$\mathbb{R}^2$

$\mathbb{R}^n$
$\downarrow$
n LI vectors
$V_k \in \mathbb{R}^n$

$\{V_1 \dots V_n\} \cup V_k = \{\overbrace{V_1 \dots V_n, V_k}^{n+1}\}$.

{A vector space has infinitely many basises vectors}

(a) Let $V_i \in \mathbb{R}^n$ and $S = \{V_i \mid V_i \in \mathbb{R}^n ; i = 1 \text{ to } 10\}$ and this set of vectors is LI, then every subset of S is LT. $\hookrightarrow$ True

$S = \{V_1, V_2 \dots V_{10}\}$

(b) Every superset of S where superset contains few additional vectors $V_k \in \mathbb{R}^n$ ~~where~~ is LI $\rightarrow$ False

(c) Let $S = \{V_i \mid V_i \in \mathbb{R}^n, i = 1 \text{ to } 10\}$ and this set of vectors S is LD,
(i) then every superset of S is LD. ~~False~~ $\rightarrow$ True

$V_k \in \mathbb{R}^n$

$V_4 = \alpha_1 V_1 + \alpha_2 V_2 + \dots \alpha_{10} V_{10} + 0 \cdot V_k$

(ii) then every subset is LD. $\rightarrow$ False

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$$

(d) A set of n vectors from $\mathbb{R}^m$ is LD if $n > m$. $\rightarrow$ True

- **DOT (INNER) PRODUCT :-**

Let us suppose, we have two vectors u and v where $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ and } v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Now, $u \cdot v = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$
$= k \in \mathbb{R}$

Eg: $u = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}, \; v = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

so, $u \cdot v = 6 - 3 + 8 = \boxed{11}$

* if $u \cdot v = 0$ then we say that both of these vectors are Orthogonal.

Q.) $u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $v = \begin{bmatrix} 6 \\ k \\ -8 \\ 2 \end{bmatrix}$ u and v are orthogonal, then find the value of k.

Ans) $6 + 2k - 24 + 8 = 0$

$2k = 10$

$\boxed{k = 5.}$

- **NORM of a VECTOR :-**

$$u \cdot v = ||u|| \cdot ||v|| \cos\theta$$

Euclidean Norm

$v_i \in \mathbb{R}^n$

$$\left\{ ||v_i|| = \sqrt{a_1^2 + a_2^2 + \dots a_n^2} \right\}$$

- **ECHLEON FORM :-**

A matrix 'A' is called as Echleon matrix if the following two conditions are satisfied :-

① All zero rows, if any, will be at last / Bottom of the matrix.

② Each leading non-zero entry in a row is to the right of the leading non-zero entry in the preceding row

$v_i \in \mathbb{R}^n$ and $i = 1$ to 10

$\{v_1, v_2 \dots v_{10}\}$

$v_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $\quad \begin{bmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \dots & v_{10} \\ | & | & | & & | \end{bmatrix}_{n \times 10}$

Eg: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Echleon form ✓

Eg: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Echleon form ✗

Eg: $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}$

Not in Echleon form

Eg: $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 3 & 6 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Echleon form ✓

* **Pivot Element** :- If a given matrix satisfies Echleon form, then the leading non-zero element ~~is~~ * in every Row is known as the Pivot Element.

Eg :
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 3 & 6 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
Pivot Element

Jo bhi Pivot Element banega uske neeche ke saare elements 0

• **Row Reduction** :-

How to convert a given matrix into Echleon form ?

Eg :
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad v_i \in \mathbb{R}^2$$

$$R_2 \leftarrow R_2 - 3R_1$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$
Pivot Element

Eg :
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$
Pivot Element

Eg :
$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \quad v_i \in \mathbb{R}^4$$

$$R_1 \longleftrightarrow R_4$$

$$A = \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_1$$
$$R_3 \leftarrow R_3 + 2R_1$$

$$A = \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \quad \cancel{\longrightarrow} \quad \cancel{R_3 \leftarrow R_3 + (-5}$$

$\rightarrow$

$R_3 \leftarrow R_3 + \left(-\frac{5}{2}\right) R_2$

$R_4 \leftarrow R_4 + \left(\frac{3}{2}\right) R_2$

$$A = \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & \checkmark & \checkmark & \checkmark \\ 0 & 0 & \checkmark & \checkmark & \checkmark \end{bmatrix}$$

After certain more Row reductions, finally we get,

$$A = \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & -1 & -2 & 3 & 3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is Echleon form.

No. of Pivot Elements $= 3$

NOTE: Row Echleon form is not unique !!

But Reduced Echleon form is unique.

In Reduced Echleon form, every pivot element is 1.

All element above & below pivot element in a column is 0.

Eg: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- **RANK of a MATRIX :-**

Rank of a matrix A is the dimension of vector space generated or spanned by its columns.

OR

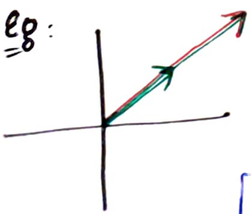
The no. of linearly independent columns of a matrix A is known as rank of that matrix.

OR

For a given matrix A in echleon form, the no. of pivot elements = rank of that matrix.

[**NOTE** : Rank of a matrix do not change after applying Row Reduction]

Eg:

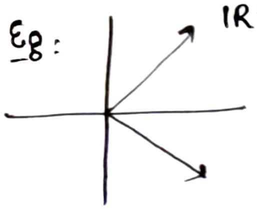


$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$

Rank = 1

Eg:



$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 \\ 0 & -4 \end{bmatrix}$

Rank = 2

* If a given Matrix is in Echleon form,

then $\rho(A)$ = # LI vectors / columns = # Pivot elements

Rank of A = Dimension of space spanned by the columns of A.

Eg: $\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\} \longrightarrow \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$ # Pivot = 1 = # LI vectors.

Eg: $\vec{v_1} \Rightarrow \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ # LI vectors ?

Solⁿ: $\Rightarrow$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow[\text{form}]{\text{Echleon}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$R_2 \leftarrow R_2 + 2R_1$

$R_3 \leftarrow R_3 - 3R_1$

$R_3 \leftarrow R_3 + R_2$

Pivot = # LI = # Rank = $\boxed{2}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ Required Echleon form.

★ Rank = Set of vectors jis space ko span kar rahe hain.★

vector $\vec{v_1}$ has 2 linearly independent vectors. Therefore, we can't represent vectors belonging to $\mathbb{R}^3$,

But we can represent vectors belonging to $\mathbb{R}^2$ and as with the help of $\vec{v_1}$ we are only able to span $\mathbb{R}^2$

$\therefore$ $\boxed{\text{RANK} = 2}$