}\right] - E[\tilde{t}] \\
&= E[\tilde{x}]\left[\frac{K}{P\left\{\mathcal{B}\right\}} - 1\right] - E[\tilde{t}] \\
&= E[\tilde{x}]\left[\frac{KE[\tilde{y}] + KE[\tilde{i}]}{E[\tilde{y}]}\right] - E[\tilde{t}]] \\
&= E[\tilde{x}]\left[K + K\frac{E[\tilde{i}]}{E[\tilde{y}]} - 1\right] + E[\tilde{t}].
\end{aligned}$$

(b) Let $\tilde{y}_K$ denote the length of a busy period for a system having K customers, and let A_i denote the event that i arrivals occur during $\tilde{x}_1$. Further, define $\tilde{y}_i^j$ t be the length of a busy period starting with i customers in the service system and j total customers in the system. Then

$$E[\tilde{y}_1^K] = E[\tilde{x}] + \sum_{i=1}^{K-1} E\left[\tilde{y}_i^K | A_i\right] P\left\{A_i\right\}.$$

In particular, for $K = 3$,

$$\begin{aligned}
E[\tilde{y}_1^3] &= E[\tilde{x}] + \sum_{i=1}^{2} E\left[\tilde{y}_i^3 | A_i\right] P\left\{A_i\right\} \\
&= E[\tilde{x}] + E[\tilde{y}_1^3]P_1 + E[\tilde{y}_2^3]P_2.
\end{aligned}$$

Now, $\tilde{y}_2^3 = \tilde{y}_1^2 + \tilde{y}_1^3$, because the service can be reordered so that one customer stands aside while the busy period of the second customer completes and then the first customer is brought back and its busy period completes. From the point of view of the second customer, the whole population is only two, but the first customer sees all three customers. Thus,

$$E[\tilde{y}_1^3] = E[\tilde{x}] + E[\tilde{y}_1^3]P_1 + \left(E[\tilde{y}_1^2] + E[\tilde{y}_1^3]\right)P_2,$$

which implies

$$E[\tilde{y}_1^3](1 - P_1 - P_2) = E[\tilde{x}] + E[\tilde{y}_1^2]P_2.$$

That is,

$$E[\tilde{y}_1^3] = \frac{E[\tilde{x}] + E[\tilde{y}_1^2]P_2}{1 - P_1 - P_2},$$

where $E[\tilde{y}_1^2]$ is the given quantity, namely the length of the busy period if the population size is two.

5-6 For the M/G/∞ queueing system, it is well known that the stochastic equilibrium distribution for the number of busy servers is Poisson with parameter $\lambda E[\tilde{x}]$, where λ is the arrival rate and $\tilde{x}$ is the holding time.

(a) Suppose that $\tilde{x} = i$ with probability p_i for $i = 1, 2, 3$ with $p_1 + p_2 + p_3 = 1$. Determine the equilibrium distribution of the number of servers that are busy serving jobs of length i for $i = 1, 2, 3$, and the distribution of the number of servers that are busy serving all jobs.

(b) Determine the probability that a job selected at random from all of the jobs in service at an arbitrary point in time in stochastic equilibrium will have service time i, $i = 1, 2, 3$.

(c) Calculate the mean length of an arbitrary job that is in service at an arbitrary point in time in stochastic equilibrium.

(d) Suppose that job service times are drawn from an arbitrary distribution $F_{\tilde{x}}(x)$. Repeat part (c).

(e) What can be concluded about the distribution of remaining service time of a customer in service at an arbitrary point in time in stochastic equilibrium for the M/G/∞ system?

Solution:

(a) From the problem statement, the equilibrium distribution of the number of busy servers is Poisson with parameter $\lambda E[\tilde{x}]$. Now, if $\tilde{x} = i$ with probability p_i, then the arrival process consists of 3 independent arrival processes with arrival rates λp_i, $i = 1, 2, 3$. Since there are an infinite number of servers, we may consider each arrival stream as being served by an independent, infinite set of servers. Thus, if we denote by $\tilde{n}_i$ the number of busy servers for class i, then

$$P\left\{\tilde{n}_i = n\right\} = \frac{(\lambda p_i i)^{n_i} e^{-\lambda p_i i}}{n_i!}.$$

The number of jobs in service at an arbitrary point in time is simply $\tilde{n} = \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3$. Since this is the sum of three independent Poisson random variables, then by Property 1 of the Poisson process (page 44 of the text) we have that $\tilde{n}$ is Poisson distributed with parameter λ. i.e.,

$$P\left\{\tilde{n} = n\right\} = \frac{\left(\lambda \sum_{i=1}^{3} i p_i\right)^n}{n!} e^{\frac{-\lambda \sum_{i=1}^{3} i p_i}{n!}}$$

$$= \frac{(\lambda E[\tilde{x}])^n}{n!} e^{-\lambda E[\tilde{x}]},$$

as in the problem statement.

(b) Since $\tilde{n}$ is a Poisson random variable that is obtained by the addition of 3 independent Poisson random variables, it follows that the proportion of calls of type i is simply

$$\frac{\lambda i p_i}{\lambda \sum_{i=1}^{3} i p_i} = \frac{i p_i}{E[\tilde{x}]}.$$

That is, it is as though there is a Poisson random variable with parameter β, and each item is marked with color i with probability $\frac{\beta_i}{\beta}$.

(c) From part (b), the probability that a job is type i is $ip_i/E[\tilde{x}]$. Therefore, if $\tilde{x}_0$ denotes the length of an observed job, $P\{\tilde{x}_0 = x\} = ip_i/E[\tilde{x}]$, and

$$\begin{aligned} E[\tilde{x}_0] &= \sum_{i=1}^{n} i \frac{[ip_i]}{E[\tilde{x}]} = \sum_{i=1}^{n} \frac{[i^2 p_i]}{E[\tilde{x}]} \\ &= \frac{E[\tilde{x}^2]}{E[\tilde{x}]}. \end{aligned}$$

(d) By the same argument as in part (c),

$$P\{x \le \tilde{x}_0 \le x + dx\} = \frac{x dF_{\tilde{x}}(x)}{E[\tilde{x}]},$$

so that

$$E[\tilde{x}_0] = \int_0^\infty x \left[\frac{x dF_{\tilde{x}}(x)}{E[\tilde{x}]} \right] = \frac{E[\tilde{x}^2]}{E[\tilde{x}]}.$$

(e) The distribution of the remaining service time is the same as the distribution of the remaining service time of a renewal interval in a renewal process whose interval distribution is $F_{\tilde{x}}(x)$.

Chapter 6

THE M/G/1 QUEUEING SYSTEM WITH PRIORITY

EXERCISE 6.1 Argue the validity of (6.1).

Solution.

EXERCISE 6.2 Derive an expression for the Laplace-Stieltjes transform of the sojourn-time distribution for the M/G/1 system under the LCFS-PR discipline conditional on the customer's service time requirement. [*Hint*: See Exercise 5.13].

Solution. For ease of notation, let $\tilde{s}$ represent *in this solution only* the sojourn time of an arbitrary customer for a system having the LCFS-PR discipline. Then, conditioning the Laplace-Stieltjes transform of this variable on the service time requirement of the customer,

$$F_{\tilde{s}}^*(s) = \int_0^\infty E[e^{-s\tilde{s}}|\tilde{x} = x]dF_{\tilde{x}}(x).$$

But this conditional may in turn be conditioned on the number of customers who arrive during the system time of the arbitrary customer. i.e.,

$$\begin{aligned} E[e^{-s\tilde{s}}|\tilde{x} = x] &= \sum_{v=0}^\infty E[e^{-s\tilde{s}}|\tilde{x} = x, \tilde{v} = v]P\{\tilde{v} = v\} \\ &= \sum_{v=0}^\infty E[e^{-s\tilde{s}}|\tilde{x} = x, \tilde{v} = v]\frac{e^{-\lambda x}(\lambda x)^v}{v!}. \end{aligned}$$

Now, if v customers arrive during the system time of the arbitrary customer, then the sojourn time will be the service time requirement plus the busy periods generated by those v customers. But, each of the v customers have the same busy period distribution $\tilde{y}$, with $\tilde{y} = 0$ with probability 1. Hence,

$$E[e^{-s\tilde{s}}|\tilde{x} = x] = \sum_{v=0}^\infty E\left[e^{-s(x+\sum_{i=1}^v \tilde{y})}\right]\frac{e^{-\lambda x}(\lambda x)^v}{v!}$$

$$\begin{aligned}
&= \sum_{v=0}^{\infty} e^{-sx} E\left[\prod_{i=1}^{v} e^{-s\tilde{y}}\right] \frac{e^{-\lambda x}(\lambda x)^v}{v!} \\
&= \sum_{v=0}^{\infty} e^{-sx} \left(F_{\tilde{y}}^*(s)\right)^v \frac{e^{-\lambda x}(\lambda x)^v}{v!} \\
&= e^{-sx} e^{-\lambda x} \sum_{v=0}^{\infty} \frac{\left(\lambda x F_{\tilde{y}}^*(s)\right)^v}{v!} \\
&= e^{-sx} e^{-\lambda x} e^{\lambda x F_{\tilde{y}}^*(s)} \\
&= e^{-x\left[s+\lambda-\lambda F_{\tilde{y}}^*(s)\right]}.
\end{aligned}$$

Therefore,

$$F_{\tilde{s}}^*(s) = F_{\tilde{x}}^*\left(s + \lambda - \lambda F_{\tilde{y}}^*(s)\right).$$

EXERCISE 6.3 Compare the means and variances of the sojourn times for the ordinary M/G/1 system and the M/G/1 system under the LCFS-PR discipline.

Solution. To compute the mean of $F_{\tilde{s}_{LCFS-PR}}^*(s)$, recall that

$$F_{\tilde{s}_{LCFS-PR}}^*(s) = F_{\tilde{y}}^*(s).$$

Hence, by (5.22),

$$E[\tilde{s}_{LCFS-PR}] = \frac{E[\tilde{x}]}{1-\rho}.$$

Now, since

$$F_{\tilde{y}}^*(s) = F_{\tilde{x}}^*\left(s + \lambda - \lambda F_{\tilde{y}}^*(s)\right),$$

we may rewrite the LST of $\tilde{s}_{LCFS-PR}$ as

$$F_{\tilde{s}_{LCFS-PR}}^*(s) = F_{\tilde{x}}^*\left(u(s)\right),$$

where

$$u(s) = s + \lambda - \lambda F_{\tilde{y}}^*(s).$$

Thus,

$$\begin{aligned}
\frac{d^2}{ds^2} F_{\tilde{y}}^*(s) &= \frac{d}{ds}\left\{\frac{d}{du} F_{\tilde{x}}^*\left(u(s)\right) \frac{d}{ds} u(s)\right\} \\
&= \frac{d^2}{du^2} F_{\tilde{x}}^*\left(u(s)\right)\left[\frac{d}{ds} u(s)\right]^2 + \frac{d}{du} F_{\tilde{x}}^*\left(u(s)\right) \frac{d^2}{ds^2} u(s) \\
&= \frac{d^2}{du^2} F_{\tilde{x}}^*\left(u(s)\right)\left[1 - \lambda \frac{d}{ds} F_{\tilde{y}}^*(s)\right]^2
\end{aligned}$$

$$+\frac{d}{du}F^*_{\tilde{x}}\left(u(s)\right)\left[-\lambda\frac{d^2}{ds^2}F^*_{\tilde{y}}(s)\right].$$

We then evaluate this expression at $s = 0$ to find

$$\begin{aligned}
\left.\frac{d^2}{ds^2}F^*_{\tilde{y}}(s)\right|_{s=0} &= E[\tilde{x}^2]\left(1+\lambda E[\tilde{y}]\right)^2 + \left(-E[\tilde{x}]\right)\left(-\lambda E[\tilde{y}^2]\right) \\
&= E[\tilde{x}^2]\left(1+\frac{\rho}{1-\rho}\right)^2 - E[\tilde{x}]\left(-\lambda E[\tilde{y}^2]\right) \\
&= E[\tilde{y}^2],
\end{aligned}$$

where we have used the above result for $E[\tilde{y}]$. Then solving for $E[\tilde{y}^2]$,

$$E[\tilde{y}^2] = \frac{E[\tilde{x}^2]}{(1-\rho)^3},$$

so that

$$\begin{aligned}
\mathrm{Var}(\tilde{s}_{LCFS-PR}) &= \mathrm{Var}(\tilde{y}) \\
&= E[\tilde{y}^2] - E^2[\tilde{y}] \\
&= \frac{E[\tilde{x}^2]}{(1-\rho)^3} - \left(\frac{E[\tilde{x}]}{1-\rho}\right)^2 \\
&= \frac{E[\tilde{x}^2] - E^2[\tilde{x}] + \rho E^2[\tilde{x}]}{(1-\rho)^3} \\
&= \frac{E^2[\tilde{x}]}{(1-\rho)^3}\left[C^2_{\tilde{x}} + \rho\right].
\end{aligned}$$

It remains to compute $\mathrm{Var}(\tilde{s})$. Observe that $\tilde{s} = \tilde{w} + \tilde{x}$, where $\tilde{w}$ and $\tilde{x}$ are independent since the service time in question occurs after the waiting period is over. Thus $\mathrm{Var}(\tilde{s}) = \mathrm{Var}(\tilde{w}) + \mathrm{Var}(\tilde{x})$. Rewrite $F^*_{\tilde{w}}(s)$ as

$$F^*_{\tilde{w}}(s) = \frac{1-\rho}{1-\rho F^*_{\tilde{x}_r}(s)}.$$

Then

$$\begin{aligned}
\frac{d^2}{ds^2}F^*_{\tilde{w}}(s) &= \frac{d}{ds}\left\{(1-\rho)\left[1-\rho F^*_{\tilde{x}_r}(s)\right]^{-2}\rho\frac{d}{ds}F^*_{\tilde{x}_r}(s)\right\} \\
&= 2(1-\rho)\left[1-\rho F^*_{\tilde{x}_r}(s)\right]^{-3}\left[\rho\frac{d}{ds}F^*_{\tilde{x}_r}(s)\right]^2 \\
&\qquad +(1-\rho)\left[1-\rho F^*_{\tilde{x}_r}(s)\right]^{-2}\rho\frac{d^2}{ds^2}F^*_{\tilde{x}_r}(s).
\end{aligned}$$

Upon evaluating this expression at $s = 0$ and using the relation $E[\tilde{x}_r^n] = E[\tilde{x}^{n+1}]/(n+1)E[\tilde{x}]$, we find that

$$\begin{aligned}
E[\tilde{w}^2] &= 2\frac{1-\rho}{(1-\rho)^3}\rho^2 E^2[\tilde{x}_r] + \frac{1-\rho}{(1-\rho)^2}\rho E[\tilde{x}_r^2] \\
&= 2\frac{\rho^2}{(1-\rho)^2}\left(\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}\right)^2 + \left(\frac{\rho}{1-\rho}\right)\frac{E[\tilde{x}^3]}{3E[\tilde{x}]} \\
&= 2\left[\frac{\rho}{1-\rho}\frac{E[\tilde{x}^2]}{2E[\tilde{x}]}\right]^2 + \left(\frac{\rho}{1-\rho}\right)\frac{E[\tilde{x}^3]}{3E[\tilde{x}]} \\
&= 2\left[\frac{\rho E[\tilde{x}]}{1-\rho}\left(\frac{C_{\tilde{x}}^2+1}{2}\right)\right]^2 + \left(\frac{\rho}{1-\rho}\right)\frac{E[\tilde{x}^3]}{3E[\tilde{x}]}
\end{aligned}$$

