Ques: Let $Ax = b$ be a system of linear equations where $A$ is an $m \times n$ matrix and $b$ is a $m \times 1$ column vector and $X$ is an $n \times 1$ column vector of unknowns. Which of the following is False??

(A.) The system has a solution if and only if, both $A$ and the augmented matrix $[Ab]$ have the same rank. T

(B.) If $m < n$ and $b$ is the zero vector, then the system has $\infty$ely many solutions. T

(C) If $m = n$ and $b$ is a non-zero vector, then the system has unique solution. F

(D.) The system will have only trivial solution when $m = n$, $b$ is the zero vector and rank$(A) = n$. T

Solⁿ: $Ax = b$

$\rho(A:B) = \rho(A) = n \longrightarrow$ Unique Solⁿ

$\rho(A:B) = \rho(A) = r < n. \longrightarrow$ $\infty$ely many Solⁿ.

for option (A): for solⁿ to exist $\rho(A:B) = \rho(A)$

$\therefore$ Solution exists { unique or $\infty$ }

But exists. $\therefore$ True.

Option (B): $m < n$

Let $m = 2$ and $n = 3$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right]$$

$$\downarrow R_2 \leftarrow R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

$Ax = b = 0$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 3y + 4z = 0$

$-2y - 2z = 0$

As pivot = 2, we have 2 L.I. and let $z$ be the 'scaling factor of $\vec{Ld}$', and as its $\vec{Ld}$ on other 2 vectors

So, they can represent $\vec{Ld}$ for any value of $z$, $\therefore$ $z$ is called as 'free' variable.

as $z$ can be anything implies that we can have $\infty$ely many solution.

Eg: $z = -1$ $\quad -2y + 2 = 0$ & $x + 3 - 4 = 0$

$y = 1 \qquad x = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Similarly $\infty$ely many solⁿ as $z$ could be anything

$z \in \mathbb{R}$

$\therefore$ True.

**Option (C)** :

$$\begin{array}{cccc} v_1 & v_2 & v_3 & B \\ \end{array}$$
$$\left[\begin{array}{ccc|c} O & - & - & - \\ - & O & - & - \\ - & - & - & - \end{array}\right] \quad \begin{array}{l} m=3 \\ n=3 \end{array}$$

Let pivot in only $v_1$ & $v_2$ → $\boxed{\infty\text{ely many sol}^n}$

Let Pivot be present in $v_1, v_2$ and B → No Solution

$$\begin{array}{cccc} v_1 & v_2 & v_3 & B \\ \end{array}$$
$$\left[\begin{array}{ccc|c} O & - & - & - \\ - & O & - & - \\ - & - & - & O \end{array}\right]$$

According to statement, system of L.E. have unique sol$^n$ but no sol$^n$ & $\infty$ely many sol$^n$ are possible too. → ∴ False

**Option (D)** : Statement holds true as, when $m=n$, $b=0$ and $\rho(A)=n$, then the given system of Linear Equation is Homogeneous.

$$\left[\begin{array}{ccc|c} O & - & - & 0 \\ - & O & - & 0 \\ - & - & O & 0 \end{array}\right] \quad \rho(A)=n$$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ iff $\alpha_i = 0$ ∴ True

**Ques** : Consider the systems, each consisting of $m$ linear equations in $n$-variables.

I. If $m<n$, then all such systems have a solution. F

II. If $m>n$, then none of these systems has a solution. F

III. If $m=n$, then there exists a system which has a solution. T

Which is correct ?

(a.) I, II and III are true

(b.) Only II and III are true

(c) Only III is true. ✓

(d.) None of them is true

**Sol$^n$**: (I) Let $m=2$ and $n=3$ $(m<n)$

$$\left[\begin{array}{ccc|c} O & - & - & - \\ - & - & - & O \end{array}\right]$$
A : B

we found a case where $m<n$ but still it can have no solution.
Hence, false.

(II) Let $m=3$ and $n=2$ $(m>n)$

$$\left[\begin{array}{cc|c} O & - & - \\ - & - & - \\ - & O & - \end{array}\right] \quad \text{OR} \quad \left[\begin{array}{cc|c} O & - & - \\ - & O & - \\ - & - & - \end{array}\right]$$

Here, also we found a case where sol$^n$ exists $(r<m)$
∴ $\infty$ely many Sol$^n$.
Hence, false.

(III) Let $m=3$ and $n=3$ $(m=n=3)$

$$\left[\begin{array}{ccc|c} O & - & - & - \\ - & O & - & - \\ - & - & O & - \end{array}\right] \qquad \left[\begin{array}{ccc|c} O & - & - & - \\ - & O & - & - \\ - & - & - & - \end{array}\right]$$

→ unique solution

→ $r<3$ $(r<m)$
∴ $\infty$ely many sol$^n$.

∴ Statement is True that ∃ a system (we found 2 of them) which has a solution.

