Practical 10 : Computation of Eigen Value and Eigen Vector for dimensionality reduction

i) Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$$

Sol^n :- Characteristic eq^n of A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{vmatrix} = 0$$

$$= 2-\lambda \begin{vmatrix} 1-\lambda & 3 \\ 2 & -4-\lambda \end{vmatrix} + 3 \begin{vmatrix} 3 & 3 \\ -5 & -4-\lambda \end{vmatrix} + 1 \begin{vmatrix} 3 & 1-\lambda \\ -5 & 2 \end{vmatrix}$$

$$= 2-\lambda[-10-\lambda+4\lambda+\lambda^2] + 3[-3\lambda+3] + [6+5-5\lambda]$$

$$= 20-2\lambda+8\lambda+2\lambda^2+10\lambda+\lambda^2-4\lambda^2-\lambda^3-9\lambda+9+6+5-5\lambda$$

$$= -\lambda^3-\lambda^2+2\lambda+0$$

$$= -(\lambda^3+\lambda^2-2\lambda)$$

ii) Find the eigen values of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Sol^n : $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$= 1-\lambda \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ 0 & 1-\lambda \end{vmatrix} + 0$$

$$= 1-\lambda[1-3\lambda+\lambda^2] - 1 + \lambda$$

$$= 1-3\lambda+\lambda^2-\lambda+3\lambda^2-\lambda^3-1+\lambda$$

$$= -\lambda^3+4\lambda^2-3\lambda$$

The eigen values for the given matrix are

$\lambda = 3$, $\lambda = 1$, $\lambda = 0$

iii) Find the eigen value and eigen vector of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol^n : Characteristic eq^n of A

$|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$= -2-\lambda \begin{vmatrix} 1-\lambda & -6 \\ -2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & -6 \\ 1 & -\lambda \end{vmatrix} - 3 \begin{vmatrix} 2 & 1-\lambda \\ -1 & -2 \end{vmatrix}$$

$$= -2-\lambda\left[-\lambda+\lambda^2-12\right] - 2(-2\lambda-6) - 3(-4+1-\lambda)$$

$$= +2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 + 12 - 3 + 3\lambda$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45$$

eigen values

$\lambda = 5, \quad \lambda = -3$

for $\lambda = 5$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = \begin{vmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ 1 & -2 & -5 \end{vmatrix}$$

$-7x + 2y - 3z = 0 \quad - ①$

$2x - 4y - 6z = 0 \quad - ②$

$1x - 2y - 5z = 0 \quad - ③$

using Cramer's rule ① & ②

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$x = -24 \qquad y = -48 \qquad z = 24$

for $\lambda = 5$ $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

for $\lambda = -3$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{vmatrix}$$

$$1x + 2y - 3z = 0 \quad - ①$$
$$2x + 4y - 6z = 0 \quad - ②$$
$$-1x - 2y + 3z = 0 \quad - ③$$

using Cramer's rule ① & ②

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ 4 & -6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}}$$

$x = 0$ $\quad y = 0$ $\quad z = 0$

for $\lambda = -3$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The eigen values for the given matrix is 5 and -3 and eigen vectors are $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ respectively

iv) Find all the eigen values and eigen vector for the following matrix A

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

sol^n: Characteristic eq^n of A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$$

$$= 1-\lambda \begin{vmatrix} 5-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 3 & 1-\lambda \end{vmatrix} + 3 \begin{vmatrix} 1 & 5-\lambda \\ 3 & 1 \end{vmatrix}$$

$= 1-\lambda(4-5\lambda-\lambda+\lambda^2) - (1-\lambda+-3) + 3\ (1-15+3\lambda)$

$= 4-5\lambda-\lambda+\lambda^2-4\lambda+5\lambda^2+\lambda^2-\lambda^3-1+\lambda+3+3-45+9\lambda$

