$V(\mathbb{F})$.

• VECTOR SUBSPACE :-

→ Let $V(\mathbb{F})$ be a vector space over a field $\mathbb{F}$, then a non-empty set $W \subseteq V$, is known as vector Subspace.

→ If $W(\mathbb{F})$ is a vector space.

✱ Conditions for W to be a subspace of $V(\mathbb{F})$:-

① $w_1 + w_2 \in W, \ \forall \ w_1, w_2 \in W$

② $\alpha \cdot w \in W \ \forall \ \alpha \in \mathbb{F}, \ w \in \text{Subspace}(W)$.

both above condition can be combined as :-

~~$\alpha w_1 + w_2 \in W$~~

$\alpha w_1 + w_2 \in W \ \forall \ \alpha \in \mathbb{F}, \ w_1, w_2 \in \text{Subspace}(W)$

Eg: $\mathbb{R}^2(\mathbb{R})$ is a vector space, wotf is a subspace of $\mathbb{R}^2(\mathbb{R})$

① $W_1 : \{ (x, y) \in \mathbb{R}^2 \ ; \ x^2 + y^2 = 0 \}$

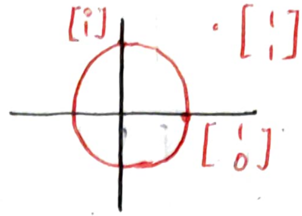
$w_a = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W_1 \quad ; \quad w_b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W_1$

$w_a + w_b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W_1$ ✓

$\alpha \cdot w_a$ OR $\alpha \cdot w_b = 5 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W_1$ ✓

$\therefore$ W_1 is a subspace of $\mathbb{R}^2(\mathbb{R})$.

② $W_2 = \{ (x,y) \in \mathbb{R}^2 : \underbrace{x^2+y^2=1}_{\text{Circle with } r=1} \}$



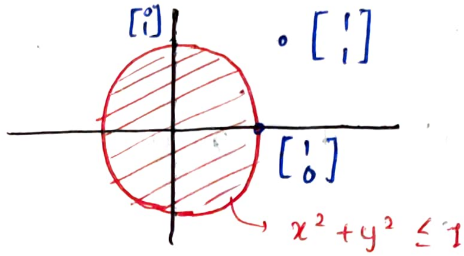
$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W_2$

$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W_2$

$v_1 + v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin W_2$

$\therefore$ Not a subspace of $\mathbb{R}^2(\mathbb{R})$

③ $W_3 = \{ (x,y) \in \mathbb{R}^2 ; x^2+y^2 \leq 1 \}$



$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W_3$

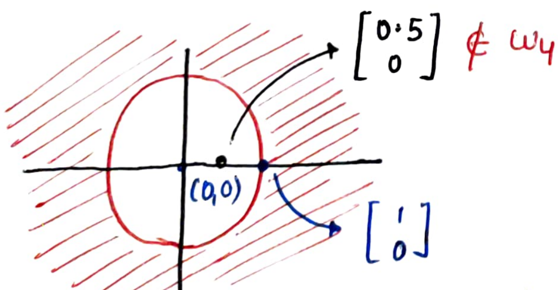
$v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W_3$

$v_1 + v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin W_2$

$\therefore$ Not a subspace of $\mathbb{R}^2(\mathbb{R})$

$\cancel{\alpha v_1}$ $\alpha[v_1] = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \notin W_3$

④ $W_4 = \{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 ; x^2+y^2 \geq 1 \}$



First of all,
zero vector is part of every subspace,
but not here.

$\therefore$ Not a subspace (Zero vector Existence)

i.e. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $0^2+0^2 \geq 1$

$0 \not\geq 1$

$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \alpha \cdot a = 0.5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \notin W_2$

(5) $W_5 = \langle (x,y) \in \mathbb{R}^2 \; ; \; x^2 - y^2 = 0 \rangle$

$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W_5$ | $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in W_5$

$a + b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin W_5$

$\hookrightarrow 2^2 - 0^2 = 0$

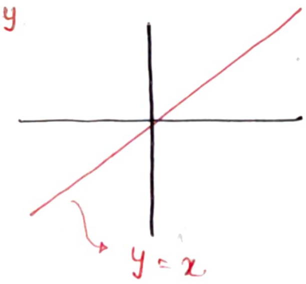
$4 \neq 0 \quad \therefore \notin W_5$

Not a Subspace

NOTE: Any line passing through origin is a subspace of $\mathbb{R}^2(\mathbb{R})$ vector space.

