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CHAPTER 1

INTRODUCTION TO COMPUTATIONAL TECHNIQUES

1.1 INTRODUCTION

Engineering problems require a lot of calculation works to get the solution. When the domain is very large or complex, getting exact analytical solution becomes extremely difficult and impractical. However, with the advent of the computer and various computational techniques, approximate solutions, reliable enough, can be obtained easily and efficiently. Computational techniques are the methods of obtaining approximate solution of the problems using numerical methods. Engineers now-a-days use various software like SAP2000, STAADpro, HEC-RAS, MikeShe, etc. which employ various computational techniques like finite element method, finite difference method, etc. In actual analysis works, these analysis software deal with equations having matrices even larger than 1000 × 1000. However, this book illustrates the procedure of the computational techniques via small and handy problems to enable students to understand how these software work. This book deals mainly on fundamental idea of the most popular solution techniques: finite element method and finite difference method.

1.2 SOME BASIC TERMS

1.2.1 Domain

Domain refers to the region/geometry where our study is concerned.

1.2.2 Boundary Condition

The known or given condition on some parts of the domain are the boundary conditions.

1.2.3 Initial Condition

Conditions at the starting of the simulation are called initial condition.

1.3 STEPS FOR NUMERICAL SIMULATION

i) Various laws are formed to describe the physical phenomenon.
ii) The laws are represented in the form of physical equations.
iii) The domain is discretized into mesh/grid, cell, node or particle.
iv) Algorithm is prepared for the numerical solution of the physical equation.
v) The program is coded.
vi) Numerical simulation is carried out.

1.4 VARIOUS SOLUTION TECHNIQUES

Some of the common solution techniques are described below.

1.4.1 Finite Element Method (FEM)

It is also called finite element analysis (FEA). FEM is one of the most powerful problem solving techniques in civil engineering which incorporates wide range of applications and even can handle irregular domains. In this method of analysis, a complex region defining a continuum is discretized into simple geometric shapes called finite elements.

The material properties and the governing relationships are considered over these elements and expressed in terms of unknown values at element nodes. An assembly process duly considering the loading and constraints results in a set of equations. Solution of these equations gives us the approximate behavior of the continuum.

Applications of FEM

FEM can be used to analyze both structural and non-structural problems. Structural analysis includes computation of deflection, stress, strain, force, energy, vibration, etc. Non-structural problem include heat transfer, fluid flow, seepage, distribution of electric and magnetic potential. Softwares like SAP2000, STAADpro, etc. use FEM.

Advantages

i) It has wide range of applications.
ii) It can handle even irregular domain.
iii) It can model dynamic behaviour of structure.
iv) It can be applied to any loading system.
v) It can model even non-linearity of the structure.
vi) It can handle very complex structures.

Disadvantages

i) It can't model discontinuous structures.

Basic Steps in Finite Element Analysis

The following steps are performed for the finite element analysis:

i) Discretization of the continuum into small interconnected elements/mesh called finite elements connected at nodes.
ii) Identification/defining of unknown displacement at nodes.
iii) Choice of approximating function.
iv) Formation of element stiffness matrix.
v) Formation of overall stiffness matrix.
vi) Formation of the element loading matrix.
vii) Formation of the overall loading matrix.
viii) Incorporation of boundary conditions.
ix) Solution of simultaneous equations.
x) Calculation of stresses or stress resultants.
xi) Interpolation of field variable at any point of domain using shape functions.

1.4.2 Finite Difference Method (FDM)

Finite difference method is the solution technique in which differential equations are solved by converting them into difference equation.

Applications

FDM is used for the solution of problems related to hydraulics, surface water, ground water, hydrology, etc. Software like HEC-RAS, MikeShe, etc. use this method.

Advantages

i) Simple and easy method to apply.
ii) It can be applied in the study of hydrology, hydraulics, groundwater, etc.

Disadvantages

i) It is not as powerful technique as compared to FEM.
ii) It is difficult to apply for irregular geometry.

Similarities between FEM and FDM

i) Grid points in FDM are analogous to nodal points in FEM.
ii) In both of them, differential equation is converted into algebraic equation.

Difference between FEM and FDM

i) Field variables in FDM is calculated at grid points only whereas in FEM, variable is an integral part of the problem formulation.
ii) FDM needs larger number of nodes than FEM to get better results.
iii) FDM can handle fairly complicated problems but FEM can handle all complicated problems.
iv) FDM makes stair type approximation to sloping and curved boundaries but FEM can handle both of them exactly.
v) FDM models differential equations only whereas FEM models entire domain of the system.

1.4.3 Discrete Element Method (DEM)

A discrete element method (DEM), also called a distinct element analysis, is a numerical method for computing the motion and effect of a large number of small particles.

The fundamental assumption of this method is that the material consists of separate, discrete particles of different shapes and properties. A DEM-simulation is started by first generating a model, which results in spatially orienting all particles and assigning an initial velocity. The initial data, the relevant laws and contact models are used to compute the forces acting on each particle. These forces are added up. Change in position and velocity of particles in next time step due to the force is evaluated by integrating method. Next position are evaluated and the forces again. The iteration is repeated until simulation completes.

Advantages

i) It is extensively used for calculating forces, stress, strain, displacements of a granular material containing large number of particles.

Disadvantages

i) Since, the computation of change in position and velocity of each particle needs to be computed during different time step, it involves a number of loop, hence it is somewhat tedious.

1.4.4 Boundary Element Method (BEM)

The boundary element method (BEM) is a numerical computational method of solving linear partial differential equations which have been formulated as integral equations (i.e., in boundary integral form). It can be applied in many areas of engineering including fluid mechanics.

The integral equation may be regarded as an exact solution of the governing partial differential equation. The boundary element method attempts to use the given boundary conditions to fit boundary values into the integral equation, rather than values throughout the space defined by a partial differential equation. Once this is done, in the post-processing stage, the integral equation can then be used again to calculate numerically the solution directly at any desired point in the interior of the solution domain.

Advantages

i) Since, it involves the discretization of domain only on the boundary, it saves processing time and reduces dimension.

Disadvantages

i) Solution obtained through such techniques may not be as good as FEM, where the entire domain is discretized to elements.

1.4.5 Smoothed Particle Hydrodynamics (SPH)

The smoothed-particle hydrodynamics (SPH) method is a computational method used for simulating fluid flow. It is a mesh-free Lagrangian method (where the co-ordinates move with the fluid). This method works by dividing the fluid into a set of discrete elements, referred as particles. These particles have a spatial distance known as smoothing length. The physical quantity of any particle can be obtained by summing the relevant properties of all the neighbouring particles. The contributions of each particle to a property are weighted according to their distance from the particle of interest.

Advantages

i) It can be used widely in fluid flow analysis.
ii) Particle approximation of all terms in PDE to convert into ODE is related with respect to time only.
iii) It is suitable for deformable boundary problems and moving surface.

Disadvantages

i) Since, it is only related to fluid dynamics, it can't be used in versatile field related to structural analysis like FEM.

1.5 HISTORY OF NUMERICAL COMPUTATIONS

1.5.1 FEM History

- Foundation of FEM has been laid independently by scientists, engineers and physicists.
- In 1943, Courant developed method to use piecewise continuous functions defined over triangular domain.
- In 1950s, Polya, Hersh and Weinberger showed interest in Courant concept.
- In 1956, Turner developed stiffness matrix for truss, beam and other elements.
- Clough (1960) used the term FEM for the first time in plane stress analysis.
- In late 1960s and 1970s, improvements in the speed and memory capacity of computers contributed to the success in the field of FEM.

- The elastic analysis of the plane stress and plain strain problem has been successfully used in the analysis of 3D problems, non-linear, fluid flow and heat transfer.

1.5.2 FDM History

- FDM is the oldest computational technique.
- Foundation of FDM was laid by Courant, Friedrichs and Lewy in 1928 by publishing Fundamental paper on the solutions of problems on mathematical physics.
- After emergence of high-speed computer after 1980, FDM was widely used.

Just Think

- Why might be the course of "Computational techniques in civil engineering" introduced in final year, final part of bachelor degree in civil engineering? What might be its course objective?
- Is it appropriate to evaluate and use only approximate solution instead of the exact solution?
- What might be the role of computer in the history of development of computational techniques?

WORKED OUT PROBLEMS

Problem 1

With the help of mechanics, explain various numerical methods for civil engineering problems. Give their advantages and disadvantages. [2070 Bhadra]

Solution: See the definition part 1.4

Problem 2

Explain foundation of finite element method. Why this method is less appropriate for large deformation problem? How do you choose numerical method for different problems? Illustrate with examples. [2070 Magh]

Solution:

Finite element methods involve dividing the physical systems, such as structures, solid or fluid continua, into small sub-regions or elements. Each element is an essentially simple unit, the behaviour of which can be readily analyzed. The complexities of the overall systems are accommodated by using large numbers of elements, rather than by resorting to the sophisticated mathematics required by many analytical solutions. The elements are assumed to be connected at their intersecting points referred to as nodal points. After continuum is discretised with desired element shapes, the individual element stiffness matrix is formulated in global coordinates and are assembled to form the overall stiffness matrix. The assembly is done through the nodes which are common to adjacent elements. The overall stiffness matrix [K] is symmetric and banded. The loading inside an element is transferred at the nodal points and consistent element matrix is formed. Like the overall stiffness matrix, the element loading matrices are assembled to form the overall loading matrix[F]. The boundary restraint conditions are imposed in the stiffness matrix. The matrix equation of the form $\{F\}=[K]\{u\}$ is solved to get the unknown nodal displacements. Nodal displacements are utilized for the calculation of stresses or stress-resultants. Shape functions are used to get the value of the variables at any point of the element.

One of the main attractions of finite element methods is the ease with which they can be applied to problems involving geometrically complicated systems. The price that must be paid for flexibility and simplicity of individual elements is in the amount of numerical computation required. Very large sets of simultaneous algebraic equations have to be solved, and this can only be done economically with the aid of digital computers. Fortunately, finite element methods may be readily programmed for this purpose.

FEM is just an approximate method. Even though it can be applied to non-linear problems, large deformation problems involving geometric non-linearity cannot be handled well. Being an approximate method, results obtained from FEM is quite poor and tends to divert away from actual result. Hence FEM is less appropriate for large deformation problems.

Different Numerical techniques like FEM, FDM, BEM, DEM, SPH, etc. are available to deal with various engineering problems. All methods have their advantages and disadvantages. Appropriate numerical method is chosen for different problems. *For example;* as FEM is most versatile computational technique and can be used in solid mechanics, fluid mechanics, heat transfer, etc. But FDM and BEM are more efficient in dealing with problems related to fluid.

Problem 3

List the computational techniques used in civil engineering. Why FEM is predominating others? Explain briefly the steps involved in FEM. [2071 Bhadra]

Solution: See the definition part 1.4.1

Problem 4

Write the basic steps in finite element analysis. How this method is advantageous over other method of structural analysis? [2071 Magh]

Solution: See the definition part 1.4.1

Problem 5

Describe briefly the various solution techniques used for solving civil engineering problems. Also give their advantages and disadvantages. [2072 Ashwin]

Solution: See the definition part 1.4

Problem 6

Explain importance of numerical computations of civil engineering problems. How finite element method works in a structural analysis. [2072 Magh]

Solution:

For the first part

In various field of civil engineering, we encounter various physical processes described by differential equations. These equations are to be solved to obtain value of variables. Though analytical solutions are exact solution for simplified equations, however, for complex problems, it is a tedious task to do. Therefore, solutions of such equations are obtained by approximate method using computer. Numerical approach involving different software and programs are used in computation. These software involve assessment of large matrices even greater than size (1000 × 1000) which gives approximate but reliable solutions.

For the second part

See the steps and advantages in short

Problem 7

Describe briefly the different solution techniques used for numerical computations of civil engineering problems. Also mention their merits and demerits. [2073 Bhadra]

Solution: See the definition part 1.4

Problem 8

Describe the different solution techniques in civil engineering and list their suitability. [2073 Magh]

Solution: See the definition part 1.4

CHAPTER 2

SOLUTIONS OF LINEAR EQUATIONS

2.1 SYSTEM OF LINEAR EQUATIONS

General form of a system of linear equations is:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= C_1 \\
a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= C_2 \\
\vdots \quad\quad \vdots \quad\quad\quad\quad\quad \vdots \quad &\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n &= C_n
\end{aligned}
$$

where, a_{ij} are coefficients of unknown x_j's and C_j's are known right side terms. The system of linear equation can be represented in matrix form as:

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{Bmatrix}
$$

The unique solution of the system of linear equations presented above is possible only if the determinant of the square coefficient matrix is not equal to zero. If the determinant is zero, the solutions are either not unique or doesn't exist.

2.2 SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Simultaneous linear algebraic equations occur quite often in various engineering fields. Matrix inversion method or Cramer's rule can be used to solve them but when the number of unknowns and equations is very large, numerical methods would be better approach, which is suited for computer operations. Numerical method solutions are of two types.

i) Direct methods

Gauss elimination method, Gauss-Jordan method, Triangulation method, Crout's method, etc.

ii) Iterative methods

Gauss-Jacobi method, Gauss-Seidel method, Relaxation method, etc.

Some of the common methods of solving simultaneous linear equations are described below:

2.2.1 Cramer's Rule

This method is rather suitable for longhand solution of small number of simultaneous equations.

Consider a system of linear equations in three unknowns x_1, x_2 and x_3 be:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_1 &= C_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_2 &= C_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= C_3
\end{aligned}
$$

In matrix form;

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix}
$$

Let the determinant of the coefficients be;

$$
\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

Δ_1, Δ_2 and Δ_3 be determinant of matrix formed by replacing first, second and third column by the elements of constant matrix $\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix}$.

i.e.,
$$
\Delta_1 = \begin{vmatrix} C_1 & a_{12} & a_{13} \\ C_2 & a_{22} & a_{23} \\ C_3 & a_{32} & a_{33} \end{vmatrix}
$$

$$
\Delta_2 = \begin{vmatrix} a_{11} & C_1 & a_{13} \\ a_{21} & C_2 & a_{23} \\ a_{31} & C_3 & a_{33} \end{vmatrix}
$$

and,
$$
\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & C_1 \\ a_{21} & a_{22} & C_2 \\ a_{31} & a_{32} & C_3 \end{vmatrix}
$$

Now, solutions are given by;

$$x_1 = \frac{\Delta_1}{\Delta}$$

$$x_2 = \frac{\Delta_2}{\Delta}$$

and, $$x_3 = \frac{\Delta_3}{\Delta}$$

NOTE

1. Unique solution of the system of linear equations is possible only if $\Delta \neq 0$.
2. If $\Delta = 0$ and at least one of Δ_1, Δ_2 and Δ_3 is non-zero, the system has no solution.
3. If $\Delta_1 = \Delta_2 = \Delta_2 = 0$, the system has infinite number of solution.

EXAMPLE 2.1

Solve by Cramer's rule:

$$
\begin{aligned}
x - 3y + z &= 2 \\
3x + y + z &= 6 \\
5x + y + 3z &= 3
\end{aligned}
$$

Solution:

In matrix form:

$$\begin{bmatrix}1 & -3 & 1\\3 & 1 & 1\\5 & 1 & 3\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{Bmatrix}2\\6\\3\end{Bmatrix}$$

$$\Delta=\begin{vmatrix}1 & -3 & 1\\3 & 1 & 1\\5 & 1 & 3\end{vmatrix}=12\neq 0$$

$$\Delta_1=\begin{vmatrix}2 & -3 & 1\\6 & 1 & 1\\3 & 1 & 3\end{vmatrix}=52$$

$$\Delta_2=\begin{vmatrix}1 & 2 & 1\\3 & 6 & 1\\5 & 3 & 3\end{vmatrix}=-14$$

$$\Delta_3=\begin{vmatrix}1 & -3 & 2\\3 & 1 & 6\\5 & 1 & 3\end{vmatrix}=-70$$

$$x=\frac{\Delta_1}{\Delta}=\frac{52}{12}=\frac{13}{3}$$

$$y=\frac{\Delta_2}{\Delta}=-\frac{14}{12}=-\frac{7}{6}$$

and, $$z=\frac{\Delta_3}{\Delta}=-\frac{70}{12}=-\frac{35}{6}$$

2.2.2 Gauss–Elimination Method

In this method, the square coefficient matrix is transformed to either uppe triangular or lower triangular matrix by elementary row operations and th unknowns are evaluated by back substitution or forward substitution respectively.

Consider the system with unknowns x, y and z as:

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

In matrix form;

$$\begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}$$

or, $[A][X]=[B]$

Consider the augmented matrix;

$$[A|B]=\left[\begin{array}{ccc|c}a_1 & b_1 & c_1 & d_1\\a_2 & b_2 & c_2 & d_2\\a_3 & b_3 & c_3 & d_3\end{array}\right]$$

After series of row operations, the augmented matrix is transformed into form:

$$\left[\begin{array}{ccc|c}a_1 & b_1 & c_1 & d_1\\0 & b_2' & c_2' & d_2'\\0 & 0 & c_3'' & d_3''\end{array}\right]$$

Now, by using backward substitution in:

$$\begin{bmatrix}a_1 & b_1 & c_1\\0 & b_2' & c_2'\\0 & 0 & c_3''\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}d_1\\d_2'\\d_3''\end{bmatrix};$$ **x, y and z are determined.**

However, if the pivots a_1, b_2' and c_3'' are zero, this method fails. To eliminate these problems, pivoting can be done. In general, the pivot should be selected as the largest (in absolute value) of the elements in any column. This process is called **partial pivoting**. Even better results can be obtained by choosing pivot as the largest element in the whole matrix. This is called **complete pivoting.**

EXAMPLE 2.2

Solve the following set of simultaneous equations by Gauss–elimination method.

$$2x_1+2x_2+x_3=9$$
$$2x_1+x_2=4$$
$$x_1+x_2+x_3=6$$

Solution:

The system in matrix form be;

$$\begin{bmatrix}2 & 2 & 1\\2 & 1 & 0\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}9\\4\\6\end{bmatrix}$$

∴ Augmented matrix is $\left[\begin{array}{ccc|c}2 & 2 & 1 & 9\\2 & 1 & 0 & 4\\1 & 1 & 1 & 6\end{array}\right]$

$$\begin{array}{l}R_2\to R_2-R_1\\R_3\to R_3-\frac{1}{2}R_1\end{array}\sim\left[\begin{array}{ccc|c}2 & 2 & 1 & 9\\0 & -1 & -1 & -5\\0 & 0 & \frac{1}{2} & \frac{3}{2}\end{array}\right]$$

Now, by backward substitution; we have,

$$\frac{1}{2}x_3=\frac{3}{2}$$

or, $x_3=3$

$$-1x_2+(-1)\times(3)=-5$$

or, $x_2=2$

$$2x_1+(2)\times(2)+(1)\times(3)=9$$

or, $x_1=1$

EXAMPLE 2.3

Use Gauss elimination with partial pivoting to solve the system of linear equations given:

$$0.143\,x_1+0.357\,x_2+2.01\,x_3=-5.173$$
$$-1.31\,x_1+0.911\,x_2+1.99\,x_3=-5.458$$
$$11.2\,x_1-4.30\,x_2-0.605\,x_3=4.415$$

Solution:

In matrix form:

$$\begin{bmatrix} 0.143 & 0.357 & 2.01 \\ -1.31 & 0.911 & 1.99 \\ 11.2 & -4.30 & -0.605 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5.17 \\ -5.46 \\ 4.42 \end{bmatrix}$$

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 0.143 & 0.357 & 2.01 & -5.17 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 11.2 & -4.30 & -0.605 & 4.42 \end{array}\right]$$

$$R_1 \leftrightarrow R_3 \sim \left[\begin{array}{ccc|c} 11.2 & -4.30 & -0.605 & 4.42 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 0.143 & 0.357 & 2.01 & -5.17 \end{array}\right]$$

$$R_1 \to \frac{R_1}{11.2} \sim \left[\begin{array}{ccc|c} 1 & -0.384 & -0.0540 & 0.395 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 0.143 & 0.357 & 2.01 & -5.17 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 + 1.31R_1 \\ R_3 \to R_3 - 0.143R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1.00 & -0.384 & -0.054 & 0.395 \\ 0.00 & 0.408 & 1.92 & -4.94 \\ 0.00 & 0.412 & 2.02 & -5.23 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \sim \left[\begin{array}{ccc|c} 1.00 & -0.384 & -0.054 & 0.395 \\ 0.00 & 0.412 & 2.02 & -5.23 \\ 0.00 & 0.408 & 1.92 & -4.94 \end{array}\right]$$

$$R_2 \to \frac{R_2}{0.412} \sim \left[\begin{array}{ccc|c} 1.00 & -0.384 & -0.054 & 0.395 \\ 0.00 & 1.00 & 4.90 & -12.7 \\ 0.00 & 0.408 & 1.92 & -4.94 \end{array}\right]$$

$$R_3 \to R_3 - 0.408 \times R_2 \sim \left[\begin{array}{ccc|c} 1.00 & -0.384 & -0.054 & 0.395 \\ 0.00 & 1.00 & 4.90 & -12.7 \\ 0.00 & 0.00 & -0.08 & 0.240 \end{array}\right]$$

$$R_3 \to \frac{R_3}{-0.08} \sim \left[\begin{array}{ccc|c} 1.00 & -0.384 & -0.054 & 0.395 \\ 0.00 & 1.00 & 4.90 & -12.7 \\ 0.00 & 0.00 & 1.00 & -3.00 \end{array}\right]$$

Thus, $x_3 = -3.00$ and using back substitution; we have,

$$x_2 = 2.00$$

and, $x_1 = 1.00$

2.2.3 Gauss–Jordan Method

In Gauss–Jordan method, the former part of augmented matrix is converted into unit matrix by row, column operations and then the values of later part give the solution.

If $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ is the augmented matrix and by row, column operations transformed into:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & d_1' \\ 0 & 1 & 0 & d_2' \\ 0 & 0 & 1 & d_3' \end{array}\right];$$

where, d_1', d_2' and d_3' are the solutions.

2.2.4 Gauss–Seidel Method

It is an iterative method to solve linear equations. Though it can be applied to any matrix with non-zero elements on the diagonal, convergence is only guaranteed if matrix is either diagonally dominant or symmetric and positive definite.

System of linear equation be:

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = C_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = C_2$$
$$\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{nn}x_n = C_n$$

Re-arrange the equations as:

$$x_1 = \frac{1}{a_{11}}\left[C_1 - \Sigma_{i=2}^{n} a_{1i}x_i\right] \qquad (a)$$

$$x_2 = \frac{1}{a_{22}}\left[C_2 - \Sigma_{i=1, i\neq 2}^{n} a_{2i}x_i\right] \qquad (b)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}\left[C_n - \Sigma_{i=1, i\neq n}^{n} a_{ni}x_i\right] \qquad (c)$$

STEPS

- Assign initial values of $x_2, x_3, \dots\dots\dots, x_n$.
- Substitute the values of variables you have in R.H.S. of the equations one by one. While substituting, always use the latest value you have gained. Repeat the iterations until the difference between new value and previous values are insignificant.

NOTE

A real matrix, M is said to be positive definite if the scalar $Z^T MZ$ is positive for every non-zero column vector Z.

EXAMPLE 2.4

Solve the following equations by Gauss-Seidal iteration method.

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 35$$

Solution:

Rearrange the equation as:

$$x = \frac{1}{8}(20 + 3y - 2z) \qquad (a)$$

$$y = \frac{1}{11}(33 - 4x + z) \qquad (b)$$

$$z = \frac{1}{12}(35 - 6x - 3y) \qquad (c)$$

Assign $y = z = 0$ and substitute in equation (a); we have,

$$x_1 = \frac{20}{8} = 2.5$$

Assign x = 2.5, z = 0 and substitute equation (b); we have,

$$y_1 = \frac{1}{11} \times (33 - 4 \times 2.5) = 2.09$$

Assign x = 2.5, y = 2.09 and substitute equation (c); we have,

$$z_1 = \frac{1}{12}[35 - 6 \times 2.5 - 3 \times 2.09] = 1.14$$

NOTE

Program the calculator

$$x = \frac{20 + 3y - 2M}{8}; y = \frac{33 - 4x + M}{11}; z = \frac{35 - 6x - 3y}{12}.$$

Second approximation

$$x_2 = \frac{1}{8}[20 + 3y_1 - 2z_1] = 2.10$$

$$y_2 = \frac{1}{11}[33 - 4x_2 + z_1] = 2.01$$

$$z_2 = \frac{1}{11}[35 - 6x_2 - 3y_2] = 0.91$$

Third approximation

$x_3 = 3.03$
$y_3 = 1.98$
$z_3 = 0.90$

Fourth approximation

$x_4 = 3.02$
$y_4 = 1.99$
$z_4 = 0.91$

Fifth approximation

$x_5 = 3.02$
$y_5 = 1.99$
$z_5 = 0.91$

Since the values of x, y and z in the fourth and fifth approximation are same, the solutions are:

x = 3.02
y = 1.99
z = 0.91

2.3 DATA STORAGE AND MEMORY OPTIMIZATION

Large size of matrix is very common in engineering problems which occupy large amount of computer memory if not optimized. Some techniques to reduce memory requirements are discussed below:

i) Use of symmetry

For symmetric matrix, storing only upper triangular part or lower triangular part is sufficient.

ii) Use of sparse nature

In FEM and FDM, many matrix we deal have most of their elements 0 and very few non-zero terms. To reduce the memory requirements only non-zero terms and their locations are stored for such sparse matrixes.

iii) Use of banded nature

A matrix with its non-zero elements very close to its main diagonal within a small bandwidth and all other elements being zero is called band matrix. Band matrices are usually stored by storing elements of diagonals in a band.

For example: A tri-diagonal 6 × 6 matrix

$$\begin{bmatrix} B_{11} & B_{12} & 0 & 0 & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 & 0 \\ 0 & B_{32} & B_{33} & B_{34} & 0 & 0 \\ 0 & 0 & B_{43} & B_{44} & B_{45} & 0 \\ 0 & 0 & 0 & B_{54} & B_{55} & B_{56} \\ 0 & 0 & 0 & 0 & B_{65} & B_{66} \end{bmatrix}$$

is stored as 6 × 3 matrix:

$$\begin{bmatrix} 0 & B_{11} & B_{12} \\ B_{21} & B_{22} & B_{23} \\ B_{32} & B_{33} & B_{34} \\ B_{43} & B_{44} & B_{45} \\ B_{54} & B_{55} & B_{56} \\ B_{65} & B_{66} & 0 \end{bmatrix}$$

Further saving is possible when the matrix is symmetric and can be stored as:

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{22} & B_{23} \\ B_{33} & B_{34} \\ B_{44} & B_{45} \\ B_{55} & B_{56} \\ B_{66} & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & B_{11} \\ B_{21} & B_{22} \\ B_{32} & B_{33} \\ B_{43} & B_{44} \\ B_{54} & B_{55} \\ B_{65} & B_{66} \end{bmatrix}$$

iv) Partitioning matrix

The matrices are partitioned into small triangular sub-matrices. Only few such triangular matrices are stored in a computer core at a given time with remaining portions stored in peripheral storage.

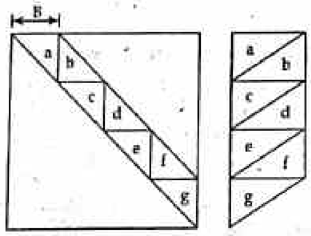


Figure: Partisioning of matrix

v) Skyline storage

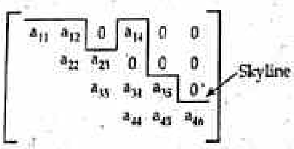


Figure: Skyline

The line separating top zeros from non-zero elements in a column are called a skyline. In skyline storage technique, only terms below skyline are stored.

2.4 CONJUGATE GRADIENT METHOD

If for a system of linear equations $AX = B$, the coefficient matrix A is symmetric ($A^T = A$) and positive definite ($Z^T AZ > 0$), the equations can be solved by conjugate gradient method, another iterative method of higher convergence.

Algorithm

Step 1 : Assign $i = 0$ and initial value to $X_i = X_0$.

Step 2 : Evaluate initial residual value;

$$r_i = r_0 = B - AX_0$$

Step 3 : Assign initial search direction;

$$P_i = P_0 = r_0$$

Step 4 : Evaluate the scalar α_i;

$$\alpha_i = \frac{r_i^T r_i}{P_i^T A P_i}$$

Step 5 : Evaluate x_{i+1} and r_{i+1};

$$x_{i+1} = x_i + \alpha_i P_i$$
$$r_{i+1} = r_i - \alpha_i A P_i$$

Step 6 : If r_{i+1} is within permissible range, go to step 11, otherwise go to **step 7.**

Step 7 : Evaluate the scalar β_i.

$$\beta_i = \frac{r_{i+1}^T r_{i+1}}{r_i^T r_i}$$

Step 8 : Evaluate new search direction;

$$P_{i+1} = r_{i+1} + \beta_i P_i$$

Step 9 : Assign $i = i + 1$

Step 10: Go to step 4.

Step 11: The solution is x_{i+1}.

Step 12: Close

Preference of conjugate gradient method over Gaussian methods

i) Conjugate gradient method has higher convergence.
ii) It needs less calculation.
iii) It consumes less memory space.

EXAMPLE 2.5

Solve the following set of linear equations by conjugate gradient method.

$3X_1 - 4X_2 = 2$
$4X_1 - X_2 = 9$

Solution:

The system in matrix form:

$$\begin{bmatrix} 3 & -4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

The coefficient matrix is not a symmetric matrix but by multiplying the first equation by −1, it can be made symmetric.

i.e. $\begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix} \Rightarrow AX = B$

Now,

Assign $X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Then,

$$r_0 = B - AX_0 = \begin{bmatrix} -2 \\ 9 \end{bmatrix} - 0 = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$P_0 = r_0 = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$\alpha_0 = \frac{r_0^T r_0}{P_0^T A P_0} = \frac{\begin{bmatrix} -2 \\ 9 \end{bmatrix}^T \begin{bmatrix} -2 \\ 9 \end{bmatrix}}{\begin{bmatrix} -2 \\ 9 \end{bmatrix}^T \begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 9 \end{bmatrix}} = \frac{85}{-237} = -0.359$$

$$X_1 = X_0 + \alpha_0 P_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 0.359\begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 0.718 \\ -3.231 \end{bmatrix}$$

$$r_1 = r_0 - \alpha_0 A P_0 = \begin{bmatrix} -2 \\ 9 \end{bmatrix} + 0.359\begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 13.07 \\ 2.89 \end{bmatrix}$$

$$\beta_0 = \frac{\begin{bmatrix} 13.07 \\ 2.89 \end{bmatrix}^T \begin{bmatrix} 13.07 \\ 2.89 \end{bmatrix}}{\begin{bmatrix} -2 \\ 9 \end{bmatrix}^T \begin{bmatrix} -2 \\ 9 \end{bmatrix}} = 2.107$$

$$P_1 = r_1 + \beta_0 P_0 = \begin{bmatrix} 13.07 \\ 2.89 \end{bmatrix} + 2.107\begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 8.856 \\ 21.853 \end{bmatrix}$$

$$\alpha_1 = \frac{r_1^T r_1}{P_1^T A P_1} = \frac{\begin{bmatrix} 13.07 \\ 2.89 \end{bmatrix}^T \begin{bmatrix} 13.07 \\ 2.89 \end{bmatrix}}{\begin{bmatrix} 8.856 \\ 21.853 \end{bmatrix}^T \begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 8.856 \\ 21.853 \end{bmatrix}}$$

$$= \frac{179.177}{835.401}$$

$$= 0.21448$$

$$X_2 = X_1 + \alpha_1 P_1$$

$$= \begin{bmatrix} 0.718 \\ -3.231 \end{bmatrix} + 0.21448\begin{bmatrix} 8.856 \\ 21.853 \end{bmatrix} = \begin{bmatrix} 2.6174 \\ 1.4560 \end{bmatrix}$$

$$r_2 = r_1 - \alpha_1 A P_1$$

$$= \begin{bmatrix} 13.07 \\ 2.89 \end{bmatrix} - 0.21448\begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 8.856 \\ 21.853 \end{bmatrix} = \begin{bmatrix} 0.02 \\ -0.02 \end{bmatrix} \approx 0$$

Therefore, the solutions are: $x_1 = 2.62$ and $x_2 = 1.46$.

NOTE

While doing numerical of CGM, try to make use of matrix mode and memory of the calculator carefully to save time. However, multiple of transpose and its matrix is just the sum of squares of the elements and it is faster to calculate without use of matrix mode.

EXAMPLE 2.6

Solve the following set of linear equations using Gauss Siedal iteration or conjugate gradient method.

$$6x + 3.8y + 9z = 12$$
$$1.2x + 2y - 3.8z = 16$$
$$-4x + 5y + 6z = -15$$

Solution:

The system of linear equations in matrix form is:

$$\begin{bmatrix} 6 & 3.8 & 9 \\ 1.2 & 2 & -3.8 \\ -4 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \\ -15 \end{bmatrix}$$

The coefficient matrix is not a symmetric matrix and can't be solved by conjugate gradient method. Since the coefficient matrix is not symmetric positive definite, the solution for conjugate gradient method.

Though the coefficient matrix is not symmetric positive definite, the solution for the equations will be possible if it is diagonally dominant.

Checking for diagonal dominant condition

$$\left.\begin{aligned} |6| &\ngtr |3.8| + |9| \\ |2| &\ngtr |1.2| + |-3.8| \\ |6| &\ngtr |-4| + |5| \end{aligned}\right\}; \text{ it is neither diagonally dominant}$$

so, convergence is neither possible by Gauss-Seidal method.

EXAMPLE 2.7

Solve the following equation by using conjugate gradient method (maximum 3 iterations)

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}$$

Solution:

We have,

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$$

$$B = \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}$$

Assign,

$$X_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Then,

$$r_0 = B - AX_0 = \begin{Bmatrix} 1 \\ -12 \\ 3 \end{Bmatrix} - \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}$$

Assign,

$$P_0 = r_0 = \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}$$

NOTE

Calculation by calculator

Mode

6 : Matrix

1 : Mat A

1 : 3 × 3

$$A\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$$

AC

Shift 4

1 : Dim

2 : Mat B

3 : 3 × 1

$$B = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$$

AC

Now,

$$\alpha_0 = \frac{r_0^T r_0}{P_0^T A P_0} = \frac{\begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}^T \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}}{\begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}^T \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}} = \frac{149}{-281} = -0.530$$

NOTE

Calculation by calculator

Shift 4

3 : Trn

Shift 4

4 : Mat B

*

Shift 4

4 : Mat B

Trn (Mat B) * Mat B = 149

Similarly,

Trn (Mat B) * Mat A * Mat B = − 281

149 ÷ −281 = −0.530

$$X_1 = X_0 + \alpha_0 P_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + (-0.530)\begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix} = \begin{Bmatrix} -0.53 \\ 6.36 \\ -1.06 \end{Bmatrix}$$

NOTE

Calculation by calculator

$$-0.530 \times \text{Mat B} = \begin{Bmatrix} -0.53 \\ 6.36 \\ -1.06 \end{Bmatrix}$$

$$r_1 = r_0 - \alpha_0 A P_0 = \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix} + 0.530 \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix}$$

NOTE

Calculation by calculator

$$\text{Mat B} + 0.530 \times \text{Mat A} \times \text{Mat B} = \begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix}$$

Shift + STO + C

$$\beta_0 = \frac{r_1^T r_1}{r_0^T r_0} = \frac{\begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix}^T \begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix}}{\begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}^T \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}} = 1.968$$

NOTE

Calculation by calculator

Trn (Mat C) * Mat C = 293.27

Trn (Mat B) * Mat B = 149

293.27 ÷ 149 = 1.968

$$P_1 = r_1 + \beta_0 P_0 = \begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix} + 1.968 \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 5.618 \\ -26.076 \\ -12.614 \end{Bmatrix}$$

NOTE

Calculation by calculator

$$\text{Mat C} + 1.968 \times \text{Mat B} = \begin{Bmatrix} 5.618 \\ -26.076 \\ -12.614 \end{Bmatrix}$$

Shift + STO + B

$$\alpha_0 = \frac{r_1^T r_1}{P_1^T A P_1} = \frac{\begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix}^T \begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix}}{\begin{Bmatrix} 5.618 \\ -26.076 \\ -12.614 \end{Bmatrix}^T \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{Bmatrix} 5.618 \\ -26.076 \\ -12.614 \end{Bmatrix}} = \frac{293.277}{1246.533} = 0.235$$

NOTE

Calculation by calculator

$r_1^T r_1 = 293.27$ (already calculated)

i.e., Trn (Mat C) * Mat C = 293.27

Or

$(3.65)^2 + (-2.46)^2 + (-16.55)^2 = 293.277$

Trn (Mat B) * Mat A * Mat B = 1246.533

293.277 ÷ 1246.533 = 0.235

$$X_2 = X_1 + \alpha_1 P_1 = \begin{Bmatrix} -0.53 \\ 6.36 \\ -1.06 \end{Bmatrix} + 0.235 \begin{Bmatrix} 5.618 \\ -26.076 \\ -12.614 \end{Bmatrix} = \begin{Bmatrix} -0.295 \\ 0.232 \\ -4.024 \end{Bmatrix}$$

NOTE

Calculation by calculator

$-0.53 + 0.235 \times 5.618 = -0.295$

$6.36 - 0.235 \times 26.076 = 0.23214$

$-1.06 - 0.235 \times 12.614 = -4.024$

$$r_2 = r_1 - \alpha_1 A P_1 = \begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix} - 0.235 \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{Bmatrix} 5.618 \\ -26.076 \\ -12.614 \end{Bmatrix} = \begin{Bmatrix} 2.654 \\ 0.305 \\ 0.513 \end{Bmatrix}$$

NOTE

Calculation by calculator

$$\text{Mat C} - 0.235 * \text{Mat A} * \text{Mat B} = \begin{Bmatrix} 2.654 \\ 0.305 \\ 0.513 \end{Bmatrix}$$

Shift + STO + C

$$\beta_1 = \frac{r_2^T r_2}{r_1^T r_1} = \frac{\begin{Bmatrix} 2.654 \\ 0.305 \\ 0.510 \end{Bmatrix}^T \begin{Bmatrix} 2.654 \\ 0.305 \\ 0.51 \end{Bmatrix}}{\begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix}^T \begin{Bmatrix} 3.65 \\ -2.46 \\ -16.55 \end{Bmatrix}} = 0.0252$$

NOTE

Calculation by calculator

Trn (Mat C) * Mat C = 7.398

Already calculated, $r_1^T r_1 = 293.27$

$7.398 \div 293.27 = 0.0252$

$$P_2 = r_2 + \beta_1 P_1 = \begin{Bmatrix} 2.654 \\ 0.305 \\ 0.513 \end{Bmatrix} + 0.0252 \begin{Bmatrix} 5.618 \\ -26.076 \\ -12.614 \end{Bmatrix} = \begin{Bmatrix} 2.795 \\ -0.352 \\ 0.195 \end{Bmatrix}$$

NOTE

Calculation by calculator

$$\text{Mat C} + 0.0252 * \text{Mat B} = \begin{Bmatrix} 2.795 \\ -0.352 \\ 0.195 \end{Bmatrix}$$

Shift + STO + B

$$\alpha_2 = \frac{r_2^T r_2}{P_2^T A P_2} = \frac{\begin{Bmatrix} 2.654 \\ 0.305 \\ 0.510 \end{Bmatrix}^T \begin{Bmatrix} 2.654 \\ 0.305 \\ 0.510 \end{Bmatrix}}{\begin{Bmatrix} 2.795 \\ -0.352 \\ 0.195 \end{Bmatrix}^T \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{Bmatrix} 2.795 \\ -0.352 \\ 0.195 \end{Bmatrix}} = \frac{7.398}{23.994} = 0.308$$

NOTE

Calculation by calculator

Trn (Mat C) * Mat C = 7.398

Trn (Mat B) * Mat A * Mat B = 0.308

$7.398 \div 0.308 = 24.019$

$$X_3 = X_2 + \alpha_2 P_2 = \begin{Bmatrix} -0.295 \\ 0.232 \\ -4.024 \end{Bmatrix} + 0.308 \begin{Bmatrix} 2.795 \\ -0.352 \\ 0.195 \end{Bmatrix} = \begin{Bmatrix} 0.566 \\ 0.124 \\ -3.964 \end{Bmatrix}$$

NOTE

Calculation by calculator

$-0.295 + 0.308 \times 2.795 = 0.566$

$0.232 - 0.308 \times 0.352 = 0.124$

$-4.024 + 0.308 \times 0.195 = -3.964$

$$r_3 = r_2 - \alpha_2 A P_2 = \begin{Bmatrix} 2.654 \\ 0.305 \\ 0.510 \end{Bmatrix} - 0.308 \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{Bmatrix} 2.795 \\ -0.352 \\ 0.195 \end{Bmatrix} = \begin{Bmatrix} 0.010 \\ 0.010 \\ -0.015 \end{Bmatrix}$$

NOTE

Calculation by calculator

$$\text{Mat C} - 0.308 * \text{Mat A} * \text{Mat B} = \begin{Bmatrix} 0.010 \\ 0.010 \\ -0.015 \end{Bmatrix}$$

Since, the residual is negligible, we stop the iteration and the solution is;

$$X = \begin{Bmatrix} 0.566 \\ 0.124 \\ -3.964 \end{Bmatrix}$$

2.5 FOURIER TRANSFORM

Consider an arbitrary periodic function p(t) with period $\overline{T}$. By definition;

$$p(t + n\overline{T}) = p(t), \; n = \pm 1, \pm 2, \ldots\ldots; \qquad [2.5.1]$$

The frequency of the periodic function p(t) is $\overline{w}$, given by;

$$\overline{w} = \frac{2\pi}{\overline{T}} = \text{fundamental frequency} \qquad [2.5.2]$$

The function p(t) can be expressed as a series of trigonometric functions called Fourier series.

$$p(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\overline{w}t + \sum_{n=1}^{\infty} b_n \sin n\overline{w}t; \qquad [2.5.3]$$

where, $a_0 = \frac{1}{\overline{T}} \int_0^{\overline{T}} p(t)\, dt$

$$a_n = \frac{2}{\overline{T}} \int_0^{\overline{T}} p(t) \cos n\overline{w}t \, dt$$

$$b_n = \frac{2}{\overline{T}} \int_0^{\overline{T}} p(t) \sin n\overline{w}t \, dt$$

Equation (a) is trigonometric form of Fourier series and can be represented in exponential form as:

$$p(t) = \sum_{n=-\infty}^{\infty} P_n e^{in\overline{w}t} \qquad [2.5.4]$$

where, $P_n = \frac{1}{\overline{T}} \int_0^{\overline{T}} p(t) e^{-in\overline{w}t}\, dt$; [2.5.5]

$$n = 0, \pm 1, \pm 2, \ldots\ldots$$

p(t) is a function of time whereas P_n is a function of frequency. R.H.S. of the expression [2.5.4] is called Fourier transform of p(t). Similarly, R.H.S. of expression [2.5.5] is called inverse fourier transform of P_n. From the above expressions, it is obvious that a system in time domain can be transformed into frequency domain and vice versa using fourier and inverse fourier series respectively.

2.5.1 Discrete Fourier Transform (DFT)

The value of variables in an experiment is sampled at constant time interval Δt. p(t) function representing the acquired values and we can write,

$$P_m = p(t_m) = p(m\Delta t), ; \; m = 0, \pm 1, \ldots\ldots$$

where, Δt is the sampling period.

$f_s = \frac{1}{\Delta t}$ is the sampling frequency.

We assume that the function is periodic with period $\overline{T} = N\Delta t$; where, N is the number of intervals. The smallest frequency that can be considered in the analysis is:

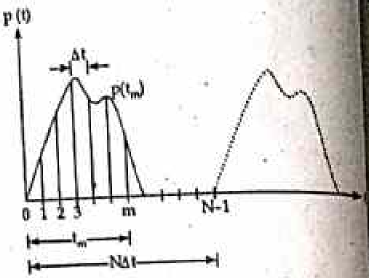
$$\overline{w}_1 = \Delta\overline{w} = \frac{2\pi}{\overline{T}} = \frac{2\pi}{N\Delta T}$$

The exponential form of Fourier transform for infinite domain can be expressed as:

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{(n)} e^{iw_n t}\, dw$$

and, $P_{(n)} = \int_{-\infty}^{\infty} p(t) e^{-iw_n t}\, dt$

where, $w_n = n \cdot \Delta\overline{w}$



We approximate;

$$dt = \Delta t$$
$$t_m = m\Delta t$$
$$\overline{T} = N\Delta t$$

Then,

$$p(t_m) = \frac{1}{\overline{T}\Delta\overline{w}}\sum_{n=0}^{N-1} p(n) e^{i\left(\frac{2\pi n}{N\Delta t}\right)m\Delta t}\Delta\overline{w}$$

$$= \frac{1}{\overline{T}}\sum_{n=0}^{N-1} p(n) e^{i\left(\frac{2\pi}{N}\right)mn};\ m = 0 \rightarrow N-1$$

and, $$P(n) = \Delta t \sum_{m=0}^{N-1} p(t_m) e^{-i\left(\frac{2\pi}{N}\right)mn} \qquad [2.5.6]$$

2.5.2 Algorithm for DFT

From equation [2.5.6]; we have,

$$\text{DFT: } P(n) = \Delta t \sum_{m=0}^{N-1} p(m) e^{-i\left(\frac{2\pi}{N}\right)mn}$$

Also, we know that;

$$e^{-ix} = \cos x - i\sin x$$

$$\therefore\ P(n) = \Delta t \sum_{m=0}^{N-1} p(m)\left[\cos\left(\frac{2\pi mn}{N}\right) - i\sin\left(\frac{2\pi mn}{N}\right)\right]$$

$$= \Delta t \sum_{m=0}^{N-1} p(m)\cos\left(\frac{2\pi mn}{N}\right) - i\Delta t\sum_{m=0}^{N-1} p(m)\sin\left(\frac{2\pi mn}{N}\right)$$

$$= C - iS$$

Algorithm

1. Define library functions
2. Define variables
3. Input sample data in time domain and read the data
4. Calculate:

$$C = \Delta t \sum_{m=0}^{N-1} p(m)\cos\left(\frac{2\pi mn}{N}\right)$$

$$S = \Delta t \sum_{m=0}^{N-1} p(m)\sin\left(\frac{2\pi mn}{N}\right)$$

For n = 0, 1, 2, ..., N − 1 .

5. Display result as: P(n) = C − iS
6. End

2.5.3 Fast Fourier Transform (FFT)

The expression of DFT is:

$$p(iw_n) = \Delta t \sum_{m=0}^{N-1} p_m e^{-i\left(\frac{2\pi mn}{N}\right)};\ n = 0, 1, 2, \ldots\ldots\ldots$$

or, $$P(n) = \Delta t \sum_{m=0}^{N-1} p_m W_N^{mn} \qquad [2.5.7]$$

where, $W_N = e^{-i\left(\frac{2\pi}{N}\right)}$.

It is easy to realize that the same value of W_N^{mn} are calculated many times as the computation proceeds. Firstly the integer product mn repeats for different combinations of m and n; secondly W_N^{mn} is a periodic function with only N distinct values.

Computation of DFT is slow as a large number of multiplications and additions are required for the calculations. For N-point DFT, there will be N^2 multiplications and N(N − 1) additions.

The FFT is a discrete Fourier transform algorithm which reduces the number of computations needed for N points from $2N^2$ to $2N\log_2 N$.

FFT algorithm generally falls into two classes: decimination in time and decimination in frequency. Decimination in time is elaborated below:

In expression [2.5.7], let us split the single summation over N samples into 2 summations each with $\frac{N}{2}$ samples, one for even m and other for odd m.

Substitute $S = \frac{m}{2}$ for m even and $S = \frac{m-1}{2}$ for m odd.

$$P(n) = \Delta t \sum_{S=0}^{\frac{N}{2}-1} P_{2S} W_N^{2Sn} + \Delta t \sum_{S=0}^{\frac{N}{2}-1} P_{(2S+1)} W_N^{(2S+1)n} \qquad [2.5.8]$$

Note that;

$$W_N^{2Sn} = e^{-i\left(\frac{2\pi}{N}\right)2Sn} = e^{-i\left(\frac{2\pi}{\frac{N}{2}}\right)Sn} = W_{\frac{N}{2}}^{Sn}$$

$$\therefore\ P(n) = \Delta t \sum_{S=0}^{\frac{N}{2}-1} P_{2S} W_{\frac{N}{2}}^{Sn} + \Delta t W_N^{n} \sum_{S=0}^{\frac{N}{2}-1} P_{(2S+1)} W_{\frac{N}{2}}^{Sn} \qquad [2.5.9]$$

Algorithm of 2.5.9 is much more convenient than algorithm of 2.5.7.