$= -\lambda + 7\lambda^2 - 36$

eigen values

$\lambda = -2$, $\lambda = 6$, $\lambda = 3$

for $\lambda = -2$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$3x + y + 3z = 0 \quad \cdots ①$

$1x + 7y + 1z = 0 \quad \cdots ②$

$3x + 1y + 2z = 0 \quad \cdots ③$

using cramer's rule on eq^n ① & ②

$$\begin{array}{ccccc} x & & -y & & z \\ \begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix} & = & \begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix} & = & \begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix} \end{array}$$

$x = -20$, $-y = 0$, $z = +20$

for $\lambda = -2$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

for $\lambda = 6$

$$\begin{vmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{vmatrix} = \begin{array}{ll} -5x + y + 3z = 0 & \cdots ① \\ x - y - z = 0 & \cdots ② \\ 3x + y - 5z = 0 & \cdots ③ \end{array}$$

using cramer's rule on eq^n ① & ②

$$\begin{array}{ccccc} x = & & -y & & z \\ \begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} & = & \begin{vmatrix} -5 & 3 \\ 1 & -1 \end{vmatrix} & = & \begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix} \end{array}$$

$x = 4$, $+y = +8$, $z = 4$

for $\lambda = 6$ $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Practical 11 : Calculate Maxima and Minima for the given function

i) Find the absolute extreme points of a function
$f(x) = 2x^3 - 3x^2 + 5$ in the interval [-2, 2]

Sol^n: $f(x) = 2x^3 - 3x^2 + 5$

$f'(x) = 6x^2 - 6x$

$f'(x) = 0$

$\therefore$ $6x^2 - 6x = 0$

$6x(x-1) = 0$

$6x = 0$ or $x = 1$

$x = 0$

for $x = 0$

$f(x) = 2x^3 - 3x^2 + 5$

$f(0) = 0 - 0 + 5$

$f(0) = 5$

for $x = 1$

$f(x) = 2(1)^3 - 3(1)^2 + 5$

$f(1) = 4$

Checking endpoints of interval

$x = -2$	$x = 2$
$f(-2) = 2(-2)^3 - 3(-2)^2 + 5$	$f(2) = 2(2)^3 - 3(2)^2 + 5$
$f(-2) = -23$	$f(2) = 9$

The minimum value is -23 at $x = -2$ &
maximum value is 9 at $x = 2$

ii) Find the absolute maxima and minima of function $f(x) = 2\sin x + \cos x$ in the interval $[-1, 1]$

Sol^n: $f(x) = 2\sin x + \cos x$

$f'(x) = 2\cos x - \sin x$

$f'(x) = 0$

$2\cos x - \sin x = 0$

$2\cos x = \sin x$

$2 = \tan x$

$x = \tan^{-1}(x)$

$x \approx 1.1071$

for $x \approx 1.1071$

$f(x) = 2\sin x + \cos x$

$f(1.1071) = 2\sin(1.1071) + \cos(1.1071)$

$= 2(0.8940) + 0.4472$

$= 1.788 + 0.4472$

$= 2.2352$

Checking endpoints of interval

for $x = -1$

$f(-1) = 2\sin(-1) + \cos(-1)$

$= 2(-0.8415) + 0.5403$

$= -1.683 + 0.5403$

$= -1.1427$

for $x = 1$

$f(1) = 2\sin(1) + \cos(1)$

$= 2(0.8415) + 0.5403$

$= 1.683 + 0.5403$

$= 2.2233$

The minimum value is -1.1427 at $x = -1$ & maximum value is 2.2352 at $x \approx 1.1071$

iii) Find the absolute extreme points of function $f(x) = \log x$ in the interval $[-2, 2]$

Sol^n:- $f(x) = \log x$

$f'(x) = \dfrac{1}{x}$

$f'(x) = 0$

$\dfrac{1}{x} = 0$

$\therefore$ No solution for x, because $\dfrac{1}{x}$ is never ~~negati~~ equal to zero for any value of x.

