04/08

$mx = 0$ HOMOGENOUS

Q. $A = \begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix}$ $\lambda^2 - 4\lambda + 4 = 0$

$(\lambda-2)(\lambda-2) = 0$

$\lambda_1 = 2$

$\lambda_2 = 2$

ALGEBRIC MULTIPLICITY :-

The AM of an eigen value is the no. of times it appears as a root of the char. poly.

It is represented by $\mu(\lambda_k)$ where λ_k is the eigen value.

Eg. $A = \begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix}$ $\mu(\lambda_k) = 2$

$A \to \lambda_1 = 2$, $\lambda_2 = 2$

$\mu(\lambda_k) = 1$

$\mu(\lambda_k) = 1$

Total roots me kitna count hai

$A \to \lambda_1 = 1$, $\lambda_2 = 3$

If distinct values of λ_1 & λ_2 and so on are present. then the count of each will be 1. corresponding to each value.

$\lambda_1 = 2+i$

$\lambda_2 = 2-i$ $\mu(\lambda_1) = 1$

$\lambda_3 = 4$

$\lambda_4 = 4$ $\mu(\lambda_3) = 3$

$\lambda_5 = 3$

$\lambda_6 = 4$

for real ~~ssa~~ matrix if one eigen value is $a+ib$ then $a-ib$ is also an eigen value where $a-ib$ is complex conjugate of $a+ib$

$a+\sqrt{b}$ Eigen value
$a-\sqrt{b}$ Eigen value

Q. $A_{4\times4}$ is a matrix having the following 3 eigen values $3+i$, 2, 2 then the det of matrix A is ____

product of ~~ss~~ eigen value = ~~pro~~ det (A)

eigen $3-i$, $3+i$, 2, 2

$$\text{pro} = (3+i)(3-i)(2)(2) = (3^2-i^2)(4) = (9+1)(4) = 10\times4 = 40.$$

eg $\begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix}$ → $\lambda_1 = 2$, $\lambda_2 = 2$ $\quad AM(\lambda_1) = 2$

$$\begin{bmatrix} 4-\lambda & -1 \\ 4 & -\lambda \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x - y = 0$
$2x = y$

$$\begin{bmatrix} y/2 \\ y \end{bmatrix} = y\begin{bmatrix} 1/2 \\ 2 \end{bmatrix}$$

↳ one value there can be more values

OR x ki terms me bhi

$$x\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

eg $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $\quad \lambda_1 = 5$, $\lambda_2 = 5$ $\quad$ Trace (A) = 10, $\lambda_1+\lambda_2 = 10$, $\lambda_1\lambda_2 = 25 \cdot 25$

$\eta(A-\lambda I) = ? \longrightarrow 2.$

$$\begin{bmatrix} 5-\lambda & 0 \\ 0 & 5-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \eta(A-\lambda I) = 2$$

BOTH FREE VARIABLE

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0x + 0y = 0$ infinite many vectors possible.

$x \in R$ $y \in R$.

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \Rightarrow x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\eta(A-\lambda I)$ = # LI eigen vectors corresponding to an eigen value.

* $\lambda_1 = \lambda_2 = 2 \longrightarrow \eta(A-\lambda I) = 1 \longrightarrow$ 1 LI eigen vector
* $\lambda_1 = \lambda_2 = 5 \longrightarrow \eta(A-\lambda I) = 2 \rightarrow$ 2 LI eigen vectors

$(A-\lambda I)X = 0$

iss system ke set of sol^n honge unka khudh ka space hoga.

sol^n = $\{x \in \mathbb{R}^2$, # LI = nullity $\}$

Q $A_{2\times2}$

$\lambda_1 = \lambda_2 = 2$

How many linear independent eigen.

Can't say because we don't known about nullity.

Q $A_{3\times3}$

$\lambda_1, \lambda_2, \lambda_3$ are eigen values then how many LI eigen vectors are these.

