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Brownian Motion and Its Applications In The Stock Market

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ABSTRACT. Wilfrid Kendall notes on the complexity of the paths of Brownian motion:

> *If you run Brownian motion in two dimensions for a positive amount of time, it will write your name.* [7]

The oddness and complexity of *Brownian motion* reveal a really deep subject in the field of mathematics that cannot be fully understood and explained even until now. The purpose of this paper is to introduce the *Brownian motion* with its properties and to explain how it is applied in an everyday but totally unpredictable environment like the stock market.

1. INTRODUCTION AND HISTORY OF BROWNIAN MOTION

Brownian motion refers to either the physical phenomenon that minute particles immersed in a fluid move around randomly or the mathematical models used to describe those random movements [11], which will be explored in this paper.

History: *Brownian motion* was discovered by the biologist Robert Brown [2] in 1827. While Brown was studying pollen particles floating in water in the microscope, he observed minute particles in the pollen grains executing the jittery motion. After repeating the experiment with particles of dust, he was able to conclude that the motion was due to pollen being "alive" but the origin of the motion remained unexplained. The first one to give a theory of *Brownian motion* was Louis Bachelier in 1900 in his PhD thesis "The theory of speculation". However, it was only in 1905 that Albert Einstein, using a probabilistic model, could sufficiently explain *Brownian motion*. He observed that if the kinetic energy of fluids was right, the molecules of water moved at random. Thus, a small particle would receive a random number of impacts of random strength and from random directions in any short period of time. This random bombardment by the molecules of the fluid would cause a sufficiently small particle to move exactly just how Brown described it [11].

2. Definitions of concepts relevant to Brownian Motion

Definition 2.1. [10] *A σ-**algebra** $\mathcal{F}$ on Ω is a family of subsets of Ω such that*

(1) $\varnothing, \ \Omega \ \in \ \mathcal{F}$
(2) *if $A \ \in \ \mathcal{F}$, then so does the complement set $\Omega - A$*
(3) *if $A_1, A_2, ...A_\infty$ is a sequence of sets in $\mathcal{F}$, then their union or intersection of countably many members of the algebra $\bigcup_{i=1}^{\infty} A_i$, $\bigcap_{i=1}^{\infty} A_i \ \in \ \mathcal{F}$.*

Definition 2.2. [10] *A **probability space** is the triple $(\Omega, \mathcal{F}, P)$, where Ω is a nonempty set, F is a σ-algebra of subsets of Ω and P is a function that, to every set $A \in \mathcal{F}$, assigns a number in [0,1], called the probability of A and written P(A). It is required that:*

- $P(\Omega) = 1$
- *(Countable additivity) whenever $A_1, A_2,...$ is a sequence of disjoint sets in $\mathcal{F}$, then*

$$P(\textstyle\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n).$$

Definition 2.3. ***Borel algebra*** [4] *A $\mathcal{B}$ denotes the collection of Borel subsets of $\mathbb{R}^n$, which is the smallest σ-algebra of subsets of $\mathbb{R}^n$ containing all open sets.*

Definition 2.4. [4] *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A mapping*

$$X : \Omega \rightarrow \mathbb{R}^n$$

*is called an n-dimensional **random variable** if for each $B \ \epsilon \ \mathcal{B}$, it is true that*

$$X^{-1}(B) \ \in \ \mathcal{F}$$

Equivalently, X is said to be $\mathcal{F} - measurable$.

Definition 2.5. [8] *A **stochastic process** is a parameterized collection of random variables*

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.

3. Properties of Brownian Motion

Brownian motion is a Wiener stochastic process. A **Wiener process** [3] is a stochastic process W(t) with values in $\mathbb{R}$ defined for $t \in [0, \infty)$ such that the following conditions hold:

(1) $W(0) = 0$.
(2) If $0 < s < t$ then $W(t)$-$W(s)$ has a normal distribution $\sim N(0, t-s)$ with mean 0 and variance $(t-s)$.
(3) If $0 \leq s \leq t \leq u \leq v$ (i.e., the two intervals [s, t] and [u, v] do not overlap) then $W(t)$-$W(s)$ and $W(v)$-$W(u)$ are independent random variables. In fact, the Wiener process is the only time-homogeneous stochastic process with independent increments that has continuous trajectories.
(4) The sample paths $t \mapsto W(t)$ are almost surely continuous.

The probability density function of $W(t)$ is $f_{W(t)}(\mathrm{x}) = \frac{1}{\sqrt{2\pi t}} \ e^{-\frac{x^2}{2t}}$ [8].