conclusion:

Thus, there are no critical points in the given interval $[-1, 1]$

iv) Find the applied maxima & minima of

$f(x) = x \log x$

Sol^n :- $f(x) = x \log x$

$f'(x) = \cancel{x}\, 1 \cdot (\log x) + x \cdot \left(\frac{1}{x}\right)$

$= \log x + 1$

$f''(x) = \log x + 1$

$= \frac{1}{x}$

$f'(x) = 0$

$\log x + 1 = 0$

$\log x = -1$

$x = e^{-1}$ .... $(\log_a x = y \ : a^y = x)$

$x = 0.3678$

for $x = e^{-1}$ (i.e $x = \frac{1}{e}$)

$f''(x) = \frac{1}{e^{-1}}$

$= 2.7182 > 0$

$f''(x) > 0$

$f(x) = x \log x$

$= e^{-1} \log_e e^{-1}$

$= -0.3678$

The function $f(x)$ has minimum value ~~2.7182~~ -0.3678 at $x = e^{-1}$

The function has no maximum value

v) Find applied maxima & minima of

$f(x) = x^2 + \frac{16}{x}$

Sol^n:-

$f'(x) = 2x + \left(-\frac{16}{x^2}\right)$

$= 2x - \frac{16}{x^2}$

$f''(x) = 2 - \left(-\frac{(16 \times 2)}{x^3}\right)$ .... $\left(\because \frac{1}{x^n} = \frac{-n}{x^{n+1}}\right)$

$= 2 + \frac{32}{x^3}$

$f'(x) = 0$

$2x - \frac{16}{x^2} = 0$

$2x = \frac{16}{x^2}$ or $x = 0$

$2x^3 = 16$

$x^3 = 8$

$x = 2$

for $x = 2$

$f''(x) = 2 + \frac{32}{x^3}$

$= 2 + \frac{32}{8}$

$f''(x) = 6$

$f''(x) > 0$

$\therefore$ $f(x)$ attains minimum at $x=2$

$f(x) = x^2 + \frac{16}{x}$

$= 4 + \frac{16}{2}$

$= 12$

$\therefore$ The function $f(x)$ has minimum value 12 at $x=2$

The function has no maximum value.

Practical 12: Impute an example based on rolle's and Mean value Theorem

a) Verify rolle's theorem

i) $f(x) = \sin x + \cos x + 7$ at interval $x \in [0, 2\pi]$

Sol^n :- The given f^n $f(x)$ is a polynomial function hence it is continuous at $[0, 2\pi]$ and differentiable at $(0, 2\pi)$

$f(0) = \sin x + \cos x + 7$
$= \sin 0 + \cos 0 + 7$
$= 0 + 1 + 7$

$f(0) = 8$

$f(2\pi) = \sin x + \cos x + 7$
$= \sin 2\pi + \cos 2\pi + 7$
$= 0 + 1 + 7$

$f(2\pi) = 8$

$\therefore f(0) = f(2\pi)$

Thus the function $f(x)$ satisfy all the condition of rolle's theorem.

Their exist $c \in (1,4)$ such that $f'(c) = 0$

$f'(x) = \sin x + \cos x + 7$

$f(c) = \sin c + \cos c + 7$

$f'(c) = \cos c - \sin c + 0$

$f'(c) = 0$

$\cos c - \sin c = 0$

$\cos c = \sin c$

they are equal at $c = \frac{\pi}{4}$

$\therefore c = \frac{\pi}{4} \in [0, 2\pi]$

Hence the rolle's theorem is verified.

ii) $f(x) = \sin \frac{x}{2}$; $x \in [0, 2\pi]$

Sol^n :- The given function $f(x)$ is a polynomial function hence it is continuous at $[0, 2\pi]$ and differentiable on $[0, 2\pi]$