Now, by (5.17),

$$E[\tilde{w}] = \frac{\rho E[\tilde{x}]}{1-\rho}\left(\frac{C_{\tilde{x}}^2+1}{2}\right),$$

so that the variance of $\tilde{w}$ is then

$$\begin{aligned}
\mathrm{Var}(\tilde{w}) &= E[\tilde{w}^2] - E^2[\tilde{w}] \\
&= 2\left[\frac{\rho E[\tilde{x}]}{1-\rho}\left(\frac{C_{\tilde{x}}^2+1}{2}\right)\right]^2 + \left(\frac{\rho}{1-\rho}\right)\frac{E[\tilde{x}^3]}{3E[\tilde{x}]} \\
&\qquad - \left[\frac{\rho E[\tilde{x}]}{1-\rho}\left(\frac{C_{\tilde{x}}^2+1}{2}\right)\right]^2 \\
&= \left[\frac{\rho E[\tilde{x}]}{1-\rho}\frac{C_{\tilde{x}}^2+1}{2}\right]^2 + \left(\frac{\rho}{1-\rho}\right)\frac{E[\tilde{x}^3]}{3E[\tilde{x}]}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Var}(\tilde{s}) &= \mathrm{Var}(\tilde{w}) + \mathrm{Var}(\tilde{x}) \\
&= \left[\frac{\rho E[\tilde{x}]}{1-\rho}\frac{C_{\tilde{x}}^2+1}{2}\right]^2 + \left(\frac{\rho}{1-\rho}\right)\frac{E[\tilde{x}^3]}{3E[\tilde{x}]} + E[\tilde{x}^2] - E^2[\tilde{x}] \\
&= E^2[\tilde{x}]\left\{\left[\frac{\rho}{1-\rho}\frac{(1+C_{\tilde{x}}^2}{2}\right]^2 + C_{\tilde{x}}^2\right\} + \left(\frac{\rho}{1-\rho}\right)\frac{E[\tilde{x}^3]}{3E[\tilde{x}]}.
\end{aligned}$$

Now,

$$E[\tilde{s}_{FCFS}] - E[\tilde{s}_{LCFS-PR}] = E[\tilde{x}]\left[1 + \frac{\rho}{1-\rho}\frac{1+C_{\tilde{x}}^2}{2}\right] - \frac{E[\tilde{x}]}{1-\rho}$$

$$
\begin{aligned}
&= E[\tilde{x}]\left[1+\frac{\rho}{1-\rho}\frac{1+C_{\tilde{x}}^2}{2}-\frac{1}{1-\rho}\right] \\
&= E[\tilde{x}]\left[\frac{\rho}{1-\rho}\frac{1+C_{\tilde{x}}^2}{2}-\frac{\rho}{1-\rho}\right] \\
&= \frac{\rho E[\tilde{x}]}{1-\rho}\left[\frac{1+C_{\tilde{x}}^2}{2}-1\right] \\
&= \frac{\rho E[\tilde{x}]}{1-\rho}\left[\frac{C_{\tilde{x}}^2-1}{2}\right].
\end{aligned}
$$

Thus, if $C_{\tilde{x}}^2 < 1$, then $E[\tilde{s}_{LCFS-PR}] < E[\tilde{s}_{FCFS}]$, and if $C_{\tilde{x}}^2 > 1$, then $E[\tilde{s}_{LCFS-PR}] > E[\tilde{s}_{FCFS}]$. If $C_{\tilde{x}}^2 = 1$, then $E[\tilde{s}_{LCFS-PR}] = E[\tilde{s}_{FCFS}]$. This result is clear for exponential service because of Little's result, which is reasoned as follows. Since the service time is memoryless, then upon each arrival, the service time of the customer in service is redrawn again from the same distribution as all other customers. Thus, all customers are alike, and it does not matter which order the service occurs; the distribution of the number of customers in the system is unaffected by order of service. Therefore, the average number of customers is unaffected by service order. Thus, by Little's result, $E[\tilde{s}_{LCFS-PR}] = E[\tilde{s}_{FCFS}]$.We now compare the variances of the two distributions.

$$
\begin{aligned}
\Delta_{\tilde{s}}\left(C_{\tilde{x}}^2\right) &= \mathrm{Var}(\tilde{s}_{FCFS}) - \mathrm{Var}(\tilde{s}_{LCFS-PR}) \\
&= E^2[\tilde{x}]\left\{\left[\frac{\rho}{1-\rho}\left(\frac{1+C_{\tilde{x}}^2}{2}\right)\right]^2 + C_{\tilde{x}}^2\right\} + \frac{\rho}{1-\rho}\frac{E[\tilde{x}^3]}{3E[\tilde{x}]} \\
&\quad -\frac{E^2[\tilde{x}]}{(1-\rho)^3}\left[C_{\tilde{x}}^2+\rho\right].
\end{aligned}
$$

That is,

$$
\begin{aligned}
\Delta_{\tilde{s}}\left(C_{\tilde{x}}^2\right) &= \frac{E^2[\tilde{x}]}{(1-\rho)^3}\left\{\frac{\rho^2(1-\rho)}{4}\left(1+C_{\tilde{x}}^2\right)^2\right. \\
&\quad \left. +(1-\rho)^3C_{\tilde{x}}^2 - C_{\tilde{x}}^2 - \rho\right\} + \frac{\rho}{1-\rho}\frac{E[\tilde{x}^3]}{3E[\tilde{x}]}.
\end{aligned}
$$

At $C_{\tilde{x}}^2 = 0$, $E[\tilde{x}^3] = E^3[\tilde{x}]$, so that for all ρ,

$$
\begin{aligned}
\Delta_{\tilde{s}}\left(C_{\tilde{x}}^2\right) &= \frac{E^2[\tilde{x}]}{(1-\rho)^3}\left\{\frac{\rho^2(1-\rho)}{4}-\rho\right\} + \frac{\rho}{1-\rho}\frac{E[\tilde{x}^3]}{3E[\tilde{x}]} \\
&= -\frac{E^2[\tilde{x}]\rho}{12(1-\rho)^3}\left\{8+4\rho+\rho(1-\rho)\right\} < 0.
\end{aligned}
$$

This means that var$(\tilde{s}_{LCFS-PR}) >$ var$(\tilde{s}_{FCFS})$ if $C^2_{\tilde{x}} = 0$ for deterministic service.
At $C^2_{\tilde{x}} = 1$, with exponential service,

$$\frac{E[\tilde{x}^2]}{3E[\tilde{x}]} = \frac{6E^3[\tilde{x}]}{3E[\tilde{x}]} = 2E^2[\tilde{x}].$$

Hence, at $C^2_{\tilde{x}} = 1$ and for all ρ,

$$\begin{aligned}
\Delta_{\tilde{s}}\left(C^2_{\tilde{x}}\right) &= \frac{E^2[\tilde{x}]}{(1-\rho)^3}\left\{\frac{\rho(1-\rho)4}{4} + (1-\rho)^3 - 1 - \rho + 2\rho(1-\rho)^2\right\} \\
&= \frac{E^2[\tilde{x}]}{(1-\rho)^3}\Bigg\{\rho - \rho^2 + 1 - 3\rho + 3\rho^2 - \rho^3 - 1 - \rho \\
&\qquad +2\rho - 4\rho^2 + 2\rho^3\Bigg\} \\
&= \frac{\rho E^2[\tilde{x}]}{(1-\rho)^3}\left\{-1 - 2\rho + \rho^2\right\} \\
&= -\frac{\rho E^2[\tilde{x}]}{(d1-\rho)^3}\left\{1 + 2\rho - \rho^2\right\} < 0.
\end{aligned}$$

Again, this means that var$(\tilde{s}_{LCFS-PR}) >$ var$(\tilde{s}_{FCFS})$ at $C^2_{\tilde{x}} = 1$. i.e., with exponential service. The question then arises as to whether $\mathrm{Var}(\tilde{s}_{LCFS-PR}) <$ var$(\tilde{s}_{FCFS})$ for any value of $C^2_{\tilde{x}}$. This question can be answered in part by considering

$$\Delta_{\tilde{s}}(C^2_{\tilde{x}} = \frac{E^2[\tilde{x}]}{(1-\rho)^3} f(C^2_{\tilde{x}}) + \frac{\rho}{1-\rho}\frac{E[\tilde{x}^3]}{3E[\tilde{x}]},$$

where

$$f(C^2_{\tilde{x}}) = \frac{\rho^2(1-\rho)}{4}(1 + C^2_{\tilde{x}}) + (1-\rho)^3 C^2_{\tilde{x}} - C^2_{\tilde{x}} + \rho.$$

Since $\tilde{x}$ is a nonnegative random variable, we conjecture that $E[\tilde{x}^3]$ is an increasing function of $C^2_{\tilde{x}}$. Now,

$$\begin{aligned}
\frac{d}{C^2_{\tilde{x}}} f(C^2_{\tilde{x}}) &= \frac{\rho}{2}\left\{\rho(1-\rho)C^2_{\tilde{x}} - [6 - 7\rho + 3\rho^2]\right\} \\
&= \frac{\rho^2(1-\rho)}{2}\left\{C^2_{\tilde{x}} - \frac{6 - 7\rho + 3\rho^2}{\rho(1-\rho)}\right\}.
\end{aligned}$$

This shows that for sufficiently large $C^2_{\tilde{x}}$, $f(C^2_{\tilde{x}})$ is an increasing function. That is, if

$$C^2_{\tilde{x}} > \frac{6 - 7\rho + 3\rho^2}{\rho(1-\rho)},$$

then $f(C^2_{\tilde{x}})$ is an increasing function. However, the minimum value of $(6 - 7\rho + 3\rho^2)/\rho(1-\rho)$ can be shown to be at $\rho = (3-\sqrt{3})/2$, and at this point, $C^2_{\tilde{x}} > 5 + 4\sqrt{3}$. Therefore, one would expect that var$(\tilde{s}_{LCFS-PR}) >$ var$(\tilde{s}_{FCFS})$ for $C^2_{\tilde{x}} < 5 + 4\sqrt{3}$, and that var$(\tilde{s}_{LCFS-PR}) <$ var$(\tilde{s}_{FCFS})$ for $C^2_{\tilde{x}} < 5 + 4\sqrt{3}$.

EXERCISE 6.4 Compare the probability generating function for the class 1 occupancy distributions for the HOL system to that of the M/G/1 system with set-up times discussed in Section 6.2. Do they have exactly the same form? Explain why or why not intuitively.

Solution. Recall Equation (6.36)

$$\mathcal{F}_{\tilde{n}_1}(z) = \left(\frac{1-\rho_1-\rho_2}{1-\rho_1} + \frac{\rho_2 F^*_{\tilde{x}_{2r}}(\lambda_1[1-z])}{1-\rho_1}\right)\frac{(1-\rho_1)F^*_{\tilde{x}_1}(\lambda_1[1-z])}{1-\rho_1 F^*_{\tilde{x}_{1r}}(\lambda_1[1-z])},$$

and (6.24)

$$\mathcal{F}_{\tilde{n}_s}(z) = \left(\frac{1}{1+\rho_s}F^*_{\tilde{x}_s}(\lambda[1-z]) + \frac{\rho_s}{1+\rho_s}F^*_{\tilde{x}_{sr}}(\lambda[1-z])\right)\mathcal{F}_{\tilde{n}}(z),$$

where (6.36) and (6.24) represent the probability generating functions for the occupancy distribution of HOL Class 1 customers and M/G/1 customers with set-up times, respectively. These probability generating functions have the exactly the same form in that they both are the probability generating function of the sum of two random variables, one of which is the probability generating function for the number left in the system in the ordinary M/G/1 system. In each case, the first part of the expression represents the number left due to arrivals that occur prior to the time an ordinary customer begins service in a busy period.

In the case of the priority system, either there is a Class 2 customer in service or there isn't. If so, then the probability generating function for the number of Class 1 customers who arrive is the same as for those who arrive during the residual service time of the Class 2 customer in service, namely $F^*_{\tilde{x}_{2_r}}(\lambda_1[1-z])$. Otherwise, the probability generating function is 1. The probability of no Class 2 customers in service is $1 - \frac{\rho_2}{1-\rho_1}$.

In the case of set-up, since this is before ordinary customers are serviced, either the customer in question in the first customer of the busy period or not. If so, the customer will leave behind all customers who arrive in the set-up time; otherwise the customer will leave behind only those who arrive after and during the set-up time. i.e., during the residual of the set-up time. The result is that in the case of the priority system, the set-up time is either 0 or $\tilde{x}_{2_r}$, depending upon whether or not a Class 2 customer is in service or not upon arrival of the first Class 1 customer of a Class 1 busy period. Recall Equation

(6.36)

$$\mathcal{F}_{\tilde{n}_1}(z) = \left(\frac{1-\rho_1-\rho_2}{1-\rho_1} + \frac{\rho_2 F^*_{\tilde{x}_{2r}}(\lambda_1[1-z])}{1-\rho_1} \right) \frac{(1-\rho_1)F^*_{\tilde{x}_1}(\lambda_1[1-z])}{1-\rho_1 F^*_{\tilde{x}_{1r}}(\lambda_1[1-z])},$$

and Equation (6.24)

$$\mathcal{F}_{\tilde{n}_s}(z) = \left(\frac{1}{1+\rho_s} F^*_{\tilde{x}_s}(\lambda[1-z]) + \frac{\rho_s}{1+\rho_s} F^*_{\tilde{x}_{sr}}(\lambda[1-z]) \right) \mathcal{F}_{\tilde{n}}(z),$$

where (6.36) and (6.24) represent the probability generating functions for the occupancy distribution of HOL Class 1 customers and M/G/1 customers with set-up times, respectively. These probability generating functions have the exactly the same form in that they both are the probability generating function of the sum of two random variables, one of which is the probability generating function for the number left in the system in the ordinary M/G/1 system. In each case, the first part of the expression represents the number left due to arrivals that occur prior to the time an ordinary customer begins service in a busy period.

In the case of the priority system, either there is a Class 2 customer in service or there isn't. If so, then the probability generating function for the number of Class 1 customers who arrive is the same as for those who arrive during the residual service time of the Class 2 customer in service, namely $F^*_{\tilde{x}_{2_r}}(\lambda_1[1-z])$. Otherwise, the probability generating function is 1. The probability of no Class 2 customers in service is $1 - \frac{\rho_2}{1-\rho_1}$.

In the case of set-up, since this is before ordinary customers are serviced, either the customer in question in the first customer of the busy period or not. If so, the customer will leave behind all customers who arrive in the set-up time; otherwise the customer will leave behind only those who arrive after and during the set-up time. i.e., during the residual of the set-up time. The result is that in the case of the priority system, the set-up time is either 0 or $\tilde{x}_{2_r}$, depending upon whether or not a Class 2 customer is in service or not upon arrival of the first Class 1 customer of a Class 1 busy period.

EXERCISE 6.5 Derive the expression for $\mathcal{F}_{\tilde{n}_2}(z)$ for the case of the HOL-PR discipline with $I = 2$.

