25/07

DETERMINANTS

$$\begin{bmatrix} a & x & m \\ b & y & n \\ c & z & p \end{bmatrix} = a[yp-nz] - x[bp-nc] + m[bz-cy]$$

$$\begin{matrix} a & x & m & a & x \\ b & y & n & b & y \\ c & z & p & c & z \end{matrix}$$

$$\boxed{\begin{matrix}(ayp + xnc + mbz) \\ - \\ (cym + zna + pbx)\end{matrix}}$$

Eg. $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 2 & 5 \\ 3 & 4 & 3 \end{vmatrix} \begin{matrix} 1 & -1 \\ 2 & 2 \\ 3 & 4 \end{matrix}$ $(6 - 15 + 0) - (0 + 20 - 6)$

det $= 12 - 35 = -23$

eg $\begin{vmatrix} 1 & 4 & 3 \\ 2 & 5 & 3 \\ 3 & 7 & 4 \end{vmatrix} \begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & 7 \end{matrix}$ $(20 + 36 + 52) - (45 + 21 + 32)$

det $= 0$

$\Rightarrow$ there will be LD vectors.

MINORS AND COFACTORS

a_{ij} is an element of $[a_{ij}]_{n \times n}$

Minor of a_{ij} = M_{ij}

Eg: $[a_{ij}]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

minor of a_{21} = M_{21} :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

minor of a_{21} = M_{21} = $(a_{12} \cdot a_{33} - a_{13} \cdot a_{32})$ $\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$

eg. $A = \begin{vmatrix} 5 & 6 & 7 \\ 2 & 0 & 5 \\ 3 & 1 & 3 \end{vmatrix}$ = M_{22} ?

$M_{22} = (5 \times 3 - 7 \times 3) = (15 - 21) = -6$

we will use co-factor to ~~to~~ find out inverse

COFACTOR

Cofactor are minors with sign included.

- Do the same step which you have done to find minor & ~~test condition~~ first consider the sign corresponding to that element.

⇒ COFACTORS for particular elements

$[a_{ij}]_{m \times n}$

COFACTOR OF a_{ij} ← $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$ → MINOR OF a_{ij}

★ $(-1)^{even} = 1$

★ $(-1)^{odd} = -1$

$C_{12} = (-1)^{1+2} [2 \times 3 - 5 \times 3]$

$= (-1)(-9)$

$C_{12} = 9$

PROPERTIES OF DETERMINANTS

1. If 2 rows / col of a matrix are proportional then $\Delta = 0$

Eg. $\begin{vmatrix} 2 & 3 & 5 \\ 6 & 9 & 15 \\ 7 & 5 & 0 \end{vmatrix}$ $R_2 = 3R_1$

FULL RANK MATRIX

⇒ Rank = # columns

2. If all the elements of any row or col is all zero then $\Delta = 0$.

Row wise vector $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ 0 & 0 & 0 \end{vmatrix}$ ← $\{V_1\ V_2\ V_3\}$

$R_3 \longrightarrow$

$0(\Delta) + 0[\Delta] + 0(\Delta) = 0$

3. If two rows or two columns of a matrix are interchanged then sign of the det changes.

eg. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det(A) = -2$

$A' = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $\det(A') = 2$.

TRANSPOSE : A^T

- $|A| = |A^T|$
- $|A \cdot B| = |A||B|$ if A and B are of same order
- $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \Delta_1$ ⇒ $k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = k\Delta_1$

$$k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} ka_1 & b_1 \\ ka_2 & b_2 \end{vmatrix} = ka_1b_2 - ka_2b_1$$
$$= k(a_1b_2 - a_2b_1)$$
$$= k\Delta_1$$

$$\begin{vmatrix} ka_{11} & a_{12} & a_{13} & \cdots \\ ka_{21} & a_{22} & a_{23} & \cdots \\ ka_{31} & a_{32} & a_{33} & \cdots \end{vmatrix} = k\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \end{bmatrix}$$

