What is Numerical method ? Why we use numerical method ?

Ans:
Numerical method : Numerical method is a technique by which mathematical problem are formulated. So that they can be solved with arithmatic operations.
Digital computer, calculators are used to solve numerical method manually.

Reasons of using numerical method

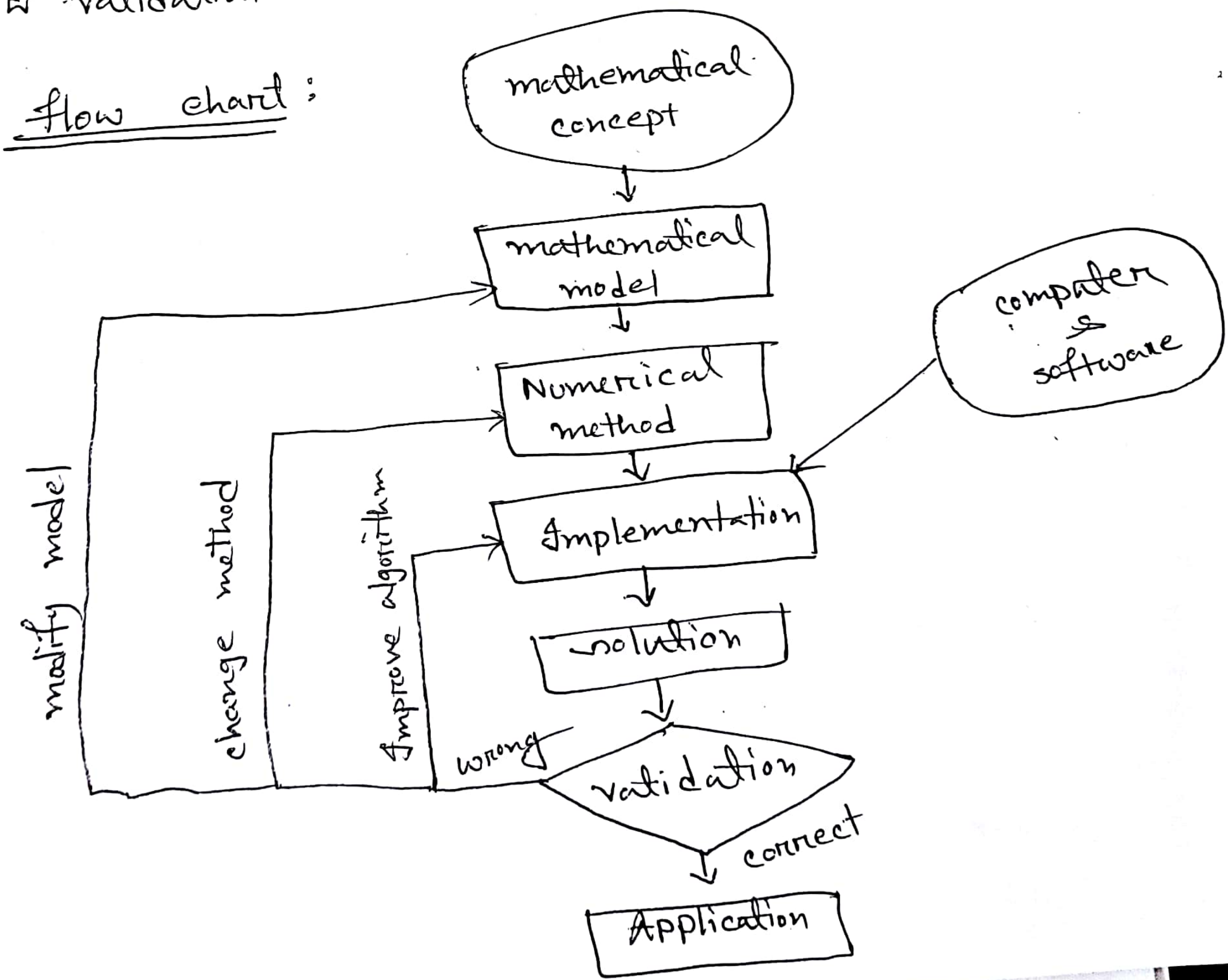
- Numerical methods are extremely powerful problem solving tools. They are capable of handling large number of equations, non-linearities and complicated geometry.
- Numerical method is an efficient tools to find the error.
- It is a vehicle to understand mathematics.
- We can design program to solve problem if we are good in numerical method.