$\lambda_1 \neq \lambda_2 \neq \lambda_3$

eigen vectors = 3

$\lambda_1 \rightarrow$ 1 eigen vector

$\lambda_2 \rightarrow 1$

$\lambda_3 \rightarrow 1$

Q $A_{3\times3}$ λ_1, λ_1 & λ_2 are eigen values then # LI eigen vectors.

a) $\eta(A-\lambda_1 I)+1$

b) $\eta(A-\lambda_1 I)+1$

c) 3

d) $\eta(A-\lambda I)+2$

Q. $A_{5\times5}$ matrix & the eigen values are $\lambda_1, \lambda_2, \lambda_3$ where

$AM(\lambda_1) = 2$ } same

$AM(\lambda_2) = 2$ } distinct

$AM(\lambda_3) = 1$

The # LI eigen vectors are — same nullity

~~A)~~ $\eta(A-\lambda_1 I) + \eta(A-\lambda_2 I) + \eta(A-\lambda_3 I)$ same A

~~B)~~ $\eta(A-\lambda_1 I) + \eta(A-\lambda_2 I) + 1$ (2n msq) → NO of LI vectors = nullity

C) # 5

~~D)~~ n where $n \geq 1$ (2n MSQ)

$n\times n \rightarrow$ n distinct eigen values.

$\lambda_i \rightarrow |A-\lambda_i I| \rightarrow$ 1

$n \times n$

$\downarrow$

n E values

$\lambda_i \rightarrow |A-\lambda I| \rightarrow$ 1

2 X 2

$\lambda^2 - (\text{Trace})\lambda + \det = 0$

V.Imp $m \pm \sqrt{m^2 - p}$ m - mean of trace, p → det.

$A_{3\times3}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $|A-\lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

In case of diagonal & triangular matrix eigen values are the principle diagonal elements.

$(1-\lambda)(1-\lambda)(1-\lambda) = 0$

$\lambda = 1, 1, 1$

3x3 $\lambda^3 - \text{Trace}(\lambda^2) + (\text{MOD})\lambda - |A| = 0$

↳ minor of diagonal elements.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \lambda^3 - 3\lambda^2 + (1+1+1)\lambda - 1 = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

product of roots $= -\frac{d}{a}$

Gate 2017 2.52

3x3 Real matrix

$\lambda^3 - 4\lambda^2 + a\lambda + 30$ $\quad a \in \mathbb{R}$

& one of the eigen value is 2 then the largest abs. eigen value is ___

$\lambda^3 - \text{Trace}(\lambda^2) + (\text{MOD})\lambda - |A| = 0$

$\lambda^3 - 4\lambda^2 + a\lambda + 30 = 0$

$2^3 - 4(2)^2 + a \times 2 + 30 = 0$

$8 - 16 + 2a + 30 = 0$

$a = -11$

$\lambda^3 - 4\lambda^2 - 11\lambda + 30 = 0.$

$(\lambda - 2) \rightarrow (\lambda^2 - 2\lambda - 15)(\lambda - 2)$

$\lambda = -3$

$\lambda = 5$

$\lambda^3 - 4\lambda^2 + a\lambda + 30 = 0$

$\lambda_1 + \lambda_2 + \lambda_3 = \text{Trace} = 4$

$\lambda_1 \lambda_2 \lambda_3 = -30$

$\lambda_2 + \lambda_3 = 2$, $\quad \lambda_2 \cdot \lambda_3 = -15$

$m \pm \sqrt{m^2 - p}$

$m = 1$, $\quad p = 15$

$1 \pm \sqrt{1+15}$

$1 + 4 = 5$

$1 - 4 = -3$

eg. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1.$

eigen vectors.

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\eta(A - \lambda I) = 2$

$0x + 1 \cdot y + 0 \cdot z = 0$

$y = 0$

$0 \cdot x + \boxed{0 \cdot y} + 0 \cdot z = 0$

+100 $\quad$ +100

x, z free variable.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$$

$$x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 1, 1, 1$

$d(\lambda_1) = 3$

$\eta(A-\lambda I) = 2$

Q. No. of LI eigen vectors corresponding to this matrix A

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = 2.$

$\eta(A - \lambda I) = 1.$

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

$\eta(A-\lambda I) = 1.$

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2y = 0$

$2z = 0$

$$\hookrightarrow x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

PROPERTIES.

1. Sum of eigen values = Trace of the matrix
2. prod. of EV's = det of matrix
3. If det (A) = 0, then atleast one of the eigen value is 0.
4. If $\rho(A) < n$, then $|A| = 0$, so atleast one of the eigen value is 0.
5. Diagonal or Triang. matrix → EV's are diagonal ele
6.

