# Spline

cubic spline (what we are going to do)

- polynomial btw each pair of points
- smooth (≡ continuous + differentiable)
- Avoids oscillations
- require continuity not only for the first at data points, but also for 2nd derivative

$$p(x) = a_j (x - x_j)^3 + b_j (x - x_j)^2 + c_j (x - x_j) + d_j \quad \text{— (1)}$$

$$\text{for} \quad x_j \leq x \leq x_{j+1}$$

this is the polynomial for the region btn $j^{th}$ & $(j+1)^{th}$ data points

require $p(x)$ be exact at $x_j$

$$p(x_j) = \boxed{f(x_j) = d_j}$$

It should also be exact at

$x_{j+1}$

$$p(x_{j+1}) = p_{j+1} = a_j h_j^3 + b_j h_j^2 + c_j h_j + p_j \quad \text{—} \ (2)$$

where $h_j = x_{j+1} - x_j$

the first derivative

$$p'(x) = 3a_j(x-x_j)^2 + 2b_j(x-x_j) + c_j \quad \text{——} \; 3$$

$$p''(x) = 6a_j(x-x_j) + 2b_j \quad \text{——} \; 4$$

For the $2^{nd}$ derivative at $x = x_j$,

$$p''(x_j) = p''_j = 2b_j$$

$$\Rightarrow \boxed{b_j = \frac{p''_j}{2}}$$

$$p''(x_{j+1}) = p''_{j+1}$$

$$= 6a_j h_j + 2b_j$$

$$a_j = \frac{p''_{j+1} - 2b_j}{6h_j}$$

Take eqn (2), substitute for $a_j$ and $b_j$, so that we get

$$c_j = \frac{p_{j+1} - p_j}{h_j} - \frac{h_j p''_{j+1} + 2h_j p''_j}{6}$$

$$p(x) = p_j + \left[ \frac{p_{j+1} - p_j}{h_j} - \frac{h_j p''_{j+1}}{6} - \frac{h_j p_j''}{3} \right] (x - x_j) + \frac{p_j''}{2} (x - x_j)^2 + \frac{p''_{j+1} - p_j''}{6 h_j} (x - x_j)^3$$

for $x_j \leq x \leq x_{j+1}$

eqn 5

the first derivative

$$p'(x) = \frac{p_{j+1} - p_j}{h_j} - \frac{h_j \, p''_{j+1}}{6}$$

eqn(6) $$- \frac{h_j \, p''_j}{3} + p''_j (x - x_j)$$

$$+ \frac{p''_{j+1} - p''_j}{2h_j} (x - x_j)^2$$

for $x_j \leq x \leq x_{j+1}$

Use the previous expr of the first derivative for $\boxed{x_{j-1} \le x \le x_j}$

eqn (*)

$$p'(x) = \frac{p_j - p_{j-1}}{h_{j-1}} - \frac{h_{j-1} p_j''}{6} - \frac{h_{j-1} p_{j-1}''}{3} + p_{j-1}''(x - x_{j-1}) + \frac{(p_j'' - p_{j-1}'')(x - x_{j-1})^2}{2h_{j-1}}$$

We now say that (6) and (7) should match at $x = x_j$

Equate (6) and (7) at $x_j$ and rearrange terms

$$h_{j-1} \boxed{p''_{j-1}} + (2h_j + 2h_{j-1}) \boxed{p''_j} + h_j \boxed{p''_{j+1}} = 6\left(\frac{p_{j+1} - p_j}{h_j} - \frac{p_j - p_{j-1}}{h_{j-1}}\right)$$

for $j = 2, 3, \ldots (n-1)$.