2.5.4 Algorithm of FFT

From equation [2.5.8]; we have,

$$P(n) = \Delta t \sum_{S=0}^{\frac{N}{2}-1} P_{2S} W_N^{2Sn} + \Delta t \sum_{S=0}^{\frac{N}{2}-1} P_{(2S+1)} W_N^{(2S+1)n}$$

$$= \Delta t \left\{ \sum_{S=0}^{\frac{N}{2}-1} P_{2S} e^{-\frac{4\pi Sn}{N}} + \sum_{S=0}^{\frac{N}{2}-1} P_{(2S+1)} e^{-i\frac{2\pi(2S+1)n}{N}} \right\}$$

$$= \Delta t \left[\sum_{S=0}^{\frac{N}{2}-1} P_{2S} \cos\left(\frac{4\pi Sn}{N}\right) + \sum_{m=0}^{\frac{N}{2}-1} P_{(2S+1)} \cos\left(\frac{2\pi(2S+1)n}{N}\right) \right]$$

$$-i\Delta t \left[\sum_{S=0}^{\frac{N}{2}-1} P_{2S} \sin\left(\frac{4\pi Sn}{N}\right) + \sum_{S=0}^{\frac{N}{2}-1} P_{(2S+1)} \sin\left(\frac{2\pi(2S+1)n}{N}\right) \right]$$

$$= \Delta t[(C_1 + C_2) - i(S_1 + S_2)]$$

$$= \Delta t[C - iS]$$

Algorithm

1. Define library functions
2. Declare variables
3. Input and read sample data in time domain
4. Calculate:

$$C_1 = \sum_{S=0}^{\frac{N}{2}-1} P_{2S} \cos\left(\frac{4\pi Sn}{N}\right)$$

$$C_2 = \sum_{m=0}^{\frac{N}{2}-1} P_{(2S+1)} \cos\left(\frac{2\pi(2S+1)n}{N}\right)$$

$$S_1 = \sum_{S=0}^{\frac{N}{2}-1} P_{2S} \sin\left(\frac{4\pi Sn}{N}\right)$$

$$S_2 = \sum_{S=0}^{\frac{N}{2}-1} P_{(2S+1)} \sin\left(\frac{2\pi(2S+1)}{N}\right)$$

$$C = C_1 + C_2$$

$$S = S_1 + S_2$$

for $n = 0, 1, 2, \ldots\ldots\ldots, N-1$

5. Display result as: $P(n) = C + iS$
6. End

WORKED OUT PROBLEMS

Problem 1

Solve the following equation by using conjugate gradient method. *(maximum 5 iterations)*

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -12 \\ 2 \end{Bmatrix}$$

[2070 Bhadra]

Solution: See the solution of example 2.7

Problem 2

Write an algorithm and a program (C or Fortan or Matlab) for fast Fourier transform. With a suitable example, explain what parameters can be identified with the help of time domain and frequency domain. *[2070 Magh]*