Solution. The probability generating function for the number left in the system by a departing class 2 customer is readily derived from (6.45). As noted earlier, the term

$$\frac{1-\gamma_2}{1+\gamma_1} \left(\frac{(1-z) + \frac{\lambda_1}{\lambda_2}\{1 - F^*_{\tilde{y}_{11}}(\lambda_2[1-z])\}}{F^*_{\tilde{x}_{2c}}(\lambda_2[1-z]) - z} \right)$$

is the probability generating functionfor the distribution of the number of customers who arrive during the waiting time of the departing customer while the term

$$F^*_{\tilde{x}_2}(\lambda_2[1-z])$$

is the probability generating functionfor the distribution of the number of customers who arrive during the service time of the departing customer. Now, the distribution of the number of customers who arrive during the waiting time is the same whether or not servicing of the class 2 customer is preemptive. On the other hand, the number of class 2 customers who arrive after the time at which the class 2 customer enters service and the same customer's time of departure is equal to the number of class 2 customers who arrive during a completion time of the class 2 customer, the distribution of which is $F_{\tilde{x}_2c}(x)$. The probability generating functionof the number of class 2 arrivals during this time is then

$$\begin{aligned} F^*_{\tilde{x}_2c}(s)|_{s=\lambda_2[1-z]} &= F^*_{\tilde{x}_2}(s+\lambda_1-\lambda_1 F^*_{\tilde{y}_{11}}(s))|_{s=\lambda_2[1-z]} \\ &= F^*_{\tilde{x}_2}(\lambda_2[1-z]+\lambda_1-\lambda_1 F^*_{\tilde{y}_{11}}(\lambda_2[1-z])) \end{aligned}$$

The required probabiltiy generating function is then

$$\begin{aligned} \mathcal{F}_{\tilde{n}_2}(z) &= \frac{1-\gamma_2}{1+\gamma_1}\left(\frac{(1-z)+\frac{\lambda_1}{\lambda_2}\{1-F^*_{\tilde{y}_{11}}(\lambda_2[1-z])\}}{F^*_{\tilde{x}_{2c}}(\lambda_2[1-z])-z}\right) \\ &\qquad F^*_{\tilde{x}_2}(\lambda_2[1-z]+\lambda_1-\lambda_1 F^*_{\tilde{y}_{11}}(\lambda_2[1-z])) \end{aligned}$$

EXERCISE 6.6 Derive expressions for $\mathcal{F}_{\tilde{n}_1}(z)$, $\mathcal{F}_{\tilde{n}_2}(z)$, and $\mathcal{F}_{\tilde{n}_3}(z)$ for the ordinary HOL discipline with $I = 3$. Extend the analysis to the case of arbitrary I.

Solution. Consider the number of customers left by a class j departing customer by separating the class j customers remaining into two sub-classes: those who arrive during a sub-busy period started by a customer of their own class are Type 1; all others are Type 2. As before, the order of service is that the service takes place as a sub-busy period of a sequence of sub-busy periods, each being started by a Type 2 customer and generated by Type 1 and higher priority customers. Then, $\tilde{n}_j = \tilde{n}_{j1} + \tilde{n}_{j2}$. Since these customers arrive in non-overlapping intervals and are due to Poisson processes, $\tilde{n}_{j1}$ and $\tilde{n}_{j2}$ are independent.

As we have argued previously, $\tilde{n}_{j1}$ is the sum of two independent random variables. The first is the number of Type 1 customers who arrive during the waiting time of the tagged customer, which has the same distribution as the corresponding quantity in an ordinary M/G/1 queueing system having traffic intensity γ_j and service time equivalent to the completion time of a class j

customer. The pgf for the distribution of the number of Type 1 customers in this category is

$$\frac{(1-\gamma_j)}{1-\gamma_j F^*_{\tilde{x}_{jcr}}\left(\lambda_j[1-z]\right)}. \tag{6.6.1}$$

In turn, $F^*_{\tilde{x}_{jc}}(s) = F^*_{\tilde{y}_{j_H}}(s)$. That is, the completion time of a class j customer is simply the length of a sub-busy period started by a class j customer and generated by customers having priority higher than j. Furthermore,

$$F^*_{\tilde{y}_{jH}}(s) = F^*_{\tilde{x}_j}\left(s + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(s)\right) \qquad (6.6.2),$$

where $F^*_{\tilde{y}_{HH}}(s)$ satisfies

$$F^*_{\tilde{y}_{HH}}(s) = F^*_{\tilde{x}_H}\left(s + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(s)\right), \qquad (6.6.3),$$

which is the LST of the length of a busy period started by a high priority customer and generated by customers whose priority is higher than j. Specifically,

$$\lambda_H = \sum_{k=1}^{j-1} \lambda_k, \tag{6.6.4}$$

and

$$F^*_{\tilde{x}_H}(s) = \frac{1}{\lambda_H} \sum_{k=1}^{j-1} \lambda_k F^*_{\tilde{x}_k}(s), \tag{6.6.5}$$

with $\lambda_H = 0$ and $F^*_{\tilde{x}_H}(s)$ undefined if $j < 2$. In addition,

$$F_{\tilde{x}_{jcr}}(s) = \frac{1 - F^*_{\tilde{x}_{jc}}(s)}{sE[\tilde{x}_{jc}]} \tag{6.6.6}$$

and

$$E[\tilde{x}_{jc}] = \frac{E[\tilde{x}_j]}{1-\sigma_{j-1}}, \tag{6.6.7}$$

where

$$\sigma_j = \sum_{k=1}^{j} P_k.$$

Since $\gamma_j = \frac{\lambda_j E[\tilde{x}_j]}{1-\sigma_{j-1}}$,

$$\begin{aligned} \gamma_j F^*_{\tilde{x}_{jcr}}(s) &= \frac{\lambda_j E[\tilde{x}_j]}{1-\sigma_{j-1}} \frac{1 - F^*_{\tilde{x}_{jc}}(s)}{\frac{sE[\tilde{x}_j]}{1-\sigma_{j-1}}} \\ &= \frac{\lambda_j \left[1 - F^*_{\tilde{x}_{jc}}(s)\right]}{s} \end{aligned}$$

$$= \frac{\lambda_j \left[1 - F^*_{\tilde{y}_{jH}}(s)\right]}{s} \qquad (6.6.8)$$

Returning to the second component of $\tilde{n}_{j1}$, this is just the number of class j customers who arrive during the service time of the class j customer. The pgf for the distribution of the number of such customers is simply $F^*_{\tilde{x}_j}\left(\lambda_j[1-z]\right)$. Thus we find on making appropriate substitutions that

$$\begin{aligned}
\mathcal{F}_{\tilde{n}_{j1}}(z) &= \frac{(1-\gamma_j)F^*_{\tilde{x}_j}\left(\lambda_j[1-z]\right)}{1-\gamma_j F^*_{\tilde{x}_{jcr}}\left(\lambda_j[1-z]\right)} \\
&= \frac{(1-\gamma_j)F^*_{\tilde{x}_j}\left(\lambda_j[1-z]\right)}{1-\frac{\lambda_j\left[1-F^*_{\tilde{y}_{jH}}(\lambda_j[1-z])\right]}{\lambda_j[1-z]}} \\
&= \frac{(1-\gamma_j)(1-z)F^*_{\tilde{x}_j}\left(\lambda_j[1-z]\right)}{F^*_{\tilde{y}_{jH}}\left(\lambda_j[1-z]\right)-z} \\
&= \frac{(1-\gamma_j)(z-1)F^*_{\tilde{x}_j}\left(\lambda_j[1-z]\right)}{z-F^*_{\tilde{x}_j}\left(\lambda_j[1-z]+\lambda_H-\lambda_H F^*_{\tilde{y}_{HH}}(\lambda_j[1-z])\right)}
\end{aligned}$$

by (6.6.2). We now turn to the distribution of the number of Type 2 customers left in the system. Such customers arrive to the system when one of the following three events is occuring:

I, the event that the system is idle.

L, the event that the system is in a completion time of a lower priority customer excluding periods of time covered by event E.

H, the event that a higher priority customer is in service excluding periods covered by events E and L. We also define

E, the event that the system is in a completion time of a class j customer. The completion time of class j customers is equal to the length of a sub-busy period of higher priority customers started by a class j customer. Similarly, the completion time of a lower priority customer excluding periods of completion times of class j customers is equal to the length of a sub-busy period of a higher priority customers started by a lower priority customer. Since there can be at most 1 completion time in progress at any given time, the event probabilities are readily calculated by applying Little's result. Hence

$$P\{E\} = \frac{\lambda_j E[\tilde{x}_j]}{1-\rho_H} = \frac{\rho_j}{1-\rho_H} = \gamma_j,$$

and

$$P\{L\} = \frac{\lambda_L E[\tilde{x}_L]}{1-\rho_H} = \frac{\rho_L}{1-\rho_H}.$$

The proportion of time spent during events E and L serving higher priority customers is readily computed by simply subtracting ρ_j and ρ_L from $P\{E\}$ and $P\{L\}$ respectively. We then find

$$\begin{aligned} P\{H\} &= \rho_H - [P\{E\} - \rho_j] - [P\{L\} - \rho_L] \\ &= \rho_H - \left[\frac{\rho_j}{1-\rho_H} - \rho_j\right] - \left[\frac{\rho_L}{1-\rho_H} - \rho_H\right], \end{aligned}$$

$$P\{H\} = \frac{(1-\rho)\rho_H}{1-\rho_H}.$$

Next we find the probabilities

$$P\{I|\bar{E}\} = \frac{1-\rho}{1-\gamma_j}$$

$$P\{H|\bar{E}\} = \frac{(1-\rho)\rho_H}{(1-\rho_H)(1-\gamma_j)}$$

$$P\{L|\bar{E}\} = \frac{\rho_L}{(1-\rho_H)(1-\gamma_j)}$$

Now, the number of Type 2 customers left in the system by an arbitrary departure from the system is equal to the number of Type 2 customers left in the system by an arbitrary Type 2 arrival. This is because each Type 2 arrival is associated with exactly one Type 1 sub-busy period. Now, the number of Type 2 customers left behind if a Type 1 customer arrives to an empty system is 0. The number left behind for the event $\{H|\bar{E}\}$ is simply the number that arrive in the residual time of a high priority busy period started by a high priority customer, the pfg of which is given by $F^*_{\tilde{y}_{HHr}}(\lambda_j[1-z])$. Similarly, the number left behind for the event $\{L|\bar{E}\}$ is simply the number that arrive in the residual life of a high priority busy period started by a low priority customer, the pgf of which is given by $F^*_{\tilde{y}_{LHr}}(\lambda_j[1-z])$. Now, from exceptional first service results,

$$F^*_{\tilde{y}_{HH}}(s) = F^*_{\tilde{x}_H}\left(s + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(s)\right),$$

$$F^*_{\tilde{y}_{LH}}(s) = F^*_{\tilde{x}_L}\left(s + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(s)\right).$$

Also,

$$\begin{aligned} F^*_{\tilde{y}_{HHr}}(s) &= \frac{1 - F^*_{\tilde{y}_{HH}}(s)}{sE[\tilde{y}_{HH}]} \\ &= \frac{1 - F^*_{\tilde{y}_{HH}}(s)}{s\frac{E[\tilde{x}_H]}{1-\rho_H}} \\ &= \frac{(1-\rho_H)\left[1 - F^*_{\tilde{y}_{HH}}(s)\right]}{sE[\tilde{x}_H]}. \end{aligned}$$

Similarly,

$$F^*_{\tilde{y}_{LH}}(s) = \frac{(1-\rho_H)\left[1 - F^*_{\tilde{x}_L}\left(s + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(s)\right)\right]}{sE[\tilde{x}_L]}.$$

We then find that

$$\begin{aligned}\mathcal{F}_{\tilde{n}_{j2}}(z) &= \frac{1-\rho}{1-\gamma_j} + \frac{(1-\rho)\rho_H}{(1-\gamma_j)}\left\{\frac{\left[1 - F^*_{\tilde{y}_{HH}}\left(\lambda_j[1-z]\right)\right]}{\lambda_j[1-z]E[\tilde{x}_H]}\right\} + \\ &\quad \frac{\rho_L}{(1-\gamma_j)}\left\{\frac{\left[1 - F^*_{\tilde{x}_L}\left(\lambda_j[1-z] + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(\lambda_j[1-z])\right)\right]}{\lambda_j[1-z]E[\tilde{x}_L]}\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{F}_{\tilde{n}_j}(z) &= \left\{(1-\rho) + \frac{(1-\rho)\lambda_H\left[1 - F^*_{\tilde{y}_{HH}}\left(\lambda_j[1-z]\right)\right]}{\lambda_j[1-z]} + \right. \\ &\quad \left.\frac{\lambda_L\left[1 - F^*_{\tilde{x}_L}\left(\lambda_L[1-z] + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(\lambda_j[1-z])\right)\right]}{\lambda_j[1-z]}\right\} \\ &\quad \frac{(z-1)F^*_{\tilde{x}_j}\left(\lambda_j[1-z]\right)}{z - F^*_{\tilde{y}_{jH}}\left(\lambda_j[1-z]\right)}.\end{aligned}$$

We also note that since

$$\mathcal{F}_{\tilde{n}_j}(z) = F^*_{\tilde{s}_j}\left(\lambda_j[1-z]\right),$$

$$\begin{aligned}F^*_{\tilde{s}}(s) &= \left((1-\rho)\left\{1 + \frac{\lambda_H\left[1 - F^*_{\tilde{y}_{HH}}(s)\right]}{s}\right\} + \right. \\ &\quad \left.\lambda_L\left[\frac{1 - F^*_{\tilde{x}_L}\left(s + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(s)\right)}{s}\right]\right)\frac{F^*_{\tilde{x}_j}(s)}{1 - \lambda_j[1 - F^*_{\tilde{y}_{jH}}(s)]/s}.\end{aligned}$$

That is,

$$\begin{aligned}F^*_{\tilde{s}_j}(s) &= \left((1-\rho)\left\{s + \lambda_H\left[1 - F^*_{\tilde{y}_{HH}}(s)\right]\right\} + \right. \\ &\quad \left.\lambda_L\left[1 - F^*_{\tilde{x}_L}\left(s + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(s)\right)\right]\right)\frac{F^*_{\tilde{x}_j}(s)}{s - \lambda_j + \lambda_j F^*_{\tilde{y}_{jH}}(s)}\end{aligned}$$

EXERCISE 6.7 Extend the analysis of the previous case to the case of HOL-PR.

