Project - 03

We have

$$a = \sqrt{\frac{m\omega}{2}}\, x + i \frac{1}{\sqrt{2m\omega}}\, p$$

$$a^* = \sqrt{\frac{m\omega}{2}}\, x - i \frac{1}{\sqrt{2m\omega}}\, p$$

Therefore

$$x = \frac{1}{\sqrt{2m\omega}}(a + a^*)$$

and

$$p = -i\sqrt{\frac{m\omega}{2}}(a - a^*)$$

Now the Hamiltonian for the oscillator is given by

$$H_{os} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

Now putting the values of p and x we have

$$H_{os} = \frac{1}{2m}\left(-\frac{m\omega}{2}\right)(a - a^*)^2 + \frac{1}{2} m\omega^2 \frac{1}{2m\omega}(a + a^*)^2$$

or

$$H_{os} = -\frac{\omega}{4}(a - a^*)^2 + \frac{1}{4}\omega (a + a^*)^2$$

$$= -\frac{\omega}{4}\left[(a - a^*)^2 - (a + a^*)^2\right]$$

$$= -\frac{\omega}{4}\left[a^2 + (a^*)^2 - 2aa^* - a^2 - (a^*)^2 - 2aa^*\right]$$

$$= \omega a a^*$$

Hence $\underline{H_{os} = \omega a a^*}$

Now the total Hamiltonian is given by

$$H = \omega a a^* + \frac{\omega_s}{2}\sigma_z + \frac{g}{2}(a\sigma_+ + a^*\sigma_-)$$

Now the quantum mechanical part of the Hamiltonian can be treated separately by

$$\hat{h}|\psi\rangle = i|\dot{\psi}\rangle \quad \text{———} \; (I)$$

$$\text{with } |\dot{\psi}\rangle = \frac{\partial \psi}{\partial t}$$

$$\text{and } \quad |\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$$

Now we have

$$\hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $\hat{h}$ takes the form

$$\hat{h} = \frac{\omega_s}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{g}{2}\left[\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^* & 0 \end{pmatrix}\right]$$

for $g = 2$ we have

$$\hat{h} = \begin{pmatrix} \omega_s/2 & \frac{g a}{2} \\ a^* \frac{g}{2} & -\omega_s/2 \end{pmatrix}$$

Hence from equation (I) we have

$$\hat{h}|\psi\rangle = i|\dot{\psi}\rangle$$

or
$$\begin{pmatrix} \omega_s/2 & \frac{g}{2}a \\ a^*\frac{g}{2} & -\omega_s/2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = i\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix}$$

Hence we have

$$\frac{\omega_s}{2}c_1 + \frac{g}{2}a\,c_2 = i\dot{c}_1 \quad \text{—— (A)}$$

$$\frac{g}{2}a^* c_1 - \frac{\omega_s}{2}c_2 = i\dot{c}_2 \quad \text{—— (B)}$$

with the initial conditions

$c_1(0) = 1$, $c_2(0) = 0$ and $|c_1|^2 + |c_2|^2 = 1$ all the times.

Now for the third equation we have the effective Hamiltonian

$$H_{eff} = H_{os} + \frac{\omega_s}{2}\langle\psi|\hat{\sigma}_z|\psi\rangle + a^*\langle\sigma_-\rangle_\psi + a\langle\sigma_+\rangle_\psi$$

Now $\underline{a^*\langle\sigma_-\rangle_\psi = a^*\langle\psi|\sigma_-|\psi\rangle}$

$$= a^*\begin{pmatrix} c_1^* & c_2^* \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= a^*\begin{pmatrix} c_1^* & c_2^* \end{pmatrix}\begin{pmatrix} 0 \\ c_1 \end{pmatrix} = \underline{a^* c_2^* c_1}$$

And $a\langle\sigma_+\rangle_\psi = a\langle\psi|\sigma_+|\psi\rangle$

$$= a\begin{pmatrix} c_1^* & c_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = a\begin{pmatrix} c_1^* & c_2^* \end{pmatrix}\begin{pmatrix} c_2 \\ 0 \end{pmatrix}$$

$$= \underline{a\,c_1^* c_2}$$

And $\langle\psi|\sigma_z|\psi\rangle = (c_1^* \; c_2^*)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$

$$= (c_1^* \; c_2^*)\begin{pmatrix}c_1\\ -c_2\end{pmatrix} = \underline{c_1^* c_1 - c_2^* c_2}$$

Now the effective Hamiltonion can be written as

$$H_{eff} = \omega a a^* + \frac{\omega_s}{2}(c_1^* c_1 - c_2^* c_2)$$
$$+ \frac{g}{2}\left[a^* c_2^* c_1 + a c_1^* c_2\right]$$

$\underline{\text{with } g = \text{coupling Parameter}}$

Now for the third differential equation we write

$$\dot{a} = \{a, H_{eff}\}$$

or $\left(\frac{da}{dt}\right) = \{a, H_{eff}\}$

$$= \{a, \omega a a^* + \frac{\omega_s}{2}(c_1^* c_1 - c_2^* c_2) + \frac{g}{2}(a^* c_2^* c_1 + a c_1^* c_2)\}$$

$$= \{a, \omega a a^*\} + \{a, \frac{g}{2}(a^* c_2^* c_1 + a c_1^* c_2)\}$$

$$= \{a, \omega a a^*\} + \{a, \frac{g}{2} a^* c_2^* c_1\}$$

$$= \omega a \{a, a^*\} + \frac{g}{2} c_2^* c_1 \{a, a^*\}$$

$$= \left(\omega a + \frac{g}{2} c_2^* c_1\right) \{a, a^*\}$$

$$= \left(\omega a + \frac{g}{2} c_2^* c_1\right) \left\{\sqrt{\frac{m\omega}{2}}\, x + i\frac{p}{\sqrt{2m\omega}},\ \sqrt{\frac{m\omega}{2}}\, x - i\frac{p}{\sqrt{2m\omega}}\right\}$$

$$= -i\left(\omega a + \frac{g}{2} c_2^* c_1\right)$$

Hence we have the three coupled equations:

$$\frac{da}{dt} = -i\left(\omega a + \frac{g}{2} c_2^* c_1\right) \quad \text{(A)}$$

$$i\left(\frac{dc_1}{dt}\right) = \frac{\omega_s}{2} c_1 + a c_2 \frac{g}{2} \quad \text{(B)}$$

$$i\left(\frac{dc_2}{dt}\right) = \frac{g}{2} a^* c_1 - \frac{\omega_s}{2} c_2 \quad \text{(C)}$$

with the initial conditions

$c_1(0) = 1$

$c_2(0) = 0$, $|c_1|^2 + |c_2|^2 = 1$ All the time

and $x = x_0$ at $t = 0$

$p = p_0$ at $t = 0$

Now these three coupled equations can be solved to obtain a, c_1 and c_2.