Describe the process of solving engineering prob using Numerical method.

The process of solving engineering problem usin numerical methods are given below -

- Formation of mathematical model.
- construction of an appropriate numerical method.
- Implementation the method to obtain a solution.
- validation of the solution.

<u>flow chart</u>:



Numerical error :

There are different types of numerical error .

1. Round off error
2. Truncation error
3. True error
4. Relative error
5. Approximate relative error
6. Total error

Round off error : we omitt some ~~som~~ number that carry some error which called round off error .

As we write $2.37602 = 2.38$.

Truncation error : The error which result when approximation are used to represent exact mathematical procedures are known as truncation error .

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \quad \cdots (1)$$

1st order; now, $e^{0.5}$ when $x = 0.5$

$= 1.6487$

when, $e^x = 1$

$1.6487 = 1$

error $= 0.6487$

2^{nd} error :

$$e^x = 1 + x$$

$$\Rightarrow 1.6487 = 1 + 0.5$$

error $= 0.1487$

3^{rd} order :

$$e^x = 1 + x + \frac{x^2}{2!}$$

$$\Rightarrow 1.6487 = 1 + 0.5 + \frac{2.5}{2}$$

error $= 0.0237$

4^{th} order :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\Rightarrow 1.6487 = 1 + 0.5 + \frac{2.5}{2} + \frac{(.5)^3}{6}$$

error $= 0.0029$

Find the truncation error in the result of the following function for $x = 1/5$ when we use

a) First three terms b) First four terms

c) First five terms.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

a) $e^{0.2} = 1 + 0.2 + \frac{(0.2)^2}{2!}$

$\Rightarrow 1.2214 = 1.2200$

error $= 0.0014$

b) $e^{0.2} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\Rightarrow 1.2214 = 1 + .2 + \frac{0.04}{2} + \frac{0.008}{6}$

error $= 0.000067$

c) $1.2214 = 1 + .2 + \frac{0.04}{2} + \frac{0.008}{6} + \frac{0.0016}{24}$

error $= 0.0000028$

True error :

We know that the true value is equal to the summation of approximation and the True error. That means.

True error = True value − Approximation

Relative error :

$$\text{Relative error} = \frac{\text{True value} - \text{Approximation}}{\text{True value}}$$

Approximate Relative error :

$$\text{Approximate Relative error} = \frac{\text{Current Appx} - \text{Previous Apprx}}{\text{Current Apprx}}$$

What are the advantages of numerical technique in solving problem instead of using conventional mathematical method?

- With the help of numerical method higher order equation can be easily solved where as it is very difficult by using convensional mathematical method.
- In this method a sequence of steps is represented again and again Through which better solutions can be obtained.
- Easily solvable with the help of a computer.
- Many problem may be solved with the help of a some algorithm.
- Rounding error are negligible.

Root of eq^n

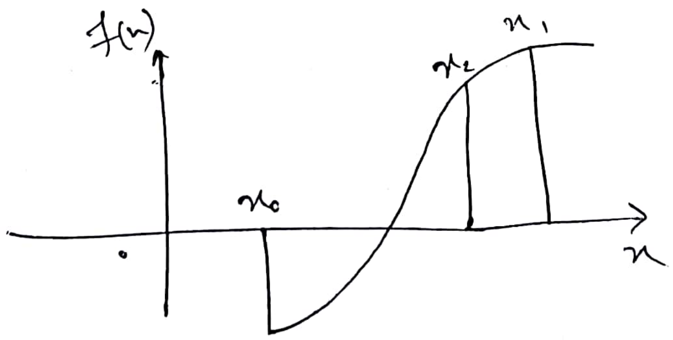
Breaking method : This method starts with the guess of the bracket that contain the root then systemitically reduce the weidth of the bracket.

Bracketing method are classified into two categories.

① Bisection method (II) False position method.

Bisection method :

The bisection method is a alternatively called chopping method. It is one kind of incremental search method in which the interval is always divided by two. If a f^n changes value over an interval, the function value at the mid point is evaluated.



Algorithm for Bisection method :

An algorithm could be defined as follows. suppose we need a root for $f(x) = 0$ and we have an error tolarence ε.

Step 1 ; Find two numbers a and b at which f has different sign.

Step 2 : Define $c = \frac{a+b}{2}$

Step 3 ; If $b - c \leq \varepsilon$ then accept c as the new root and stop.