Solution. The solution to this problem is similar to that of Exercise 6.5, and proceeds as follows. The probability generating function for the number left in the system by a departing class j customer is readily derived from the results of Exercise 6.6, which are as follows

$$\begin{aligned}\mathcal{F}_{\tilde{n}_j}(z) &= \left\{(1-\rho) + \frac{(1-\rho)\lambda_H\left[1 - F^*_{\tilde{y}_{HH}}\left(\lambda_j[1-z]\right)\right]}{\lambda_j[1-z]} + \right. \\ &\qquad \left. \frac{\lambda_L\left[1 - F^*_{\tilde{x}_L}\left(\lambda_L[1-z] + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(\lambda_j[1-z])\right)\right]}{\lambda_j[1-z]}\right\} \\ &\qquad \frac{(z-1)F^*_{\tilde{x}_j}\left(\lambda_j[1-z]\right)}{z - F^*_{\tilde{y}_{jH}}\left(\lambda_j[1-z]\right)},\end{aligned}$$

or

$$\begin{aligned}\mathcal{F}_{\tilde{n}}(z) &= \left\{(1-\rho) + \frac{(1-\rho)\lambda_H\left[1 - F^*_{\tilde{y}_{HH}}\left(\lambda_j[1-z]\right)\right]}{\lambda_j[1-z]} + \right. \\ &\qquad \left. \frac{\lambda_L\left[1 - F^*_{\tilde{x}_L}\left(\lambda_L[1-z] + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(\lambda_j[1-z])\right)\right]}{\lambda_j[1-z]}\right\} \\ &\qquad \left\{\frac{(z-1)}{z - F^*_{\tilde{y}_{jH}}\left(\lambda_j[1-z]\right)}\right\} F^*_{\tilde{x}_j}\left(\lambda_j[1-z]\right)\end{aligned}$$

Now, the term $F^*_{\tilde{x}_j}(\lambda_j[1-z])$ is the probability generating functionfor the distribution of the number of customers who arrive during the service time of the departing class j customer, while the remainder of the expression represents the probability generating function for the distribution of the number of customers who arrive during the waiting time of the class j customer. The distribution of the number of customers who arrive during the waiting time is the same whether or not servicing of the class j customer is preemptive. On the other hand, the number of class 2 customers who arrive after the time at which the class 2 customer enters service and the same customer's time of departure is equal to the number of class j customers who arrive during a completion time of the class j customer, the distribution of which is $F_{\tilde{x}_{jc}}(x)$. The probability generating function of the number of class j arrivals during this time is

$$\begin{aligned}F^*_{\tilde{x}_{jc}}(s)\Big|_{s=\lambda_j[1-z]} &= F^*_{\tilde{x}_j}(s + \lambda_H - \lambda_H F^*_{\tilde{y}_{11}}(s))\Big|_{s=\lambda_j[1-z]} \\ &= F^*_{\tilde{x}_j}(\lambda_j[1-z] + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(\lambda_j[1-z])).\end{aligned}$$

The required probabiltiy generating function is thus

$$\begin{aligned}\mathcal{F}_{\tilde{n}}(z) &= \Bigg\{(1-\rho) + \frac{(1-\rho)\lambda_H\left[1 - F^*_{\tilde{y}_{HH}}\left(\lambda_j[1-z]\right)\right]}{\lambda_j[1-z]} + \\ &\qquad \frac{\lambda_L\left[1 - F^*_{\tilde{x}_L}\left(\lambda_L[1-z] + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(\lambda_j[1-z])\right)\right]}{\lambda_j[1-z]}\Bigg\} \\ &\qquad \left\{\frac{(z-1)F^*_{\tilde{x}_j}(\lambda_j[1-z])}{z - F^*_{\tilde{y}_{jH}}(\lambda_j[1-z])}\right\} + \lambda_H - \lambda_H F^*_{\tilde{y}_{HH}}(\lambda_j[1-z]).\end{aligned}$$

EXERCISE 6.8 Suppose that the service time of the customers in an M/G/1 system are drawn from the distribution $F_{\tilde{x}_i}(x)$ with probability p_i such that $\sum_{i=1}^{I} p_i = 1$. Determine $E[\tilde{w}]$ for this system.

Solution. Let S_i denote the even that distribution i is selected. Then, given $\tilde{x}$ is drawn from $F_{\tilde{x}_i}(x)$ with probability p_i, we find

$$\begin{aligned}E[\tilde{x}] &= \sum_{i=1}^{I} E[\tilde{x}|S_i]P\left\{S_i\right\} \\ &= \sum_{i=1}^{I} E[\tilde{x}_i]p_i.\end{aligned}$$

Similarly,

$$E[\tilde{x}^2] = \sum_{i=1}^{I} E[\tilde{x}_i^2]p_i.$$

Therefore,

$$E[\tilde{x}_r] = \frac{\sum\limits_{i=1}^{I} E[\tilde{x}_i^2]p_i}{2\sum\limits_{i=1}^{I} E[\tilde{x}_i]p_i}.$$

From (5.96), we then have

$$\begin{aligned}E[\tilde{w}] &= \frac{\rho}{1-\rho}E[\tilde{x}_r] \\ &= \frac{\rho}{1-\rho} \cdot \frac{\sum\limits_{i=1}^{I} E[\tilde{x}_i^2]p_i}{2\sum\limits_{i=1}^{I} E[\tilde{x}_i]p_i}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda}{2(1-\rho)} \sum_{i=1}^{I} E[\tilde{x}_i^2] p_i \\
&= \frac{1}{2(1-\rho)} \sum_{i=1}^{I} \lambda_i E[\tilde{x}_i^2] \\
&= \frac{1}{1-\rho} \sum_{i=1}^{I} \lambda_i E[\tilde{x}_i] \frac{E[\tilde{x}_i^2]}{2E[\tilde{x}_i]} \\
&= \frac{1}{1-\rho} \sum_{i=1}^{I} \rho_i E[\tilde{x}_{r_i}],
\end{aligned}
$$

where $\lambda_i = \lambda p_i$ and $\rho_i = \lambda_i E[\tilde{x}_i]$.

EXERCISE 6.9 **Conservation Law (Kleinrock [1976]**) Under the conditions of Exercise 6.8, suppose the customers whose service times are drawn from the distribution $F_{\tilde{x}_i}(x)$ are assigned priority i and the service discipline is HOL. Show that $\sum_{i=1}^{I} \rho_i E[\tilde{w}_i] = \rho E[\tilde{w}]$ where $E[\tilde{w}]$ is as determined in Exercise 5.9. Explain the implications of this result. Does the result imply that the expected waiting time is independent of the priority assignment? Why or why not? If not, under what conditions would equality hold?

Solution. This exercise refers to nonpreemptive service or ordinary HOL, with

$$
\begin{aligned}
E[\tilde{w}_j] &= \frac{\sum_{i=1}^{I} E[\tilde{x}_{r_i}]\rho_i}{(1-\sigma_j)(1-\sigma_{j-1})} \\
&= \frac{\rho E[\tilde{x}_r]}{(1-\sigma_j)(1-\sigma_{j-1})}.
\end{aligned}
$$

We wish to show

$$
E[\tilde{w}_{\mathrm{FCFS}}] = \frac{\rho E[\tilde{x}_r]}{1-\rho},
$$

so that

$$
\begin{aligned}
\rho E[\tilde{w}_{\mathrm{FCFS}}] &= \rho \sum_{i=1}^{I} \rho_i E[\tilde{w}_i] \\
&= \rho E[\tilde{x}_r] \sum_{i=1}^{I} \frac{\rho_i}{(1-\sigma_i)(1-\sigma_{i-1})}.
\end{aligned}
$$

Clearly, it is sufficient to show

$$
\sum_{i=1}^{I} \frac{\rho_i}{(1-\sigma_i)(1-\sigma_{i-1})} = \frac{\rho}{1-\rho}.
$$

We prove the following proposition by induction, and the above result follows as a special case. *Proposition:* For all j,

$$\sum_{i=1}^{j} \frac{\rho_i}{(1-\sigma_i)(1-\sigma_{i-1})} = \frac{\sigma_j}{1-\sigma_j},$$

with $\sigma_0 = 0$. *proof:* Let T denote the truth set for the proposition. Clearly $1 \in T$. Suppose $j-1 \in T$. That is,

$$\sum_{i=1}^{j-1} \frac{\rho_i}{(1-\sigma_i)(1-\sigma_{i-1})} = \frac{\sigma_{j-1}}{1-\sigma_{j-1}}.$$

Then

$$\begin{aligned}
\sum_{i=1}^{j} \frac{\rho_i}{(1-\sigma_i)(1-\sigma_{i-1})} &= \sum_{i=1}^{j-1} \frac{\rho_i}{(1-\sigma_i)(1-\sigma_{i-1})} + \frac{\rho_j}{(1-\sigma_j)(1-\sigma_{j-1})} \\
&= \frac{\sigma_{j-1}}{(1-\sigma_{j-1})} + \frac{\rho_j}{(1-\sigma_j)(1-\sigma_{j-1})} \\
&= \frac{1}{(1-\sigma_{j-1})}\left[\frac{\sigma_{j-1}(1-\sigma_j)+\rho_j}{(1-\sigma_j)(1-\sigma_{j-1})}\right] \\
&= \frac{1}{(1-\sigma_{j-1})}\left[\frac{\sigma_j(1-\sigma_{j-1})}{1-\sigma_j}\right] \\
&= \frac{\sigma_j}{1-\sigma_j}.
\end{aligned}$$

Thus $j \in T$ for all j, and with $j = I$, we have

$$\sum_{i=1}^{I} \frac{\rho_i}{(1-\sigma_i)(1-\sigma_{i-1})} = \frac{\sigma_I}{1-\sigma_I} = \frac{\rho}{1-\rho}.$$

This proves the proposition. Continuing with the exercise, we have

$$\rho E[\tilde{w}_{\text{FCFS}}] = \sum_{j=1}^{I} \rho_j E[\tilde{w}_j],$$

or

$$E[\tilde{w}_{\text{FCFS}}] = \sum_{j=1}^{I} \frac{\rho_j}{\rho} E[\tilde{w}_j].$$

That is, $E[\tilde{w}]$ can be expressed as a weighted sum of the expected waiting times of individual classes, where the weighting factors are equal to the proportion of all service times devoted to the particular class. Thus,

$$E[\tilde{w}_{\text{FCFS}}] = \sum_{j=1}^{I} \frac{\lambda_j E[\tilde{x}_j]}{\lambda E[\tilde{x}]} E[\tilde{w}_j].$$

Now, $\lambda E[\tilde{w}_{\rm FCFS}] = E[\tilde{n}_q]$, and $\lambda_j E[\tilde{w}_j] = E[\tilde{n}_{q_j}]$. Hence

$$E[\tilde{n}_{q_{\rm FCFS}}] = \sum_{j=1}^{I} \frac{E[\tilde{x}_j]}{E[\tilde{x}]} E[\tilde{n}_{q_j}].$$

But

$$E[\tilde{n}_{q_{\rm HOL}}] = \sum_{j=1}^{I} E[\tilde{n}_{q_j}], \tag{6.6.1}$$

so that a sufficient condition for $E[\tilde{n}_{\rm HOL}] = E[\tilde{n}_{q_{\rm FCFS}}]$ is that $E[\tilde{x}_j] = E[\tilde{x}]$ for all j. Therefore, by Little's result, a sufficient condition for $E[\tilde{w}_{\rm HOL}] = E[\tilde{w}_{FCFS}]$ is that $E[\tilde{x}_j] = E[\tilde{x}]$ for all j. From (6.6.1), we see that

$$\lambda E[\tilde{w}_{q_{\rm HOL}}] = \sum_{j=1}^{I} \lambda_j E[\tilde{w}_j],$$

which implies

$$E[\tilde{w}_{q_{\rm HOL}}] = \sum_{j=1}^{I} \frac{\lambda_j}{\lambda} E[\tilde{w}_j] \neq \sum_{j=1}^{I} \frac{\rho_j}{\rho} E[\tilde{w}_j] = E[\tilde{w}_{q_{\rm FCFS}}].$$

That is, if the service times are not identical, the average waiting time may not be conserved.

Chapter 7

VECTOR MARKOV CHAIN ANALYSIS: THE M/G/1 AND G/M/1 PARADIGMS

EXERCISE 7.1 Suppose that $P\{\tilde{x} = 1\} = 1$, that is, the service time is deterministic with mean 1. Determine $\{a_k, k = 0, 1, \ldots\}$ as defined by (7.6).

Solution: if $P\{\tilde{x} = 1\} = 1$, then we find that $dF_{\tilde{x}}(x) = \delta(x-1)dx$, where $\delta(x)$ is the Dirac delta function. Thus, from (5.6),

$$\begin{aligned} a_k &= \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dF_{\tilde{x}}(x) \\ &= \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} \delta(x-1) dx \\ &= \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } k = 0, 1, 2, \cdots \end{aligned}$$

EXERCISE 7.2 Suppose that $\tilde{x}$ is a discrete valued random variable having support set $\mathcal{X} = \{x_0, x_1, \ldots, x_K\}$, where K is an integer. Define $\alpha_k = P\{\tilde{x} = x_k\}$ for $x_k \in \mathcal{X}$. Determine $\{a_k, k = 0, 1, \ldots\}$ as defined by (7.6). In order to get started, let $dF_{\tilde{x}}(x) = \sum_{x \in \mathcal{X}} \alpha_k \delta(x - x_k)$, where $\delta(x)$ is the Dirac delta function.

Solution: By (7.6),

$$\begin{aligned} a_k &= \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dF_{\tilde{x}}(x) \\ &= \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} \sum_{x_j \in \mathcal{X}} \alpha_j \cdot \delta(x - x_j) dx \\ &= \sum_{x_j \in \mathcal{X}} \alpha_j \int_0^\infty \frac{(\lambda x)^k e^{-\lambda x}}{k!} e^{-\lambda x} \delta(x - x_j) dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x_j \in \mathcal{X}} \alpha_j \frac{(\lambda x_j)^k e^{-\lambda x_j}}{k!} \\
&= \sum_{j=0}^{J} \alpha_j \frac{(\lambda x_j)^k e^{-\lambda x_j}}{k!} \quad \text{for } k = 0, 1, \cdots
\end{aligned}
$$

EXERCISE 7.3 Suppose that $\tilde{x}$ is an exponential random variable with parameter μ. Determine $\{a_k, k = 0, 1, \ldots\}$ as defined by (7.6)).

Solution: By (7.6),

$$
\begin{aligned}
a_k &= \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dF_{\tilde{x}}(x) \\
&= \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} \mu e^{-\lambda x} dx \\
&= \mu \left(\frac{\lambda}{\lambda+\mu}\right)^k \int_0^\infty \frac{[(\lambda+\mu)x]^k}{k!} e^{-(\lambda+\mu)x} dx \\
&= \frac{\mu}{\lambda+\mu} \left(\frac{\lambda}{\lambda+\mu}\right)^k \int_0^\infty \frac{(\lambda+\mu)\,[(\lambda+\mu)x]^k}{k!} e^{-(\lambda+\mu)x} dx
\end{aligned}
$$

But, the expression inside the integral is just the density of the Erlang-(k+1) distribution with parameter $(\lambda + \mu)$, so that the expression integrates to unity. Therefore,

$$
a_k = \frac{\mu}{\lambda+\mu} \left(\frac{\lambda}{\lambda+\mu}\right)^k, \quad \text{for } k = 0, 1, \cdots
$$

This result is obvious since a_k is just the probability that in a sequence of independent trials, there will be k failures prior to the first success.

EXERCISE 7.4 Suppose that $\tilde{x} = \tilde{x}_1 + \tilde{x}_2$, where $\tilde{x}_1$ and $\tilde{x}_2$ are exponential random variables with parameter μ_1 and μ_2, respectively. Determine $\{a_k, k = 0, 1, \ldots\}$ as defined by (7.6)).