VERY IMP

$A = [a_{ij}]_{m\times n}$

$B = kA = [ka_{ij}]_{n\times n}$

$$\boxed{|B| = |kA| = k^n|A|}$$

$$= \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

$$= k\,k\,k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= k^3|A|$$

- $$\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

imp

* If in a determinant

$R_i \to R_i + kR_j$ }
$C_i \to C_i + kC_j$ } Det remains unchanged

eg. $$\begin{vmatrix} 1 & 2 & 5 \\ 1 & 4 & 7 \\ 1 & 6 & 5 \end{vmatrix}$$ det $= (20+14+30)-(20+52+10)$
$= 64 - 82 = -18$

(i) $R_2 \to R_2 - R_1$

$$\begin{vmatrix} 1 & 2 & 5 \\ 0 & 2 & 2 \\ 1 & 6 & 5 \end{vmatrix}$$

(ii) $R_3 \to R_3 - R_1$

$$\begin{vmatrix} 1 & 2 & 5 \\ 0 & 2 & 2 \\ 0 & 4 & 0 \end{vmatrix}$$

ADJOINT OF A MATRIX A

$A = [a_{ij}]_{n\times n}$

$C = [c_{ij}]_{n\times n}$ where c_{ij} is cofactor of a_{ij}

$\text{Adj } A = C^T$ Adjoint is transpose of cofactor matrix.

eg. $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

$$C = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\text{adj } A = C^T = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ → sign change → interchange

If A is a sq. matrix of order 'n' then
$A(\text{adj } A) = (\text{adj } A)A = |A|I$

Q multiplication of a sq. matrix with its adjoint commutes T/F

$A.B = B.A$ where B is adjoint.

* A sq. matrix is called singular if $|A| = 0$
* A sq. matrix is called non-singular if $|A| \neq 0$
*

Eg. find the det of adj A if order of that matrix A is n

$|\text{adj}(A)| = ?$

$|\text{adj}(A).A| = ||A|I| = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|\text{adj } A| . |A| \{I\}$

$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$

$= |A|^3 I$

LHS

$$|\text{adj} A|\,|A| = |A|^n\,|I| \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det(I) = 1$$

$$|\text{adj} A| \cdot |A| = |A|^n \cdot 1$$

$$|\text{adj} A| = \frac{|A|^n}{|A|} = |A|^{n-1}$$

Q. Find the value of $|\text{adj}(\text{adj} A)|$ if matrix A is of order 3 and $|A| = 15$.

$$|\text{adj}(\text{adj} A)| = |A|^{(n-1)^2}$$

$$|\text{adj}(\text{adj}(\text{adj} A))| = |A|^{(n-1)^3}$$

$$|\mathrm{adj}(\underbrace{\mathrm{adj}(\mathrm{adj}\,A)}_{X})| = |\mathrm{adj}(X)|$$

$$= |X|^{n-1}$$

$$= |\mathrm{adj}(\mathrm{adj}\,A)|^{n-1}$$

$$= \left(|A|^{(n-1)^2}\right)^{n-1}$$

$$= \left(|A|^{(n-1)^3}\right)$$

INVERSE OF A SQ MATRIX.

Inverse exist only for non-singular matrix.

$$\boxed{A \cdot B = B \cdot A = I}$$

B is known as inverse of A $\Rightarrow A^{-1} = B$

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

$|A^{-1}| = ?$

$$|A \cdot A^{-1}| = |I|$$

$$|A| \cdot |A^{-1}| = 1$$

$$\boxed{|A^{-1}| = \frac{1}{|A|}}$$ IMP.

$$A \cdot (\text{adj} A) = (\text{adj } A) \cdot A = |A| I.$$

$$A \cdot (\text{adj } A) = |A| I$$

$$\frac{1}{|A|} \cdot (A \cdot \text{adj} A) = I$$

$$(\text{adj} A) \cdot (A) = |A| I$$

$$\frac{(\text{adj} A)(A)}{|A|} = I$$

$$A \cdot \underbrace{\frac{(\text{adj } A)}{|A|}}_{A^{-1}} = \underbrace{\frac{(\text{adj} A)}{|A|}}_{A^{-1}}(A) = I.$$

$$A A^{-1} = A^{-1} A = I$$

$$\boxed{A^{-1} = \frac{\text{adj } A}{|A|}}$$