Step 4 ; If $f(a) \cdot f(c) \leq 0$ then set c as new b otherwise set c as new a and return to step 1.

Prob 1:

$f(x) = x^2 - 2x - 3$ with initial estimate $[a,b] = [2.0, 3.2]$

Use the bisection method to solve it.

Soln: we know,

$$f(x) = x^2 - 2x - 3$$

$$f(2.0) = -3$$

$$f(3.2) = 0.84$$

No of iteration	a	b	c	f(c)	ε%
1	2.0	3.2	2.6	-1.44	100%
2	2.6	3.2	2.9	-0.39	10.34%
3	2.9	3.2	3.05	0.2025	4.91%
4	2.9	3.5	2.975	-0.099	-2.52%
5	2.975	3.05	3.0125	0.05	1.2448%
6	2.975	3.0125	2.993	-0.027	-0.651%
7	2.993	3.0125	3.0027	0.0108	0.323%

2. $f(x) = x^3 + x^2 - 1$ with initial guesses $[a,b] = [0,1]$. Use bisection method to solve it. [max 7 iteration]

we know,

$$f(x) = x^3 + x^2 - 1$$

$$f(0) = -1$$

$$f(1) = 1$$

No of iteration	a	b	$c = \frac{a+b}{2}$	f(c)	ε %
1	0	1	0.5	-0.625	100%
2	0.5	1	0.75	-0.015625	33.33%
3	0.75	1	0.875	0.435525	14.29%
4	0.75	0.875	0.8125	0.196526	-7.69%
5	0.75	0.8125	0.78125	0.08715	-4%
6	0.75	0.78125	0.7656	0.0348	-2.044%
7	0.75	0.7656	0.7578	0.0094	-1.0292%

the root is 0.7578.

Advantages of Bisection method :

1. The method is guaranted to converge

2. The error bound to decrease by half with each iteration.

3. It is easy ~~an~~ always convergent.

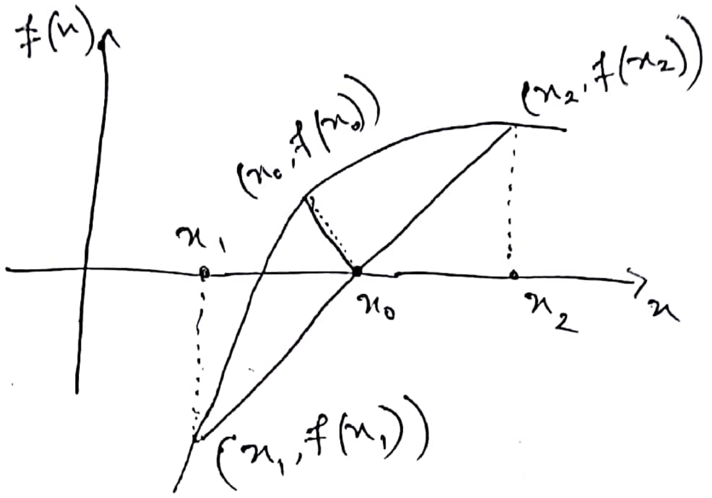
Disadvantages :

1. The main disadvantages of bisection method is like that compared with Newton - Raphson method and secant method, it requires a lot of iteration to get answer with very small error.

2. It converges very slowly.

3. The bisection method cant detect multiple roots.

False position method :



A graphical representation of the false position method is shown in the figure we know that the eqⁿ of the line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are given below;

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y - f(x_1)}{x - x_1} \quad \text{---} \quad (1)$$

Since the line intersects the x axis at x_0, so when $x = x_0$ then $y = 0$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-f(x_1)}{x_0 - x_1}$$

or, $$(x_0 - x_1) = -\frac{f(x_1)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

then, we have,

$$x_0 = x_1 - \frac{f(x_1)\,(x_2 - x_1)}{f(x_2) - f(x_1)} \quad \text{--- (11)}$$

So, eq^n (11) is known as false position method.

False position method algorithm :

let, $x_0 = x_1 - \dfrac{f(x_1)\,(x_2 - x_1)}{f(x_2) - f(x_1)}$

if, $f(x_0) \times f(x_1) < 0$

set $x_2 = x_0$.

otherwise, $x_1 = x_0$.

