Part-I HW-02 Answers

① $T(n) = T(n-3) + 3\lg n$. Our guess $T(n) = O(n\lg n)$

We need to show that $T(n) \leq cn\lg n$ $(\forall c>0, n>n_0)$

So, using the guessing function in the above Recurrence equation.

We get, $T(n) = T(n-3) + 3\lg n$

$$T(n) \leq c(n-3)\lg(n-3) + 3\lg n$$

$$T(n) \leq c(n-3)\lg n + 3\lg n$$

Because, logn is monotonically increasing for $n>0$, we can deduce $\lg(n-3)$ as $\lg n$. $[\lg n > \lg(n-3)]$

$$T(n) \leq cn\lg n - 3c\lg n + 3\lg n$$

Removing lower order terms like $-3c\lg n$ and $3\lg n$

$$T(n) \leq cn\lg n.$$

Hence, $T(n) \leq cn\lg n$, Hence our guess was correct.

Thus, Proved. $\therefore T(n) = O(n\lg n)$

② $T(n) = 4T\left(\frac{n}{3}\right)+n$. Our guess: $T(n) = O(n^{\log_3 4})$.

We need to show that $T(n) \leq cn^{\log_3 4}$ $(\forall c>0, n>n_0)$

So,

$$T(n) = O(n^{\lg_3 4})$$

$$\leq cn^{\lg_3 4} \quad \forall c>0 \quad \text{—— ①}$$

$$T(n) = 4T\left(\frac{n}{3}\right)+n$$ ⟨Using equation ① in the recurrence given above⟩

$$T(n) \leq 4c\left(\frac{n}{3}\right)^{\lg_3 4}+n$$

$$T(n) \leq \frac{4}{3^{\lg_3 4}} cn^{\lg_3 4}+n$$

$$T(n) \leq \frac{\cancel{4}}{\cancel{4}} cn^{\lg_3 4}+n$$

$$T(n) \leq cn^{\lg_3 4}+n$$

Because $n^{\lg_3 4} > n$

$$T(n) \leq cn^{\lg_3 4}$$

Thus, $T(n) \leq cn^{\lg_3 4}$. Hence, our initial guess was correct.

Thus, Proved. $\therefore T(n) = O(n^{\log_3 4})$

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③ $T(n) = T\left(\frac{n}{2}\right)+T\left(\frac{n}{4}\right)+T\left(\frac{n}{8}\right)+n$. Our guess: $T(n) = O(n)$

We need to show that $T(n) \leq cn$ $(\forall c>0, n>n_0)$ —①

So,

$$T(n) = T\left(\frac{n}{2}\right)+T\left(\frac{n}{4}\right)+T\left(\frac{n}{8}\right)+n$$

$$T(n) \leq c\left(\frac{n}{2}\right)+c\left(\frac{n}{4}\right)+c\left(\frac{n}{8}\right)+n$$ ⟨Using equation ① in the recurrence given above⟩

$$T(n) \leq \left(\frac{c}{4}+\frac{c}{16}+\frac{c}{64}+1\right)n$$

$$T(n) \leq c'n \qquad \text{let } c'\left(\frac{c}{4}+\frac{c}{16}+\frac{c}{64}+1\right)$$

let $c = 64$

$c' = \left(\frac{c}{4}+\frac{c}{16}+\frac{c}{64}+1\right) = (16+4+1+1) = 22$, which is positive.

Hence, $T(n) \leq c'n$. [proved] $\therefore T(n) = O(n)$

④ $T(n) = 4T\left(\frac{n}{2}\right) + n^2$. Our guess $T(n) \leq O(n^2)$

We need to show that $T(n) \leq cn^2$ [$\forall\, c > 0, n > n_0$]

So,

$$T(n) \leq O(n^2)$$

$$T(n) \leq cn^2 \quad \forall\, c > 0 \quad \text{———} \quad (1)$$

Also,

$$T(n) \leq 4T\left(\frac{n}{2}\right) + n^2$$

⟨ using equation (1) in the recurrence given above ⟩

$$\leq 4c \cdot \left(\frac{n}{2}\right)^2 + n^2$$

$$\leq (c+1)n^2$$

Let $c' = (c+1)$

$$T(n) \leq c'n^2$$

Hence, $T(n) \leq c'n^2$ for constant $c' > 1$. Our guess is correct.

Thus, Proved. $\therefore T(n) \leq O(n^2)$