(6) $W_6 = \langle (x,y) \in \mathbb{R}^2 \; ; \; x - y = 0 \rangle$

$\hookrightarrow x = y$



$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W_6$

$b = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \in W_6$

$a + b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \in W_6$

$\alpha \cdot a = 5 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \in W_6$

$\therefore$ a valid Subspace of $\mathbb{R}^2(\mathbb{R})$

(7) $W_7 = \langle (x,y) \in \mathbb{R}^2 ; \; x = 3y \rangle$

Equation of a line passing through origin

$y = mx$

$x = 3y$

$y = \underbrace{\frac{1}{3}}_{m} x$

$\therefore$ Valid Subspace.

$a = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \in W_7$ | $b = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \in W_7$

$a + b = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix} \in W_7$

$\hookrightarrow 9 = 3 \cdot 3$ ✓

$$\alpha \cdot a = 5 \times \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ 5 \end{bmatrix} \in W_7.$$

$\hookrightarrow 15 = 3 \cdot 5$

$\therefore$ we may conclude that for $\mathbb{R}^2(\mathbb{R})$ space following would always a sub-space.

1. Zero vector
2. Line passing through origin
3. $\mathbb{R}^2(\mathbb{R})$ itself

(8) $W_8 = \{ (x,y) \in \mathbb{R}^2 ;\ xy = 0 \}$

$$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W_8$$

$$a + b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W_8$$

$\hookrightarrow 1 \cdot 1 \neq 0$

$\therefore$ Not a Subspace

(9) $W_9 = \{ (x,y) \in \mathbb{R}^2 ;\ x + y = 0 \}$

$\hookrightarrow y = -x$

Line passing through origin $\leftarrow$ $y = -1x$

$\hookrightarrow m$

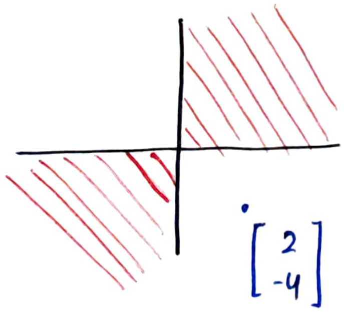
$\therefore$ A valid Subspace of $\mathbb{R}^2(\mathbb{R})$

(10) $W_{10} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x, y \geq 0 \right\}$

$\hookrightarrow$ 1st and 3rd Quadrant.

as $(1,1) \geq 0$. and

$(-1,-1) = -1 \cdot -1 = 1 \geq 0$



$$\begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$a = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \in W_{10} \quad \Big| \quad b = \begin{bmatrix} 0 \\ -4 \end{bmatrix} \in W_{10}$$

$$a + b = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \in W_{10}.$$

$\hookrightarrow 2 \times -4 \geq 0$

$8 \geq 0$: False.

(11) $w_{11} = \{ (x,y) \in \mathbb{R}^2 \mid x = 0 \}$

↳ Equation of y-axis.

⇒ Line passing through origin. ∴ Subspace.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} \in \mathbb{R}^2$$

$$a = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in w_{11} \qquad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \in w_{11}$$

$$a + b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \in w_{11}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in w_{11} \quad (y = 0)$$

↳ Zero Vector Existence also satisfied.

$$\alpha [a] = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \in w_{11} \checkmark$$

↳ $x = 0$.

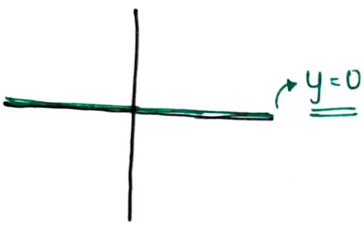
∴ A valid subspace.

(12) $w_{12} = \{ (x,y) \in \mathbb{R}^2 \mid y = 0 \}$

↳ Equation of x-axis

⇒ Line passing through origin

∴ Valid Subspace



(13) $w_{13} = \{ (x,y) \in \mathbb{R}^2 \mid y^3 = x \}$

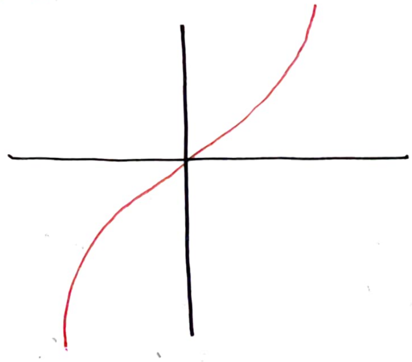
↳ $y = \sqrt[3]{mx}$

$$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$a + b = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \in w_{13}$$

