INFSCI 2251 FALL 2024 HW #4

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Problem 1. *QUESTION 5: Use Floyd's algorithm for the Shortest Paths problem 2 (Algorithm 3.4) to construct the matrix D, which contains the lengths of the shortest paths, and the matrix P, which contains the highest indices of the intermediate vertices on the shortest paths, for the following graph. Show the actions step by step.*

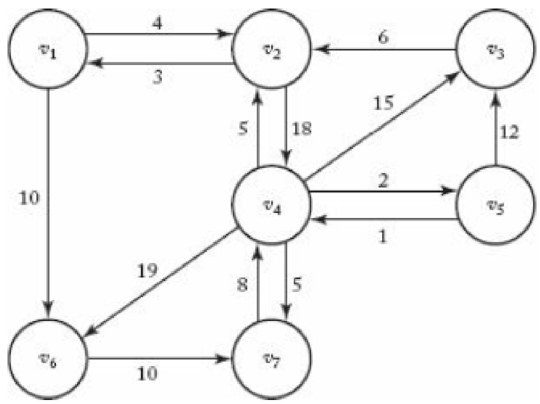

Figure 1: Question 5

Solution.

$D^{(0)} =$

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	4	∞	∞	∞	10	∞
v_2	3	0	∞	18	∞	∞	∞
v_3	∞	6	0	∞	∞	∞	∞
v_4	∞	5	15	0	2	19	5
v_5	∞	∞	12	1	0	∞	∞
v_6		∞	∞	∞	∞	0	10
v_7	∞	∞	∞	8	∞	∞	0

$P^{(0)} =$

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	0	0	0	0	0	0
v_2	0	0	0	0	0	0	0
v_3	0	0	0	0	0	0	0
v_4	0	0	0	0	0	0	0
v_5	0	0	0	0	0	0	0
v_6	0	0	0	0	0	0	0
v_7	0	0	0	0	0	0	0

i=1

$$D^{(1)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 4 & \infty & \infty & \infty & 10 & \infty \\ v_2 & 3 & 0 & \infty & 18 & \infty & 13 & \infty \\ v_3 & \infty & 6 & 0 & \infty & \infty & \infty & \infty \\ v_4 & \infty & 5 & 15 & 0 & 2 & 19 & 5 \\ v_5 & \infty & \infty & 12 & 1 & 0 & \infty & \infty \\ v_6 & \infty & \infty & \infty & \infty & \infty & 0 & 10 \\ v_7 & \infty & \infty & \infty & 8 & \infty & \infty & 0 \end{array}$$

$$P^{(1)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

i=2

$$D^{(2)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 4 & \infty & 22 & \infty & 10 & \infty \\ v_2 & 3 & 0 & \infty & 18 & \infty & 13 & \infty \\ v_3 & 9 & 6 & 0 & 24 & \infty & 19 & \infty \\ v_4 & 8 & 5 & 15 & 0 & 2 & 18 & 5 \\ v_5 & \infty & \infty & 12 & 1 & 0 & \infty & \infty \\ v_6 & \infty & \infty & \infty & \infty & \infty & 0 & 10 \\ v_7 & \infty & \infty & \infty & 8 & \infty & \infty & 0 \end{array}$$

$$P^{(2)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_3 & 2 & 0 & 0 & 2 & 0 & 2 & 0 \\ v_4 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

i=3

$$D^{(3)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 4 & \infty & 22 & \infty & 10 & \infty \\ v_2 & 3 & 0 & \infty & 18 & \infty & 13 & \infty \\ v_3 & 9 & 6 & 0 & 24 & \infty & 19 & \infty \\ v_4 & 8 & 5 & 15 & 0 & 2 & 18 & 5 \\ v_5 & 21 & 18 & 12 & 1 & 0 & 31 & \infty \\ v_6 & \infty & \infty & \infty & \infty & \infty & 0 & 10 \\ v_7 & \infty & \infty & \infty & 8 & \infty & \infty & 0 \end{array}$$

$$P^{(3)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_3 & 2 & 0 & 0 & 2 & 0 & 2 & 0 \\ v_4 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ v_5 & 3 & 3 & 0 & 0 & 0 & 3 & 0 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

i=4

$$D^{(4)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 4 & 37 & 22 & 24 & 10 & 27 \\ v_2 & 3 & 0 & \infty & 18 & 20 & 13 & 23 \\ v_3 & 9 & 6 & 0 & 24 & 26 & 19 & 29 \\ v_4 & 8 & 5 & 15 & 0 & 2 & 18 & 5 \\ v_5 & 9 & 6 & 12 & 1 & 0 & 19 & 6 \\ v_6 & \infty & \infty & \infty & \infty & \infty & 0 & 10 \\ v_7 & 16 & 13 & 23 & 8 & 10 & 26 & 0 \end{array}$$