MATRIX	
A	$\lambda_1, \lambda_2, \lambda_3$
A^m	$\lambda_1^m, \lambda_2^m, \lambda_3^m$
A^{-1}	$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$
$(\mathrm{adj}\,A)$	$\frac{\lvert A\rvert}{\lambda_1}, \frac{\lvert A\rvert}{\lambda_2}, \frac{\lvert A\rvert}{\lambda_3}$
KA	$K\lambda_1, K\lambda_2, K\lambda_3$

$\mathrm{adj}(A) = |A| . A^{-1}$

Eg. $A \rightarrow 1, 2, 3$

$A^2 \rightarrow 1, 4, 9$

$A^{-1} \rightarrow \frac{1}{1}, \frac{1}{2}, \frac{1}{9}$

$\mathrm{adj}\,(A) = 6, 3, 2$ $\quad |A| = 6.$

$A + 5I \longrightarrow 6, 7, 8$

$(A+5I)^{-1} \rightarrow \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$

$(A - 5I) \rightarrow \frac{-1}{4}, \frac{-1}{3}, \frac{-1}{2}$

$$\begin{array}{ccc} 1 & 2 & 3 \\ -5 & -5 & -5 \\ \hline -4 & -3 & -2 \end{array}$$

(1) eigen value corr to 3×3 real matrix A is $1, 2, 4$ find the determinant of $A^3 - 2A$?

$A^3 \rightarrow 1^3, 2^3, 4^3 = 1, 8, 64$

$-2A \rightarrow -2, -4, -8$

$A^3 - 2A = (1, 8, 64) - (-2, -4, -8)$

$= (-1, 4, 56)$

$(\det A) = 4 \times 56 \times -1 = -224$

* If in a sq. matrix, r rows are identical then $(r-1)$ eigen values will be 0. we can say ~~othe~~ about other eigen value.

eg. $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

$= (0, 0, 6)$

eg. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ and one eigen value is 1 then find the other eigen value

$(1, 0, x)$
↳ 2 rows identical

trace $= 6$

$x + 1 = 6$

$x = 5$

eg. A is non-singular 2×2 matrix and $Tr(A) = 6$, $Tr(A^2) = 12$ find $\det(A)$ =

$\lambda_1 + \lambda_2 = 6$

$\lambda_1^2 + \lambda_2^2 = 12$

$\lambda_1 \cdot \lambda_2 = \det|A|$

$(\lambda_1 + \lambda_2)^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2$

$(6)^2 = 12 + 2\lambda_1\lambda_2$

$36 - 12 = 2\lambda_1\lambda_2$

$\frac{24}{2} = \lambda_1\lambda_2$

$\det A = \lambda_1\lambda_2 = 12$

eg. 1, 4 are eigen values corresponding to a matrix A & $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are the respective eigen vectors find $A_{2\times2}$?

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ — (1)

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ — (2)

$\cancel{a - b = 1}$

$\cancel{c - d = -1}$

$\begin{bmatrix} a + b \\ -c + d \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$a - b = 1$

$c - d = -1$ } — (1)

$2a + b = 8$

$2c + d = 4$ } — (2)

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

CALEY'S HAMILTON THEOREM

Every Matrix satisfies its char eq^n.

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad \lambda_1 = 1, \quad \lambda_2 = 2$$

$$\boxed{\lambda^2 - 3\lambda + 2 = 0}$$

$$\boxed{\lambda^2 - 3A + 2I = 0}$$ ← Always true

OTHER REPRES.

$A^{10} - 50A^5 + 25A = 0$ ⟶ always true.

$A \longrightarrow A^{100}$

$\underbrace{A \cdot A \cdot A - - - A}_{100}$

$\boxed{A^n = C_1 A + C_2 I}$

$\boxed{ax^2 + bx + c = 0}$ 2 degree poly. eq^n

$\boxed{aA^2 + bA + c = 0}$ → matrix ki eigen value satisfy karegi (True)

$A^n - C_1 A + C_2 I = 0$

ANY MATRIX OF HIGHER POWER CAN BE EXPRESSED AS LINEAR COMBINATION OF LINEAR DEGREE MATRIX AND IDENTITY MATRIX.