$f(0) = \sin \frac{x}{2}$

$= \sin \frac{0}{2}$

$f(0) = 0$

$f(2\pi) = \sin \frac{x}{2}$

$= \sin \frac{2\pi}{2}$

$f(2\pi) = 0$

$f(0) = f(2\pi)$

Thus the function satisfy all the condition of rolle's theorem their exist $c \in (0, 2\pi)$ such that $f'(c) = 0$

$f(c) = \sin \frac{c}{2}$

$f'(c) = \cos \frac{c}{2}$

$f'(c) = 0$

$\cos \frac{c}{2} = 0$

$\left(\because \cos \frac{\pi}{2} = 0 \right)$

$\therefore \; c = \pi \quad \in [0, 2\pi]$

Hence the rolle's theorem is verified

b) Verify Lagrange's Mean Value Theorem

i) $f(x) = x^2 - 3x - 1$; $x \in \left[-\frac{11}{7}, \frac{13}{7}\right]$

Sol^n:- The given function $f(x)$ is a polynomial function it is continuous at $\left[-\frac{11}{7}, \frac{13}{7}\right]$ and differentiable at $\left[-\frac{11}{7}, \frac{13}{7}\right]$

$$f(a) = f\left(-\frac{11}{7}\right) = x^2 - 3x - 1$$

$$= \left(-\frac{11}{7}\right)^2 - 3\left(-\frac{11}{7}\right) - 1$$

$$f(a) = 6.1836$$

$$f(b) = f\left(\frac{13}{7}\right) = x^2 - 3x - 1$$

$$= \left(\frac{13}{7}\right)^2 - 3\left(\frac{13}{7}\right) - 1$$

$$f(b) = -3.1224$$

$$f(c) = c^2 - 3c - 1$$

$$f'(c) = 2c - 3$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2c - 3 = \frac{(-3.1225) - 6.1836}{\left(\frac{13}{7} - \left(-\frac{11}{7}\right)\right)}$$

$$2c - 3 = \frac{-9.306}{24/7}$$

$$14c - 21 = \frac{-9.306}{24}$$

$$14c - 21 = -0.38775$$

$$14c = -0.38775 + 21$$

$$c = \frac{20.61225}{14}$$

$$c = 1.4723 \quad \in \left[-\frac{11}{7}, \frac{13}{7}\right]$$

ii) $f(x) = 2x - x^2$; $x \in [0,1]$

Sol^n:- The given function $f(x)$ is a polynomial function it's continuous at $[0,1]$ and differentiable at $(0,1)$

$$f(0) = 2x - x^2$$
$$= 2(0) - 0$$
$$f(a) = f(0) = 0$$

$$f(b) = f(1) = 2x - x^2$$
$$= 2 - 1^2$$
$$f(b) = f(1) = 1$$

$$f(c) = 2c - c^2$$
$$f'(c) = 2 - 2c$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2 - 2c = \frac{1 - 0}{1 - 0}$$

$$2 - 2c = 1$$
$$-2c = 1 - 2$$
$$2c = 1$$
$$c = \frac{1}{2} \quad \in [0,1]$$

Practical 13: Dot product and Norm of the Vector

i) Consider the Two vector $u = (1,2,-1)$ & $v = (0,1,1)$

a) determine the dot product of u & v.

$u \cdot v = (1 \cdot 0) + (2 \cdot 1) + (-1 \cdot 1)$

$= 0 + 2 - 1$

$u \cdot v = 1$

b) determine the Norm of 'v'.