Solution: See algorithm in definition part 2.5.4

Program

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
Void main()
      {
      int m,n,y,N,i;
      float F[1000],FFT[1000][2],pi=3.14,C1,C2,S1,S2,C,S;
      clrscr();
printf(\n Enter the number of sample data");
      scanf('%d",&y);
      N=power(2,y);
      Printf("\n Enter %d sample data in time domain:",N);
      for(n=0;n<N;n++)
      {
      C1=0;
      C2=0;
      S1=0;
      S2=0;
      for(m=0;m<N/2;m++)
      (
      C1=C1+F[2*m]*cos(4*pi*m*n/N);
      C2=C2+F[2*m+1]*cos(4*pi*m*n/N);
      S1=S1+F[2*m]*sin(4*pi*m*n/N);
      S1=S2+F[2*m+1]*sin(4*pi*m*n/N); S2=7
      }
      C=C1+C2;
      S=S1+S2;
      FFT[n][0]=C;
      FFT[n][1]=S;
      }
printf("Given sequence in frequency domain is:");
      for(n=0;n<N;n++)
printf("\n %f+%fi" , FFT[n][0],FFT[n][1]);
      getch();
      }
```

Time domain analysis

- Analysis of data over a period of time is called time domain analysis. Response of structure to dynamic loading is an example of time domain analysis.

Frequency domain analysis

- Analysis of frequency of event is frequency domain analysis.
- Study of occurance of flood, earthquake etc. are done by frequency domain analysis.

Problem 3

Solve the given system of equations using conjugate gradient method. [2071 Bhadra]

$$\begin{bmatrix}3 & 0 & 2\\0 & 1 & 1\\2 & 1 & 3\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix}=\begin{Bmatrix}1\\0\\-1\end{Bmatrix}$$

Solution:

We have,

$$A=\begin{bmatrix}3 & 0 & 2\\0 & 1 & 1\\2 & 1 & 3\end{bmatrix}$$

and, $B=\begin{Bmatrix}1\\0\\-1\end{Bmatrix}$

Assign,

$$X_0=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

Then,

$$r_0=B-AX_0=\begin{Bmatrix}1\\0\\-1\end{Bmatrix}-\begin{bmatrix}3 & 0 & 2\\0 & 1 & 1\\2 & 1 & 3\end{bmatrix}\begin{Bmatrix}0\\0\\0\end{Bmatrix}=\begin{Bmatrix}1\\0\\-1\end{Bmatrix}$$

Assign,

$$P_0=\begin{Bmatrix}1\\0\\-1\end{Bmatrix}$$

Now,

$$\alpha_0=\frac{r_0^Tr_0}{P_0^TAP_0}=\frac{\begin{Bmatrix}1\\0\\-1\end{Bmatrix}^T\begin{Bmatrix}1\\0\\-1\end{Bmatrix}}{\begin{Bmatrix}1\\0\\-1\end{Bmatrix}^T\begin{bmatrix}3 & 0 & 2\\0 & 1 & 1\\2 & 1 & 3\end{bmatrix}\begin{Bmatrix}1\\0\\-1\end{Bmatrix}}=\frac{2}{2}=1$$

$$X_1=X_0+\alpha_0P_0=\begin{Bmatrix}0\\0\\0\end{Bmatrix}+1\begin{Bmatrix}1\\0\\-1\end{Bmatrix}=\begin{Bmatrix}1\\0\\-1\end{Bmatrix}$$

$$r_1=r_0-\alpha_0AP_0=\begin{Bmatrix}1\\0\\-1\end{Bmatrix}-1\begin{bmatrix}3 & 0 & 2\\0 & 1 & 1\\2 & 1 & 3\end{bmatrix}\begin{Bmatrix}1\\0\\-1\end{Bmatrix}=\begin{Bmatrix}0\\1\\0\end{Bmatrix}$$

$$\beta_0=\frac{r_1^Tr_1}{r_0^Tr_0}=\frac{\begin{Bmatrix}0\\1\\0\end{Bmatrix}^T\begin{Bmatrix}0\\1\\0\end{Bmatrix}}{2}=\frac{1}{2}=0.5$$

$$P_1=r_1+\beta_0P_0=\begin{Bmatrix}0\\1\\0\end{Bmatrix}+0.5\begin{Bmatrix}1\\0\\-1\end{Bmatrix}=\begin{Bmatrix}0.5\\1\\-0.5\end{Bmatrix}$$

$$\alpha_1=\frac{r_1^Tr_1}{P_1^TAP_1}=\frac{1}{\begin{Bmatrix}0.5\\1\\-0.5\end{Bmatrix}^T\begin{bmatrix}3 & 0 & 2\\0 & 1 & 1\\2 & 1 & 3\end{bmatrix}\begin{Bmatrix}0.5\\1\\-0.5\end{Bmatrix}}=\frac{1}{0.5}=2$$

$$X_2=X_1+\alpha_1P_1=\begin{Bmatrix}1\\0\\-1\end{Bmatrix}+2\begin{Bmatrix}0.5\\1\\-0.5\end{Bmatrix}=\begin{Bmatrix}2\\2\\-2\end{Bmatrix}$$

$$r_2=r_1-\alpha_1AP_1=\begin{Bmatrix}0\\1\\0\end{Bmatrix}-2\begin{bmatrix}3 & 0 & 2\\0 & 1 & 1\\2 & 1 & 3\end{bmatrix}\begin{Bmatrix}0.5\\1\\-0.5\end{Bmatrix}=\begin{Bmatrix}-1\\0\\-1\end{Bmatrix}$$

$$\beta_1=\frac{r_2^Tr_2}{r_1^Tr_1}=\frac{\begin{Bmatrix}-1\\0\\-1\end{Bmatrix}^T\begin{Bmatrix}-1\\0\\-1\end{Bmatrix}}{1}=\frac{2}{1}=2$$

$$P_2=r_2+\beta_1P_1=\begin{Bmatrix}-1\\0\\-1\end{Bmatrix}+2\begin{Bmatrix}0.5\\1\\-0.5\end{Bmatrix}=\begin{Bmatrix}0\\2\\-2\end{Bmatrix}$$

$$\alpha_2=\frac{r_2^Tr_2}{P_2^TAP_2}=\frac{2}{\begin{Bmatrix}0\\2\\-2\end{Bmatrix}^T\begin{bmatrix}3 & 0 & 2\\0 & 1 & 1\\2 & 1 & 3\end{bmatrix}\begin{Bmatrix}0\\2\\-2\end{Bmatrix}}=\frac{2}{8}=0.25$$

$$X_3=X_2+\alpha_2P_2=\begin{Bmatrix}2\\2\\-2\end{Bmatrix}+0.25\begin{Bmatrix}0\\2\\-2\end{Bmatrix}=\begin{Bmatrix}2\\2.5\\-2.5\end{Bmatrix}$$

$$r_3=r_2-\alpha_2AP_2=\begin{Bmatrix}-1\\0\\-1\end{Bmatrix}-0.25\begin{bmatrix}3 & 0 & 2\\0 & 1 & 1\\2 & 1 & 3\end{bmatrix}\begin{Bmatrix}0\\2\\-2\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

Since, residual is 0, the required solution is $X=\begin{Bmatrix}2\\2.5\\-2.5\end{Bmatrix}$.

Problem 4

Write the algorithm for conjugate gradient method. [2071 Bhadra]

Solution:

$r_0 := b - Ax_0$

$p_0 := r_0$

$k := 0$

Repeat

$$\alpha_k := \frac{r_k^Tr_k}{p_k^TAp_k}$$

$x_{k+1} := x_k + \alpha_k p_k$

$r_{k+1} := r_k - \alpha_k A p_k$

If r_{k+1} is sufficiently small then exit loop.

$\beta_k := \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}$

$p_{k+1} := r_{k+1} + \beta_k p_k$

$k := k + 1$

End repeat

The result is x_{k+1}.

Problem 5

Explain solution of linear equations. Write an algorithm for conjugate gradient method. **[2071 Magh]**

Solution:

General form of a system of linear equations is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= C_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= C_2 \\ \vdots \qquad \vdots \qquad & \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n &= C_n \end{aligned}$$

where, a_{ij} are coefficients of unknown x_i's and C_i's are known right side terms. The system of linear equation can be represented in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots\dots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{Bmatrix}$$

The unique solution of the system of linear equations presented above is possible only if the determinant of the square coefficient matrix is not equal to zero. If the determinant is zero, the solutions are either not unique or doesn't exist.

i) Direct methods

Gauss elimination method, Gauss-Jordan method, Triangulation method, Crout's method, etc.

ii) Iterative methods

Gauss-Jacobi method, Gauss-Seidel method, Relaxation method, Conjugate gradient method, etc.

For the second part

See the definition part 2.4

Problem 6

Explain different solution techniques of linear equations. Write the algorithm for conjugate gradient method. **[2072 Ashwin]**

Solution: See the definition part 2.2

Problem 7

Explain the concept of banded matrices. Also, explain data storage and memory optimization techniques using banded matrix. **[2072 Magh]**

Solution: *See the definition part 2.3 (ii)*

Problem 8

Solve the following set of linear equations using Gauss Siedal iteration or conjugate gradient method. **[2072 Magh]**

$6x + 3.5y + 9z = 12$

$12x + 2y - 3.5z = 16$

$-4x + 5y + 6z = -15$

Solution: *See the solution of example 2.5*

Problem 9

What do you mean by banded matrix and how do you optimize the memory? Write the algorithm for conjugate gradient method. **[2073 Bhadra]**

Solution:

For the first part

See the definition part 2.3

For the second part

See the definition part 2.4

Problem 10

Write down the algorithm for conjugate gradient method. Consider the system

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 6 & -2 \\ 4 & -3 & 8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -4 \\ 5 \end{Bmatrix}$$

Solve the system by using Gauss-Seidal iteration starting with $x^{(0)} = (0, 0, 0)^t$. **[2073 Magh]**

Solution:

For the first part

See the definition part 2.4

For the second part

The given system can be written in the equation form as;

$2x_1 - x_2 = 2$

$x_1 + 6x_2 - 2x_3 = -4$

$4x_1 - 3x_2 + 8x_3 = 5$

Rearranging the equation as;

$x_1 = \frac{1}{2}(2 + x_2)$ (1)

$x_2 = \frac{1}{6}(-4 - x_1 + 2x_3)$ (2)

$x_3 = \frac{1}{8}(5 - 4x_1 + 3x_2)$ (3)

Assign $x_2 = x_3 = 0$ and substitute in the equation (1); we have,

$x_1 = \frac{1}{2}(2 + 0) = 1$

Assign $x_1 = 1$, $x_3 = 0$ and substituting in the equation (2); we have,

$x_2 = \frac{1}{8}(-4 - 1 + 2 \times 0) = -\frac{5}{8}$

Assign $x_1 = 1$, $x_2 = -\frac{5}{8}$ and substituting in the equation (3); we have,

$x_3 = \frac{1}{8}\left(5 - 4 \times 1 + 3 \times \left(-\frac{5}{8}\right)\right) = -\frac{5}{[illegible]}$

Second approximation

$x_1 = \frac{1}{2}(2 + x_2) = \frac{7}{[illegible]}$

$x_2 = \frac{1}{8}(-4 - x_1 + 2x_3) = -\frac{119}{144}$

$x_3 = \frac{1}{8}(5 - 4x_1 + 3x_2) = \frac{5}{128}$

Third approximation

$x_1 = \frac{169}{[illegible]}$

$x_2 = -\frac{2615}{3456}$

$x_3 = \frac{49}{1024}$

Fourth approximation

$x_1 = 0.6217$

$x_2 = -0.7543$

$x_3 = 0.0313$

Fifth approximation

$x_1 = 0.6228$

$x_2 = -0.76$

$x_3 = 0.0286$

Sixth approximation

$x_1 = 0.62$

$x_2 = -0.76$

$x_3 = 0.029$

Since, the values of x_1, x_2 and x_3 in the fifth and sixth approximation are almos[t] same, the solutions are:

$x_1 = 0.62$

$x_2 = -0.76$

$x_3 = 0.029$

CHAPTER 3

ELASTICITY IN SOLIDS

3.1 EQUILIBRIUM EQUATIONS

Consider 3-D element of size dx × dy × dz as shown in the *figure* (a). Assume stresses acting in all the faces as shown in the figure (b) and *figure* (c). Note that the stress is positive when it acts in positive direction on positive face and acts in negative direction on negative face. Material is homogenous and isotropic. σ_x represents direct stress acting in x-direction and similar do σ_y and σ_z.

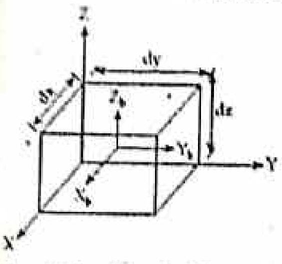


Figure: (a) Three dimensional element and body forces acting in it

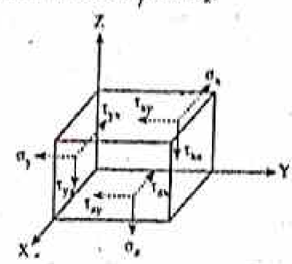


Figure: (b) Stresses in negative faces

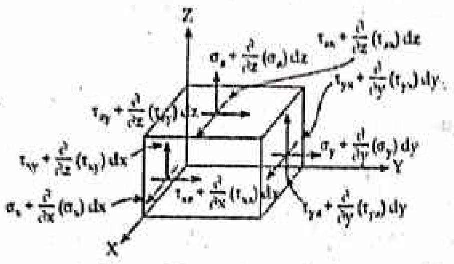


Figure: (c) Stresses in positive faces

τ_{xy} represents shear stress acting on plane x in y direction *i.e.*, first subscript of τ represent the plane and the second represents the direction.

X_b, Y_b and Z_b represent the intensity of body forces acting in x, y and z directions respectively as shown in the figure (a).

The equation of equilibrium in X-direction is:

$$\Sigma F_X = 0$$

or, $\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy\,dz - \sigma_x\,dy\,dz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx\,dz - \tau_{yx}\,dx\,dz$

$$+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dx\,dy - \tau_{zx}\,dx\,dy + X_b\,dx\,dy\,dz = 0$$

or, $\left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X_b\right] dx\,dy\,dz = 0$

or, $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X_b = 0$ [3.1.1(a)]

Similarly,

$\sum F_Y = 0$

or, $\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + Y_b = 0$ [3.1.1(b)]

and, $\sum F_Z = 0$

or, $\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + Z_b = 0$ [3.1.1(c)]

Since the element is in equilibrium, moment about line parallel to x-axis through the centroid would also add up to zero.

i.e., $\tau_{yz} \times dx\,dz \times \frac{dy}{2} + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy\right) \times dx\,dz \times \frac{dy}{2} - \tau_{zy} \times dx\,dz \times \frac{dy}{2}$

$$- \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz\right) \times dx\,dy \times \frac{dz}{2} = 0$$

or, $\frac{1}{2}\left[2\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy\right] dx\,dy\,dz - \frac{1}{2}\left[2\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz\right] dx\,dy\,dz = 0$

or, $2\tau_{yz}\,dx\,dy\,dz + \frac{\partial \tau_{yz}}{\partial y} \times dx \cancelto{0}{(dy)^2}\,dz - 2\tau_{zy}(dx\,dy\,dz) - \frac{\partial \tau_{zy}}{\partial z} dx \cancelto{0}{(dz)^2}\,dy = 0$

Neglecting the higher order terms and eliminating the common terms;

$\tau_{yz} - \tau_{zy} = 0$

$\therefore \quad \tau_{yz} = \tau_{zy}$ [3.1.2(a)]

Similarly, we can obtain,

$\tau_{xy} = \tau_{yx}$ [3.1.2(b)]

and, $\tau_{zx} = \tau_{xz}$ [3.1.2(c)]

So, there are not 9 stress terms but only 6 and can be represented in matrix form as

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$ [3.1.3]

and, the equilibrium equations are:

$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X_b = 0$ [3.1.4(a)]

$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y_b = 0$ [3.1.4(b)]

$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z_b = 0$ [3.1.4(c)]

3.2 STRAIN-DISPLACEMENT RELATIONSHIP

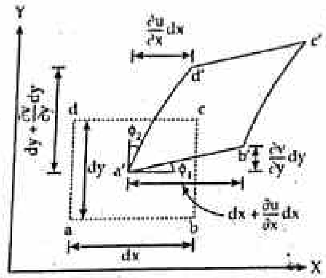


Figure: Differential element before and after deformation

abcd represents a section of an element. a'b'c'd' is its deformed shape.

Consider line element ab in x-direction. After straining it becomes a'b'. u and v represents the displacement in x-direction and y-direction respectively.

By definition of normal strain,

$$\epsilon_x = \frac{a'b' - ab}{ab}$$ [3.2.1]

$$(a'b')^2 = \left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2$$

Using binomial theorem of expansion and neglecting higher order terms $\left(\frac{\partial u}{\partial x}\right)^2$ and $\left(\frac{\partial v}{\partial x}\right)^2$,

$$a'b' = dx + \frac{\partial u}{\partial x} dx$$ [3.2.2]

Clearly,

$$ab = dx$$ [3.2.3]

Substituting equation [3.2.2] and [3.2.3] into [3.2.1]; we get,

$$\epsilon_y = \frac{\left(dx + \frac{\partial u}{\partial x} dx\right) - dx}{dx} = \frac{\partial u}{\partial x}$$ [3.2.4 (a)]

Similarly; we get,

$$\epsilon_y = \frac{\partial v}{\partial y}$$ [3.2.4 (b)]

The shear strain γ_{xy} can be defined as the change in angle between two lines, such as ab and ad, that originally formed right angles. Hence, from the figure, we can see γ_{xy} is the sum of two angles and given by;

$$\gamma_{xy} = \phi_1 + \phi_2 = \frac{\frac{\partial v}{\partial x} dx}{dx} + \frac{\frac{\partial u}{\partial y} dy}{dy}$$

& $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ [3.2.4 (c)]

Equations [3.2.4 (a)], [3.2.4 (b)] and [3.2.4 (c)] represent strain-displacement relationship for in-plane behaviour.

Likewise, we can also derive for 3-D

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

Dilation at a point in the body is given by:

$$\epsilon = \epsilon_x + \epsilon_y + \epsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad [3.2.5]$$

3.3 CONSTITUTIVE RELATIONS (STRESS-STRAIN RELATIONS)

σ represents the direct stress with its suffix denoting the direction as well as the plane. τ is for shear stress with its first suffix representing plane and second representing the direction.

Similarly, ϵ is for direct strain with suffix representing direction and γ is for shear strain its suffixes representing the plane and direction.

Material is homogenous and isotropic.

E is the modulus of elasticity.

G is the modulus of rigidity.

υ is the Poisson's ratio.

Now, from strength of material; we have,

$$\epsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y - v\sigma_z) \qquad [3.3.1 (a)]$$

$$\epsilon_y = \frac{1}{E}(-v\sigma_x + \sigma_y - v\sigma_z) \qquad [3.3.1 (b)]$$

$$\epsilon_z = \frac{1}{E}(-v\sigma_x - v\sigma_y + \sigma_z) \qquad [3.3.1 (c)]$$

Similarly,

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{2(1+v)}{E}\tau_{yz}\left[\because G = \frac{E}{2(1+v)}\right] \qquad [3.3.2 (a)]$$

$$\gamma_{zx} = \frac{2(1+v)}{E}\tau_{zx} \qquad [3.3.2 (b)]$$

$$\gamma_{xy} = \frac{2(1+v)}{E}\tau_{xy} \qquad [3.3.2 (c)]$$

All above equations can be represented in matrix form as:

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -v & -v & 0 & 0 & 0\\ -v & 1 & -v & 0 & 0 & 0\\ -v & -v & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+v) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+v) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+v)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{Bmatrix} \qquad [3.3.3]$$

Equation [3.3.1 (a)] can be rearranged as:

$$\epsilon_x = \frac{1}{E}\left((1+v)\sigma_x - v\sigma_x - v\sigma_y - v\sigma_z\right)$$

or, $$\epsilon_x = \frac{1}{E}\left((1+v)\sigma_x - v\theta_\sigma\right) \qquad [3.3.4 (a)]$$

where, $\theta_\sigma = \sigma_x + \sigma_y + \sigma_z$.

Similarly,

$$\epsilon_y = \frac{1}{E}[(1+v)\sigma_y - v\theta_\sigma] \qquad [3.3.4 (b)]$$

$$\epsilon_z = \frac{1}{E}[(1+v)\sigma_z - v\theta_\sigma] \qquad [3.3.4 (c)]$$

Adding equations [3.3.4]; we get,

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{1}{E}[(1+v)(\sigma_x + \sigma_y + \sigma_z) - 3v\theta_\sigma]$$

or, $$\theta_\epsilon = \frac{1}{E}[(1+v)\theta_\sigma - 3v\theta_\sigma]$$

$\therefore$ $$\theta_\epsilon = \frac{1}{E}(1-2v)\theta_\sigma \qquad [3.3.5]$$

where, $\theta_\epsilon = \epsilon_x + \epsilon_y + \epsilon_z$.

Expression [3.3.5] is called a volumetric form of Hook's law.

From expression [3.3.5]; we have,

$$\theta_\sigma = \frac{E}{1-2v}\theta_\epsilon$$

Substituting this value into [3.3.4 (a)]; we get,

$$\epsilon_x = \frac{1}{E}\left\{(1+v)\sigma_x - \frac{Ev}{1-2v}\theta_\epsilon\right\}$$

or, $$\sigma_x = \frac{Ev}{(1+v)(1-2v)}\theta_\epsilon + \frac{E}{(1+v)}\epsilon_x$$

or, $$\sigma_x = \lambda\theta_\epsilon + 2\mu\epsilon_x \qquad [3.3.6]$$

where, $\lambda = \frac{Ev}{(1+v)(1-2v)}$

and, $\mu = \frac{E}{2(1+v)}$; are called Lames constants.

Further,

$$\sigma_x = \frac{Ev}{(1+v)(1-2v)}(\epsilon_x + \epsilon_y + \epsilon_z) + \frac{E}{(1+v)}\epsilon_x$$

or, $$\sigma_x = \frac{E}{(1+v)(1-2v)}[v\epsilon_x + v\epsilon_y + v\epsilon_z + (1-2v)\epsilon_x]$$

or, $$\sigma_x = \frac{E}{(1+v)(1-2v)}[(1-v)\epsilon_x + v\epsilon_y + v\epsilon_z] \qquad [3.3.7 (a)]$$

Likewise,

$$\sigma_y = \frac{E}{(1+v)(1-2v)}[v\epsilon_x + (1-v)\epsilon_y + v\epsilon_z] \qquad [3.3.7 (b)]$$

or, $$\sigma_z = \frac{E}{(1+v)(1-2v)}[v\epsilon_x + v\epsilon_y + (1-v)\epsilon_z] \qquad [3.3.7 (c)]$$

Rearranging the equation [3.3.2 (a)]; we have,

$$\tau_{yz} = \frac{E}{(1+v)(1-2v)} \times \left(\frac{1}{2} - v\right)\gamma_{yz} \qquad [3.3.8 (a)]$$

Likewise,

$$\tau_{zx} = \frac{E}{(1+v)(1-2v)} \times \left(\frac{1}{2} - v\right)\gamma_{zx} \qquad [3.3.8\text{ (b)}]$$

$$\tau_{xy} = \frac{E}{(1+v)(1-2v)} \times \left(\frac{1}{2} - v\right)\gamma_{xy} \qquad [3.3.8\text{ (c)}]$$

Representing equations from [3.3.7 (a)] to [3.3.8 (c)] in the matrix form,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-v \end{bmatrix} \times \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} \qquad [3.3.9]$$

This is the constitutive relation in 3-D. The equation is often represented as:

$\{\sigma\} = [D]\{\epsilon\}$;

where, $\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$,

$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}$

and, $$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-v \end{bmatrix} \qquad [3.3.10]$$

3.4 PLANE STRESS ELEMENTS

Thin plate elements subjected to forces in their plane only are called plane stress elements. *For example*; cloth of tents, gusset plates, etc.

Consider a plate element as shown in figure.

The plate element would be a plane stress element when there is no force acting in z-direction.

i.e., $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

$\tau_{xz} = 0$ implies $\gamma_{xz} = 0$

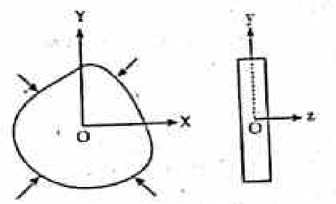


Figure: Plate element

$\tau_{yz} = 0$ implies $\gamma_{yz} = 0$

and, $\sigma_z = 0$ implies $\frac{E}{(1+v)(1-2v)}\left[v\epsilon_x + v\epsilon_y + (1-v)\epsilon_z\right] = 0$

From equation [3.3.7 (c)]

or, $$\epsilon_z = \frac{-v}{1-v}(\epsilon_x + \epsilon_y) \qquad [3.4.1]$$

From equation [3.3.6] of section 3.3; we have,

$\sigma_x = \lambda\theta_\epsilon + 2\mu\epsilon_x$

or, $\sigma_x = \frac{Ev}{(1+v)(1-2v)}\{\epsilon_x + \epsilon_y + \epsilon_z\} + \frac{E}{(1+v)}\epsilon_x$

Substituting into it the value of ϵ_z from [3.4.1]; we have,

$$\sigma_x = \frac{Ev}{(1+v)(1-2v)}\left\{\epsilon_x + \epsilon_y + \frac{-v}{1-v}(\epsilon_x + \epsilon_y)\right\} + \frac{E}{(1+v)}\epsilon_x$$

$$= \frac{Ev}{(1+v)(1-2v)}\left\{(\epsilon_x + \epsilon_y)\left(1 - \frac{v}{1-v}\right)\right\} + \frac{E}{(1+v)}\epsilon_x$$

$$= \frac{Ev}{(1+v)(1-2v)}\left\{(\epsilon_x + \epsilon_y)\frac{(1-2v)}{1-v}\right\} + \frac{E}{(1+v)}\epsilon_x$$

$$= \frac{Ev}{1-v^2}(\epsilon_x + \epsilon_y) + \frac{E(1-v)}{1-v^2}\epsilon_x$$

$$= \frac{E}{1-v^2}\epsilon_x(v + 1 - v) + \frac{Ev}{1-v^2} \times \epsilon_y$$

$$\therefore \sigma_x = \frac{E}{1-v^2}\left[\epsilon_x + v\epsilon_y\right] \qquad [3.4.2\text{ (a)}]$$

Similarly; we get,

$$\sigma_y = \frac{E}{1-v^2}\left[v\epsilon_x + \epsilon_y\right] \qquad [3.4.2\text{ (b)}]$$

Obviously;

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+v)}\gamma_{xy}$$

or, $$\tau_{xy} = \frac{E}{1-v^2} \times \frac{1-v}{2}\gamma_{xy} \qquad [3.4.3]$$

Representing equation [3.4.2] and [3.4.3] in the matrix form;

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [3.4.4]$$

This is the constitutive relation for plane stress problems.

3.5 PLANE STRAIN ELEMENTS

Long body subjected to significant lateral forces but negligible longitudinal forces are plain strain elements.

Pipes, long strip footings, retaining walls, gravity dams and tunnels etc. are the examples of plane strain elements.

For 3-D element to be plain strain element, the strain in any of the orthogonal axis direction must be zero. For our operation, consider the strains are 'zero' in z-direction. This means $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$.

$\gamma_{xz} = 0$ implies $\tau_{xz} = 0$

$\gamma_{yz} = 0$ implies $\tau_{yz} = 0$

$\epsilon_z = 0$ implies $\frac{\sigma_z}{E} - v\frac{(\sigma_x + \sigma_y)}{E} = 0$ [From (3.3.1 (c) of section 3.3)]

or, $\sigma_z = v(\sigma_x + \sigma_y)$ [3.5.1]

To get the remaining values of stresses, equation [3.3.9] of section 3.3 can be reduced into;

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad [3.5.2]$$

3.6 AXI-SYMMETRIC ELEMENTS

Elements having symmetry of geometry as well as loading about an axis is called axi-symmetric elements. Soil mass subjected to circular footing loads, thick walled pressure vessels, etc. are some examples of axi-symmetric element. The symmetry can be used to simplify the analysis of such elements.

Consider an axi-symmetric element symmetric about z-axis. The radial displacements of element develop circumferential strains that induce stresses σ_r, σ_θ, σ_z and τ_{rz}; where, r, θ and z indicate the radial, circumferential and longitudinal directions respectively.

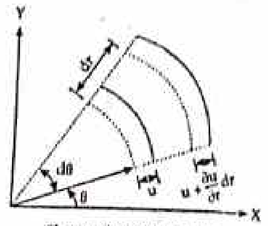


Figure: Plane cross-section

u be the displacement in radial direction and w in the longitudinal direction (in the direction of z).

Due to the symmetry of element about z-axis, the stresses are independent of the θ-co-ordinate i.e., all derivatives with respect to θ vanish.

i.e., $\gamma_{r\theta} = \gamma_{\theta z} = \tau_{r\theta} = \tau_{\theta z} = 0$

Radial strain, $\epsilon_r = \frac{\partial u}{\partial r}$

Tangential strain, , $\epsilon_\theta = \frac{(r+u)\,d\theta - r\,d\theta}{r\,d\theta} = \frac{u}{r}$

Longitudinal strain, $\epsilon_z = \frac{\partial w}{\partial z}$

Shear strain, $\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$

Constitutive relation for an axis symmetric case can be written as:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{Bmatrix} \quad [3.6.1]$$

WORKED OUT PROBLEMS

Problem 1

Derive an expression for Lame constants. [2070 Bhadra]

Solution: See the definition part 3.3

Problem 2

Define plane stress and plane strain problems with necessary conditions and suitable examples. [2070 Bhadra]

Solution:

Plane stress and plane strain

See the definition part 3.4 and 3.5

Necessary conditions

i) Plane stress problems
- The body under consideration should necessarily be thin planner body.
- The loading should be in plane and on the edge surface.

ii) Plane strain problems
- The body being subjected to significant lateral fore but very less longitudinal force.
- The displacement in longitudinal direction (z-direction) is zero in typical strip

Problem 3

Derive the equilibrium equations for 3-D state of stress in a solid. [2070 Magh]

Solution: See the definition part 3.1

Problem 4

What do you mean by axisymmetric problem? Write the constitutive relation and strain displacement relation for axismmetric condition. [2070 Magh]

Solution: See the definition part 3.6

Problem 5

Derive the constitutive relation $\{\sigma\} = [D]\{\epsilon\}$ for an elastic isotropic material. [2071 Bhadra]

Solution: See the definition part 3.3

Problem 6

What are the conditions at which axisymmetric stress exists? Write the stress-strain relations for axisymmetric condition. [2071 Bhadra]

Solution: See the definition part 3.6

Problem 7

Derive constitutive law for a two dimensional plain strain problems of isotropic materials. Explain the terms plane stress and plane strain problem with examples. [2071 Magh]

Solution: See the definition part 3.3, 3.4 and 3.5

Problem 8

Explain the terms axi-symmetric problem with examples. Derive strain displacement and constitutive relationships that exist in plane stress problem for isotropic material. [2072 Kartik]

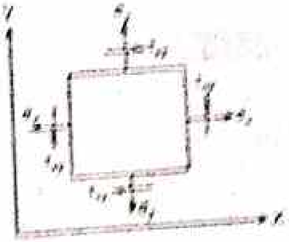
Solution: See the definition part 3.4 and 3.6

Problem 9

Explain constitutive relations for a two dimensional problems of isotropic materials. Explain plain stress, plain strain and axi-symmetric problems with examples. [2072 Magh]

Solution:

Consider a two dimensional element in XY plane. σ_x be stress in X-direction, σ_y be stress in Y-direction and τ_{xy} be shear stress. Material be homogenous and isotropic with modulus of elasticity (E), modulus of rigidity (G) and Poisson's ratio (ν).



Now, from strength of material,

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

and, $\epsilon_y = \frac{1}{E}(-\nu\sigma_x + \sigma_y)$

The equations can be rearranged as:

$$\epsilon_x = \frac{1}{E}\{(1+\nu)\sigma_x - \nu(\sigma_x + \sigma_y)\} \quad [a]$$

and, $\epsilon_y = \frac{1}{E}\{(1+\nu)\sigma_y - \nu(\sigma_x + \sigma_y)\}$ [b]

Adding equations [a] and [b], we have,

$$\epsilon_x + \epsilon_y = \frac{1}{E}\{(1+\nu)(\sigma_x + \sigma_y) - 2\nu(\sigma_x + \sigma_y)\}$$

or, $\epsilon_x + \epsilon_y = \frac{1}{E}\{(1-\nu)(\sigma_x + \sigma_y)\}$

or, $(\sigma_x + \sigma_y) = \frac{E}{(1-\nu)}(\epsilon_x + \epsilon_y)$

Substituting this value into equation [a], we get,

$$\epsilon_x = \frac{1}{E}\left\{(1+\nu)\sigma_x - \frac{E\nu}{(1-\nu)}(\epsilon_x + \epsilon_y)\right\}$$

or, $\sigma_x = \frac{E\epsilon_x}{(1+\nu)} + \frac{E\nu}{(1+\nu)(1-\nu)}(\epsilon_x + \epsilon_y)$

or, $\sigma_x = \frac{E}{(1-\nu^2)}\{(1-\nu)\epsilon_x + \nu\epsilon_x + \nu\epsilon_y\} = \frac{E}{(1-\nu^2)}(\epsilon_x + \nu\epsilon_y)$ [c]

Similarly; we can obtain, [d]

$$\sigma_y = \frac{E}{(1-\nu^2)}(\nu\epsilon_x + \epsilon_y)$$

We know that;

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+v)}\gamma_{xy} \qquad \left[\because G = \frac{E}{2(1+v)}\right]$$

or, $$\tau_{xy} = \frac{E}{(1-v^2)}\frac{(1-v)}{2}\gamma_{xy} \qquad [e]$$

Representing equation [c], [d] and [e] in the matrix form; we get,

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{(1-v^2)}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{(1-v)}{2}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

This is the constitutive relation for a two dimensional problem.

For the second part

See the definition part 3.4, 3.5 and 3.6

Problem 10

Differentiate the plain stress and plain strain problem with suitable example. **[2073 Bhadra]**

Solution:

The difference between plain stress and plain strain problem are as follows:

	Plain stress problems	Plain strain problems
i)	The problems involving thin plate elements which are subjected to the forces in their plain only are referred as plain stress problems.	The problems involving long body which is subjected to significant lateral forces but negligible longitudinal forces are referred as plain strain problem.
ii)	For any problem involving plate element to be a plane stress problem, the force acting in the z-direction must be zero. i.e., $\sigma_z = \tau_{xz} = \tau_{yz} = 0$	For any problem involving a 3D-element, to be plain strain problem, the strain in any of the orthogonal axis direction must be zero. i.e., either, $\varepsilon_x = \gamma_{xy} = \gamma_{xz} = 0$ or, $\varepsilon_y = \gamma_{xy} = \gamma_{yz} = 0$ or, $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$
iii)	The constitutive relation for plane stress problems is given by; $\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$	The constitutive relation for plain strain problems is given by; $\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{(1+v)(1-2v)}\begin{bmatrix}1-v & v & 0\\ v & 1-v & 0\\ 0 & 0 & \frac{1-2v}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$
iv)	The problems including cloth of tents, gusset plates etc. are good examples of plain stress problem.	The problems including pipes, long strip footings, retaining walls, gravity dams, tunnels, etc. are good examples of plain strain problems.

Problem 11

Derive the equilibrium equations for solid element and also define the axisymmetric problems. **[2073 Bhadra]**

Solution: See the definition part 3.1 and 3.6 respectively

Problem 12

Define plane stress and plane strain problems. Derive the differential equation of equilibrium for three dimensional problems. **[2073 Magh]**

Solution:

For the first part

See the definition part 3.4 and 3.5

For the second part

See the definition part 3.1

CHAPTER 4

FINITE ELEMENT METHOD

4.1 ELEMENT SHAPES

i) One dimensional elements

Two noded bar element, three noded bar element, etc.

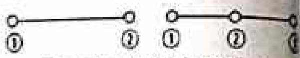


Figure: One-dimensional elements

ii) Two dimensional elements

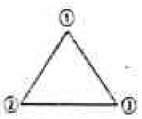


Figure: (a) Constant strain triangle (Linear displacement triangle)

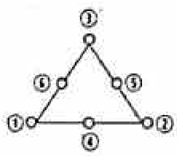


Figure: (b) Linear strain triangle (Quadratic displacement triangle)

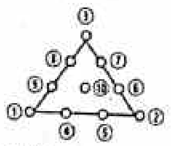


Figure: (c) Quadratic strain triangle (Cubic displacement triangle)

NOTE

One can also use higher order triangular elements like cubic strain triangles (15 nodes) and quartic strain triangles (21 nodes).

d) Lagrange elements

Elements in which nodes are in grid point form.

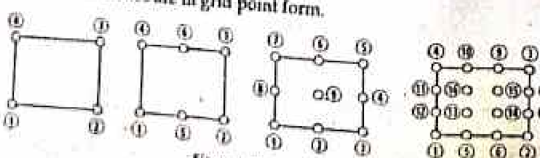


Figure: Lagrange elements

e) Serendipity elements

Elements in which nodes are provided in boundaries only.

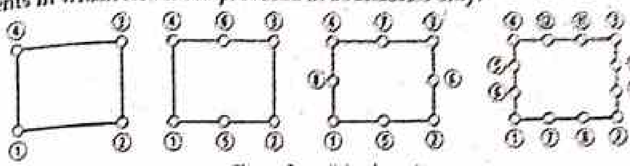


Figure: Serendipity elements

f) Curved 2-D elements

Development of iso-parametric concept has facilitated using curved elements too.

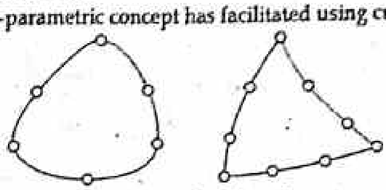
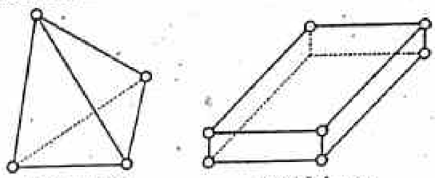


Figure: Curved 2-D elements

iii) Axi-symmetric elements

Discussed in section 3.6

iv) Three dimensional elements



(a) Tetrahedron (b) Brick element

Figure: Examples of 3-D elements

4.2 CO-ORDINATE SYSTEM

4.2.1 Global co-ordinate system

The co-ordinate system used to define the points in the entire structure is called global co-ordinate system. For the randomly oriented bar element in the figure X and Y are global co-ordinates axis.

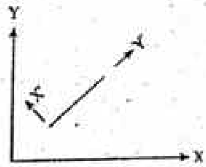


Figure: Global and local co-ordinates of an element

4.2.2 Local co-ordinate system

For the convenience of deriving element properties, in FEM many times, for each element a separate co-ordinate system is used, called local co-ordinate system. In the figure, X' and Y' are the local co-ordinates of the bar element.

4.2.3 Natural co-ordinate system

A natural co-ordinate system is a co-ordinate system, which permits the specification of a point within the element by a set of dimensionless number whose value never exceeds unity. Natural co-ordinates are nothing but weightage to nodal co-ordinates.

Natural co-ordinate system can be applied in two ways:

i) For the element with 'n' nodes, 'n' variables are used to define location of point such that the value of i^{th} term at node 'i' is unity with all other terms value zero. In this method, total weightage at a point is unity.

Consider a two noded bar element as shown in the figure (a). Location of point P is defined by co-ordinates (L_1, L_2) where the value of L_1 is 1 at node (1) and O at node (2). Similarly, value of L_2 is O at node (1) and 1 at node (2).

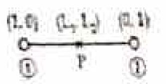


Figure: (a) Natural co-ordinates of two noded bar

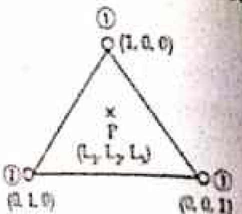


Figure: (b) Natural co-ordinate in a C.S.T.

For a CST element as shown in the figure (b), with three nodes, three variables are used in the form (L_1, L_2, L_3) to define the location of a point P, where,

at node (1): $L_1 = 1, L_2 = L_3 = 0$

at node (2): $L_2 = 1, L_1 = L_3 = 0$

at node (3): $L_3 = 1, L_1 = L_2 = 0$

ii) In another method, the value of co-ordinate ξ in a direction varies from -1 to $+1$ from one extreme to another. If the element is two-dimensional, two variables are required.

Figure: (a) Natural co-ordinates of three noded bar element in terms of ξ

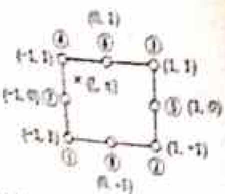


Figure: (b) Natural co-ordinates of eight noded serendipity element represented in form of (ξ, η)

4.3 RELATION BETWEEN NATURAL CO-ORDINATE AND CARTESIAN CO-ORDINATE

4.3.1 For two noded beam with natural co-ordinate represented in (L_1, L_2) form

The total weightage of co-ordinates at a point is unity.

i.e., $L_1 + L_2 = 1$ [4.3.1]

From similar triangles; we have,

$$\frac{L_1}{x_2 - x} = \frac{1}{x_2 - x_1}$$

and, $$\frac{L_2}{x - x_1} = \frac{1}{x_2 - x_1}$$

or, $$\frac{L_1}{x_2 - x} = \frac{L_2}{x - x_1}$$

or, $L_1 x - L_1 x_1 = L_2 x_2 - L_2 x$

or, $L_1x_1 + L_2x_2 = (L_1 + L_2)x = x$

∴ $L_1x_1 + L_2x_2 = x$ [4.3.2]

Representing equations [4.3.1] and [4.3.2] combinedly in matrix form; we have,

$$\begin{bmatrix}1 & 1\\ x_1 & x_2\end{bmatrix}\begin{Bmatrix}L_1\\ L_2\end{Bmatrix} = \begin{Bmatrix}1\\ x\end{Bmatrix}$$

or, $$\begin{Bmatrix}L_1\\ L_2\end{Bmatrix} = \begin{bmatrix}1 & 1\\ x_1 & x_2\end{bmatrix}^{-1}\begin{Bmatrix}1\\ x\end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1}\begin{bmatrix}x_2 & -1\\ -x_1 & 1\end{bmatrix}\begin{Bmatrix}1\\ x\end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1}\begin{Bmatrix}x_2 - x\\ -x_1 + x\end{Bmatrix}$$

But, $x_2 - x_1 = l$

∴ $$\begin{Bmatrix}L_1\\ L_2\end{Bmatrix} = \begin{Bmatrix}\dfrac{x_2 - x}{l}\\ \dfrac{x - x_1}{l}\end{Bmatrix}$$ [4.3.3]

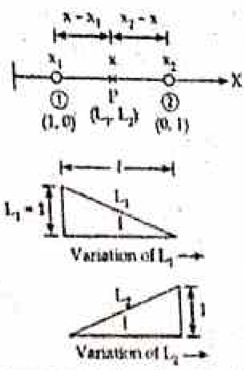


Figure: Two noded bar element in cartesian and natural co-ordinate system

4.3.2 For two noded beam with natural co-ordinate represented in the form $\xi(-1 \to +1)$

By similarity of triangles; we have,

$$\frac{\xi}{1} = \frac{x - x_c}{\frac{l}{2}}$$

$$= \frac{2}{l}\left\{x - \frac{x_1 + x_2}{2}\right\} \quad \left[\because x_c = \frac{x_1 + x_2}{2}\right]$$

$$= \frac{2}{l}\left\{x - \frac{x_2 - x_1 + 2x_1}{2}\right\}$$

$$= \frac{2}{l}\left\{x - \frac{l + 2x_1}{2}\right\}$$

$$= \frac{2}{l}\left\{x - x_1 - \frac{l}{2}\right\}$$

∴ $\frac{l}{2}(1 + \xi) = x - x_1$ [4.3.4]

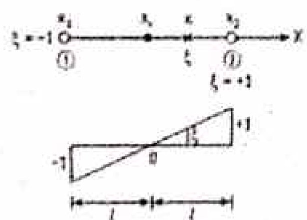


Figure: Two noded bar in cartesian and natural co-ordinate system in terms of ξ

4.3.3 For CST with natural co-ordinate represented in the form (L_1, L_2, L_3)

From the definition of natural co-ordinate, we have,

$L_1 + L_2 + L_3 = 1$

$L_1x_1 + L_2x_2 + L_3x_3 = x$

$L_1y_1 + L_2y_2 + L_3y_3 = y$

In the matrix form;

$$\begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix}\begin{Bmatrix}L_1\\ L_2\\ L_3\end{Bmatrix} = \begin{Bmatrix}1\\ x\\ y\end{Bmatrix}$$

or, $$\begin{bmatrix}L_1\\ L_2\\ L_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix}^{-1}\begin{Bmatrix}1\\ x\\ y\end{Bmatrix}$$

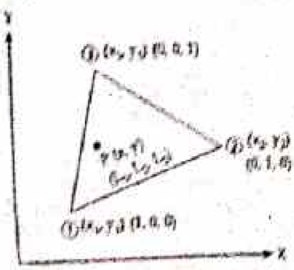


Figure: CST element in Cartesian as well as natural co-ordinate system

$$= \frac{1}{2A}\begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}^T \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

where, $2A = \text{determinant} = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 2 \times$ Area of triangle.

or, $$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$= \frac{1}{2A}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

where, $a_1 = x_2y_3 - x_3y_2$ $\quad a_2 = x_3y_1 - x_1y_3$ $\quad a_2 = x_1y_2 - x_2y_1$

$b_1 = y_2 - y_3$ $\quad b_2 = y_3 - y_1$ $\quad b_3 = y_1 - y_2$

$c_1 = x_3 - x_2$ $\quad c_2 = x_1 - x_3$ $\quad c_3 = x_2 - x_1$

$$\therefore \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{a_1 + b_1x + c_1y}{2A} \\ \dfrac{a_2 + b_2x + c_2y}{2A} \\ \dfrac{a_3 + b_3x + c_3y}{2A} \end{Bmatrix} \qquad [4.3.5]$$

4.4 DIRECT STIFFNESS METHOD

Direct stiffness method is a technique in which total stiffness for an assemblage can be directly obtained by superimposing the stiffness matrices of individual elements. The method is also called displacement method and matrix stiffness method.

In applying this method, the system must be modeled as a set of simpler, idealized elements interconnected at the nodes. The matrix stiffness properties of the elements are then, through matrix mathematics, compiled into a single matrix equation which governs the behaviour of entire idealized structure. The structures' unknown displacements and forces can then be determined by solving equation $\{F\} = [k]\{u\}$.

4.5 BAR ELEMENT

4.5.1 Stiffness matrix of a bar element in local co-ordinate system

For a bar of length (l) and cross-section area (A) of a homogenous isotropic material of modulus of elasticity (E) has elongation (δ) after application of load (P) longitudinally. Then,

$$\delta = \frac{Pl}{AE}$$

or, $$\frac{P}{\delta} = \frac{AE}{l} = \text{Stiffness } (K)$$

Consider a bar element as shown in the figure. Forces F_1 and F_2 are subjected in nodes (1) and (2) respectively in the direction of positive X'. u_1' and u_2' be the displacement of the nodes in same direction.

Figure: A bar element

Obviously,

$$\text{Elongation } (\delta) = u_2 - u_1$$

For equilibrium,

$$\sum F_{x'} = 0$$

or, $$F_1' + F_2' = 0$$

$\therefore$ $$F_1' = -F_2'$$

Since,

$$\text{Force} = K\delta$$

$$F_1' = -K(u_2' - u_1') \qquad \text{[– sign for being compressive force]}$$

and, $$F_2' = K(u_2' - u_1')$$

In matrix form; we have,

Figure: Spring element

$$\begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$$

or, $$\begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = K\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \qquad [4.5.1]$$

Expression 4.5.1 can be used for analysis of spring element as shown in the figure.

For the bar element,

$$\begin{Bmatrix} F_1' \\ F_2' \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} \qquad [4.5.2]$$

or, $$\{F'\} = [K_e']\{u'\} \qquad [4.5.3]$$

The term $\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = K_e'$ is the stiffness of bar element in local co-ordinate system.

4.5.2 Stiffness matrix of a bar / truss element in global co-ordinate system

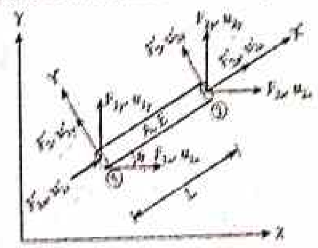


Figure: Bar element in global co-ordinate

For the bar element in the local co-ordinate system is:

$$\{F'\} = [K_e']\{u'\}$$

Expanding the expression [4.5.2] as:

$$\begin{Bmatrix} F_{1x}' \\ F_{1y}' \\ F_{2x}' \\ F_{2y}' \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{1x}' \\ u_{1y}' \\ u_{2x}' \\ u_{2y}' \end{Bmatrix} \qquad [4.5.4]$$

θ be an angle made by bar element with positive x-axis in anti-clockwise direction.

Let, $C = \cos\theta$

and, $S = \sin\theta$

Then,

$$u'_{1x} = Cu_{1x} + Su_{1y}$$
$$u'_{1y} = -Su_{1x} + Cu_{1y}$$
$$u'_{2x} = Cu_{2x} + Su_{2y}$$
$$u'_{2y} = -Su_{2x} + Cu_{2y}$$

In the matrix form;

$$\begin{Bmatrix} u'_{1x} \\ u'_{1y} \\ u'_{2x} \\ u'_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{bmatrix} \quad [4.5.5]$$

or, $\{u'\} = [T]\{u\}$ [4.5.6]

where, $\{u'\}$ is the displacement matrix in local co-ordinate.

[T] is the transformation matrix.

[u] is the displacement matrix in global co-ordinate.

In equation [4.5.5], u'_{1y} and u'_{2y} are obviously 0. So, for practical purpose, it can be reduced into:

$$\begin{Bmatrix} u'_{1x} \\ u'_{2x} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{Bmatrix} \quad [4.5.7]$$

We can get in the similar way,

$[F'] = [T][F_e]$ [4.5.8]

From equation [4.5.3]; we have,

$\{F'\} = [K'_e]\{u'\}$

or, $[T][F_e] = [K'_e][T]\{u'\}$ (From equation [4.5.6] and [4.5.8])

or, $[F_e] = [T]^{-1}[K'_e][T]\{u'\}$

Comparing with general form; we have,

$[F_e] = [K_e]\{u'\}$

We get,

$[K_e] = [T]^{-1}[K'_e][T]$

It can be shown that [T] is an orthogonal matrix and $[T]^{-1} = [T]^T$.

$\therefore$ $[K_e] = [T]^T[K'_e][T]$

$$= \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix} \times \frac{AE}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

or, $$[K_e] = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \quad [4.5.9]$$

Equation [4.5.9] is the element stiffness matrix of bar / truss element in 2-D.

4.5.3 Computation of stress and strain for a bar in X-Y plane

From equation [4.5.2]; we have,

$$\begin{Bmatrix} F'_{1x} \\ F'_{2x} \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_{1x} \\ u'_{2x} \end{Bmatrix}$$

By definition; we have,

$$\sigma = \frac{F'_{2x}}{A} = \frac{-F'_{1x}}{A}$$

(– sign while taking force at node 1 because it is compressive force as assumed in section 4.5.1)

or, $$\sigma = \frac{1}{A}\frac{AE}{l}[-1 \quad 1]\begin{Bmatrix} u'_{1x} \\ u'_{2x} \end{Bmatrix}$$

$\therefore$ $$\sigma = \frac{E}{l}[-1 \quad 1]\{u'_e\} \quad [4.5.10]$$

This is the expression to evaluate stress in a bar element using nodal displacements in local co-ordinate.

Also, using equation [4.5.7]; we have,

$$\sigma = \frac{E}{l}[-1 \quad 1]\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{Bmatrix}$$

or, $$\sigma = \frac{E}{l}[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{Bmatrix} \quad [4.5.11]$$

This is the expression to evaluate stress in a bar element using nodal displacement in the global co-ordinate.

Since, $\epsilon = \frac{\sigma}{E}$; we can get the value of strain using:

$$\epsilon = \frac{1}{l}[-1 \quad 1]\begin{Bmatrix} u'_{1x} \\ u'_{2x} \end{Bmatrix} \quad [4.5.12]$$

and, $$\epsilon = \frac{1}{l}[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{Bmatrix} \quad [4.5.13]$$

4.5.4 Equivalent nodal loads

Any type of load acting at any part of the element should be transferred to node as equivalent nodal loads in FEM.

i) **Body force**

A body force is a force that acts throughout the volume of a body. *For example;* gravity force, electromagnetic force, etc.

Here, γ is the specific weight of material.

A is the cross-section area.

l is the length.

W is the weight.

Now,

Total weight of the bar, $(W) = \gamma Al$

Dividing equally to the two nodes; we have,

$$\begin{Bmatrix} f_{1b} \\ f_{2b} \end{Bmatrix} = \frac{\gamma Al}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix};$$

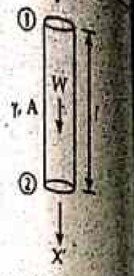


Figure: Body force of a bar

had the direction of X' chosen upward, the nodal value of body weight would be:

$$\begin{Bmatrix} f_{1b} \\ f_{2b} \end{Bmatrix} = \frac{\gamma Al}{2}\begin{Bmatrix} -1 \\ -1 \end{Bmatrix}$$

If γ is not given in a problem, self weight of the element may be neglected.

ii) Surface load

If X_s is the intensity of surface load, P is the perimeter of the section and l is the length of the bar, total surface load would be:

$$F_s = X_s Pl$$

Dividing the values equally to the nodes; we have,

$$\begin{Bmatrix} f_{s1} \\ f_{s2} \end{Bmatrix} = \frac{X_s Pl}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

iii) Point load

Point loads are directly added to nodal force vectors.

4.5.5 Use of symmetry in truss

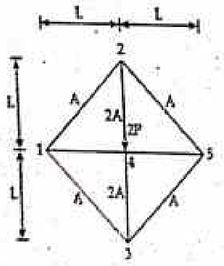


Figure: (a) A plane truss

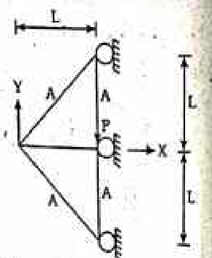


Figure: (b) Truss reduced by symmetry

- Vertical plane perpendicular to the plane of paper passing through nodes ②, ④ and ③ is the plane of reflective symmetry.
- For the load 2P occurring at the original truss, only half load P is applied in reduced truss.
- For elements occurring in the plane of symmetry, half of the c/s area should be taken in the reduced truss.
- For nodes in the plane of symmetry, the displacement components normal to the plane of symmetry are restricted.

4.6.6 Bar in 3-D

i)
$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

where, $\{u'\} = \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$ is the displacement matrix in the local co-ordinate system.

$u = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$ is the displacement matrix in the global co-ordinate system.

$[T] = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$ is the transformation matrix.

$$C_x = \frac{x_2 - x_1}{L}$$

$$C_y = \frac{y_2 - y_1}{L}$$

and, $C_z = \dfrac{z_2 - z_1}{L}$

ii) Element stiffness matrix [K]

$$[K] = \frac{AE}{L}\begin{bmatrix} C_x^2 & C_xC_y & C_xC_z & -C_x^2 & -C_xC_y & -C_xC_z \\ & C_y^2 & C_yC_z & -C_xC_y & -C_y^2 & -C_yC_z \\ & & C_z^2 & -C_xC_y & -C_zC_y & -C_z^2 \\ & & & C_x^2 & C_xC_y & C_xC_z \\ & & & & C_y^2 & C_yC_z \\ & \text{sym} & & & & C_z^2 \end{bmatrix}$$

iii) Element stress

$$(\sigma) = \frac{E}{L}\begin{bmatrix} -C_x & -C_y & -C_z & C_x & C_y & C_z \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{Bmatrix}$$

4.5.7 Truss with skew-support

At node i, u_i and v_i be the displacement in the global direction and u_i' and v_i', the displacement in the local co-ordinate system. θ be the orientation of $x' - y'$ plane with respect to x-y plane; then,

$$\begin{Bmatrix} u_i' \\ v_i' \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}\begin{Bmatrix} u_x \\ v_x \end{Bmatrix};$$

or, $\{u_i'\} = [t]\{u_i\}$

where, $C = \cos\theta$ and $S = \sin\theta$.

Consider a truss with skew support as shown in the figure.

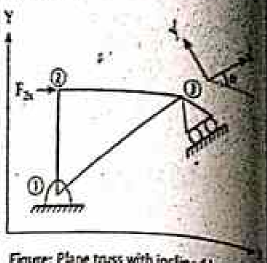


Figure: Plane truss with inclined boundary conditions at node ③

For node ③;

$$\{u_3'\} = [t_3]\{u_3\}$$

or, $$\{u\} = [t_3]^T\{u_3'\}$$

For the whole truss structure;

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3]^T \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3' \\ v_3' \end{Bmatrix}$$

or, $$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & -S \\ 0 & 0 & 0 & 0 & S & C \end{bmatrix}}_{[T]^T} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3' \\ v_3' \end{Bmatrix}$$

Furthermore,

$$[T]\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x}' \\ f_{3y}' \end{Bmatrix}$$

We have,

$$\{f\} = [K]\{u\}$$

or, $$\{f\} = [K][T]^T\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3' \\ v_3' \end{Bmatrix}$$

Pre-multiplying both sides by [T]; we get,

$$[T]\{f\} = [T][K][T]^T\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3' \\ v_3' \end{Bmatrix}$$

or, $$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x}' \\ f_{3y}' \end{Bmatrix} = [T][K][T]^T\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3' \\ v_3' \end{Bmatrix}$$

4.5.8 Effect of change in temperature

We know that;

$$\Delta l = \alpha l \Delta t$$

or, $$\frac{\Delta l}{l} = \epsilon_t = \alpha\Delta t = \frac{\sigma}{E} = \frac{f}{AE}$$

∴ $$f = AE\alpha\Delta t$$

Element load vector due to temperature change $\{f_T^e\} = \begin{Bmatrix} f_{1T}^e \\ f_{2T}^e \end{Bmatrix} = AE\alpha\Delta t\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

Stress in an element $= \frac{E}{l}[-1 \quad 1]\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} - E\alpha\Delta t$

Strain in an element $= \frac{1}{l}[-1 \quad 1]\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} - \alpha\Delta t$

4.5.9 Effect of lack of fit

$$\{f_L^e\} = \begin{Bmatrix} f_{L1}^e \\ f_{L2}^e \end{Bmatrix} = AE\frac{\delta}{l}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

δ is +ve when element is very long and −ve when element is short.

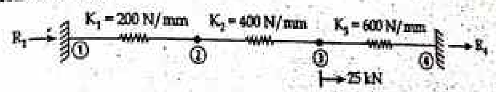
Stress in an element $(\sigma) = \frac{E}{l}[-1 \quad 1]\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} - E\frac{\delta}{l}$

Strain in an element $(\delta) = \frac{1}{l}[-1 \quad 1]\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} - \frac{\delta}{l}$

EXAMPLE 4.1

Evaluate:

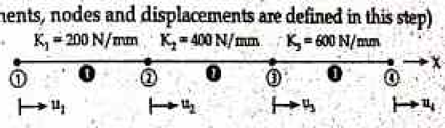
i) **General stiffness matrix**
ii) **Displacement at node ② and ③**
iii) **Reaction at node ① and ④**
iv) **Element forces**



Solution:

Step 1: Discretizing and modeling

(Elements, nodes and displacements are defined in this step)



Step 2: Force vector

$$\{F'\} = \begin{Bmatrix} R_1 \\ 0 \\ 25\text{ kN} \\ R_4 \end{Bmatrix}$$

Step 3: Displacement vector

$$\{u'\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Step 4: Boundary condition

$$u_1 = u_4 = 0$$

$$\therefore \quad \{u'\} = \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix}$$

Step 5: Stiffness matrix

For element ❶:

$$[K^1] = K_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 200\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ u_2 \end{array}$$

For element ❷:

$$[K^2] = K_2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 400\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 200\begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}\begin{array}{c} u_2 \\ u_3 \end{array}$$

For element ❸:

$$[K^3] = 600\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 200\begin{array}{c} \begin{array}{cc} u_3 & u_4 \end{array} \\ \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \end{array}\begin{array}{c} u_3 \\ u_4 \end{array}$$

NOTE

Common value 200 is taken out from all three matrices so that assessment of superposition would be easier.

Superposing the individual element stiffness matrices to get single structural stiffness matrix.

$$[K] = 200\begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & \boxed{1+2} & -2 & 0 \\ 0 & -2 & \boxed{2+3} & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} = 200\begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}$$

Step 6: [F] = [K][u]

$$\begin{Bmatrix} R_1 \\ 0 \\ 25\text{ kN} \\ R_4 \end{Bmatrix} = 200\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix}$$

NOTE

Cancel out the rows to which the value of displacement is 0 and the corresponding columns in stiffness matrix.

The reduced matrix is:

$$\begin{array}{cc} [B] & [A] \end{array}$$

$$\begin{Bmatrix} 0 \\ 25 \end{Bmatrix} = 200\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

or,
$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{200}\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ 25 \end{Bmatrix} = \begin{Bmatrix} 0.0227\text{ mm} \\ 0.0340\text{ mm} \end{Bmatrix}$$