Ques: Suppose $\alpha, \beta, \gamma \in \mathbb{R}$, consider the following system of linear equation

IIT JAM.

$$x + y + z = \alpha$$
$$x + \beta y + z = \gamma$$
$$x + y + \alpha z = \beta$$

If the system has atleast one sol$^n$, then which of the following statements are True?

$\beta = 1$

(a.) if $\alpha = 1$ then $\gamma = 1$
(b) if $\beta = 1$ then $\gamma = \alpha$
(c.) if $\beta \neq 1$ then $\alpha = 1$
(d) if $\gamma = 1$, then $\alpha = 1$

Sol$^n$:

interprets as
(a) if $\alpha = 1$ then for a solution to exist $\gamma = 1$.

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \beta & 1 \\ 1 & 1 & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \gamma \\ \beta \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1$
$R_3 \leftarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 0 & \beta - 1 & 0 & \gamma - \alpha \\ 0 & 0 & \alpha - 1 & \beta - \alpha \end{array}\right]$$

Echleon form

Just put the values given in the options to get the answer !!

Ques: $w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $w_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$

Which of the following is True / False ?

(a) $\langle w_1, w_2 \rangle$ spans $\mathbb{R}^2$ True

(b) $\langle w_1, w_2, w_3 \rangle$ spans $\mathbb{R}^2$ True

(c) $\langle w_1, w_2 \rangle$ spans $\mathbb{R}$ False

(d) $\langle w_1, w_2 \rangle$ spans $\mathbb{R}^3$ False

Ans:
$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$w_1$ and $w_2$ are Linearly independent, $w_3$ is Linearly dependent

$\rho(A) = 2$

$\therefore$ 2 LI exists

Ques: $\langle \ln x, \ln x^2, \ln x^3 \rangle$. is this set LI / LD ?

Ans. $\Rightarrow$ $\langle \ln x, 2\ln x, 3\ln x \rangle$

~~$\alpha \ln x$~~ $\alpha \ln x + \beta 2 \ln x + \gamma 3 \ln x = 0$

$(\alpha + 2\beta + 3\gamma) \ln x = 0$

$\ln \neq 0$, so $\alpha + 2\beta + 3\gamma = 0$

$\alpha = 1$, $\beta = 1$, $\rho = -1$

$\alpha, \beta, \gamma \neq 0$ and sol$^n$ exists apart from $\alpha_i = 0$, $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

i.e. $\exists$ non-zero scalars, hence Linearly Dependent.

Ques: Let $\alpha, \beta$ and $\gamma$ be real no's. Consider the following system of linear equations

$$x + 2y + z = 7$$
$$x + \alpha z = 11$$
$$2x - 3y + \beta z = \gamma$$

(a) $P \to 3$ ; $Q = 1$ ; $R \to 2$

(b) $P \to 2$ ; $Q = 1$ ; $R \to 3$

(c) $P \to 3$ ; $Q = 2$ ; $R \to 1$ ✓

(d) $P \to 1$ ; $Q = 1$ ; $R \to 2$

(P): if $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma = 28$, then the system has

(Q): if $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma \neq 28$, then the system has

(R): If $\beta \neq \frac{1}{2}(7\alpha - 3)$, where $\alpha = 1$ and $\gamma \neq 28$, then system has

(1) ~~Unique~~ No Sol$^n$

(2) Unique Sol$^n$

(3) Infinitely many Sol$^n$

Sol$^n$:

$$\begin{array}{ccc} A & x & B \\ \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & \alpha \\ 2 & -3 & \beta \end{bmatrix} & \begin{bmatrix} x \\ y \\ z \end{bmatrix} = & \begin{bmatrix} 7 \\ 11 \\ y \end{bmatrix} \end{array}$$

$A : B$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 0 & \alpha & 11 \\ 2 & -3 & \beta & y \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & \alpha-1 & 4 \\ 0 & -7 & \beta-2 & y-28 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + \frac{7}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & \alpha-1 & 4 \\ 0 & 0 & \beta-2-\frac{7}{2}(\alpha+1) & y-28 \end{array}\right]$$

(Echleon form)

(iii) $\alpha = 1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & \alpha-1 & 4 \\ 0 & 0 & \beta-2-\frac{7}{2}(\alpha+1) & y-28 \end{array}\right]$$

On $\alpha = 1$, $\beta - 2 - \frac{7}{2}(\alpha+1) = \beta - 2 - \frac{14}{2} = \beta - 2 - 7 = \beta - 9$.

$\beta = 9$

On $\alpha = 1$ and $\beta = 9$, $\beta - 2 - \frac{7}{2}(\alpha+1)$

$= 9 - 2 - \frac{7}{2}(1+1)$

$= 7 - 7 = 0$

$\therefore$ Matrix on $\alpha = 1$ and $\beta = 9$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & y-28 \end{array}\right]$$

$y \neq 28$, Let $y = 29$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\therefore$ No Solution

Option (c)