Solution: First, we determine $dF_{\tilde{x}}(x) = f_{\tilde{x}}(x)dx$.

$$
\begin{aligned}
f_{\tilde{x}}(x) &= \int_0^x f_{\tilde{x}_2}(x-y) f_{\tilde{x}_1}(y) dy \\
&= \int_0^x \mu_2 e^{-\mu_2(x-y)} \mu_1 e^{-\mu_1 y} dy \\
&= \mu_1 \mu_2 e^{-\mu_2 x} \int_0^x e^{-(\mu_1-\mu_2)y} dy
\end{aligned}
$$

If $\mu_1 = \mu_2$, we find

$$
f_{\tilde{x}}(x) = \mu_1 \mu_2 e^{-\mu_2 x} x
$$

$$\begin{aligned} &= \mu_2^2 x e^{-\mu_2 x} \\ &= \mu_2(\mu_2 x)e^{-\mu_2 x}. \end{aligned}$$

If $\mu_1 \neq \mu_2$, we have

$$\begin{aligned} f_{\tilde{x}}(x) &= \frac{\mu_1\mu_2}{\mu_1-\mu_2} e^{-\mu_2 x} \int_0^x (\mu_1-\mu_2)\, e^{-(\mu_1-\mu_2)y} dy \\ &= \frac{\mu_1\mu_2}{\mu_1-\mu_2} e^{-\mu_2 x} \left[1 - e^{-(\mu_1-\mu_2)x}\right] \\ &= \frac{\mu_1\mu_2}{\mu_1-\mu_2} \left[e^{-\mu_2 x} - e^{-\mu_1 x}\right] \end{aligned}$$

Then, from Exercise 7.3, we find

$$\begin{aligned} a_k &= \frac{\mu_1\mu_2}{\mu_1-\mu_2}\left[\frac{1}{\mu_2}\left(\frac{\mu_2}{\lambda+\mu_2}\right)\left(\frac{\lambda}{\lambda+\mu_2}\right)^k - \frac{1}{\mu_1}\left(\frac{\mu_1}{\lambda+\mu_1}\right)\left(\frac{\lambda}{\lambda+\mu_1}\right)^k\right] \\ &= \frac{\mu_1}{\mu_1-\mu_2}\left(\frac{\mu_2}{\lambda+\mu_2}\right)\left(\frac{\lambda}{\lambda+\mu_2}\right)^k \\ &\qquad -\frac{\mu_2}{\mu_1-\mu_2}\left(\frac{\mu_1}{\lambda+\mu_1}\right)\left(\frac{\lambda}{\lambda+\mu_1}\right)^k \quad \text{for } k = 0, 1, \cdots \end{aligned}$$

EXERCISE 7.5 Show that the quantity $\mathcal{F}_{\tilde{q}}(1)$ corresponds to the stationary probability vector for the phase process.

Solution: From (5.37), we find

$$\mathcal{F}_{\tilde{q}}(z)\left[Iz - \mathcal{P}\mathcal{F}_{\tilde{a}}(z)\right] = \pi_0\left[z-1\right]\mathcal{P}\mathcal{F}_{\tilde{a}}(z)$$

So, with $z = 1$,

$$\mathcal{F}_{\tilde{q}}(1)\left[I - \mathcal{P}\mathcal{F}_{\tilde{a}}(1)\right] = 0$$

But, $\mathcal{F}_{\tilde{a}}(1) = I$, so the above equation becomes

$$\mathcal{F}_{\tilde{q}}(1)\left[I - P\right] = 0$$

i.e.,

$$\mathcal{F}_{\tilde{q}}(1) = \mathcal{F}_{\tilde{q}}(1)\mathcal{P}$$

Therefore, $\mathcal{F}_{\tilde{q}}(1)$ is the stationary probability vector for the Markov Chain having transition matrix $\mathcal{P}$; that is, for the phase process.

EXERCISE 7.6 Derive equation (7.38).

Solution: From equation (5,37), we have that

$$\mathcal{F}_{\tilde{q}}(z)\left[Iz - \mathcal{P}\mathcal{F}_{\tilde{a}}(z)\right] = \pi_0\left[z-1\right]\mathcal{P}\mathcal{F}_{\tilde{z}}(z)$$

Differentiating with respect to z, we find

$$\mathcal{F}_{\tilde{q}}'(z)\left[Iz - \mathcal{P}\mathcal{F}_{\tilde{a}}(z)\right] + \mathcal{F}_{\tilde{q}}(z)\left[I - \mathcal{P}\mathcal{F}_{\tilde{a}}'(z)\right] = \pi_0\mathcal{P}\mathcal{F}_{\tilde{z}}(z) + \pi_0\left[z-1\right]\mathcal{P}\mathcal{F}_{\tilde{z}}'(z)$$

Taking the limit as $z \to 1$, and recalling that $\mathcal{F}_{\tilde{a}}(1) = I$,

$$\mathcal{F}_{\tilde{q}}'(1)\left[I - \mathcal{P}\right] + \mathcal{F}_{\tilde{q}}(1)\left[I - \mathcal{P}\mathcal{F}_{\tilde{a}}'(1)\right] = \pi_0\mathcal{P}$$

Postmultiplying by $\mathbf{e}$,

$$\mathcal{F}_{\tilde{q}}'(1)\left[I - \mathcal{P}\right]\mathbf{e} + \mathcal{F}_{\tilde{q}}(1)\left[I - \mathcal{P}\mathcal{F}_{\tilde{a}}'(1)\right]\mathbf{e} = \pi_0\mathcal{P}\mathbf{e}$$

That is,

$$1 - \mathcal{F}_{\tilde{q}}(1)\mathcal{P}\mathcal{F}_{\tilde{a}}'(1)\mathbf{e} = \pi_0\mathbf{e}$$

EXERCISE 7.7 Suppose there are N identical traffic sources, each of which has an arrival process that is governed by an M-state Markov chain. Suppose the state of the combined phase process is defined by an M-vector in which the ith element is the number of sources currently in phase i. First, argue that this state description completely characterizes the phase of the arrival process. Next, show that the number of states of the phase process is given by $\binom{N+M-1}{N}$.

Solution: First note that the distribution of the number of arrivals from a given source depends solely upon the current phase of the source. Now, since all sources operate independently of each other, the distribution of the total number of arrivals due to all sources is simply the convolution of the distribution of the number of arrivals from the individual sources. Since the latter depends solely upon the phase of the individual sources, the totals number of sources in each phase is sufficient to determine the distribution of the number of arrivals. That is, if there are n_i sources in phase i, then the pgf for the number of arrivals is simply $\prod_{i=1}^{M} \mathcal{F}_i^{n_i}(z)$. Also, since the dynamics of the sources are identical, the future evolution depends only upon the number of sources in each phase and not upon the which sources are in which phase.

Now, if there are a total of N sources and M phases, then a typical state is represented by the M-tuple $(n_1, n_2, \cdots, n_M)$ with $n_i \geq 0$ for $i = 1, 2, \cdots, M$ and $\sum_{i=1}^{M} n_i = N$. We wish to show that the number of such states is

$$C_M^N = \binom{N+M-1}{N},$$

which we shall do by induction on M. That is, we have the proposition $C_M^N = \binom{N+M-1}{N}$ for all M, N positive integers. Let $\mathcal{T}$ denote the truth set of the proposition. Then

$$C_1^N = \binom{N+1-1}{N} = 1 \quad \text{for all } N,$$

so $M = 1 \in \mathcal{T}$. Assume

$$C_M^N = \binom{N+M-1}{N},$$

for all $n = 0, 1, 2, \cdots, N$, and $m = 1, 2, \cdots, M-1$. i.e., assume $1, 2, \cdots, M-1 \in \mathcal{T}$. If the last element of the M-tuple, x_M,$= i$, then the remaining elements sum to $N - i$ so that $\sum_{j=1}^{M-1} x_j = N - i$. Thus,

$$\begin{aligned} C_M^N &= \sum_{i=0}^{N} C_{M-1}^{N-i} \\ &= \sum_{i=0}^{N} \binom{N-i+M-1-1}{N-i} \\ &= \sum_{i=1}^{N} \binom{N-i+(M-1)-1}{N-i} + \binom{N+(M-1)-1}{N} \\ &= \sum_{i=0}^{N-1} \binom{N-i+(M-1)-1}{N-i} + \binom{N-(M-1)-1}{N} \end{aligned}$$

Now, we must show

$$\sum_{i=0}^{N} \binom{N-i+(M-1)-1}{N-i} = \binom{N+M-1}{N}$$

To do this, we prove

$$\sum_{i=0}^{N} \binom{N-i+x}{N-i} = \binom{N+x+1}{N},$$

by induction on N.

If $N = 0$, we have

$$\binom{0+x}{0} = 1 = \binom{x+1}{0}$$

Now assume

$$\sum_{i=0}^{N-1} \binom{N-1-i+x}{N-1-i} = \binom{N-1+x+1}{N-1}$$

Then

$$\begin{aligned}
\sum_{i=0}^{N}\binom{N-i+x}{N-i} &= \sum_{i=0}^{N}\binom{N-i+x}{N-i}+\binom{N+x}{N} \\
&= \sum_{i=0}^{N-1}\binom{N-1-i+x}{N-1-i}+\binom{N+x}{N} \\
&= \binom{N-1+x+1}{N-1}+\binom{N+x}{N} \\
&= \binom{N+x}{N-1}+\binom{N+x}{N} \\
&= \frac{(N+x)!}{(N-1)!\,(x+1)!}+\frac{(N+x)!}{N!x!} \\
&= \frac{(N+x)!}{(N-1)!\,(x+1)!}\left[\frac{1}{x+1}+\frac{1}{N}\right] \\
&= \frac{(N+x+1)!}{N!\,(x+1)!} \\
&= \binom{N+x+1}{N}
\end{aligned}$$

The result now follows with $x = M - 2$.

EXERCISE 7.8 Define the matrices $\mathcal{P}$ and $\mathcal{F}_{\tilde{a}}(a)$ for the model defined in Example 7.1 for the special case of $N = 3$, where the states of the phase process have the following interpretations shown in Table 7.2. In the table, if the phase vector is ijk, then there are i sources in phase 0, j sources in phase 1, and k sources in phase 2.

Solution: If the phase vector is (ijk) then there are i sources in phase 0, j sources in phase 1, and k sources in phase 2. for $i, j, k = 0, 1, 2, 3$. Phase 0 corresponds to the phase vector 300; i.e., all sources are in phase 0. Each source acts independently of all others, and the only transitions from phase 0 are to phase 0 or to phase 1 for individual sources to $0, 1, 2$, or 3. For transition $(0- > 0)$, all sources must remain in the 'off' state. Therefore, this transition has probability β^3. For $(0- > 1)$, 2 sources remain off and there is one transition to the 'on' state. Thus, the transition probability is $\binom{3}{1}\beta^2(1 - \beta) = 3\beta^2(1 - \beta)$. Continuing in this manner results in the following table of transition probabilities:

From		To	Probability
3 0 0	$->$	3 0 0	β^3
	$->$	2 1 0	$2\beta(1-\beta)$
	$->$	1 2 0	$3\beta(1-\beta)$
	$->$	0 3 0	$(1-\beta)^3$
2 1 0	$->$	2 0 1	β^2
	$->$	1 1 1	$2\beta(1-\beta)$
	$->$	0 2 1	$(1-\beta)^2$
1 2 0	$->$	1 0 2	β
	$->$	0 1 2	$(1-\beta)$
0 3 0	$->$	0 0 3	1
2 0 1	$->$	3 0 0	$(1-\alpha)\beta^2$
	$->$	2 1 0	$2\beta(1-\beta)(1-\alpha)+\beta^2\alpha$
	$->$	1 2 0	$2\beta(1-\beta)\alpha+(1-\beta)^2(1-\alpha)$
	$->$	0 3 0	$(1-\beta)^2\alpha$
1 1 1	$->$	2 0 1	$\beta(1-\alpha)$
	$->$	1 1 1	$(1-\beta)(1-\alpha)+\beta\alpha$
	$->$	0 2 1	$(1-\beta)\alpha$
0 2 1	$->$	0 1 2	α
	$->$	1 0 2	$(1-\alpha)$
1 0 2	$->$	3 0 0	$(1-\alpha)^2\beta$
	$->$	2 1 0	$2\beta(1-\alpha)\alpha+(1-\beta)(1-\alpha)^2$
	$->$	1 2 0	$\beta\alpha^2+2\alpha(1-\beta)(1-\alpha)$
	$->$	0 3 0	$(1-\beta)\alpha^2$
0 1 2	$->$	2 0 1	$(1-\alpha)^2$
	$->$	1 1 1	$2\alpha(1-\alpha)$
	$->$	0 2 1	α^2
0 0 3	$->$	3 0 0	$(1-\alpha)^3$
	$->$	2 1 0	$3\alpha(1-\alpha)^2$
	$->$	1 2 0	$3\alpha^2(1-\alpha)$
	$->$	0 3 0	α^3

EXERCISE 7.9 In the previous example, we specified $\mathcal{F}_{\tilde{a}}(z)$ and $\mathcal{P}$. From these specifications, we have

$$\mathcal{P}\mathcal{F}_{\tilde{a}}(z) = \sum_{i=0}^{2} \mathcal{A}_i z^i.$$

Therefore,

$$\mathcal{K}(1) = \sum_{i=0}^{2} \mathcal{A}_i \left[\mathcal{K}(1)\right]^i,$$

with

$$\mathcal{A}_0 = \begin{bmatrix} \beta^2 & 2\beta(1-\beta) & (1-\beta)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta(1-\alpha) & \beta\alpha + (1-\beta)(1-\alpha) & \alpha(1-\beta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ (1-\alpha)^2 & 2\alpha(1-\alpha) & \alpha^2 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & (1-\beta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-\alpha) & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathcal{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, suppose we compute $\mathcal{K}(1)$ iteratively; that is, we use the formula

$$\mathcal{K}_j(1) = \sum_{i=0}^{2} \mathcal{A}_i \left[\mathcal{K}_{j-1}(1)\right]^i \quad \text{for } j \geq 1, \tag{7.1}$$

with $\mathcal{K}_0(1) = 0$. Prove that the final three columns of $\mathcal{K}_j(1)$ are zero columns for all j.

Solution: The proof is by induction. Consider the proposition that the last 3 columns of $\mathcal{K}_j(1)$ are zero columns for all $j \geq 0$. Let $\mathcal{T}$ denote the truth set for this proposition. Since $\mathcal{K}_0(1) = 0$, $0 \in \mathcal{T}$. Also, $\mathcal{K}_1(1) = \mathcal{A}_0$, so that $1 \in \mathcal{T}$.

Now suppose $(j-1) \in \mathcal{T}$. Then,

$$\mathcal{K}_j(1) = \mathcal{A}_0 + \mathcal{A}_1\mathcal{K}_{j-1}(1) + \mathcal{A}_2\left[\mathcal{K}_{j-1}(1)\right]^2.$$

Now, if $\mathcal{D} = \mathcal{A}\mathcal{B}$, and the k-th column of $\mathcal{B}$ is a zero column, then the k-th column of $\mathcal{D}$ is also 0 because $c_{lk} = \sum_t a_{lt}b_{tk}$, and $b_{tk} = 0$ for all k. Therefore, with $\mathcal{A} = \mathcal{B} = \mathcal{K}_{j-1}(1)$, we have $[\mathcal{K}_{j-1}(1)]^2$ with its last 3 columns equal to zero, and with $\mathcal{A} = \mathcal{A}_2$, $\mathcal{B} = [\mathcal{K}_{j-1}(1)]^2$, we have $\mathcal{A}_2\left[\mathcal{K}_{j-1}(1)\right]^2$ with its last 3 columns equal to zero. Also, with $\mathcal{A} = \mathcal{A}_1$, $\mathcal{B} = \mathcal{K}_{j-1}(1)$, we have the last 3 columns of $\mathcal{A}_1\mathcal{K}_{j-1}(1) = 0$. Since this is also true of $\mathcal{A}_0$, the sum leading to $\mathcal{K}_j(1)$ also has its last 3 columns equal to zero. Hence, $j \in \mathcal{T}$, and the proof is complete.