$$P^{(4)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 0 & 4 & 2 & 4 & 0 & 4 \\ v_2 & 0 & 0 & 4 & 0 & 4 & 1 & 4 \\ v_3 & 2 & 0 & 0 & 2 & 4 & 2 & 4 \\ v_4 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ v_5 & 4 & 4 & 0 & 0 & 0 & 4 & 4 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_7 & 4 & 4 & 4 & 0 & 4 & 4 & 0 \end{array}$$

i=5

$$D^{(5)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 4 & 36 & 22 & 24 & 10 & 27 \\ v_2 & 3 & 0 & 32 & 18 & 20 & 13 & 23 \\ v_3 & 9 & 6 & 0 & 24 & 26 & 19 & 29 \\ v_4 & 8 & 5 & 14 & 0 & 2 & 18 & 5 \\ v_5 & 9 & 6 & 12 & 1 & 0 & 19 & 6 \\ v_6 & \infty & \infty & \infty & \infty & \infty & 0 & 10 \\ v_7 & 16 & 13 & 22 & 8 & 10 & 26 & 0 \end{array}$$

$$P^{(5)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 0 & 5 & 2 & 4 & 0 & 4 \\ v_2 & 0 & 0 & 5 & 0 & 4 & 1 & 4 \\ v_3 & 2 & 0 & 0 & 2 & 4 & 2 & 4 \\ v_4 & 2 & 0 & 5 & 0 & 0 & 2 & 0 \\ v_5 & 4 & 4 & 0 & 0 & 0 & 4 & 4 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_7 & 4 & 4 & 5 & 0 & 4 & 4 & 0 \end{array}$$

i=6

$$D^{(6)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 4 & 36 & 22 & 24 & 10 & 20 \\ v_2 & 3 & 0 & 32 & 18 & 20 & 13 & 23 \\ v_3 & 9 & 6 & 0 & 24 & 26 & 19 & 29 \\ v_4 & 8 & 5 & 14 & 0 & 2 & 18 & 5 \\ v_5 & 9 & 6 & 12 & 1 & 0 & 19 & 6 \\ v_6 & \infty & \infty & \infty & \infty & \infty & 0 & 10 \\ v_7 & 16 & 13 & 22 & 8 & 10 & 26 & 0 \end{array}$$

$$P^{(6)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 0 & 5 & 2 & 4 & 0 & 6 \\ v_2 & 0 & 0 & 5 & 0 & 4 & 1 & 4 \\ v_3 & 2 & 0 & 0 & 2 & 4 & 2 & 4 \\ v_4 & 2 & 0 & 5 & 0 & 0 & 2 & 0 \\ v_5 & 4 & 4 & 0 & 0 & 0 & 4 & 4 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_7 & 4 & 4 & 5 & 0 & 4 & 4 & 0 \end{array}$$

i=7

$$D^{(7)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 4 & 36 & 22 & 24 & 10 & 20 \\ v_2 & 3 & 0 & 32 & 18 & 20 & 13 & 23 \\ v_3 & 9 & 6 & 0 & 24 & 26 & 19 & 29 \\ v_4 & 8 & 5 & 14 & 0 & 2 & 18 & 5 \\ v_5 & 9 & 6 & 12 & 1 & 0 & 19 & 6 \\ v_6 & 26 & 23 & 32 & 18 & 20 & 0 & 10 \\ v_7 & 16 & 13 & 22 & 8 & 10 & 26 & 0 \end{array}$$

$$P^{(7)} = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 0 & 5 & 2 & 4 & 0 & 6 \\ v_2 & 0 & 0 & 5 & 0 & 4 & 1 & 4 \\ v_3 & 2 & 0 & 0 & 2 & 4 & 2 & 4 \\ v_4 & 2 & 0 & 5 & 0 & 0 & 2 & 0 \\ v_5 & 4 & 4 & 0 & 0 & 0 & 4 & 4 \\ v_6 & 7 & 7 & 7 & 7 & 7 & 0 & 0 \\ v_7 & 4 & 4 & 5 & 0 & 4 & 4 & 0 \end{array}$$

