Runge-Kutta-Fehlberg (RKF) adaptive step size method for solving ordinary differential equations.

Remember the RK4 algorithm we used, where the derivative is evaluated at the mid-point multiple times and with each time it is used for improving the final derivative used for propagating the solution to the edge of the interval.

Fehlberg modified the Runge-Kutta methods such that two different orders (for example 4th and 5th order RK) of function evaluation use the same function evaluations for the derivative inside the interval. This methods helps reduce the computing load in developing an adaptive step-size algorithm using the 4th and 5th order methods. The algorithm is called RKF45.

Let us start of with a differential equation $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$. Let us define a interval of size h. We now write down the equations for $y(x_0 + h)$, the RK4 evaluation of the function at $x_0 + h$, and $\hat{y}(x_0 + h)$ the RK5 evaluation of the function at the same point.

$$y = y_0 + h\left(\frac{25}{216}f_0 + \frac{1408}{2565}f_2 + \frac{2197}{4104}f_3 - \frac{1}{5}f_4\right) \tag{1}$$

$$\hat{y} = y_0 + h\left(\frac{16}{135}f_0 + \frac{6656}{12825}f_2 + \frac{28561}{56430}f_3 - \frac{9}{50}f_4 + \frac{2}{55}f_5\right)$$

Therefore,

$$\hat{y} - y = h\left(\frac{1}{360}f_0 - \frac{128}{4275}f_2 - \frac{2197}{75240}f_3 + \frac{1}{50}f_4 + \frac{2}{55}f_5\right) \tag{2}$$

Following the method we learnt in class for the new step size h_{new} that allows function evaluation at $x_0 + h$ with a tolerance of ϵh_{new}, where ϵ is an appropriate numerical value of your choice,

$$h_{new} = 0.9h\sqrt[4]{\frac{|h|\varepsilon}{|y(x_0+h) - \hat{y}(x_0+h)|}}. \tag{3}$$

Notice that you need not evaluate $\hat{y}$ but can use the equation-2 to obtain eqn-3 directly.

Problem. One dimensional projectile motion with air resistance.

Equation for rate of change of momentum,

$$\frac{dp}{dt} = mg - kv^2$$

Find the velocity as a function of time, of a ball released from rest at $t = 0$, for $0 < t < 10$ s. The values of other parameters are $g = 9.8$ ms^{-2}, $k = 10^{-4}$ kg/m, and $m = 10^{-2}$ kg. Tolerance $\epsilon = 10^{-5}$.