Introduction to Differential Equations – Math 286 X1
Fall 2009
Homework 9 — due November 11

1. Determine the fundamental period of the following functions:

   (a) $\cos(2t)$

   (b) $\sin(2\pi t)$

   (c) $\sin^2(t)$

   (d) $\cos(t) + \sin(t)$

   Solution: We know from trig that the fundamental period of $\cos(t)$ or $\sin(t)$ is 2π. Clearly, then, if we consider $\cos(\alpha t)$ or $\sin(\alpha t)$ for any real α, then this should have fundamental period $2\pi/\alpha$. To see this, notice that $\cos(\alpha t)$ has a period of $2\pi/\alpha$:

   $$\cos(\alpha(t + 2\pi/\alpha)) = \cos(\alpha t + 2\pi) = \cos(\alpha t),$$

   and, moreover, if p is a period for $\cos(\alpha t)$, then we have

   $$\cos(\alpha(t + p)) = \cos(\alpha t + \alpha p) = \cos(\alpha t)$$

   for all t, and therefore $\alpha p = 2k\pi$ for some integer k, and thus $p = 2k\pi/\alpha$ for some integer k. The smallest of these is choosing $k = 1$, or $2\pi/\alpha$.

   Once we know all this, solving (a,b) is straightforward, and we obtain π and 1, respectively.

   For (c), we see that clearly any period of $\sin(t)$ is also a period of $\sin^2(t)$, i.e. if we have a p with $\sin(t + p) = \sin(t)$ for all t, then clearly, also $\sin^2(t + p) = \sin^2(t)$ for all t. However, it is possible that $\sin^2(t)$ has a smaller period, because if we require that

   $$\sin^2(t + p) = \sin^2(t)$$

   then this means that

   $$\sin(t + p) = \pm \sin(t).$$

   If it is possible to solve this equation with the minus sign with a p smaller than 2π, then we have a smaller fundamental period. But note that

   $$\sin(t + \pi) = -\sin(t)$$

   for all t, and thus

   $$\sin^2(t + \pi) = \sin^2(t)$$

   and we have a smaller fundamental period.

   Part (d) is a bit trickier. We need a p which solves

   $$\cos(t + p) + \sin(t + p) = \cos(t) + \sin(t)$$

   for all t. Now, of course, any multiple of 2π will work, but can we do it with a smaller p? We use the trig identities

   $$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$
   $$\sin(A + B) = \cos A \sin B + \sin A \cos B.$$

Thus we have

$$\begin{aligned}\cos(t+p)+\sin(t+p) &= \cos t\cos p - \sin t\sin p + \cos t\sin p + \sin t\cos p\\ &= (\cos p+\sin p)\cos t + (\cos p - \sin p)\sin t.\end{aligned}$$

Thus we need

$$\begin{aligned}\cos p + \sin p &= 1,\\ \cos p - \sin p &= 0,\end{aligned}$$

or $\cos p = 1, \sin p = 0$. This is solved by exactly $p = 2k\pi$, and thus 2π is the fundamental period.

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2. Is the function $f(t) = \cos(t) + \cos(4t)$ periodic? If yes, demonstrate this by finding a period of the function. Same questions for $g(t) = \cos(t) + \cos(\pi t)$.

   Solution: Yes it is, and the way to see that is to note that the periods of $\cos(t)$ are $2k\pi$, and the periods of $\cos(4t)$ are $l\pi/2$, where k, l are integers. These sets of numbers share a common element, namely 2π (choose $k = 1, l = 4$) and thus $f(t)$ is periodic with period 2π.

   On the other hand, this will not work for g; notice that the periods of $\cos(t)$ are $2k\pi$, but the periods of $\cos(\pi t)$ are $2l$, and

   $$2k\pi = 2l$$

   only if $\pi = l/k$, which would mean π is rational, which it is not.

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3. Let $f(t)$ be a 2π-periodic function defined by

   $$f(t) = \begin{cases} 3, & -\pi < t < 0,\\ -4, & 0 < t < \pi,\\ 132, & t = 0, \pi.\end{cases}$$

   Compute its Fourier series.

   Solution: We know that the 132 in the formula will not matter at all, so we can ignore it. Here we have $L = \pi$.