Final Values

$$D = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 4 & 36 & 22 & 24 & 10 & 20 \\ v_2 & 3 & 0 & 32 & 18 & 20 & 13 & 23 \\ v_3 & 9 & 6 & 0 & 24 & 26 & 19 & 29 \\ v_4 & 8 & 5 & 14 & 0 & 2 & 18 & 5 \\ v_5 & 9 & 6 & 12 & 1 & 0 & 19 & 6 \\ v_6 & 26 & 23 & 32 & 18 & 20 & 0 & 10 \\ v_7 & 16 & 13 & 22 & 8 & 10 & 26 & 0 \end{array}$$

$$P = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 0 & 5 & 2 & 4 & 0 & 6 \\ v_2 & 0 & 0 & 5 & 0 & 4 & 1 & 4 \\ v_3 & 2 & 0 & 0 & 2 & 4 & 2 & 4 \\ v_4 & 2 & 0 & 5 & 0 & 0 & 2 & 0 \\ v_5 & 4 & 4 & 0 & 0 & 0 & 4 & 4 \\ v_6 & 7 & 7 & 7 & 7 & 7 & 0 & 0 \\ v_7 & 4 & 4 & 5 & 0 & 4 & 4 & 0 \end{array}$$

□

Problem 2. *QUESTION 6: Use the Print Shortest Path algorithm (Algorithm 3.5) to find the shortest path from vertex v7 to vertex v3 , in the graph of Exercise 5, using the matrix P found in that exercise. Show the actions step by step.*

Solution.

$$P = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 0 & 5 & 2 & 4 & 0 & 6 \\ v_2 & 0 & 0 & 5 & 0 & 4 & 1 & 4 \\ v_3 & 2 & 0 & 0 & 2 & 4 & 2 & 4 \\ v_4 & 2 & 0 & 5 & 0 & 0 & 2 & 0 \\ v_5 & 4 & 4 & 0 & 0 & 0 & 4 & 4 \\ v_6 & 7 & 7 & 7 & 7 & 7 & 0 & 0 \\ v_7 & 4 & 4 & 5 & 0 & 4 & 4 & 0 \end{array}$$

The shortest path algorithm is recursive.

- **CHECK P[7][3]**
  - P[7][3] = 5
  - Path from v7 to v3 is via v5
- **CHECK P[7][5]**
  - P[7][5] = 4
  - Path from v7 to v5 is via v4
- **CHECK P[7][4]**

- P[7][4] = 0
- There is no intermediate vertex from v7 to v4
- Go back to P[7][5] and the path consists of vertex v4

- **CHECK P[7][5]**
  - Check if there is a path between v4 and v5
- **CHECK P[4][5]**
  - P[4][5] =0
  - There is no intermediate vertex between v4 and v5
  - We go back to P[7][3]
  - The path now consists of V4, V5
- **CHECK P[7][3]**
  - Check the path for v5 to v3
- **CHECK P[5][3]**
  - P[5][3] =0
  - There is no intermediate vertex between v3 and v5
  - Print output as all intermediate vertices are done

Output: v7 – v4 – v5 – v3
The Shortest Distance is 22. □

Problem 3. *QUESTION 21: How many different binary search trees can be constructed using six distinct keys?*

Solution. The number of binary search trees that can be constructed using n distinct keys can be calculated using the formula for calculating a Catalan Number.
The number of BSTs will be a part of the Catalan Number series.

This number is often used in different combinatorial problems, like polygon triangulation or valid brackets sequences. The Catalan number sometimes can describe the number of objects, which are defined recursively. In our case, the Catalan number is $\boldsymbol{F(n)}$.
So, the introduced recursive formula F is:

$$F(n) = \text{Catalan}_n = \frac{(2n)!}{(n+1)! \cdot n!} = \frac{1}{n+1}\binom{2n}{n}$$

For n = 6,
1. **Plugging in the value of $n = 6$ into the formula:**

$$F(6) = \frac{(2 \times 6)!}{(6+1)! \cdot 6!}$$

This simplifies to:

$$F(6) = \frac{12!}{7! \cdot 6!}$$

2. **Calculating** $12!$**:**

$$12! = 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479,001,600$$

3. **Calculating** $7!$**:**

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5,040$$

4. **Calculating** $6!$**:**

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

5. **Calculating the denominator** $7! \cdot 6!$**:**

$$7! \cdot 6! = 5,040 \times 720 = 3,628,800$$

6. **Calculating the Catalan number** $F(6)$**:**

$$F(6) = \frac{12!}{7! \cdot 6!} = \frac{479,001,600}{3,628,800} = 132$$

Therefore, the Catalan number for $n = 6$ is **132**. Therefore, there are 132 BSTS $\square$