# Problem-4 : Neural Networks

Given in the question, the neural network contains two hidden layers. Considering all the assumptions, we can write the equation as —

$$W^1,\ x \rightarrow \underbrace{W^1 x}_{z^1} \quad \underbrace{\sigma(z^1)}_{a^1} \longrightarrow \underbrace{W^2 a^1}_{z^2} \quad \underbrace{\sigma(z^2)}_{a^2} \longrightarrow \underbrace{W^3 a^2}_{z^3} \quad \underbrace{\frac{e^{z_i^3}}{\sum_j e^{z_j^3}}}_{a^3 = \hat{y}_i}$$

Using Backpropagation procedure,

1. Break up equations

- $a^3 = \hat{y}_i = \dfrac{e^{z_i^3}}{\sum_j e^{z_j^3}}$.

- $z^3 = W^3 a^2$

- $a^2 = \sigma(z^2)$

- $z^2 = W^2 a^1$

- $a^1 = \sigma(z^1)$

- $z^1 = W^1 x$.

Here, $\sigma(z) = \dfrac{1}{1+e^{-z}}$

$\Rightarrow$ The loss function is given by $L = \|y - \hat{y}\|^2 = (y_1 - \hat{y}_1)^2 + (y_2 - \hat{y}_2)^2$

$\Rightarrow$ We need to derive $\dfrac{\partial L}{\partial W^3}$, $\dfrac{\partial L}{\partial W^2}$ and $\dfrac{\partial L}{\partial W^1}$

i) To find $\dfrac{\partial L}{\partial W^3}$

$$\frac{\partial L}{\partial W^3} = \frac{\partial L}{\partial a^3} \cdot \frac{\partial a^3}{\partial z^3} \cdot \frac{\partial z^3}{\partial W^3} \quad \text{--- (1)}$$

Here

$$\frac{\partial L}{\partial a^3} = \frac{\partial L}{\partial \hat{y}} = 2(y - \hat{y}) \cdot (-1) = \underline{\underline{-2(y - \hat{y})}}$$

For $\dfrac{\partial a^3}{\partial z^3}$, WKT, $a^3 = \hat{y} = \dfrac{e^{z_i^3}}{\sum_j e^{z_j^3}}$

There are two cases,

if $i \neq j$,

$$\frac{\partial a^3}{\partial z^3} = \frac{\partial}{\partial z_j^3}\left(\frac{e^{z_i^3}}{e^{z_j^3} + \sum_{k \neq j} e^{z_k^3}}\right) = \frac{\left(e^{z_j^3} + \sum e^{z_k^3}\right)0 - e^{z_i^3}\left(e^{z_j^3}\right)}{\left(e^{z_j^3} + \sum_{k \neq j} e^{z_k^3}\right)^2}$$

$$= \frac{-e^{z_i^3}}{\left(\sum e^{z_k^3}\right)} \times \frac{e^{z_j^3}}{\left(\sum e^{z_k^3}\right)}$$

$$= -a_i^3 a_j^3 \quad [\text{if } i \neq j]$$

if $i = j$,

$$\frac{\partial a^3}{\partial z^3} = \frac{\partial}{\partial z_i^3}\left(\frac{e^{z_i^3}}{e^{z_i^3} + \sum_{k \neq i} e^{z_k^3}}\right) = \frac{\left(e^{z_i^3} + \sum_{k \neq i} e^{z_k^3}\right)\frac{\partial}{\partial z_i}\left(e^{z_i^3}\right) - e^{z_i^3} \times \frac{\partial}{\partial z_i}\left(e^{z_i^3} + \sum e^{z_k^3}\right)}{\left(e^{z_i^3} + \sum e^{z_k^3}\right)^2}$$

$$= \frac{\left(e^{z_i^3} + \sum e^{z_k^3}\right)\left(e^{z_i^3}\right) - e^{z_i^3} \cdot e^{z_i^3}}{\left(e^{z_i^3} + \sum e^{z_k^3}\right)^2}$$

$$= \frac{\cancel{e^{z_i^3} \cdot e^{z_i^3}} + e^{z_i^3} \sum_{k \neq i} e^{z_k^3} - \cancel{e^{z_i^3} \cdot e^{z_i^3}}}{\left(e^{z_i^3} + \sum_{k \neq i} e^{z_k^3}\right)^2}$$

$$= \frac{e^{z_i^3}}{\left(\sum e^{z_k^3}\right)} \times \frac{\sum_{k \neq i} e^{z_k^3}}{\left(\sum e^{z_k^3}\right)} = \frac{e^{z_i^3}}{\left(\sum e^{z_k^3}\right)} \times \left[1 - \frac{e^{z_i^3}}{\sum e^{z_k^3}}\right]$$

$$= a_i(1 - a_i)$$

Therefore,

$$\frac{\partial a^3}{\partial z^3} = \begin{cases} -a_i^3 a_j^3 & i \neq j \\ a_i(1 - a_i) & i = j \end{cases}$$

For $\frac{\partial z^3}{\partial w^3} = a^2$

Substituting the values into the equation ①,

$$\frac{\partial L}{\partial w^3} = \begin{cases} -2(y-\hat{y}) \times (-a_i^3 a_j^3) \times a^2 & i \neq j \\ -2(y-\hat{y}) \times (a_i(1-a_j)) \times a^2 & i = j \end{cases}$$

ii) To find $\frac{\partial L}{\partial w^2}$

$$\frac{\partial L}{\partial w^2} = \frac{\partial L}{\partial a^3} \frac{\partial a^3}{\partial z^3} \frac{\partial z^3}{\partial a^2} \frac{\partial a^2}{\partial z^2} \frac{\partial z^2}{\partial w^2} \quad \text{——— ②}$$

For $\frac{\partial z^3}{\partial a^2} = w^3$

For $\frac{\partial a^2}{\partial z^2} = a^2(1-a^2)$

For $\frac{\partial z^2}{\partial w^2} = a^1$

Substituting the values are found out previously into equation ②,

$$\frac{\partial L}{\partial w^2} = \begin{cases} -2(y-\hat{y}) \times (-a_i^3 a_j^3) \times w^3 \times (a^2(1-a^2)) \times a^1 & i \neq j \\ -2(y-\hat{y}) \times (a_i(1-a_i)) \times w^3 \times (a^2(1-a^2)) \times a^1 & i = j. \end{cases}$$

iii) To find $\frac{\partial L}{\partial w^{1}}$

$$\frac{\partial L}{\partial w^1} = \frac{\partial L}{\partial a^3} \cdot \frac{\partial a^3}{\partial z^3} \cdot \frac{\partial z^3}{\partial a^2} \cdot \frac{\partial a^2}{\partial z^2} \cdot \frac{\partial z^2}{\partial a^1} \cdot \frac{\partial a_1}{\partial z_1} \cdot \frac{\partial z_1}{\partial w^1} \quad - (3)$$

For $\frac{\partial z^2}{\partial a^1} = w^2$

For $\frac{\partial a_1}{\partial z_1} = a^1(1-a^1)$

For $\frac{\partial z_1}{\partial w^1} = x$

Substituting the values we found previously into equation (3),

$$\frac{\partial L}{\partial w_1} = \begin{cases} -2(y-\hat{y})(-a_i^3 a_j^3)\, w^3 \left(a^2(1-a^2)\right) w^2 \cdot a^1(1-a^1)\, x & i \neq j \\ -2(y-\hat{y})\left(a_i(1-a_i)\right) w^3 \left(a^2(1-a^2)\right) w^2 \cdot a^1(1-a^1)\, x & i = j \end{cases}$$