Problem 1 :

Use the false position method to find a root of the function, $f(x) = x^2 - x - 2 = 0$, in the range $1 < x < 3$.

Iteration 1 :

Given $f(x) = x^2 - x - 2$ and $x_1 = 1$, $x_2 = 3$

$f(x_1) = -2$

$f(x_2) = 4$

we know,

$$x_0 = x_1 - \frac{f(x_1)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 1 - \frac{-2(3-1)}{4+2}$$

$$= 1 + \frac{2}{3}$$

$$= 1.6667$$

$f(x_0) = (1.6667)^2 - 1.6667 - 2 = 2.7778 - 1.667 - 2 = -0.888811)$

$f(x_0) \times f(x_1) = -0.8888 \times -2 = 1.7776$

So, $f(x_0) \times f(x_1) > 0$

so, update $x_1 = x_0$

<u>Iteration 2:</u>

we know,

$$x_0 = 1.667 - \frac{-0.8888(3 - 1.6667)}{4 + 0.8888}$$

$$= 1.9094$$

$f(x_0) = (1.9094)^2 - 1.9094 - 2 = -0.2636$

$1.98^2 - 1.98 - 2$

$1.9763^2 - 1.9768 - 2$

$3.9204 - 1.98 - 2$

$f(x_0) \times f(x_1) = -0.2636 \times -0.8888$

$= 0.2343$

So, $f(x_0) \times f(x_1) > 0$

So update $x_1 = x_0$

Iteration 3:

We know,

$$x_0 = 1.9094 - \frac{-0.2636\,(3 - 1.9094)}{4 + 0.2636}$$

$= 1.9768$

So, the root is 1.9768.

Prob 2: Find an appropriate value of the root of the eq^n $x^3 + x - 1 = 0$ near $x = 1$, by the method of falsi using the formula twice, suppose $x_1 = 0.5$, $x_2 = 1$

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Sol^n: Iteration 1:

Given,

$f(x) = x^3 + x - 1$

and $x_1 = 0.5$, $x_2 = 1$.

$f(x_1) = (0.5)^3 + 0.5 - 1 = -0.375$

$f(x_2) = 1^3 + 1 - 1 = 1$

We know, $x_0 = 0.5 - \frac{-0.375\,(1-0.5)}{1+0.375} = 0.636$

$f(x_0) = -0.107$

$f(x_0) \times f(x_1) = -0.107 \times -0.375 = 0.04$

So, $f(x_0) \times f(x_1) > 0$

So, update $x_1 = x_0$

Iteration 2 :

We know,

$$x_0 = 0.636 - \frac{-0.107\,(1-0.636)}{1+0.107}$$

$$= 0.671$$

So, the root is 0.671.

Prob 3: Find the real root of the eq^n $x\log_{10}x - 1.2 = 0$ correct to five decimal places by Falsi method using the formula 4 times. suppose $x_1 = 2$, $x_2 = 3$.

Sol^n: **Iteration 1:**

Given, $f(x) = x\log_{10} x - 1.2$

and $x_1 = 2$, $x_2 = 3$.

$f(x_1) = 2\log_{10} 2 - 1.2 = -0.59794$

$f(x_2) = 0.23136$

$$x_0 = 2 - \frac{-0.59794\,(3-2)}{0.23136 + 0.59794} = 2.72102$$

$f(x_0) = -0.01709$

$f(x_0) \times f(x_1) > 0$

So, update $x_1 = x_0$

Iteration 2: $$x_0 = 2.72102 - \frac{-0.01709\,(3 - 2.72102)}{0.23136 + 0.01709}$$

$$= 2.74021$$

$f(x_0) = -0.00038$

So, $f(x_0) \times f(x_1) < = 0.0000064942$

So, $f(x_0) \times f(x_1) > 0$

So, update $x_1 = x_0$

<u>Iteration 3:</u>

$$x_0 = 2.74021 - \frac{-0.00038\,(3-2.74021)}{0.23136 + 0.00038}$$

$$= 2.74064$$

$$f(x_0) = 2.74064 \log_{10} 2.74064 - 1.2 = -0.00000531658$$

$$f(x_0) \times f(x_1) = -0.00000\,1736$$

So, $f(x_0) \times f(x_1) < 0$

So, update $x_2 = x_0$

<u>Iteration 4:</u>

$$x_0 = 2.7064 - \frac{-0.00000531658\,(3-2.74064)}{0.00031 + 0.000005368}$$

$$= 2.74D2\cancel{5}2$$

So, the root is 2.74022.

