# Tournaments and Kirkman's Schoolgirls

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# Outline

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# Problem Description

## Problem Statement

*In a tournament with N teams, what is the least number of rounds required to play all possible matches ? Design a tournament schedule that achieves the minimum number of rounds.*

# Number of Rounds

- *$N$ is even:* In each round, $N/2$ matches are played, and there are $(N-1)$ rounds
- *$N$ is odd:* In each round, $(N-1)/2$ matches are played with one player on standby, and there are $N$ rounds

Total number of rounds $= N(N-1)/2$

# Tournament Schedule

*N is odd:* Draw a regular $N$-gon. In each round, pick one player (vertex) as standby. Players on the $(N-1)/2$ parallel diagonals compete.

In round $i$, players $j, k$ such that $j + k = 2i \ mod \ n$ will compete.

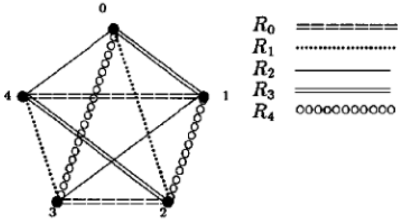


**Fig. 8.7. Tournament schedule: five teams**

# Kirkman's Schoolgirls Problem



## Problem Statement

*In a boarding school there are fifteen schoolgirls who always take their daily walks in row of threes. How can it be arranged so that each schoolgirl walks in the same row with every other schoolgirl exactly once a week?*

# Description

- The problem was posed in the Lady's and Gentleman's Diary for 1850, by the English mathematician T.P. Kirkman.
- Several constructions have been presented (Forst, Pierce)
- Steiner Systems solve a more general version of this problem

# Outline

# Steiner Problem



## Problem Statement

*Given integers $l < m < n$, what is the greatest number of m-element subsets of the n-element set such that every l-element subset lies in atmost one of the m-element subsets ?*

- Posed in *Lady's and Gentlemans Diary* in 1845.
- Very difficult problem, specialized to $l = 2, m = 3$.

## Bound on number of subsets



Let $B$ be the family of $m$-subsets of an $n$-set $X$ satisfying this condition. It can be shown that

$$|B| \leq \frac{\binom{n}{l}}{\binom{m}{l}} \tag{1}$$

The pair $(X, B)$ is a Steiner System $S(l, m, n)$.

A Steiner Triple System $STS(n) = S(2, 3, n)$

# Examples



An example $STS(7)$ is

$$
\begin{aligned}
X &= \{1,2,3,4,5,6,7\} \\
B &= \{123,145,167,246,257,347,356\}
\end{aligned}
$$

An $STS(15)$ with the right ordering can solve the 15-schoolgirls problem

# Existence

**Theorem**

*There exists an $STS(n)$ if and only if either*

- $n = 0$ *or*
- $n \equiv 1$ *or* $3 \pmod 6$

Forward proof uses Double Counting Arguments.

Backward proof uses construction. Larger STSes can be built recursively from smaller ones

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# Kirkman's Schoolgirls Problem



## Problem Statement

*In a boarding school there are fifteen schoolgirls who always take their daily walks in row of threes. How can it be arranged so that each schoolgirl walks in the same row with every other schoolgirl exactly once a week?*

# Example Solution

| Sun. | Mon. | Tues. | Wed. | Thurs. | Fri. | Sat. |
|---|---|---|---|---|---|---|
| 01-06-11 | 01-02-05 | 02-03-06 | 05-06-09 | 03-05-11 | 05-07-13 | 11-13-04 |
| 02-07-12 | 03-04-07 | 04-05-08 | 07-08-11 | 04-06-12 | 06-08-14 | 12-14-05 |
| 03-08-13 | 08-09-12 | 09-10-13 | 12-13-01 | 07-09-15 | 09-11-02 | 15-02-08 |
| 04-09-14 | 10-11-14 | 11-12-15 | 14-15-03 | 08-10-01 | 10-12-03 | 01-03-09 |
| 05-10-15 | 13-15-06 | 14-01-07 | 02-04-10 | 13-14-02 | 15-01-04 | 06-07-10 |

Can generate Solutions using Steiner Triple Systems.

# Constructing an STS



We begin by constructing an $STS(15)$

Divide into 2 sets

- $X = x_0, \ldots, x_6$, as the point set of an STS(7)
- Y as the 8 players in a tournament with 7 rounds $R_0, \ldots, R_6$ rounds.

Each round has 4 disjoint pairs of girls (matches), and we can add $x_i$ to form a triple.

# Constructing an STS



4 such matched triples in every round, and each round has 1 unmatched triple in $X$. So total of 35 triples, which is the same as the upper bound on $STS(15)$.

We can verify this construction actually gives an STS(15).

# Solving the Schoolgirls Problem

It remains to divide the triples into 5 walking groups a day for the 7 days

.

Let the triplets in STS(7) be $\{B_0, \ldots, B_6\}$ where $B_i = \{x_{i+1}, x_{i+2}, x_{i+4}\}$ (a valid STS(7)).

Let $Y = \{y_0, \ldots, y_6, z\}$. Round $R_i$ of the tournament has $y_i, z$ and $y_j, y_k$ such that $j + k = 2i \ mod \ 7$ matched up.

# Solving the Schoolgirls Problem



Consider Round 0.

Attach an $x$ to each of the above matchings. These 4, along with $\{x_1, x_2, x_4\}$ form the 5 groups for Day 0.

Say the groups in day 0 are
$\{x_0, y_0, z\}, \{y_1, y_6, x_5\}, \{y_2, y_5, x_3\}, \{y_3, y_4, x_6\}, \{x_1, x_2, x_4\}$.

In Round $i$, we can just add $i$ to each index. No repeat grouping occurs, since it was constructed from an STS(15).

# Outline

# Applications of STS



Steiner Tripe Systems are used in a variety of applications:

- Coding Theory
- DNA transcription
- Latin Squares
- Speed Networking event design.
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# Conclusion





We have used the solution of the Tournament problem to construct an STS, which helped us solve the Kirkman's Schoolgirls problem.

# Conclusion

Questions ?

# Conclusion

Thank You !