* $A = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Li columns = C_1 and C_3, corresponding sf = u, w

Ld columns = C_2 and C_4, corresponding sf = v, y.

$\boxed{\text{Scaling Factor.}}$

Corresponding to pivot

Corresponding to LD (free-variable)

Imp: To find solution we assign arbitrary value of free-variable.

① Write vectors corresponding to Li columns as equations.

$$u + 3v - y = 0$$
$$w + y = 0$$

Now write above equations, pivot wale s.f ko free variables ke terms mein.

$$\left.\begin{aligned} u &= y - 3v \\ w &= -y \end{aligned}\right\} ♡$$

Now in solution vector, replace u and w with ♡

$$\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3v + y \\ v \\ -y \\ y \end{bmatrix}$$

$$\Rightarrow v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} & \begin{array}{c} u \\ v \\ w \\ y. \end{array} \end{array}$$

* RANK - NULLITY THEOREM :-

Rank : → # Number of linearly independent columns.

→ Number of pivot elements in reduced row echleon form

Nullity : # free variables / # columns without pivot

$A_{m \times n}$ ↷ columns.

$r(A) = r \xrightarrow{\text{rank}} r$ L.I. vectors.

$(n-r)$ free variables = nullity.

$\therefore \quad r + n - r = n.$

$\rho(A) + \text{Nullity}(A) = \#$ of columns.

⇒ Nullity is dimension of Null Space.

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad V_i \in \mathbb{R}^2$$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \mathbb{R}^2 \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \begin{matrix} x \in \mathbb{R} \\ y \in \mathbb{R} \end{matrix} \right\}$$

$$Ax = 0$$

$A_{m \times n}$

↳ no. of variables

no. of equations.

① $n > m$ ⇛ If ∃ more variables than equations, then free variables would always exist.

Eg: $x + y = 0$ [equation = 1, variables = 2]

↳ $y = 1, 2 \ldots$

To satisfy $x = -1, -2 \ldots$

∴ if $n > m$, then free variables always exist

Rank$(A) = m$

Nullity $= n - m$ (free variables).

⇒ at least $n-m$ free variables

⇒ $(n-m)$ ke alaawa rest LI vectors.

Ques: The columns of A are linearly independent exactly when N(A) = {zero vector}

↳ Null Space of A.

⇒ True.

Sol^n: as N(A) = zero vector

i.e. # of free variables = 0.

→ This implies all 'n' columns are Linearly Independent

→ So, if all vectors are L.I., there must exists a unique. trivial solution i.e.

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = 0$$

$$\forall \alpha_i = 0$$

- **VECTOR SPACE and SUB-SPACE :-**

* **Field.:** Field is a set, denoted by "F".

For F to be a field, it should satisfy the following properties :- (Closure Property)

$\forall a, b \in F$

① $a + b \in \mathbb{F}$

② $a - b \in \mathbb{F}$

③ $a * b \in \mathbb{F}$

④ $\frac{a}{b} \in \mathbb{F}$ and $b \neq 0$

Eg: for $\mathbb{R}$, if $a = 1$ $b = 2$

① $1 + 2 = 3 \in \mathbb{R}$

② $1 - 2 = -1 \in \mathbb{R}$

③ $1 \times 2 = 2 \in \mathbb{R}$

④ $1/2 = 0.5 \in \mathbb{R}$

∴ a and b $\in \mathbb{R}$ (i.e. $\mathbb{F}$)

↳ here field = $\mathbb{R}$

* **Vector Space** ⇒ Let V be a non-empty set and $\mathbb{F}$ be a field then V together with two operations.

1. '+' : $v + v \in V$ [Binary Opr]

↳ $v_1, v_2 \in V(\mathbb{F})$

then $v_1 + v_2 \in V(\mathbb{F})$ [Closure Property]

2. Scalar multiplication.

$v \in$ Vector Space and $\alpha \in \mathbb{F}$

then $\alpha \cdot v \in V(\mathbb{F})$, $\forall v \in V(\mathbb{F})$

Set jaha se vector aa rahe hain

★★★ $V(\#)$, $\langle F \subseteq V \rangle$ ★★

Scalar.

Ques.: $\mathbb{R}^2(\mathbb{R})$, is it a vector space ??

Sol^n: Yes, where $\mathbb{R}$ is $\mathbb{F}$

NOTE: If $V(\mathbb{F})$ is a vector space, then $V \supseteq \mathbb{F}$

Super set

- Check whether the following are vector space or not

For $\mathbb{R}^n(\mathbb{R})$ to be valid, $\mathbb{R}$ should be proper subset of $\mathbb{R}^n$.

