- OPERATIONS on MATRICES :-

① Addition of Matrices :-

⇒ Two matrices can only be added if they are of the same order.

- $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ ⇒ $A + B = C_{m\times n}$

$$\boxed{C_{ij} = a_{ij} + b_{ij} \quad \forall\, i,j}$$

NOTE: Complexity of Matrix Addition & Matrix Subtraction ⇒ $\boxed{O(n^2)}$

Eg: $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times 3} + \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix}_{3\times 3}$

↳ Here we have to perform (3×3) times Arithmetic oper (Add^n).

② Multiplication of a Matrix by Scalar :-

We define it as follows :-

if $A_{m\times n} = [a_{ij}]_{m\times n}$

and k is a scalar, then

$k\cdot a_{ij} \;\forall\, i,j \longrightarrow k\cdot A_{m\times n}$

OR

$k\,[a_{ij}] \;\forall\, i,j$

OR

$[k\cdot a_{ij}]$

Lecture-2

NOTE: $[0]_{1\times 1}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ → Valid Diagonal Matrix.

* Null Matrix is not necessarily should be a Square Matrix, it can also be Rectangular Matrix.

• PROPERTIES of MATRIX :-

① **Commutative Property** :-

$$[A]_{m\times n} + [B]_{m\times n} = [C]_{m\times n} \qquad \boxed{A+B = B+A}$$

$$a_{ij} + b_{ij} = c_{ij} \ \forall\, i,j$$

② **Associative Property** :-

$$[a_{ij}]_{m\times n} + \left([b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}\right) = \left([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}\right) + [c_{ij}]_{m\times n}$$

$$\boxed{A+(B+C) = A+(B+C)}$$

③ **Additive Identity** :-

$$[a_{ij}]_{m\times n} + ? = [a_{ij}]_{m\times n} \qquad \boxed{A+0=A}$$

$$? \rightarrow [0]_{m\times n}$$

④ **Additive Inverse** :-

$$[a_{ij}]_{m\times n} + ? = [0]_{m\times n} \qquad \boxed{A+(-A)=0}$$

$$? \rightarrow [-a_{ij}]_{m\times n}.$$

*** Properties of Scalar Multiplication of a Matrix** :-

(i) $(k+l)[a_{ij}]_{m\times n} = k[a_{ij}]_{m\times n} + l[a_{ij}]_{m\times n}$

(ii) $k\left([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}\right) = k[a_{ij}]_{m\times n} + k[b_{ij}]_{m\times n}$

Ques: Find x and y if $x+y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$

$$x - y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

Solⁿ: $x = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

Multiplication of Matrices :-

$A = [a_{ij}]_{m \times n}$

$B = [b_{ij}]_{n \times p}$

Eg: $\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \times \begin{bmatrix} 5 & 4 \\ 50 & 40 \end{bmatrix}$

A (LHS) B (RHS.)

No. of columns in 1st matrix should be equal to no. of rows in 2nd matrix.

$5 \begin{bmatrix} 2 \\ 8 \end{bmatrix} + 50 \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 260 \\ 540 \end{bmatrix}$

$4 \begin{bmatrix} 2 \\ 8 \end{bmatrix} + 40 \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 208 \\ 432 \end{bmatrix}$

$= \begin{bmatrix} 260 & 208 \\ 540 & 432 \end{bmatrix}$

NOTE : $[\ ]_{m \times n} \cdot [\ ]_{n \times p} = [\ ]_{m \times p}$

$m \times \boxed{n \cdot n} \times p$

$\Downarrow$

$m \times p$

The product of matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ is a matrix C of order $m \times p$. ($C = [c_{ij}]_{m \times p}$)

Eg: *

$$\underset{3\times3}{\begin{bmatrix} a & b & c \\ d & e & f \\ x & y & z \end{bmatrix}} \quad \underset{3\times2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}$$

a_{11}, a_{12}, a_{13}; b_{11}, b_{21}, b_{31}

$m \times n \quad n \times p$

$\to m\,p\,n$

$$\underset{3\times2}{\begin{bmatrix} \frac{3}{—} & \frac{3}{—} \\ \frac{3}{—} & \frac{3}{—} \\ \frac{3}{—} & \frac{3}{—} \end{bmatrix}}$$

$a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$

$a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$

Total Multiplications $= mpn \cong O(n^3)$

Total Summations $= mp(n-1)$

Eg: Matrix A is of order 20 × 30 and matrix B is of order 30 × 10, find how many multiplications and additions are needed to perform A·B ?