↳ $2^3 = 2$

$8 \neq 2$



↳ Passing through origin but curve, and not a line

∴ Not a Valid Subspace

- Subspaces for $\mathbb{R}^3(\mathbb{R})$ vector space :-

1. 0 vector is always a subspace of $\mathbb{R}^3(\mathbb{R})$.
2. Line through origin is always a subspace of $\mathbb{R}^3(\mathbb{R})$.
3. Plane through origin is always a subspace of $\mathbb{R}^3(\mathbb{R})$.
4. $\mathbb{R}^3(\mathbb{R})$ itself.

* Important Results :-

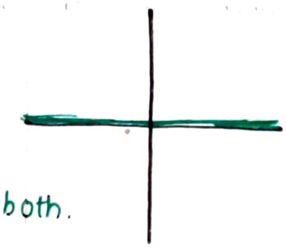
→ Let $V_1(\mathbb{F})$ and $V_2(\mathbb{F}) \subseteq V(\mathbb{F})$ be two subspaces, then,

$V_1 \cap V_2$ is always a subspace and it is the largest subspace in V_1 and V_2

Eg: for $\mathbb{R}^2(\mathbb{R})$ vector space let $V_1 \to x$-axis, $V_2 \to x$-y plane $(\mathbb{R}^2(\mathbb{R}))$

$\Rightarrow$ $V_1 \cap V_2 = x$ axis.

Largest isliye kyuki "$\cap$" contain vector which are part of subspace V_1 and V_2 both.



→ If V_1 and V_2 are subspaces of $V(\mathbb{F})$, then $V_1 \cup V_2$ need not be subspace.

Eg: (1) $V(\mathbb{F}) = \mathbb{R}^2(\mathbb{R})$

$V_1 \longrightarrow x$-axis

$V_2 \longrightarrow y$-axis.

$V_1 \cup V_2$

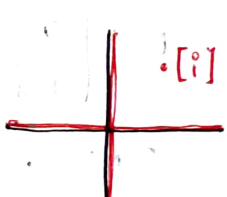
$$V_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdots \right\}$$

$$V_2 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \cdots \right\}$$

$$V_1 \cup V_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

$V_1 \cup V_2$ Not a Subspace as.

$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V_1 \cup V_2$



$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V_1 \cup V_2$

$a + b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V_1 \cup V_2$

$\therefore$ if V_1 is a subspace and V_2 is a subspace, their union need not to be a subspace as well.

Eg: ② $V(\mathbb{F}) = \mathbb{R}^2(\mathbb{R})$

$V_1 \longrightarrow x\text{-axis}$

$V_2 \longrightarrow \mathbb{R}^2(\mathbb{R})$.

$V_1 \cup V_2 = x\text{-}y$ Plane $(\mathbb{R}^2(\mathbb{R}))$

$\therefore$ $V_1 \cup V_2$ is a Valid Subspace.

From above examples, we may say that.

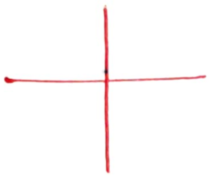
★ $V_1 \cup V_2$ is a subspace iff $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$

* $V_1 + V_2$ is always a subspace of $V(\mathbb{F})$ and $V_1 + V_2$ is the smallest subspace containing V_1 and V_2.

Eg: $V(\mathbb{F}) = \mathbb{R}^2(\mathbb{R})$

$V_1 \longrightarrow x\text{-axis}$

$V_2 \longrightarrow y\text{-axis}$.



X-Y Plane.

$a = \begin{bmatrix} x \\ 0 \end{bmatrix} \in x\text{-axis}$. $\qquad b = \begin{bmatrix} 0 \\ y \end{bmatrix} \in y\text{-axis}$

$a + b = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2(\mathbb{R})$ / X-Y Plane

$\therefore$ Subspace.

$$V_1 = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \cdots\cdots \right\rangle$$

$$V_2 = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \cdots\cdots \right\rangle$$

$$V_1 + V_2 = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \cdots\cdots \right\rangle$$

$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V_1$ | $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V_2$

$a + b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in V_1 + V_2$

$\therefore$ Valid Subspace

- **Dimension of a Sub Space :-**

→ Number of vectors in a Basis of that space

OR

Minimum no. of L.I. vectors needed to span sub-space.

Eg : $\mathbb{R}^2(\mathbb{R})$ is a vector space, let w = x-axis be a sub-space, find w's dimension.