Newton - Raphson method / Newton Iteration method :

This is also known as iteration method and is used to find the isolated root of an equation $f(x) = 0$, when $f(x)$ derivation is a simple equation.

let, $x = x_0$ be an approximate value of one root of the eq^n $f(x) = 0$, if $x = x_1$ is the exact root then $f(x_1) = 0$ — ①

where the difference bet^n x_0 and x_1 is very small and if h denotes the differences $x_1 - x_0 = h$

then, $x_1 = x_0 + h$ — ⑪

substituting the value of x_1 in equation ① we get,

$$f(x_1) = f(x_0 + h) = 0 \quad \text{— (III)}$$

Expanding eq^n (III) by Taylors series method we get,

$$f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0 \quad \text{— (IV)}$$

since h is very small so neglecting all the power of h in eq^n (IV) we get,

$$f(x_0) + h f'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

from eq^n (ii) we get,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{---} \text{(v)}$$

The above value of x_1 is a closer approximation of the root $f(x) = 0$

#1 Using Newton - Raphson method, correct to four decimal places the root betn 0 and 1 of the eq^n $x^3 - 6x + 4 = 0$

Soln : We have,

$$f(x) = x^3 - 6x + 4$$

$$f(0) = 4$$

$$f(1) = -1$$

So, A root of $f(x) = 0$, lies betn 0 and 1.

The value of the root is nearer to 1.

let, $x_0 = 0.7$ be an approximate of the root

Now, $f(x) = x^3 - 6x + 4$

$$f'(x) = 3x^2 - 6$$

$f(x_0) = 0.143$

$f'(x_0) = 3\times(0.7)^2 - 6 = -4.53$

From Newton - Raphson method we know,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.7 - \frac{0.143}{-4.53} = 0.7316$$

$f(x_1) = 0.0019805$

$f'(x_1) = -4.39428$

From Newton Raphson method we know,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.7321 \quad (\text{Appr})$$

The root of the eqⁿ is 0.7321 (Apprx).

\# 2 By applying Newtons method twice, find the real root near 2 of the equation $x^4 - 12x + 7 = 0$

Solⁿ:

We have,

$$f(x) = x^4 - 12x + 7$$

$$f'(x) = 4x^3 - 12$$

let, $x_0 = 2$ be an approximation of the root.

Now,

$f(x_0) = -1$

$f'(x_0) = 20$

from Newton Raphson method,

$$x_1 = 2 - \frac{-1}{20} = 2.05$$

$f(x_1) = 0.061$

$f'(x_1) = 22.4605$

$$\therefore x_2 = 2.05 - \frac{0.061}{22.4605} = 2.047 \text{ (Apprx)}$$

The root of the eqn is 2.047 (Apprx).

\# 3 Find by Newton's method, the root of the $e^x = 4x$, which is approximately 2, correct to three places of decimal.

Soln:

we have, $e^x = 4x$

$\Rightarrow e^x - 4x = 0$

$\therefore f(x) = e^x - 4x$

$f'(x) = e^x - 4$

let, $x_0 = 2$ be an approximation of the root.

Now, $f(x_0) = -0.611$

$f'(x_0) = 3.389$

from Newton Raphson,

$$x_1 = 2 - \frac{-0.611}{3.389} = 2.18$$

$f(x_1) = 0.126$

$f'(x_1) = 4.846$

from Newton Raphson method,

$$x_2 = 2.18 - \frac{0.126}{4.846}$$

$= 2.154$ (Apprx)

The root is 2.154 (Apprx)

Finite difference

Forward difference operator :

let $y = f(x)$ be any function given by the values $y_0, y_1, y_2 \cdots y_n$ which it takes for the equidistant values $x_0, x_1, x_2 \cdots x_n$ of the independent variable x.

Then, $y_1 - y_0, y_2 - y_1, y_3 - y_2 \cdots y_{n+1} - y_n$ are called the first difference of the function y. So, we can denoted them as $\Delta y_0, \Delta y_1, \Delta y_2 \cdots$ etc.

$\therefore$ we have,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2$$
$$\vdots \qquad \vdots$$
$$\Delta y_n = y_{n+1} - y_n$$

the symbol Δ is called the difference operator.

The differences of the first differences is denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2 \cdots \Delta^2 y_n$.