Soln: Multiplications = $20 \times 30 \times 10 = 6000$

Additions = $20 \times 10 \times (30 - 1) = 5800$.

* Code for Matrix Multiplication :-

$a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}_{3\times 3}$$

$m \times \boxed{n \quad n} \times p \rightarrow m \times p$

```
for(int i=1; i<=3; i++)
{
  for(int j=1; j<=3; j++)
  {
     int sum = 0;
     for(int k=1; k<=3; k++)
     {
        sum = sum + a[i][k] · b[k][j]
     }
     c[i][j] = sum;
  }
}
```

Imp: If A·B is possible then B·A may be or may not be possible.

↳ "Matrix Multiplication is not Commutative."

→ "Matrix Multiplication is Associative."

$A(BC) = (AB)C$

NOTE : $IA = AI = A$

where I = Identity Matrix

& A = Square Matrix

Ques: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ find $B \neq 0$ such that $AB = 0$.

Sol^n: $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\times 3 \left(\begin{array}{l} a + 2c = 0 \\ 3a + 6c = 0 \end{array} \right.$ $\quad$ $\left. \begin{array}{l} b + 2d = 0 \\ 3b + 6d = 0 \end{array} \right) \times 3$

$a = -2c$ $\quad$ $b = -2d$

~~if d = 1~~

if $\boxed{\begin{array}{l} c = 1 \\ a = -2 \end{array}}$ $\quad$ if $\boxed{\begin{array}{l} d = 1 \\ b = -2 \end{array}}$

A matrix whose determinant is 0, is called Singular Matrix.

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}} \underset{B}{\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\det(A) = 0$ $\quad$ $\det(B) = 0$.

{ If $A \cdot B = 0$, then $\det(A) = 0$ & $\det(B) = 0$, that is matrix are singular. }

- **TRANSPOSE of a MATRIX :-**

$A = [a_{ij}]_{m \times n}$, then transpose of A is represented by A^T or A'.

$$\boxed{A^T = [a_{ji}]_{n \times m}}$$

Eg: $\begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 1 & 8 \end{bmatrix} \xrightarrow{\text{Transpose.}} \begin{bmatrix} 2 & 5 & 1 \\ 3 & 7 & 8 \end{bmatrix}$

Properties :

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(AB)^T = B^T A^T$

(iv) $(kA)^T = k(A^T)$
where k is constant

- Symmetric & Skew Symmetric Matrices :-

① Symmetric Matrix : $\boxed{A^T = A}$ $\boxed{[a_{ij}]^T_{n\times n} = [a_{ji}]_{n\times n}}$

only for Square Matrix.

② Skew Symmetric Matrix : $\boxed{A^T = -A}$ $\boxed{[a_{ij}]^T_{n\times n} = -[a_{ji}]_{n\times n}}$

only for Square Matrix.

NOTE : $[a_{ii}] = -[a_{ii}]$

$2[a_{ii}] = 0$

$[a_{ii}] = 0$

→ All diagonal elements are 0 in Skew Symmetric Matrix.

Proof: For any square matrix A, (i) $A + A^T$ is always symmetric. ($a_{ij} \in R$).

↳ Real No.

(ii) $A - A^T$ is always Skew - Symmetric

Sol^n: (i) $(A + A^T)^T$

$= A^T + (A^T)^T$

$= A^T + A$

$= A + A^T$

∴ Symmetric

(ii) $(A - A^T)^T$

$= A^T - (A^T)^T$

$= A^T - A$

$= -(A - A^T)$

∴ Skew Symmetric

Theorem : Any square matrix A can be represented as sum of symmetric and skew-symmetric matrix.

$$\left\{ A = \frac{2A}{2} = \frac{1}{2}\left[(A+A^T) + (A-A^T)\right] \right\}$$

Ques: If A and B are symmetric matrices, of same order, then $AB - BA$ is :

✓ (a.) Skew Symmetric   (b.) Symmetric.   (c.) Identity Matrix   (d.) Zero Matrix

Sol^n: $(AB-BA)^T = (AB)^T - (BA)^T$

$= B^TA^T - A^TB^T$

$= BA - AB$

$= -(AB-BA)$.

Given: $A^T = A$
$B^T = B$

$\therefore$ Skew Symmetric Matrix

Ques: If matrix A is both symmetric & skew-symmetric, then :

(a.) Diagonal   (b.) Skew Symmetric   (c.) Symmetric   ✓ (d.) Null

Sol^n: A → symmetric $\Rightarrow A^T = A$

A → Skew Symmetric $\Rightarrow A^T = -A$

only possible for Null Matrix