EXERCISE 7.10 Suppose $\mathcal{K}(1)$ has the form

$$\mathcal{K}(1) = \begin{bmatrix} \mathcal{K}_{00} & 0 \\ \mathcal{K}_{10} & 0 \end{bmatrix}, \tag{7.2}$$

where $\mathcal{K}_{00}$ is a square matrix. Bearing in mind that $\mathcal{K}(1)$ is stochastic, prove that $\mathcal{K}_{00}$ is also stochastic and that κ has the form $[\,\kappa \quad 0\,]$, where κ_0 is the stationary probability vector for $\mathcal{K}_{00}$.

Solution: Recall that $\mathcal{K}(1)$ stochastic means that $\mathcal{K}(1)$ is a one-step transition probability matrix for a discrete parameter Markov chain. That is, all entries are nonnegative and the elements of every row sum to unity. Thus, $\mathcal{K}(1)\mathbf{e} = \mathbf{e}$. But

$$\mathcal{K}(1) = \begin{bmatrix} \mathcal{K}_{00} & 0 \\ \mathcal{K}_{10} & 0 \end{bmatrix},$$

and

$$\begin{aligned} \mathcal{K}(1)\mathbf{e} &= \begin{bmatrix} \mathcal{K}_{00} & 0 \\ \mathcal{K}_{10} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{e} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{K}_{00}\mathbf{e} \\ \mathcal{K}_{10}\mathbf{e} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{e} \\ \mathbf{e} \end{bmatrix} \end{aligned}$$

Therefore, the element of the rows of $\mathcal{K}_{00}$ sum to unity and are nonnegative because they come from $\mathcal{K}(1)$. Since $\mathcal{K}_{00}$ is square it is a legitimate one-step transition probability matrix for a discrete paramter Markov chain.

Now, recall that κ is the stationary vector of $\mathcal{K}(1)$; thus

$$\kappa \cdot \mathcal{K}(1) = \kappa,$$

with $\kappa\mathbf{e} = 1$. Therefore,

$$[\,\kappa_0 \quad \kappa_1\,] \begin{bmatrix} \mathcal{K}_{00} & 0 \\ \mathcal{K}_{10} & 0 \end{bmatrix} \quad = \quad [\,\kappa_0\mathcal{K}_{00} + \kappa_1\mathcal{K}_{10} \quad 0\,]$$

$$= \begin{bmatrix} \kappa_0 & \kappa_1 \end{bmatrix},$$

so that $\kappa_1 = 0$. Thus, $\kappa_0 \mathcal{K}_{00} = \kappa_0$, and we find that κ has the form $\begin{bmatrix} \kappa_0 & 0 \end{bmatrix}$. Furthermore, since $\kappa \mathbf{e} = \begin{bmatrix} \kappa_0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{e} & \mathbf{e} \end{bmatrix}^T = 1$, we find $\kappa_0 \mathbf{e} = 1$, so that κ_0 is the stationary probability vector for $\mathcal{K}_{00}$.

EXERCISE 7.11 Prove Theorem 7.3.

Solution: By definition,

$$\mathcal{PF}_{\tilde{a}}(z) = \sum_{i=0}^{\infty} \mathcal{A}_i z^i$$

Then, with $z = 1$,

$$\mathcal{PF}_{\tilde{a}}(1) = \mathcal{P} = \sum_{i=0}^{\infty} \mathcal{A}_i,$$

so that the latter sum is stochastic.

First, note that $[\mathcal{K}_0(1)]^n$ represents an n-step transition probability matrix for a discrete parameter Markov chain so that this matrix is stochastic. We may also show $[\mathcal{K}_0(1)]^n$ is stochastic by induction on n. Let $\mathcal{T}$ be the truth set for the propostion. Then $0 \in \mathcal{T}$ since $[\mathcal{K}_0(1)]^0 = I$, which is stochastic. Now assume that $0, 1, \cdots, n-1 \in \mathcal{T}$. Hence $[\mathcal{K}_0(1)]^{n-1}$ is nonnegative and $[\mathcal{K}_0(1)]^{n-1} \mathbf{e} = \mathbf{e}$. It follows that

$$\begin{aligned} [\mathcal{K}_0(1)]^n &= \mathcal{K}_0(1) \left[\mathcal{K}_0(1)\right]^{n-1} \\ &= \mathcal{K}_0(1)\mathbf{e} = \mathbf{e}. \end{aligned}$$

And clearly, $[\mathcal{K}_0(1)]^n$ is nonnegative for all n. Therefore, if $[\mathcal{K}_0(1)]^{n-1}$ is stochastic, then so is $\mathcal{K}^n(1)$. This proves the proposition.

Now, $\mathcal{A}_i$ is a matrix of probabilities and is therefore nonnegative. Thus,

$$\mathcal{K}_j(1) = \sum_{i=0}^{\infty} \mathcal{A}_i \left[\mathcal{K}_{j-1}\right]^i,$$

so that $\mathcal{K}_j(1)$ is nonnegative for all $j \geq 0$. Now,

$$\mathcal{K}_j(1)\mathbf{e} = \sum_{i=0}^{\infty} \mathcal{A}_i \left[\mathcal{K}_{j-1}\right]^i \mathbf{e},$$

and by the above proposition, $\left[\mathcal{K}_{j-1}\right]^i \mathbf{e} = \mathbf{e}$. Thus,

$$\mathcal{K}_j(1)\mathbf{e} = \sum_{i=0}^{\infty} \mathcal{A}_i \mathbf{e}$$

$$= \quad \mathcal{P}\mathbf{e} = \mathbf{e},$$

so that $\mathcal{K}_j(1)$ is also stochastic.

EXERCISE 7.12 Suppose that $\mathcal{P}$ is the one-step transition probability matrix for an irreducible discrete-valued, discrete-parameter Markov chain. Define ϕ to be the stationary probability vector for the Markov chain. Prove that $\mathbf{e}\phi\,[I - \mathcal{P} + \mathbf{e}\phi] = \mathbf{e}\phi$ and that, therefore, $\mathbf{e}\phi = \mathbf{e}\phi\,[1 - \mathcal{P} + \mathbf{e}\phi]^{-1}$.

Solution: Observe that

$$\mathbf{e}\phi\,[I - \mathcal{P} + \mathbf{e}\phi] = \mathbf{e}\phi - \mathbf{e}\phi\mathcal{P} + \mathbf{e}\phi\mathbf{e}\phi$$

But ϕ is the stationary probability vector for $\mathcal{P}$, so that $\phi\mathcal{P} = \phi$ and $\phi\mathbf{e} = 1$. Therefore,

$$\mathbf{e}\phi\,[I - \mathcal{P} + \mathbf{e}\phi] = \mathbf{e}\phi - \mathbf{e}\phi + \mathbf{e}\phi = \mathbf{e}\phi.$$

Thus, given that $[I - \mathcal{P} + \mathbf{e}\phi]$ is nonsingular, we find

$$\mathbf{e}\phi = \mathbf{e}\phi\,[I - \mathcal{P} + \mathbf{e}\phi]^{-1}$$

as required.

EXERCISE 7.13 Define

$$\mathcal{F}_{\tilde{q}}^{(1)}(z) = \frac{1}{z}\left[\mathcal{F}_{\tilde{q}}(z) - \pi_0\right] \text{ and } \mathcal{F}_{\tilde{q}}^{(i+1)}(z) = \frac{1}{z}\left[\mathcal{F}_{\tilde{q}}^{(i)}(z) - \pi_i\right], i \geq 1.$$

Starting with (7.49), substitute a function of $\mathcal{F}_{\tilde{q}}^{(1)}(z)$ for $\mathcal{F}_{\tilde{q}}(z)$, then a function of $\mathcal{F}_{\tilde{q}}^{(2)}(z)$ for $\mathcal{F}_{\tilde{q}}^{(1)}(z)$, and continue step by step until a function of $\mathcal{F}_{\tilde{q}}^{(C)}(z)$ is substituted for $\mathcal{F}_{\tilde{q}}^{(C-1)}(z)$. Show that at each step, one element of

$$\sum_{j=0}^{C-1} \pi_j z^j \mathcal{A}(z)$$

is eliminated, resulting in (7.51).

Solution: Starting with

$$\mathcal{F}_{\tilde{q}}^{(1)}(z) = \frac{1}{z}\left[\mathcal{F}_{\tilde{q}}(z) - \pi_0\right],$$

we find

$$\mathcal{F}_{\tilde{q}}(z) = z\mathcal{F}_{\tilde{q}}^{(1)}(z) + \pi_0.$$

For $i > 0$, we find

$$\mathcal{F}_{\tilde{q}}^{(i)}(z) = z\mathcal{F}_{\tilde{q}}^{(i+1)}(z) + \pi_i.$$

Given

$$\mathcal{F}_{\tilde{q}}(z)\left[z^C I - \mathcal{A}(z)\right] = \sum_{j=0}^{C-1} \pi_j \left[z^C \mathcal{B}_j(z) - z^j \mathcal{A}(z)\right], \tag{7.49}$$

we find

$$\left[z\mathcal{F}_{\tilde{q}}^{(1)}(z) + \pi_0\right]\left[z^C I - \mathcal{A}(z)\right] = \sum_{j=0}^{C-1} \pi_j \left[z^C \mathcal{B}_j(z) - z^j \mathcal{A}(z)\right],$$

or

$$z\mathcal{F}_{\tilde{q}}^{(1)}(z)\left[z^C I - \mathcal{A}(z)\right] = -\pi_0\left[z^C I - \mathcal{A}(z)\right] + \sum_{j=0}^{C-1} \pi_j \left[z^C \mathcal{B}_j(z) - z^j \mathcal{A}(z)\right].$$

Upon simplifying, we find

$$z\mathcal{F}_{\tilde{q}}^{(1)}(z)\left[z^C I - \mathcal{A}(z)\right] = -\pi_0\left[z^C I - \mathcal{A}(z)\right] + \sum_{j=0}^{C-1} \pi_j z^C \mathcal{B}_j(z) - \sum_{j=0}^{C-1} z^j \mathcal{A}(z).$$

or

$$z\mathcal{F}_{\tilde{q}}^{(1)}(z)\left[z^C I - \mathcal{A}(z)\right] = -\pi_0 z^C I + \sum_{j=0}^{C-1} \pi_j z^C \mathcal{B}_j(z) - \sum_{j=1}^{C-1} z^j \mathcal{A}(z).$$

We now substitute $z\mathcal{F}_{\tilde{q}}^{(2)}(z) + \pi_1$ for $\mathcal{F}_{\tilde{q}}^{(1)}(z)$ in the previous equation to find

$$z\left[z\mathcal{F}_{\tilde{q}}^{(2)}(z) + \pi_1\right]\left[z^C I - \mathcal{A}(z)\right] = -\pi_0 z^C I + \sum_{j=0}^{C-1} \pi_j z^C \mathcal{B}_j(z) - \sum_{j=1}^{C-1} z^j \mathcal{A}(z).$$

Simplifying, we get

$$z^2\mathcal{F}_{\tilde{q}}^{(2)}(z)\left[z^C I - \mathcal{A}(z)\right] = -\pi_0 z^C I - z\pi_1 z^C + \sum_{j=0}^{C-1} \pi_j z^C \mathcal{B}_j(z) - \sum_{j=2}^{C-1} z^j \mathcal{A}(z).$$

Continuing in this manner, we get

$$z^C\mathcal{F}_{\tilde{q}}^{(C)}(z)\left[z^C I - \mathcal{A}(z)\right] = -z^C \sum_{j=0}^{C-1} \pi_j z^j + \sum_{j=0}^{C-1} \pi_j z^C \mathcal{B}_j(z),$$

or

$$\mathcal{F}_{\tilde{q}}^{(C)}(z)\left[z^C I - \mathcal{A}(z)\right] = \sum_{j=0}^{C-1} \pi_j \left[\mathcal{B}_j(z) - z^j\right],$$

which is the required result.

EXERCISE 7.14 Beginning with (7.68), develop an expression for the first and second moments of the queue length distribution.

Solution: Starting with

$$\mathcal{F}_{\tilde{q}}(z) = \sum_{i=0}^{C-1} z^i \pi_i + z^C g \left[I - Fz\right]^{-1} H, \tag{7.68}$$

we just use the properties of moment generating functions to compute the required moments. Recall that for any integer valued, nonnegative random variable, $\tilde{x}$ we have $E\left[\tilde{x}\right] = \frac{d}{dz}\mathcal{F}_{\tilde{x}}(z)\|_{z=1}$ and $E\left[\tilde{x}^2\right] = \frac{d^2}{dz^2}\mathcal{F}_{\tilde{x}}(z)\|_{z=1} + \frac{d}{dz}\mathcal{F}_{\tilde{x}}(z)\|_{z=1}$. In this case, $E\left[\tilde{q}\right] = \frac{d}{dz}\mathcal{F}_{\tilde{q}}(1)\mathbf{e}$ because $\mathcal{F}_{\tilde{q}}(z)e$ is the generating function for the occupancy distribution. The appropriate derivatives are

$$\frac{d}{dz}\mathcal{F}_{\tilde{q}}(z) = \sum_{i=1}^{C-1} iz^{i-1}\pi_i + Cz^{C-1}g\left[I - Fz\right]^{-1} H + z^C gF\left[I - Fz\right]^{-2} H,$$

and

$$\begin{aligned}\frac{d^2}{dz^2}\mathcal{F}_{\tilde{q}}(z) = \sum_{i=2}^{C-1} i(i-1)z^{i-2}\pi_i + C(C-1)z^{C-2}g\left[I - Fz\right]^{-1} H \\ +2Cz^{C-1}gF\left[I - Fz\right]^{-2} H + z^C gF^2\left[I - Fz\right]^{-3} H.\end{aligned}$$

We then have

$$E\left[\tilde{q}\right] = \frac{d}{dz}\mathcal{F}_{\tilde{q}}(1)\mathbf{e} = \sum_{i=1}^{C-1} i\pi_i\mathbf{e} + Cg\left[I - F\right]^{-1} H\mathbf{e} + gF\left[I - F\right]^{-2} H\mathbf{e},$$

and

$$\begin{aligned}E\left[\tilde{q}^2\right] = \sum_{i=2}^{C-1} i(i-1)\pi_i\mathbf{e} + C(C-1)g\left[I - F\right]^{-1} H \\ +2CgF\left[I - F\right]^{-2} H\mathbf{e} + gF^2\left[I - F\right]^{-3} H\mathbf{e} + E\left[\tilde{q}\right].\end{aligned}$$

EXERCISE 7.15 Suppose $C = 2$. Define

$$\hat{A}_i = \begin{bmatrix} A_{2i} & A_{2i+1} \\ A_{2i-1} & A_{2i} \end{bmatrix}, i = 1, 2, \dots,$$

$$\hat{\mathcal{A}}(z) = \sum_{i=0}^{\infty} \hat{A}_i z^i.$$

and ϕ_2 to be the stationary vector of $\hat{\mathcal{A}}(1)$. Suppose

$$\rho = \phi_2 \left[\sum_{i=0}^{\infty} i\hat{A}_i \right] \mathbf{e}.$$

Show that

(a) $\phi_2 = \frac{1}{2} \left[\phi \quad \phi \right]$, where ϕ is the stationary vector of $\mathcal{A}(1)$.