NOTE

Solve directly using calculator:
M at A^{-1} × M at B ÷ 200

Step 7: Reaction calculation

$$\begin{Bmatrix} R_1 \\ 0 \\ 25\text{ kN} \\ R_4 \end{Bmatrix} = 200\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.0227 \\ 0.0340 \\ 0 \end{Bmatrix}$$

or, $R_1 = 200(-1 \times 0.0227 + 0 \times 0.0227) = -4.54\text{ N}$

and, $R_4 = 200(0 \times 0.0227 - 3 \times 0.034) = -20.4\text{ N}$

Step 8: Force in each spring

For element ❶:

$$[F_1] = [K_1][u_1]$$

or, $$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 200\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.0227 \end{Bmatrix}$$

or, $$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} -4.54\text{ N} \\ +4.54\text{ N} \end{Bmatrix}$$

∴ Element ❶ has member force 4.54 N (tension).

For element ❷:

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 200\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{Bmatrix} 0.0227 \\ 0.0340 \end{Bmatrix}$$

∴ $$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} -4.52\text{ N} \\ +4.52\text{ N} \end{Bmatrix}$$

∴ Element ❷ has tensile force of 4.52 N.

For element ❸:

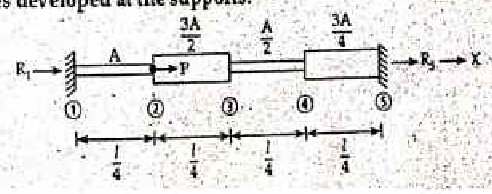
$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 200\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{Bmatrix} 0.0340 \\ 0 \end{Bmatrix}$$

∴ $$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{Bmatrix} +20.4 \\ -20.4 \end{Bmatrix}$$

∴ Element ❸ has compressive force of 20.4 N.

EXAMPLE 4.2

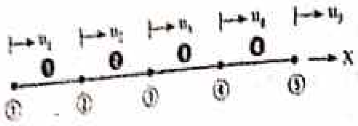
For the given stepped bar obtain nodal displacements at nodes 2, 3 and 4. Also obtain forces developed at the supports.



Take: E = Constant and cross-sectional area as indicated in figure [2071 ...

Solution:

Step 1: Modeling



NOTE

● represents element and ○ represents node.

Step 2: Force matrix

$$F = \begin{Bmatrix} R_1 \\ P \\ 0 \\ 0 \\ R_5 \end{Bmatrix}$$

Step 3: Displacement matrix

$$u = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

Step 4: Boundary condition

$$u_1 = u_5 = 0$$

$$\therefore \quad u = \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \\ 0 \end{Bmatrix}$$

Step 5: Stiffness matrix

For element ❶:

$$[K_1] = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{A \times E}{\left(\frac{l}{4}\right)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{4AE}{l}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ u_2 \end{array}$$

For element ❷:

$$[K_2] = \frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{\left(\frac{3A}{2}\right)E}{\left(\frac{l}{4}\right)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{4AE}{l}\begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \end{array}\begin{array}{c} u_2 \\ u_3 \end{array}$$

For element ❸:

$$[K_3] = \frac{\left(\frac{A}{2}\right)E}{\left(\frac{l}{4}\right)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{4AE}{l}\begin{array}{c} \begin{array}{cc} u_3 & u_4 \end{array} \\ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}\begin{array}{c} u_3 \\ u_4 \end{array}$$

For element ❹:

$$[K_4] = \frac{4AE}{l}\begin{array}{c} \begin{array}{cc} u_4 & u_5 \end{array} \\ \begin{bmatrix} \frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{bmatrix} \end{array}\begin{array}{c} u_4 \\ u_5 \end{array}$$

Assembling element stiffness matrices to get generalized stiffness matrix.

$$[K] = \frac{4AE}{l}\begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1+\frac{3}{2} & -\frac{3}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2}+\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2}+\frac{3}{4} & -\frac{3}{4} \\ 0 & 0 & 0 & -\frac{3}{4} & \frac{3}{4} \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}$$

NOTE

If you can visualize how matrices are overlapped, assembling becomes more easier.

$$\therefore \quad [K] = \frac{4AE}{l}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ & \frac{5}{2} & -\frac{3}{2} & 0 & 0 \\ & & 2 & -\frac{1}{2} & 0 \\ & \text{sym} & & \frac{5}{4} & -\frac{3}{4} \\ & & & & \frac{3}{4} \end{bmatrix}$$

NOTE

Writing only upper triangular matrix and stating symmetrical in lower part would be helpful in saving time of calculation.

Step 6: [F] = [K][u]

$$\text{or,} \quad \begin{Bmatrix} R_1 \\ P \\ 0 \\ 0 \\ R_5 \end{Bmatrix} = \frac{4AE}{l}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ & \frac{5}{2} & -\frac{3}{2} & 0 & 0 \\ & & 2 & -\frac{1}{2} & 0 \\ & \text{sym} & & \frac{5}{4} & -\frac{3}{4} \\ & & & & \frac{3}{4} \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \\ 0 \end{Bmatrix}$$

NOTE

Eliminate the rows for which displacement value in the displacement matrix and also eliminate corresponding columns in stiffness matrix.

The reduced matrix is:

$$\begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} = \frac{4AE}{L}\begin{bmatrix} \frac{5}{2} & -\frac{3}{2} & 0 \\ & 2 & -\frac{1}{2} \\ \text{sym} & & \frac{5}{4} \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

or, $$\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \frac{L}{4AE}\begin{bmatrix} \frac{5}{2} & -\frac{3}{2} & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{5}{4} \end{bmatrix}^{-1}\begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} = \frac{L}{4AE}\begin{bmatrix} 0.76 & 0.6 & 0.12 \\ 0.6 & 1 & 0.2 \\ 0.12 & 0.2 & 0.44 \end{bmatrix}\begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{L}{4AE}\begin{Bmatrix} 0.76\,P \\ 0.6\,P \\ 0.12\,P \end{Bmatrix}$$

Step 7: Rewriting [F] = [K][u]

$$\begin{Bmatrix} R_1 \\ P \\ 0 \\ 0 \\ R_5 \end{Bmatrix} = \frac{4AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ & \frac{5}{2} & -\frac{3}{2} & 0 & 0 \\ & & 2 & -\frac{1}{2} & 0 \\ & & & \frac{5}{4} & -\frac{3}{4} \\ & & & & \frac{3}{4} \end{bmatrix} \times \frac{L}{4AE}\begin{Bmatrix} 0 \\ 0.76\,P \\ 0.6\,P \\ 0.12\,P \\ 0 \end{Bmatrix}$$

or, $R_1 = -0.76\,P$

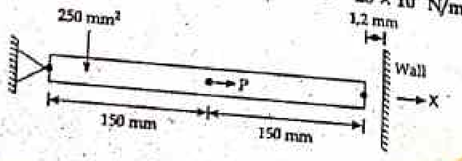
$R_5 = -\frac{3}{4} \times 0.12\,P = -0.09\,P$

NOTE

To get R_1, you multiplying first row to displacement matrix. Utilizing presence of 0 values, you can easily get the result. To get R_5, you multiply the displacement matrix with the last column of stiffness matrix. It should have been last row but since [K] is symmetrical, last row and last column are the same.

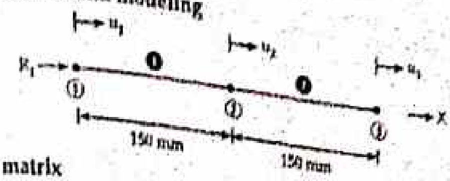
EXAMPLE 4.3

In given figure, a load $P = 60 \times 10^3$ N is applied as shown. Determine displacement field stress and support reactions in the body. Take: $E = 20 \times 10^3\ N/mm^2$



Solution:

Step 1: Discretization and modeling



Step 2: Force matrix

As the wall at right side is not in touch, we assume it has no effect in the bar system.

$$F = \begin{Bmatrix} R_1 \\ P \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 60 \times 10^3 \\ 0 \end{Bmatrix}$$

Step 3: Displacement matrix

$$[u] = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}$$; applying boundary condition $u_1 = 0$

Step 4: Stiffness matrix

For element 1:

$$[K_1] = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{250 \times 20 \times 10^3}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{100000}{3}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ u_1 \\ u_2 \end{array}$$

For element 2:

$$[K_2] = \frac{100000}{3}\begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ u_2 \\ u_3 \end{array}$$

Assembling the element stiffness matrices to get generalized stiffness matrix.

$$[K] = \frac{100000}{3}\begin{bmatrix} 1 & -1 & 0 \\ -1 & \boxed{1+1} & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{100000}{3}\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array}$$

Step 5: [F] = [K][u]

$$\begin{Bmatrix} \cancel{R_1} \\ 60 \times 10^3 \\ 0 \end{Bmatrix} = \frac{100000}{3}\begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} \\ \cancel{-1} & 2 & -1 \\ \cancel{0} & -1 & 1 \end{bmatrix}\begin{Bmatrix} \cancel{0} \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{3}{100000}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} 60 \times 10^3 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1.8 \\ 1.8 \end{bmatrix} \text{mm}$$

NOTE

Calculation in calculator:

Mode ↵

6 : Matrix ↵

1 : Mat A ↵

5 : 2 × 2 ↵

$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ ↵

AC ↵

Shift ↵

4 ↵

1 : Dim ↵

2 : Mat B ↵

6 : 2 × 1 ↵

$\begin{bmatrix} 60 \times 10^3 \\ 0 \end{bmatrix}$ ↵

AC ↵

Shift ↵

4 ↵

3 : Mat A ↵

x^{-1} ↵

Shift 4 ↵

4 : Mat B ↵

* 3 ÷ 100000 ↵

Ans

$\begin{bmatrix} 1.8 \\ 1.8 \end{bmatrix}$

Calculation shows right end displaces by 1.8 mm. But after it moves 1.2 mm, it is blocked. It can't further displace and a new reaction is produced by wall (say in positive x-direction). So, we have new force matrix and displacement matrix.

$$[F] = \begin{Bmatrix} R_1 \\ 60 \times 10^3 \\ R_3 \end{Bmatrix},$$

$$[u] = \begin{Bmatrix} 0 \\ u_2 \\ 1.2 \end{Bmatrix}$$

With these values

$$[F] = [K][u]$$

or, $$\begin{Bmatrix} R_1 \\ 60 \times 10^3 \\ R_3 \end{Bmatrix} = \frac{100000}{3}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 1.2 \end{Bmatrix}$$

or, $$60 \times 10^3 = \frac{10^5}{3}[2 \times u_2 - 1.2]$$

or, $$u_2 = 1.5 \text{ mm}$$

and, $$R_3 = \frac{10^5}{3}[-\overset{1.5}{\cancel{u_2}} + 1.2] = -10^4 \text{ N} = 10^4 \text{ N}(\leftarrow)$$

Likewise,

$$R_1 = \frac{100000}{3}(-1.5) = -50000 \text{ N} = 50000 \text{ N}(\leftarrow)$$

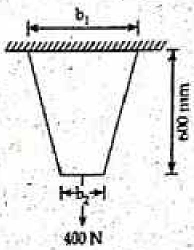
EXAMPLE 4.4

Determine the extension of the bar shown in the figure due to self weight and a concentrated load of 400 N applied at its end.

Given: $b_1 = 150$ mm and $b_2 = 75$ mm, $t = 25$ mm

Also determine the reaction at the support, stress and strain in the member and member forces.

$E = 2 \times 10^5$ N/mm², $\gamma = 0.8 \times 10^{-4}$ N/mm²



Solution:

Step 1: Discretization and modeling

We are trying to model the structure as two element structure. For simplicity, we model it with two rectangular bar elements.

$$\text{Width of the bar at middle section} = \frac{150 \text{ mm} + 75 \text{ mm}}{2} = 112.5 \text{ mm}$$

$$\text{Average width of bar above the middle section} = \frac{112.5 + 150}{2} = 131.25 \text{ mm}$$

$$\text{Average width of the bar below the middle section} = \frac{112.5 + 75}{2} = 93.75 \text{ mm}$$

We draw an equivalent system and its model as:

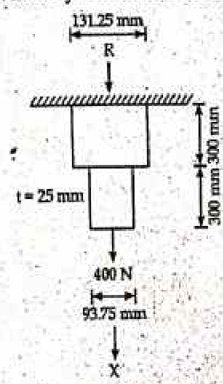


Figure: Equivalent structure

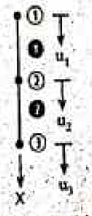


Figure: Model

Step 2: Force matrix

For element ❶, body force vector,

$$\begin{Bmatrix} f_{b1}^1 \\ f_{b2}^1 \end{Bmatrix} = \frac{\gamma A_1 l_1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{0.8 \times 10^{-4} \times (131.25 \times 25) \times 300}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0.3\begin{bmatrix} 131.25 \\ 131.25 \end{bmatrix}$$

For element ❷, body force vector,

$$\begin{Bmatrix} f_{b2}^2 \\ f_{b3}^2 \end{Bmatrix} = \frac{\gamma A_2 l_2}{2}\begin{bmatrix}1\\1\end{bmatrix} = \frac{0.8\times10^{-4}\times(93.75\times25)\times300}{2}\begin{bmatrix}1\\1\end{bmatrix} = 0.3\begin{bmatrix}93.75\\93.75\end{bmatrix}$$

Assembling to get body force vector of the structure,

$$\{f_b\} = 0.3\begin{Bmatrix}131.25\\131.25+93.75\\93.75\end{Bmatrix} = 0.3\begin{Bmatrix}131.25\\225\\93.75\end{Bmatrix}$$

Point load vector for the system is:

$$\{f_p\} = \begin{Bmatrix}R_1\\0\\400\end{Bmatrix}$$

Total load vector is:

$$[F] = 0.3\begin{Bmatrix}131.25\\225\\93.75\end{Bmatrix} + \begin{Bmatrix}R_1\\0\\400\end{Bmatrix} = \begin{Bmatrix}39.375+R\\67.5\\428.125\end{Bmatrix}$$

Step 3: Displacement vector

$$u = \begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix} = \begin{Bmatrix}0\\u_2\\u_3\end{Bmatrix};\ \text{after applying boundary condition } u_1 = 0$$

Step 4: Stiffness matrix

For element ❶,

$$[K_1] = \frac{A_1E_1}{l_1}\begin{bmatrix}1&-1\\-1&1\end{bmatrix} = \frac{(131.25\times25)\times(2\times10^5)}{300}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$$

$$= \frac{50000}{3}\begin{array}{c c} & \begin{array}{cc} u_1 & u_2 \end{array} \\ & \begin{bmatrix}131.25 & -131.25\\-131.25 & 131.25\end{bmatrix} \begin{array}{c} u_1\\u_2\end{array}\end{array}$$

For element ❷,

$$[K_2] = \frac{50000}{3}\begin{array}{c c} & \begin{array}{cc} u_2 & u_3 \end{array} \\ & \begin{bmatrix}93.75 & -93.75\\-93.75 & 93.75\end{bmatrix} \begin{array}{c} u_2\\u_3\end{array}\end{array}$$

Assembling,

$$[K] = \frac{50000}{3}\begin{array}{c c} & \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ & \begin{bmatrix}131.25 & -131.25 & 0\\-131.25 & 225 & -93.75\\0 & -93.75 & 93.75\end{bmatrix} \begin{array}{c} u_1\\u_2\\u_3\end{array}\end{array}$$

Step 5: [F] = [K][u]

$$\begin{Bmatrix}\cancel{39.375+R}\\67.5\\428.125\end{Bmatrix} = \frac{5000}{3}\begin{bmatrix}\cancel{131.25} & \cancel{-131.25} & \cancel{0}\\ \cancel{-131.25} & 225 & -93.75\\ \cancel{0} & -93.75 & 93.75\end{bmatrix}\begin{Bmatrix}\cancel{0}\\u_2\\u_3\end{Bmatrix}$$

The reduced matrix equation is:

$$\begin{Bmatrix}67.5\\428.125\end{Bmatrix} = \frac{5000}{3}\begin{bmatrix}225 & -93.75\\-93.75 & 93.75\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}$$

or,
$$\begin{Bmatrix}u_2\\u_3\end{Bmatrix} = \frac{3}{5000}\begin{bmatrix}225 & -93.75\\-93.75 & 93.75\end{bmatrix}^{-1}\begin{Bmatrix}67.5\\428.125\end{Bmatrix} = \begin{Bmatrix}2.266\times10^{-3}\\5.006\times10^{-3}\end{Bmatrix}\text{ mm}$$

Step 6: Reaction calculation

From [F] = [K][u];

$$(39.375 + R) = \frac{5000}{3}[-131.25\times(2.266\times10^{-3})]$$

or, $R = 535.06$ N

Step 7: Element stresses and strain

For element ❶,

$$\sigma_1 = \frac{E}{l}[-1\quad 1]\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \frac{2\times10^5}{300}[-1\quad 1]\begin{Bmatrix}0\\2.266\times10^{-3}\end{Bmatrix} = 1.51\text{ N/mm}^2$$

For element ❷,

$$\sigma_2 = \frac{E}{l}[-1\quad 1]\begin{Bmatrix}u_2\\u_3\end{Bmatrix} = \frac{2\times10^5}{300}[-1\quad 1]\begin{Bmatrix}2.266\times10^{-3}\\5.006\times10^{-3}\end{Bmatrix} = 1.827\text{ N/mm}^2$$

$$\epsilon_1 = \frac{\sigma_1}{E} = \frac{1.51}{2\times10^5} = 7.55\times10^{-6}$$

$$\epsilon_2 = \frac{\sigma_2}{E} = \frac{1.827}{2\times10^5} = 9.135\times10^{-6}$$

Step 8: Member force

For element ❶,

$$f_1 = \sigma_1 A = 1.51\times(131.25\times25) = 4954.687\text{ N}$$

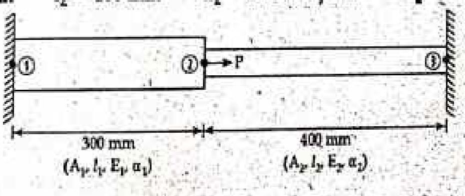
For element ❷,

$$f_2 = \sigma_2 A = 1.827\times(93.75\times25) = 4282.03\text{ N}$$

EXAMPLE 4.5

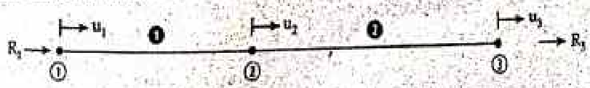
Determine nodal displacements at node ②, stresses in each material and support reactions in the bar shown in the figure, due to applied force P = 400 × 10³ N and temperature rise of 30°C. Given:

$A_1 = 2400\text{ mm}^2$	$l_1 = 300$ mm	$E_1 = 0.7\times10^5\text{ N/mm}^2$	$\alpha_1 = 22\times10^{-6}/°C$
$A_2 = 1200\text{ mm}^2$	$l_2 = 400$ mm	$E_2 = 2\times10^5\text{ N/mm}^2$	$\alpha_2 = 12\times10^{-6}/°C$



Solution:

Step 1: Discretization



Step 2: Force matrix

i) Temperature induced force in member ❶

$$\begin{Bmatrix} f_{1t}^1 \\ f_{2t}^1 \end{Bmatrix} = A_1E_1\alpha_1\Delta t\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = 2400 \times 0.7 \times 10^5 \times 22 \times 10^{-6} \times 30\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} -110880 \\ 110880 \end{Bmatrix}$$

ii) Temperature induced force in member ❷

$$\begin{Bmatrix} f_{2t}^2 \\ f_{3t}^2 \end{Bmatrix} = A_2E_2\alpha_2\Delta t\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = 1200 \times 2 \times 10^5 \times 12 \times 10^{-6} \times 30\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} -86400 \\ 86400 \end{Bmatrix}$$

iii) Force in structure due to temperature change

$$[F_t] = \begin{Bmatrix} f_{1t} \\ f_{2t} \\ f_{3t} \end{Bmatrix} = \begin{Bmatrix} -110880 \\ 110880 - 86400 \\ 86400 \end{Bmatrix} = \begin{Bmatrix} -110880 \\ 24480 \\ 86400 \end{Bmatrix}$$

iv) Point load vector / matrix

$$[F_P] = \begin{Bmatrix} f_{1p} \\ f_{2p} \\ f_{3p} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 400 \times 10^3 \\ R_3 \end{Bmatrix}$$

v) Total force matrix

$$\{F\} = \begin{Bmatrix} R_1 - 110880 \\ 400 \times 10^3 + 24480 \\ 86400 + R_3 \end{Bmatrix} = \begin{Bmatrix} R_1 - 110880 \\ 424480 \\ 86400 + R_3 \end{Bmatrix}$$

Step 3: Displacement vector

$$[u] = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix}; \text{ after applying boundary condition}$$

Step 4: Stiffness matrix

For member ❶,

$$[K_1] = \frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2400 \times 0.7 \times 10^5}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 5.6 & -5.6 \\ -5.6 & 5.6 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \end{array}$$

For element ❷,

$$[K_2] = \frac{1200 \times 2 \times 10^5}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ u_3 \end{array}$$

Assembling the member stiffness matrices to get generalized stiffness matrix:

$$[K] = 10^5\begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 5.6+6 & -6 \\ 0 & -6 & 6 \end{bmatrix} = 10^5\begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 11.6 & -6 \\ 0 & -6 & 6 \end{bmatrix}$$

Step 5: [F] = [K][u]

$$\text{or, } \begin{Bmatrix} R_1 - 110880 \\ 424480 \\ 86400 + R_3 \end{Bmatrix} = 10^5\begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 11.6 & -6 \\ 0 & -6 & 6 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix}$$

The reduced form is:

$$424480 = 10^5 \times 11.6 \times u_2$$

or, $u_2 = 0.3659$ mm

Step 6: Stresses

$$\sigma = \frac{E}{l}[-1 \quad 1]\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} - E\alpha\Delta T$$

$$\therefore \quad \sigma_1 = \frac{0.7 \times 10^5}{300}[-1 \quad 1]\begin{Bmatrix} 0 \\ 0.3659 \end{Bmatrix} - 0.7 \times 10^5 \times 22 \times 10^{-6} \times 30$$

$$= 39.18 \text{ N/mm}^2$$

Similarly,

$$\sigma_2 = \frac{2 \times 10^5}{400}[-1 \quad 1]\begin{Bmatrix} 0.3659 \\ 0 \end{Bmatrix} - 2 \times 10^5 \times 12 \times 10^{-6} \times 30 = 50.965 \text{ N/mm}^2$$

Step 7: Reaction calculations

From [F] = [K][u] in step 5;

$$R_1 - 110880 = 10^5 \times (-5.6) \times 0.3659$$

$$\therefore \quad R_1 = -94024 \text{ N}$$

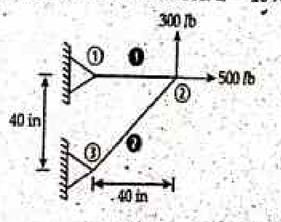
Similarly,

$$R_3 + 86400 = 10^5 \times (-6) \times 0.3659$$

$$\therefore \quad R_3 = -305959 \text{ N}$$

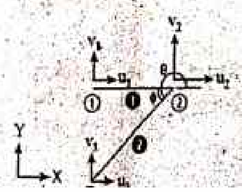
EXAMPLE 4.6

In the given structure, determine displacements at node 2, reaction forces at supports, element strain, stress and member force. $E = 10 \times 10^6$ lb/in^2, $A = 1.5$ in^2



Solution:

Step 1: Discretization



Step 2: Force matrix

$$\{F\} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ 500 \\ 300 \\ R_{3x} \\ R_{3y} \end{Bmatrix}$$

Step 3: Displacement matrix

$$\{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

Step 4: Stiffness matrix

For element ❶

$$[K_1] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1.5 \times 10 \times 10^6}{40}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 1.5 \times 10^6 \begin{bmatrix} 0.25 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & 0 \\ -0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For element ❷

$$l = \sqrt{(40)^2 + (40)^2} = 56.57$$

$$\phi = \tan^{-1}\left(\frac{40}{40}\right) = 45°$$

$$\theta = 180° + 45° = 225°$$

$$C = \cos(225°) = -\frac{1}{\sqrt{2}}$$

$$S = \sin(225°) = -\frac{1}{\sqrt{2}}$$

$$C^2 = 0.5$$

$$S^2 = 0.5$$

$$CS = 0.5$$

$$[K_2] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{sym} & & & S^2 \end{bmatrix} = \frac{1.5 \times 10 \times 10^6}{56.57}\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ & 0.5 & -0.5 & -0.5 \\ & & 0.5 & 0.5 \\ \text{sym} & & & 0.5 \end{bmatrix}$$

$$= 1.5 \times 10^6 \begin{bmatrix} 0.0884 & 0.0884 & -0.0884 & -0.0884 \\ & 0.0884 & -0.0884 & -0.0884 \\ & & 0.0884 & 0.0884 \\ \text{sym} & & & 0.0884 \end{bmatrix}$$

Assembling,

$$[K] = 1.5 \times 10^6 \begin{bmatrix} 0.25 & 0 & -0.25 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & (0.25+0.0884) & (0+0.0884) & -0.0884 & -0.0884 \\ & & & (0+0.0884) & -0.0884 & -0.0884 \\ & & & & 0.0884 & 0.0884 \\ & \text{sym} & & & & 0.0884 \end{bmatrix}$$

$$= 1.5 \times 10^6 \begin{bmatrix} 0.25 & 0 & -0.25 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0.3384 & 0.0884 & -0.0884 & -0.0884 \\ & & & 0.0884 & -0.0884 & -0.0884 \\ & & & & 0.0884 & 0.0884 \\ & \text{sym} & & & & 0.0884 \end{bmatrix}$$

Step 5:

$$[F] = [K][u]$$

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ 500 \\ 300 \\ R_{3x} \\ R_{3y} \end{Bmatrix} = 1.5 \times 10^6 \begin{bmatrix} 0.25 & 0 & -0.25 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0.3384 & 0.0884 & -0.0884 & -0.0884 \\ & & & 0.0884 & -0.0884 & -0.0884 \\ & & & & 0.0884 & 0.0884 \\ & \text{sym} & & & & 0.0884 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

or, $$\begin{Bmatrix} 500 \\ 300 \end{Bmatrix} = 1.5 \times 10^6 \begin{bmatrix} 0.3384 & 0.0884 \\ 0.0884 & 0.0884 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$$

or, $$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{1}{1.5 \times 10^6} \begin{bmatrix} 0.3384 & 0.0884 \\ 0.0884 & 0.0884 \end{bmatrix}^{-1} \begin{Bmatrix} 500 \\ 300 \end{Bmatrix} = \begin{Bmatrix} 5.33 \times 10^{-4} \\ 1.73 \times 10^{-3} \end{Bmatrix}$$

Step 6: From [F] = [K][u]

$$R_{1x} = 1.5 \times 10^6[-0.25 \times 5.33 \times 10^{-4} + 0] = -199.87\ lb$$

$$R_{1y} = 1.5 \times 10^6(0) = 0$$

$$R_{2x} = 1.5 \times 10^6[-0.0884 \times 4 - 0.0884 \times v] = -300\ lb$$

$$R_{2y} = 1.5 \times 10^6[-0.0884 \times 4 - 0.0884 \times v] = -300\ lb$$

NOTE

Program in calculator:

$$A = 1.5 \times 10^6(5.33 \times 10^{-4} \times X + 1.73 \times 10^{-3} \times Y)$$

Put value of x and y, the number at second and third column of corresponding row or the number at the second and third row of corresponding row.

Step 7: Element strain

$$\epsilon = \frac{1}{l}[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\epsilon_1 = \frac{1}{40}[-1 \quad -0 \quad 1 \quad 0]\begin{Bmatrix} 0 \\ 0 \\ 5.33 \times 10^{-4} \\ 1.73 \times 10^{-3} \end{Bmatrix} = 1.33 \times 10^{-5}$$

$$\epsilon_2 = \frac{1}{56.57}[0.7071 \quad 0.7071 \quad -0.7071 \quad -0.7071]\begin{Bmatrix} 5.33\times10^{-4} \\ 1.73\times10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$= 2.829 \times 10^{-5}$$

Step 8: Element stress

$$\sigma = E\epsilon$$

$$\sigma_1 = 10\times10^6 \times (1.33\times10^{-5})$$

$$= 133\ \text{lb/sq. in}$$

$$\sigma_2 = 10\times10^6 \times (2.829\times10^{-5}) = 282.9\ \text{lb/sq. in}$$

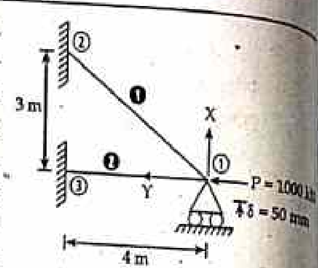
Step 9: Element force

$$f_1 = \sigma_1 A = 133 \times 1.5 = 199.5\ \text{lb}$$

$$f_2 = \sigma_2 A = 282.9 \times 1.5 = 424.35\ \text{lb}$$

EXAMPLE 4.7

For the two-bar truss shown in the figure, determine the displacement in the y-direction of node 1 and the axial force in each element. A force of P = 1000 kN is applied at node 1 in the positive y-direction while node 1 settles an amount δ = 50 mm in the negative x-direction. Let E = 210 GPa and A = 6×10^{-4} m² for each element. The length of the element is shown in the figure.



Solution:

NOTE

The solution of the problem will be shown in two processes. Students are requested to follow the process they feel easier.

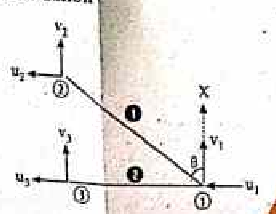
Process I

Step 1: Unit conversion

$$E = 210\ \text{GPa} = 210\times10^6\ \text{kN/m}^2$$

$$\delta = -50\ \text{mm} = -0.05\ \text{m}$$

Step 2: Modeling and discretization



Step 3: Force matrix

$$[F] = \begin{Bmatrix} R_{x1} \\ 1000 \\ R_{x2} \\ R_{y2} \\ R_{x3} \\ R_{y3} \end{Bmatrix}$$; R refer to the reactions at the supports in positive direction.

Step 4: Displacement matrix

$$[u] = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -0.05 \\ v_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix};$$

by applying boundary conditions $u_2 = v_2 = u_3 = v_3 = 0$ and $u_1 = -0.05$ m

Step 5: Stiffness matrix

For element ❶,

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13°$$

$$S = \sin(53.13°) = 0.8$$

$$C = \cos(53.13°) = 0.6$$

$$CS = 0.48$$

$$C^2 = 0.36$$

$$S^2 = 0.64$$

$$[K_1] = \frac{A_1E_1}{L_1}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ & \text{sym} & & S^2 \end{bmatrix}$$

$$= \frac{6\times10^{-4}\times210\times10^6}{5}\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ & 0.64 & -0.48 & -0.64 \\ & & 0.36 & 0.48 \\ & \text{sym} & & 0.64 \end{bmatrix}$$

$$= 252\ \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 36 & 48 & -36 & -48 \\ & 64 & -48 & -64 \\ & & 36 & 48 \\ & \text{sym} & & 64 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

Similarly, for element ❷,

$$[K_2] = \frac{A_2E_2}{L_2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \frac{6\times10^{-4}\times210\times10^6}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$
= 252\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} \boxed{\begin{matrix} 0 & 0 \\ 0 & 125 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & -125 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & -125 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 125 \end{matrix} \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}
$$

Assembling the element matrices to get generalized stiffness matrix:

$$
[K] = 252 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 36+0 & 48+0 & -36 & -48 & 0 & 0 \\ & 64+125 & -48 & -64 & 0 & -125 \\ & & 36 & 48 & 0 & 0 \\ & & & 64 & 0 & 0 \\ & & & & 0 & 0 \\ & \text{sym} & & & & 125 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}
$$

$$
= 252 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 36 & 48 & -36 & -48 & 0 & 0 \\ & 189 & -48 & -64 & 0 & -125 \\ & & 36 & 48 & 0 & 0 \\ & & & 64 & 0 & 0 \\ & & & & 0 & 0 \\ & \text{sym} & & & & 125 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}
$$

Step 5: [F] = [K][u]

$$
\begin{Bmatrix} R_{x1} \\ 1000 \\ R_{x2} \\ R_{y2} \\ R_{x3} \\ R_{y3} \end{Bmatrix} = 252 \begin{bmatrix} 36 & 48 & -36 & -48 & 0 & 0 \\ & 189 & -48 & -64 & 0 & -125 \\ & & 36 & 48 & 0 & 0 \\ & & & 64 & 0 & 0 \\ & & & & 0 & 0 \\ & \text{sym} & & & & 125 \end{bmatrix} \begin{Bmatrix} -0.05 \\ v_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

By elimination reduced matrix is:

$$
\begin{Bmatrix} R_{x1} \\ 1000 \end{Bmatrix} = 252 \begin{bmatrix} 36 & 48 \\ 48 & 189 \end{bmatrix} \begin{Bmatrix} -0.05 \\ v_1 \end{Bmatrix}
$$

or, $1000 = 252 \times \{48 \times (-0.05) + 189 \times v_1\}$

$\therefore$ $v_1 = 0.0337$ m

Step 6: Member force calculation

We know that;

$$
\sigma = \frac{E}{l}[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}
$$

and, $f = \sigma A$

$$
\therefore \quad f = \frac{EA}{l}[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}
$$

For element ❶,

$$
f_1 = \frac{210 \times 10^6 \times 6 \times 10^{-4}}{5}[-0.6 \quad -0.8 \quad 0.6 \quad 0.8]\begin{Bmatrix} -0.05 \\ 0.0337 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix}
$$

$= 76.608$ kN

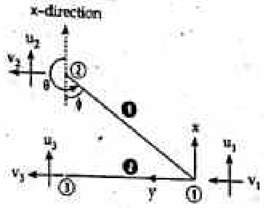
For element ❷, $\theta = 90°$, $C = 0$, $S = 1$

$$
f_2 = \frac{210 \times 10^6 \times 6 \times 10^{-4}}{5}[0 \quad -1 \quad 0 \quad -1]\begin{Bmatrix} -0.05 \\ 0.0337 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{matrix}
$$

$= -1061.55$ kN

Process II

Step 2: Modeling and discretization



Step 3: Force matrix

$$
[F] = \begin{Bmatrix} R_{2x} \\ R_{2y} \\ R_{1x} \\ 1000 \\ R_{3x} \\ R_{3y} \end{Bmatrix}
$$; R refer to the reactions at the supports in positive direction.

NOTE

In this solution technique, you can see order is changed. The order is taken from one end of the structure to another.

Step 4: Displacement matrix

$$
[u] = \begin{Bmatrix} u_2 \\ v_2 \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.05 \\ v_1 \\ 0 \\ 0 \end{Bmatrix}
$$

NOTE

Displacements are also taken in the same order of force.

Step 5: Stiffness matrix

For element **①**.

$$\phi = \tan^{-1}\left(\frac{4}{3}\right) = 53.13°$$

$$\theta = 180° + 53.13° = 233.13°$$

$$C = \cos(233.13°) = -0.6$$

$$S = \sin(233.13°) = -0.8$$

$$C^2 = 0.36$$

$$S^2 = 0.64$$

$$CS = 0.48$$

NOTE

Here, we are going from node ② to ①, so we take angle at node ①, anticlockwise direction from x-direction.

$$[K_1] = \frac{A_1E_1}{L_1}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{sym} & & & S^2 \end{bmatrix}$$

$$= 252\begin{array}{c} \begin{matrix} u_2 & v_2 & u_1 & v_1 \end{matrix} \\ \begin{bmatrix} 36 & 48 & -36 & -48 \\ & 64 & -48 & -64 \\ & & 36 & 48 \\ \text{sym} & & & 64 \end{bmatrix} \end{array}\begin{matrix} u_2 \\ v_2 \\ u_1 \\ v_1 \end{matrix}$$

Similarly,

$$[K_2] = \frac{A_2E_2}{L_2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = 252\begin{array}{c} \begin{matrix} u_1 & v_1 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 125 & 0 & -125 \\ & & 0 & 0 \\ \text{sym} & & 0 & 125 \end{bmatrix} \end{array}\begin{matrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{matrix}$$

Assembling the matrices; we have,

$$[K] = 252\begin{array}{c} \begin{matrix} u_2 & v_2 & u_1 & v_1 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 36 & 48 & -36 & -48 & 0 & 0 \\ & 64 & -48 & -64 & 0 & 0 \\ & & 36+0 & 48+0 & 0 & 0 \\ & & & 64+125 & 0 & -125 \\ & & & & 0 & 0 \\ \text{sym} & & & & & 125 \end{bmatrix} \end{array}\begin{matrix} u_2 \\ v_2 \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{matrix}$$

$$= 252\begin{bmatrix} 36 & 48 & -36 & -48 & 0 & 0 \\ & 64 & -48 & -64 & 0 & 0 \\ & & 36 & 48 & 0 & 0 \\ & & & 189 & 0 & -125 \\ & & & & 0 & 0 \\ \text{sym} & & & & & 125 \end{bmatrix}$$

Step 6: [F] = [K][u]

or,

$$\begin{Bmatrix} R_{2x} \\ R_{2y} \\ R_{1x} \\ 1000 \\ R_{3x} \\ R_{3y} \end{Bmatrix} = 252\begin{bmatrix} 36 & 48 & -36 & -48 & 0 & 0 \\ & 64 & -48 & -64 & 0 & 0 \\ & & 36 & 48 & 0 & 0 \\ & & & 189 & 0 & -125 \\ & & & & 0 & 0 \\ \text{sym} & & & & & 125 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.05 \\ v_1 \\ 0 \\ 0 \end{Bmatrix}$$

By elimination reduced matrix is:

$$\begin{Bmatrix} R_{x1} \\ 1000 \end{Bmatrix} = 252\begin{bmatrix} 36 & 48 \\ 48 & 189 \end{bmatrix}\begin{Bmatrix} -0.05 \\ v_1 \end{Bmatrix}$$

$$1000 = 252 \times \{48 \times (-0.05) + 189 \times v_1\}$$

or,

$$v_1 = 0.0337 \text{ m}$$

∴ **Step 7: Member force calculation**

$$f_1 = \frac{EA}{l_1}[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_2 \\ v_2 \\ u_1 \\ v_1 \end{Bmatrix}$$

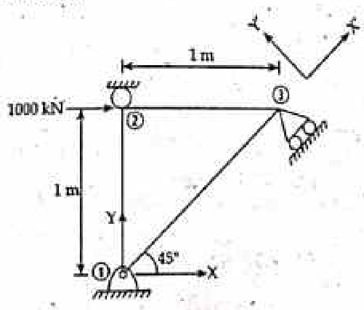
$$= \frac{210 \times 10^6 \times 6 \times 10^{-4}}{5}[0.6 \quad 0.8 \quad -0.6 \quad -0.8]\begin{Bmatrix} 0 \\ 0 \\ -0.05 \\ 0.0337 \end{Bmatrix} = 76.608$$

NOTE

Compare the values in the matrices from previous technique.

EXAMPLE 4.8

For the plane truss shown in the figure, determine the displacements and reactions. Let E = 210 GPa, $A = 6.00 \times 10^{-4}$ m² for element 1 and 2, and $A = 6\sqrt{2} \times 10^{-4}$ m² for element 3.



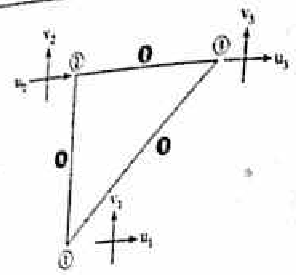
Solution:

Step 1: Unit conversion

$$E = 210 \text{ GPa} = 210 \times 10^6 \text{ kN/m}^2$$

$$l_{①-③} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Step 2: Discretization



Step 3: Force matrix

$$[\bar{F}] = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ 1000 \\ R_{2y} \xrightarrow{} 0 \\ \cancel{R_{3x}} \\ R'_{3y} \end{Bmatrix}; \text{ R refer to reactions acting in positive directions.}$$

Step 4: Displacement matrix

$$\{\bar{u}\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u'_3 \\ v'_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ u'_3 \\ 0 \end{Bmatrix}; \text{ after applying boundary conditions.}$$

Step 5: Stiffness matrix

For element ❶, [From ① and ②]

$$[K_1] = \frac{A_1E_1}{L_1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

NOTE

For bars oriented parallel to y-axis, stiffness in x-direction are 0 and corresponding rows and columns are 0. In remaining positions, just fill up $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

$$\therefore \quad [K_1] = \frac{61 \times 10^{-4} \times 210 \times 10^6}{1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= 1260 \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc|c} u_1 & v_1 & u_2 & v_2 & \\ \end{array}$$

$$= 1260\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & -100 \\ 0 & 0 & 0 & 0 \\ 0 & -100 & 0 & 100 \end{bmatrix}\begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix}$$

Similarly, for element ❷; we have,

$$[K_2] = 1260\begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 0 & 0 & 0 \\ -100 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix}$$

(columns: u_2, v_2, u_3, v_3)

Similarly, for element ❸;

From ③ to ①,

$\theta = 180° + 45° = 225°$

$C = \cos(225°) = -0.707$

$S = \sin(225°) = -0.707$

$C^2 = 0.5$

$S^2 = 0.5$

$CS = 0.5$

$$[K_3] = \frac{A_3E_3}{L_3}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{sym} & & & S^2 \end{bmatrix}$$

$$= \frac{6\sqrt{2} \times 10^{-4} \times 210 \times 10^6}{\sqrt{2}}\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ & 0.5 & -0.5 & -0.5 \\ & & 0.5 & 0.5 \\ \text{sym} & & & 0.5 \end{bmatrix}$$

$$= 1260\begin{bmatrix} 50 & 50 & -50 & -50 \\ & 50 & -50 & -50 \\ & & 50 & 50 \\ \text{sym} & & & 50 \end{bmatrix}\begin{matrix} u_3 \\ v_3 \\ u_1 \\ v_1 \end{matrix}$$

(columns: u_3, v_3, u_1, v_1)

Assembling to get generalized stiffness matrix:

$$[K] = 1260\begin{bmatrix} 0+50 & 0+50 & 0 & 0 & -50 & -50 \\ & 100+50 & 0 & -100 & -50 & -50 \\ & & 0+100 & 0+0 & -100 & 0 \\ & & & 100+0 & 0 & 0 \\ & \text{sym} & & & 100+50 & 0+50 \\ & & & & & 0+50 \end{bmatrix}\begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix}$$

(columns: u_1, v_1, u_2, v_2, u_3, v_3)

$$\therefore \quad [K] = 1260\begin{bmatrix} 50 & 50 & 0 & 0 & -50 & -50 \\ & 150 & 0 & -100 & -50 & -50 \\ & & 100 & 0 & -100 & 0 \\ & & & 100 & 0 & 0 \\ & \text{sym} & & & 150 & 50 \\ & & & & & 50 \end{bmatrix}$$

Step 6: $[T][K][T]^T$

For node 3,

$\theta = 45°$

$C = \cos(45°) = \frac{1}{\sqrt{2}}$

$S = \sin(45°) = \frac{1}{\sqrt{2}}$

$$[T] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$[T][K][T]^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 50 & 50 & 0 & 0 & -50 & -50 \\ & -150 & 0 & -100 & -50 & -50 \\ & & 100 & 0 & -100 & 0 \\ & & & 100 & 0 & 0 \\ & \text{sym} & & & 150 & 50 \\ & & & & & 50 \end{bmatrix} [T]^T$$

$$= 1260 \begin{bmatrix} 50 & 50 & 0 & 0 & -50 & -50 \\ 50 & -150 & 0 & -100 & -50 & -50 \\ 0 & 0 & 100 & 0 & -100 & 0 \\ 0 & -100 & 0 & 100 & 0 & 0 \\ -\frac{100}{\sqrt{2}} & -\frac{100}{\sqrt{2}} & -\frac{100}{\sqrt{2}} & 0 & \frac{200}{\sqrt{2}} & \frac{100}{\sqrt{2}} \\ 0 & 0 & \frac{100}{\sqrt{2}} & 0 & -\frac{100}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} [I] & [0] & [0] & \\ [0] & [I] & [0] & \\ [0] & [0] & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$[T][K][T]^T = 1260 \begin{bmatrix} 50 & 50 & 0 & 0 & -\frac{100}{\sqrt{2}} & 0 \\ & -50 & 0 & -100 & -\frac{100}{\sqrt{2}} & 0 \\ & & 100 & 0 & -\frac{100}{\sqrt{2}} & \frac{100}{\sqrt{2}} \\ & & & 100 & 0 & 0 \\ & \text{sym} & & & 150 & -25 \\ & & & & & 50 \end{bmatrix}$$

Step 7: $[\bar{F}] = [T][K][T]^T[\bar{u}]$

or,

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ 1000 \\ R_{2y} \\ 0 \\ R_{3y} \end{Bmatrix} = 1260 \begin{bmatrix} 50 & 50 & 0 & 0 & -\frac{100}{\sqrt{2}} & 0 \\ & 150 & 0 & 100 & \frac{100}{\sqrt{2}} & 0 \\ & & 100 & 0 & -\frac{100}{\sqrt{2}} & \frac{100}{\sqrt{2}} \\ & & & 100 & 0 & 0 \\ & \text{sym} & & & 150 & -50 \\ & & & & & 50 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ u_3' \\ 0 \end{Bmatrix}$$

Reduced matrix after elimination is:

$$\begin{Bmatrix} 1000 \\ 0 \end{Bmatrix} = 1260 \begin{bmatrix} 100 & -\frac{100}{\sqrt{2}} \\ -\frac{100}{\sqrt{2}} & 25 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3' \end{Bmatrix}$$

or,
$$\begin{Bmatrix} u_2 \\ u_3' \end{Bmatrix} = \frac{1}{1260} \begin{bmatrix} 100 & -\frac{100}{\sqrt{2}} \\ -\frac{100}{\sqrt{2}} & 150 \end{bmatrix}^{-1} \begin{Bmatrix} 1000 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 11.90 \times 10^{-3} \\ 5.612 \times 10^{-3} \end{Bmatrix} \text{m}$$

Step 8: Reaction calculation

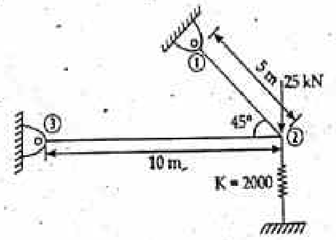
$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2y} \\ R_{3y}' \end{Bmatrix} = 1260 \begin{array}{c} \begin{matrix} u_2 & \quad u_3' \end{matrix} \\ \begin{bmatrix} 0 & -\frac{100}{\sqrt{2}} \\ 0 & -\frac{100}{\sqrt{2}} \\ 0 & 0 \\ \frac{100}{\sqrt{2}} & -50 \end{bmatrix} \end{array} \begin{Bmatrix} 11.90 \times 10^{-3} \\ 5.612 \times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} -500 \text{ kN} \\ -500 \text{ kN} \\ 706 \text{ kN} \end{Bmatrix}$$

NOTE

Program in calculator: $A = 1260 \times (11.90 \times 10^{-3}X + 5.612 \times 10^{-3}Y)$.

EXAMPLE 4.9

Evaluate the nodal displacements, support reactions and member forces in the system shown in the figure below. Assume $A = 5 \times 10^{-4}$ m² and $E = 210$ GPa for the bar members.

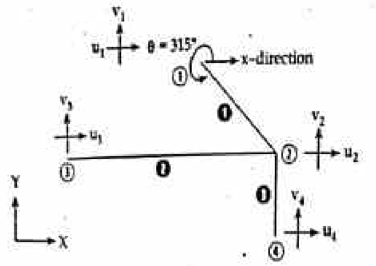


NOTE

Here, more than two elements are connected at the same node.

Solution:

Step 1: Discretization



Step 2: Force matrix

$$\{F\} = \begin{Bmatrix} R_{x1} \\ R_{1y} \\ 0 \\ -25 \\ R_{x3} \\ R_{y3} \\ R_{x4} \\ R_{y4} \end{Bmatrix};$$ assuming R represents reactions acting in positive directions.

Step 3: Displacement matrix

$$[u] = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix};$$

after applying boundary conditions $u_1 = v_1 = u_3 = v_3 = u_4 = v_4 = 0$.

Step 4: Stiffness matrix

For element ❶,

$\theta = 315°$

$C = \cos\theta = \cos(315°) = \frac{1}{\sqrt{2}}$

$S = \sin\theta = \sin(315°) = -\frac{1}{\sqrt{2}}$

$C^2 = 0.5$

$S^2 = 0.5$

$CS = -0.5$

$$[K_1] = \frac{A_1E_1}{L_1}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{sym} & & & S^2 \end{bmatrix}$$

$$= \frac{5\times10^{-4}\times210\times10^{6}}{5}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ & 0.5 & 0.5 & -0.5 \\ & & 0.5 & -0.5 \\ \text{sym} & & & 0.5 \end{bmatrix}$$

$$= 1000\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 10.5 & -10.5 & -10.5 & 10.5 \\ & 10.5 & 10.5 & -10.5 \\ & & 10.5 & -10.5 \\ \text{sym} & & & 10.5 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

For element ❷,

$$[K_2] = \frac{5\times10^{-4}\times210\times10^{6}}{10}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 1000\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 10.5 & 0 & -10.5 & 0 \\ 0 & 0 & 0 & 0 \\ -10.5 & 0 & 10.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

For element ❸,

$$[K_3] = 2000\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = 1000\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} \end{array}\begin{array}{c} u_2 \\ v_2 \\ u_4 \\ v_4 \end{array}$$

Assembling element matrices to get generalized stiffness matrix:

$$[K] = 1000\begin{bmatrix} 10.5 & -10.5 & -10.5 & 0 & 0 & 0 & 0 & 0 \\ & 10.5 & 10.5 & 0 & 0 & 0 & 0 & 0 \\ & & (10.5+10.5+0) & (-10.5+0+0) & -10.5 & 0 & 0 & 0 \\ & & & 10.5+0+2 & 0 & 0 & 0 & -2 \\ & & & & 10.5 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & \text{sym} & & & & & 0 & 0 \\ & & & & & & & 2 \end{bmatrix}$$

$$\therefore \quad [K] = 1000 \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 10.5 & -10.5 & -10.5 & 10.5 & 0 & 0 & 0 & 0 \\ & 10.5 & 10.5 & -10.5 & 0 & 0 & 0 & 0 \\ & & \boxed{21} & \boxed{-10.5} & -10.5 & 0 & 0 & 0 \\ & & & \boxed{12.5} & 0 & 0 & 0 & -2 \\ & & & & 10.5 & 0 & 0 & 0 \\ & & \text{sym} & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 2 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}$$

Step 5: [F] = [K][u]

$$\begin{Bmatrix} R_{x1} \\ R_{y1} \\ 0 \\ -25 \\ R_{x3} \\ R_{y3} \\ R_{x4} \\ R_{y4} \end{Bmatrix} = 1000 \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 10.5 & -10.5 & -10.5 & 10.5 & 0 & 0 & 0 & 0 \\ & 10.5 & 10.5 & -10.5 & 0 & 0 & 0 & 0 \\ & & 21 & -10.5 & -10.5 & 0 & 0 & 0 \\ & & & 12.5 & 0 & 0 & 0 & -2 \\ & & & & 10.5 & 0 & 0 & 0 \\ & & \text{sym} & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 2 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

After elimination reduced matrix is:

$$\begin{Bmatrix} 0 \\ -25 \end{Bmatrix} = 1000 \begin{bmatrix} 21 & -10.5 \\ -10.5 & 12.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$$

$$\therefore \quad \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{1}{1000} \begin{bmatrix} 21 & -10.5 \\ -10.5 & 12.5 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -25 \end{Bmatrix} = \begin{Bmatrix} -1.724 \times 10^{-3} \\ -3.44 \times 10^{-3} \end{Bmatrix} \text{m}$$

Step 6: Reaction calculation

$$\begin{Bmatrix} R_{x1} \\ R_{y1} \\ R_{x3} \\ R_{y3} \\ R_{x4} \\ R_{y4} \end{Bmatrix} = 1000 \times \begin{bmatrix} -10.5 & 10.5 \\ 10.5 & -10.5 \\ -10.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{Bmatrix} -1.724 \times 10^{-3} \\ -3.44 \times 10^{-3} \end{Bmatrix}$$

NOTE

Program in calculator:

$$A = 1000[-1.724 \times 10^{-3}X + (-3.44 \times 10^{-3})Y]$$

$$\begin{Bmatrix} R_{x1} \\ R_{y1} \\ R_{x3} \\ R_{y3} \\ R_{y4} \end{Bmatrix} = \begin{Bmatrix} -18.018 \\ 18.018 \\ 18.102 \\ 0 \\ 6.88 \end{Bmatrix} \text{kN}$$

Step 7: Member force

For element ❶,

$$\theta = 315^\circ$$

$$S = -\frac{1}{\sqrt{2}}$$

$$C = \frac{1}{\sqrt{2}}$$

$$f_1 = \frac{A_1E_1}{L_1}[-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$= \frac{5 \times 10^{-4} \times 210 \times 10^6}{5} \left[-\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}\right] \begin{Bmatrix} 0 \\ 0 \\ -1.724 \times 10^{-3} \\ -3.44 \times 10^{-3} \end{Bmatrix}$$

$$= 25.48 \text{ kN}$$

For element ❷,

$$f_2 = \frac{A_2E_2}{L_2}[-1 \quad 1] \begin{Bmatrix} u_3 \\ u_2 \end{Bmatrix}$$

$$= \frac{5 \times 10^{-4} \times 210 \times 10^6}{10}[-1 \quad 1] \begin{Bmatrix} 0 \\ -1.724 \times 10^{-3} \end{Bmatrix}$$

$$= -18.102 \text{ kN}$$

NOTE

Element ❷ is oriented parallel to x-axis. So, displacements in the local co-ordinate system coincide with displacements in x-direction in the global co-ordinate system *i.e.*, $u'_3 = u_3$ and $u'_2 = u_2$. So, this property can be advantageously used to use formula of local co-ordinate system.

For element ❸,

$$f_3 = K[-1 \quad 1] \begin{Bmatrix} v_4 \\ v_2 \end{Bmatrix} = 2000[-1 \quad 1] \begin{Bmatrix} 0 \\ -3.44 \times 10^{-3} \end{Bmatrix} = -6.88 \text{ kN}$$

NOTE

Displacements in the global direction are taken in the sequence toward positive direction like $\begin{Bmatrix} u_3 \\ u_2 \end{Bmatrix}$, $\begin{Bmatrix} v_4 \\ v_2 \end{Bmatrix}$.

EXAMPLE 4.10

For the given structure:

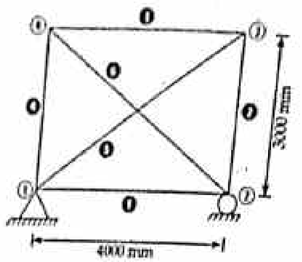
i) Assemble global stiffness matrix

ii) Determine load vector if temperature of member 1-3 increases by 25°C, $\alpha = 12 \times 10^{-6}/°C$

iii) Determine load vector if the member 1-3 is longer by 0.2 mm.

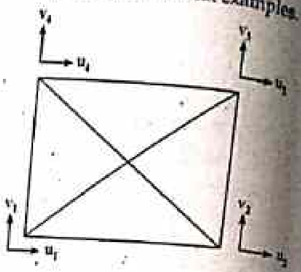
$A = 2000 \text{ mm}^2$ for diagonal members, $A = 1000 \text{ mm}^2$ for other members

$E = 2000 \text{ kN/mm}^2$



Solution:

Determine the element stiffness matrices as in the previous numerical examples.



$$\begin{array}{cc} & \begin{matrix} u_1 & v_1 & u_2 & v_2 \end{matrix} \\ [K_1] = & \begin{bmatrix} 50 & 0 & -50 & 0 \\ 0 & 0 & 0 & 0 \\ -50 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} u_2 & v_2 & u_3 & v_3 \end{matrix} \\ [K_2] = & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 66.67 & 0 & -66.67 \\ 0 & 0 & 0 & 0 \\ 0 & -66.67 & 0 & 66.67 \end{bmatrix} \begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} u_3 & v_3 & u_4 & v_4 \end{matrix} \\ [K_3] = & \begin{bmatrix} 50 & 0 & -50 & 0 \\ 0 & 0 & 0 & 0 \\ -50 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{matrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} u_4 & v_4 & u_1 & v_1 \end{matrix} \\ [K_4] = & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 66.67 & 0 & -66.67 \\ 0 & 0 & 0 & 0 \\ 0 & -66.67 & 0 & 66.67 \end{bmatrix} \begin{matrix} u_4 \\ v_4 \\ u_1 \\ v_1 \end{matrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} u_1 & v_1 & u_3 & v_3 \end{matrix} \\ [K_5] = & \begin{bmatrix} 51.2 & 38.4 & -51.2 & -38.4 \\ 38.4 & 28.8 & -38.4 & -28.8 \\ -51.2 & -38.4 & 51.2 & 38.4 \\ -38.4 & -28.8 & 38.4 & 28.8 \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{matrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} u_2 & v_2 & u_4 & v_4 \end{matrix} \\ [K_6] = & \begin{bmatrix} 51.2 & -38.4 & -51.2 & 38.4 \\ -38.4 & 28.8 & 38.4 & -28.8 \\ -51.2 & 38.4 & 51.2 & -38.4 \\ 38.4 & -28.8 & -38.4 & 28.8 \end{bmatrix} \begin{matrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{matrix} \end{array}$$

Assembling, we have,

u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	
50 + 0 + 51.2	0 + 0 + 38.4	− 50	0	0 + 51.2	0 − 38.4	0	0	u_1
50 + 0 + 38.4	0 + 66.67 + 28.87	0	0	− 38.4	− 28.8	0	− 66.67	v_1
− 50	0	50 + 0 + 51.2	0 + 0 − 38.4	0	0	− 51.2	38.4	u_2
0	0	0 + 0 + 38.4	0 + 66.67 + 28.80	0	− 66.67	38.4	− 28.8	v_2
− 51.2	− 38.4	0	0	0 + 50 + 51.2	0 + 0 + 38.4	− 50	0	u_3
− 38.4	− 28.8	0	− 66.67	0 + 0 + 38.4	66.67 + 0 + 28.8	0	0	v_3
0	0	− 51.2	38.4	− 50	0	50 + 0 + 51.2	0 0 − 38.4	u_4
0	− 66.67	38.4	− 28.8	0	0	0 + 0 − 38.4	0 + 66.67 + 28.8	v_4

$$K = \begin{bmatrix} 101.2 & 38.4 & -50 & 0 & -51.2 & -38.4 & 0 & 0 \\ 38.4 & 95.47 & 0 & 0 & -38.4 & -28.8 & 0 & -66.67 \\ -50 & 0 & 101.2 & -38.4 & 0 & 0 & -51.2 & 38.4 \\ 0 & 0 & -38.4 & 95.47 & 0 & -66.67 & 38.4 & -28.8 \\ -51.2 & -38.4 & 0 & 0 & 101.2 & 38.4 & -50 & 0 \\ -38.4 & -28.8 & 0 & -66.67 & 38.4 & 95.47 & 0 & 0 \\ 0 & 0 & -51.3 & 38.4 & -50.0 & 0 & 101.2 & -38.4 \\ 0 & -66.67 & 38.4 & -28.8 & 0 & 0 & -38.4 & 95.47 \end{bmatrix}$$

Load vector for the temperature forces

$$\begin{Bmatrix} f^t_{1x} \\ f^t_{2y} \\ f^t_{3x} \\ f^t_{3y} \end{Bmatrix} = E_5 A_5 \alpha_5 \Delta T \begin{Bmatrix} -C \\ -S \\ C \\ S \end{Bmatrix} = 200 \times 2000 \times 12 \times 10^{-6} \times 25 \begin{Bmatrix} -0.8 \\ -0.6 \\ 0.8 \\ 0.6 \end{Bmatrix} = \begin{Bmatrix} -96.0 \\ -72.0 \\ 96.0 \\ 72.0 \end{Bmatrix}$$

Similarly, load vector if the member 5 (1 → 3) is longer by 0.2 mm

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = E_5 A_5 \alpha_5 \frac{t}{l} \begin{Bmatrix} -C \\ -S \\ C \\ S \end{Bmatrix} = 200 \times 2000 \times \frac{0.2}{500} \begin{Bmatrix} -0.8 \\ -0.6 \\ 0.8 \\ 0.6 \end{Bmatrix} = \begin{Bmatrix} -128.0 \\ -96.0 \\ 128.0 \\ 96.0 \end{Bmatrix}$$

4.5.10 Transformation matrix and stiffness matrix for a bar in 3-D space

The relation between displacement matrix in the local co-ordinate system and displacement matrix in the global co-ordinate system is:

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

where, $\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \{u'\}$ is the displacement matrix in the local co-ordinate system.

$\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix} = \{u\}$ is the displacement matrix in the global co-ordinate system.

$\begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix} = [T] =$ is the transformation matrix.

$$C_x = \frac{x_2 - x_1}{l}$$

$$C_y = \frac{y_2 - y_1}{l}$$

$$C_z = \frac{z_2 - z_1}{l}$$

l is the length of the element.

The stiffness matrix [K] for truss in 3-D is given by;

$$[K] = \frac{AE}{L}\begin{bmatrix} C_x^2 & C_xC_y & C_xC_z & -C_x^2 & -C_xC_y & -C_xC_z \\ & C_y^2 & C_yC_z & -C_xC_y & -C_y^2 & -C_yC_z \\ & & C_z^2 & -C_xC_z & -C_yC_z & -C_z^2 \\ & & & C_x^2 & C_xC_y & C_xC_z \\ & \text{sym} & & & C_y^2 & C_yC_z \\ & & & & & C_z^2 \end{bmatrix}$$

4.6 BEAM ELEMENT

4.6.1 Stiffness matrix

Forces and nodal displacement of a beam element can be related by:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L^2 & 2L^2 \\ & & 12 & -6L \\ \text{sym} & & & 4L^2 \end{bmatrix}\begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{Bmatrix}$$

or, $\{F\} = [K]\{u\}$

where, $\{F\} = \begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix}$ is the force matrix.

$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L^2 & 2L^2 \\ & & 12 & -6L \\ \text{sym} & & & 4L^2 \end{bmatrix}$ is the stiffness matrix.

$\{u\} = \begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{Bmatrix}$ is the displacement matrix.

If axial force and axial displacements are also considered,

$$[K] = \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \phi_1 & u_2 & v_2 & \phi_2 \end{array} \\ \left[\begin{array}{cccccc} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{12IE}{l^3} & \frac{6IE}{l^2} & 0 & -\frac{12IE}{l^3} & \frac{6IE}{l^2} \\ 0 & \frac{6IE}{l^2} & \frac{4IE}{l} & 0 & -\frac{6IE}{l^2} & \frac{2IE}{l} \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{12IE}{l^3} & -\frac{6IE}{l^2} & 0 & \frac{12IE}{l^3} & -\frac{6IE}{l^2} \\ 0 & \frac{6IE}{l^2} & \frac{2IE}{l} & 0 & -\frac{6IE}{l^2} & \frac{4IE}{l} \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{array}$$

4.6.2 Transformation matrix

$$\underbrace{\begin{Bmatrix} V_1' \\ \theta_1' \\ V_2' \\ \theta_2' \end{Bmatrix}}_{\{u'\}} = \underbrace{\begin{bmatrix} -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{[T]} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}}_{\{u\}}$$

4.8.3 Nodal loads

Loads in beams are transferred to the beam as equivalent nodal loads.

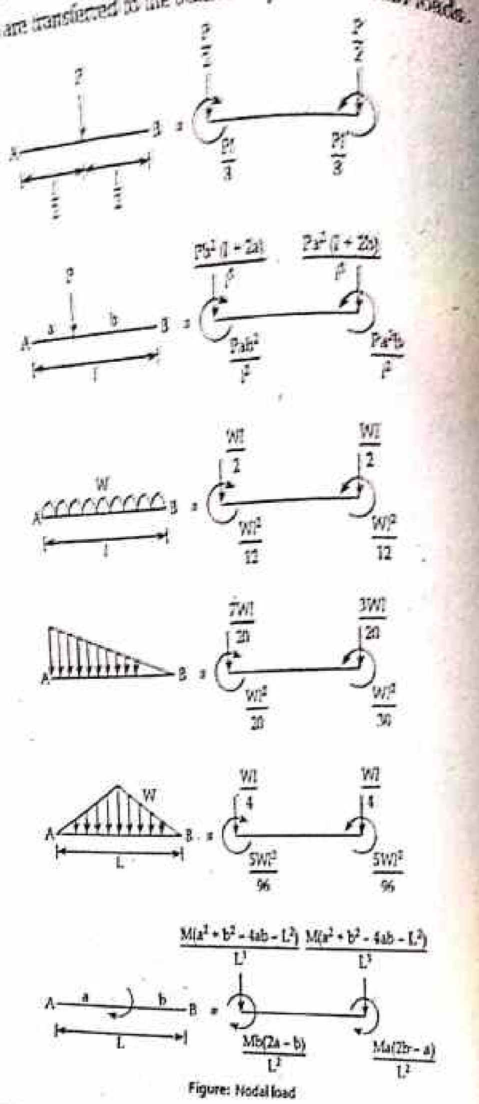
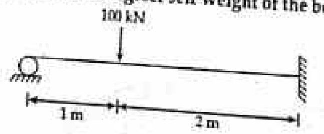


Figure: Nodal load

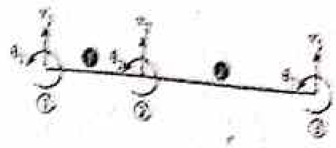
EXAMPLE 4.11

For the beam shown in the figure below, calculate displacement and slope 1 m from the simply supported end. Neglect self weight of the beam. [2072 Magh]



Solution:

Step 1: Discretization



Step 2: Force matrix

$$\{F\} = \begin{Bmatrix} R_1 \\ 0 \\ -100 \\ 0 \\ R_3 \\ M_3 \end{Bmatrix}$$

where, R and M are the reaction force and moments produced at the supports.

Step 3: Displacement matrix

$$\{u\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \theta_1 \\ v_2 \\ \theta_2 \\ 0 \\ 0 \end{Bmatrix}$$; after applying boundary conditions $\theta_3 = v_3 = v_1 = 0$.

Step 4: Stiffness matrix

For element ❶,

$$[K_1] = \frac{IE}{(l)^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym} & & & 4l^2 \end{bmatrix}$$

$$= IE \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array}$$

$$= IE\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ \text{sym} & & & 4 \end{bmatrix}\begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix}$$

For element ❷,

$$[K_2] = \frac{IE}{(l)^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym} & & & 4l^2 \end{bmatrix}$$

$$= \frac{IE}{8}\begin{bmatrix} 12 & 12 & -12 & 12 \\ & 16 & -12 & 8 \\ & & 12 & -12 \\ \text{sym} & & & 16 \end{bmatrix}$$