↳ $\boxed{A^n = C_1 A + C_2 I}$

Eg. $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ $A^{20} = ?$

diagonal

$\boxed{\lambda_1 = 2 \qquad \lambda_2 = 3}$

$(\lambda - 2)(\lambda - 3) = 0$

$\boxed{\lambda^2 - 5\lambda + 6 = 0}$

↓

$\boxed{A^2 - 5A + 6I = 0}$

$\boxed{A^{20} = C_1 A + C_2 I}$

$\boxed{\lambda^{20} = C_1 \lambda + C_2}$ ⇒

$2^{20} = 2C_1 + C_2$

$3^{20} = 3C_1 + C_2$

Now value of C_1 & C_2.

$3^{20} - 2^{20} = C_1$

$C_1 = 3^{20} - 2^{20}$

$C_2 = 2^{20} - 2(3^{20} - 2^{20})$

$C_2 = 2^{20}(1 - 2 \times 3^{20})$

or

$C_2 = 2^{20} - 2 \cdot 3^{20} + 2^{21}$

Q. Eigen values of a 4x4 matrix is $1, -1, -i, i$ then find A^4, $|A^4|$

$A \rightarrow 1, -1, -i, i$

$A^4 \rightarrow (1)^4, (-1)^4, (-i)^4, (i)^4$

$= 1, 1, 1, 1$

$|A^4| = 1.$

$A^4 = C_1 A + C_2 I$ → 4 eq^n aayegi solve karenge

$(\lambda-1)(\lambda+1)(\lambda+i)(\lambda-i) = 0$

$(\lambda^2-1)(\lambda^2+1)$

$\lambda^4 - 1 = 0$

$\boxed{\lambda^4 = 1}$

$\boxed{A^4 = I}$

nth root of unity padhna hai.

$\boxed{x^4 + 1 = 0}$

Q. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ & A satisfies $\lambda^2 + 3\lambda + 2 = 0$

× A) $A^2 + 2A + 2I = 0$

B) $(A+I)(A+3I) = 0$

C) $A + 3I + 2A^{-1} = 0$

D) $(A+3I)(A+2I) = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$A^2 + 3A + 2 = 0$

C) $A^2 A^{-1} \longrightarrow A$ $\qquad$ $A^k \cdot A^{-1} = A^{k-1}$

$A^2 A^{-1} + 3AA^{-1} + 2IA^{-1} = 0$

$A + 3I + 2A^{-1} = 0$

LU DECOMPOSITION.

LOWER TRIANGULAR MATRIX $\qquad$ UPPER TRIANGULAR MATRIX.

for any given sq. matrix A, I can always express it in the form $A = LU$ & it means we have decomposed A into product of 2 matrices.

To solve system of linear eq^n.

$Ax = B$ ← system of eqⁿ

$LUx = B$

$LY = B$

find y

$Ux = B$

find value of x.

$$\begin{bmatrix} \_ & 0 & 0 \\ x & x & 0 \\ x & \_ & x \end{bmatrix} \qquad \begin{bmatrix} x \end{bmatrix}$$

Eg. $A = \begin{bmatrix} 3 & 2 \\ 9 & 4 \end{bmatrix} \rightarrow LU$

1. form a structure of lower triangular matrix of same order having diagonal elements as 1. keep the values below the diagonal empty

$$\begin{bmatrix} 1 & 0 \\ \_ & 1 \end{bmatrix}$$

2. Try to convert A into an upper triangular matrix using Row reduction operation.

3. whatever operation we are applying reduction, reverse the scaling factor and write in place of the empty LT matrix.

eg. $A = \begin{bmatrix} 3 & 2 \\ 9 & 4 \end{bmatrix}$ $\qquad L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ ↪ LOWER TRIANG. MATRIX.

$(2,1)$

0

$R_2 \to R_2 - 3R_1$ $\qquad$ isko reverse karke

$$\begin{bmatrix} 3 & 2 \\ 0 & -2 \end{bmatrix}$$

$(2,1)$ UPPER TRIANGULAR MATRIX.

$$L \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 9 & 4 \end{bmatrix}$$

eg. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$

LU

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/3 & 1 \end{bmatrix}$$

$\downarrow R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 2 & 3 & 1 \end{bmatrix}$$

$\downarrow R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

$\downarrow R_3 \leftarrow R_3 + 1/3 R_2$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 13/3 \end{bmatrix}$$

$$3 - \frac{4}{3} = \frac{9+4}{3} = \frac{13}{3}$$