① $\mathbb{R}^n(\mathbb{R})$ $\quad \mathbb{R} \subseteq \mathbb{R}^n$: ✓

② $\mathbb{C}(\mathbb{R})$ $\quad \mathbb{R} \subseteq \mathbb{C}$: ✓

③ $\mathbb{C}^n(\mathbb{R})$ $\quad \mathbb{R} \subseteq \mathbb{C}^n$: ✓

~~④ $\mathbb{R}(\mathbb{R})$ $\quad \mathbb{R} \subseteq \mathbb{R}$:~~

④ $\mathbb{R}^n(\mathbb{C})$ $\quad \mathbb{C} \subseteq \mathbb{R}^n$: ✗

⑤ $\mathbb{R}(\mathbb{R})$ $\quad \mathbb{R} \subseteq \mathbb{R}$: ✓

$\mathbb{C}$: Complex No's Set

* $\mathbb{R}^2(\mathbb{C})$

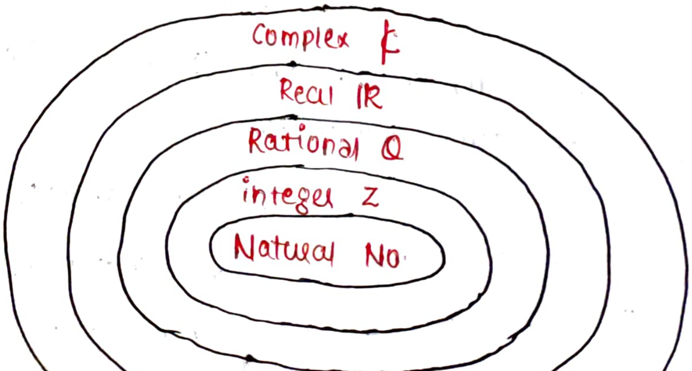
Real $\quad$ Complex

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$

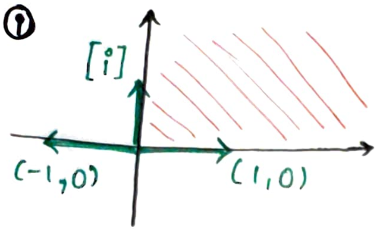
$$i \in \mathbb{C}$$

$$i \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} i \\ 2i \end{bmatrix} \notin \mathbb{R}^2$$

∴ Not a valid vector Space



1.



Is the Shaded area a valid vector space over field $\mathbb{R}$?

$\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix} \in \mathbb{R}^2$

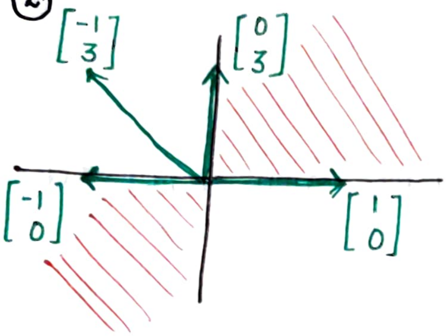
(i) $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \in \mathbb{R}^2$ ✓

(ii) $-1\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix} \rightarrow \notin$ Shaded Area

$-1 \in \mathbb{R}^2$

∴ Not a valid Vector Space

2.

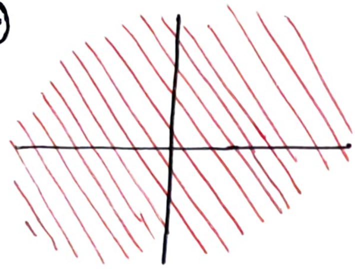


$\begin{bmatrix}-1\\0\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$

and $\begin{bmatrix}-1\\3\end{bmatrix} \notin$ 1^{st} OR 3^{rd} Quadrant.

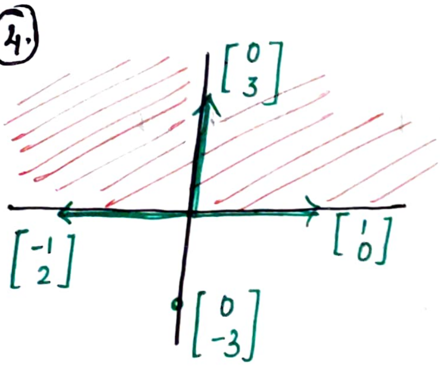
∴ Not a valid Vector Space

3.



$\mathbb{R}^2(\mathbb{R})$ is a valid vector space

4.



$\begin{bmatrix}0\\3\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix} \in$ 1^{st} Quadrant.

$-1 \cdot \begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}0\\-3\end{bmatrix} \notin$ 1^{st} OR 3^{rd} Quadrant.

$-1 \in \mathbb{R}$.

∴ Not a Valid Vector Space.

* If $V(\mathbb{F})$ is a vector space, then $0 \in V(\mathbb{F})$

↳ Zero vector existence.

→ Zero vector is always present in the vector space.

Proof :

$$v_1 \in V(\mathbb{F})$$

$$\alpha = -1, \quad -1 \in \mathbb{R} \text{ (here } \mathbb{F}\text{)}$$

$$v_2 = -v_1$$

$$v_1 + v_2 \in V(\mathbb{F})$$

$$v_1 - v_1 \in V(\mathbb{F})$$

$$0 \in V(\mathbb{F})$$