$$\begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array}$$

$$= IE\begin{bmatrix} 1.5 & 1.5 & -1.5 & 1.5 \\ & 2 & -1.5 & 1 \\ & & 1.5 & -1.5 \\ \text{sym} & & & 2 \end{bmatrix}\begin{matrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

Assembling; we have,

$$[K] = IE \begin{array}{c} \begin{array}{cccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ & 4 & -6 & 2 & 0 & 0 \\ & & (12+1.5) & (-6+1.5) & -1.5 & 1.5 \\ & & & (12+2) & -1.5 & 1 \\ & & & & 1.5 & -1.5 \\ & \text{sym} & & & & 2 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

$$= IE \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ & 4 & -6 & 2 & 0 & 0 \\ & & 13.5 & -4.5 & -1.5 & 1.5 \\ & & & 14 & -1.5 & 1 \\ & & & & 1.5 & -1.5 \\ & \text{sym} & & & & 2 \end{bmatrix}$$

Step 5: {F} = [K]{u}

$$\text{or, } \begin{Bmatrix} R_1 \\ 0 \\ -100 \\ 0 \\ R_3 \\ M_3 \end{Bmatrix} = IE \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ & 4 & -6 & 2 & 0 & 0 \\ & & 13.5 & -4.5 & -1.5 & 1.5 \\ & & & 14 & -1.5 & 1 \\ & & & & 1.5 & -1.5 \\ & \text{sym} & & & & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_1 \\ v_2 \\ \theta_2 \\ 0 \\ 0 \end{Bmatrix}$$

After elimination; we have,

$$\begin{Bmatrix} 0 \\ -100 \\ 0 \end{Bmatrix} = IE \begin{bmatrix} 4 & -6 & 2 \\ -6 & 13.5 & -4.5 \\ 2 & -4.5 & 14 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\text{or, } \begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{IE} \begin{bmatrix} 4 & -6 & 2 \\ -6 & 13.5 & -4.5 \\ 2 & -4.5 & 14 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -100 \\ 0 \end{Bmatrix}$$

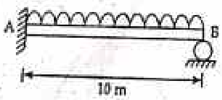
$$= \frac{1}{IE} \begin{Bmatrix} -33.33 \\ -23.11 \\ -2.67 \end{Bmatrix}$$

Hence, displacement and slope at position of load are $-\frac{23.1}{IE}$ m and $-\frac{2.67}{IE}$ radian.

EXAMPLE 4.12

Draw BMD and SFD of the beam shown in the figure considering (i) single element, (ii) two element. Also, describe why the result obtained by FEM is different from exact BMD and SFD and state how the exact solution can be approached.

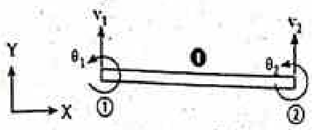
[2071 Magh]



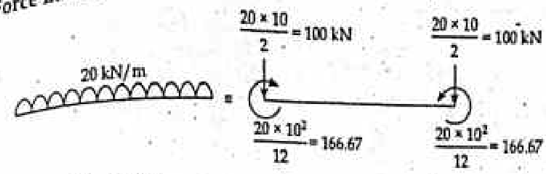
Solution:

i) **Discretizing into single element**

Step 1: Discretization



Step 2: Force matrix



$$\{F\} = \begin{Bmatrix} R_1 - 100 \\ M_1 - 166.67 \\ R_2 - 100 \\ 166.67 \end{Bmatrix}$$

Step 3: Displacement matrix

$$\{u\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \end{Bmatrix}$$

Step 4: Stiffness matrix

$$[K] = \frac{IE}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym} & & & 4l^2 \end{bmatrix} = \frac{IE}{1000} \begin{bmatrix} 12 & 60 & -12 & 60 \\ & 400 & -60 & 200 \\ & & 12 & -60 \\ \text{sym} & & & 400 \end{bmatrix}$$

Step 5: {F} = [K]{u}

$$\begin{Bmatrix} R_1 - 100 \\ M_1 - 166.67 \\ R_2 - 100 \\ 166.67 \end{Bmatrix} = \frac{IE}{1000} \begin{bmatrix} 12 & 60 & -12 & 60 \\ & 400 & -60 & 200 \\ & & 12 & -60 \\ \text{sym} & & & 400 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \end{Bmatrix}$$

After elimination; we have,

$$166.67 = \frac{IE}{1000} \times 400\,\theta_2$$

$$\text{or, } \theta_2 = \frac{416.67}{IE} \text{ rad}$$

Step 6: B.M.D. and S.F.D.

For element 1, element forces

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{IE}{1000} \begin{bmatrix} 12 & 60 & -12 & 60 \\ & 400 & -60 & 200 \\ & & 12 & -60 \\ \text{sym} & & & 400 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{416.67}{IE} \end{Bmatrix} = \begin{Bmatrix} 25 \\ 83.334 \\ -25 \\ 166.67 \end{Bmatrix}$$

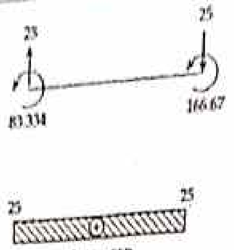


Figure: SFD

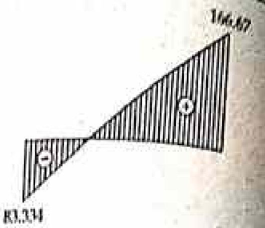
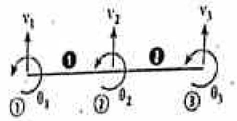


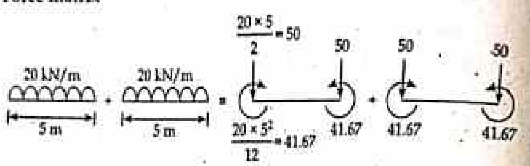
Figure: BMD

ii) **Discretizing into two elements**

Step 1: Discretization



Step 2: Force matrix



$$\{F\} = \begin{Bmatrix} R_1 - 50 \\ M_1 - 41.67 \\ -50-50 \\ 0 \\ R_3 - 50 \\ 41.67 \end{Bmatrix}$$

Step 3: Displacement matrix

$$\{u\} = \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ \theta_2 \\ 0 \\ \theta_3 \end{Bmatrix}$$; after applying boundary conditions $u_1 = \theta_1 = u_3 = 0$.

Step 4: Stiffness matrix

$$[K] = \frac{IE}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym} & & & 4l^2 \end{bmatrix}$$

$$[K_1] = \frac{IE}{125}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 30 & -12 & 30 \\ & 100 & -30 & 50 \\ & & 12 & -30 \\ \text{sym} & & & 100 \end{bmatrix} \end{array}\begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

$$[K_2] = \frac{IE}{125}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 30 & -12 & 30 \\ & 100 & -30 & 50 \\ & & 12 & -30 \\ \text{sym} & & & 100 \end{bmatrix} \end{array}\begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

Assembling; we have,

$$[K] = \frac{IE}{125}\begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ & 100 & -30 & 50 & 0 & 0 \\ & & 24 & 0 & -12 & 30 \\ & & & 200 & -30 & 50 \\ & \text{sym} & & & 12 & -30 \\ & & & & & 100 \end{bmatrix}$$

Step 5: $\{F\} = [K]\{u\}$

or,
$$\begin{Bmatrix} R_1 - 50 \\ M_1 - 41.67 \\ -50-50 \\ 0 \\ R_3 - 50 \\ 41.67 \end{Bmatrix} = \frac{IE}{125}\begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ & 100 & -30 & 50 & 0 & 0 \\ & & 24 & 0 & -12 & 30 \\ & & & 200 & -30 & 50 \\ & \text{sym} & & & 12 & -30 \\ & & & & & 100 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ 0 \\ \theta_3 \end{Bmatrix}$$

After elimination, the reduced matrix is:

$$\begin{Bmatrix} -100 \\ 0 \\ 41.67 \end{Bmatrix} = \frac{IE}{125}\begin{bmatrix} 24 & 0 & 30 \\ 0 & 200 & 50 \\ 30 & 50 & 100 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

or,
$$\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{125}{IE}\begin{bmatrix} 24 & 0 & 30 \\ 0 & 200 & 50 \\ 30 & 50 & 100 \end{bmatrix}^{-1}\begin{Bmatrix} -100 \\ 0 \\ 41.67 \end{Bmatrix} = \frac{125}{IE}\begin{Bmatrix} -8.33 \\ -0.833 \\ 3.33 \end{Bmatrix}$$

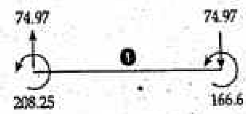
Step 6: S.F.D. and B.M.D.

For element ❶,

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{IE}{125}\begin{bmatrix} 12 & 30 & -12 & 30 \\ & 100 & -30 & 50 \\ & & 12 & -30 \\ \text{sym} & & & 100 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$= \frac{IE}{125}\begin{bmatrix} 12 & 30 & -12 & 30 \\ & 100 & -30 & 50 \\ & & 12 & -30 \\ \text{sym} & & & 100 \end{bmatrix} \times \frac{125}{IE}\begin{Bmatrix} 0 \\ 0 \\ -8.33 \\ -0.833 \end{Bmatrix}$$

$$= \begin{bmatrix} -12 & 30 \\ -30 & 50 \\ 12 & -30 \\ -30 & 100 \end{bmatrix}\begin{Bmatrix} -8.33 \\ -0.833 \end{Bmatrix} = \begin{Bmatrix} 74.97 \\ 208.25 \\ -74.97 \\ 166.6 \end{Bmatrix}$$



NOTE

Program in calculator: $A = -8.33\,X - 0.833\,Y$

For element ❷,

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{IE}{125}\begin{bmatrix} 12 & 30 & -12 & 30 \\ & 100 & -30 & 50 \\ & & 12 & -30 \\ sym & & & 100 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

$$= \frac{IE}{125}\begin{bmatrix} 12 & 30 & -12 & 30 \\ & 100 & -30 & 50 \\ & & 12 & -30 \\ sym & & & 100 \end{bmatrix} \times \frac{125}{IE}\begin{Bmatrix} -8.33 \\ -0.833 \\ 0 \\ 3.33 \end{Bmatrix}$$

$$= \begin{bmatrix} 12 & 30 & 30 \\ 30 & 100 & 50 \\ -12 & -30 & -30 \\ 30 & 50 & 100 \end{bmatrix}\begin{Bmatrix} -8.33 \\ -0.833 \\ 3.33 \end{Bmatrix}$$

$$= \begin{Bmatrix} -25.05 \\ -166.7 \\ 25.05 \\ 41.45 \end{Bmatrix}$$

NOTE

Program in calculator: $A = -8.33X - 0.833Y + 3.33M$

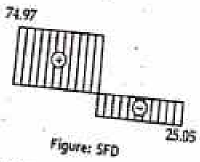


Figure: SFD

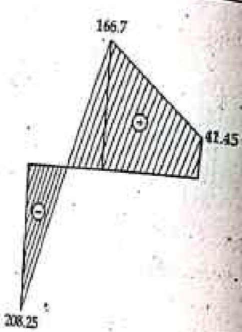


Figure: BMD

Discussion

Exact B.M.D. of the given beam would be like:

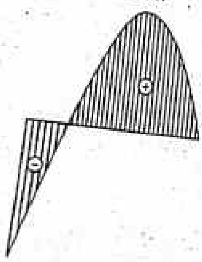
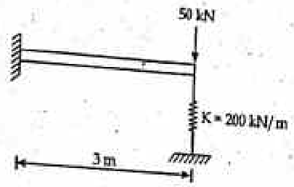


Figure: Exact BMD

As discussed in the beginning, FEM gives approximate solution, not exact solution. The more number of finite element we break the structure into, more the S.F.D. and B.M.D. approaches the exact solution.

EXAMPLE 4.13

Determine the nodal displacement and rotation at node 2 of the given structure. E = 210 GPa, I = 2 × 10^{-4} m^4

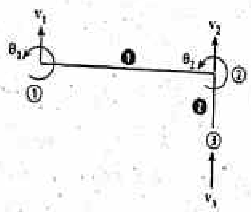


Solution:

Step 1: Unit conversion

$$E = 210 \text{ GPa} = 210 \times 10^6 \text{ kN/m}^2$$

Step 2: Modeling



Step 3: Force matrix

$$[F] = \begin{Bmatrix} R_1 \\ M_1 \\ -50 \\ 0 \\ R_3 \end{Bmatrix}$$

Step 4: Displacement matrix

$$\{u\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ 0 \end{Bmatrix}$$; after applying boundary condition.

Step 5: Stiffness matrix

$$[K_1] = \frac{IE}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ Sym & & & 4L^2 \end{bmatrix} = \frac{210 \times 10^6 \times 2 \times 10^{-4}}{3^3}\begin{bmatrix} 12 & 18 & -12 & 18 \\ & 36 & -18 & 18 \\ & & 12 & -18 \\ Sym & & & 36 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 18666.67 & 28000 & -18666.67 & 28000 \\ & 56000 & -28000 & 28000 \\ & & 18666.67 & -28000 \\ \text{sym} & & & 56000 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

$$[K_2] = K\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 200\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} v_2 & v_3 \end{array} \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \end{array} \begin{array}{c} v_2 \\ v_3 \end{array}$$

$$[K] = [K_1] + [K_2] = \begin{bmatrix} 18666.67 & 28000 & -18666.67 & 28000 & 0 \\ & 56000 & -28000 & 28000 & 0 \\ & & \boxed{18866.67} & -28000 & \boxed{-200} \\ & & & 56000 & 0 \\ \text{sym} & & & & \boxed{200} \end{bmatrix}$$

Step 6: [F] = [K][u]

$$\begin{Bmatrix} \cancel{R_1} \\ \cancel{M_1} \\ -50 \\ 0 \\ \cancel{R_3} \end{Bmatrix} = \begin{bmatrix} \cancel{18666.67} & \cancel{28000} & \cancel{-18666.67} & \cancel{28000} & \cancel{0} \\ & \cancel{56000} & \cancel{-28000} & \cancel{28000} & \cancel{0} \\ & & 18866.67 & -28000 & -200 \\ & & & 56000 & 0 \\ \text{sym} & & & & \cancel{200} \end{bmatrix} \begin{Bmatrix} \cancel{0} \\ \cancel{0} \\ v_2 \\ Q_2 \\ \cancel{0} \end{Bmatrix}$$

The reduced matrix is:

$$\begin{Bmatrix} -50 \\ 0 \end{Bmatrix} = \begin{bmatrix} 18866.67 & -28000 \\ -28000 & 56000 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}$$

or, $$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 18866.67 & -28000 \\ -28000 & 56000 \end{bmatrix}^{-1} \begin{Bmatrix} -50 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.0102 \\ -0.005 \end{Bmatrix}$$

Therefore, displacement and rotation at node 2 are −0.0102 m and −0.005 radian.

4.6.4 Use of symmetry in beam

Like in truss, reflective symmetry in beam element can be advantageously used to reduce calculations.

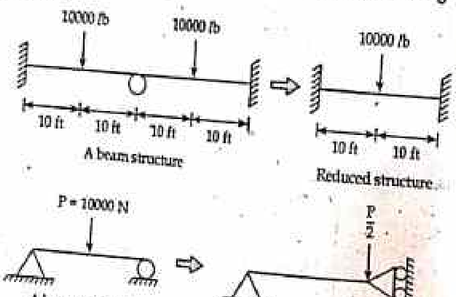
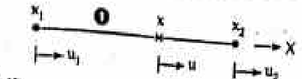


Figure: Use of symmetry in beam

4.7 INTERPOLATION FUNCTIONS

The function, which relates the field variable at any point within the element to the field variables of nodal points, is called shape function/interpolation function/approximation function.



$$u = N_1u_1 + N_2u_2$$

where, N_1 and N_2 are the shape functions.

4.7.1 Polynomial functions

Polynomials are generally used to represent displacement because:

i) They are easy to handle mathematically.
ii) Polynomials can approximate the function reasonably well. If the function is highly non-linear, higher order polynomials can be used to represent it.

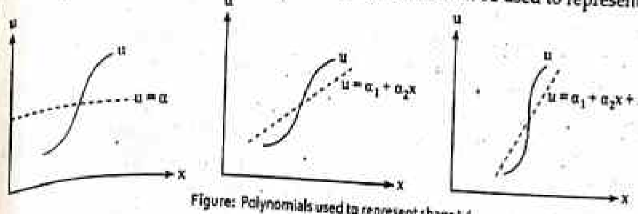


Figure: Polynomials used to represent shape 'u'

For 1-D elements,

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots\dots\dots + \alpha_{n+1}x^n$$

where, x represents position,

and, $\alpha_1, \alpha_2, \alpha_3, \dots\dots\dots, \alpha_{n+1}$ are generalized co-ordinates.

In one dimensional n^{th} order polynomials, there are $m = n + 1$ terms.

For 2-D elements,

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \dots\dots\dots + \alpha_m y^n$$

$$v(x, y) = \alpha_{m+1} + \alpha_{m+2} x + \dots\dots\dots + \alpha_{2m} y^n$$

In two dimensional n order polynomial, there are $m = \frac{(n+1)(n+2)}{2}$ terms.

Pascal triangle is very much helpful to remember the terms of polynomials of various degrees in two co-ordinates as shown in the figure.

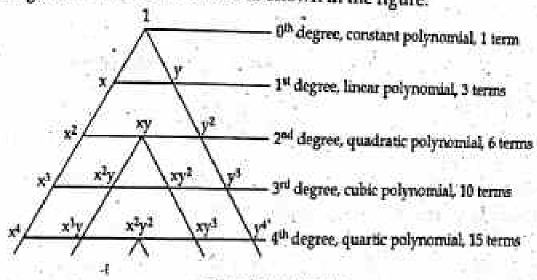


Figure: Pascal triangle

4.7.2 Convergence requirements of shape functions

I) Compatibility requirement

The displacement models must be continuous within the element and between the adjacent elements.

ii) Rigid body displacement

The displacement models should be such that all points of the element experience equal relative displacement.

iii) Constant strain state

The displacement models should be such that all points in the element experience equal strain.

iv) Geometric isotropy / spatial isotropy / geometric invariance

The displacement models described by the shape function should be such that it is independent of orientation of local co-ordinates.

4.7.3 Derivation of shape functions using polynomials

This method of deriving shape function is also called generalized co-ordinate approach. The major steps include:

Step 1:

Select a polynomial with number of constants equal to nodal degree of freedom of the elements.

Step 2:

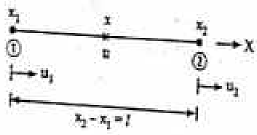
Using nodal values, equations are formed equal in the number of constants and those equations are solved to get the value of constants.

Step 3:

Shape functions are identified.

EXAMPLE 4.14

Evaluate the shape function of two noded bar / truss element using generalized co-ordinate approach.



Solution:

Nodal degree of freedom is two. Therefore, we choose a polynomial with two constants.

i.e., $u = \alpha_1 + \alpha_2 x$

In matrix form,

$$u = [1 \quad x]\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix} \qquad [4.7.1]$$

Applying boundary conditions; we have,

At $x = x_1$,

$$u = u_1$$

or, $u_1 = \alpha_1 + \alpha_2 x_1$

At $x = x_2$,

$$u = u_2$$

or, $u_2 = \alpha_1 + \alpha_2 x_2$

$$\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{bmatrix}1 & x_1\\ 1 & x_2\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix}$$

$\therefore$
$$\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix} = \begin{bmatrix}1 & x_1\\ 1 & x_2\end{bmatrix}^{-1}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \frac{1}{x_2 - x_1}\begin{bmatrix}x_2 & -x_1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

or,
$$\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix} = \frac{1}{l}\begin{bmatrix}x_2 & -x_1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} \qquad [4.7.2]$$

$\therefore$ Substituting the value of $\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix}$ into equation [4.7.1]; we have,

$$u = [1 \quad x] \times \frac{1}{l}\begin{bmatrix}x_2 & -x_1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \frac{1}{l}[x_2 - x \quad -x_1 + x]\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$= \begin{bmatrix}\frac{x_2 - x}{l} & \frac{x - x_1}{l}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = [N_1 \quad N_2]\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = N_1u_1 + N_2u_2$$

where, $N_1 = \frac{x_2 - x}{l}$ and $N_2 = \frac{x - x_1}{l}$ are the shape functions.

The value of N_1 and N_2 varies linearly as shown in the figure below.

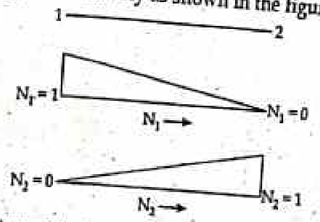
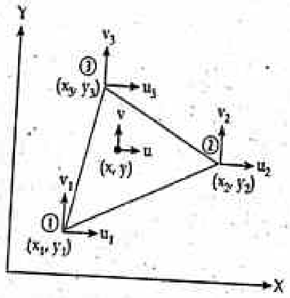


Figure: Variation of shape function in two-noded bar

EXAMPLE 4.15

Evaluate the shape function of a constant strain triangle (CST) using polynomial function.



Solution:

There are 6 nodal degree of freedom and so there should be 6 constants in the displacement polynomials. The selected polynomials are:

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

At $x = x_1$,

$$u = u_1$$

At $x = x_2$,

$$u = u_2$$

At $x = x_3$,

$$u = u_3$$

so,

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

and

$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

i.e.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

or,

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

where,

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{Base} \times \text{Height}$$

or,

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

where, $a_1 = x_2y_3 - x_3y_2 \qquad b_1 = y_2 - y_3 \qquad c_1 = x_3 - x_2$

$a_2 = x_3y_1 - x_1y_3 \qquad b_2 = y_3 - y_1 \qquad c_2 = x_1 - x_3$

$a_3 = x_1y_2 - x_2y_1 \qquad b_3 = y_1 - y_2 \qquad c_3 = x_2 - x_1$

Now,

$$u = [1 \quad x \quad y] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = [1 \quad x \quad y] \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \left[\frac{a_1 + b_1x + c_1y}{2A} \quad \frac{a_2 + b_2x + c_2y}{2A} \quad \frac{a_3 + b_3x + c_3y}{2A} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

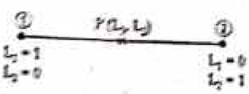
where, N_1, N_2 and N_3 are the shape functions.

Similarly, we get,

$$v = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

EXAMPLE 4.16

Determine the interpolation function for a two noded bar element using natural co-ordinate system.



Solution:

$$u = \alpha_1 L_1 + \alpha_2 L_2 = [L_1 \quad L_2] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

Since, $L_1 = 1$ and $L_2 = 0$ at node 1 and, $L_1 = 0$ and $L_2 = 1$ at node 2.

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$\therefore$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

or,

$$u = [L_1 \quad L_2] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = [L_1 \quad L_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [L_1 \quad L_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$\therefore$ Comparing with $u = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$, shape function are: $N_1 = L_1$ and $N_2 = L_2$.

EXAMPLE 4.17

Derive the shape function for a two noded bar element taking natural co-ordinate ξ varying from −1 to 1.

Solution:

The selected polynomial is:

$$u = \alpha_1 + \alpha_2 \xi = [\alpha_1 \quad \alpha_2] \begin{Bmatrix} 1 \\ \xi \end{Bmatrix}$$

Boundary conditions at $\xi = -1$;

$$u = u_1$$

Boundary conditions at $\xi = +1$;

$$u = u_2$$

i.e.

$$u_1 = \alpha_1 - \alpha_2$$

and

$$u_2 = \alpha_1 + \alpha_2$$

In the matrix form; we have,

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$\therefore$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{1+1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$\therefore$

$$u = [1 \quad \xi] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = [1 \quad \xi] \times \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

where, $N_1 = \dfrac{1-\xi}{2}$ and $N_2 = \dfrac{1+\xi}{2}$ are the shape functions.

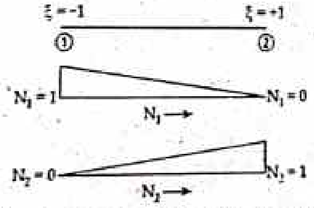
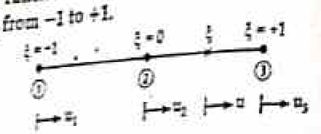


Figure: Variation of shape function in two-noded bar element in natural co-ordinate system

EXAMPLE 4.18

Derive the shape function for three noded bar element taking natural co-ordinate ξ varying from -1 to $+1$.



Solution:

The three noded elements possesses 3 nodal D.O.F. so we choose a polynomial with three constants

i.e., $u(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2$

At $\xi = -1$,

$$u(\xi) = u_1$$

or, $u_1 = \alpha_1 - \alpha_2 + \alpha_3$

At $\xi = 0$,

$$u(\xi) = u_2$$

or, $u_2 = \alpha_1$

At $\xi = +1$,

$$u(\xi) = u_3$$

or, $u_3 = \alpha_1 + \alpha_2 + \alpha_3$

$$\therefore \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\text{or, } \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 = [1 \quad \xi \quad \xi^2]\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = [1 \quad \xi \quad \xi^2] \times \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \left[\frac{-\xi+\xi^2}{2} \quad 1-\xi^2 \quad \frac{\xi+\xi^2}{2}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \left[\frac{\xi(\xi-1)}{2} \quad 1-\xi^2 \quad \frac{\xi(\xi+1)}{2}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

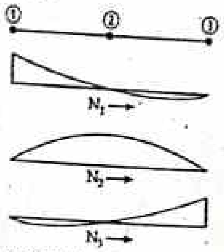
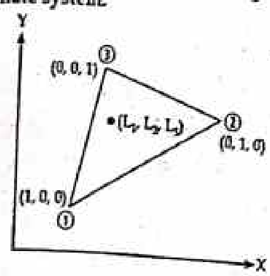


Figure: Variation of shape function in three noded bar element

where, $N_1 = \frac{\xi(\xi-1)}{2}$, $N_2 = 1 - \xi^2$ and $N_3 = \frac{\xi(\xi+1)}{2}$ are the shape functions.

EXAMPLE 4.19

Find the shape function for a constant strain triangular (CST) element in the term of natural co-ordinate system.



Solution:

The displacement function be:

$$u = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 = [L_1 \quad L_2 \quad L_3]\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

The boundary conditions at ①; (1, 0, 0)

$$u = u_1$$

or, $u_1 = \alpha_1$

The boundary conditions at ②; (0, 1, 0)

$$u = u_2$$

or, $u_2 = \alpha_2$

The boundary conditions at ③; (0, 0, 1)

$$u = u_3$$

or, $u_3 = \alpha_3$

In the matrix form; we have,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\text{or, } \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\therefore \quad u = [L_1 \quad L_2 \quad L_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [L_1 \quad L_2 \quad L_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

where, $N_1 = L_1$, $N_2 = L_2$ and $N_3 = L_3$ are the shape functions.

Similarly; we get,

$$v = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

4.7.4 Lagrange polynomial

If only continuity of basic unknown (displacement in the stress analysis) is to be satisfied, Lagrange polynomials can be used to derive shape functions.

Lagrange polynomial in one dimension is defined by;

$$L_K(x) = \pi_{m=1, m\neq K}^{n} \frac{x - x_m}{x_K - x_m}$$

For example: if n = 5 and K = 2

$$L_2(x) = \frac{(x - x_1)(x - x_3)(x - x_4)(x - x_5)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)}$$

4.7.4.1 Shape function of 1-D elements using Lagrange polynomial

Lagrange polynomials can be directly used as interpolation functions.

EXAMPLE 4.20

Find shape functions using Lagrange polynomial for:

i) **two noded bar element**
ii) **three noded bar element and**
iii) **five noded bar element**

Solution:

i) **Two noded bar element**

$$N_1 = L_1(x) = \frac{x - x_2}{x_1 - x_2} = \frac{x_2 - x}{x_2 - x_1}$$

$$N_2 = L_2(x) = \frac{x - x_1}{x_2 - x_1}$$

You can note that result obtained is same as obtained in the example 4.14.

ii) **Three noded bar element**

$$N_1 = L_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

$$N_2 = L_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$N_3 = L_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

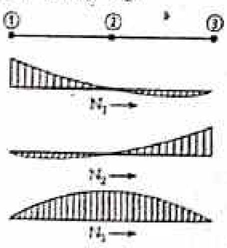


Figure: Variation of shape function in three noded bar element

iii) **Five noded bar element**

$$N_1 = L_1(x) = \frac{(x - x_2)(x - x_3)(x - x_4)(x - x_5)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)}$$

$$N_2 = L_2(x) = \frac{(x - x_1)(x - x_3)(x - x_4)(x - x_5)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)}$$

$$N_3 = L_3(x) = \frac{(x - x_1)(x - x_2)(x - x_4)(x - x_5)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5)}$$

$$N_4 = L_4(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_5)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)(x_4 - x_5)}$$

$$N_5 = L_2(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}$$

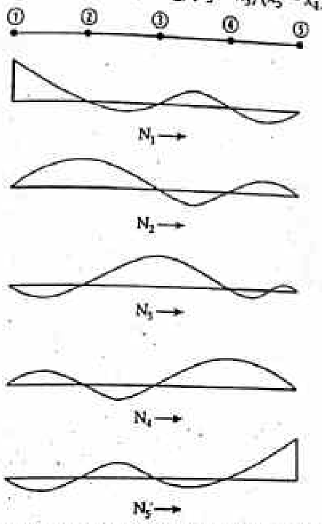


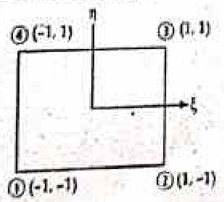
Figure: Variation of shape function in five noded bar element

4.7.4.2 Shape function of 2-D elements using Lagrange element

Though Lagrange interpolation functions exist for one dimension only, the concept can be extended for 2-D and 3-D. For 2-D, the function for a point would be product of two individual Lagrange polynomial in two orthogonal directions.

EXAMPLE 4.21

Write shape function for four noded rectangular element using Lagrange function.



[illegible] and ①-④. In line ①-②, [illegible]

[illegible] Lagrange polynomial is:

$$L_1(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{\xi - 1}{-1-1} = \frac{-1}{2}(\xi - 1)$$

[illegible] and the respective Lagrange polynomial is:

$$L_1(\eta) = \frac{\eta - \eta_4}{\eta_1 - \eta_4} = \frac{\eta - 1}{-1-1} = \frac{-1}{2}(\eta - 1)$$

The shape function for node ① would be:

$$N_1 = L_1(\xi)L_1(\eta) = \frac{-1}{2}(\xi - 1) \times \frac{-1}{2}(\eta - 1) = \frac{1}{4}(\xi - 1)(\eta - 1)$$

$$= \frac{1}{4}(1 - \xi)(1 - \eta)$$

Similarly,

$$N_2 = L_2(\xi)L_2(\eta) = \frac{\xi - \xi_1}{\xi_2 - \xi_1}\frac{\eta - \eta_3}{\eta_2 - \eta_3} = \frac{\xi - (-1)}{1 - (-1)}\frac{\eta - 1}{-1-1} = \frac{\xi + 1}{2}\frac{\eta - 1}{-2}$$

$$= \frac{1}{4}(1 + \xi)(1 - \eta)$$

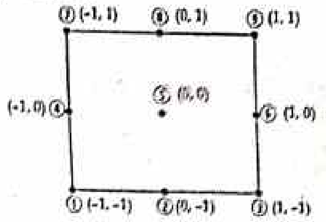
Likewise, we get,

$$N_3 = \frac{(1 + \xi)(1 + \eta)}{4}$$

and $\quad N_4 = \frac{(1 - \xi)(1 + \eta)}{4}$

EXAMPLE 4.22

Write the shape function for nine noded rectangular element as shown in the figure using Lagrange functions.



Solution:

$$N_1 = L_1(\xi)L_1(\eta) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}\frac{(\eta - \eta_4)(\eta - \eta_7)}{(\eta_1 - \eta_4)(\eta_1 - \eta_7)}$$

$$= \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)}\frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)} = \frac{\xi(\xi - 1)\,\eta(\eta - 1)}{4}$$

$$N_2 = L_2(\xi)L_2(\eta) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)}\frac{(\eta - \eta_5)(\eta - \eta_8)}{(\eta_2 - \eta_5)(\eta_2 - \eta_8)}$$

$$= \frac{(\xi + 1)(\xi - 1)}{(0 + 1)(0 - 1)}\frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)} = \frac{(\xi + 1)(\xi - 1)\,\eta(\eta - 1)}{-2}$$

Likewise, we get,

$$N_3 = \frac{(\xi + 1)\xi\,\eta(\eta - 1)}{4}$$

$$N_4 = \frac{\xi(\xi - 1)(\eta + 1)(\eta - 1)}{-2}$$

$$N_5 = \frac{(\xi + 1)(\xi - 1)(\eta + 1)(\eta - 1)}{1}$$

$$N_6 = \frac{(\xi + 1)\xi(\eta + 1)(\eta - 1)}{-2}$$

$$N_7 = \frac{\xi(\xi - 1)(\eta + 1)\eta}{4}$$

$$N_8 = \frac{(\xi + 1)(\xi - 1)(\eta + 1)\eta}{-2}$$

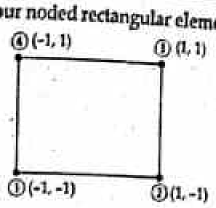
$$N_9 = \frac{(\xi + 1)\xi(\eta + 1)\eta}{4}$$

4.7.5 Shape function for serendipity family elements

Derivation of shape functions for the serendipity family elements are illustrated viz examples.

EXAMPLE 4.23

Derive shape functions for four noded rectangular element using serendipity concept.



Solution:

The value of N_1 should be 1 at node ① and 0 at other nodes *i.e.,* ②, ③ and ④ nodes. These conditions can be expressed as:

$N_1 = 1$ at $\xi = -1$ and $\eta = -1$

$N_1 = 0$ along line ②-③,

i.e., line $\xi = 1$

or, $1 - \xi = 0$

$N_1 = 0$ along line ③-④,

i.e., line $\eta = 1$

or, $1 - \eta = 0$

Let $N_1 = C(1 - \xi)(1 - \eta)$; where, C is an arbitrary constant.

$\because$ $N_1 = 1$ at $\xi = -1$ and $\eta = -1$,

$1 = C\{1 - (-1)\}\{1 - (-1)\}$

or, $1 = C \times 2 \times 2$

$C = \frac{1}{4}$

$\therefore \quad N_1 = \frac{1}{4}(1-\xi)(1-\eta)$

Similarly; we get,

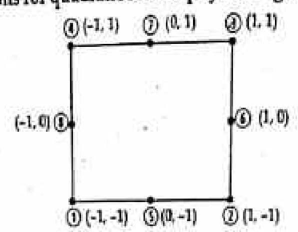
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4}$$

$$N_4 = \frac{(1-\xi)(1-\eta)}{4}$$

EXAMPLE 4.24

Find the shape functions for quadratic serendipity rectangular element.[2072 Magh]



Solution:

The value of N_1 should be 1 at node ① and 0 at all the other nodes. This condition can be expressed as:

$N_1 = 1$ at $\xi = -1$ and $\eta = -1$

$N_1 = 0$ along line ②-③,

i.e., line $\xi = 1$

or, $1 - \xi = 0$

$N_1 = 0$ along line ③-④,

i.e., line $\eta = 1$

or, $1 - \eta = 0$

$N_1 = 0$ along line ⑤-⑧,

i.e., Line $\xi + \eta + 1 = 0$

Let, $N_1 = C(1-\xi)(1-\eta)(1+\xi+\eta)$; where, C is an arbitrary constant.

Since, at $\xi = -1$ and $\eta = -1$, $N_1 = 1$

$1 = C(2)(2)(-1)$

$\therefore \quad C = -\frac{1}{4}$

$\therefore \quad N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$

Similarly; we can find,

$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

N_5 should fulfill the following conditions:

$N_5 = 1$ at $\xi = 0$ and $\eta = -1$

$N_5 = 0$ along line ②-③,

i.e., line $\xi = 1$

or, $1 - \xi = 0$

$N_5 = 0$ along line ③-④,

i.e., line $\eta = 1$

or, $1 - \eta = 0$

$N_5 = 0$ along line ①-④,

i.e., $\xi = -1$

or, $1 + \xi = 0$

Let, $N_5 = C(1-\xi)(1-\eta)(1+\xi)$

Since, $N_5 = 1$ at $(0, -1)$,

$1 = C(1)(2)(1)$

or, $C = \frac{1}{2}$

$\therefore \quad N_5 = \frac{1}{2}(1-\xi)(1-\eta)(1+\xi) = \frac{1}{2}(1-\xi^2)(1-\eta)$

Similarly; we can get,

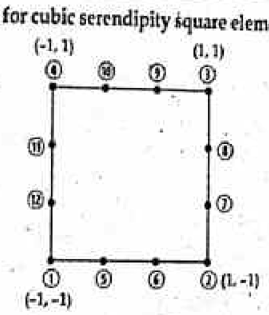
$$N_6 = \frac{(1+\xi)(1-\eta^2)}{2}$$

$$N_7 = \frac{(1-\xi^2)(1+\eta)}{2}$$

$$N_8 = \frac{(1-\xi)(1-\eta^2)}{2}$$

EXAMPLE 4.25

Find the shape functions for cubic serendipity square element as shown in the figure.



Solution:

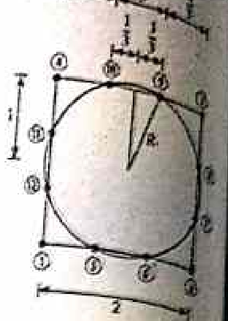
Draw a circle with its centre at the centre of the serendipity element.

From the figure; we have,

$$R = \sqrt{\left(\frac{1}{3}\right)^2 + 1^2} = \sqrt{\frac{10}{9}}$$

Hence, the equation of the circle is:

$$\xi^2 + \eta^2 = \frac{10}{9}$$



N_1 should fulfill the following conditions:

$N_1 = 0$ along line $1 - \xi = 0$

$N_1 = 0$ along line $1 - \eta = 0$

$N_1 = 0$ along circle $\xi^2 + \eta^2 - \frac{10}{9} = 0$

$N_1 = 1$ at point $(-1, -1)$

Let, $N_1 = C(1-\xi)(1-\eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$

Since, $N_1 = 1$ at $(-1, -1)$

$$1 = C(1-(-1))(1-(-1))\left\{(-1)^2 + (-1)^2 - \frac{10}{9}\right\}$$

or, $C = \frac{9}{32}$

$\therefore$ $N_1 = \frac{9}{32}(1-\xi)(1-\eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$

Similarly, we get,

$$N_2 = \frac{9}{32}(1+\xi)(1-\eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$$

$$N_3 = \frac{9}{32}(1+\xi)(1+\eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$$

$$N_4 = \frac{9}{32}(1-\xi)(1+\eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$$

N_5 should fulfill the following conditions:

$N_5 = 0$ along line $1 - \xi = 0$

$N_5 = 0$ along line $1 - \eta = 0$

$N_5 = 0$ along line $1 + \xi = 0$

$N_5 = 0$ along line $1 - 3\xi = 0$

and, $N_5 = 1$ at $\left(-\frac{1}{3}, -1\right)$

Let, $N_5 = C(1-\xi)(1-\eta)(1+\xi)(1-3\xi)$

Substituting $N_5 = 1$ at $(\xi, \eta) = \left(-\frac{1}{3}, -1\right)$, we get,

$$1 = C \times \frac{4}{3} \times 2 \times \frac{2}{3} \times 2$$

$\therefore$ $C = \frac{9}{32}$

$\therefore$ $N_5 = \frac{9}{32}(1-\xi)(1-\eta)(1+\xi)(1-3\xi) = \frac{9}{32}(1-\xi^2)(1-\eta)(1-3\xi)$

Similarly, we get,

$$N_6 = \frac{9}{32}(1-\xi^2)(1-\eta)(1+3\xi)$$

$$N_7 = \frac{9}{32}(1-\eta^2)(1+\xi)(1-3\eta)$$

$$N_8 = \frac{9}{32}(1-\xi^2)(1+\eta)(1+3\xi)$$

$$N_9 = \frac{9}{32}(1-\xi^2)(1+\eta)(1+3\xi)$$

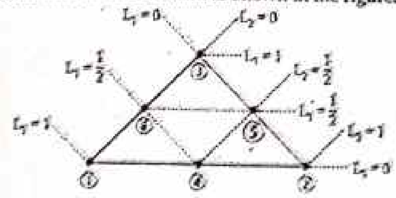
$$N_{10} = \frac{9}{32}(1-\xi^2)(1+\eta)(1-3\xi)$$

$$N_{11} = \frac{9}{32}(1-\eta^2)(1-\xi)(1+3\eta)$$

$$N_{12} = \frac{9}{32}(1-\eta^2)(1-\xi)(1-3\eta)$$

EXAMPLE 4.28

Derive the shape function for the element as shown in the figure. **[2072 Ashwin]**



Solution:

N_1 fulfills the following condition:

$N_1 = 0$ along $\frac{1}{2} - L_1 = 0$

$N_1 = 0$ along $L_1 = 0$

$N_1 = 1$ at $L_1 = 1$

Let, $N_1 = C\left(\frac{1}{2} - L_1\right)L_1$

Since, $N_1 = 1$ at $L_1 = 1$,

$$1 = C\left(-\frac{1}{2}\right)(1)$$

or, $C = -2$

$\therefore$ $N_1 = 2\left(L_1 - \frac{1}{2}\right)L_1$

$$N_2 = 2\left(L_2 - \frac{1}{2}\right)L_2$$

and, $N_3 = 2\left(L_3 - \frac{1}{2}\right)L_3$

N_4 should fulfill the following conditions:

$N_4 = 0$ along $L_1 = 0$

$N_4 = 0$ along $L_2 = 0$

$N_4 = 1$ at $L_1 = \frac{1}{2}, L_2 = \frac{1}{2}$

Let $N_4 = CL_1L_2$

Since, $N_4 = 1$ at $L_1 = \frac{1}{2}, L_2 = \frac{1}{2}$

$$1 = C \times \frac{1}{2} \times \frac{1}{2}$$

$$C = 4$$

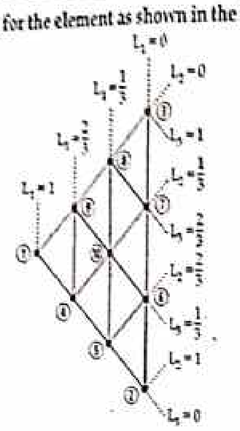
$\therefore$ $N_4 = 4L_1L_2$

$N_5 = 4L_2L_3$

and, $N_6 = 4L_1L_3$

EXAMPLE 4.27

Derive the shape function for the element as shown in the figure. **[2070 Bhadra]**



Solution:

N_1 fulfills the following condition:

$$N_1 = 1 \text{ at } L_1 = 0$$

$$N_1 = 0 \text{ at } L_1 - \frac{2}{3} = 0$$

$$N_1 = 0 \text{ along } L_1 - \frac{1}{3} = 0$$

$$N_1 = 0 \text{ along } L_1 = 0$$

Let, $N_1 = C\left(L_1 - \frac{2}{3}\right)\left(L_1 - \frac{1}{3}\right)L_1$

Since, $N_1 = 1$ at $L_1 = 1$,

$$1 = C\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(1)$$

or, $C = \frac{9}{2}$

$\therefore$ $N_1 = \frac{9}{2}\left(L_1 - \frac{2}{3}\right)\left(L_1 - \frac{1}{3}\right)L_1$

Similarly; we get,

$$N_2 = \frac{9}{2}\left(L_2 - \frac{2}{3}\right)\left(L_2 - \frac{1}{3}\right)L_2$$

and, $N_3 = \frac{9}{2}\left(L_3 - \frac{2}{3}\right)\left(L_3 - \frac{1}{3}\right)L_3$

N_4 fulfills the following conditions:

$$N_4 = 0 \text{ along } L_2 = 0$$

$$N_4 = 0 \text{ along } L_1 = \frac{1}{3}$$

$$N_4 = 0 \text{ along } L_1 = 0$$

$N_4 = 1$ at $L_1 = \frac{2}{3}, L_2 = \frac{1}{3}$

Let, $N_4 = C(L_2)\left(L_1 - \frac{1}{3}\right)(L_1)$

Since, $N_4 = 1$ at $L_1 = \frac{2}{3}$ and $L_2 = \frac{1}{3}$,

$$1 = C\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$$

$\therefore$ $C = \frac{27}{2}$

$\therefore$ $N_4 = \frac{27}{2}L_1L_2\left(L_1 - \frac{1}{3}\right)$

Similarly, we get,

$$N_5 = \frac{27}{2}L_1L_2\left(L_2 - \frac{1}{3}\right)$$

$$N_6 = \frac{27}{2}L_2L_3\left(L_2 - \frac{1}{3}\right)$$

$$N_7 = \frac{27}{2}L_2L_3\left(L_3 - \frac{1}{3}\right)$$

$$N_8 = \frac{27}{2}L_1L_3\left(L_3 - \frac{1}{3}\right)$$

$$N_9 = \frac{27}{2}L_1L_3\left(L_1 - \frac{1}{3}\right)$$

N_{10} fulfills the conditions:

$$N_{10} = 0 \text{ at } L_1 = 0$$

$$N_{10} = 0 \text{ at } L_2 = 0$$

$$N_{10} = 0 \text{ at } L_3 = 0$$

$$N_{10} = 1 \text{ at } L_1 = \frac{1}{3}, L_2 = \frac{1}{3}, L_3 = \frac{1}{3}$$

Let, $N_{10} = CL_1L_2L_3$

$\therefore$ $1 = C \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$

or, $C = 27$

$\therefore$ $N_{10} = 27L_1L_2L_3$

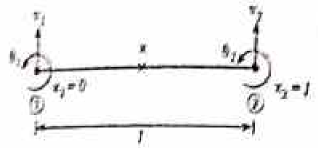
4.7.6 Hermite shape functions

When not only the function but also its derivative are interpolated for a point from the nodal values, the interpolation functions are termed as Hermite shape functions. Both the Hermite shape function and its derivative are continuous throughout the structure.

In case of beam analysis, not only the transverse displacement but also the slope/rotations are evaluated which are the derivatives of transverse displacements. Hence, the shape functions of beam element are Hermite shape functions.

4.7.6.1 Shape function of beam in Cartesian system

Consider a beam element as shown in the figure.



Polynomial describing the displacement is:

$$v = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \qquad [4.7.1]$$

and, $$\theta = \frac{\partial v}{\partial x} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2 \qquad [4.7.2]$$

Assigning $x_1 = 0$ and $x_2 = l$;

$$v_1 = \alpha_1$$
$$\theta_1 = \alpha_2$$
$$v_2 = \alpha_1 + \alpha_2 l + \alpha_3 l^2 + \alpha_4 l^3$$
$$\theta_2 = \alpha_2 + 2\alpha_3 l + 3\alpha_4 l^2$$

In the matrix form; we have,

$$\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{3l^4 - 2l^4} \begin{bmatrix} l^4 & 0 & -3l^2 & 2l \\ 0 & l^4 & -2l^3 & l^2 \\ 0 & 0 & 3l^2 & -2l \\ 0 & 0 & -l^3 & l^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l^2} & -\frac{2}{l} & \frac{3}{l^2} & -\frac{1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & -\frac{2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

Since, $v = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$

$$= [1 \quad x \quad x^2 \quad x^3] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = [1 \quad x \quad x^2 \quad x^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l^2} & -\frac{2}{l} & \frac{3}{l^2} & -\frac{1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & -\frac{2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$= \left[\left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) \left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right) \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right) \right] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$= [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

where,
$$\left.\begin{aligned} N_1 &= 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \\ N_2 &= x - \frac{2x^2}{l} + \frac{x^3}{l^2} \\ N_3 &= \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \\ N_4 &= -\frac{x^2}{l} + \frac{x^3}{l^2} \end{aligned}\right\} \qquad [4.7.3]$$

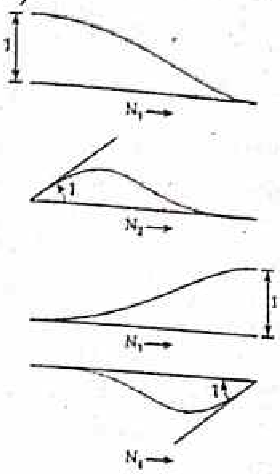


Figure: Variation of shape functions in a beam element

Deriving the shape functions in alternative way

Polynomial describing the displacement is:

$$v = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

and, $$\theta = \frac{\partial v}{\partial x} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2$$

Substituting the condition that at $x = x_1 = 0$, $v = v_1$ and $\theta = \theta_1$; we have,

$$v_1 = \alpha_1$$

or, $$\alpha_1 = v_1 \qquad [4.7.4]$$

$$\theta_1 = \alpha_2$$

or, $$\alpha_2 = \theta_1 \qquad [4.7.5]$$

At $x = x_2 = l$, $v = v_2$ and $\theta = \theta_2$;

$$v_2 = \alpha_1 + \alpha_2 l + \alpha_3 l^2 + \alpha_4 l^3$$

or, $$v_2 = v_1 + \theta_1 l + \alpha_3 l^2 + \alpha_4 l^3$$

$$\therefore \quad \alpha_3 = \frac{v_2 - v_1 - \theta_1 l - \alpha_4 l^3}{l^2} \qquad [4.7.6]$$

and, $\theta_2 = \alpha_2 + 2\alpha_3 l + 3\alpha_4 l^2$

or, $\theta_2 = \theta_1 + 2\left(\frac{v_2 - v_1 - \theta_1 l - \alpha_4 l^3}{l^2}\right) l + 3\alpha_4 l^2$

$$\therefore \quad \alpha_4 = \frac{2}{l^3}(v_1 - v_2) + \frac{\theta_1 + \theta_2}{l^2} \qquad [4.7.7]$$

Substituting this value into equation [4.7.6]; we have,

$$\alpha_3 = \frac{3}{l^2}(v_2 - v_1) - \frac{(2\theta_1 + \theta_2)}{l} \qquad [4.7.8]$$

Substituting the values of α_1, α_2, α_3 and α_4 into equation [4.7.1]; we have,

$$v(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

$$= v_1 + \theta_1 x + \left\{\frac{3}{l^2}(v_2 - v_1) - \frac{(2\theta_1 + \theta_2)}{l}\right\} x^2 + \left\{\frac{2}{l^3}(v_1 - v_2) + \frac{\theta_1 + \theta_2}{l^2}\right\} x^3$$

$$= \left(1 - \frac{3}{l^2}x^2 + \frac{2}{l^3}x^3\right) v_1 + \left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right)\theta_1 + \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) v_2 + \left(\frac{x^3}{l^2} - \frac{x^2}{l}\right)\theta_2$$

$$= N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2$$

4.7.6.2 Shape function of beam in natural co-ordinate system ranging from 0 to +1

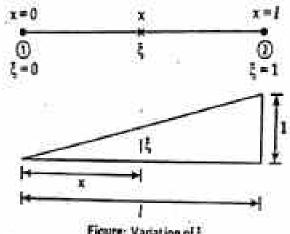


Figure: Variation of ξ

By similarity of triangles; we have,

$$\frac{x}{l} = \frac{\xi}{1}$$

Substituting this value into equation [4.7.3] to get,

$$\left.\begin{aligned} N_1 &= 1 - 3\xi^2 + 2\xi^3 \\ N_2 &= l\xi(\xi - 1)^2 \\ N_3 &= \xi^2(3 - 2\xi) \\ N_4 &= l\xi^2(\xi - 1) \end{aligned}\right\} \qquad [4.7.9]$$

4.7.6.3 Shape function of beam in natural co-ordinate system ranging from −1 to +1

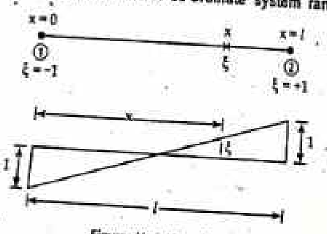


Figure: Variation of ξ

By similarity of triangles; we have,

$$\frac{\xi}{1} = \frac{x - \frac{l}{2}}{\frac{l}{2}} = \frac{2x}{l} - 1$$

or, $\frac{x}{l} = \frac{(\xi + 1)}{2}$

Substituting this value into equation [4.X3] to get,

$$\left.\begin{aligned} N_1 &= \frac{2 - 3\xi + \xi^3}{4} \\ N_2 &= (1 - \xi - \xi^2 + \xi^3)\frac{l}{8} \\ N_3 &= \frac{1}{4}(2 + 3\xi - \xi^3) \\ N_4 &= \frac{l}{8}(-1 - \xi + \xi^2 + \xi^3) \end{aligned}\right\} \qquad [4.7.10]$$

For the natural co-ordinate ξ ranging from −1 to +1, bending moment and shear forces can be directly calculated using the following formulae.

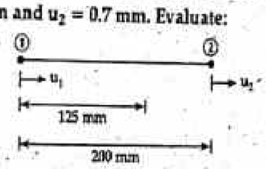
$$M = \frac{EI}{l_e^3}\begin{bmatrix} 6\xi & -(1-3\xi)l_e & -6\xi & (1+3\xi)l_e \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$v = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

EXAMPLE 4.28

Consider a bar element of 200 mm length, c/s area of 2500 mm^2 and E = 2 × 10^5 N/mm^2. Nodal displacements be u_1 = 0.5 mm and u_2 = 0.7 mm. Evaluate:

i) **displacement at point P at 125 mm from first node.**
ii) **strain and stress in the element.**
iii) **element stiffness matrix.**
iv) **strain energy in the element.**



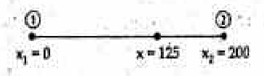
Solution:

We have,

l = 200 mm

A = 2500 mm^2

E = 2 × 10^5 N/mm^2



i) Shape functions (in Cartesian form)

$$N_1 = \frac{x_2 - x}{l} = \frac{200 - 125}{200} = 0.375$$

$$N_2 = \frac{x - x_1}{l} = \frac{125 - 0}{200} = 0.625$$

Shape function (in natural co-ordinate system)

From equation [4.3.4], we have,

$$\frac{l}{2}(1+\xi) = x - x_1$$

or, $\frac{200}{2}(1+\xi) = 125 - 0$

$\therefore$ $\xi = 0.25$

$$N_1 = \frac{1-\xi}{2} = \frac{1-0.25}{2} = 0.375$$

$$N_2 = \frac{1+\xi}{2} = \frac{1+0.25}{2} = 0.625$$

Hence, in both co-ordinate system; we get the same value for N_1 and N_2.

Now,

$$u_{(x=125)} = N_1u_1 + N_2u_2 = 0.375 \times 0.5 + 0.625 \times 0.7 = 0.625 \text{ mm}$$

ii) Here,

$$\epsilon = [B]\{u\} = \frac{1}{l}[-1 \quad 1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{200}[-1 \quad 1]\begin{Bmatrix} 0.5 \\ 0.7 \end{Bmatrix} = 10^{-3}$$

and, $\sigma = E\epsilon = 2 \times 10^5 \times 10^{-3} = 2 \times 10^2 \text{ N/mm}^2$

iii) Here,

$$[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2500 \times 2 \times 10^5}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 25 \times 10^5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

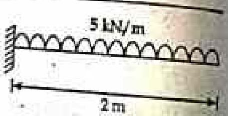
iv) Strain energy $(u_e) = \frac{1}{2}\{u\}^T[K]\{u\}$

$$= \frac{1}{2}\begin{Bmatrix} 0.5 \\ 0.7 \end{Bmatrix}^T 25 \times 10^5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 0.7 \end{Bmatrix}$$

$$= 50000 \text{ N-mm}$$

EXAMPLE 4.29

Evaluate the slope and deflection at the midpoint of the beam shown in the figure considering one element structure. $IE = 2 \times 10^{12}$ Nmm^2

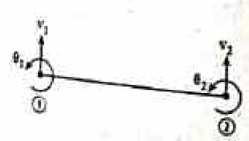


Solution:

Step 1: Unit conversion

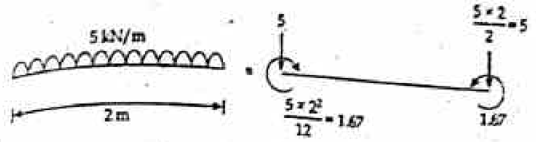
$$IE = 2 \times 10^{12} \text{ Nmm}^2 = 2.5 \times 10^3 \text{ kNm}^2$$

Step 2: Modeling



Step 3: Force matrix

$$[F] = \begin{Bmatrix} R_1 - 5 \\ M_1 - 1.67 \\ -5 \\ 1.67 \end{Bmatrix}; \text{ } R_1 \text{ and } M_1 \text{ are reactions developed at node } ①.$$



Step 4: Displacement matrix

$$[u] = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \end{Bmatrix}; \text{ applying boundary condition } v_1 = \theta_1 = 0.$$

Step 5: Stiffness matrix

$$[K] = \frac{IE}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym} & & & 4l^2 \end{bmatrix} = \frac{2.5 \times 10^3}{2^3}\begin{bmatrix} 12 & 12 & -12 & 12 \\ & 16 & -12 & 8 \\ & & 12 & -12 \\ \text{sym} & & & 16 \end{bmatrix}$$

Step 6: [F] = [K][u]

Here,

$$\begin{Bmatrix} \cancel{R_1 - 5} \\ \cancel{M_1 - 1.67} \\ -5 \\ 1.67 \end{Bmatrix} = 312.5\begin{bmatrix} \cancel{12} & \cancel{12} & \cancel{-12} & \cancel{12} \\ & \cancel{16} & \cancel{-12} & \cancel{8} \\ & & 12 & -12 \\ \text{sym} & & & 16 \end{bmatrix}\begin{Bmatrix} \cancel{0} \\ \cancel{0} \\ v_2 \\ \theta_2 \end{Bmatrix}$$

By elimination, the reduced matrix is:

$$\begin{Bmatrix} -5 \\ 1.67 \end{Bmatrix} = 312.5\begin{bmatrix} 12 & -12 \\ -12 & 16 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{312.5}\begin{bmatrix} 12 & -12 \\ -12 & 16 \end{bmatrix}^{-1}\begin{Bmatrix} -5 \\ 1.67 \end{Bmatrix} = \begin{Bmatrix} -0.004 & \text{m} \\ -0.003 & \text{rad} \end{Bmatrix}$$

Now,

$$v_{(x)} = N_1v_1 + N_2\theta_1 + N_3v_2 + N_4\theta_2$$

$$= \left[1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right]v_1 + \left[x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right]\theta_1 + \left[\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right]v_2 + \left[\frac{-x^2}{l} + \frac{x^3}{l^2}\right]\theta_2$$

At midpoint,

$x = 1$ m

$l = 2$ m

$$\therefore v_{(1)} = \left[1 - \frac{3}{4} + \frac{1}{4}\right]v_1 + \left[1 - 1 + \frac{1}{4}\right]\theta_1 + \left[\frac{3}{4} - \frac{1}{4}\right]v_2 + \left[\frac{-1}{2} + \frac{1}{4}\right]\theta_2$$

$$= \frac{1}{2}v_1 + \frac{1}{4}\theta_1 + \frac{1}{2}v_2 - \frac{1}{4}\theta_2 = 0 + 0 + \frac{1}{2} \times (-0.004) - \frac{1}{4}(-0.003)$$

$$= -1.25 \times 10^{-3} \text{ m}$$

Similarly; we have,

$$\theta_{(x)} = \frac{\partial v_{(x)}}{\partial x}$$

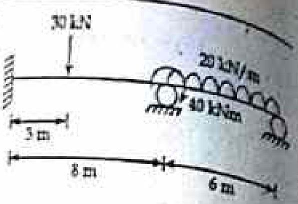
$$= \left[-\frac{6x}{l^2} + \frac{6x^2}{l^3}\right]\cancelto{0}{v_1} + \left[1 - \frac{4x}{l} + \frac{3x^2}{l^2}\right]\cancelto{0}{\theta_1} + \left[\frac{6x}{l^2} - \frac{6x^2}{l^3}\right]v_2 + \left[\frac{-2x}{l} + \frac{3x^2}{l^2}\right]\theta_2$$

or, $\theta_{(3)} = \left[\frac{6}{l^2} - \frac{6}{l^2}\right]v_2 + \left[\frac{-2}{l} + \frac{3}{4}\right]\theta_2$

or, $\theta_{(3)} = \frac{3}{4}v_2 - \frac{1}{4}\theta_2 = \frac{3}{4} \times (-0.004) - \frac{1}{4}(-0.003) = -0.0022$ **rad**

EXAMPLE 4.30

Determine the support reactions and deflections at mid span for the given structure. Also draw the bending moment diagram. $E = 20 \times 10^5$ MPa, $I = 5 \times 10^6$ mm^4 [2072 Ashwin]



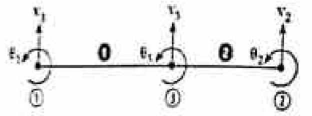
Solution:

Step 1: Unit conversion

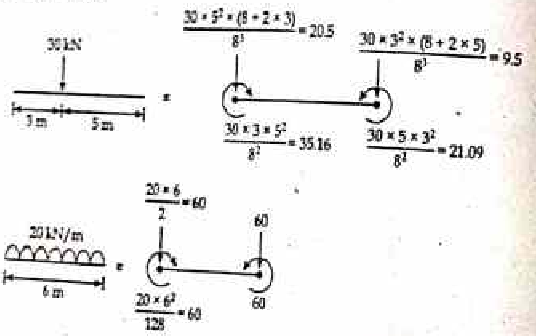
$E = 20 \times 10^5$ MPa $= 20 \times 10^8$ kN/m^2

$I = 5 \times 10^6$ mm$^4 = 5 \times 10^{-6}$ m^4

Step 2: Modeling



Step 3: Force matrix



$$\{F\} = \begin{Bmatrix} R_1 - 20.5 \\ M_1 - 35.16 \\ R_3 - 9.5 - 60 \\ 21.09 - 60 \\ R_2 - 60 \\ 60 \end{Bmatrix}$$

; R and M are support reactions at positive direction.

Step 4: Displacement matrix

$$\{u\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_3 \\ \theta_3 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_3 \\ 0 \\ \theta_2 \end{Bmatrix}$$

; applying boundary condition $v_1 = \theta_1 = v_3 = v_2 = 0$.

Step 5: Stiffness matrix

$$[K_1] = \frac{IE}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym} & & & 4l^2 \end{bmatrix}$$

$$= \frac{20 \times 10^8 \times 5 \times 10^{-6}}{8^3}\begin{bmatrix} 12 & 48 & -12 & 48 \\ & 256 & -48 & 128 \\ & & 12 & -48 \\ \text{sym} & & & 256 \end{bmatrix}$$

$$= 100\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 2.34 & 9.36 & -2.34 & 9.36 \\ & 50 & -9.36 & 25 \\ & & 2.34 & -9.36 \\ \text{sym} & & & 50 \end{bmatrix} \end{array}\begin{array}{c} v_1 \\ \theta_1 \\ v_3 \\ \theta_3 \end{array}$$

$$[K_2] = \frac{100 \times 10^2}{6^3}\begin{bmatrix} 12 & 36 & -12 & 36 \\ & 144 & -36 & 72 \\ & & 12 & -36 \\ \text{sym} & & & 144 \end{bmatrix}$$

$$= 100\begin{array}{c} \begin{array}{cccc} v_3 & \theta_3 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 5.55 & 16.67 & -5.55 & 16.67 \\ & 66.67 & -16.67 & 33.33 \\ & & 5.55 & -16.67 \\ \text{sym} & & & 66.67 \end{bmatrix} \end{array}\begin{array}{c} v_3 \\ \theta_3 \\ v_2 \\ \theta_2 \end{array}$$

Assembling; we have,

$$[K] = 100\begin{array}{c} \begin{array}{cccccc} v_1 & \theta_1 & v_3 & \theta_3 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 2.34 & 9.36 & -2.34 & 9.36 & & \\ & 50 & -9.36 & 25 & & \\ & & 7.89 & 7.31 & -5.55 & 16.67 \\ & & & 116.67 & -16.67 & 33.33 \\ & & & & 5.55 & -16.67 \\ \text{sym} & & & & & 66.67 \end{bmatrix} \end{array}\begin{array}{c} v_1 \\ \theta_1 \\ v_3 \\ \theta_3 \\ v_2 \\ \theta_2 \end{array}$$

Step 6: $[F] = [K][u]$

Here,

$$\begin{Bmatrix} \cancel{R_1 - 20.5} \\ \cancel{M_1 - 35.16} \\ \cancel{R_3 - 69.5} \\ -38.91 \\ \cancel{R_2 - 60} \\ 60 \end{Bmatrix} = 100\begin{bmatrix} \cancel{2.34} & \cancel{9.36} & \cancel{-2.34} & \cancel{9.36} & \cancel{0} & \cancel{0} \\ & \cancel{50} & \cancel{-9.36} & \cancel{25} & \cancel{0} & \cancel{0} \\ & & \cancel{7.89} & \cancel{7.31} & \cancel{-5.55} & \cancel{16.67} \\ & & & 116.67 & -16.67 & 33.33 \\ & & & & \cancel{5.55} & \cancel{-16.67} \\ \text{sym} & & & & & 66.67 \end{bmatrix}\begin{Bmatrix} \cancel{0} \\ \cancel{0} \\ \cancel{0} \\ \theta_3 \\ \cancel{0} \\ \theta_2 \end{Bmatrix}$$

By elimination, the reduced matrix is:

$$\begin{Bmatrix} -38.91 \\ 90 \end{Bmatrix} = 100\begin{bmatrix} 116.67 & 33.33 \\ 33.33 & 66.67 \end{bmatrix}\begin{Bmatrix} \theta_3 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \theta_3 \\ \theta_2 \end{Bmatrix} = \frac{1}{100}\begin{bmatrix} 116.67 & 33.33 \\ 33.33 & 66.67 \end{bmatrix}^{-1}\begin{Bmatrix} -38.91 \\ 90 \end{Bmatrix} = \begin{Bmatrix} -8.389 \times 10^{-3} \\ 0.0176 \end{Bmatrix}$$

Step 7: Reaction calculation

From $[F] - [K][u]$

$$\begin{Bmatrix} R_1 - 20.5 \\ M_1 - 35.16 \\ R_3 - 69.5 \\ R_2 - 60 \end{Bmatrix} = 100 \begin{bmatrix} 9.36 & 0 \\ 25 & 0 \\ 7.31 & 16.67 \\ -16.67 & -16.67 \end{bmatrix} \begin{Bmatrix} -8.389 \times 10^{-3} \\ 0.0176 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 - 20.5 \\ M_1 - 35.16 \\ R_3 - 69.5 \\ R_2 - 60 \end{Bmatrix} = \begin{Bmatrix} -7.852 \\ -20.972 \\ 23.207 \\ -15.355 \end{Bmatrix}$$

> **NOTE**
>
> Program the calculator
>
> $A = 100 \times \left(-8.389 \times 10^{-3} \times X + 0.0176 \times Y\right)$

$$\therefore \begin{Bmatrix} R_1 \\ M_1 \\ R_3 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 12.648 \\ 14.188 \\ 92.707 \\ 44.645 \end{Bmatrix}$$

Step 8: Deflections at mid span of the elements

Deflection at mid span of element ❶ is:

$$v_{(1)} = \left[\frac{2-3\xi+\xi^3}{4}\right]\cancel{v_1}^{0} + \left[\frac{l}{8}\left(1-\xi-\xi^2+\xi^3\right)\right]\cancel{\theta_1}^{0} + \left[\frac{2+3\xi-\xi^3}{4}\right]\cancel{v_3}^{0}$$

$$+ \left[\frac{l}{8}\left(-1-\xi+\xi^2+\xi^3\right)\right]\theta_3$$

$$= -\theta_3 \qquad [\because \xi = 0 \text{ at mid-span}]$$

$$= -8.389 \times 10^{-3}$$

Deflection at mid span of element ❷ is:

$$v_{(2)} = \left[\frac{2-3\xi+\xi^3}{4}\right]v_3 + \left[\frac{l}{8}\left(1-\xi-\xi^2+\xi^3\right)\right]\theta_3 + \left[\frac{2+3\xi-\xi^3}{4}\right]v_2$$

$$+ \left[\frac{l}{8}\left(-1-\xi+\xi^2+\xi^3\right)\right]\theta_2$$

$$= \frac{6}{8}\theta_3 - \frac{6}{8}\theta_2 \qquad [\text{Since, } \xi = 0,\ v_3 = 0, v_2 = 0]$$

$$= \frac{6}{8}(\theta_3 - \theta_2) = \frac{6}{8}\left(-8.389 \times 10^{-3} - 0.0176\right)$$

$$= -0.0195 \text{ m}$$

Step 9: B.M.D.

For element ❶:

$$\begin{Bmatrix} f_1 \\ m_1 \\ f_3 \\ m_3 \end{Bmatrix} = 100 \begin{bmatrix} 2.34 & 9.36 & -2.34 & 9.36 \\ & 50 & -9.36 & 25 \\ & & 2.34 & -9.36 \\ \text{sym} & & & 50 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -8.389 \times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} -7.852 \\ -20.972 \\ 7.852 \\ -41.945 \end{Bmatrix}$$

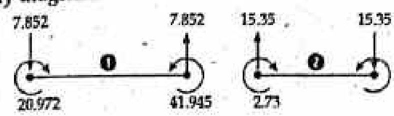
> **NOTE**
>
> Program the calculator
>
> $A = 100 \times X \times 8.389 \times 10^{-3}$
>
> and substitute,
>
> $x = 9.36, 25, -9.36, 50$

For element ❷:

$$\begin{Bmatrix} f_3 \\ m_3 \\ f_2 \\ m_2 \end{Bmatrix} = 100 \begin{bmatrix} 5.55 & 16.67 & -5.55 & 16.67 \\ & 66.67 & -16.67 & 33.33 \\ & & 5.55 & -16.67 \\ \text{Sym} & & & 66.67 \end{bmatrix} \begin{Bmatrix} 0 \\ -8.389 \times 10^{-3} \\ 0 \\ 0.0176 \end{Bmatrix} = \begin{Bmatrix} 15.35 \\ 2.73 \\ -15.35 \\ 89.37 \end{Bmatrix}$$

> **NOTE**
>
> Program in calculator