Basic properties of *Brownian motion* are:

- B_t is a Gaussian process, i.e., for all $0 \leq t_1 \leq t_2 \leq ... \leq t_k$ the random vector $\mathbb{Z} = (B_{t_1}, ..., B_{t_k}) \in \mathbb{R}$ has a multinormal distribution.
- B_t has stationary increments; i.e., the process $(B_{t+h} - B_t)_{h \geq 0}$ has the same distribution for all t; thus, $E(B_{t+h} - B_t) = 0$ and $\mathrm{Var}(B_{t+h} - B_t) = h$.
- B_t has continuous paths but is not differentiable anywhere at all.
- *Brownian motion* is a martingale:

    Definition 3.1. *The process M_t is a martingale if*
    $- \mid E(M_t) \mid < \infty$ for all t;
    $- M_t$ is F_t measurable for all t;
    $-$ the conditional expectation $E(M_t|\mathcal{F}_s) = M_s$ a.s. if $s < t$ [1].

- $\mathrm{Cov}(B_s, B_t) = \min(\mathrm{s, t})$ [7].

Distributional properties of *Brownian motion*: [7]

- **Spatial Homogeniety:**
  $B_t + x$ for any $x \in \mathbb{R}$ is a *Brownian motion* started at x.
- **Symmetry:**
  $-B_t$ is also a *Brownian motion*.

- **Scaling:**
  $\sqrt{c}B_{t/c}$ for any $c > 0$ is a *Brownian motion*. No matter what scale you examine *Brownian motion* on, it looks just the same.
- **Time inversion:**

(1)
$$Z_t = \begin{cases} 0, & t = 0, \\ tB_{\frac{1}{t}}, & t > 0 \end{cases} \text{is a Brownian motion.}$$

- **Time reversibility:**   For a given $t > 0$
$$\{B_s : 0 \leq s \leq t\} \sim \{B_{t-s} - B_t : 0 \leq s \leq t\}.$$

Continuity and differentiability:
It is really astonishing how *Brownian motion* behaves. Even though it is continuous everywhere, it is nowhere differentiable.

Ideas of proof [7]: Consider a small increment $B(t + \Delta t) - B(t)$ that is normally distributed with mean 0 and variance Δt as defined earlier. Then,
$$E(|B(t + \Delta t) - B(t)|^2) = \Delta t;$$
that is, the usual size of an increment $|B(t+\Delta t)-B(t)|$ is about $\sqrt{\Delta t}$. Now, as $\Delta t \to 0, \sqrt{\Delta t} \to 0$, which is consistent with the continuity of the paths. However, if we take the derivative
$$\frac{\partial B_t}{\partial t} = \lim_{\Delta t \mapsto 0} \frac{B(t+\Delta t)-B(t)}{\Delta t}$$
then we can see that when Δt is small, the absolute value of the numerator looks like $\sqrt{\Delta t}$ which is much larger than Δt. Therefore the limit does not exist. From that it is concluded that the path of a *Brownian motion* B_t is nowhere differentiable.

Local path properties of *Brownian motion*:

Local Maxima and Minima: For a continuous function $f : [0, \infty) \mapsto \mathbb{R}$, a point t is a local (strict) maximum if
$$\exists \, \epsilon > 0 \text{s.t.} \forall \, s \in (t - \epsilon, t + \epsilon), f(s) \leq f(t).$$

For almost all paths, the set of local maxima for the *Brownian path* B is countable (a set which is either finite or denumerable) and dense [7]. The same theory applies to minima. A Brownian motion path has a local max or min within any interval. This means that the set of local maxima and minima is dense. There is a local max or min arbitrarily close to any given number [7].

Points of increase and decrease: [7]

A point t is a point of increase if
$$\exists\ \epsilon > 0 \text{ s.t. } \forall\ s\ \epsilon\ (0, \epsilon), f(t - s) \leq f(t) \leq f(t + s).$$
For almost all paths, the *Brownian motion* path has no points of increase or decrease.

What makes *Brownian motion* so odd and unique: [7]

- *Brownian motion* is nowhere differentiable despite the fact that it is continuous everywhere.
- It is self-similar; i.e., any small piece of a *Brownian motion* trajectory, if expanded, looks like the whole trajectory, like fractals [5].
- *Brownian motion* will eventually hit any and every real value, no matter how large or how negative. It may be really high above the axis, but it will be back down again to 0 at some later time.
- Once *Brownian motion* hits 0 or any particular value, it will hit it again infinitely often, and then again from time to time in the future.

4. APPLICATION TO THE STOCK MARKET:

Background: The mathematical theory of *Brownian motion* has been applied in contexts ranging far beyond the movement of particles in fluids. Until recently, stock market researchers have confronted the same problem. While they can chart the path of the market on a minute-by-minute basis it is very hard for them to observe who buys, who sells and how demand and supply affects price movements. There exist many interesting theories about how the behavior of different investors makes the prices move, but there is no empirical evidence to support the critical link between the investor decisions and the price dynamics [6].

