Problem-1. Before a patient undergoes a minor operation, a certain anaesthesia is injected in the muscle of the upper arm. From there it slowly flows into the blood where it exerts its sedating effect. From the blood it is picked up by the liver, where it is ultimately degraded. We write the following model for the amount of anaesthesia in the muscle M, blood B and liver L:

$$\frac{dM}{dt} = -eM, \quad \frac{dB}{dt} = eM - cB, \quad \text{and} \quad \frac{dL}{dt} = cB - \delta L$$

where the parameter e is the efflux from the muscle, c is the clearance from the blood, and δ is the degradation in the liver. All parameters are rates per hour. The degradation in the muscle and blood is assumed to be negligible.

The initial amount of anaesthesia injected is M (0): 315 mg, the amount in the muscle at time zero.

Let the parameter values of e = 0.5, c = 0.3 and $\delta = 0.4$.

(a) Sketch the amounts of anaesthesia in the muscle, M (t), and in the blood, B (t), as a function of time.

(b) How long does it take before half of the injected amount has flown from the muscle to the blood?

(c) Suppose the degradation rate is slow, i.e., if δ tends to 0, how much anaesthesia will ultimately end up in the liver?

Problem-2. 1. The famous Lotka-Volterra predator-prey model in non-dimensional form is given by

$$\frac{du}{d\tau} = u(1-v)$$
$$\frac{dv}{d\tau} = \alpha v(u-1)$$

where $u(\tau)$ is the non-dimensional prey population, $v(\tau)$ is the non-dimensional predator population and α is a positive rate constant.

(i) Using R, solve this differential equation for $u(\tau) = v(\tau) = 1.3$ and $\alpha = 0.1$. Set $\tau_{end} = 100$. Plot u and v as functions of τ. Then plot $u(\tau)$ versus $v(\tau)$ (phase space trajectory) and see if you get a closed curve in the phase space.

(ii) The system has two sets of fixed points, $(0,0)$ and $(1,1)$. Linearize the differential equation about these fixed points and show that the fixed point $(0,0)$ is a saddle point and the fixed point $(1,1)$ is a center (purely imaginary eigen values). Use R to do it.

Problem-3. A population dynamics model with predation can be written in a dimensionless form as

$$\frac{dx}{d\tau} = \alpha x(1 - \frac{x}{K}) - \frac{x}{1+x} = f(x)$$

where α is the growth rate and K is the carrying capacity. Let us choose $K = 20$.

(a) For $\alpha = 1.3$, $\alpha = 0.3$ and $\alpha = 0.1$, plot $f(x)$ vs x using R. Geometrically identify the fixed points and comment on their stability.

(b) For $\alpha = 0.3$, numerically solve the above differential equation for two different initial conditions $x(\tau = 0) = 1.2$ and $x(\tau = 0) = 3.5$. Use $\tau_{end} = 100s$. Do you see any difference in the steady state values? If so, what is the reason for this difference? If not, explain why there should not be any difference