   We use the formula for

   $$\begin{aligned}A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt\\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} 3\cos(nt)\,dt + \int_0^{\pi} -4\cos(nt)\,dt\right),\\ &= \frac{1}{\pi}\left(\frac{3}{n}\sin(nt)\bigg|_{t=-\pi}^{t=0} + \frac{-4}{n}\sin(nt)\bigg|_{t=0}^{t=\pi}\right)\\ &= \frac{1}{\pi}(0-0+0-0) = 0,\end{aligned}$$

   but this formula only works for $n > 0$ since we divided by n. For

   $$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{p}i(-\pi) = -1.$$

For B_n, we compute

$$
\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} 3\sin(nt)\,dt + \int_{0}^{\pi} -4\sin(nt)\,dt\right), \\
&= \frac{1}{\pi}\left(\frac{-3}{n}\cos(nt)\Big|_{t=-\pi}^{t=0} + \frac{4}{n}\cos(nt)\Big|_{t=0}^{t=\pi}\right) \\
&= \frac{1}{n\pi}(-3(1-(-1)^n) + 4((-1)^n - 1)),
\end{aligned}
$$

which is 0 when n is even, but $-14/n$ when n is odd. Therefore the Fourier series of f is

$$-\frac{1}{2} + \sum_{n \text{odd}} \frac{-14}{n}\sin(nt).$$

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4. Let $f(t)$ be a 2π-periodic function defined by $f(t) = |t|$ for $t \in [-\pi, \pi]$ and extended periodically elsewhere. Compute its Fourier series.

   Solution: We first note that f is even and therefore all of the B_n are zero. Moreover, we can use the Fourier cosine series coefficient formula for A_n and save a bit of writing, so we have

$$
\begin{aligned}
A_n &= \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos(nt)\,dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} t\cos(nt)\,dt, \\
&= \frac{2}{\pi}\left(\frac{t\sin(nt)}{n}\Big|_{t=0}^{t=\pi} - \int_{0}^{\pi}\frac{1}{n}\sin(nt)\,dt\right) \\
&= \frac{2}{\pi}\left(\frac{t\sin(nt)}{n} + \frac{\cos(nt)}{n^2}\Big|_{t=0}^{t=\pi}\right) = \frac{2}{\pi}\begin{cases} 0, & n \text{ even}, \\ \frac{-2}{n^2}, & n \text{ odd}, \end{cases}
\end{aligned}
$$

   where, again, this does not work if $n = 0$. For A_0 we compute

$$A_0 = \frac{2}{\pi}\int_{0}^{\pi} t\,dt = \pi.$$

   Thus the Fourier series is

$$\frac{\pi}{2} + \sum_{n \text{odd}} \frac{-4}{n^2}\cos(nt).$$

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5. Define f to be the function with period 3 defined as

$$f(t) = t^2, \quad -3/2 < t < 3/2.$$

   Compute its Fourier series.

Solution: Again note that t^2 is even so we need not compute B_n. We have

$$A_n = \frac{2}{3}\int_{-3/2}^{3/2} t^2 \cos(2\pi n t/3)\, dt.$$

To simplify notation, we first compute

$$\int t^2 \cos(n\pi t/L)\, dt,$$

and after two integrations by parts, we obtain

$$\int t^2 \cos(n\pi t/L)\, dt = -\frac{t^2 L}{n\pi}\sin(n\pi t/L) + \frac{2tL^2}{n^2\pi^2}\cos(n\pi t/L) - \frac{2L^3}{n^3\pi^3}\sin(n\pi t/L).$$

Computing the definite integral (evaluating all of these terms at L and $-L$) gives

$$0 - 0 + \frac{2L^3}{n^2\pi^2}\cos(n\pi) + \frac{2L^3}{n^2\pi^2}\cos(-n\pi) = (-1)^n \frac{4L^3}{n^2\pi^2}.$$

Of course, we also have to do the separate calculation

$$A_0 = \frac{1}{L}\int_{-L}^{L} t^2\, dt = \frac{1}{L}\left.\frac{t^3}{3}\right|_{t=-L}^{t=L} = \frac{2}{3}L^2.$$

Plugging in $L = 3/2$ gives

$$\begin{aligned} A_0 &= \frac{3}{2}, \\ A_n &= (-1)^n \frac{27}{2n^2\pi^2}. \end{aligned}$$

So we have a Fourier series of

$$\frac{3}{4} + \sum_{n=1}^{\infty} (-1)^n \frac{27}{2n^2\pi^2}\cos(2n\pi t/3).$$

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6. Prove that

$$\int_{-\pi}^{\pi} \cos(nt)\sin(mt) = 0$$

for any integers n, m. **Hint:** Think about even and odd functions.

Solution: cos is even and sin is odd, therefore their product is odd, therefore the integral over any symmetric interval is zero.

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7. Prove that

$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt) = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases}$$

Hint: For $m \neq n$, use the trig identity

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)).$$

Why does this calculation fail when $m = n$?

Solution: We compute

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nt)\cos(mt) &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((m+n)t) + \cos((m-n)t)\,dt \\ &= \frac{1}{2}\ \frac{\sin((m+n)t)}{m+n} + \frac{\sin((m-n)t)}{m-n}\Bigg|_{t=-\pi}^{t=\pi} \\ &= \frac{1}{2}(0-0+0-0) = 0,\end{aligned}$$

but notice that we divided by $m-n$, so this formula is invalid for $m=n$.