However, stock markets, the foreign exchange markets, commodity markets and bond markets are all assumed to follow *Brownian motion*, where assets are changing continually over very small intervals of time and the position, namely the change of state on the assets, is being altered by random amounts. More importantly, the mathematical models used to describe *Brownian motion* are the fundamental tools on which all financial asset pricing and derivatives pricing models are based.

These models are of key importance to the work that is being done on market models and risk analysis.

Geometric Brownian motion as a basis for options pricing:

A stochastic process S_t is said to follow a **Geometric *Brownian motion*** if it satisfies the following stochastic differential equation
$$dS_t = S_t(\mu dt + \sigma dB_t)$$
where μ is the percentage drift and σ the percentage volatility [11].

This equation has an analytic solution [11]:

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma dB_t}$$

for an arbitrary initial value S_0. This model is used in options pricing.

Definition 4.1. [9] ***Options*** *in the financial world are generally defined as a contract between two parties in which one party has the right but not the obligation to do something, usually to buy or sell some underlying asset.*

Having rights without obligations has financial value, so option holders must purchase these rights, making them assets. These assets derive their value from some other asset, so they are called derivative assets. Modern option pricing techniques, with roots in stochastic calculus, are often considered among the most mathematically complex of all applied areas of finance [9].

For an exercise price K, and an exercise date T, one has the right to buy stocks with price K and sell it with S_T in the market if $S_T > K$. If not, one has no obligation to buy. This option is called an European call option and we define claim C (payoff at time T) by

$$C = (S_T - K)^+ = max(S_T - K, 0).$$

So, if $S_T > K$ then the owner of the option will obtain the payoff C at time T while if $S_T \leq K$ then the owner will not exercise his option and the payoff is 0 [8].

How much would a person be willing to pay for such an option?

Fischer Black, Myron Scholes and Robert Merton solved this problem in 1973 using stochastic analysis and an equilibrium argument to compute a theoretical value for the price. This is now called the Black & Scholes option price formula or Black & Scholes model [8]. Geometric *Brownian motion* is the basis of the Black & Scholes Model.

Assumptions of the Black & Scholes Model:

(1) **The stock pays no dividends during the option's life.**
This may seem to be a serious limitation to the model since higher dividend yields elicit lower call premiums. A usual way of adjusting the model for this situation is to subtract the discounted value of a future dividend from the stock price [9].

(2) **European exercise terms are used.**
According to European exercise terms the option can only be used on the expiration date (American exercise terms allow the option to be exercised at any time during the life of the option). This limitation is not a major concern because very few call options are ever exercised before the last few days of their life because of the remaining time value on the call option that would be lost [9].

(3) **Market is efficient.**
When market is efficient, people cannot predict the direction of the market or an individual stock. The market operates continuously with share prices following a continuous Itô process, a Markov process in continuous time [9].

(4) **No commissions are charged.**
Usually market participants have to pay a commission to buy or sell options or give some kind of fee to floor traders but in this case the fees that the investor's pay are more substantial [9].

(5) **Interest rates remain constant and known.**
In this model the risk-free rate represents this constant and known interest rate even though there is no such thing in reality [9].

(6) **Returns are normally distributed.**
Returns on the underlying stocks are normally distributed, which is reasonable for most assets that offer options [9].

According to the Black & Scholes Model the current fair price of the claim is:

Theorem 4.2. *Black* & *Scholes Formula*

(2) $$C = S_0\Phi(T_+) - Ke^{-rT}\Phi(T_-).$$

where

$$T_+ = [\log(\tfrac{S_0}{K}) + (r + \tfrac{1}{2}\sigma^2 T)]/[\sigma\sqrt{T}],$$
$$T_- = [\log(\tfrac{S_0}{K}) + (r - \tfrac{1}{2}\sigma^2 T)]/[\sigma\sqrt{T}].$$

$\Phi(x)$ is a cumulative standard normal distribution $\mathbf{P}[Z < x]$ for

$$Z \sim N(0,1) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du,$$

C is the Call premium(amount per share that an option buyer pays to the seller), T is time until expiration, K is the Option exercise price, and r is the Risk free interest rate.

The first part of the model $S_0\Phi(T_+)$ derives the expected benefit from acquiring the stock outright. This is found by multiplying the initial price by the change in the call premium with respect to a change in the underlying stock price [9].

The second part of the model $Ke^{-rT}\Phi(T_-)$ gives the present value of paying the exercise price on the expiration day. The fair market value of the call option is then calculated by taking the difference between the two parts [9].

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