>
> $A = 100 \times \left(-8.389 \times 10^{-3} \times X + 0.0176 \times Y\right)$
>
> and substitute,
>
> | X | Y |
> |---|---|
> | 16.67 | 16.67 |
> | 66.67 | 33.33 |
> | −16.67 | −16.67 |
> | 33.33 | 66.67 |

Drawing free body diagrams of the elements



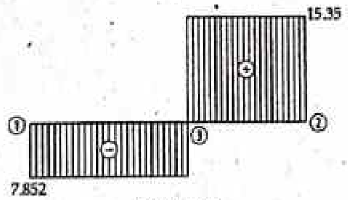


Figure: SFD

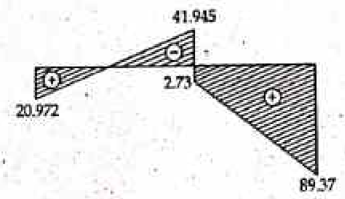


Figure: BMD

4.8 CONSTANT STRAIN TRIANGLE (CST)

4.8.1 Strain-displacement relation for C.S.T.

Consider a C.S.T. element in x-y plane as shown in the figure. Define nodes and displacements as shown.

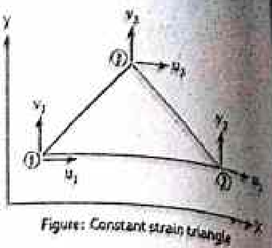


Figure: Constant strain triangle

Now, from section 3.2; we have,

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{aligned}\right\} \qquad [4.8.1]$$

From example 4.15; we have,

$$u = N_1u_1 + N_2u_2 + N_3u_3$$
$$v = N_1v_1 + N_2v_2 + N_3v_3$$

where, $N_1 = \frac{a_1 + b_1x + c_1y}{2A}$

$$N_2 = \frac{a_2 + b_2x + c_2y}{2A}$$
$$N_3 = \frac{a_3 + b_3x + c_3y}{2A}$$

$$a_1 = x_2y_3 - x_3y_2 \qquad b_1 = y_2 - y_3 \qquad c_1 = x_3 - x_2$$
$$a_2 = x_3y_1 - x_1y_3 \qquad b_2 = y_3 - y_1 \qquad c_2 = x_1 - x_3$$
$$a_3 = x_1y_2 - x_2y_1 \qquad b_3 = y_1 - y_2 \qquad c_3 = x_2 - x_1$$

$$\therefore \quad \epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{2A}\{b_1u_1 + b_2u_2 + b_3u_3\}$$
$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{2A}\{c_1v_1 + c_2v_2 + c_3v_3\}$$

and, $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{2A}\{c_1u_1 + c_2u_2 + c_3u_3\} + \frac{1}{2A}\{b_1v_1 + b_2v_2 + b_3v_3\}$

Representing the equations in the matrix form; we have,

$$[\epsilon] = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

or, $[\epsilon] = [B]\{u\}$

where, $[B] = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$. [4.8.2]

It is easier to represent [B] as:

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \qquad [4.8.3]$$

where, $y_{23} = y_2 - y_3 = b_1$

$x_{32} = x_3 - x_2 = c_1$ and so on.

4.8.2 Stiffness matrix for C.S.T.

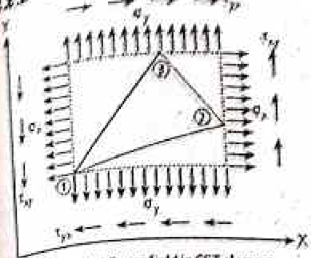


Figure: (a) Stress field in CST element

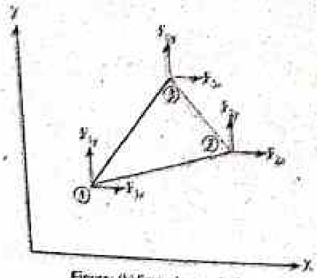


Figure: (b) Equivalent nodal forces

Consider the typical C.S.T. element as shown in the figure (a) subjected to constant stresses σ_x, σ_y, $\tau_{xy} = \tau_{yx}$. F_{1x}, F_{1y}, ..., F_{3y} in the figure (b) are equivalent nodal forces whereas F_{m1x}, F_{m1y}, ..., F_{m3y} are equivalent mid-side forces for the element subjected to the constant stresses. Let, t be the thickness of element. Now,

$$F_{m1x} = \sigma_x(y_3 - y_2)t + \tau_{xy}(x_2 - x_3)t$$

or, $F_{m1x} = \sigma_x y_{32}t + \tau_{xy}x_{23}t$

where, $y_{32} = y_3 - y_2$

and, $x_{23} = x_2 - x_3$

Similarly; we have,

$$F_{m1y} = \sigma_y x_{23}t + \tau_{xy}y_{32}t$$
$$F_{m2x} = -\sigma_x y_{31}t + \tau_{xy}x_{31}t$$
$$F_{m2y} = \sigma_y x_{31}t - \tau_{xy}y_{31}t$$
$$F_{m3x} = \sigma_x y_{21}t - \tau_{xy}x_{21}t$$
$$F_{m3y} = -\sigma_y x_{21}t + \tau_{xy}y_{21}t$$

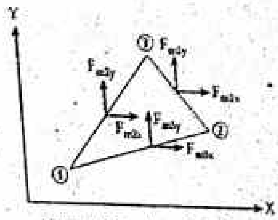


Figure: (c) Equivalent mid-side forces

Now, we evaluate equivalent nodal forces as:

$$F_{1x} = \frac{1}{2}(F_{m2x} + F_{m3x}) = \frac{1}{2}\left[-\sigma_x y_{31}t + \tau_{xy}x_{31}t + \sigma_x y_{21}t - \tau_{xy}x_{21}t\right]$$
$$= \frac{t}{2}\left[\sigma_x(y_{21} - y_{31}) + \tau_{xy}(x_{31} - x_{21})\right] = \frac{t}{2}\left[\sigma_x y_{23} + \tau_{xy}x_{32}\right]$$

Likewise; we get,

$$F_{1y} = \frac{t}{2}\left[\sigma_y x_{32} + \tau_{xy}y_{23}\right]$$
$$F_{2x} = \frac{t}{2}\left[\sigma_x y_{31} + \tau_{xy}x_{13}\right]$$
$$F_{2y} = \frac{t}{2}\left[\sigma_y x_{13} + \tau_{xy}y_{31}\right]$$
$$F_{3x} = \frac{t}{2}\left[\sigma_x y_{12} + \tau_{xy}x_{21}\right]$$
$$F_{3y} = \frac{t}{2}\left[\sigma_y x_{21} + \tau_{xy}y_{12}\right]$$

Representing all nodal force terms in the matrix form,

$$[F] = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \frac{t}{2}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{12} \\ 0 & x_{12} & y_{12} \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad [4.8.4]$$

From equation [4.8.3]; we have,

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

or, $$[B]^T = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{12} \\ 0 & x_{12} & y_{12} \end{bmatrix}$$

or, $$\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} = 2A[B]^T$$

Substituting this value into equation [4.8.4]; we have,

$$[F] = \frac{t}{2}2A[B]^T[\sigma]$$

or, $[F] = tA[B]^T[D][\epsilon]$ Since, $[\sigma] = [D][\epsilon]$

or, $[F] = tA[B]^T[D][B][u]$ Since, $[\epsilon] = [B][u]$

Comparing with $[F] = [K][u]$; we get,

$$[K] = t\,A[B]^T[D][B] \qquad [4.8.5]$$

4.8.3 Forces in C.S.T. element

I) Body force

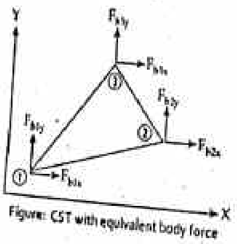


Figure: CST with equivalent body force

Here, t is the thickness.

A is the area of the plate.

γ is the unit weight.

γ_{AT} is the weight of the element.

Dividing the weight equally to all the three nodes; we have,

$$f_b = \frac{\gamma_{AT}}{3}$$

$$\therefore \quad [F_b] = \begin{Bmatrix} F_{b1x} \\ F_{b1y} \\ F_{b2x} \\ F_{b2y} \\ F_{b3x} \\ F_{b3y} \end{Bmatrix} = -\frac{\gamma_{AT}}{3}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix} \qquad [4.8.6]$$

Negative sign is introduced because weight always acts downward.

II) Traction force

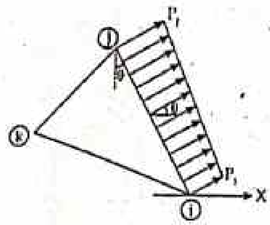


Figure: (a) CST with traction force

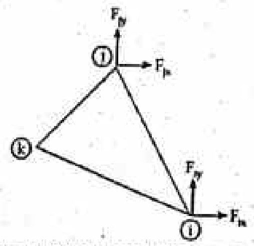


Figure: (a) CST with traction force transformed into equivalent nodal force

$$\begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{jx} \\ f_{jy} \end{Bmatrix} = \frac{tl_{i-j}}{6}\begin{Bmatrix} 2P_i\cos\theta + P_j\cos\theta \\ 2P_i\sin\theta + P_j\sin\theta \\ P_i\cos\theta + 2P_j\cos\theta \\ P_i\sin\theta + 2P_j\sin\theta \end{Bmatrix}$$

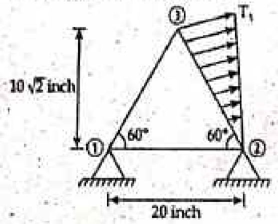
where, t is the thickness of the plate.

l_{i-j} is the length of side $i-j$.

θ is the orientation of traction force with respect to positive x-direction.

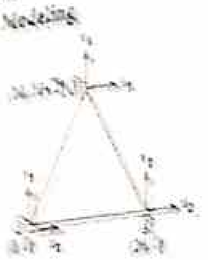
EXAMPLE 4.31

Considering plane stress condition, find out the nodal displacements and stresses of the C.S.T. elements as shown in the figure. $E = 30 \times 10^6$ psi, $t = 0.3$ in, $\gamma = 460$ lb/in^3, $\nu = 0.3$, $T_3 = 360$ psi with usual notations. **[2070 Bhadra]**

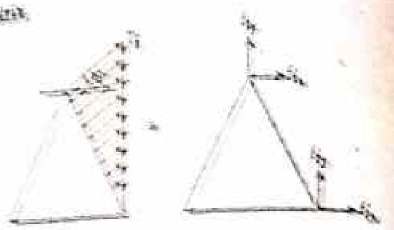


Solution:

Step 1: Modeling



Step 2: Force matrix



$$\begin{Bmatrix} f_{x2}\\ f_{y2}\\ f_{x3}\\ f_{y3}\end{Bmatrix} = \frac{0.5\times 20}{?}\begin{Bmatrix} (1-360\cos 30^\circ \\ (1-360\sin 30^\circ \\ (1-2\times 360\cos 30^\circ \\ (1-2\times 360\sin 30^\circ \end{Bmatrix} = \begin{Bmatrix} 311.77\\ 180\\ 623.54\\ 360\end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} R_{x1}\\ R_{y1}\\ R_{x2}-f_{x2}\\ R_{y2}-f_{y2}\\ f_{x3}\\ f_{y3}\end{Bmatrix} = \begin{Bmatrix} R_{x1}\\ R_{y1}\\ R_{x2}-311.77\\ R_{y2}-180\\ 623.54\\ 360\end{Bmatrix}$$

Step 3: Displacement matrix

$$\{u\} = \begin{Bmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\end{Bmatrix} = \begin{Bmatrix} 0\\ 0\\ 0\\ 0\\ u_3\\ v_3\end{Bmatrix}$$

after applying boundary condition $u_1 = v_1 = u_2 = v_2 = 0$.

Step 4: Stiffness matrix

$$2A = \begin{vmatrix} 1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix} = \begin{vmatrix} 1 & 1 & 1\\ 0 & 20 & 10\\ 0 & 0 & 10\sqrt{2}\end{vmatrix} = 20\times 10\sqrt{2} = 200\sqrt{2}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0\\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21}\\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12}\end{bmatrix}$$

$$= \frac{1}{200\sqrt{2}}\begin{bmatrix} -10\sqrt{2} & 0 & 10\sqrt{2} & 0 & 0 & 0\\ 0 & -10 & 0 & -10 & 0 & 20\\ -10 & -10\sqrt{2} & -10 & 10\sqrt{2} & 20 & 0\end{bmatrix}$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} = \frac{30\times 10^6}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix}$$

$$[K] = tA[B]^T[D][B] = tA[B]^T\times\frac{30\times 10^6}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix}$$

$$\times\frac{1}{200\sqrt{2}}\begin{bmatrix} -10\sqrt{2} & 0 & 10\sqrt{2} & 0 & 0 & 0\\ 0 & -10 & 0 & -10 & 0 & 20\\ -10 & -10\sqrt{2} & -10 & 10\sqrt{2} & 20 & 0\end{bmatrix}$$

NOTE

Program the calculator

$X:Y:M:A = X+0.3Y:B = 0.3X+Y:C = 0.35M$

and substitute the value of X, Y and M from columns of [B] matrix.

$$[K] = \frac{tA}{2A}\begin{bmatrix} -10\sqrt{2} & 0 & -10\\ 0 & -10 & -10\sqrt{2}\\ 10\sqrt{2} & 0 & -10\\ 0 & -10 & 10\sqrt{2}\\ 0 & 0 & 20\\ 0 & 20 & 0\end{bmatrix}$$

$$\times 116556\begin{bmatrix} -10\sqrt{2} & -3 & 14.14 & -3 & 0 & 6\\ 3\sqrt{2} & -10 & 4.24 & -10 & 0 & 20\\ -3.5 & -4.95 & -3.5 & 4.95 & 7 & 0\end{bmatrix}$$

NOTE

Program the calculator in two steps, first for first 3 rows and second for last 3 rows

$X:Y:M:A = -10\sqrt{2}X-10M:B = -10Y-10\sqrt{2}M:C = 10\sqrt{2}X-10M$

$X:Y:M:A = -10Y+10\sqrt{2}M:B = 20M:C = 20Y$

$$[K] = \frac{0.5\times 116556}{2}\begin{bmatrix} 242.43 & 142.43 & -202.37 & 142.43 & 0 & -284.85\\ & 170.00 & 7.10 & 29.99 & -98.99 & -200\\ & & 234.97 & -91.92 & -70 & 84.85\\ & & & 170 & 98.99 & -200\\ & \text{sym} & & & 140 & 0\\ & & & & & 400\end{bmatrix}$$

Step 5: $\{F\} = [K]\{u\}$

Here,

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x}+[illegible] \\ F_{2y}+[illegible] \\ 623.54 \\ 360 \end{Bmatrix} = \frac{87417}{3}\begin{bmatrix} 242.43 & 142.43 & -242.37 & 142.43 & 0 & -284.85 \\ & 170.00 & 7.10 & 29.99 & -98.99 & -200 \\ & & 234.97 & -91.92 & -70 & 84.85 \\ & & & 170 & 98.99 & -200 \\ & \text{sym} & & & 140 & 0 \\ & & & & & 400 \end{bmatrix}\begin{Bmatrix} [illegible] \\ [illegible] \\ [illegible] \\ [illegible] \\ u_3 \\ v_3 \end{Bmatrix}$$

After elimination, the reduced matrix is:

$$\begin{Bmatrix} 623.54 \\ 360 \end{Bmatrix} = \frac{87417}{3}\begin{bmatrix} 140 & 0 \\ 0 & 400 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \frac{3}{87417}\begin{bmatrix} 140 & 0 \\ 0 & 400 \end{bmatrix}^{-1}\begin{Bmatrix} 623.54 \\ 360 \end{Bmatrix} = \begin{Bmatrix} 2.55\times10^{-4} \\ 5.15\times10^{-5} \end{Bmatrix} \text{ inch}$$

Step 6: Element stresses

$$[\sigma] = [D][\epsilon] = [D][B][u]$$

or,
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 116556\begin{bmatrix} -10\sqrt{2} & -3 & 14.14 & -3 & 0 & 6 \\ 3\sqrt{2} & -10 & 4.24 & -10 & 0 & 20 \\ -3.5 & -4.95 & -3.5 & 4.95 & 7 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2.55\times10^{-4} \\ 5.15\times10^{-5} \end{Bmatrix}$$

$$= 116556\begin{bmatrix} 0 & 6 \\ 0 & 20 \\ 7 & 0 \end{bmatrix}\begin{Bmatrix} 2.55\times10^{-4} \\ 5.15\times10^{-5} \end{Bmatrix}$$

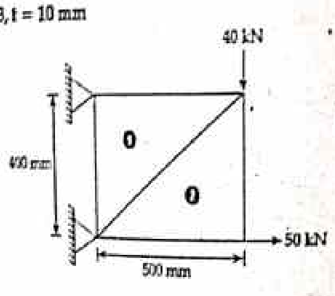
$$= \begin{Bmatrix} 36.01 \text{ psi} \\ 120.05 \text{ psi} \\ 208.05 \text{ psi} \end{Bmatrix}$$

NOTE

The value of [D][B] is imported from step ***. Matrix multiplication is done directly in calculator in the matrix mode.

EXAMPLE 4.32

Determine the stiffness matrices for the element as shown in the figure.
E = 200 GPa, ν = 0.3, t = 10 mm

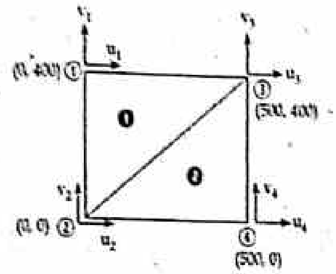


Solution:

Step 1: Unit conversion

$E = 200 \text{ GPa} = 200 \text{ kN/mm}^2$

Step 2: Modeling



Step 3: Element stiffness matrix of 1

$$2A = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 500 \\ 400 & 0 & 400 \end{vmatrix} = 400\times500 = 20\times10^4$$

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{2A}\begin{bmatrix} -400 & 0 & 0 & 0 & 400 & 0 \\ 0 & 500 & 0 & -500 & 0 & 0 \\ 500 & -400 & -500 & 0 & 0 & 400 \end{bmatrix}$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{200}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$[K_1] = tA[B]^T[D][B] = tA[B]^T\times\frac{200}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\times\frac{1}{2A}\begin{bmatrix} -400 & 0 & 0 & 0 & 400 & 0 \\ 0 & 500 & 0 & -500 & 0 & 0 \\ 500 & -400 & -500 & 0 & 0 & 400 \end{bmatrix}$$

$$= 10\times\frac{200}{0.91}\times\frac{1}{4A}\begin{bmatrix} -400 & 0 & 500 \\ 0 & 500 & -400 \\ 0 & 0 & -500 \\ 0 & -500 & 0 \\ 400 & 0 & 0 \\ 0 & 0 & 400 \end{bmatrix}\begin{bmatrix} -400 & 150 & 0 & -150 & 400 & 0 \\ -120 & 500 & 0 & -500 & 120 & 0 \\ 175 & -140 & -175 & 0 & 0 & 140 \end{bmatrix}$$

$$= \frac{1}{364}\begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 100000 & 190000 & 0 & -190000 & -100000 & 0 \\ & 306000 & 70000 & -250000 & 60000 & 0 \\ & & 87500 & 0 & 0 & -56000 \\ & & & 250000 & -60000 & 0 \\ & \text{sym} & & & 160000 & 0 \\ & & & & & 56000 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

Step 4: Element stiffness matrix of ❶

$$2A = \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 500 & 500 \\ 0 & 400 & 0 \end{vmatrix} = 400 \times 500 = 20 \times 10^4$$

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{34} & 0 & y_{42} & 0 & y_{23} & 0 \\ 0 & x_{43} & 0 & x_{24} & 0 & x_{32} \\ x_{43} & y_{34} & x_{24} & y_{42} & x_{32} & y_{23} \end{bmatrix}$$

$$= \frac{1}{2A}\begin{bmatrix} 400 & 0 & 0 & 0 & -400 & 0 \\ 0 & 0 & 0 & -500 & 0 & 500 \\ 0 & 400 & -500 & 0 & 500 & -400 \end{bmatrix}$$

$$[K_e] = tA[B]^T[D][B]$$

$$= 10 \times A[B]^T \times \left(\frac{200}{0.91} \times \frac{1}{2A}\right)\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} 400 & 0 & 0 & 0 & -400 & 0 \\ 0 & 0 & 0 & -500 & 0 & 500 \\ 0 & 400 & -500 & 0 & 500 & -400 \end{bmatrix}$$

$$= \frac{200}{0.91 \times 4A}\begin{bmatrix} 400 & 0 & 0 \\ 0 & 0 & 400 \\ 0 & 0 & -500 \\ 0 & -500 & 0 \\ -400 & 0 & 500 \\ 0 & 500 & -400 \end{bmatrix} \times \begin{bmatrix} 400 & 0 & 0 & -150 & -400 & 150 \\ 120 & 0 & 0 & -500 & -120 & 500 \\ 0 & 140 & -175 & 0 & 175 & -140 \end{bmatrix}$$

$$= \frac{1}{364}\begin{array}{c} \begin{array}{cccccc} u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 160000 & 0 & 0 & -60000 & -160000 & 60000 \\ & 56000 & -70000 & 0 & 70000 & -56000 \\ & & 87500 & 0 & -87500 & 70000 \\ & & & -250000 & 60000 & -250000 \\ & \text{sym} & & & 247500 & -130000 \\ & & & & & 306000 \end{bmatrix} \end{array}\begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}$$

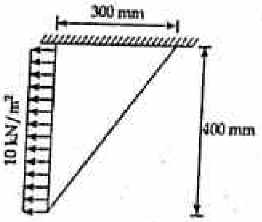
Step 5: Generalized stiffness matrix

$$[K] = \frac{1}{364}\begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 20000 & 19000 & 0 & -19000 & -10000 & 0 & & \\ & 306000 & 7000 & -25000 & 60000 & 0 & & \\ & & 247500 & 0 & 0 & -116000 & -116000 & 60000 \\ & & & 306000 & -130000 & 0 & 70000 & -56000 \\ & & & & 247500 & 0 & -87500 & 70000 \\ & & & & & -194000 & 60000 & -250000 \\ & \text{sym} & & & & & 247500 & -130000 \\ & & & & & & & 306000 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}$$

EXAMPLE 4.33

A steel plate of thickness 10 mm is being loaded in the structural system as shown in the figure. Calculate stresses at the centroid of the plate. E = 200 GPa, ν = 0.3

[2072 Ashwin]



Solution:

Step 1: Unit conversion

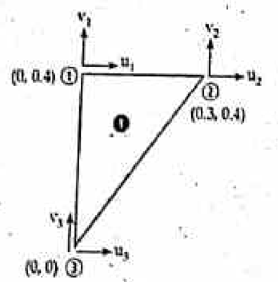
300 mm = 0.3 m

10 mm = 0.01 m

400 mm = 0.4 m

$E = 200\text{ GPa} = 200 \times 10^6\text{ kN/m}^2$

Step 2: Modeling



Step 3: Force matrix

$$\text{Traction force}\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \frac{t \times l_{1-3}}{6}\begin{bmatrix} 2T_1\cos\theta + T_3\cos\theta \\ 2T_1\sin\theta + T_3\cos\theta \\ T_1\cos\theta + 2T_3\cos\theta \\ T_1\sin\theta + 2T_3\cos\theta \end{bmatrix}$$

$$= \frac{0.3 \times 0.4}{6}\begin{bmatrix} 2 \times (-10) - 10 \\ 0 + 0 \\ -10 - 2 \times 10 \\ 0 + 0 \end{bmatrix} = \begin{Bmatrix} -0.6 \\ 0 \\ -0.6 \\ 0 \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} R_{1x} - 0.6 \\ R_{1y} + 0 \\ R_{2x} \\ R_{2y} \\ -0.6 \\ 0 \end{Bmatrix}$$ 'R' are reactions at supports at positive direction

Step 4: Displacement matrix

$$[u]=\begin{Bmatrix}u_1\\v_1\\u_2\\v_2\\u_3\\v_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\u_3\\v_3\end{Bmatrix}$$

Step 5: Stiffness matrix

$$2A=\begin{vmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{vmatrix}=\begin{vmatrix}1&1&1\\0&0.3&0\\0.4&0.4&0\end{vmatrix}=0.4\times0.3=0.12$$

$$[B]=\frac{1}{2A}\begin{bmatrix}y_{23}&0&y_{31}&0&y_{12}&0\\0&x_{32}&0&x_{13}&0&x_{21}\\x_{32}&y_{23}&x_{13}&y_{31}&x_{21}&y_{12}\end{bmatrix}$$

$$=\frac{1}{2A}\begin{bmatrix}0.4&0&-0.4&0&0&0\\0&-0.3&0&0&0&0.3\\-0.3&0.4&0&-0.4&0.3&0\end{bmatrix}$$

$$[D]=\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\\nu&1&0\\0&0&\frac{1-\nu}{2}\end{bmatrix}; \text{ considering plane stress condition.}$$

$$=\frac{200\times10^6}{1-0.3^2}\begin{bmatrix}1&0.3&0\\0.3&1&0\\0&0&0.35\end{bmatrix}$$

$$[K]=tA[B]^T[D][B]$$

$$=tA[B]^T\times2.197\times10^8\begin{bmatrix}1&0.3&0\\0.3&1&0\\0&0&0.35\end{bmatrix}\frac{1}{2A}\begin{bmatrix}0.4&0&-0.4&0&0&0\\0&-0.3&0&0&0&0.3\\-0.3&0.4&0&-0.4&0.3&0\end{bmatrix}$$

$$=t\times A\times\frac{1}{2A}\begin{bmatrix}0.4&0&-0.3\\0&-0.3&0.4\\-0.4&0&0\\0&0&-0.4\\0&0&0.3\\0&0.3&0\end{bmatrix}\times1.83\times10^9\begin{bmatrix}0.4&-0.09&-0.4&0&0&0.9\\0.12&-0.3&-0.12&0&0&0.3\\-0.105&0.14&0&-0.14&0.105&0\end{bmatrix}$$

$$=\frac{0.01\times1.83\times10^9}{2}\begin{bmatrix}0.191&-0.078&-0.16&0.042&-0.031&0.036\\&0.146&0.036&-0.056&0.042&-0.09\\&&0.16&0&0&-0.036\\&&&0.056&-0.042&0\\&\text{sym}&&&0.0315&0\\&&&&&0.09\end{bmatrix}$$

Step 6: $\{F\}=[K]\{u\}$

$$\begin{Bmatrix}R_{1x}\\R_{1y}\\R_{2x}\\R_{2y}\\-0.6\\0\end{Bmatrix}=9.15\times10^6\begin{bmatrix}0.191&-0.078&-0.16&0.042&-0.031&0.036\\&0.146&0.036&-0.056&0.042&-0.09\\&&0.16&0&0&-0.036\\&&&0.056&-0.042&0\\&\text{sym}&&&0.0315&0\\&&&&&0.09\end{bmatrix}\begin{Bmatrix}0\\0\\0\\0\\u_3\\v_3\end{Bmatrix}$$

Reduced matrix after elimination is:

$$\begin{Bmatrix}-0.6\\0\end{Bmatrix}=9.15\times10^6\begin{bmatrix}0.315&0\\0&0.9\end{bmatrix}\begin{Bmatrix}u_3\\v_3\end{Bmatrix}$$

or, $$\begin{Bmatrix}u_3\\v_3\end{Bmatrix}=\frac{1}{9.15\times10^6}\begin{bmatrix}0.315&0\\0&0.9\end{bmatrix}^{-1}\begin{Bmatrix}-0.6\\0\end{Bmatrix}=\begin{Bmatrix}-2.08\times10^{-7}\\0\end{Bmatrix}\text{ m}$$

Step 7: Element stress

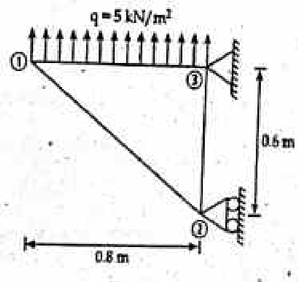
$$[\sigma]=[D][\epsilon]=[D][B][u]$$

or, $$\begin{Bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{Bmatrix}=1.83\times10^9\begin{bmatrix}0.4&-0.09&-0.4&0&0&0.9\\0.12&-0.3&-0.12&0&0&0.3\\-0.105&0.14&0&-0.14&0.105&0\end{bmatrix}\begin{Bmatrix}0\\0\\0\\0\\-2.08\times10^{-7}\\0\end{Bmatrix}$$

or, $$\begin{Bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{Bmatrix}=1.83\times10^9\times(-2.08\times10^{-7})\times\begin{bmatrix}0\\0\\0.105\end{bmatrix}=\begin{bmatrix}0\\0\\-39.96\end{bmatrix}\text{ kN/m}^2$$

EXAMPLE 4.34

Using plane stress condition, calculate displacements and stresses of the C.S.T. element as shown in the figure. Given: E = 200 GPa, t = 10 mm, γ = 78.5 kN/m³, ν = 0.3, q = 5 kN/m².



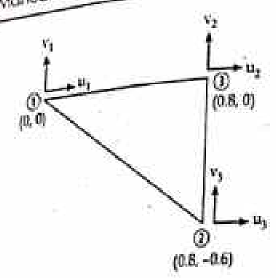
Solution:

Step 1: Unit conversion

t = 10 mm = 0.1 m

E = 200 × 10⁶ kN/m²

Step 2: Modeling



Step 3: Force matrix

Body force $\begin{Bmatrix} f_{y1}^{b} \\ f_{y2}^{b} \\ f_{y3}^{b} \end{Bmatrix} = \frac{\gamma AT}{3}\begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix} = \frac{78.5 \times \left(\frac{1}{2} \times 0.8 \times 0.6\right) \times 0.01}{3}\begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix}$

$$= \begin{Bmatrix} -0.0628 \\ -0.0628 \\ -0.0628 \end{Bmatrix}$$

Traction force $\begin{Bmatrix} f_{1x}^{T} \\ f_{1y}^{T} \\ f_{3x}^{T} \\ f_{3y}^{T} \end{Bmatrix} = \frac{tl_{③-①}}{6}\begin{bmatrix} 2T_1 \cos\theta + T_3 \cos\theta \\ 2T_1 \sin\theta + T_3 \cos\theta \\ T_1 \cos\theta + 2T_3 \cos\theta \\ T_1 \sin\theta + 2T_3 \cos\theta \end{bmatrix}$

$$= \frac{0.01 \times 0.8}{6}\begin{bmatrix} 0 \\ 2 \times 5 + 5 \\ 0 \\ 5 + 2 \times 5 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0.02 \\ 0 \\ 0.02 \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} 0 \\ 0.02 - 0.0628 \\ R_{2x} \\ -0.0628 \\ R_{3x} \\ R_{3y} + 0.02 - 0.0628 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.0428 \\ R_{2x} \\ -0.0628 \\ R_{3x} \\ R_{3y} - 0.0428 \end{Bmatrix}$$

R refers to the reaction force acting in the positive direction.

Step 4: Displacement matrix

$\{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$; applying boundary condition.

Step 5: Stiffness matrix

$$2A = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0.8 & 0.8 \\ 0 & -0.6 & 0 \end{vmatrix} = 0.6 \times 0.8 = 0.48$$

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{2A}\begin{bmatrix} -0.6 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & -0.8 & 0 & 0.8 \\ 0 & -0.6 & -0.8 & 0 & 0.8 & 0.6 \end{bmatrix}$$

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} = \frac{200 \times 10^6}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= 10^6\begin{bmatrix} 18 & 5.4 & 0 \\ 5.4 & 18 & 0 \\ 0 & 0 & 6.3 \end{bmatrix}$$

$$[K] = tA[B]^T[D][B]$$

$$= tA[B]^T \times 10^6\begin{bmatrix} 18 & 5.4 & 0 \\ 5.4 & 18 & 0 \\ 0 & 0 & 6.3 \end{bmatrix} \times \frac{1}{2A}\begin{bmatrix} -0.6 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & -0.8 & 0 & 0.8 \\ 0 & -0.6 & -0.8 & 0 & 0.8 & 0.6 \end{bmatrix}$$

$$= tA \times \frac{1}{2A}\begin{bmatrix} -0.6 & 0 & 0 \\ 0 & 0 & -0.6 \\ 0 & 0 & -0.8 \\ 0 & -0.8 & 0 \\ 0.6 & 0 & 0.8 \\ 0 & 0.8 & 0.6 \end{bmatrix} \times \frac{10^6}{0.48}\begin{bmatrix} -10.8 & 0 & 0 & -4.32 & 10.8 & 4.32 \\ -3.24 & 0 & 0 & -14.4 & 3.24 & 14.4 \\ 0 & -3.78 & -5.04 & 0 & 5.04 & 3.78 \end{bmatrix}$$

NOTE

Program in calculator:

A = 18 X + 5.4 Y : B = 5.4 X + 18 Y : C = 6.3 M

$$= \frac{0.01}{2}\frac{10^6}{0.48}\begin{bmatrix} 6.48 & 0 & 0 & 2.592 & -6.48 & -2.592 \\ & 2.27 & 3.02 & 0 & -3.02 & -2.27 \\ & & 4.03 & 0 & -4.03 & -3.02 \\ & & & 11.52 & -2.59 & -11.52 \\ & \text{sym} & & & 10.51 & 5.61 \\ & & & & & 5.72 \end{bmatrix}$$

NOTE

Program in calculator in two steps: first step for first three rows and second step for last three rows.

A = −0.6 X + 0 × Y + 0 × M : B = −0.6 M : C = −0.8 M

A = 0 × X − 0.8 Y + 0 × M : B = 0.6 X + 0.8 M : C = 0.8 X + 0.6 M

Step 6: [F] = [K][u]

or,

$$\begin{Bmatrix} 0 \\ -0.0428 \\ R_{2x} \\ -0.0628 \\ R_{3x} \\ R_{3y} - 0.0428 \end{Bmatrix} = 1.04 \times 10^4\begin{bmatrix} 6.48 & 0 & 0 & 2.592 & -6.48 & -2.592 \\ & 2.27 & 3.02 & 0 & -3.02 & -2.27 \\ & & 4.03 & 0 & -4.03 & -3.02 \\ & & & 11.52 & -2.59 & -11.52 \\ & \text{sym} & & & 10.51 & 5.61 \\ & & & & & 5.72 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

Reduced matrix after elimination is:

$$\begin{Bmatrix} 0 \\ -0.0428 \\ -0.0628 \end{Bmatrix} = 1.04 \times 10^{4} \begin{bmatrix} 6.45 & 0 & 2.592 \\ & 2.27 & 0 \\ sym & & 11.52 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_2 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} u_1 \\ v_1 \\ v_2 \end{Bmatrix} = \frac{1}{1.04 \times 10^{4}} \begin{bmatrix} 6.45 & 0 & 2.592 \\ & 2.27 & 0 \\ sym & & 11.52 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -0.0428 \\ -0.0628 \end{Bmatrix} = \begin{Bmatrix} 2.3 \times 10^{-7} \\ -1.81 \times 10^{-6} \\ -5.76 \times 10^{-7} \end{Bmatrix}$$

Step 7: Strain and stress

$$\{\epsilon\} = [B]\{u\}$$

$$\text{or,} \quad \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{0.48} \begin{bmatrix} -0.6 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & -0.8 & 0 & 0.8 \\ 0 & -0.6 & -0.8 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \frac{1}{0.48} \begin{bmatrix} -0.6 & 0 & 0 \\ 0 & 0 & -0.8 \\ 0 & -0.6 & 0 \end{bmatrix} \begin{Bmatrix} 2.3 \times 10^{-7} \\ -1.81 \times 10^{-6} \\ -5.76 \times 10^{-7} \end{Bmatrix} = \begin{Bmatrix} -2.87 \times 10^{-7} \\ 9.6 \times 10^{-7} \\ 2.26 \times 10^{-7} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\epsilon\}$$

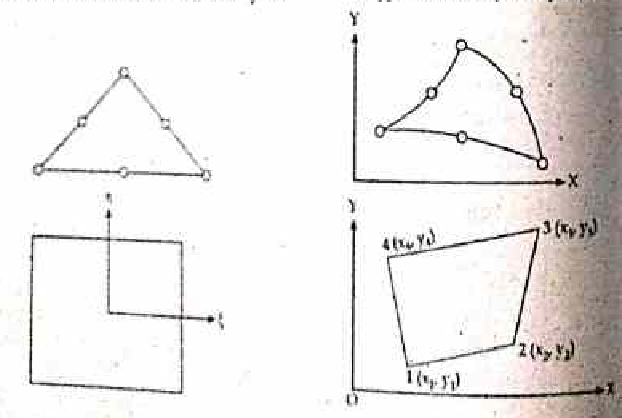
$$\text{or,} \quad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 10^{6} \begin{bmatrix} 18 & 5.4 & 0 \\ 5.4 & 18 & 0 \\ 0 & 0 & 6.3 \end{bmatrix} \begin{Bmatrix} -2.87 \times 10^{-7} \\ 9.6 \times 10^{-7} \\ 2.26 \times 10^{-7} \end{Bmatrix} = \begin{Bmatrix} 9 \times 10^{-3} \\ 15.72 \\ 14.25 \end{Bmatrix} \approx \begin{Bmatrix} 0 \\ 15.72 \\ 14.25 \end{Bmatrix}$$

4.9 ISOPARAMETRIC FORMULATION

One of the major advantages of using finite element method is that it can model and analyze irregular shapes also very effectively and this is possible due to development of iso-parametric formulation method.

Shape functions of regular element in natural co-ordinate system can also be used to locate the co-ordinate of any arbitrary point in an irregular element from global co-ordinates of nodal points. Such shape functions are also termed as mapping functions and the method is known as iso-parametric formulation.

Parent element in natural co-ordinate system | **Mapped element in global system**



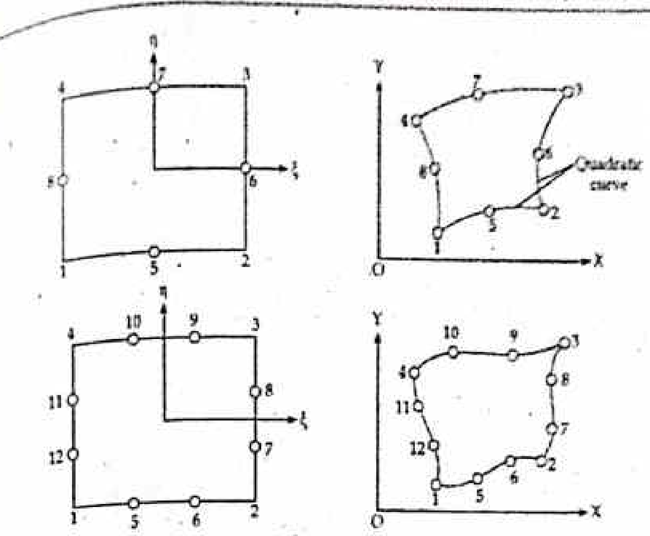


Figure: Concept of mapping in isoparametric elements

4.9.1 Iso-parametric, super-parametric and sub-parametric elements

If same shape functions are used to define the geometry and displacements in an element, the element is called iso-parametric element.

If more numbers of nodes are used to define the geometry of an element than in defining displacements, the element is termed as super-parametric element.

If more numbers of nodes are used to define the displacements in an element than in defining the geometry, the element is termed as sub-parametric element.

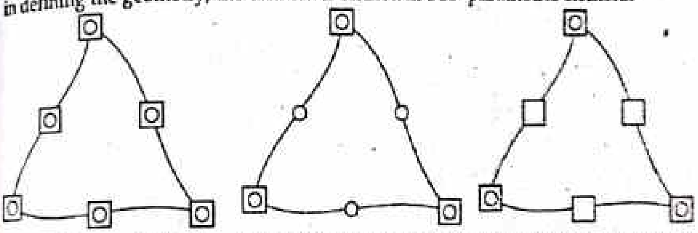


Figure: (a) Isoparametric element Figure: (b) Superparametric element Figure: (c) Subparametric element

Here, ○ and □ nodes used to define geometry and displacement respectively.

4.9.2 Jacobian matrix

Jacobian matrix relates the derivative of a function in local co-ordinate system to derivative in global co-ordinate system.

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \qquad [4.9.1]$$

By chain rule of partial differentiation; we have,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} \times \frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi} \times \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta} \times \frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi} \times \frac{\partial}{\partial \eta}$$

In matrix form; we have,

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

$$\therefore \quad [J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad [4.9.2]$$

By iso-parametric formulation at any point; we have,

$$x = \sum_{i=1}^{n} N_i x_i$$

$$\therefore \quad \frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i$$

Similarly; we can evaluate $\frac{\partial y}{\partial \xi}, \frac{\partial x}{\partial \eta}$ and $\frac{\partial y}{\partial \eta}$

$$\therefore \quad [J] = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \qquad [4.9.3]$$

4.9.3 Strain–displacement matrix for iso–parametric element (strain–displacement matrix for irregular elements)

From equation [4.9.1]; we have,

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

or,

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

Let, $[J]^{-1} = \begin{bmatrix} J^*_{11} & J^*_{12} \\ J^*_{21} & J^*_{22} \end{bmatrix}$

From section 3.2; we have,

$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y}$$

and, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

$$\therefore \quad \epsilon_x = J^*_{11} \frac{\partial u}{\partial \xi} + J^*_{12} \frac{\partial u}{\partial \eta}$$

$$\epsilon_y = J^*_{21} \frac{\partial v}{\partial \xi} + J^*_{22} \frac{\partial v}{\partial \eta}$$

$$\gamma_{xy} = J^*_{21} \frac{\partial u}{\partial \xi} + J^*_{22} \frac{\partial u}{\partial \eta} + J^*_{11} \frac{\partial v}{\partial \xi} + J^*_{12} \frac{\partial v}{\partial \eta}$$

In the matrix form; we have,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} \qquad [4.9.4]$$

But, $u = \sum_{i=1}^{n} N_i u_i$, $v = \sum_{i=1}^{n} N_i v_i$

$\therefore \quad \frac{\partial u}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} u_i$ and so on.

In the matrix form; we have,

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & \cdots\cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & \cdots\cdots & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \cdots\cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \cdots\cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ \vdots \\ u_n \\ uv_n \end{Bmatrix} \qquad [4.9.5]$$

Substituting into equation [4.9.4]; we have,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & \cdots\cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & \cdots\cdots & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \cdots\cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \cdots\cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ \vdots \\ u_n \\ uv_n \end{Bmatrix}$$

Comparing with $[\epsilon] = [B][u]$; we get,

$$[B] = \begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & \cdots\cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & \cdots\cdots & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \cdots\cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \cdots\cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix} \qquad [4.9.6]$$

4.9.4 Stiffness matrix of iso–parametric element

We know that;

$$[K] = \iiint [B]^T [D][B] \, dV$$

For plates,

$$dV = \partial x \, \partial y \, t$$

$$\therefore \quad [K] = t \iint [B]^T [D][B] \, \partial x \, \partial y$$

It can be shown that:

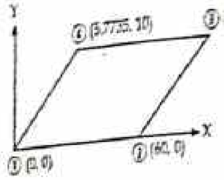
$$\partial x\,\partial y = |J|\,\partial\xi\,\partial\eta$$

$$\therefore \quad [K] = t\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]|J|\,\partial\xi\,\partial\eta \qquad [4.9.7]$$

Equation [4.9.7] is the expression for the stiffness matrix of iso-parametric element. Numerical integration can be used to solve the equation for convenience.

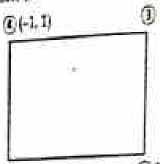
EXAMPLE 4.35

Form the Jacobi matrix and strain-displacement matrix corresponding to the Gauss point (0.57735, 0.57735) for element shown in the figure.



Solution:

Parent element for the element is:



and, the shape functions are:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Now, evaluating $\frac{\partial N_i}{\partial\xi}$ and $\frac{\partial N_i}{\partial\eta}$ at (0.57735, 0.57735)

$$\frac{\partial N_1}{\partial\xi} = -\frac{1}{4}(1-\eta) = -\frac{1}{4}(1-0.57735) = -0.10566$$

$$\frac{\partial N_2}{\partial\xi} = \frac{1}{4}(1-\eta) = 0.10566$$

$$\frac{\partial N_3}{\partial\xi} = \frac{1}{4}(1+\eta) = 0.39438$$

$$\frac{\partial N_4}{\partial\xi} = -\frac{1}{4}(1+\eta) = -0.39438$$

$$\frac{\partial N_1}{\partial\eta} = -\frac{1}{4}(1+\xi) = -0.10566$$

$$\frac{\partial N_2}{\partial\eta} = -\frac{1}{4}(1+\xi) = -0.39438$$

$$\frac{\partial N_3}{\partial\eta} = \frac{1}{4}(1+\xi) = 0.39438$$

$$\frac{\partial N_4}{\partial\eta} = \frac{1}{4}(1-\xi) = 0.10566$$

We know that:

$$[J] = \begin{bmatrix} \sum_{i=1}^{n}\frac{\partial N_i}{\partial\xi}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial\xi}y_i \\ \sum_{i=1}^{n}\frac{\partial N_i}{\partial\eta}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial\eta}y_i \end{bmatrix}$$

Now,

$$\sum_{i=1}^{n}\frac{\partial N_i}{\partial\xi}x_i = -0.10566\times 0 + 0.10566\times 60 + 0.39438\times 65.7735 - 0.39438 \times 5.7735 = 30.00$$

$$\sum_{i=1}^{n}\frac{\partial N_i}{\partial\eta}x_i = 0 - 0.39438\times 60 + 0.39438\times 65.7735 + 0.10566\times 5.7735 = 2.88698$$

$$\sum_{i=1}^{n}\frac{\partial N_i}{\partial\xi}y_i = 0 + 0 + 0.39438\times 10 - 0.39438\times 10 = 0$$

$$\sum_{i=1}^{n}\frac{\partial N_i}{\partial\eta}y_i = 0 + 0 + 0.39438\times 10 + 0.10566\times 10 = 5.0$$

$$\therefore \quad [J] = \begin{bmatrix} 30.0 & 0 \\ 2.89 & 5.0 \end{bmatrix}$$

Now,

$$[J]^{-1} = \begin{bmatrix} J^*_{11} & J^*_{12} \\ J^*_{21} & J^*_{22} \end{bmatrix} = \begin{bmatrix} 30.0 & 0 \\ 2.89 & 5.0 \end{bmatrix}^{-1} = \begin{bmatrix} 0.033 & 0 \\ -0.019 & 0.2 \end{bmatrix}$$

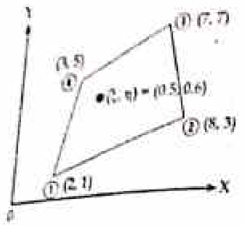
Now, the strain-displacement matrix is:

$$[B] = \begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial\xi} & 0 & \frac{\partial N_2}{\partial\xi} & 0 & \frac{\partial N_3}{\partial\xi} & 0 & \frac{\partial N_4}{\partial\xi} & 0 \\ \frac{\partial N_1}{\partial\eta} & 0 & \frac{\partial N_2}{\partial\eta} & 0 & \frac{\partial N_3}{\partial\eta} & 0 & \frac{\partial N_4}{\partial\eta} & 0 \\ 0 & \frac{\partial N_1}{\partial\xi} & 0 & \frac{\partial N_2}{\partial\xi} & 0 & \frac{\partial N_3}{\partial\xi} & 0 & \frac{\partial N_4}{\partial\xi} \\ 0 & \frac{\partial N_1}{\partial\eta} & 0 & \frac{\partial N_2}{\partial\eta} & 0 & \frac{\partial N_3}{\partial\eta} & 0 & \frac{\partial N_4}{\partial\eta} \end{bmatrix}$$

$$= \begin{bmatrix} 0.033 & 0 \\ -0.019 & 0.2 \end{bmatrix} \begin{bmatrix} -0.106 & 0 & 0.106 & 0 & 0.394 & 0 & -0.394 & 0 \\ -0.106 & 0 & -0.394 & 0 & 0.394 & 0 & 0.106 & 0 \\ 0 & -0.106 & 0 & 0.106 & 0 & 0.394 & 0 & -0.394 \\ 0 & -0.106 & 0 & -0.394 & 0 & 0.394 & 0 & 0.106 \end{bmatrix}$$

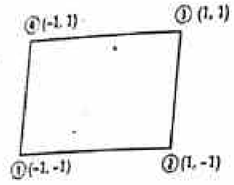
EXAMPLE 4.36

Find Cartesian co-ordinate of Gaussian point (0.5, 0.6) for element shown in the figure



Solution:

The parent element is:



and, the shape functions are:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

At $\xi = 0.5, \eta = 0.6$

$$N_1 = \frac{1}{4}(1-0.5)(1-0.6) = 0.05$$

$$N_2 = 0.15$$

$$N_3 = 0.6$$

$$N_4 = 0.2$$

Now,

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$
$$= 0.05 \times 2 + 0.15 \times 8 + 0.6 \times 7 + 0.2 \times 3$$
$$= 6.1$$

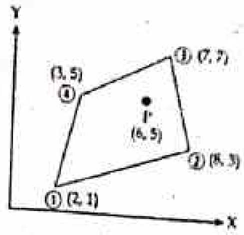
and, $y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$

$$= 0.05 \times 1 + 0.15 \times 3 + 0.6 \times 7 + 0.2 \times 5$$
$$= 5.7$$

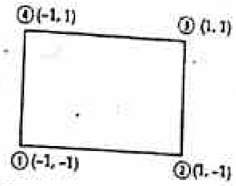
EXAMPLE 4.37

The Cartesian co-ordinate of a point is (6, 5). Find the value of ξ and η at that point.

Solution:



The parent element is:



and, the shape functions are:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Now,

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

or, $6 = \frac{1}{4}\{(1-\xi)(1-\eta)2 + (1+\xi)(1-\eta)8 + (1+\xi)(1+\eta)7 + (1-\xi)(1+\eta)3\}$

or, $24 = 2(1-\xi-\eta+\xi\eta) + 8(1+\xi-\eta-\xi\eta) + 7(1+\xi+\eta+\xi\eta) + 3(1-\xi+\eta-\xi\eta)$

or, $24 = 20 + 10\xi + 0 \times \eta - 2\xi\eta$

or, $2 = 5\xi - \xi\eta$ [a]

Also,

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

or, $5 = \frac{1}{4}\{1(1-\xi-\eta+\xi\eta) + 3(1+\xi-\eta-\xi\eta) + 7(1+\xi+\eta+\xi\eta) + 5(1-\xi+\eta-\xi\eta)\}$

or, $1 = \xi - 2\eta$

or, $\eta = \frac{1-\xi}{2}$ [b]

Substituting this value into equation [a]; we get,

$$2 = 5\xi - \xi\left(\frac{1-\xi}{2}\right)$$

or, $4 = 9\xi + \xi^2$

or, $\xi^2 + 9\xi - 4 = 0$

$\therefore$ $\xi = \dfrac{-9 \pm \sqrt{9^2 + 4 \times 4}}{2}$

Neglecting –ve sign, we have,

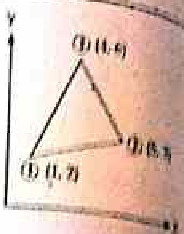
$\xi = 0.424$

From equation [b], we have,

$$\eta = \frac{1 - 0.424}{2} = 0.288$$

EXAMPLE 4.38

For the C.S.T. element as shown in the figure, $N_1 = 0.3$ x-co-ordinate point P is 3.3. Find the value of N_2, N_3 of the element and y-co-ordinate of point P.



Solution:

We know that,

$N_1 + N_2 + N_3 = 1$

or, $0.3 + N_2 + N_3 = 1$

or, $N_2 + N_3 = 0.7$ [a]

Also,

$x = N_1x_1 + N_2x_2 + N_3x_3$

or, $3.3 = 0.3 \times 0.1 + N_2 \times 5 + N_3 \times 4$

or, $5N_2 + 4N_3 = 3$ [b]

Solving equation [a] and [b], we get,

$N_2 = 0.2$

$N_3 = 0.5$

Now,

$$y = N_1y_1 + N_2y_2 + N_3y_3 = 0.3 \times 2 + 0.2 \times 3 + 0.5 \times 6 = 4.2$$

4.10 GENERAL INTRODUCTION TO PRE AND POST PROCESSING

A typical finite element program consists of three basic units.

i) Preprocessor

In preprocessor, the program takes necessary informations from the user via graphical user interface and develop data files required by the processor unit. The major tasks involved in preprocessor unit are:

a) read the input data of the problems like geometry, material properties, loads, boundary conditions, initial conditions, etc.
b) generate finite element mesh
c) number the elements and nodes automatically.
d) generate the nodal co-ordinates.

e) develop nodal connectivity details.
f) develop standard tables specifying various loads and load conditions.
g) develop the tables of material numbers, material properties, boundary conditions and other details.

ii) Processor

The processors perform the analysis work using the data files prepared by preprocessor and stores the final results in the form of temporary files. The major tasks performed by processor are:

a) generate element matrices.
b) assemble element matrices to get generalized matrices.
c) solve the equations to get nodal displacements, calculated value of stresses, strains, moments, etc.

iii) Post processors

The output of the processor includes nodal displacements, stresses, strains, bending moment, shear force, axial force, etc. for each and every element at each and every point. It is time consuming to go through entire output file to get the design value. Hence, the user friendly, graphic users interface unit, called post processors assists the users to:

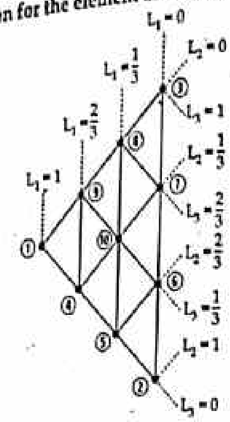
a) pick up absolute maximum stress resultants.
b) plot the graphs of deflection, bending moment, shear force, axial force, torsion, etc.
c) pick up value for specified element or point.

Various FEA packages are available in the market and there is competition among the software developers to make pre and post processors as user friendly as possible.

WORKED OUT PROBLEMS

Problem 1

Derive the shape function for the element as shown in the figure. [2070 Bhadra]



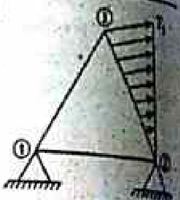
Solution: See the solution of example 4.27

Problem 2

Consider plane stress condition, find out the nodal displacements and stresses of the CST element as shown in the figure.

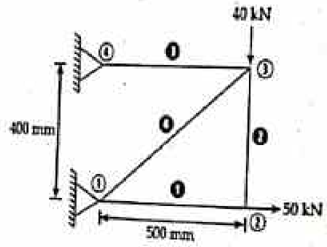
$E = 30 \times 10^6$ psi, $t = 0.3$ in, $\gamma = 460$ lb/in^3, $\nu = 0.3$, $T_3 = 360$ psi with usual notations **[2070 Bhadra]**

Solution: See the solution of example 4.31



Problem 4

Determine the stiffness matrix for the element as shown in the figure. A = 300 mm² and E = 2.1 × 10⁵ MPa [2070 Magh]



Solution:

Step 1: Unit conversion

$$E = 2.1 \times 10^5\ \text{MPa} = 2.1 \times 10^5 \times 10^{-3}\ \text{kN/mm}^2 = 2.1 \times 10^2\ \text{kN/mm}^2$$

Step 2: Stiffness matrix of elements

For element ❶, ① to ②

$$[K_1] = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{300 \times 2.1 \times 10^2}{500}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 2.1 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 60 & 0 & -60 & 0 \\ 0 & 0 & 0 & 0 \\ -60 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

For element ❷, ② to ③

$$[K_2] = \frac{A_2E_2}{L_2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \frac{300 \times 2.1 \times 10^2}{400}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= 2.1 \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 75 & 0 & -75 \\ 0 & 0 & 0 & 0 \\ 0 & -75 & 0 & 75 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

For element ❸, ③ to ④

$$[K_3] = 2.1 \begin{array}{c} \begin{array}{cccc} u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 60 & 0 & -60 & 0 \\ 0 & 0 & 0 & 0 \\ -60 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}$$

NOTE

$[K_3]$ same as $[K_1]$. Only difference is u_1, v_1, u_2, v_2 are replaced by u_3, v_3, u_4 and v_4.

For element ❹, ① to ③

$$\theta = \tan^{-1}\left(\frac{400}{500}\right) = 38.659°$$

$$C = \cos\theta = \cos(38.659°) = 0.7809$$

$$S = \sin\theta = \sin(38.659°) = 0.6247$$

$$C^2 = 0.6098$$

$$S^2 = 0.3902$$

$$CS = 0.4878$$

$$L_4 = \sqrt{(400)^2 + (500)^2} = 640.3124\ \text{m}$$

Now,

$$[K_4] = \frac{A_4 E_4}{L_4}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{sym} & & & S^2 \end{bmatrix}$$

$$= \frac{300 \times 2.1 \times 10^7}{640.3124}\begin{bmatrix} 0.6098 & 0.4878 & -0.6098 & -0.4878 \\ & 0.3902 & -0.4878 & -0.3902 \\ & & 0.6098 & 0.4878 \\ \text{sym} & & & 0.3902 \end{bmatrix}$$

$$= 2.1 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 28.57 & 22.85 & -28.57 & -22.85 \\ & 18.28 & -22.85 & -18.28 \\ & & 28.57 & 22.85 \\ \text{sym} & & & 18.28 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

Step 3: Assembling

$[K] = 2.1$

u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	
60 28.57	0 22.85	−60	0	−28.57	−22.85			u_1
	0 18.28	0	0	−22.85	−18.28			v_1
		60 0	0 0	0	0			u_2
			0 75	0	−75			v_2
				0 60 28.57	0 0 22.85	−60	0	u_3
	SYMMETRIC				75 0 18.28	0	0	v_3
						60	0	u_4
							0	v_4

NOTE

Step to get above table:

i) Number of nodes = 4
so, size of [K] = 2 × 4 = 8
Draw table of size of 8 × 8

ii) First fill the values of $[K_1]$ in respective position.

iii) Similarly, fill the value of $[K_2]$, $[K_3]$ and $[K_4]$ one by one in right position.

$$[K] = 2.1 \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 88.57 & 22.85 & -60 & 0 & -28.57 & -22.85 & 0 & 0 \\ & 18.28 & 0 & 0 & -22.85 & -18.28 & 0 & 0 \\ & & 60 & 0 & 0 & 0 & 0 & 0 \\ & & & 75 & 0 & -75 & 0 & 0 \\ & & & & 88.57 & 22.85 & -60 & 0 \\ & & & & & 93.28 & 0 & 0 \\ & & & & & & 60 & 0 \\ \text{sym} & & & & & & & 0 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}$$

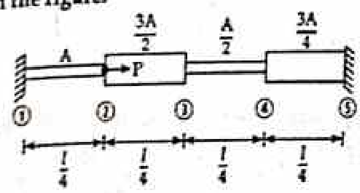
Problem 5

Formulate the stiffness matrix for a bar element. Rotate the same bar element and formulate stiffness matrix for 2-D truss element. **[2070 Magh]**

Solution: See the definition part 4.5.1 and 4.5.2

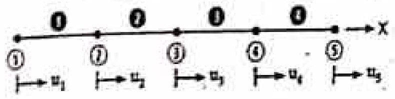
Problem 6

For the given stepped bar obtain nodal displacements at node 2, 3 and 4. Also, obtain forces developed at the supports. Take: E = Constant and cross-section areas as indicated in the figure. **[2071 Bhadra]**



Solution:

Step 1: Modeling



Step 2: Displacement matrix

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \\ 0 \end{Bmatrix}; \text{ applying boundary condition } u_1 = u_5 = 0$$

Step 3: Force matrix

$$\{F\} = \begin{Bmatrix} R_1 \\ P \\ 0 \\ 0 \\ R_5 \end{Bmatrix};$$

where, R_1 and R_5 are reactions at supports acting in positive direction.

Step 4: Stiffness matrix

For element ❶; we have,

$$[K_1] = \frac{A_1E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{AE}{\frac{l}{4}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{AE}{l}\begin{bmatrix}4 & -4\\ -4 & 4\end{bmatrix}$$

Similarly, for element ❷; we have,

$$[K_2] = \frac{\left(\frac{3A}{2}\right)E}{\frac{l}{4}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{AE}{l}\begin{bmatrix}6 & -6\\ -6 & 6\end{bmatrix}$$

and, for element ❸ and ❹; we have,

$$[K_3] = \frac{\left(\frac{A}{2}\right)E}{\frac{l}{4}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{AE}{l}\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$$

and, $$[K_4] = \frac{\left(\frac{3A}{4}\right)E}{\frac{l}{4}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{AE}{l}\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$$

Assembling element matrices,

$$[K] = \frac{AE}{l}\begin{bmatrix}4 & -4 & 0 & 0 & 0\\ & 4+6 & -6 & 0 & 0\\ & & 6+2 & -2 & 0\\ & & & 2+3 & -3\\ & \text{sym} & & & 3\end{bmatrix} = \frac{AE}{l}\begin{bmatrix}4 & -4 & 0 & 0 & 0\\ & 10 & -6 & 0 & 0\\ & & 8 & -2 & 0\\ & & & 5 & -3\\ & \text{sym} & & & 3\end{bmatrix}$$

Step 5: [F] = [K][u]

or, $$\begin{Bmatrix}R_1\\ P\\ 0\\ 0\\ R_5\end{Bmatrix} = \frac{AE}{l}\begin{bmatrix}4 & -4 & 0 & 0 & 0\\ & 10 & -6 & 0 & 0\\ & & 8 & -2 & 0\\ & & & 5 & -3\\ & \text{sym} & & & 3\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ u_3\\ u_4\\ 0\end{Bmatrix} \quad [a]$$

Reduced matrix after elimination is;

$$\begin{Bmatrix}P\\ 0\\ 0\end{Bmatrix} = \frac{AE}{l}\begin{bmatrix}10 & -6 & 0\\ & 8 & -2\\ & & 5\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ u_4\end{Bmatrix}$$

or, $$\begin{Bmatrix}u_2\\ u_3\\ u_4\end{Bmatrix} = \frac{l}{AE}\begin{bmatrix}10 & -6 & 0\\ -6 & 8 & -2\\ 0 & -2 & 5\end{bmatrix}^{-1}\begin{Bmatrix}P\\ 0\\ 0\end{Bmatrix} = \frac{l}{AE}\begin{bmatrix}0.2 & 0.167 & 0.067\\ 0.167 & 0.278 & 0.111\\ 0.067 & 0.111 & 0.244\end{bmatrix}\begin{Bmatrix}P\\ 0\\ 0\end{Bmatrix}$$

$$= \frac{l}{30AE}\begin{bmatrix}6\\ 5\\ 2\end{bmatrix} \times P = \frac{Pl}{30AE}\begin{bmatrix}6\\ 5\\ 2\end{bmatrix}$$

Step 6: Support reactions

From equation [a]; we have,

$$\begin{Bmatrix}R_1\\ R_5\end{Bmatrix} = \frac{AE}{l}\begin{bmatrix}-4 & 0 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ u_4\end{Bmatrix} = \frac{AE}{l}\begin{bmatrix}-4 & 0 & 0\\ 0 & 0 & -3\end{bmatrix} \times \frac{Pl}{30AE}\begin{bmatrix}6\\ 5\\ 2\end{bmatrix}$$

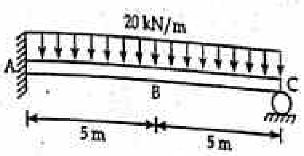
$$= \frac{P}{30}\begin{Bmatrix}-24\\ -6\end{Bmatrix} = \frac{P}{5}\begin{Bmatrix}-4\\ -1\end{Bmatrix}$$

$$R_1 = -\frac{4P}{5} = \frac{4P}{5}(\leftarrow)$$

and, $$R_5 = -\frac{P}{5} = \frac{4P}{5}(\leftarrow)$$

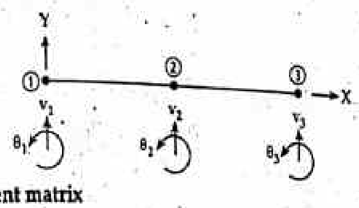
Problem 7

For the given beam, find the deflection at point B and rotations at point B and C. Take EI as constant throughout the beam. Discretize the beam into two elements. **[2071 Bhadra]**



Solution:

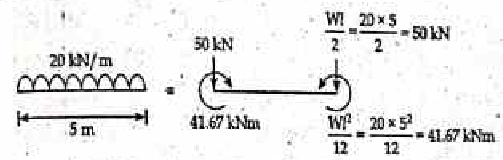
Step 1: Modeling



Step 2: Displacement matrix

$$\{u\} = \begin{Bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ v_2\\ \theta_2\\ 0\\ v_3\end{Bmatrix}$$

Step 3: Force matrix



$$\{F\} = \begin{Bmatrix}R_1 - 50\\ M_1 - 41.67\\ -50 - 50\\ 41.67 - 41.67\\ R_3 - 50\\ 41.67\end{Bmatrix};$$

where, R and M are reaction and moment developed in the positive direction.

Step 4: Stiffness matrix

For element ❶ we have

$$[k_1] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{sym} & & & 4L^2 \end{bmatrix}$$

$$= \frac{EI}{125}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 30 & -12 & 30 \\ & 100 & -12 & 50 \\ & & 12 & -30 \\ & \text{sym} & & 100 \end{bmatrix} \end{array}\begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

Similarly

$$[k_2] = \frac{EI}{125}\begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 30 & -12 & 30 \\ & 100 & -30 & 50 \\ & & 12 & -30 \\ & \text{sym} & & 100 \end{bmatrix} \end{array}\begin{array}{c} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

Assembling element stiffness matrices, we have,

$$[K] = \frac{EI}{125}\begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ & 100 & -30 & 50 & 0 & 0 \\ & & 24 & 0 & -12 & 30 \\ & & & 200 & -30 & 50 \\ & & & & 12 & -30 \\ & \text{sym} & & & & 100 \end{bmatrix}$$

Step 5: $\{F\} = [K]\{u\}$

$$\begin{Bmatrix} R_1 - 50 \\ M_1 - 41.67 \\ -100 \\ 0 \\ R_3 - 50 \\ 41.67 \end{Bmatrix} = \frac{EI}{125}\begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ & 100 & -30 & 50 & 0 & 0 \\ & & 24 & 0 & -12 & 30 \\ & & & 200 & -30 & 50 \\ & & & & 12 & -30 \\ & \text{sym} & & & & 100 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ 0 \\ \theta_3 \end{Bmatrix}$$

After elimination, the reduced matrix is;

$$\begin{Bmatrix} -100 \\ 0 \\ 41.67 \end{Bmatrix} = \frac{EI}{125}\begin{bmatrix} 24 & 0 & 30 \\ & 200 & 50 \\ \text{sym} & & 100 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

or, $$\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{125}{EI}\begin{bmatrix} 24 & 0 & 30 \\ 0 & 200 & 50 \\ 30 & 50 & 100 \end{bmatrix}^{-1}\begin{Bmatrix} -100 \\ 0 \\ 41.67 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} -1041.67 \\ -104.168 \\ 416.675 \end{Bmatrix}$$

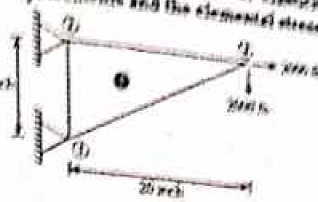
Deflection at point B = $\frac{-1041.67}{EI}$ m

Rotation at point B = $\frac{104.168}{EI}$ radian

Rotation at point C = $\frac{416.675}{EI}$ radian

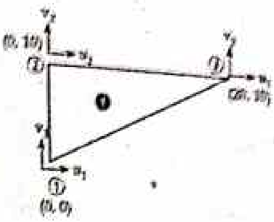
Problem 8

A thin plate is subjected to the loads as shown in the figure below. The plate thickness is 0.3 inch and the other dimensions are shown in the figure. Given that the Poisson's ration = 0.3 and the modulus of elasticity E = 30 × 10^6 psi. Determine nodal load displacements and the elemental stresses. [2071 Bhadra]



Solution:

Step 1: Modeling



Step 2: Force matrix

$$\{F\} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 2000 \\ -2000 \end{Bmatrix};$$

where, R represent reactions developed at the support in the positive direction.

Step 3: Displacement matrix

$$\{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

Step 4:

$$2A = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 20 \\ 0 & 10 & 10 \end{vmatrix} = 20 \times 10 = 200$$

Step 5:

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where $y_{23} = y_2 - y_3$ and so on.

$$[B] = \frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & 0 & -20 & 10 & 0 & -10 \end{bmatrix}$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{30 \times 10^6}{1-(0.3)^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= 32.97 \times 10^6\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Step 6:

$$[K] = tA[B]^T[D][B]$$

$$= tA[B]^T \times 32.97 \times 10^6\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \times \frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & 0 & -20 & 10 & 0 & -10 \end{bmatrix}$$

$$= tA \times \frac{1}{200}\begin{bmatrix} 0 & 0 & 20 \\ 0 & 20 & 0 \\ 10 & 0 & -20 \\ 0 & -20 & 10 \\ -10 & 0 & 0 \\ 0 & 0 & -10 \end{bmatrix} \times 1.65 \times 10^5 \times \begin{bmatrix} 0 & 6 & 10 & -6 & -10 & 0 \\ 0 & 20 & 3 & -20 & -3 & 0 \\ 7 & 0 & -7 & 3.5 & 0 & -3.5 \end{bmatrix}$$

$$= 24750\begin{bmatrix} 140 & 0 & -140 & 70 & 0 & -70 \\ & 400 & 60 & -400 & -60 & 0 \\ & & 240 & -130 & -100 & 70 \\ & & & 435 & 60 & -35 \\ & \text{sym} & & & 100 & 0 \\ & & & & & 35 \end{bmatrix}$$

Step 7: $[F] = [K][u]$

or,
$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ 2000 \\ -2000 \end{Bmatrix} = 24750\begin{bmatrix} 140 & 0 & -140 & 70 & 0 & -70 \\ & 400 & 60 & -400 & -60 & 0 \\ & & 240 & -130 & -100 & 70 \\ & & & 435 & 60 & -35 \\ & \text{sym} & & & 100 & 0 \\ & & & & & 35 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

Reduced matrix after elimination is;

$$\begin{Bmatrix} 2000 \\ -2000 \end{Bmatrix} = 24750\begin{bmatrix} 100 & 0 \\ 0 & 35 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}$$

or,
$$\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \frac{1}{24750}\begin{bmatrix} 100 & 0 \\ 0 & 35 \end{bmatrix}^{-1}\begin{Bmatrix} 2000 \\ -2000 \end{Bmatrix} = \begin{Bmatrix} 8.08 \times 10^{-4} \\ -2.308 \times 10^{-3} \end{Bmatrix} \text{ in}$$

Step 8: Stresses

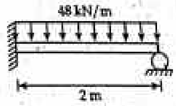
$$[\sigma] = [D][B][u]$$

or,
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 1.65 \times 10^5\begin{bmatrix} 0 & 6 & 10 & -6 & -10 & 0 \\ 0 & 20 & 3 & -20 & -3 & 0 \\ 7 & 0 & -7 & 3.5 & 0 & -3.5 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