(b)

$$\phi \left[\sum_{i=0}^{\infty} iA_i \right] \mathbf{e} = 2\rho.$$

Solution:

(a) Given

$$\hat{A}_i = \begin{bmatrix} A_{2i} & A_{2i+1} \\ A_{2i-1} & A_{2i} \end{bmatrix}, i = 1, 2, \dots,$$

we have

$$\hat{A}(z) = \begin{bmatrix} \sum_{i=0}^{\infty} A_{2i} z^i & \sum_{i=0}^{\infty} A_{2i+1} z^i \\ \sum_{i=0}^{\infty} A_{2i-1} z^i & \sum_{i=0}^{\infty} A_{2i} z^i \end{bmatrix}.$$

Upon setting $z = 1$, we have

$$\hat{A}(1) = \begin{bmatrix} \sum_{i=0}^{\infty} A_{2i} & \sum_{i=0}^{\infty} A_{2i+1} \\ \sum_{i=0}^{\infty} A_{2i-1} & \sum_{i=0}^{\infty} A_{2i} \end{bmatrix}.$$

Define

$$A_e = \sum_{i=0}^{\infty} A_{2i} \quad \text{and} \quad A_o = \sum_{i=0}^{\infty} A_{2i+1}.$$

We then have

$$\hat{A}(1) = \begin{bmatrix} A_e & A_o \\ A_o & A_e \end{bmatrix}.$$

Now, suppose $\phi_2 = \left[\phi_{2e} \quad \phi_{2o} \right]$ is the stationary vector of $\hat{\mathcal{A}}(1)$. Then, we have $\phi_2 = \phi_2 \hat{A}(1)$ and $\phi_2 \mathbf{e} = 1$. From $\phi_2 = \phi_2 \hat{A}(1)$, we have

$$\left[\phi_{2e} \quad \phi_{2o} \right] = \left[\phi_{2e} \quad \phi_{2o} \right] \begin{bmatrix} A_e & A_o \\ A_o & A_e \end{bmatrix}$$

from which we have

$$\phi_{2e} = \phi_{2e}A_e + \phi_{2o}A_o \quad \text{and} \quad \phi_{2o} = \phi_{2e}A_o + \phi_{2o}A_e.$$

Now suppose we set $\phi_{2e} = \phi_{2o} = \frac{1}{2}\phi$. We then have

$$\frac{1}{2}\phi = \frac{1}{2}\phi A_e + \frac{1}{2}\phi A_o = \frac{1}{2}\phi\left[A_e + A_o\right].$$

But, $A_e + A_o = \mathcal{A}(1)$, so we have $\phi\mathcal{A}(1) = \phi$ and $\phi\mathbf{e} = 1$ because ϕ is the stationary vector of $\mathcal{A}(1)$. Therefore, if we set $\phi_2 = \frac{1}{2}\left[\phi \quad \phi\right]$, then ϕ_2 satisfies $\phi_2 = \phi_2\hat{\mathcal{A}}(1)$, $\phi_2\mathbf{e} = 1$. That is, ϕ_2 is the unique stationary vector of $\hat{\mathcal{A}}(1)$.

(b) Given

$$\hat{A}(z) = \begin{bmatrix} \sum_{i=0}^{\infty} A_{2i}z^i & \sum_{i=0}^{\infty} A_{2i+1}z^i \\ \sum_{i=0}^{\infty} A_{2i-1}z^i & \sum_{i=0}^{\infty} A_{2i}z^i \end{bmatrix},$$

we have

$$\frac{d}{dz}\hat{A}(z) = \begin{bmatrix} \sum_{i=0}^{\infty} iA_{2i}z^{(i-1)} & \sum_{i=0}^{\infty} iA_{2i+1}z^{(i-1)} \\ \sum_{i=1}^{\infty} iA_{2i-1}z^{(i-1)} & \sum_{i=0}^{\infty} iA_{2i}z^{(i-1)} \end{bmatrix},$$

So that

$$\frac{d}{dz}\hat{A}(z)\|_{z=1} = \begin{bmatrix} \sum_{i=0}^{\infty} iA_{2i} & \sum_{i=0}^{\infty} iA_{2i+1} \\ \sum_{i=1}^{\infty} iA_{2i-1} & \sum_{i=0}^{\infty} iA_{2i} \end{bmatrix},$$

Hence,

$$\frac{d}{dz}\hat{A}(z)\Big|_{z=1} = \begin{bmatrix} \frac{1}{2}\sum_{i=0}^{\infty} 2iA_{2i} & \frac{1}{2}\sum_{i=0}^{\infty}(2i+1)A_{2i+1} - \frac{1}{2}\sum_{i=0}^{\infty} A_{2i+1} \\ \frac{1}{2}\sum_{i=1}^{\infty}(2i-1)A_{2i-1} + \frac{1}{2}\sum_{i=1}^{\infty} A_{2i-1} & \frac{1}{2}\sum_{i=0}^{\infty} 2iA_{2i} \end{bmatrix}$$

If we now postmultiply by $\mathbf{e}$, we find

$$\frac{d}{dz}\hat{A}(z)\Big|_{z=1}\mathbf{e} = \begin{bmatrix} \frac{1}{2}\sum_{i=0}^{\infty} 2iA_{2i}\mathbf{e} + \frac{1}{2}\sum_{i=0}^{\infty}(2i+1)A_{2i+1}\mathbf{e} - \frac{1}{2}\sum_{i=0}^{\infty} A_{2i+1}\mathbf{e} \\ \frac{1}{2}\sum_{i=1}^{\infty} iA_{2i-1}\mathbf{e} + \frac{1}{2}\sum_{i=1}^{\infty} A_{2i-1}\mathbf{e} + \frac{1}{2}\sum_{i=0}^{\infty} 2iA_{2i}\mathbf{e} \end{bmatrix}$$

Then, upon changing the index of summation, we find

$$\frac{d}{dz}\hat{A}(z)\Big|_{z=1}\mathbf{e} = \begin{bmatrix} \frac{1}{2}\sum_{i=0}^{\infty} iA_i\mathbf{e} - \frac{1}{2}\sum_{i=0}^{\infty} A_{2i+1}\mathbf{e} \\ \frac{1}{2}\sum_{i=0}^{\infty} iA_i\mathbf{e} + \frac{1}{2}\sum_{i=1}^{\infty} A_{2i-1}\mathbf{e} \end{bmatrix}$$

Then premultiplying by $\phi_2 = \frac{1}{2}\left[\phi \quad \phi\right]$ yields

$$\phi_2\frac{d}{dz}\hat{A}(z)\Big|_{z=1}\mathbf{e} = \frac{1}{2}\phi\sum_{i=0}^{\infty} iA_i\mathbf{e}.$$

Therefore,

$$\phi\sum_{i=0}^{\infty} iA_i\mathbf{e} = 2\phi_2\frac{d}{dz}\hat{A}(z)\|_{z=1}\mathbf{e} = 2\rho.$$

7.1 Supplementary Problems

7-1 Consider a slotted time, single server queueing system having unit service rate. Suppose the arrival process to the system is a Poisson process modulated by a discrete time Markov chain on $\{0, 1\}$ that has the following one-step transition probability matrix:

$$\mathcal{P} = \begin{bmatrix} \beta & 1-\beta \\ 1-\alpha & \alpha \end{bmatrix}.$$

While the arrival process is in phase i, arrivals to the system occur according to a Poisson process with rate λ_i, $i = 0, 1$.

(a) Argue that $\mathcal{A}(z) = \mathcal{B}(z)$ for this system.

(b) Show that

$$\mathcal{A}(z) = \begin{bmatrix} \beta & 1-\beta \\ 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} e^{-\lambda_0(1-z)} & 0 \\ 0 & e^{-\lambda_1(1-z)} \end{bmatrix}.$$

(c) Determine $\mathcal{A}_i$ for all $i \geq 0$.

(d) Suppose that at the end of a time slot, the system occupancy level is zero and the phase of the arrival process is i, $i = 0, 1$. Determine the probability that the system occupancy level at the end of the following time slot will be k, $k = \{0, 1, \cdots\}$ and the phase of the arrival process will be j, $j = 0, 1$.

(e) Suppose that at the end of a time slot, the system occupancy level is $k > 0$ and the phase of the arrival process is i, $i = 0, 1$. Determine the probability that the system occupancy level at the end of the following time slot will be l, $l = \{k-1, k, k+1\}$ and the phase of the arrival process will be j, $j = 0, 1$.

(f) Let $\alpha = 0.5$, $\beta = 0.75$, $\lambda_0 = 1.2$, and $\lambda_1 = 0.3$. Determine the equilibrium probability vector for the phase process and ρ for the system.

(g) Write a computer program to determine $\mathcal{K}(1)$ for the parameters given in part (f), and then determine the equilibrium probability vector for the Markov chain for which $\mathcal{K}(1)$ is the one-step transition probability matrix.

(h) Compute $E[\tilde{q}]$, the expected number of packets in the system at the end of an arbitrary time slot. Compare the result to the equivalent mean value $E[\tilde{q}]$ for the system in which $\lambda_0 = \lambda_1 = \rho$ as computed in part (f). What can be said about the effect of burstiness on the average system occupancy?

Solution:

(a) The matrices $\mathcal{B}_j$ represent the number of arrivals that occur in the first slot of a busy period while the matrices $\mathcal{A}_j$ represent the number of arrivals that occur in an arbitrary slot. Since arrivals are Poisson and the slots are of fixed length, there is no difference between the two.

(b) The matrix $\mathcal{A}(z)$ represents the generating function for the number of arrivals that occur during a time slot. In particular, if the phase of the phase process is 0 at the end of a given time slot, then the phase will be 0 at the end of the next time slot with probability β, and the distribution of the number of arrivals will be Poisson with parameter λ_0. Similarly, if the phase is initially 1, then it will be zero with probability $(1-\alpha)$ and the distribution of the number of arrivals will be Poisson with parameter λ_0. Hence

$$\begin{aligned}
\mathcal{A}(z) &= \begin{bmatrix} \beta e^{-\lambda_0(1-z)} & (1-\beta)e^{-\lambda_1(1-z)} \\ (1-\alpha)e^{-\lambda_0(1-z)} & (1-\beta)e^{-\lambda_1(1-z)} \end{bmatrix} \\
&= \begin{bmatrix} \beta & 1-\beta \\ 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} e^{-\lambda_0(1-z)} & 0 \\ 0 & e^{-\lambda_1(1-z)} \end{bmatrix}.
\end{aligned}$$

(c) From the formula for the Poisson distribution,

$$\mathcal{A}_i = \begin{bmatrix} \beta & 1-\beta \\ 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} \frac{\lambda_0^i}{i!} & 0 \\ 0 & \frac{\lambda_1^i}{i!} \end{bmatrix}.$$

(d)

$$\begin{aligned}
P\{\tilde{n}=k, \tilde{\wp}=0|\tilde{n}=0, \tilde{\wp}=0\} &= \beta\frac{\lambda_0^k}{k!} \\
P\{\tilde{n}=k, \tilde{\wp}=1|\tilde{n}=0, \tilde{\wp}=0\} &= (1-\beta)\frac{\lambda_1^k}{k!} \\
P\{\tilde{n}=k, \tilde{\wp}=0|\tilde{n}=0, \tilde{\wp}=0\} &= (1-\alpha)\frac{\lambda_0^k}{k!} \\
P\{\tilde{n}=k, \tilde{\wp}=1|\tilde{n}=0, \tilde{\wp}=1\} &= \alpha\frac{\lambda_1^k}{k!}
\end{aligned}$$

(e) The level wil be $k-1$ if there are no arrivals, this probability matrix is given by

$$\mathcal{A}_0 = \begin{bmatrix} \beta & 1-\beta \\ 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} e^{-\lambda_0} & 0 \\ 0 & e^{-\lambda_1} \end{bmatrix}.$$

Now, $l=k$ if exactly one arrival given by

$$\mathcal{A}_1 = \begin{bmatrix} \beta & 1-\beta \\ 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} \lambda_0 e^{-\lambda_0} & 0 \\ 0 & \lambda_1 e^{-\lambda_1} \end{bmatrix},$$

and $l = k + 1$ if exactly two arrivals given by

$$\mathcal{A}_2 = \begin{bmatrix} \beta & 1-\beta \\ 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} \frac{\lambda_0^2 e^{-\lambda_0}}{2!} & 0 \\ 0 & \frac{\lambda_1^2 e^{-\lambda_1}}{2!} \end{bmatrix}$$

(f) The equilibrium distribution for the phase process is found by solving

$$x = x \begin{bmatrix} \beta & 1-\beta \\ 1-\alpha & \alpha \end{bmatrix},$$

where $xe = 1$. Also, we can simply take ratios of the mean time in each phase to the mean of the cycle time. Let $\tilde{n}_i$ be the number of slots in phase i, $i = 0, 1$. Thus, $P\{\tilde{n}_0 = k\} = (1-\beta)\beta^{k-1}$ implies $E[\tilde{n}_0] = \frac{1}{1-\beta}$. Similarly, $E[\tilde{n}_1] = \frac{1}{1-\alpha}$. Therefore,

$$\begin{aligned} P\{\text{phase } 0\} &= \frac{1/(1-\beta)}{1/(1-\beta) + 1/(1-\alpha)} \\ &= \frac{1-\alpha}{2-\alpha-\beta} \\ &= \frac{0.5}{2-0.5-0.75} = \frac{2}{3}, \end{aligned}$$

and

$$\begin{aligned} P\{\text{phase } 1\} &= \frac{1/(1-\alpha)}{1/(1-\beta) + 1/(1-\alpha)} \\ &= \frac{1-\beta}{2-\alpha-\beta} \\ &= \frac{0.25}{2-0.5-0.75} = \frac{1}{3}. \end{aligned}$$

Thus,

$$\begin{aligned} \rho &= \frac{2}{3}\lambda_0 + \frac{1}{3}\lambda_1 \\ &= \frac{2}{3}(1.2) + \frac{1}{3}(0.3) = 0.9 \end{aligned}$$

(g) Since $\mathcal{A}(z)$ and $\mathcal{B}(z)$ are identical, $G(1)$ and $\mathcal{K}(1)$ are also identical. We may then solve for $\mathcal{K}(1)$ by using (5.39). The result is

$$\mathcal{K}(1) = \begin{bmatrix} 0.486555 & 0.513445 \\ 0.355679 & 0.644321 \end{bmatrix}.$$

Note that the result is a legitimate one-step transition probability matrix for a discrete parameter Markov chain. The equilibrium may be obtained by normalizing any row of the adjoint of $I - \mathcal{K}(1)$. The relevant adjoint, in turn, is

$$\text{adj } I - \mathcal{K}(1) = \begin{bmatrix} 0.355679 & 0.513445 \\ 0.355679 & 0.513445 \end{bmatrix}.$$

and, if normalized so that the row sums are both unity, the result is

$$\kappa = [\, 0.409239 \quad 0.590761 \,].$$

(h) We may compute $E[\tilde{q}]$ from (5.47). In order to do this, we must first define each term in the equation. From part g, $\mathcal{F}_{\tilde{q}}(1) = [\, 0.6667 \quad 0.3333 \,]$. Also,

$$\mathcal{F}_{\tilde{a}}(z) = \begin{bmatrix} e^{-\lambda_0[1-z]} & 0 \\ 0 & e^{-\lambda_1[1-z]} \end{bmatrix}$$

so that, in general,

$$\mathcal{F}_{\tilde{a}}^{(j)}(z) = \begin{bmatrix} \lambda_0^j e^{-\lambda_0[1-z]} & 0 \\ 0 & \lambda_1^j e^{-\lambda_1[1-z]} \end{bmatrix}.$$

In particular,

$$\mathcal{F}_{\tilde{a}}^{(1)}(1) = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

and

$$\mathcal{F}_{\tilde{a}}^{(2)}(1) = \begin{bmatrix} \lambda_0^2 & 0 \\ 0 & \lambda_1^2 \end{bmatrix} = \begin{bmatrix} 1.44 & 0 \\ 0 & 0.09 \end{bmatrix}.$$

From (5.41),

$$\begin{aligned} \pi_0 &= \left[1 - \mathcal{F}_{\tilde{q}}(1)\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e}\right] \kappa \\ &= [\, 0.0409239 \quad 0.0590761 \,]. \end{aligned}$$

All of the terms are now defined. Substituting these into (5.41) yeilds $E[\tilde{q}] = 6.319$. For the M/G/1 case, the appropriate formula is (6.10), which is

$$\begin{aligned} E[\tilde{n}] &= \rho \left(1 + \frac{\rho}{1-\rho} \frac{C_{\tilde{x}}^2 + 1}{2} \right) \\ &= 4.95. \end{aligned}$$

From this, it can be seen that the effect of burstiness is to increase the average system occupancy. As has been observed many times before, a Poisson assumption is not valid in many cases where an observed phenomena shows that the traffic process is not smooth.