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdots \right\rangle$$

dimension $(w) = 1$ as only 1 vector needed to represent w.

→ $\langle 0 \rangle$ is a sub-space of $\mathbb{R}^2(\mathbb{R})$, its dimension ??

Vectors needed to represent zero vector = 0.

Dimension = 0.

Ques: $V(\mathbb{F}) = \mathbb{R}^3(\mathbb{R})$, w = line ~~passin~~ through origin is a subspace, dimension ??

⇒ dimension $(w) = 2$.

Ques: Consider the set of (column) vectors defined by

$$X = \{ x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \text{, where } x^T = [x_1, x_2, x_3]^T \}$$

Which of the following is True ??

(a) $\langle [1, -1, 0]^T, [1, 0, -1]^T \rangle$ is a basis for subspace X.

(b) $\langle [1, -1, 0]^T, [1, 0, -1]^T \rangle$ is a LI set, but it does not span X and ∴ is not basis of X.

(c) X is not a subspace of $\mathbb{R}^3$

(d) None of the above.

$$a = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} , \quad a, b \in X$$

$$a + b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \in X$$

$\hookrightarrow 0 - 1 + 1 = 0$ ✓

$$\alpha \cdot b = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix}$$

$\hookrightarrow 0 + 5 - 5 = 0$ ✓

$\therefore$ Option c is wrong.

$$a = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; \quad x_1 + x_2 + x_3 = 0$$

$$x_1 = -(x_2 + x_3)$$

$$a = \begin{bmatrix} -(x_2 + x_3) \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$a = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We are able to write a, $\vec{a} \in X$ in terms of other $\vec{v} \in X$. which means $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are L.I. and as we are able to write $a \in X$ in ~~the~~ terms of these LI.

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are Basis Vectors.

Ques.: If V_1 and V_2 are 4-dimensional subspace of a 6-dimensional vector space V, then the smallest possible dimension of $V_1 \cap V_2$ is $\underline{2}$.

Sol^n: ⇒ as V_1 is a 4-dimensional subspace, we need 4 vectors to represent an element of V_1.

Similarly, 4 vectors for V_2

Vector space is 6 dimensional which means we need 6 vectors.

let $V = \langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$ be those vectors to represent V.

V_1 as 4d would have 4 vectors. Similarly, V_2

$$V_1 = \langle a_1, a_2, a_3, a_4 \rangle$$

$$V_2 = \langle a_1, a_2, a_3, a_4 \rangle$$

$$V_1 \cap V_2 = 4 \; (\langle a_1, a_2, a_3, a_4 \rangle)$$

we need to find smallest possible dimension of $V_1 \cap V_2$

$$V_1 = \langle a_1, a_2, a_3, a_4 \rangle$$

$$V_2 = \langle a_1, a_2, a_5, a_6 \rangle$$

$$V_1 \cap V_2 = \cancel{4}\; 2 \; (\langle a_1, a_2 \rangle)$$

↳ 2 is the smallest possible dimension.

NOTE: $\dim(V_1 \cup V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$

Ques: let V_1 and V_2 be 4-dimensional sub-space of 6-dimensional vector space V, what is the min. dimension of $V_1 \cap V_2$?

Sol^n: $\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cup V_2)$

$$\dim(V_1 \cap V_2) = 4 + 4 - \dim(V_1 \cup V_2)$$

we need to minimise this.

→ for minimising the $\dim(V_1 \cap V_2)$, the dimension $(V_1 \cup V_2)$ should be maximum.

maximum is possible when

$V_1 \cup V_2 = V$ or $V_1 \subseteq V_2 = V$

or

$V_2 \subseteq V_1 = V$

$$\dim(V_1 \cap V_2) = 4 + 4 - \dim(V_1 \cup V_2)$$

$$= 8 - \underline{\dim(V_1 \cup V_2)}.$$

↳ This max^m value is 6 when

$V_1 \subseteq V_2 = V$

or

$V_2 \subseteq V_1 = V$

So, $\min(\dim(V_1 \cap V_2))$

$= 8 - 6$

$= 2$

Que: Dim of $V = 120$, let W_1 and W_2 be two subspaces of V of dimension 70 and 80 respectively, then $\min[\text{Dim}(W_1 \cap W_2)] = ?$

Sol^n: $\min(\dim(W_1 \cap W_2)) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cup W_2)$

$= 70 + 80 - 120$

$= 150 - 120$

$= \boxed{30}$