$||x||^2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$= \sqrt{0^2 + 1^2 + 1^2}$

$= \sqrt{2}$

c) determine the norm of 'u'

$||x||^2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$= \sqrt{1^2 + 2^2 + (-1)^2}$

$= \sqrt{1+4+1}$

$= \sqrt{6}$

ii) determine the angle between $a = (3,2)$ and $b = (1,7)$

$a \cdot b = (3 \times 1) + (2 \cdot 7)$

$a \cdot b = 3 + 14$

$a \cdot b = 17$

$|\bar{a}| = \sqrt{3^2 + 2^2}$

$= \sqrt{13}$

$|\bar{b}| = \sqrt{1^2 + 7^2}$

$= \sqrt{50}$

$\theta = \cos^{-1}\left(\frac{a \cdot b}{|\bar{a}| \cdot |\bar{b}|}\right)$

$= \cos^{-1}\left(\frac{17}{\sqrt{13} \cdot \sqrt{50}}\right)$

$= \cos^{-1}(0.667)$

$\theta = 48.16°$

Practical 14 : Impute Inverse of Matrix by adjoint Method

i) Find the adjoint of Matrix

a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

Sol^n:- $A_{11} = (-1)^2 \cdot a_{22} = 2$

$A_{12} = (-1)^3 \cdot a_{12} = -2$

$A_{21} = (-1)^3 \cdot a_{21} = 0$

$A_{22} = (-1)^4 \cdot a_{11} = 1$

$\text{adj}(A) = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

Sol^n:- $C_{11} = (-1)^2 [(1 \times -4) - (3 \times 2)] = -10$

$C_{12} = (-1)^3 [(3 \times -4) - (5 \times -3)] = -3$

$C_{13} = (-1)^4 [(3 \times 2) - (1 \times -5)] = 11$

$C_{21} = (-1)^3 [(-3 \times -4) - (2 \times 1)] = -10$

$C_{22} = (-1)^4 [(2 \times -4) - (1 \times -5)] = -3$

$C_{23} = (-1)^5 [(2 \times 2) - (-5 \times -3)] = 11$

$C_{31} = (-1)^4 [(-3 \times 3) - 1] = -10$

$C_{32} = (-1)^5 [(2 \times 3) - (3)] = -3$

$C_{33} = (-1)^6 [(2) - (3 \times -3)] = 11$

Cofactor. $A = \begin{bmatrix} -10 & -3 & 11 \\ -10 & -3 & 11 \\ -10 & -3 & 11 \end{bmatrix}$

$\text{adj}(A) = A^T$

$\text{adj}(A) = \begin{bmatrix} -10 & -10 & -10 \\ -3 & -3 & -3 \\ 11 & 11 & 11 \end{bmatrix}$

ii) Find the inverse of a matrix by adjoint Method

$$A = \begin{bmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{bmatrix}$$

Sol^n:- Finding the cofactors

$C_{11} = (-1)^2 (2-0) = 2$

$C_{12} = (-1)^3 (-10-24) = 34$

$C_{13} = (-1)^4 (0+4) = 4$

$C_{21} = (-1)^3 (-4+0) = 4$

$C_{22} = (-1)^4 (-18-4) = -22$

$C_{23} = (-1)^5 (0-8) = 8$

$C_{31} = (-1)^4 (12+1) = 13$

$C_{32} = (-1)^5 (54-5) = -49$

$C_{33} = (-1)^6 (-9-10) = -19$

co-factor matrix of $A = \begin{bmatrix} 2 & 34 & 4 \\ 4 & -22 & 8 \\ 13 & -49 & -19 \end{bmatrix}$

$adj(A) = A^T$

$$adj(A) = \begin{bmatrix} 2 & 4 & 13 \\ 34 & -22 & -49 \\ 4 & 8 & -19 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} \begin{aligned} &9(+2) - 2(-10-24) + 1(4) \\ &= +18 + \overset{6}{\cancel{2}}8 + 4 \\ &= \cancel{14}90 \end{aligned}$$

$\therefore |A| = \cancel{14}\, 90$

$$A^{-1} = adj(A) \cdot \frac{1}{|A|}$$

$$= \frac{1}{\cancel{14}90} \begin{bmatrix} 2 & 4 & 13 \\ 34 & -22 & -49 \\ 4 & 8 & -19 \end{bmatrix}$$