7-2 The objective of this problem is to develop a closed-form expression for the mean queue length, $E[\tilde{q}]$, for the slotted M/D/1 system having phase-dependent arrivals and unit service times. Our point of departure is the expression for the generating function of the occupancy distribution as given in (7.37), which is now repeated for continuity:

$$\mathcal{F}_{\tilde{q}}(z)\,[Iz - \mathcal{P}\mathcal{F}_{\tilde{a}}(z)] = \pi_0\,[z-1]\,\mathcal{P}\mathcal{F}_{\tilde{a}}(z). \tag{7.69}$$

(a) Differentiate both sides of (7.69) to obtain

$$\mathcal{F}_{\tilde{q}}(z)\left[Iz - \mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(z)\right] + \mathcal{F}_{\tilde{q}}^{(1)}(z)\,[Iz - \mathcal{P}\mathcal{F}_{\tilde{a}}(z)] \\ = \pi_0[z-1]\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(z) + \pi_0\mathcal{P}\mathcal{F}_{\tilde{a}}(z) \tag{7.70}$$

(b) Take limits of both sides of (7.70) as $z \to 1$ to obtain

$$\mathcal{F}_{\tilde{q}}^{(1)}(1)\,[I - \mathcal{P}] = \pi_0\mathcal{P} - \mathcal{F}_{\tilde{q}}(1)\left[I - \mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\right]. \tag{7.71}$$

(c) Define ρ to be the marginal probability that the system is not empty at the end of a slot. Postmultiply both sides of (7.71) by $\mathbf{e}$, and show that $\rho = \mathcal{F}_{\tilde{q}}(1)\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e}$.

(d) Add $\mathcal{F}_{\tilde{q}}^{(1)}(1)\mathbf{e}\mathcal{F}_{\tilde{q}}(1)$ to both sides of (7.71), solve for $\mathcal{F}_{\tilde{q}}^{(1)}(1)$, and then postmultiply by $\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e}$ to obtain

$$\mathcal{F}_{\tilde{q}}^{(1)}(1)\mathcal{P}\mathcal{F}_{\tilde{q}}(1) = \mathcal{F}_{\tilde{q}}^{(1)}(1)\mathbf{e}\rho \\ +\{\pi_0\mathcal{P} - \mathcal{F}_{\tilde{q}}(1)[I - \mathcal{F}_{\tilde{a}}^{(1)}(1)]\} \\ [I - \mathcal{P} + \mathbf{e}\mathcal{F}_{\tilde{q}}(1)]^{-1}. \tag{7.72}$$

Use the fact that $\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\,[I - \mathcal{P} + \mathbf{e}\mathcal{F}_{\tilde{q}}(1)] = \mathbf{e}\mathcal{F}_{\tilde{q}}(1)$, as shown in Exercise 7.12 in Section 7.3.

(e) Differentiate both sides of (7.70) with respect to z, postmultiply both sides by $\mathbf{e}$, take limits on both sides as $z \to 1$, and then rearrange terms to find

$$\mathcal{F}_{\tilde{q}}^{(1)}(1)\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e} = \mathcal{F}_{\tilde{q}}^{(1)}(1)\mathbf{e} - \frac{1}{2}\mathcal{F}_{\tilde{q}}^{(1)}(1)\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e} \\ -\pi_0\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e} \tag{7.73}$$

(f) Equate right-hand sides of (7.72) and (7.73), and then solve for $\mathcal{F}^{(1)}_{\tilde{q}}(1)\mathbf{e}$ to obtain

$$\begin{aligned}
E[\tilde{q}] &= \mathcal{F}^{(1)}_{\tilde{q}}(1)\mathbf{e} \\
&= \frac{1}{1-\rho}\Big\{\frac{1}{2}\mathcal{F}_{\tilde{q}}(1)\mathcal{F}^{(2)}_{\tilde{a}}(1)\mathbf{e} + \pi_0\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e} \\
&\quad + \Big(\pi_0\mathcal{P} - \mathcal{F}_{\tilde{q}}(1)\left[I - \mathcal{F}^{(1)}_{\tilde{a}}(1)\right]\Big) \\
&\quad \times \left[I - \mathcal{P} + \mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1}\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e}\Big\}.
\end{aligned}$$

Solution:

(a) This part is accomplished by simply using the formula

$$\frac{d}{dz}u(z)v(z) = \left[\frac{d}{dz}u(z)\right]v(z) + u(z)\left[\frac{d}{dz}v(z)\right]$$

with $u(z) = \mathcal{F}_{\tilde{q}}(z)$ and $v(z) = [Iz - \mathcal{P}\mathcal{F}_{\tilde{a}}(z)]$.

(b) Given

$$\begin{aligned}
&\mathcal{F}_{\tilde{q}}(z)\left[Iz - \mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(z)\right] + \mathcal{F}^{(1)}_{\tilde{q}}(z)\left[Iz - \mathcal{P}\mathcal{F}_{\tilde{a}}(z)\right] \\
&\qquad = \pi_0[z-1]\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(z) + \pi_0\mathcal{P}\mathcal{F}_{\tilde{a}}(z)
\end{aligned} \tag{7.70}$$

from Part (a) and using the fact that $\lim_{z\to 1}\mathcal{F}_{\tilde{a}}(z) = I$, the result follows.

(c) From Part(b),

$$\mathcal{F}^{(1)}_{\tilde{q}}(1)\left[I - \mathcal{P}\right] = \pi_0\mathcal{P} - \mathcal{F}_{\tilde{q}}(1)\left[I - \mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\right]. \tag{7.71}$$

Now, $I\mathbf{e} = \mathbf{e}$, and $\mathcal{P}\mathbf{e} = \mathbf{e}$ because $\mathcal{P}$ is the one-step transition probability matrix for a DPMC. The $[I - \mathcal{P}]\,\mathbf{e} = 0$. Considering the RHS, $\pi_0\mathcal{P}\mathbf{e} = \pi_0\mathbf{e}$, which is the probability the occupancy is equal to zero. Also, $\mathcal{F}_{\tilde{q}}(1)I\mathbf{e} = \mathcal{F}_{\tilde{q}}(1)\mathbf{e} = 1$ because $\mathcal{F}_{\tilde{q}}(1)$ is a vector of probability masses covering all possible states. Considering the term $\mathcal{F}_{\tilde{q}}(1)\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e}$, $\mathcal{F}_{\tilde{q}}(1)\mathcal{P} = \mathcal{F}_{\tilde{q}}(1)$ because $\mathcal{F}_{\tilde{q}}(1)$ is the stationary vector for $\mathcal{P}$; this fact can easily be seen from (7.69) by setting $z = 1$. In summary, we have

$$0 = \pi_0\mathbf{e} - \left[1 - \mathcal{F}_{\tilde{q}}(1)\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e}\right] \quad \text{or} \quad 1 - \pi_0\mathbf{e} = \rho = \mathcal{F}_{\tilde{q}}(1)\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e}.$$

(d) Starting with

$$\mathcal{F}^{(1)}_{\tilde{q}}(1)\left[I - \mathcal{P} + \mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right] = \pi_0\mathcal{P} - \mathcal{F}_{\tilde{q}}(1)\left[I - \mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\right]. \tag{7.71}$$

we add $\mathcal{F}^{(1)}_{\tilde{q}}(1)\mathbf{e}\mathcal{F}_{\tilde{q}}(1)$ to both sides to get

$$\begin{aligned}\mathcal{F}^{(1)}_{\tilde{q}}(1) & \left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right] \\ &= \mathcal{F}^{(1)}_{\tilde{q}}(1)\mathbf{e}\mathcal{F}_{\tilde{q}}(1)+\pi_0\mathcal{P}-\mathcal{F}_{\tilde{q}}(1)\left[I-\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\right].\end{aligned}$$

We now postmultiply both sides by $\left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1}$ to obtain

$$\begin{aligned}\mathcal{F}^{(1)}_{\tilde{q}}(1) &= \mathcal{F}^{(1)}_{\tilde{q}}(1)\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1} \\ &\quad+\left\{\pi_0\mathcal{P}+\mathcal{F}_{\tilde{q}}(1)\left[I-\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\right]\right\} \\ &\quad\left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1}.\end{aligned}$$

We then note that since

$$\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]=\mathbf{e}\mathcal{F}_{\tilde{q}}(1),$$

it is also true that

$$\mathbf{e}\mathcal{F}_{\tilde{q}}(1)=\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]\left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1}.$$

In addition, $\mathcal{F}_{\tilde{q}}(1)\mathcal{P}=\mathcal{F}_{\tilde{q}}(1)$ because $\mathcal{F}_{\tilde{q}}(1)$ is the stationary vector of $\mathcal{P}$. Hence,

$$\begin{aligned}\mathcal{F}^{(1)}_{\tilde{q}}(1) &= \mathcal{F}^{(1)}_{\tilde{q}}(1)\mathbf{e}\mathcal{F}_{\tilde{q}}(1) \\ &\quad+\left\{\pi_0\mathcal{P}+\mathcal{F}_{\tilde{q}}(1)\left[I-\mathcal{F}^{(1)}_{\tilde{a}}(1)\right]\right\} \\ &\quad\left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1}.\end{aligned}$$

Upon multiplying both sides by $\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e}$, we have

$$\begin{aligned}\mathcal{F}^{(1)}_{\tilde{q}}(1)\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e} &= \mathcal{F}^{(1)}_{\tilde{q}}(1)\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e} \\ &\quad+\left\{\pi_0\mathcal{P}+\mathcal{F}_{\tilde{q}}(1)\left[I-\mathcal{F}^{(1)}_{\tilde{a}}(1)\right]\right\} \\ &\quad\left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1}\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e}.\end{aligned}$$

Again using $\mathcal{F}_{\tilde{q}}(1)\mathcal{P}=\mathcal{F}_{\tilde{q}}(1)$ and $\rho=\mathcal{F}_{\tilde{q}}(1)\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e}$ leads to

$$\begin{aligned}\mathcal{F}^{(1)}_{\tilde{q}}(1)\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e} &= \mathcal{F}^{(1)}_{\tilde{q}}(1)\mathbf{e}\rho \\ &\quad+\left\{\pi_0\mathcal{P}+\mathcal{F}_{\tilde{q}}(1)\left[I-\mathcal{F}^{(1)}_{\tilde{a}}(1)\right]\right\} \\ &\quad\left[I-\mathcal{P}+\mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1}\mathcal{P}\mathcal{F}^{(1)}_{\tilde{a}}(1)\mathbf{e}.\end{aligned}$$

(e) Upon taking the derivative of both sides of (7.70), we find

$$\mathcal{F}_{\tilde{q}}^{(2)}(z)\left[Iz - \mathcal{P}\mathcal{F}_{\tilde{a}}(z)\right] + 2\mathcal{F}_{\tilde{q}}^{(1)}(z)\left[I - \mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(z)\right] \\ -\mathcal{F}_{\tilde{q}}(z)\mathcal{P}\mathcal{F}_{\tilde{a}}^{(2)}(z) = \pi_0[z-1]\mathcal{P}\mathcal{F}_{\tilde{a}}^{(2)}(z) + 2\pi_0\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(z)$$

Then multiplying by $\mathbf{e}$ and taking the limit as $z \to 1$ yields

$$2\mathcal{F}_{\tilde{q}}^{(1)}(1)\left[I - \mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\right]\mathbf{e} - \mathcal{F}_{\tilde{q}}(1)\mathcal{P}\mathcal{F}_{\tilde{a}}^{(2)}(1)\mathbf{e} = +2\pi_0\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1),$$

where we have used the fact that $\mathcal{F}_{\tilde{q}}^{(2)}(1)\left[I - \mathcal{P}\mathcal{F}_{\tilde{a}}(1)\right]\mathbf{e} = 0$. Upon solving for $\mathcal{F}_{\tilde{q}}^{(1)}(1)\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e}$, we find

$$\mathcal{F}_{\tilde{q}}^{(1)}(1)\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e} = \mathcal{F}_{\tilde{q}}^{(1)}(1)\mathbf{e} - \frac{1}{2}\mathcal{F}_{\tilde{q}}(1)\mathcal{F}_{\tilde{a}}^{(2)}(1)\mathbf{e} - \pi_0\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e}.$$

(f) The final result is obtained by simple algebraic operations. These operations do not involve any other equations. We begin with

$$\mathcal{F}_{\tilde{q}}^{(1)}(1)\mathbf{e} - \frac{1}{2}\mathcal{F}_{\tilde{q}}(1)\mathcal{F}_{\tilde{a}}^{(2)}(1)\mathbf{e} - \pi_0\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1) = \mathcal{F}_{\tilde{q}}^{(1)}(1)\mathbf{e}\rho \\ + \left\{\pi_0\mathcal{P} + \mathcal{F}_{\tilde{q}}(1)\left[I - \mathcal{F}_{\tilde{a}}^{(1)}(1)\right]\right\}\left[I - \mathcal{P} + \mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1}\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e}.$$

This yields

$$\mathcal{F}_{\tilde{q}}^{(1)}(1)\mathbf{e}(1-\rho) = \frac{1}{2}\mathcal{F}_{\tilde{q}}(1)\mathcal{F}_{\tilde{a}}^{(2)}(1)\mathbf{e} + \pi_0\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1) \\ + \left\{\pi_0\mathcal{P} + \mathcal{F}_{\tilde{q}}(1)\left[I - \mathcal{F}_{\tilde{a}}^{(1)}(1)\right]\right\}\left[I - \mathcal{P} + \mathbf{e}\mathcal{F}_{\tilde{q}}(1)\right]^{-1}\mathcal{P}\mathcal{F}_{\tilde{a}}^{(1)}(1)\mathbf{e}.$$

The result follows by dividing both sides by $(1-\rho)$.