or,
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 1.65 \times 10^5\begin{bmatrix} -10 & 0 \\ -3 & 0 \\ 0 & -3.5 \end{bmatrix}\begin{Bmatrix} 8.08 \times 10^{-4} \\ -2.308 \times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} -1333.2 \\ -399.96 \\ 1332.87 \end{Bmatrix} \text{lb/in}^2$$

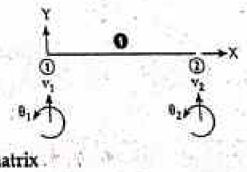
Problem 9

Draw bending moment diagram of the concrete beam given in the figure. For simplicity, neglect self weight. Calculate displacement at 0.5 m from the fixed end. **[2071 Magh]**



Solution:

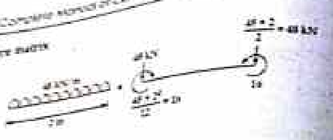
Step 1: Modeling



Step 2: Displacement matrix

$$\{u\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \end{Bmatrix}; \text{ applying boundary conditions}$$

Step 3: Force matrix



$$[F] = \begin{Bmatrix} R_1 - 48 \\ M_1 - 16 \\ R_2 - 48 \\ 16 \end{Bmatrix}$$

where, R and M represents reactions and moments developed at supports in positive direction

Step 4: Stiffness matrix

$$[K] = \frac{IE}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & \text{sym} & & 4L^2 \end{bmatrix} = \frac{IE}{8}\begin{bmatrix} 12 & 12 & -12 & 12 \\ & 16 & -12 & 8 \\ & & 12 & -12 \\ & \text{sym} & & 16 \end{bmatrix}$$

Step 5: [F] = [K][u]

or, $$\begin{Bmatrix} R_1 - 48 \\ M_1 - 16 \\ R_2 - 48 \\ 16 \end{Bmatrix} = \frac{IE}{8}\begin{bmatrix} 12 & 12 & -12 & 12 \\ & 16 & -12 & 8 \\ & & 12 & -12 \\ & \text{sym} & & 16 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \end{Bmatrix}$$

or, $16 = \frac{IE}{8} \times 16\theta_2$

or, $\theta_2 = \frac{8}{IE}$ radian

Step 6: To draw B.M.D.

$$\begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \end{Bmatrix} = \frac{IE}{8}\begin{bmatrix} 12 & 12 & -12 & 12 \\ & 16 & -12 & 8 \\ & & 12 & -12 \\ & \text{sym} & & 16 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \end{Bmatrix}$$

or, $$\begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \end{Bmatrix} = \frac{IE}{8}\begin{bmatrix} 12 \\ 8 \\ -12 \\ 16 \end{bmatrix} \times \frac{8}{IE}$$

or, $$\begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 8 \\ -12 \\ 16 \end{Bmatrix}$$

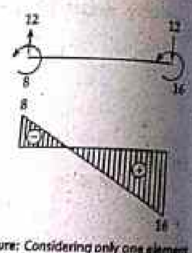


Figure: Considering only one element

NOTE

16 kNm positive moment at simply supported end is an error in the approximate solution. Higher the number of finite elements we take, the moment at free end goes on decreasing approaching the exact approximate solution i.e., 0 at the simply supported end.

We know that, displacement at any point is;

$$u(x) = H_1V_1 + H_2\theta_1 + H_3V_2 + H_4\theta_2;$$

where x is the value of position and H_1, H_2, H_3 and H_4 are Hermite shape functions at x.

$$u_{(0.5)} = H_4 \times \frac{8}{IE}$$

or Since $V_1 = \theta_1 = V_2 = 0$ and $\theta_2 = \frac{8}{IE}$;

and $$H_4 = \left[-\frac{x^2}{l} + \frac{x^3}{l^2}\right] = \left[-\frac{(0.5)^2}{2} + \frac{(0.5)^3}{(2)^2}\right] = -0.094$$

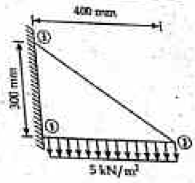
$$u_{(0.5)} = -0.094 \times \frac{8}{IE} = -\frac{0.75}{IE} \text{ m}$$

Problem 10

Draw shape functions of a three noded iso-parametric element with equations. A steel plate of thickness 10 mm is being loaded in the structural system as shown in the figure. Calculate stresses at centroid of the element using constant strain triangle. Use plane stress condition. [2071 Magh]

$E = 200$ GPa

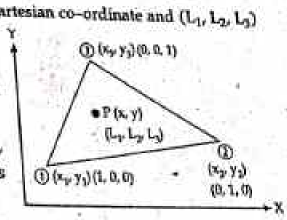
$\nu = 0.3$



Solution:

An element with 3 nodes is shown. (x, y) is Cartesian co-ordinate and (L_1, L_2, L_3) is natural co-ordinate for any point in the element such that natural co-ordinates of the node ①, ② and ③ are (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively.

Then, shape function at any point are $N_1 = L_1$, $N_2 = L_2$ and $N_3 = L_3$; where, (L_1, L_2, L_3) is the natural co-ordinate of that point.



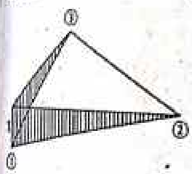


Figure: Variation of N_1

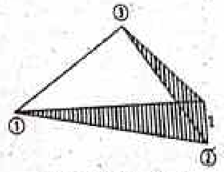


Figure: Variation of N_2

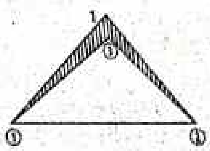


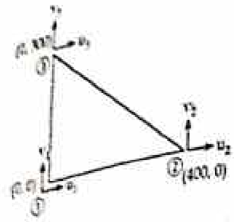
Figure: Variation of N_3

Unit conversion

$$E = 200\text{ GPa} = 200 \times 10^9 \times \frac{1}{10^3 \times 10^6}\text{ kN/mm}^2 = 200\text{ kN/mm}^2$$

$$5\text{ kN/m}^2 = 5 \times 10^{-6}\text{ kN/mm}^2$$

Step 1: Modeling



Step 2: Displacement matrix

$$\{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}; \text{ after applying boundary conditions.}$$

Step 3: Force matrix

Traction force in side ① → ②; we have,

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{t\, l_{1-2}}{6} \begin{Bmatrix} 0 \\ -2\times 5-5 \\ 0 \\ -5-2\times 5 \end{Bmatrix} \times 10^{-6}$$

$$= 666.67 \times \begin{Bmatrix} 0 \\ -15 \\ 0 \\ -15 \end{Bmatrix} \times 10^{-6} = \begin{Bmatrix} 0 \\ -0.01 \\ 0 \\ -0.01 \end{Bmatrix} \text{kN}$$

$$\therefore \quad \{F\} = \begin{Bmatrix} R_{1x} \\ R_{1y}-0.01 \\ 0 \\ -0.01 \\ R_{3x} \\ R_{3y} \end{Bmatrix}$$

Step 4:

$$2A = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 400 & 0 \\ 0 & 0 & 300 \end{vmatrix} = 120000$$

$$[B] = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where, $y_{23} = y_2 - y_3$, $y_{31} = y_3 - y_1$ and so on.

$$[B] = \frac{1}{120000} \begin{bmatrix} 300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -400 & 0 & 0 & 0 & 400 \\ -400 & -300 & 0 & 300 & 400 & 0 \end{bmatrix}$$

or,

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{200}{1-(0.3)^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= 219.78 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Step 5:

$$[K] = tA[B]^T[D][B] = tA[B]^T \times 219.78 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \times \frac{1}{120000} \begin{bmatrix} 300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -400 & 0 & 0 & 0 & 400 \\ -400 & -300 & 0 & 300 & 400 & 0 \end{bmatrix}$$

$$= 10 \times \frac{120000}{2} \times \frac{1}{1200} \times \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & -3 & 0 & 3 & 4 & 0 \end{bmatrix}^T \times \frac{219.78}{1200} \begin{bmatrix} -3 & -1.2 & 3 & 0 & 0 & 1.2 \\ -0.9 & -4 & 0.9 & 0 & 0 & 4 \\ -1.4 & -1.05 & 0 & 1.05 & 1.4 & 0 \end{bmatrix}$$

$$= 5 \times \frac{219.78}{12} \begin{bmatrix} 14.6 & 7.8 & -9 & -4.2 & -5.6 & -3.6 \\ & 19.15 & -3.6 & -3.15 & -4.2 & -16 \\ & & 9 & 0 & 0 & 3.6 \\ & & & 3.15 & 4.2 & 0 \\ & \text{sym} & & & 5.6 & 0 \\ & & & & & 16 \end{bmatrix}$$

Step 6: [F] = [K][u]

$$\begin{Bmatrix} R_{1x} \\ R_{1y}-0.01 \\ 0 \\ -0.01 \\ R_{3x} \\ R_{3y} \end{Bmatrix} = 91.575 \begin{bmatrix} 14.6 & 7.8 & -9 & -4.2 & -5.6 & -3.6 \\ & 19.15 & -3.6 & -3.15 & -4.2 & -16 \\ & & 9 & 0 & 0 & 3.6 \\ & & & 3.15 & 4.2 & 0 \\ & \text{sym} & & & 5.6 & 0 \\ & & & & & 16 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

or, $$\begin{Bmatrix} 0 \\ -0.01 \end{Bmatrix} = 91.575 \begin{bmatrix} 9 & 0 \\ 0 & 3.15 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$$

or, $$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{1}{91.575} \begin{bmatrix} 9 & 0 \\ 0 & 3.15 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -0.01 \end{Bmatrix}$$

or, $\begin{Bmatrix} 0 \\ -3.46 \times 10^{-5} \end{Bmatrix}$ mm

Step 7: Stresses at centroid

$$[\sigma] = [D][B][u]$$

or, $$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{219.78}{1200}\begin{bmatrix} -3 & -1.2 & 3 & 0 & 0 & 1.2 \\ -0.9 & -4 & 0.9 & 0 & 0 & 4 \\ -1.4 & -1.05 & 0 & 1.05 & 1.4 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

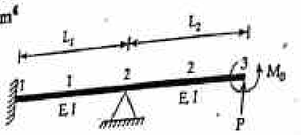
or, $$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{219.78}{1200}\begin{bmatrix} 3 & 0 \\ 0.9 & 0 \\ 0 & 1.05 \end{bmatrix}\begin{Bmatrix} 0 \\ -3.46 \times 10^{-5} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ -6.65 \times 10^{-6} \end{Bmatrix}$$

Problem 11

Determine the nodal displacements and reaction forces of the beam loaded as shown in the figure below.

Given that; $L_1 = L_2 = 3\,m$, $P = 15\,kN$, $M_0 = 40$ kN-m, $E_1 = E_2 = 2 \times 10^5$ MPa, $I_1 = I_2 = 5 \times 10^6\,mm^4$ **[2073 Bhadra]**



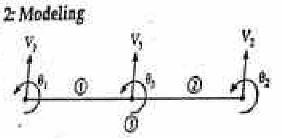
Solution:

Step 1: Unit conversion

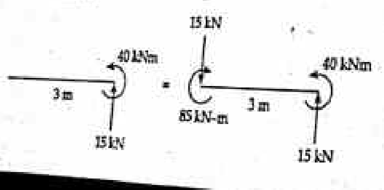
$$E_1 = E_2 = 2 \times 10^5\,MPa = 200 \times 10^6\,kN/m^2$$

$$I_1 = I_2 = 5 \times 10^6\,mm^4 = 5 \times 10^{-6}\,m^4$$

Step 2: Modeling



Step 3: Force matrix



$$\{F\} = \begin{Bmatrix} R_{x1} \\ M_1 \\ R_3 - 15 \\ M_3 - 85 \\ \overset{0}{\cancel{R_2}} + 15 \\ \overset{0}{\cancel{M_2}} + 40 \end{Bmatrix}$$; R and M are support reaction at positive directions.

Step 4: Displacement matrix

$$\{u\} = \begin{Bmatrix} V_1 \\ \theta_1 \\ V_3 \\ \theta_3 \\ V_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_2 \\ \theta_2 \end{Bmatrix}$$; applying boundary conditions.

Step 5: Stiffness matrix

$$[K_1] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & -6l \\ & 4l^2 & -6l & -2l^2 \\ & & 12 & -6l \\ & sym & & 4l^2 \end{bmatrix}$$

$$= \frac{200 \times 10^6 \times 5 \times 10^{-6}}{3^3}\begin{bmatrix} 12 & 18 & -12 & 18 \\ & 36 & -18 & 18 \\ & & 12 & -18 \\ & sym & & 36 \end{bmatrix}$$

$$= 100\begin{bmatrix} 4.44 & 6.67 & -4.44 & 6.67 \\ & 13.33 & -6.67 & 6.67 \\ & & 4.44 & -6.67 \\ & sym & & 13.33 \end{bmatrix}$$

$$[K_2] = \frac{1000}{(3)^3}\begin{bmatrix} 12 & 18 & -12 & 18 \\ & 36 & -18 & 18 \\ & & 12 & -18 \\ & sym & & 36 \end{bmatrix}$$

$$= 100\begin{bmatrix} 4.44 & 6.67 & -4.44 & 6.67 \\ & 13.33 & -6.67 & 6.67 \\ & & 4.44 & -6.67 \\ & sym & & 13.33 \end{bmatrix}$$

Assembling; we have,

$$[K] = 100\begin{bmatrix} 4.44 & 6.67 & -4.44 & 6.67 & & \\ & 13.33 & -6.67 & 6.67 & & \\ & & 8.88 & 0 & 4.44 & 6.67 \\ & & & 26.66 & 6.67 & 6.67 \\ & & & & 4.44 & 6.67 \\ & sym & & & & 13.33 \end{bmatrix}$$

Step 6: $\{F\} = [K]\{u\}$

Here,

$$\begin{Bmatrix} R_1 \\ M_1 \\ R_3-15 \\ M_3-85 \\ 15 \\ 40 \end{Bmatrix} = 100 \begin{bmatrix} 4.44 & 6.67 & -4.44 & 6.67 & 0 & 0 \\ & 13.33 & -6.67 & 6.67 & 0 & 0 \\ & & 8.88 & 0 & 4.44 & 6.67 \\ & & & 26.66 & -6.67 & 6.67 \\ & & & & 4.44 & 6.67 \\ sym & & & & & 13.33 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_2 \\ \theta_2 \end{Bmatrix}$$

By elimination reduced matrix is:

$$\begin{Bmatrix} 15 \\ 40 \end{Bmatrix} = 100 \begin{bmatrix} 4.44 & -6.67 \\ -6.67 & 13.33 \end{bmatrix} \begin{Bmatrix} V_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} V_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{100} \begin{bmatrix} 4.44 & -6.67 \\ -6.67 & 13.33 \end{bmatrix}^{-1} \begin{Bmatrix} 15 \\ 40 \end{Bmatrix} = \begin{Bmatrix} 0.318 \\ 0.189 \end{Bmatrix}$$

Step 7: Reaction calculation

From $\{F\} = [K]\{u\}$

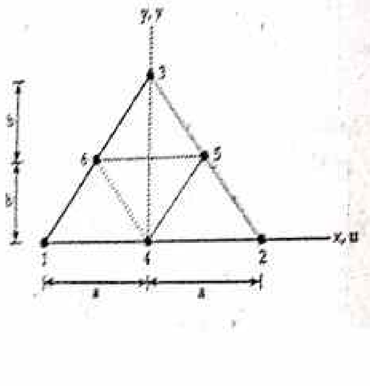
$$\begin{Bmatrix} R_1 \\ M_1 \\ R_3 - 15 \\ M_3 - 85 \end{Bmatrix} = 100 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -4.44 & 6.67 \\ -6.67 & 6.67 \end{bmatrix} \begin{Bmatrix} 0.318 \\ 0.189 \end{Bmatrix}$$

or, $$\begin{Bmatrix} R_1 \\ M_1 \\ R_3 - 15 \\ M_3 - 85 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -15.129 \\ -86.043 \end{Bmatrix}$$

or, $$\begin{Bmatrix} R_1 \\ M_1 \\ R_3 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.129 \\ -1.043 \end{Bmatrix}$$

Problem 12

Derive the shape functions Ni for the six-nodded triangular 2D element shown in the figure below. **[2073 Bhadra]**



Solution:

The value of N_1 should be 1 at node ① and 0 at other nodes, which can be expressed as;

$N_1 = 1$ at node 1

$N_1 = 0$ at line passing through ④ and ⑥

$N_1 = 0$ at line passing through ②, ⑤ and ③

Equation of line passing through ②, ⑤ and ③ is;

$$v = \frac{-2b}{a}u + 2b \qquad \left[\because \begin{array}{l} y = mx + C \\ m = \frac{y_2 - y_1}{x_2 - x_1} \end{array}\right]$$

or, $av = -2bu + 2ab$

or, $2bu + av - 2ab = 0$

Equation of line passing through ④ and ⑥ is;

$$v = \frac{-2b}{a}u \qquad \left[\because \begin{array}{l} \text{Slope is same} \\ m = 0 \end{array}\right]$$

or, $2bu + av = 0$

Let, $N_1 = C\{2bu + av - 2ab\}\{2bu + av\}$

$N_1 = 1$ at $u = -a, v = 0$

so, $1 = C\{-2ab - 2ba\}\{-2ab\}$

or, $C = \dfrac{1}{8a^2b^2}$

For node ④

$N_4 = 0$, for line passing through ②, ⑤ and ③

i.e., $2bu + av - 2ab = 0$

$N_4 = 0$, for line passing through ①, ⑥ and ③

i.e., $v = \dfrac{2b}{a} + 2b$

or, $av - 2b - 2ab = 0$

Equation of line passing through ①, ④ and ② is;

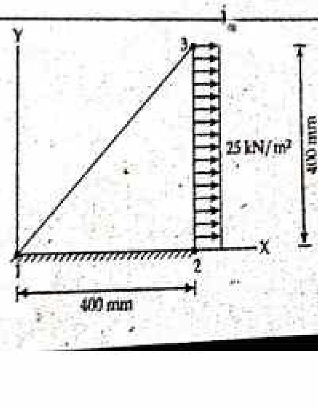
$v = 0$

Equation of line passing through ⑥ and ⑤ is;

$v = b$

Problem 13

A steel plate of thickness 8 mm is being loaded as shown in the figure. Considering the plane stress condition, determine the nodal displacements and stresses of the CST element. Take: $E = 210 \times 10^3$ MPa, $G = 105 \times 10^3$ MPa and unit weight of the steel is 78.5 kN/m³. **[2073 Bhadra]**



Solution:

Step 1: Unit conversion

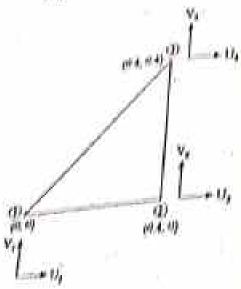
$$t = 8 \text{ mm} = 0.008 \text{ m}$$
$$E = 210 \times 10^3 \text{ MPa} = 210 \times 10^6 \text{ kN/m}^2$$
$$G = 105 \times 10^3 \text{ MPa} = 105 \times 10^6 \text{ kN/m}^2$$

We know that

$$G = \frac{E}{2(1+\nu)}$$

or, $$\text{Poisson's ratio } (\nu) = \frac{E}{2G} - 1 = 0$$

Step 2: Modeling



Step 3: Force matrix

$$\text{Body force} \begin{Bmatrix} f^b_{1y} \\ f^b_{2y} \\ f^b_{3y} \end{Bmatrix} = \frac{\gamma A t}{3} \begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix}$$

$$= \frac{78.5 \times \left(\frac{1}{2} \times 0.4 \times 0.4\right) \times 0.008}{3} \begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix}$$

$$= \begin{Bmatrix} -0.0167 \\ -0.0167 \\ -0.0167 \end{Bmatrix}$$

$$\text{Traction force} \begin{Bmatrix} f^T_{2x} \\ f^T_{2y} \\ f^T_{3x} \\ f^T_{3y} \end{Bmatrix} = \frac{t\, l_{(2)-(3)}}{6} \begin{Bmatrix} 2T_2 \cos\theta + T_3 \cos\theta \\ 2T_2 \sin\theta + T_3 \sin\theta \\ T_2 \cos\theta + 2T_3 \cos\theta \\ T_2 \sin\theta + 2T_3 \sin\theta \end{Bmatrix}$$

$$= \frac{0.008 \times 0.4}{6} \begin{Bmatrix} 2 \times 25 + 25 \\ 0 + 0 \\ 25 + 2 \times 25 \\ 0 \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} R_{1x} \\ R_{1y} - 0.0167 \\ R_{2x} + 0.04 - 0.0167 \\ R_{2y} - 0.0167 \\ 0.04 \\ -0.0167 \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} - 0.0167 \\ R_{2x} + 0.0233 \\ R_{2y} - 0.0167 \\ 0.04 \\ -0.0167 \end{Bmatrix};$$

R refers to the reaction force acting in the positive direction.

Step 4: Displacement matrix

$$\{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix};\ \text{applying boundary condition.}$$

Step 5: Stiffness matrix

$$2A = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0.4 & 0.4 \\ 0 & 0 & 0.4 \end{vmatrix} = 0.4 \times 0.4 = 0.16$$

$$B = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} -0.4 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 & 0.4 \\ 0 & -0.4 & -0.4 & 0.4 & 0.4 & 0 \end{bmatrix}$$

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{210 \times 10^6}{1-0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} = 210 \times 10^6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$[K] = tA[B]^T[D][B]$$

$$= tA[B]^T = 210 \times 10^6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \times \frac{1}{2A} \begin{bmatrix} -0.4 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 & 0.4 \\ 0 & -0.4 & -0.4 & 0.4 & 0.4 & 0 \end{bmatrix}$$

$$= tA \times \frac{1}{2A} \begin{bmatrix} -0.4 & 0 & 0 \\ 0 & 0 & -0.4 \\ 0.4 & 0 & -0.4 \\ 0 & -0.4 & 0.4 \\ 0 & 0 & 0.4 \\ 0 & 0.4 & 0 \end{bmatrix} \times \frac{210 \times 10^6}{0.16} \begin{bmatrix} -0.4 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 & 0.4 \\ 0 & -0.2 & -0.2 & 0.2 & 0.2 & 0 \end{bmatrix}$$

$$= \frac{0.009 \times 200 \times 10^6}{[illegible]} \begin{bmatrix} 0.16 & 0 & -0.16 & 0 & 0 & 0 \\ & 0.8 & 0.8 & -0.8 & -0.8 & 0 \\ & & 0.24 & -0.8 & -0.8 & 0 \\ & & & 0.24 & 0.8 & -0.16 \\ & & & & 0.8 & 0 \\ \text{sym} & & & & & 0.16 \end{bmatrix}$$

Step 6: $[F] = [K][u]$

$$\begin{Bmatrix} R_{1x} \\ R_{1y} - 0.0167 \\ R_{2x} + 0.0233 \\ R_{2y} - 0.0167 \\ 0.04 \\ -0.0167 \end{Bmatrix} = 5.25 \times 10^6 \begin{bmatrix} 0.16 & 0 & -0.16 & 0 & 0 & 0 \\ & 0.8 & 0.8 & -0.8 & -0.8 & 0 \\ & & 0.24 & -0.8 & -0.8 & 0 \\ & & & 0.24 & 0.8 & -0.16 \\ & & & & 0.8 & 0 \\ \text{sym} & & & & & 0.16 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 0.04 \\ -0.0167 \end{Bmatrix} = 5.25 \times 10^6 \begin{bmatrix} 0.8 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \frac{1}{5.25 \times 10^6} \begin{bmatrix} 0.8 & 0 \\ 0 & 0.16 \end{bmatrix}^{-1} \begin{Bmatrix} 0.04 \\ -0.0167 \end{Bmatrix}$$

$$= \begin{Bmatrix} 9.524 \times 10^{-9} \\ -1.988 \times 10^{-8} \end{Bmatrix}$$

Step 7: Strain and stress

$$\{\epsilon\} = [B]\{u\}$$

or, $$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{0.32} \begin{bmatrix} -0.4 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 & 0.4 \\ 0 & -0.4 & -0.4 & 0.4 & 0.4 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \frac{1}{0.32} \begin{bmatrix} 0 & 0 & 0 \\ -0.4 & 0 & 0.4 \\ 0.4 & 0.4 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 9.524 \times 10^{-9} \\ -1.988 \times 10^{-8} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ -2.485 \times 10^{-8} \\ 1.1905 \times 10^{-8} \end{Bmatrix}$$

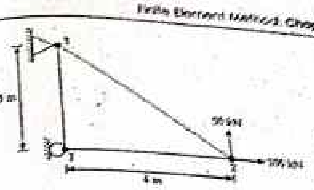
$$\{\sigma\} = [D]\{\epsilon\}$$

or, $$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 200 \times 10^6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{Bmatrix} 0 \\ -2.485 \times 10^{-8} \\ 1.1905 \times 10^{-8} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5.2185 \\ 1.25 \end{Bmatrix} \text{ kN/m}^2$$

Problem 14

Determine the nodal displacements, reaction forces, and member forces of the given truss structure, loaded as shown in the figure. Given that for each member, sectional area, $A = 2 \times 10^{-3}\ m^2$ and modulus of elasticity, $E = 2 \times 10^5$ MPa.

[2073 Magh]



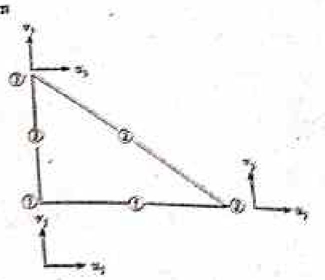
Solution:

Step 1: Unit conversion

$E = 2 \times 10^5$ MPa $= 200 \times 10^6$ kN/m²

$l_{2-3} = \sqrt{(3)^2 + (4)^2} = 5$ m

Step 2: Discretization



Step 3: Force matrix

$$[F] = \begin{Bmatrix} R_{1x} \\ \cancel{R_{1y}}\ 0 \\ 100 \\ 50 \\ R_{3x} \\ R_{3y} \end{Bmatrix}; \text{ R refer to the reactions acting in the positive direction.}$$

Step 4: Displacement matrix

$$[u] = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ v_1 \\ u_1 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}; \text{ by applying boundary conditions.}$$

Step 5: Stiffness matrix

For element ❶, [From ❶ to ❷]

$$[K_1] = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{2\times10^{-3}\times200\times10^{6}}{4}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 10^4 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

Similarly, for element 2. [From 2 to 3]

$$\tan(\phi) = \frac{3}{4}$$
$$\phi = 36.87°$$
$$\theta = 180° - 36.87° = 143.13°$$
$$S = \sin(143.13°) = 0.6$$
$$C = \cos(143.13°) = -0.8$$
$$CS = -0.48$$
$$C^2 = 0.64$$
$$S^2 = 0.36$$

Now,

$$[K_2] = \frac{A_2E_2}{L_2}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{sym} & & & S^2 \end{bmatrix}$$

$$= \frac{2\times10^{-3}\times200\times10^{6}}{5}\begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ & 0.36 & 0.48 & -0.36 \\ & & 0.64 & -0.48 \\ \text{sym} & & & 0.36 \end{bmatrix}$$

$$= 10^4 \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 5.12 & -3.84 & -5.12 & 3.84 \\ & 2.88 & 3.84 & -2.88 \\ & & 5.12 & -3.84 \\ \text{sym} & & & 2.88 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

For element 3, [From 3 to 1]

$$[K_3] = \frac{A_3E_3}{L_3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \frac{2\times10^{-3}\times200\times10^{6}}{3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= 10^4 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 13.33 & 0 & -13.33 \\ 0 & 0 & 0 & 0 \\ 0 & -13.33 & 0 & 13.33 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

Assembling the element matrices to get generalized stiffness matrix:

$$[K] = 10^4 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 10+0 & 0+0 & -10 & 0 & 0 & 0 \\ & 0+13.33 & 0 & 0 & 0 & -13.33 \\ & & 10+5.12 & 0-3.84 & -5.12 & 3.84 \\ & & & 0+2.88 & 3.84 & -2.88 \\ & & & & 5.12+0 & -3.84+0 \\ & \text{sym} & & & & 2.88+13.33 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

$$= 10^4 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ & 13.33 & 0 & 0 & 0 & -13.33 \\ & & 15.12 & -3.84 & -5.12 & 3.84 \\ & & & 2.88 & 3.84 & -2.88 \\ & & & & 5.12 & -3.84 \\ & \text{sym} & & & & 16.21 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

Step 6: $[T][K][T]^T$

For node 2;

$$\theta = 36.87°$$
$$C = 0.8$$
$$S = 0.6$$

$$[T] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [t] & [0] \\ [0] & [0] & [I] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T][K][T]^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 10^4 \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ & 13.33 & 0 & 0 & 0 & -13.33 \\ & & 15.12 & -3.84 & -5.12 & 3.84 \\ & & & 2.88 & 3.84 & -2.88 \\ & & & & 5.12 & -3.84 \\ \text{sym} & & & & & 16.21 \end{bmatrix} \times [T]^T$$

$$= 10^4 \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 13.33 & 0 & 0 & 0 & -13.33 \\ -8 & 0 & 14.4 & -4.8 & -6.4 & 4.8 \\ -6 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\ 0 & -13.33 & 3.84 & -2.88 & -3.84 & 16.21 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 10^4 \begin{bmatrix} 10 & 0 & -8 & -6 & 0 & 0 \\ & 13.33 & 0 & 0 & 0 & -13.33 \\ & & 14.4 & 4.8 & -6.4 & 4.8 \\ & & & 3.6 & 0 & 0 \\ & & & & 5.12 & -3.84 \\ \text{sym} & & & & & 16.21 \end{bmatrix}$$

Step 7: $[F] = [T][K][T]^T\{u\}$

$$\begin{Bmatrix} R_{1x} \\ 0 \\ 100 \\ 50 \\ R_{3x} \\ R_{3y} \end{Bmatrix} = 10^4 \begin{bmatrix} 10 & 0 & -8 & -6 & 0 & 0 \\ & 13.33 & 0 & 0 & 0 & -13.33 \\ & & 14.4 & 4.8 & -6.4 & 4.8 \\ & & & 3.6 & 0 & 0 \\ & & & & 5.12 & -3.84 \\ \text{sym} & & & & & 16.21 \end{bmatrix} \begin{Bmatrix} 0 \\ v_1 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

Reduced matrix after elimination is;

$$\begin{Bmatrix} 0 \\ 100 \\ 50 \end{Bmatrix} = 10^4 \begin{bmatrix} 13.33 & 0 & 0 \\ 0 & 14.4 & 4.8 \\ 0 & 4.8 & 3.6 \end{bmatrix} \begin{Bmatrix} v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

or, $$\begin{Bmatrix} v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \frac{1}{10^4} \begin{bmatrix} 13.33 & 0 & 0 \\ 0 & 14.4 & 4.8 \\ 0 & 4.8 & 3.6 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 100 \\ 50 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 4.167 \times 10^{-4} \\ 8.33 \times 10^{-4} \end{Bmatrix} \text{m}$$

Step 8: Reaction calculation

$$\begin{Bmatrix} R_{1x} \\ R_{3x} \\ R_{3y} \end{Bmatrix} = 10^4 \begin{bmatrix} 0 & -8 & -6 \\ 0 & -6.4 & 0 \\ -13.33 & 4.8 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 4.167 \times 10^{-4} \\ 8.33 \times 10^{-4} \end{Bmatrix}$$

$$= \begin{Bmatrix} -83.32 \\ -26.67 \\ 20 \end{Bmatrix} \text{kN}$$

Step 9: Member force

For element ❶;

$$f_1 = \frac{A_1E_1}{L_1}[-1 \quad 1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{2 \times 10^{-3} \times 200 \times 10^6}{4}[-1 \quad 1]\begin{Bmatrix} 0 \\ 4.167 \times 10^{-4} \end{Bmatrix}$$

$$= 41.67 \text{ kN}$$

For element ❷;

$\theta = 143.13°$

$S = 0.6$

$C = -0.8$

$$f_2 = \frac{A_2E_2}{L_2}[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \frac{2 \times 10^{-3} \times 200 \times 10^6}{5}[0.8 \quad -0.6 \quad -0.8 \quad 0.6]\begin{Bmatrix} 4.167 \times 10^{-4} \\ 8.33 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$= -13.32 \text{ kN}$$

For element ❸;

$$f_3 = \frac{A_3E_3}{L_3}[-1 \quad 1]\begin{Bmatrix} v_1 \\ v_3 \end{Bmatrix}$$

$$= \frac{2 \times 10^{-3} \times 200 \times 10^6}{3}[-1 \quad 1]\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$= 0 \text{ kN}$$

Problem 15

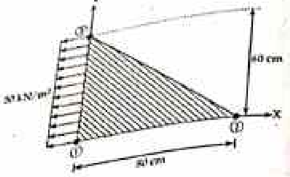
Derive the relation of strain-displacement [B] matrix for constant strain triangle. **[2073 Magh]**

Solution: See the definition part 4.8.1

Problem 16

A steel plate of 10 mm thick is loaded as shown in the figure below. For the plane stress problem, obtain the nodal deformations and the stresses in the CST

element. Take: $E = 2 \times 10^5$ MPa, $G = 105 \times 10^3$ MPa and unit weight of steel is $78.5\ kN/m^3$. [2073 Magh]



Solution: Proceed as the solution of Q. no. 13

CHAPTER 5

FINITE DIFFERENCE METHOD

5.1 FINITE DIFFERENCE METHOD

Finite difference methods (FDM) are numerical methods for solving differential equations by approximating them with difference equations. FDMs are discretization methods. The continuous variables are represented by their values at a finite set of points, and derivatives are approximated by differences between values at adjacent points.

Features of FDM

- The domain is discretized in regularly spaced grid/mesh.
- Governing equations are approximated point-wise.
- The value of a function at a point is related to nearby points.
- This method is simple and easy to implement but irregular geometry is difficult to handle with.

5.2 DERIVATION OF DIFFERENCE EQUATIONS FROM TAYLOR'S SERIES

As stated before, in FDM, we convert differential equations to finite difference equations and solve them. We can convert differential equations into difference equations by the help of Taylor's series. The standard form of Taylor's series for a function f(x) of single variable x is:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) \quad [5.2.1]$$

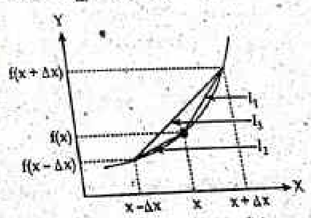


Figure: Curve representing f(x)

Here, in the space domain, we want to derive the difference equation for derivative of function at location x.

The value of function at location $(x + \Delta x)$ after applying Taylor series is:

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2!}f''(x) + \frac{\Delta x^3}{3!}f'''(x) + \ldots\ldots$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) + \frac{\Delta x}{2!}f''(x) + \frac{\Delta x^2}{3!}f'''(x) + \ldots\ldots\ldots \quad [5.2.2\,(a)]$$

or,

Neglecting terms of higher power.

$$f'(x) = \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad [5.2.3\,(a)]$$

L.H.S. of equation [5.2.3 (a)] is the derivative of function f(x) at position x. R.H.S. is the difference equation to approximate it. Likewise, if we derive the difference equation for a derivative for a function at a point (here derivative of f(x) at x) by comparing its functional value with functional value at the position forward to it (here at x + Δx), the approximation is called forward difference approximation.

Slope of line l_1 in the figure 5.1 gives forward difference approximation of the function at location x.

Again, the value of the function at the location x − Δx is:

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2!}f''(x) - \frac{\Delta x^3}{3!}f'''(x) + \ldots\ldots\ldots$$

or,

$$\frac{f(x-\Delta x)-f(x)}{\Delta x} = -f'(x) + \frac{\Delta x}{2!}f''(x) - \frac{\Delta x^2}{3!}f'''(x) + \ldots\ldots\ldots \quad [5.2.2\,(b)]$$

Neglecting the higher order terms.

$$f'(x) = \frac{f(x)-f(x-\Delta x)}{\Delta x} \quad [5.2.3\,(b)]$$

We compare the value of the function at the location x with the function value at the location (x − Δx), which is back to x, to get derivative of the function at the location x. So, it is backward difference approximation.

Slope of line l_2 in the figure 5.2 gives backward difference approximation of the function at location x.

Subtract equation [5.2.2 (b)] from equation [5.2.2 (a)] or equation [5.2.3 (b)] from [5.2.3 (a)]; we get,

$$f'(x) = \frac{f(x+\Delta x)+f(x-\Delta x)}{2\Delta x} \quad [5.2.3\,(c)]$$

As done here, if you approximate the derivative of a function at a point x by a difference equation formed by the comparison/difference of functional values at position forward (x + Δx) and backward (x − Δx) of the position of our concern i.e., x, it is called central difference approximation.

Slope of line l_3 in the figure 5.3 gives central difference approximation of the function at location x.

Add up equation [5.2.2 (a)] and [5.2.2 (b)]; we get,

$$f''(x) = \frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{\Delta x^2} \quad [5.2.4]$$

This is the difference equation to approximate the value of second derivative.

5.3 ORDER OF ACCURACY OF SCHEMES

Order of accuracy shows how closely the solution of difference equation approximates the solution of differential equation.

Order of accuracy can be quantified by the lowest power of Δx in truncation error.

Compare equation [5.2.2 (a)] and [5.2.3 (a)], the neglected terms are:

$$\frac{\Delta x}{2!}f''(x) + \frac{\Delta x^2}{3!}f'''(x) + \ldots\ldots\ldots;$$

which is the error of our approximation.

The lowest power of Δx in this truncation error is 1. Therefore, the forward difference approximation is first order accurate. Likewise, backward difference approximation is also first order accurate.

Subtracting equation [5.2.2 (b)] from [5.2.2 (a)]; we get,

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} - \frac{f(x-\Delta x)-f(x)}{\Delta x} = 2f'(x) + 2\frac{\Delta x^2}{3!}f'''(x) + \ldots\ldots\ldots$$

or,

$$f'(x) = \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x} - \{\Delta x^2 f'''(x) + \Delta x^4 f'''''(x) + \ldots\ldots\ldots\}$$

The terms in parenthesis are truncation error. The lowest power of Δx is 2. Therefore, central difference approximations are always second order accurate.

5.4 EXPLICIT AND IMPLICIT SCHEMES

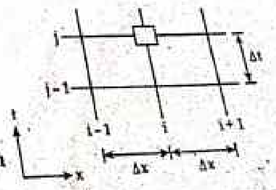
If we approximate the solution at time step n applying the conditions at time step n − 1, i.e., conditions before hand of time of our concern, the scheme is explicit.

If we approximate the solution at time step n, applying the conditions at time step n as well as time step n − 1, i.e., the condition at the time of our concern as well as the condition at the time before hand, the scheme is implicit.

	Explicit scheme	Implicit scheme
i)	If we approximate the solution of time step n applying conditions at time step n − 1, the scheme is explicit.	If we approximate the solution at time step n applying conditions at n and n − 1, the scheme is implicit.
ii)	The solution algorithm is simple.	The solution algorithm of implicit scheme requires more involved process.
iii)	The equations are solved in sequence.	Simultaneous solving of the equations is necessary.
iv)	Explicit schemes can be unstable, so puts restriction on time step for the given spatial step.	Stability can be obtained for large value of time step.
v)	It is not suitable for simulation over long time period.	It is suitable for simulation over long time period.
vi)	It can treat slightly varying channel geometry from section to section.	It can handle channel geometry varying significantly from one section to next.

EXAMPLE 5.1

Derive different forms of difference equations to approximate the value of derivative of a function f(x) in i at time step j.



□ → point of our concern.

f_i^j represents the value of function in position i at time step j.

Solution:

Forward explicit scheme

$$f' = \frac{f_{i+1}^{j-1} - f_i^{j-1}}{\Delta x}$$

Forward implicit scheme

$$f' = \frac{f_{i+1}^{j} - f_i^{j}}{\Delta x}$$

Backward explicit scheme

$$f' = \frac{f_i^{j-1} - f_{i-1}^{j-1}}{\Delta x}$$

Backward implicit scheme

$$f' = \frac{f_i^{j} - f_{i-1}^{j}}{\Delta x}$$

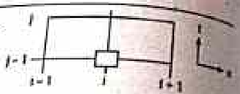
Central explicit scheme

$$f' = \frac{f_{i+1}^{j-1} - f_{i-1}^{j-1}}{2\Delta x}$$

Central implicit scheme

$$f' = \frac{f_{i+1}^{j} - f_{i-1}^{j}}{2\Delta x}$$

EXAMPLE 5.2



Values of discharge (Q) in m³/s at three points in the space-time grid are given as $Q_{i-1}^{j-1} = 39$, $Q_i^{j-1} = 39.1$ and $Q_{i+1}^{j-1} = 39.2$.

Taking $\Delta x = 700$ m and $\Delta t = 7$ minute, compute $\frac{dQ}{dx}$ **at location i using:**

i) **backward difference approximation**

ii) **Central difference approximation**

iii) **Forward difference approximation**

Also, find $\frac{\partial^2 Q}{\partial x^2}$.

Solution:

i) Backward difference $\left(\frac{\partial Q}{\partial x}\right) = \frac{Q_i^{j-1} - Q_{i-1}^{j-1}}{\Delta x} = \frac{39.1 - 39}{700} = \frac{1}{7000}$

ii) Central difference $\left(\frac{\partial Q}{\partial x}\right) = \frac{Q_{i+1}^{j-1} - Q_{i-1}^{j-1}}{2\Delta x} = \frac{39.2 - 39}{2 \times 700} = \frac{1}{7000}$

iii) Forward difference $\left(\frac{\partial Q}{\partial x}\right) = \frac{Q_{i+1}^{j-1} - Q_i^{j-1}}{\Delta x} = \frac{39.2 - 39.1}{700} = \frac{1}{7000}$

$$\therefore \quad \frac{\partial^2 Q}{\partial x^2} = \frac{Q_{i+1}^{j-1} - 2Q_i^{j-1} + Q_{i-1}^{j-1}}{\Delta x^2} = \frac{39.2 - (2 \times 39.1) + 39}{(700)^2} = 0$$

5.5 GOVERNING EQUATIONS OF FLUID MECHANICS

i) **Steady and uniform flow**

a) **Continuity equation**

$$AV = \text{Constant}$$

b) **Bernoulli's principle (energy principle)**

$$\frac{P_1}{\gamma} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_f; \text{ for pipe flow}$$

or, $$y_1 + \frac{\alpha_1 V_1^2}{2g} + z_1 = y_2 + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_f; \text{ for channel flow}$$

c) **Momentum principle**

$$\Sigma F = \rho Q(\beta_2 V_2 - \beta_1 V_1)$$

ii) **Unsteady, non-uniform flow in open channel**

a) **Continuity equation**

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q$$

b) **Momentum equation**

$$\frac{\partial Q}{\partial t} + \frac{\partial\left(\frac{\beta Q^2}{A}\right)}{\partial x} + gA\left(\frac{\partial y}{\partial x} + S_f - S_0 + S_e\right) + W_f b - \beta q V_x = 0$$

iii) **Unsteady pipe flow**

a) **Continuity equation**

$$\frac{\partial P}{\partial t} + V\frac{\partial P}{\partial x} + \rho C^2 \frac{\partial V}{\partial x} = 0$$

b) **Momentum equation**

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} + g\sin\theta + \frac{fV|V|}{2D} = 0$$

iv) **Groundwater flow**

a) **Darcy's law**

$$v = -k\frac{\partial h}{\partial x}$$

b) **Continuity equation**

Inflow − Outflow = Storage

The notations used in the equations are:

P is the pressure at a section.

v is the velocity at a section.

z is the elevation at a section.

y is the depth of the flow at a section.

h_f is the head loss.

α is the kinetic energy correction factor.

β is the momentum correction factor.

γ is the unit weight of the liquid.

ρ is the density of the liquid.

A is the area of the section.
x is the position of the section considered.
t is the time.
q is the lateral flow per unit width.
Q is the discharge at a section.
S_f is the energy slope.
S_0 is the bed slope.
S_e is the eddy loss slope.
W_f is the wind shear factor.
b is the top width of the channel.
V_x is the velocity of lateral flow in x-direction.
g is the acceleration due to gravity.
c is the celerity of the wave.
θ is the slope of the pipe.
f is the friction factor.
D is the diameter of the pipe.
k is the hydraulic conductivity.
$\frac{\partial h}{\partial x}$ is the hydraulic gradient.

5.6 CONSISTENCY AND CONVERGENCE

Consistency *talks about the equations. If the difference equation tends to be actual differential equation when Δx and Δt tends to be zero, the resulting solution is said to be consistent.*

Convergence *talks about the solution. If the solution by difference equations tends to be actual solution of the differential equation when Δx and Δt tends to be zero, solution is said to be convergent.*

5.7 UNSTEADY NON-UNIFORM FLOW EQUATION IN OPEN CHANNEL: SAINT VENANT EQUATIONS

Saint Venant equations are based on the following assumptions:

i) *The channel section is prismatic.*
ii) *The flow is one dimensional with uniform velocity in a cross-section.*
iii) *The effect of hydrostatic pressure is significant.*
iv) *Vertical acceleration can be neglected.*
v) *Channel has small bottom slope.*
vi) *We deal with incompressible fluid.*
vii) *Manning's equation is applicable.*

Full Saint venant equations are:

i) ***Continuity equation***

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=q \qquad [5.7.1\ (a)]$$

ii) ***Momentum equation***

$$\frac{\partial Q}{\partial t}++\frac{\partial\left(\frac{\beta Q^2}{A}\right)}{\partial x}+gA\left(\frac{\partial y}{\partial x}+S_f-S_0+S_e\right)+W_f b-\beta q V_x=0 \qquad [5.7.1\ (b)]$$

where, Q is the discharge at x
A is the cross-sectional area at x
q is the lateral flow per unit width,
β is the momentum correction factor,
S_0 is the bed slope,
S_f is the energy slope,
S_e is the eddy loss slope,
W_f is the wind shear factor,
b is the top width of water surface,
V_x is the velocity of lateral flow in x-direction,

Full equation is simplified assuming:

$q=0$
$\beta=1$
$S_e=0$
$W_f=0$

The simplified form is:

i) **Continuity equation:**

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=0 \qquad [5.7.2\ (a)]$$

ii) **Momentum equation:**

$$\frac{\partial Q}{\partial t}+\frac{\partial\left(\frac{Q^2}{A}\right)}{\partial x}+gA\frac{\partial y}{\partial x}-gAS_0+gAS_f=0 \qquad [5.7.2\ (b)]$$

Each terms of momentum equation represents effect of different types of forces:

- First term represents inertia force due to local acceleration
- Second term represents inertia force due to convective acceleration
- Third term represents pressure force
- Fourth term represents gravity force
- Fifth term represents friction force

Above equations are expressed in terms of Q and A. These are **conservative forms** of Saint Venant equations.

If the equations are expressed in terms of velocity, V and flow depth, y, they are said to be in **non-conservative form.** Non-conservative form of Saint Venant equations are as follows:

i) **Continuity equation**

$$\frac{\partial y}{\partial t}+\frac{A}{T}\frac{\partial V}{\partial t}+V\frac{\partial y}{\partial x}=0 \qquad [5.7.3\ (a)]$$

ii) **Momentum equation**

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+g\frac{\partial y}{\partial x}-g(S_0-S_f)=0 \qquad [5.7.3\ (b)]$$

5.8 COMMON SIMPLIFICATIONS OF SAINT VENANT EQUATIONS

5.8.1 Dynamic wave

The dynamic wave is the term used to describe the full Saint Venant equation. Equation [5.7.2 (a)] and [5.7.2 (b)] represent the dynamic wave. This form is numerically challenging to solve but is valid for all channel flow scenarios. The dynamic wave is used in modeling programs like HEC-RAS, SWMM, etc.

5.8.2 Diffusive wave model

For the diffusive wave, it is assumed that the inertial terms are negligible in comparison of gravity, friction and pressure forces. The diffusive wave can therefore be more accurately described as a non-inertia wave, and can be represented by the following expressions:

i) *Continuity equation*

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=0 \quad [5.8.1(a)]$$

ii) *Momentum equation*

$$\frac{\partial y}{\partial x}=S_0-S_f \quad [5.8.1(b)]$$

Diffusive wave is applicable in modeling backwater effect, steep slope channel, etc.

5.8.3 Kinematic wave

For the kinematic wave, it is assumed that the flow is uniform and that the friction slope is approximately equal to slope of the channel. By neglecting first three terms of momentum equations, we get kinematic wave model of Saint Venant equations as:

i) *Continuity equation*

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=q \quad [5.8.2(a)]$$

ii) *Momentum equation*

$$S_0=S_f \quad [5.8.2(b)]$$

Equation [5.8.2 (b)] shows gravity force and friction force balance each other in this model. This model can be described principally by the continuity equation. Since no force term is included; the wave model is termed as kinematic wave model.

The kinematic wave is valid when the change in wave height over distance and velocity over distance and time is negligible relative to the bed slope, e.g.; for shallow flow over steep slopes.

In kinematic wave model, A can be expressed in terms of Q as:

$$A=\alpha Q^{\beta} \quad [5.8.3]$$

Differentiating with respect to t; we get,

$$\frac{\partial A}{\partial t}=\alpha\beta Q^{\beta-1}\frac{\partial Q}{\partial t} \quad [5.8.4]$$

Substituting into equation [5.8.2 (a)]; we get,

$$\frac{\partial Q}{\partial x}+\alpha\beta Q^{\beta-1}\frac{\partial Q}{\partial t}=q \quad [5.8.5]$$

Equation [5.8.5] is another form of kinematic wave model. α and β in these expressions are obtained from Manning's equation.

$$Q=\frac{1}{n}AS^{\frac{1}{2}}R^{\frac{2}{3}} \quad [5.8.6]$$

where, R is the hydraulic radius $=\frac{A}{P}$

P is the wetted perimeter.

$$Q=\frac{1}{n}\frac{A^{\frac{5}{3}}}{P^{\frac{2}{3}}}S^{\frac{1}{2}}$$

$$A=\left(\frac{nP^{\frac{2}{3}}}{\sqrt{S}}\right)^{\frac{3}{5}}Q^{\frac{3}{5}}$$

$$A=\alpha Q^{\beta}$$

where, $\alpha=\left(\frac{nP^{\frac{2}{3}}}{\sqrt{S}}\right)^{0.6}$ [5.8.7(a)]

$\beta=0.6$ [5.8.7(b)]

5.9 STABILITY OF NUMERICAL SCHEME

A solution is stable if it doesn't diverges from the exact analytical solution due to accumulation of round-off and truncation errors.

For an explicit scheme to be stable, it should fulfill courant condition,

i.e., Courant number ≤ 1

where, $\text{Courant number}=\frac{\text{Actual wave speed}}{\text{Numerical wave speed}}=\frac{V+|c|}{\frac{\Delta x}{\Delta t}}$ [5.9.1]

[5.9.2]

V is the velocity of flow.

c is the celerity of wave.

5.9.1 Stability of explicit dynamic wave model

An explicit dynamic wave model is stable when:

$$\text{Courant number}=\frac{V+|c|}{\frac{\Delta x}{\Delta t}}\leq 1$$

where, $c=\sqrt{gy}$ [5.9.3]

y is the hydraulic depth $=\frac{A}{T}$

A is the cross-section area of the flow.

T is the top width.

5.9.2 Stability of explicit kinematic wave model

An explicit kinematic wave is stable when:

$$\text{Courant number}=\frac{c}{\frac{\Delta x}{\Delta t}}\leq 1 \quad [5.9.4(a)]$$

or, $\Delta t\leq\frac{\Delta x}{c}$ [5.9.4(b)]

Here, velocity term being negligible is omitted.

and, $c=\frac{dQ}{dA}=\frac{1}{b}\frac{dQ}{dy}$ [5.9.4(c)]

where, b is the channel width.

While doing the numerical solution in space-time domain, Δx is kept fixed and Δt is evaluated from courant condition.

EXAMPLE 5.3

A channel of width 40 m, bed slope 2%, Manning's n = 0.03 carries a discharge of 100 m³/sec. through a section. If Δx is taken as 1500 m, recommend the maximum time step for stable condition of kinematic wave routing in the condition. Assume hydraulic radius is equal to flow depth. **[2072 Ashwin]**

Solution:

Given that:

$$Q = 100\ m^3/sec.$$
$$b = 40\ m$$
$$S = 2\% = 0.02$$
$$n = 0.03$$
$$\Delta x = 1500$$
$$R = y$$
$$\Delta t = ?$$

We know that:

$$Q = \frac{1}{n} A R^{\frac{2}{3}} S^{\frac{1}{2}}$$

or, $$Q = \frac{1}{0.03} \times b \times y \times y^{\frac{2}{3}} \times (0.02)^{\frac{1}{2}}$$

or, $$Q = 4.714 b y^{\frac{5}{3}} \qquad (1)$$

or, $$100 = 4.714 \times 40 \times y^{\frac{5}{3}}$$

∴ $$y = 0.68\ m$$

Differentiating (1) with respect to A; we get,

$$\frac{dQ}{dA} = \frac{1}{b}\frac{dQ}{dy} = \frac{1}{b} \times 4.714 b \times \frac{5}{3} y^{\frac{2}{3}}$$

∴ $$c = \frac{dQ}{dA} = 4.714 \times \frac{5}{3} \times (0.68)^{\frac{2}{3}} = 6.075\ m/sec.$$

Now, courant condition

$$\Delta t \le \frac{\Delta x}{c} = \frac{1500}{6.075} = 246.91\ sec. = 4.11\ min$$

Hence, take time step (Δt) = 4 min.

5.10 Solution steps in FDM

i) Division of domain into a set of regularly spaced grid points.

ii) Conversion of partial differential equation to algebraic difference equation at grid points.

iii) Algorithm showing steps for solving algebraic equations.

5.11 NUMERICAL SCHEMES FOR SAINT VENANT EQUATIONS

5.11.1 Common schemes of finite difference approximations

A wide variety of finite difference schemes exist for solving Saint Venant equations. A few of these, which are in common use, are presented below:

i) **Diffusing scheme**

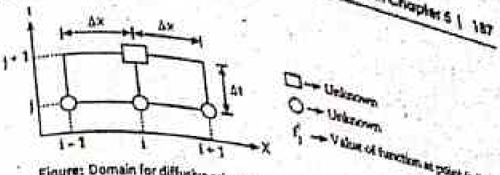


Figure: Domain for diffusive scheme

Order of accuracy: first order

Space derivative: $\frac{\partial f}{\partial x} = \frac{f_{i+1}^{j} - f_{i-1}^{j}}{2\Delta x}$

Time derivative: $\frac{\partial f}{\partial t} = \frac{f_i^{j+1} - \frac{1}{2}(f_{i+1}^{j} + f_{i-1}^{j})}{\Delta t}$

f as a coefficient: $f = \frac{f_{i-1}^{j} + f_{i+1}^{j}}{2}$

ii) **Upstream differencing scheme**

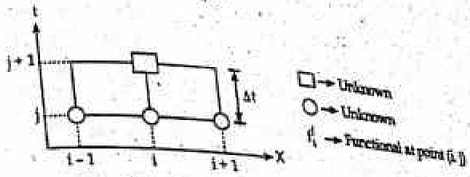


Figure: Domain for upstream differencing scheme

Order of accuracy: first order

Space derivative: $\frac{\partial f}{\partial x} = \frac{f_{i+1}^{j} - f_i^{j}}{\Delta x}$ or $\frac{f_i^{j} - f_{i-1}^{j}}{\Delta x}$

Time derivative: $\frac{\partial f}{\partial t} = \frac{f_i^{j+1} - f_i^{j}}{\Delta t}$

f as a coefficient: $f = f_i^{j}$ or $\frac{f_{i-1}^{j} + f_{i+1}^{j}}{2}$ or $\frac{f_{i-1}^{j} + f_i^{j} + f_{i+1}^{j}}{3}$

iii) **Leap-frog scheme**

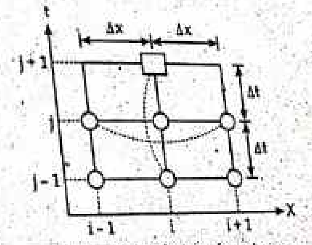


Figure: Domain for Leap-frog scheme

Order of accuracy: second order

Space derivative: $\frac{\partial f}{\partial x} = \frac{f_{i+1}^{j} - f_{i-1}^{j}}{2\Delta x}$

Time derivative: $\frac{\partial f}{\partial t} = \frac{f_i^{j+1} - f_i^{j-1}}{2\Delta t}$

f as a coefficient: $f = f_i^j$ or $\frac{f_{i-1}^j + f_{i+1}^j}{2}$ or $\frac{f_{i-1}^j + f_i^j + f_{i+1}^j}{3}$ or $\frac{f_{i-1}^{j+1} + f_i^j}{2}$

iv) Four point implicit scheme

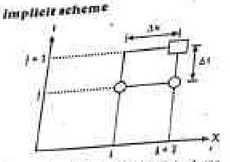


Figure: Domain for four point implicit scheme

Order of accuracy: second order

Space derivative: $\frac{\partial f}{\partial x} = \frac{1}{2\Delta x}\left(f_{i+1}^j + f_{i+1}^{j+1} - f_i^j - f_i^{j+1}\right)$

Time derivative: $\frac{\partial f}{\partial t} = \frac{1}{2\Delta t}\left(f_i^{j+1} + f_{i+1}^{j+1} - f_i^j - f_{i+1}^j\right)$

f as a coefficient: $f = \frac{1}{4}\left(f_{i+1}^{j+1} + f_i^{j+1} + f_{i+1}^j + f_i^j\right)$

5.11.2 Linear schemes for kinematic wave model

i) First order accurate implicit scheme

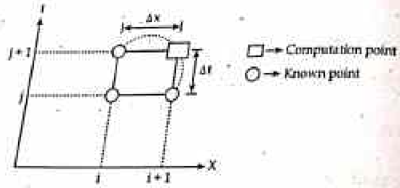


Figure: Space-time domain for first order accurate implicit kinematic scheme

Channel properties, initial and boundary conditions are given. Discretize the equation [5.8.5].

i.e., $\frac{\partial Q}{\partial x} + \alpha\beta Q^{\beta-1}\frac{\partial Q}{\partial t} = q$; into arithmetic difference equation to solve for Q_{i+1}^{j+1}

First order accurate refers to either forward or backward difference approximation. Implicit scheme refers to use condition at time step of computation point *i.e.,* $j+1$ time step.

About non-derivative terms *i.e.,* Q and q take Q average of average along diagonal. q is taken average of the values at the location of computation *i.e.,* $i+1$ position.

In summary,

$$\frac{\partial Q}{\partial x} = \frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x}$$

$$\frac{\partial Q}{\partial t} = \frac{Q_{i+1}^{j+1} - Q_{i+1}^j}{\Delta x}$$

$$Q = \frac{Q_i^{j+1} + Q_{i+1}^j}{2}$$

$$q = \frac{q_{i+1}^{j+1} + q_{i+1}^j}{2}$$

Substituting these values in $\frac{\partial Q}{\partial x} + \alpha\beta Q^{\beta-1}\frac{\partial Q}{\partial t} = q$; we have,

$$\frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x} + \alpha\beta\left(\frac{Q_i^{j+1} + Q_{i+1}^j}{2}\right)^{\beta-1}\frac{Q_{i+1}^{j+1} - Q_{i+1}^j}{\Delta t} = \frac{q_{i+1}^{j+1} + q_{i+1}^j}{2}$$

or,
$$Q_{i+1}^{j+1} = \frac{\frac{\Delta t}{\Delta x}Q_i^{j+1} + \alpha\beta Q_{i+1}^j\left(\frac{Q_{i+1}^j + Q_i^{j+1}}{2}\right)^{\beta-1} + \left(\frac{q_{i+1}^{j+1} + q_{i+1}^j}{2}\right)\Delta t}{\frac{\Delta t}{\Delta x} + \alpha\beta\left(\frac{Q_{i+1}^j + Q_i^{j+1}}{2}\right)^{\beta-1}} \quad [5.11.1]$$

Algorithm for flow routing

1. Introduce initial condition at $t = 0, j = 1$.
2. Check for Δt for every location using courant condition; choose the minimum value as Δt.
3. Move to next time step $t = t + \Delta t$, $j = j + 1$ and location $i = 2$.
4. Assign value of Q_1^j from boundary condition.
5. Evaluate Q_i^j using kinematic wave formula of Q_{i+1}^{j+1} from equation [5.11.1].
6. Go to next position $i = i + 1$.
7. Repeat step 5 until i is last point on grid at time step j.
8. Go to step (3) for remaining time period.
9. Stop.

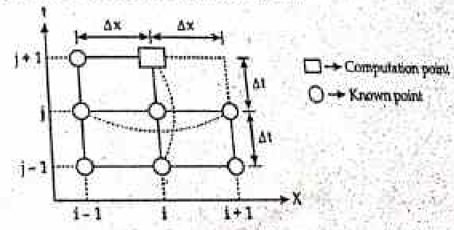
ii) First order accurate explicit scheme

Discretize $\frac{\partial Q}{\partial x}$ at j time step

i.e., $\frac{\partial Q}{\partial x} = \frac{Q_{i+1}^j - Q_i^j}{\Delta x}$

Other things, similar as in section 5.11.1 (a).

iii) Second order accurate explicit scheme



Second order accurate refers to central difference approximation.

Explicit scheme means we use conditions beforehand of time step $j+1$, *i.e.,* time step j.

For linear scheme, we solve the equation:

$$\frac{\partial Q}{\partial x} + \alpha\beta Q^{\beta-1}\frac{\partial Q}{\partial t} = q \quad [5.8.5]$$

So, we can approximate the equation by:

$$\frac{\partial Q}{\partial x} = \frac{Q_{i+1}^{j} - Q_{i-1}^{j}}{2\Delta x}$$

$$\frac{\partial Q}{\partial t} = \frac{Q_{i}^{j+1} - Q_{i}^{j-1}}{2\Delta t}$$

Similar to previous schemes, non-derivative term 'Q' is taken average of diagonal nodes. q is taken as average of q value at the location of our concern i.e., i.

i.e., $$Q = \frac{Q_{i-1}^{j+1} + Q_{i}^{j} + Q_{i+1}^{j-1}}{3}$$

$$q = \frac{q_{i}^{j-1} + q_{i}^{j} + q_{i}^{j+1}}{3}$$

Substituting these values into equation [5.8.5]; we get,

$$\frac{Q_{i+1}^{j} - Q_{i-1}^{j}}{2\Delta x} + \alpha\beta\left(\frac{Q_{i-1}^{j+1} + Q_{i}^{j} + Q_{i+1}^{j-1}}{3}\right)^{\beta-1}\left(\frac{Q_{i}^{j+1} - Q_{i}^{j-1}}{2\Delta t}\right) = \frac{q_{i}^{j-1} + q_{i}^{j} + q_{i}^{j+1}}{3}$$

or, $$Q_{i}^{j+1} = \frac{-\frac{\Delta t}{\Delta x}\left(Q_{i+1}^{j} - Q_{i-1}^{j}\right) + \alpha\beta\left(\frac{Q_{i+1}^{j-1} + Q_{i}^{j} + Q_{i-1}^{j+1}}{3}\right)^{\beta-1} + \left(\frac{q_{i}^{j+1} + q_{i}^{j} + q_{i}^{j-1}}{3}\right)2\Delta t}{\alpha\beta\left(\frac{Q_{i+1}^{j-1} + Q_{i}^{j} + Q_{i-1}^{j+1}}{3}\right)^{\beta-1}} \quad [5.11.2]$$

EXAMPLE 5.4

Water flows through a rectangular channel 25 m wide, having bed slope 0.015 and Manning's n = 0.035. The following flow rates are given: $Q_i^{j+1} = 30\ m^3/sec.$, $Q_i^j = 22\ m^3/sec.$ and $Q_{i+1}^j = 20\ m^3/sec.$;

where, i is the index for space.

j is the index for time.

Taking $\Delta x = 1500$ m and $\Delta t = 10$ min, determine Q_{i+1}^{j+1} using first order accurate linear kinetic wave model. Assume lateral flow to be zero and wetted perimeter is equal to width of the channel. **[2072 Magh]**

Solution:

Given that;

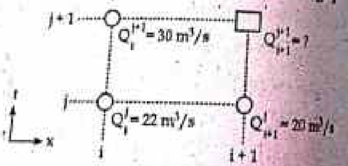
$b = 25$ m

$S = 0.015$

$n = 0.035$

$q = 0$

$P = b = 25$ m



Now, Manning's equation is given by:

$$Q = \frac{1}{n}AS^{\frac{1}{2}}R^{\frac{2}{3}} = \frac{1}{n}AS^{\frac{1}{2}}\left(\frac{A}{P}\right)^{\frac{2}{3}} = \frac{A^{\frac{5}{3}}S^{\frac{1}{2}}}{nP^{\frac{2}{3}}}$$

or, $$A = \left(\frac{nP^{\frac{2}{3}}}{S^{\frac{1}{2}}}\right)^{\frac{3}{5}}Q^{\frac{3}{5}} = \left[\frac{0.035 \times (25)^{\frac{2}{3}}}{(0.015)^{\frac{1}{2}}}\right]^{\frac{3}{5}}Q^{\frac{3}{5}}$$

Comparing with $A = \alpha Q^{\beta}$; we get,

$\alpha = 1.709$

and, $\beta = 0.6$

Again,

$$Q_{i+1}^{j+1} = \frac{\frac{\Delta t}{\Delta x}Q_i^{j+1} + \alpha\beta Q_{i+1}^{j}\left(\frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}}{\frac{\Delta t}{\Delta x} + \alpha\beta\left(\frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}} \quad [\because q = 0]$$

or, $$Q_{i+1}^{j+1} = \frac{\frac{10 \times 60}{1500} \times 30 + 1.709 \times 0.6 \times 20 \times \left(\frac{20 + 30}{2}\right)^{0.6-1}}{\frac{10 \times 60}{1500} + 1.709 \times 0.6 \times \left(\frac{20 + 30}{2}\right)^{0.6-1}} = \frac{17.66}{0.683} = 25.87\ m^3/s$$

EXAMPLE 5.5

A flood of 150 m^3/sec. peak discharge passed a gauging station at 12:00 noon on a river. There is a community adjacent to the river 7.2 km downstream. What will be the value of peak discharge at the community at 12:00 noon if the velocity of flow is 1.2 m/sec. and peak discharge at the community at 9:00 A.M. was 100 m^3/sec. Use first order accurate scheme of kinematic wave function. Take: $\Delta x = 7.2$ km and $\Delta t = 1$ hr **[2070 Magh]**

Solution:

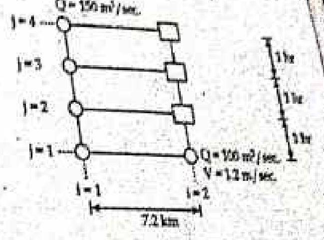
Given that;

At the community at 9:00 A.M.;

$Q = 100\ m^3/sec.$

$V = 1.2\ m/sec.$

$\therefore \quad A = \frac{Q}{V} = \frac{100}{1.2} = 83.33\ m^2$



As usual, $\beta = 0.6$

$A = \alpha Q^{\beta}$

or, $\alpha = \frac{A}{Q^{\beta}} = \frac{83.33}{(100)^{0.6}} = 5.26$

Initial conditions:

$Q_1^1 = Q_2^1 = 100\ m^3/sec.$

Boundary conditions:

$Q_1^1 = Q_1^2 = Q_1^3 = 100\ m^3/sec.$

$Q_1^4 = 150\ m^3/sec.$

12 A.M.	4	150 m^3/sec.	?
11 A.M.	3	100 m^3/sec.	?
10 A.M.	2	100 m^3/sec.	?
9 A.M.	1	100 m^3/sec.	100 m^3/sec.
Time ↑	j↑ i→	1	2
	x→	0	7.2 km

For $q = 0$,

$$Q_{i+1}^{j+1} = \frac{\frac{\Delta t}{\Delta x} Q_i^{j+1} + \alpha\beta Q_{i+1}^{j} \left(\frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}}{\frac{\Delta t}{\Delta x} + \alpha\beta \left(\frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}}$$

$$\frac{\Delta t}{\Delta x} = \frac{3600}{7.2 \times 1000} = 0.5$$

$$\alpha\beta = 5.26 \times 0.6 = 3.156$$

$$\beta - 1 = 0.6 - 1 = -0.4$$

Now, for the value of Q_2^2; we have,

$$Q_2^2 = \frac{0.5 \times Q_1^2 + 3.156 \times Q_2^1 \times \left(\frac{Q_2^1 + Q_1^2}{2}\right)^{-0.4}}{0.5 + 3.156 \times \left(\frac{Q_2^1 + Q_1^2}{2}\right)^{-0.4}} = 100\ m^3/sec.$$

Similarly, other values can be calculated.

NOTE

Program in calculator;

$$A = \frac{0.5 \times X + 3.156 \times Y \times \left(\frac{X+Y}{2}\right)^{-0.4}}{0.5 + 3.156 \times \left(\frac{X+Y}{2}\right)^{-0.4}}$$

12 A.M.	4	150 m^3/sec.	126.11 m^3/sec.
11 A.M.	3	100 m^3/sec.	100 m^3/sec.
10 A.M.	2	100 m^3/sec.	100 m^3/sec.
9 A.M.	1	100 m^3/sec.	100 m^3/sec.
Time ↑	j↑ i→	1	2
	x→	0	7.2 km

EXAMPLE 5.6

The inflow hydrograph for a rectangular channel 100 m wide and 5 km long with bed slope of 1.5% and Manning's n = 0.02 is given as:

Time (min)	0	5	10	15
Flow (m^3/sec.)	14	20	25	30

Take uniform flow of 14 m^3/sec. at the beginning of simulation. There is no lateral flow. Take $\Delta x = 1000$ m and $\Delta t = 5$ min. Route the flow in the channel using explicit linear scheme of kinematic wave model. Wetted perimeter is approximately equal to the width of the channel.

Solution:

Given that;

$b = 100$ m

$l = 5$ km

$S_0 = 1.5\% = 0.015$

$n = 0.02$

$q = 0$

$\Delta x = 1000$ m

$\Delta t = 5$ min $= 300$ sec.

$P = b = 100$

From Manning's equation; we have,

$$Q = \frac{1}{n} A S_0^{\frac{1}{2}} R^{\frac{2}{3}} = \frac{1}{n} A S_0^{\frac{1}{2}} \left(\frac{A}{P}\right)^{\frac{2}{3}}$$

or, $$Q = \frac{1}{0.02} \times A^{\frac{5}{3}} \times \frac{\sqrt{0.015}}{(100)^{\frac{2}{3}}}$$

or, $$A = (3.518)^{\frac{3}{5}} Q^{\frac{3}{5}}$$

Comparing with $A = \alpha Q^{\beta}$; we get,

$$\alpha = (3.518)^{\frac{3}{5}} = 2.13$$

and, $\beta = 0.6$

15	4	30	24.73	20.54	17.69	15.96	14.97
10	3	25	20.25	17.23	15.57	14.73	14.29
5	2	20	16.5	15	14.4	14.16	14
0	1	14	14	14	14	14	14
Time (min)	j↑ i→	1	2	3	4	5	6
	Distance (km)	0	1	2	3	4	5

For $q = 0$; we have,

$$Q_{i+1}^{j+1} = \frac{\frac{\Delta t}{\Delta x} Q_i^{j+1} + \alpha\beta Q_{i+1}^j \left(\frac{Q_{i+1}^j + Q_i^{j+1}}{2}\right)^{\beta-1}}{\frac{\Delta t}{\Delta x} + \alpha\beta\left(\frac{Q_{i+1}^j + Q_i^{j+1}}{2}\right)^{\beta-1}}$$

$$\frac{\Delta t}{\Delta x} = \frac{300}{1000} = 0.3$$

$$\alpha\beta = 2.13 \times 0.6 = 1.278$$

$$\beta - 1 = 0.6 - 1 = -0.4$$

Now, for the value of Q_2^2; we have,

$$Q_2^2 = \frac{0.3 \times Q_1^2 + 1.278 \times Q_2^1 \times \left(\frac{Q_2^1 + Q_1^2}{2}\right)^{-0.4}}{0.3 + 1.278 \times \left(\frac{Q_2^1 + Q_1^2}{2}\right)^{-0.4}} = 16.5\ m^3/sec.$$

Similarly, we can calculate the other values.

EXAMPLE 5.7

Station B is 1 km downstream of station B. Given discharge at different time are:

Time (sec.)	Station A	Station B
0	14 cumec	14 cumec
300	19 cumec	16.17 cumec
600	28 cumec	-

Evaluate the discharge at station B after 600 sec. of beginning of simulation. Take: $\beta = 0.6$. Use first order accurate implicit scheme.

Solution:

Time (sec.)	Time step (j)	Station A (i = 1)	Station B (j = 1)
0	1	14	14
300	2	19	16.17
600	3	28	$\boxed{21.65}$

$$Q_{i+1}^{j+1} = \frac{\frac{\Delta t}{\Delta x} Q_i^{j+1} + \alpha\beta\, Q_{i+1}^j \left(\frac{Q_{i+1}^j + Q_i^{j+1}}{2}\right)^{\beta-1}}{\frac{\Delta t}{\Delta x} + \alpha\beta\left(\frac{Q_{i+1}^j + Q_i^{j+1}}{2}\right)^{\beta-1}}$$

$$Q_2^2 = \frac{\frac{300}{1000} \times Q_1^2 + 0.6\alpha\, Q_2^1 \left(\frac{Q_2^1 + Q_1^2}{2}\right)^{0.6-1}}{\frac{300}{1000} + 0.6\alpha\left(\frac{Q_2^1 + Q_1^2}{2}\right)^{0.6-1}}$$

or, $$16.17 = \frac{0.3 \times 19 + 0.6\alpha \times 14\left(\frac{14+19}{2}\right)^{-0.4}}{0.3 + 0.6\alpha\left(\frac{14+19}{2}\right)^{-0.4}}$$

or, $\alpha = 2$

Now,

$$Q_2^3 = \frac{0.3 Q_1^3 + 0.6 \times 2 \times Q_2^2\left(\frac{Q_2^2 + Q_1^3}{2}\right)^{0.6-1}}{0.3 + 0.6 \times 2 \times \left(\frac{Q_2^2 + Q_1^3}{2}\right)^{0.6-1}}$$

$$= \frac{0.3 \times 28 + 0.6 \times 2 \times 16.17 \times \left(\frac{16.17 + 28}{2}\right)^{-0.4}}{0.3 + 0.6 \times 2 \times \left(\frac{16.17 + 28}{2}\right)^{-0.4}}$$

$= 21.65$ cumec

Hence, discharge at station B after 600 sec. after beginning of simulation is 21.65 cumec.

5.11.3 Non-linear scheme for kinematic wave model

i) First order accurate implicit scheme for non-linear kinematic wave

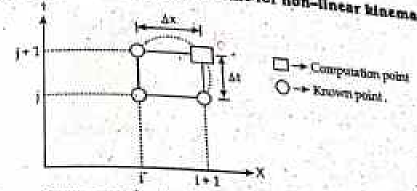


Figure: Space-time domain for first order accurate implicit scheme

For non-linear scheme, we solve equation [5.8.2 (a)].

i.e., $$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q$$

$$\frac{\partial Q}{\partial x} = \frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x}$$

$$\frac{\partial A}{\partial t} = \frac{A_{i+1}^{j+1} - A_{i+1}^j}{\Delta t}$$

$$q = \frac{q_{i+1}^{j+1} + q_{i+1}^j}{2}$$

$\therefore$ $\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q$ becomes,

$$\frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x} + \frac{A_{i+1}^{j+1} - A_{i+1}^j}{\Delta t} = \frac{q_{i+1}^{j+1} + q_{i+1}^j}{2}$$

Since, $A = \alpha Q^\beta$;

$$\frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x} + \frac{\alpha\left(Q_{i+1}^{j+1}\right)^\beta - \alpha\left(Q_{i+1}^j\right)^\beta}{\Delta t} = \frac{q_{i+1}^{j+1} + q_{i+1}^j}{2}$$

or, $$\frac{\Delta t}{\Delta x}Q_{i+1}^{j+1} + \alpha\left(Q_{i+1}^{j+1}\right)^{\beta} = \frac{\Delta t}{\Delta x}Q_i^{j+1} + \alpha\left(Q_{i+1}^{j}\right)^{\beta} + \Delta t\left(\frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2}\right) \quad [5.11.3]$$

You can see in the expression [5.11.3] that only Q_{i+1}^{j+1} is the unknown term and the equation is non-linear in it. Hence, the scheme is called as non-linear scheme and can be solved by Newton-Raphson iteration.

For equation [5.11.3] residual function is:

$$F\left(Q_{i+1}^{j+1}\right) = \frac{\Delta t}{\Delta x}Q_{i+1}^{j+1} + \alpha\left(Q_{i+1}^{j+1}\right)^{\beta} - \frac{\Delta t}{\Delta x}Q_i^{j+1} - \alpha\left(Q_{i+1}^{j}\right)^{\beta} - \Delta t\left(\frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2}\right) \quad [5.11.4\ (a)]$$

Note that only Q_{i+1}^{j+1} in above equations are variables and all other terms are constants. Derivating with respect to Q_{i+1}^{j+1}, we have,

$$F'\left(Q_{i+1}^{j+1}\right) = \frac{\Delta t}{\Delta x} + \alpha\beta\left(Q_{i+1}^{j+1}\right)^{\beta-1} \quad [5.11.4\ (b)]$$

Now, we can apply Newton-Raphson formula:

$$\left(Q_{i+1}^{j+1}\right)_{k+1} = \left(Q_{i+1}^{j+1}\right)_k - \frac{F\left(Q_{i+1}^{j+1}\right)_k}{F'\left(Q_{i+1}^{j+1}\right)_k} \quad [5.11.5]$$

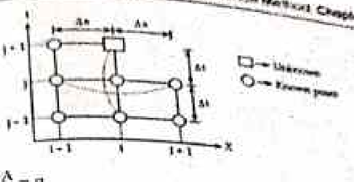
Iteration may be started taking any initial value of Q_{i+1}^{j+1} but for faster solution, you may use the solution of non-linear solution as initial value.

Algorithm for flow routing

1. Start with initial condition on time $t = 0$, $j = 1$
2. Check for Δt at every location using courant condition and choose the minimum value as Δt.
3. Move to next time step $t = t + \Delta t$, $j = j + 1$ and location $i = 2$.
4. Assign value of Q_1^j from boundary condition.
5. Evaluate $\left(Q_i^j\right)_0$ by linear scheme.
6. Evaluate residual function value $F\left(Q_i^j\right)$.
7. Evaluate the derivative $F'\left(Q_i^j\right)$.
8. Evaluate $\left(Q_i^j\right)_{k+1}$ by using equation [5.11.5].
9. Repeat steps 6, 7 and 8 until residue is negligible.
10. Proceed to next location $i = i + 1$ till the last point on the grid and go to step 5.
11. Go to step 3.
12. End.

ii) *Second order accurate explicit scheme for non-linear kinematic wave*

As said earlier, for non-linear schemes of kinematic wave model, we solve the equations:



$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q$$

To convert the differential equation into difference equation, we make the following substitutions.

Derivative term

$$\frac{\partial Q}{\partial x} = \frac{Q_{i+1}^{j} - Q_{i-1}^{j}}{2\Delta x}$$ [Second order accurate = Central difference; Explicit = Space derivative in time line j]

$$\frac{\partial A}{\partial t} = \frac{A_i^{j+1} - A_i^{j-1}}{2\Delta t}$$

Non-derivative term

$$q = \frac{q_i^{j-1} + q_i^{j} + q_i^{j+1}}{3}$$ [q is the average of q-values at location of our concern]

We get,

$$\frac{Q_{i+1}^{j} - Q_{i-1}^{j}}{2\Delta x} + \frac{A_i^{j+1} - A_i^{j-1}}{2\Delta t} = \frac{q_i^{j-1} + q_i^{j} + q_i^{j+1}}{3}$$

Since, $A = \alpha Q^{\beta}$;

$$\frac{Q_{i+1}^{j} - Q_{i-1}^{j}}{2\Delta x} + \frac{\alpha\left(Q_i^{j+1}\right)^{\beta} - \alpha\left(Q_i^{j-1}\right)^{\beta}}{2\Delta t} = \frac{q_i^{j-1} + q_i^{j} + q_i^{j+1}}{3}$$

or, $$\alpha\left(Q_i^{j+1}\right)^{\beta} = -\frac{\Delta t}{\Delta x}\left(Q_{i+1}^{j} - Q_{i-1}^{j}\right) + \alpha\left(Q_i^{j-1}\right)^{\beta} + 2\Delta t\left(\frac{q_i^{j-1} + q_i^{j} + q_i^{j+1}}{3}\right) \quad [5.11.6]$$

Obviously this equation is non-linear. However, this is directly solvable.

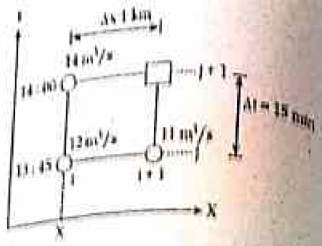
EXAMPLE 5.7

Using finite difference non-linear scheme, compute discharge at 1 km downstream of location x at time 14:00 hours for the following set of data.
Rectangular channel width = 20 m
Bed slope = 0.001
Manning's n = 0.03
Discharge at location x at time 14:00 hrs = 14 m³/sec.
Discharge at location x at time 13:45 hrs = 12 m³/sec.
Discharge at 1 km d/s of location x at time 13:45 hrs = 11 m³/sec.
Assume no lateral flow and wetted perimeter is equal to width of channel.
[2071 Bhadra]

Solution:

Given that;

$$P = b = 20\text{ m}$$
$$S_0 = 0.001$$
$$n = 0.03$$
$$\alpha = \left(\frac{nP^{\frac{2}{3}}}{\sqrt{S_0}}\right)^{\frac{3}{5}} = \left[\frac{0.03\times(20)^{\frac{2}{3}}}{\sqrt{0.001}}\right]^{0.6}$$
$$= 3.21$$
$$\beta = 0.6$$
$$\Delta x = 1000\text{ m}$$
$$\Delta t = 15\text{ min} = 900\text{ sec.}$$



From equation [5.11.4 (a)], residual function is:

$$F\left(Q_{i+1}^{j+1}\right) = \frac{\Delta t}{\Delta x}Q_{i+1}^{j+1} + \alpha\left(Q_{i+1}^{j+1}\right)^{\beta} - \frac{\Delta t}{\Delta x}Q_i^{j+1} - \alpha\left(Q_{i+1}^{j}\right)^{\beta} - \Delta t\left(\frac{q_{i+1}^{j+1}+q_{i+1}^{j}}{2}\right)$$

or, $$F\left(Q_{i+1}^{j+1}\right) = 0.9Q_{i+1}^{j+1} + 3.21\left(Q_{i+1}^{j+1}\right)^{0.6} - 0.9\times 14 - 3.21\times(11)^{0.6} = 0$$

or, $$F\left(Q_{i+1}^{j+1}\right) = 0.9Q_{i+1}^{j+1} + 3.21\left(Q_{i+1}^{j+1}\right)^{0.6} - 26.135 \quad \text{(a)}$$

Similarly,

$$F'\left(Q_{i+1}^{j+1}\right) = \frac{\Delta t}{\Delta x} + \alpha\beta\left(Q_{i+1}^{j+1}\right)^{\beta-1}$$

or, $$F'\left(Q_{i+1}^{j+1}\right) = 0.9 + 1.926\left(Q_{i+1}^{j+1}\right)^{-0.4} \quad \text{(b)}$$

Take initial value of $Q_{i+1}^{j+1} = 14\text{ m}^3/\text{sec}$.

i.e., $\left(Q_{i+1}^{j+1}\right)_0 = 14$

$\therefore$ $F\left(Q_{i+1}^{j+1}\right)_0 = 0.9\times 14 + 3.21\times(14)^{0.6} - 26.135 = 2.103$

and, $F'\left(Q_{i+1}^{j+1}\right)_0 = 0.9 + 1.926\times(14)^{-0.4} = 1.570$

In account to N-R formula;

$$\left(Q_{i+1}^{j+1}\right)_1 = \left(Q_{i+1}^{j+1}\right)_0 - \frac{F\left(Q_{i+1}^{j+1}\right)_0}{F'\left(Q_{i+1}^{j+1}\right)_0} = 12.660$$

Proceed to next step of iteration in the same way. The result is shown in the following table.

Iteration number	$\left(Q_{i+1}^{j+1}\right)_k$	$F\left(Q_{i+1}^{j+1}\right)_k$	$F'\left(Q_{i+1}^{j+1}\right)_k$	$\left(Q_{i+1}^{j+1}\right)_{k+1}$	Remarks
1	14	2.103	1.570	12.660	
2	12.660	−0.018	1.598	12.672	
3	12.672	-1.396×10^{-6}	1.597	12.672	$Q_k \approx Q_{k+1}$

$\therefore$ $Q_{i+1}^{j+1} = 12.672$

5.11.4 Numerical scheme for dynamic wave model

I) First order accurate explicit scheme

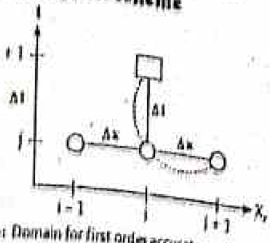


Figure: Domain for first order accurate explicit scheme

Recalling expressions [5.7.2 (a)] and [5.7.2 (b)] of dynamic wave model:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \quad [5.7.2\text{ (a)}]$$

$$\frac{\partial Q}{\partial t} + \frac{\partial\left(\frac{Q^2}{A}\right)}{\partial x} + gA\frac{\partial y}{\partial x} - gA(S_0 - S_f) = 0 \quad [5.7.2\text{ (b)}]$$

Both diffusive scheme and upstream differencing scheme discussed in section 5.11.1 can form first order accurate explicit scheme for dynamic wave model. This section illustrates the scheme by taking forward difference approximation in upward differencing scheme.

Continuity equation

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

or, $$\frac{Q_{i+1}^{j} - Q_i^{j}}{\Delta x} + \frac{A_i^{j+1} - A_i^{j}}{\Delta t} = 0$$

or, $$A_i^{j+1} = A_i^{j} - \Delta t\left(\frac{Q_{i+1}^{j} - Q_i^{j}}{\Delta x}\right) \quad [5.11.6\text{ (a)}]$$

Momentum equation

$$\frac{\partial Q}{\partial t} + \frac{\partial\left(\frac{Q^2}{A}\right)}{\partial x} + gA\frac{\partial y}{\partial x} - gA(S_0 - S_f) = 0$$

or, $$\frac{Q_i^{j+1} - Q_i^{j}}{\Delta t} + \frac{1}{\Delta x}\left\{\left(\frac{Q^2}{A}\right)_{i+1}^{j} - \left(\frac{Q^2}{A}\right)_i^{j}\right\} + gA_i^{j}\left(\frac{y_{i+1}^{j} - y_i^{j}}{\Delta x}\right) - gA_i^{j}(S_0 - S_f) = 0$$

or, $$Q_i^{j+1} = Q_i^{j} - \Delta t\left[\frac{1}{\Delta x}\left\{\left(\frac{Q^2}{A}\right)_{i+1}^{j} - \left(\frac{Q^2}{A}\right)_i^{j}\right\} + gA_i^{j}\left(\frac{y_{i+1}^{j} - y_i^{j}}{\Delta x}\right) - gA_i^{j}(S_0 - S_f)_i^{j}\right] \quad [5.11.6\text{ (b)}]$$

II) Second order accurate explicit scheme

Second order accurate explicit scheme for dynamic wave model is formed by leap-frog scheme explained in section [5.11.1]. Hence,

Continuity equation

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

or, $$\frac{Q_{i+1}^{j} - Q_{i-1}^{j}}{2\Delta x} + \frac{A_i^{j+1} - A_i^{j-1}}{2\Delta t} = 0$$

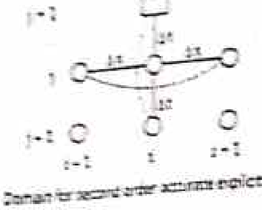


Figure: Domain for second order accurate explicit scheme

or, $$A_i^{j+1} = A_i^{j-1} - \Delta t\left(\frac{Q_{i+1}^j - Q_{i-1}^j}{\Delta x}\right) \quad [5.11.7\,(a)]$$

Momentum equation

$$\frac{\partial Q}{\partial t} + \frac{\partial\left(\frac{Q^2}{A}\right)}{\partial x} + gA\frac{\partial y}{\partial x} - gA(S_f - S_o) = 0$$

or, $$\frac{Q^{j+1} - Q^{j-1}}{2\Delta t} + \frac{1}{2\Delta x}\left[\left(\frac{Q^2}{A}\right)_{i+1}^j - \left(\frac{Q^2}{A}\right)_{i-1}^j\right] + gA_i^j\left(\frac{y_{i+1}^j - y_{i-1}^j}{2\Delta x}\right) - gA_i^j(S_f - S_o)_i^j = 0$$

[5.11.7 (b)]

iii) Implicit scheme for dynamic wave model

Preissmann scheme (Box scheme) is a four point weighted implicit model. This scheme is very advantageous since it can readily be used with unequal distance steps, which is particularly important for natural waterway where channel characteristics are highly variable even in short distances.

Similarly, the applicability of unequal time steps is another important characteristic of the Preissmann scheme, particularly in the case of hydrograph routing where channel characteristics are highly variable even in short time.

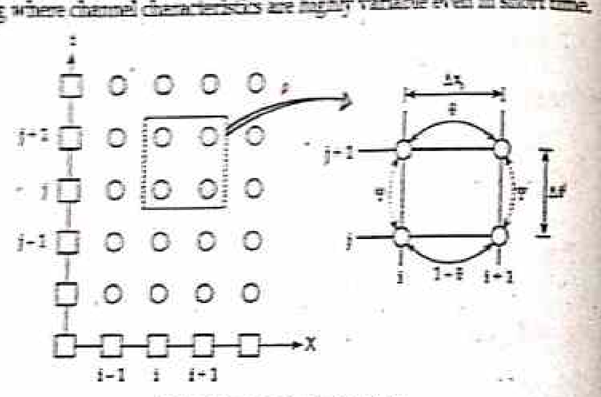


Figure: Domain for Preissmann scheme

The approximation of the derivatives and constant terms in the four-point weighted difference scheme are given as follows for a dummy parameter 'f'.

Time derivative:

$$\frac{\partial f}{\partial t} = \psi\frac{f_{i+1}^{j+1} - f_{i+1}^j}{\Delta t^j} + (1-\psi)\frac{f_i^{j+1} - f_i^j}{\Delta t^j} \quad [5.11.8\,(a)]$$

Space derivative:

$$\frac{\partial f}{\partial x} = \theta\frac{f_{i+1}^{j+1} - f_i^{j+1}}{\Delta x_i} + (1-\theta)\frac{f_{i+1}^j - f_i^j}{\Delta x_i} \quad [5.11.8\,(b)]$$

Other terms:

$$f = \theta\frac{f_{i+1}^{j+1} + f_i^{j+1}}{2} + (1-\theta)\frac{f_{i+1}^j + f_i^j}{2} \quad [5.11.8\,(c)]$$

where ψ and θ are weighing factors between 0 and 1. Many researchers prefer to take $\psi = 0.5$.

If θ value of 0.5 is used, the corresponding scheme is called box scheme. A scheme with θ value of 1.0 is known as fully-implicit scheme where as $\theta = 0$ gives fully-explicit scheme.

Preissmann scheme is unconditionally stable for any time step if $\theta = 0.5$ to 1.0. But accuracy of computation decreases as θ departs from 0.5 and approaches to 1.0. Analysis has revealed that θ value between 0.55 and 0.6 provides unconditional stability and good accuracy.

Applying Preissmann scheme on dynamic wave model of Saint Venant equations we have,

Continuity equation:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

Momentum equation:

$$\frac{\partial Q}{\partial t} + \frac{\partial\left(\frac{Q^2}{A}\right)}{\partial x} + gA\frac{\partial y}{\partial x} - gA(S_f - S_o) = 0$$

We get

Continuity equation:

$$\frac{1}{2}\left(\frac{A_{i+1}^{j+1} - A_{i+1}^j}{\Delta t^j} + \frac{A_i^{j+1} - A_i^j}{\Delta t^j}\right) + \theta\left(\frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x_i}\right) + (1-\theta)\left(\frac{Q_{i+1}^j - Q_i^j}{\Delta x_i}\right) = 0$$

[5.11.9 (a)]

Momentum equation:

$$\frac{1}{2}\left(\frac{Q_{i+1}^{j+1} - Q_{i+1}^j}{\Delta t^j} + \frac{Q_i^{j+1} - Q_i^j}{\Delta t^j}\right) + \theta\frac{\left(\frac{Q^2}{A}\right)_{i+1}^{j+1} - \left(\frac{Q^2}{A}\right)_i^{j+1}}{\Delta x_i} + (1-\theta)\frac{\left(\frac{Q^2}{A}\right)_{i+1}^j - \left(\frac{Q^2}{A}\right)_i^j}{\Delta x_i}$$

$$+g\left\{\theta\left(\frac{A_{i+1}^{j+1} + A_i^{j+1}}{2}\right) + (1-\theta)\frac{A_i^j + A_{i+1}^j}{2}\right\}\left\{\theta\left(\frac{y_{i+1}^{j+1} - y_i^{j+1}}{\Delta x_i}\right) + (1-\theta)\frac{y_{i+1}^j - y_i^j}{\Delta x_i}\right\}$$

$$-g\left\{\theta\left(\frac{A_{i+1}^{j+1} + A_i^{j+1}}{2}\right) + (1-\theta)\frac{A_{i+1}^j + A_i^j}{2}\right\}\left\{\theta\left(\frac{(S_f - S_o)_{i+1}^{j+1} + (S_f - S_o)_i^{j+1}}{2}\right)\right.$$

$$\left.+(1-\theta)\left(\frac{(S_f - S_o)_{i+1}^j + (S_f - S_o)_i^j}{2}\right)\right\} = 0 \quad [5.11.9\,(b)]$$

Two equations are formed at each points in the domain. All those equations are to be solved simultaneously. Generally, the equations are solved by Newton-Raphson iteration in matrix form.

We know, in N-R iteration,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

where, f(x) is the residual function

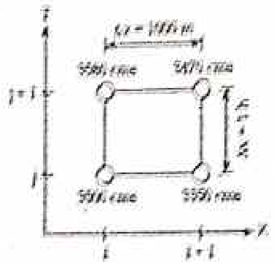
$$\therefore \quad [x_{i+1}] = [x_i] - [f'(x_i)]^{-1}[f(x_i)]$$

Solution algorithm

1. Assign initial value to $[x_i] = [x_0]$.
2. Calculate $[f(x_i)]$, $[f'(x_i)]$ and $[f'(x_i)]^{-1}$.
3. Calculate correction $[f'(x_i)]^{-1}[f(x_i)]$.
4. Calculate $[x_{i+1}] = [x_i] - [f'(x_i)]^{-1}[f(x_i)]$.
5. Assign $[x_i] = [x_{i+1}]$
6. If correction $[f'(x_i)]^{-1}[f(x_i)]$ is within permissible limit, show $[x_i]$ as answer. Else go to step 2.
7. End

EXAMPLE 5.8

The value of flow rate at four points in the space-time grid are shown in the figure below. $\Delta t = 1$ hr., $\Delta x = 1000$ m and $\theta = 0.55$; calculate the value of $\frac{\partial Q}{\partial t}$ and $\frac{\partial Q}{\partial x}$ by four point implicit method, θ = weighting factor. **[2070 Magh]**



Solution:

Here,

$$\frac{\partial Q}{\partial t} = \frac{1}{2}\left(\frac{Q_i^{j+1} - Q_i^j}{\Delta t} + \frac{Q_{i+1}^{j+1} - Q_{i+1}^j}{\Delta t}\right)$$

$$= \frac{1}{2}\left(\frac{3580 - 3500}{60 \times 60} + \frac{3470 - 3350}{60 \times 60}\right)$$

$$= 0.027 \text{ cm/sec.}$$

and,

$$\frac{\partial Q}{\partial x} = \theta\left(\frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x}\right) + (1 - \theta)\left(\frac{Q_{i+1}^j - Q_i^j}{\Delta x}\right)$$

$$= 0.55 \times \left(\frac{3470 - 3580}{1000}\right) + (1 - 0.55)\left(\frac{3350 - 3500}{1000}\right)$$

$$= -1.478 \text{ cumec/m}$$

5.12 NUMERICAL DIFFUSION AND DAMPING

Numerical diffusion and damping are two types of error occurring in FDM analysis due to even and odd order of accuracy.

Numerical diffusion refers to the amplitude damping of the numerical solution as compared to the exact solution. Numerical diffusion is caused by truncation error. Numerical diffusion arises when the approximations used [illegible] the corresponding finite difference of odd order of accuracy.

Figure: Numerical diffusion and damping

Numerical dispersion refers to a mismatch in phase between the numerical and the exact solution. Numerical dispersion emerges for finite differences of even order of accuracy.

WORKED OUT PROBLEMS

Problem 1

Write down complete equations governing the movement of fluid. [2070 Bhadra]

Solution: See the definition part 5.5

Problem 2

Derive the kinematic wave approximation for movement of fluid. [2070 Bhadra]

Solution:

St. Venant equations are:

i) **Continuity equation**

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=q \qquad [a]$$

ii) **Momentum equation**

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)+gA\frac{\partial y}{\partial x}-gAS_0+gAS_f=0 \qquad [b]$$

For kinematic wave, it is assumed that the flow is uniform and that the slope is approximately equal to slope of the channel. By neglection first three terms of momentum equation, we get kinematic wave model of St. Venant equation as:

i) **Continuity equation**

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=q \qquad [c]$$

ii) **Momentum equation**

$$S_0=S_f \qquad [d]$$

Equation [d] shows gravity force and friction force balance each other in kinematic wave model. Kinematic wave model can be described principally by the continuity equation. Since, no force term is included; the wave model is termed as Kinematic wave model.

In Kinematic wave model, A can be expressed in terms of Q as:

$$A=\alpha Q^{\beta}$$

Differentiating with respect to t, we get:

$$\frac{\partial A}{\partial t}=\alpha\beta Q^{\beta-1}\frac{\partial Q}{\partial t}$$

Substitution into equation [c]; we get,

$$\frac{\partial Q}{\partial x}+\alpha\beta Q^{\beta-1}\frac{\partial Q}{\partial t}=q \qquad [e]$$

This is another form of kinematic wave model in terms of Q.

The value of α and β in above equations are obtained from Mannings equations and their values are $\alpha=\left(\frac{nP^{2/3}}{\sqrt{S}}\right)^{0.6}$ and $\beta=0.6$.

Problem 3

Derive second order finite difference scheme of linear kinematic wave equation, which computes discharge for unknown time and location. [2070 Bhadra]

Solution: See the definition part 5.11.2 (iii)

Problem 4

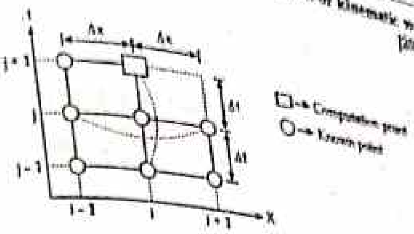
Derive first order accurate implicit finite difference equation for kinematic wave model in non-linear form. [2071 Bhadra]

Solution: See the definition part 5.11.3 (i)

Problem 5

Write an algorithm for second order accurate solution of kinematic wave used in analyzing fluid flow. [2071 Magh]

Solution:



Algorithm

Step 1 : Introduce initial condition at t = 0, j = 1.

Step 2 : Check for Δt for every location using courant condition; choose the minimum value as Δt.

Step 3 : Move to next time step t = t + Δt, j = j + Δt and location i = 1.

Step 4 : Assign value of Q_i^j from boundary condition.

Step 5 : Go to location i = i + 1.

Step 6 : Evaluate Q_i^j using formula:

$$Q_i^{j+1}=\frac{-\frac{\Delta t}{\Delta x}\left(Q_{i+1}^{j}-Q_{i-1}^{j}\right)+\alpha\beta\left(\frac{Q_{i+1}^{j-1}+Q_i^j+Q_{i-1}^{j+1}}{3}\right)^{\beta-1}+\left(\frac{q^{j-1}+q^j+q^{j+1}}{3}\right)2\Delta t}{\alpha\beta\left(\frac{Q_{i+1}^{j-1}+Q_i^j+Q_{i-1}^{j+1}}{3}\right)^{\beta-1}}$$

Step 7 : Go to step 5 till last location.

Step 8 : Go to step 3 till last time step.

Problem 6

Derive the expression for second order accurate explicit finite equation for dynamic wave model. [2072 Ashwin]

Solution: See the definition part 5.11.4 (ii)

Problem 7

Using definition sketch, discretized the following form of the saint venant equations using implicit four point method.

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0 \text{ (Continuity)}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial\left(\frac{Q^2}{A}\right)}{\partial x} + gA\frac{\partial y}{\partial x} - gA(S_0 - S_f) = 0 \text{ (Momentum)};$$

where, Q is the discharge at x.

A is the cross-sectional area at x.

y is the flow depth.

g is the acceleration due to gravity.

S_f is the energy slope.

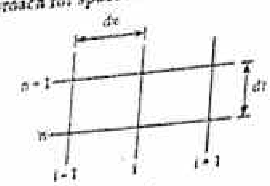
S_0 is the bed slope.

[2072 Magh]

Solution: See the definition part 5.11.4 (ii)

Problem 8

Explain the concept of finite difference method. What do you mean explicit and implicit scheme in finite difference method? For the grids given below discretize using explicit and implicit scheme using forward, backward and central difference approach for space and time derivatives. [2073 Bhadra]



Solution:

Concept of finite difference method

See the definition part 5.1

For the second part

See the definition part 5.4

For the third part

Proceed as example 5.1

NOTE

Only notation is different.

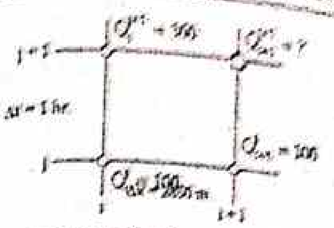
Problem 9

A river which is generalized as trapezoidal channel is 300 m wide with side slope 5 : 1, has bed slope 1% and Manning's (n) 0.04. Initially discharge through the river is 100 m³/sec. Due to a flood the discharge observed at the upstream section of the river is 300 m³/sec., compute discharge at 2650 m downstream from upstream section. Take: Δx = 2560 m and Δt = 1 hr. Use linear kinematic wave solution. [2073 Magh]

Solution:

Let us assume;

Wetted perimeter = Width of channel



Here, flow is uniform; so, before the flood;

$$Q_i^j = Q_{i+1}^j = 100$$

Given that;

$b = 300$ m

$S = 0.01$

$n = 0.04$

Side slope $= 5 : 1$

Hence, Manning's equation is given by;

$$Q = \frac{1}{n}AS_0^{\frac{1}{2}}R^{\frac{2}{3}} = \frac{1}{n}AS_0^{\frac{1}{2}}\left(\frac{A}{P}\right)^{\frac{2}{3}}$$

or, $$Q = \frac{A^{\frac{5}{3}}S^{\frac{1}{2}}}{nP^{\frac{2}{3}}}$$

or, $$A = \left(\frac{nP^{\frac{2}{3}}}{S^{\frac{1}{2}}}\right)^{\frac{3}{5}} Q^{\frac{3}{5}} = \left[\frac{0.04 \times (300)^{\frac{2}{3}}}{(0.01)^{\frac{1}{2}}}\right]^{\frac{3}{5}} Q^{\frac{3}{5}}$$

∴ $$A = 5.65Q^{\frac{3}{5}} \qquad (1)$$

Comparing equation (1) with $A = \alpha Q^{\beta}$; we get,

$\alpha = 5.65$

and, $\beta = 0.6$

Again,

$$Q_{i+1}^{j+1} = \frac{\frac{\Delta t}{\Delta x}Q_i^{j+1} + \alpha\beta\, Q_{i+1}^{j}\left(\frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}}{\frac{\Delta t}{\Delta x} + \alpha\beta\left(\frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}}$$

$$= \frac{\frac{60 \times 60}{2650} \times 300 + 5.65 \times 0.6 \times 100 \times \left(\frac{100 + 300}{2}\right)^{0.6-1}}{\frac{60 \times 60}{2650} + 5.65 \times 0.6 \times \left(\frac{100 + 300}{2}\right)^{0.6-1}}$$

$= 253.88$ m³/sec.

Problem 10

With appropriate expressions and graphs, explain first and second order accurate schemes of finite differences of partial differential equations. [2073 Magh]

Solution: See the expression 5.11.2, 5.11.3 and 5.11.4 with graph

Problem

Using any explicit finite difference scheme for full saint venant equations, compute discharge and flow depth at grid (i, n + 1) for the following data: Manning's (n) = 0.04, Rectangular channel Width = 10 m, bed slope = 0.0002, no lateral flow, Δx = 1 km and Δt = 5 min.

Discharge : Q^n_{i-1} = 40 m³/sec, Q^n = 38 m³/sec and Q^n_{i+1} = 37.5 m³/sec.

Flow depth : Y^n_{i-1} = 1.9 m, Y^n_i = 1.85 m and Y^n_{i+1} = 2.0 m [2073 Magh]

Solution:

Given that

$b = 10$ m

Bed slope $(S_0) = 0.0002$

$n = 0.04$

$\Delta x = 1$ km

$\Delta t = 5$ min

$Q^n_{i+1} = 37.5$ m³/sec

$Q^n_{i-1} = 40$ m³/sec

$Q^n_i = 38$ m³/sec

$Y^n_{i+1} = 2.0$ m

$Y^n_{i-1} = 1.9$ m

$Y^n_i = 1.85$ m

Now, calculating area of each node; we have,

$A^n_{i+1} = b \times Y^n_{i+1} = 10 \times 2 = 20$ m²

$A^n_{i-1} = b \times Y^n_{i-1} = 10 \times 1.9 = 19$ m²

$A^n_i = b \times Y^n_i = 10 \times 1.85 = 18.5$ m²

Using equation [5.11.6 (b)]; we have,

$$Q_i^{j+1} = Q_i^j - \Delta t\left[\frac{1}{\Delta x}\left\{\left(\frac{Q^2}{A}\right)^j_{i+1} - \left(\frac{Q^2}{A}\right)^j_i\right\} + gA_i^j\left(\frac{y^j_{i+1} - y^j_i}{\Delta x}\right) - gA_i^j(S_o - S_f)^j_i\right]$$

$$= 38 - 5 \times 60\left[\frac{1}{1000}\left\{\frac{(37.5)^2}{20} - \frac{(38)^2}{18.5}\right\} + 9.81 \times 18.5 \times \left(\frac{2 - 18.5}{1000}\right) - 9.81 \times 18.5 \times (0.0002 - 0)\right]$$

$= 43.04$ m³/sec

Again, using equation [5.11.6 (a)]; we have,

$$A_i^{j+1} = A_i^j - \Delta t\left(\frac{Q^j_{i+1} - Q^j_i}{\Delta x}\right) = 18.5 - 5 \times 60 \times \left(\frac{37.5 - 38}{1000}\right) = 18.65 \text{ m}^2$$

$\therefore \quad y_i^{j+1} = \frac{18.65}{10} = 1.865$ m

CHAPTER 6

METHOD OF CHARACTERISTICS

6.1 INTRODUCTION

Method of characteristics is a technique for solving partial differential equations. For a first-order partial differential equation, the method of characteristics discovers curves (called characteristic curves or just characteristics) along which the partial differential equation becomes an ordinary differential equation. Once the ODE is found, it can be solved along the characteristic curves numerically or graphically.

6.2 CHARACTERISTICS

As shown in the figure, characteristics are the plotting of curve $\frac{dx}{dt} = V \pm c$ in x-t plane.

where, v is the velocity of flow.

c is the celerity of the wave.

$$\tan\theta = \text{Slope of curve} = \frac{dx}{dt} = \frac{1}{V \pm c}$$

$$\text{Characteristics slopes} = \frac{1}{V + c} \text{ along } C^+$$

$$= \frac{1}{V - c} \text{ along } C^-$$

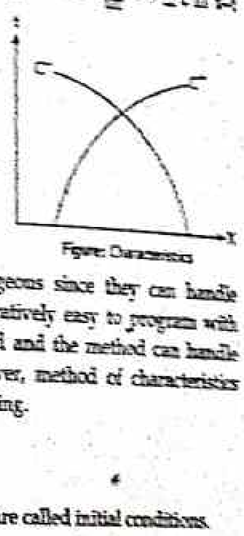


Figure: Characteristics

Method of characteristics are generally advantageous since they can handle complex problem comprehensively and is comparatively easy to program with higher accuracy. Stability can be easily established and the method can handle and program boundary conditions easily. However, method of characteristics can't be implemented without computer programming.

6.3 INITIAL AND BOUNDARY CONDITIONS

i) Initial conditions

Known conditions at the starting of the simulation are called initial conditions.

ii) Boundary conditions

Given or known conditions on some parts of a domain are known as boundary conditions.

Figure shows the characteristic plot of $\frac{dx}{dt} = V \pm c$ in x-t plane. As we see, at u/s end, only C^- line intersects the boundary whereas at d/s end, only C^+ line intersects the boundary. However, only the intersection of the two characteristics gives the solution for H.G.L. (H) and discharge (Q). Hence, we need known conditions at the boundaries, which are known as boundary conditions.

Boundary conditions may be constant discharge, constant head, varying discharge, varying head, head discharge relationship, dead end, etc.

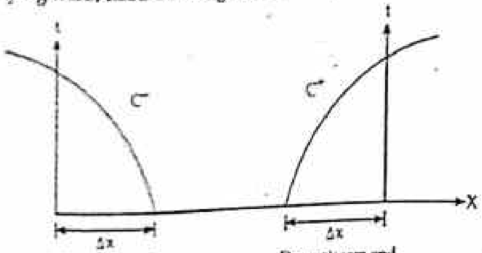


Figure: Boundary conditions

6.4 GOVERNING EQUATIONS FOR UNSTEADY PIPE FLOW

i) Momentum equation

$$\frac{\partial V}{\partial t}+\frac{V\partial V}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}+g\sin\theta+\frac{fV|V|}{2D}=0 \qquad [6.4.1\ (a)]$$

ii) Continuity equation

$$\frac{\partial P}{\partial t}+\frac{V\partial P}{\partial x}+\rho c^2\frac{\partial V}{\partial x}=0 \qquad [6.4.1\ (b)]$$

where, V is the average velocity at a section.

ρ is the fluid density.

P is the pressure at a section.

c is the celerity of a wave = $\sqrt{\dfrac{k}{\rho\left(1+\dfrac{Dk}{ET}\right)}}$

g is the acceleration due to gravity.

θ is the angle made by pipe with horizontal.

f is the friction factor.

K is the bulk modulus of elasticity of fluid.

D is the diameter of the pipe.

E is the modulus of elasticity of pipe material.

t is the thickness of pipe.

6.5 APPLICATION OF METHOD OF CHARACTERISTICS IN UNSTEADY PIPE FLOW PROBLEMS

Adding the momentum equation [6.4.1 (a)] to the continuity equation [6.4.1 (b)] multiplied by a parameter λ,

i.e., $$\frac{\partial V}{\partial t}+\frac{V\partial V}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}+g\sin\theta+\frac{fV|V|}{2D}+\lambda\left[\frac{\partial P}{\partial t}+\frac{V\partial P}{\partial x}+\rho c^2\frac{\partial V}{\partial x}\right]=0 \qquad [6.5.1]$$

We know that;

$$\frac{dV}{dt}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial V}{\partial t}$$

and, $$\frac{dP}{dt}=\frac{\partial P}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial P}{\partial t}$$

Rearranging and collecting terms that resemble $\frac{dV}{dt}$ and $\frac{dP}{dt}$ in equation [6.5.1],

$$\left[\frac{\partial V}{\partial x}(V+\lambda\rho c^2)+\frac{\partial V}{\partial t}\right]+\lambda\left[\frac{\partial P}{\partial x}\left(\frac{1}{\rho\lambda}+V\right)+\frac{\partial P}{\partial t}\right]+g\sin\theta+\frac{fV|V|}{2D}=0 \qquad [6.5.2]$$

The equation can be written as:

$$\frac{dV}{dt}+\lambda\frac{dP}{dt}+g\sin\theta+\frac{fV|V|}{2D}=0$$

where, $$\frac{\partial x}{\partial t}=V+\lambda\rho c^2 \qquad [6.5.3]$$

and, $$\frac{\partial x}{\partial t}=\frac{1}{\rho\lambda}+V \qquad [6.5.4\ (a)]$$

Equating equation [6.5.4 (a)] and [6.5.4 (b)]; we have, [6.5.4 (b)]

$$V+\lambda\rho c^2=\frac{1}{\rho\lambda}+V$$

or, $$\lambda=\pm\frac{1}{\rho c} \qquad [6.5.5]$$

Substituting equation [6.5.5] in [6.5.3]; we get,

$$\frac{dV}{dt}\pm\frac{1}{\rho c}\frac{dP}{dt}+g\sin\theta+\frac{fV|V|}{2D}=0 \qquad [6.5.6]$$

Substituting equation [6.5.5] in the condition [6.5.4 (a)] or [6.5.4 (b)] yields;

$$\frac{dx}{dt}=V\pm c \qquad [6.5.7]$$

Equation [6.5.7] represents characteristic curve. Equation [6.5.6] is the ordinary differential equation form of the partial differential equation forms which is valid only along the curve [6.5.7]. So, ODE [6.5.6] is solved along the characteristic curve [6.5.7] by graphical or numerical method to get the solution of partial differential equations [6.4.1 (a)] and [6.4.1 (b)]. This is the main concept of the method of characteristics.

Terms $\frac{V\partial V}{\partial x}$ in equation [6.4.1 (a)] and $\frac{V\partial P}{\partial x}$ in equation [6.4.1 (b)] are very much smaller than other terms and can be neglected. On doing so; we get,

$$\frac{dx}{dt}=\pm c \qquad [6.5.8]$$

6.5.1 Numerical solution of unsteady pipe flow problems using fixed grids

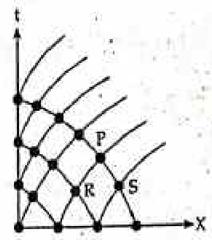


Figure: Characteristics grid

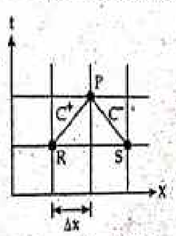


Figure: Rectangular grid

Variables at point R and S^+ are known i.e., V_R, P_R, x_R and t_R at R and V_S, P_S, x_S and t_S are given or already determined. Our goal is to evaluate the variables at point P, intersection point of two characteristics lines.

From equation [6.5.6], the ordinary differentiation equation along C^+ curve is;

$$\frac{dV}{dt} + \frac{1}{\rho c}\frac{dP}{dt} + g\sin\theta + \frac{fV|V|}{2D} = 0$$

Multiplication by $\rho c\,dt$ and integrating from R to P gives:

$$\rho c\int_R^P dV + \int_R^P dP + \int_R^P \rho g\sin\theta\, dt + \int_R^P \rho c\, dt\, \frac{fV|V|}{2D} = 0$$

Since, $\frac{dx}{dt} = c$

or, $dx = c\,dt$

Then, $\rho c\int_R^P dV + \int_R^P dP + \int_R^P \rho g\sin\theta\, dx + \int_R^P \rho\, dx\, \frac{fV|V|}{2D} = 0$

or, $\rho c(V_P - V_R) + (P_P - P_R) + \rho g\sin\theta\,\Delta x + \rho\Delta x\frac{fV_R|V_R|}{2D} = 0$ [6.5.9 (a)]

Similarly, along C^-; we get,

$$\rho c(V_P - V_S) + (P_P - P_S) + \rho g\sin\theta\,\Delta x + \frac{\rho\Delta x f V_S|V_S|}{2D} = 0 \qquad [6.5.9\ (b)]$$

Solution of equation [6.5.9 (a)] and [6.5.9 (b)] gives the value of V_P and P_P.

6.5.2 Numerical solution of unsteady pipe flow problems in terms of hydraulic gradient line (H) and discharge Q

Since, $\sin\theta = \frac{\Delta z}{\Delta x}$

or, $\Delta z = \Delta x\sin\theta$

$\because$ $P = \rho g\Delta h = \rho g(H - z)$

$P_P = \rho g(H_P - z_P)$

$P_R = \rho g(H_R - z_R)$

and, $P_S = \rho g(H_S - z_S)$

Figure: Flow in pipe

Then,

$$P_P - P_R = \rho g(H_P - H_R) - \rho g\Delta z = \rho g(H_P - H_R) - \rho g\Delta x\sin\theta \qquad [6.5.10\ (a)]$$

Similarly,

$$P_P - P_S = \rho g(H_P - H_S) - \rho g\Delta x\sin\theta \qquad [6.5.10\ (b)]$$

Substituting equation [6.5.10 (a)] in [6.5.9 (a)]; we get,

$$\rho c(V_P - V_R) + \rho g(H_P - H_R) - \rho g\Delta x\sin\theta + \rho g\Delta x\sin\theta + \frac{\rho\Delta x f V_R|V_R|}{2D} = 0$$

or, $\frac{\rho c(Q_P - Q_R)}{A} + \rho g(H_P - H_R) + \frac{\rho\Delta x f Q_R|Q_R|}{2DA^2} = 0$ [$\because Q = AV$]

or, $H_P = H_R - \frac{c(Q_P - Q_R)}{gA} - \frac{f\Delta x Q_R|Q_R|}{2gDA^2} = 0$

or, $H_P = H_R - B(Q_P - Q_R) - RQ_R|Q_R|$ [6.5.11 (a)]

where, $B = \frac{c}{gA}$ and $R = \frac{f\Delta x}{2gDA^2}$.

Similarly; we get,

$$H_P = H_S + B(Q_P - Q_S) + RQ_S|Q_S| \qquad [6.5.11\ (b)]$$

6.5.3 Solution algorithm for water hammer effect

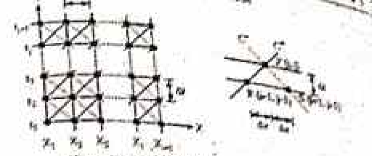


Figure: Domain discretized for method of Characteristics

Rewriting equation [6.5.11 (a)] in terms of (i, j);

$$H_i^j = H_{i-1}^{j-1} - B\left(Q_i^j - Q_{i-1}^{j-1}\right) - RQ_{i-1}^{j-1}\left|Q_{i-1}^{j-1}\right|$$

or, $H_i^j + BQ_i^j = H_{i-1}^{j-1} + Q_{i-1}^{j-1}\left(B - R\left|Q_{i-1}^{j-1}\right|\right)$

or, $H_i^j + BQ_i^j = CP$ along C^+

where, $CP = H_{i-1}^{j-1} + Q_{i-1}^{j-1}\left(B - R\left|Q_{i-1}^{j-1}\right|\right)$ is known quantity. [6.5.12 (a)]

Similarly; we get,

$H_i^j - BQ_i^j = CM$ along C^-

where, $CM = H_{i+1}^{j-1} - Q_{i+1}^{j-1}\left(B - R\left|Q_{i+1}^{j-1}\right|\right)$ is known quantity. [6.5.12 (b)]

Solving equation [6.5.12 (a)] and [6.5.12 (b)]; we get,

$$H_i^j = \frac{1}{2}(CP + CM) \qquad [6.5.13\ (a)]$$

$$Q_i^j = \frac{1}{2B}(CP - CM) \qquad [6.5.13\ (b)]$$

Algorithm steps

1. Divide the entire length of grid into N equal reaches such that $\Delta x = \frac{L}{N}$ and $\Delta t = \frac{\Delta x}{c}$.
2. Evaluate $B = \frac{c}{gA}$ and $R = \frac{f\Delta x}{2gDA^2}$.
3. Evaluate CP and CM for point (i, j) using equation [6.5.12 (a)] and [6.5.12 (b)]
4. Do similar calculations for all nodes at time step j.
5. Move to next time step.
6. Repeat the process for all time step.

6.5.4 Stability of method of characteristics

For a scheme to be stable, it should fulfill courant condition,

i.e, Courant number $(C_n) = \frac{V + |c|}{\frac{\Delta x}{\Delta t}} < 1$

or, $\Delta t < \frac{\Delta x}{V + |c|}$

or, $\Delta t < \frac{L}{N(V + |c|)}$ $\left[\because \Delta x = \frac{L}{N}\right]$

6.5.5 Method of characteristic when characteristics do not meet grid points

The value of variables at points M, N and O are known. Our goal is to determine the variables at point P which can be determined using equation [6.5.11 (a)] and [6.5.11 (b)] or [6.5.13 (a)] and [6.5.13 (b)]. But for this, we need the value of variable at points R and S. We evaluate R and S by interpolation.

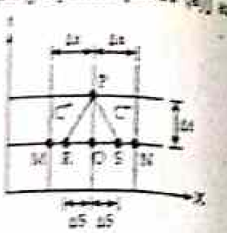


Figure: Rectangular grid when characteristics do not meet grid points

$$\frac{Q_R - Q_O}{Q_M - Q_O} = \frac{\Delta S}{\Delta x} \text{ gives } Q_R$$

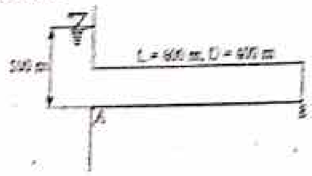
$$\frac{Q_S - Q_O}{Q_N - Q_O} = \frac{\Delta S}{\Delta x} \text{ gives } Q_S$$

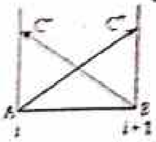
$$\frac{H_R - H_O}{H_M - H_O} = \frac{\Delta S}{\Delta x} \text{ gives } H_R$$

$$\frac{H_S - H_O}{H_N - H_O} = \frac{\Delta S}{\Delta x} \text{ gives } H_S$$

EXAMPLE 6.1

*A pipe conveys water from a reservoir as shown in the figure. Take: **f = 0.02, c = 1200 m/sec**. The H.G.L. at reservoir is given as **$H_{PA} = 100 + 3\sin(\pi t)$**. The discharge at downstream is zero at all times. By using only one reach, compute discharge from A and elevation of H.G.L. at B at 3 sec. using discretized equation of MOC.* **[2070 Magh]**





Solution:

Given that;

$L = 600$ m

$D = 400$ mm $= 0.4$ m

$H_{PA} = 100 + 3\sin(\pi t)$

$f = 0.02$

$Q_B = 0$ (all time)

$Q_A = ?$

$H_B = ?$ at 3 sec.

$N = 1$ (only one reach)

$\Delta x = \frac{L}{N} = 600$ m

$\Delta t = \frac{\Delta x}{c} = \frac{600}{1200} = 0.5$ sec.

$$B = \frac{c}{gA} = \frac{1200}{9.8 \times \frac{\pi}{4} \times (0.4)^2} = 974.42$$

$$R = \frac{f\Delta x}{2gDA^2} = \frac{0.02 \times 600}{2 \times 9.8 \times 0.4 \times \left[\frac{\pi}{4} \times (0.4)^2\right]^2} = 96.93$$

Now we write, for C^-

$$H_i^j = H_{i+1}^{j-1} + B\left(Q_i^j - Q_{i+1}^{j-1}\right) + RQ_{i+1}^{j-1}\left|Q_{i+1}^{j-1}\right|$$

$$Q_i^j = \frac{1}{974.42}\left(H_i^j - H_{i+1}^{j-1}\right) \quad [a]$$

Similarly, for C^+

$$H_{i+1}^j = H_i^{j-1} - B\left(Q_{i+1}^j - Q_i^{j-1}\right) - RQ_i^{j-1}\left|Q_i^{j-1}\right|$$

$$H_{i+1}^j = H_i^{j-1} + 974.42Q_i^{j-1} - 96.93Q_i^{j-1}\left|Q_i^{j-1}\right| \quad [b]$$

Boundary conditions:

$$H_i^j = 100 + 3\sin(\pi t) \quad [c]$$

$$Q_{i+1}^j = 0$$

Initial conditions:

at $t = 0$

$$H_i^1 = H_{i+1}^1 = 100$$

$$Q_i^1 = Q_{i+1}^1 = 0$$

t (sec.)	j	$H_i^j = 100 + 3\sin(\pi t)$	Q_i^j	H_{i+1}^j	Remarks
0	1	100	0	100	Given
0.5	2	103	0.00308	100	
1	3	100	0	106.00	
1.5	4	97	−0.00924	100	
2	5	100	0	88.00919	
2.5	6	103	0.015384	100	
3	7	100	0	117.967	

Sample calculation

$$H_i^2 = 100 + 3\sin(\pi \times 0.5) = 103$$

NOTE

Work in radian mode of calculator.

$$Q_i^2 = \frac{1}{974.42}\left(H_i^2 - H_{i+1}^1\right) = \frac{1}{974.42}(103 - 100) = 0.00308$$

$$H_{i+1}^2 = H_i^1 + 974.42Q_i^1 - 96.93Q_i^1\left|Q_i^1\right| = 100 + 974.42 \times 0 - 96.93 \times 0 = 100$$

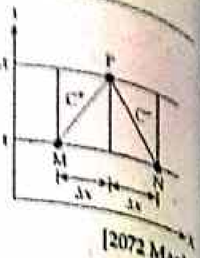
Hence, at t = 3 sec.;

$H_{P8} = 117.967$ m

and $Q_{P8} = ?$

EXAMPLE 6.2

Following are the data given at two points M and N along a pipe of diameter 15 cm carrying water. The discharge are 0.25 m³/sec. and 0.28 m³/sec. respectively at M and N, heads being 18.5 m and 18.0 m at M and N. Compute discharge and head at P using finite difference form of characteristics equation. $\Delta x = 100$ m, $\Delta t = 10$ sec., $f = 0.02$, $c = 1200$ m/sec., elevation difference for 100 m = 1.5 m



[2072 Magh]

Solution:

Given that

$D = 0.15$ m

$\therefore A = \frac{\pi}{4} \times (0.15)^2 = 0.0176 \text{ m}^2$

$Q_M = 0.25 \text{ m}^3/\text{sec.}$

$Q_N = 0.28 \text{ m}^3/\text{sec.}$

$H_M = 18.5$ m

$H_N = 18$ m

$H_P = ?$

$Q_P = ?$

$\Delta x = 100$ m

$\Delta t = 10$ sec.

$f = 0.02$

$C = 1200$ m/sec.

Now,

$$B = \frac{C}{gA} = \frac{1200}{9.8 \times 0.0176} = 6957.33$$

and,
$$R = \frac{f\Delta x}{2gDA^2} = \frac{0.02 \times 100}{2 \times 9.8 \times 0.15 \times (0.0176)^2} = 2196.13$$

Again,

$$H_P = H_M - B(Q_P - Q_M) - RQ_M|Q_M|$$

or, $H_P = 18.5 - 6957.33 \times (Q_P - 0.25) - 2196.13 \times (0.25)^2$

or, $H_P = 18.5 - 6957.33Q_P + 1739.33 - 137.26$

or, $H_P + 6957.33Q_P = 1620.57$ [a]

Also,

$$H_P = H_N + B(Q_P - Q_N) + RQ_N|Q_N|$$

or, $H_P = 18 + 6957.33 \times (Q_P - 0.28) + 2196.13 \times (0.28)^2$

or, $H_P = 18 + 6957.33Q_P - 1948.05 + 172.18$

or, $H_P - 6957.33Q_P = -1757.87$ [b]

Solving equation [a] and [b]; we get,

$H_P = -68.65$ m

$Q_P = 0.24 \text{ m}^3/\text{sec.}$

And

Alternative method

$$CP = H_M + Q_M(B - R|Q_M|)$$
$$= 18.5 + 0.25(6957.33 - 2196.13 \times 0.25) = 1620.57$$

$$CM = H_N - Q_N(B - R|Q_N|)$$
$$= 18 - 0.25(6957.33 - 2196.13 \times 0.28) = -1757.87$$

$$H = \frac{1}{2}(CP + CM) = -68.65 \text{ m}$$

$$Q = \frac{1}{2B}(CP - CM) = 0.24 \text{ m}^3/\text{sec.}$$

NOTE

The question is unrealistic because of error in the question. For c = 1200 m/sec., $\Delta t = 10$ sec., $\Delta x = C\Delta t = 1200 \times 10 = 12000$ m but we have $\Delta x = 100$ m.

EXAMPLE 6.3

If MOC is applied for $t_1 = 1$ sec. and $t_2 = 1$ sec., time levels for a pipe with a diameter 30 cm carrying water. If $Q_A = 0.7$ m³/sec., $Q_B = 0.76$ m³/sec., $Q_C = 0.74$ m³/sec., $H_A = 20$ m, $H_B = 20.6$ m, $H_C = 20.4$ m are values at grid points. Find values of Q and H at $t_1 = 1$ sec. that will be required for finding Q and H at P. When characteristics do not lie on diagonal, use $\Delta x = 1000$ m, $\Delta t = 1$ sec., $f = 0.02$ and $c = 800$ m/sec.

[2072 Ashwin]

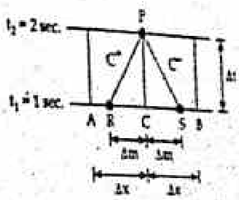
Solution:

Given that;

$Q_A = 0.7 \text{ m}^3/\text{sec.}$

$Q_B = 0.76 \text{ m}^3/\text{sec.}$

$Q_C = 0.74 \text{ m}^3/\text{sec.}$



$H_A = 20$ m

$H_B = 20.6$ m

$H_C = 20.4$ m

$\Delta x = 1000$ m

$\Delta t = 1$ sec.

$f = 0.02$

$c = 800$ m/sec.

$\therefore \Delta m = c \cdot \Delta t = 800 \times 1 = 800$ m

$Q_R = ?$

$H_R = ?$

$Q_S = ?$

$H_S = ?$

Now, interpolation gives,

$$\frac{Q_R - Q_C}{Q_A - Q_C} = \frac{\Delta m}{\Delta x}$$

or,
$$\frac{Q_R - 0.74}{0.7 - 0.74} = \frac{800}{1000}$$

∴
$$Q_R = 0.708\ m^3/sec.$$

NOTE

Using calculator

$$\frac{X - 0.74}{0.7 - 0.74} = \frac{800}{1000}$$

Shift + Calc

Similarly,

$$\frac{H_R - 20.4}{20 - 20.4} = \frac{800}{1000}$$

∴
$$H_R = 20.08\ m$$

$$\frac{Q_S - 0.74}{0.76 - 0.74} = \frac{800}{1000}$$

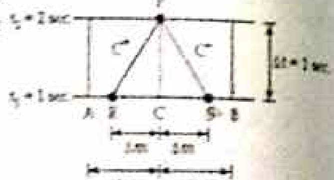
∴
$$Q_S = 0.756\ m^3/sec.$$

$$\frac{H_S - 20.4}{20.6 - 20.4} = \frac{800}{1000}$$

∴
$$H_S = 20.56\ m$$

EXAMPLE 6.4

Velocity and pressure at points A and B are given. Evaluate velocity and pressure at point P.



$D = 0.3\ m$ $\quad V_A = 4\ m/sec.$

$V_B = 6\ m/sec.$ $\quad P_R = 90\ kPa$

$P_S = 120\ kPa$ $\quad \Delta x = 800\ m$

$\Delta t = 1\ sec.$ $\quad f = 0.02$

$c = 800\ m/sec.$

Elevation difference for 100 m distance = 1 m. What should be done, if $\Delta x = 1000$ m?

Solution:

Here,

$$\Delta m = c\Delta t = 800 \times 1 = 800\ m$$

$$\Delta x = \Delta m$$

∴
$$V_R = V_A = 4\ m/sec.$$

$$V_S = V_B = 6\ m/sec.$$

$$P_R = P_A = 90\ kPa$$

$$P_S = P_B = 120\ kPa$$

For C^+; from equation [6.5.9 (a)]; we have,

$$\rho c(V_P - V_R) + (P_P - P_R) + \rho g \sin\theta\,\Delta x + \frac{\rho f \Delta x V_R|V_R|}{2D} = 0$$

or,
$$1000 \times 800(V_P - 4) + (P_P - 90 \times 1000) + 1000 \times 9.81 \times \frac{1}{100} \times 800 + \frac{1000 \times 800 \times 0.02 \times 4 \times 4}{2 \times 0.3} = 0$$

or,
$$8 \times 10^5 V_P - 32 \times 10^5 + P_P - 9 \times 10^4 + 505146.66 = 0$$

or,
$$8 \times 10^5 V_P + P_P = 2784853.34 \qquad [a]$$

For C^-; from equation [6.5.9 (b)]; we have,

$$\rho c(V_P - V_S) - (P_P - P_S) + \rho g \sin\theta\,\Delta x + \frac{\rho f \Delta x V_S|V_S|}{2D} = 0$$

or,
$$1000 \times 800(V_P - 6) - (P_P - 120) + 1000 \times 9.81 \times \frac{1}{100} \times 800 + \frac{1000 \times 800 \times 0.02 \times 6 \times 6}{2 \times 0.3} = 0$$

or,
$$8 \times 10^5 V_P - 48 \times 10^5 - P_P + 120 + 1038480 = 0$$

or,
$$8 \times 10^5 V_P - P_P = 3761400 \qquad [b]$$

Solving equation [a] and [b]; we get,

$$V_P = 4.09\ m/sec.$$

and
$$P_P = -488273.33\ Pa = -48.83\ kPa$$

If $\Delta x = 1000$ m, $\Delta x \neq \Delta m$ so value of V_R, V_S, P_R and P_S have to be interpolated as described in the section 6.5.5 and proceed as above.

6.6 APPLICATION OF METHOD OF CHARACTERISTICS IN GRADUALLY VARIED UNSTEADY FLOW IN OPEN CHANNEL

Recalling equation [5.7.3 (a)] and [5.7.3 (b)], non-conservative form of Saint Venant equations:

Continuity equation

$$\frac{\partial y}{\partial t} + \frac{A}{T}\frac{\partial V}{\partial t} + V\frac{\partial y}{\partial x} = 0 \qquad [5.7.3\,(a)]$$

Momentum equation

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} - g(S_0 - S_f) = 0 \qquad [5.7.3\,(b)]$$

Multiplying equation [5.7.3 (a)] by an arbitrary parameter λ and add it to equation [5.7.3 (b)] and rearrange the terms to obtain,

$$\left[\frac{\partial V}{\partial t} + \left(V + \frac{\lambda A}{T}\right)\frac{\partial V}{\partial x}\right] + \lambda\left[\frac{\partial y}{\partial t} + \left(\frac{g}{\lambda} + V\right)\frac{\partial y}{\partial x}\right] = g(S_0 - S_f) \qquad [6.6.1]$$

The total derivatives for V and y can be written as:

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{dx}{dt} \qquad [6.6.2\,(a)]$$

and,
$$\frac{dy}{dt} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x}\frac{dx}{dt} \qquad [6.6.2\,(b)]$$

Now, choose λ such that;

$$V + \frac{\lambda A}{T} = \frac{g}{\lambda} + V = \frac{dx}{dt} \qquad [6.6.3]$$

Then equation [6.5.1] becomes,

$$\frac{dV}{dt} \pm \lambda \frac{dy}{dt} = g(S_0 - S_f) \qquad [6.6.4]$$

and $$\lambda = \pm\sqrt{\frac{gT}{A}} \qquad [6.6.5]$$

& $$\frac{g}{\lambda} = \pm\sqrt{\frac{gA}{T}} \qquad [6.6.6]$$

It can be seen that $\frac{g}{\lambda}$ represents the celerity and $\frac{dx}{dt}$ represents the absolute wave velocity. Thus,

$$\frac{dV}{dt} + \frac{g}{c}\frac{dy}{dt} = g(S_0 - S_f); \qquad [6.6.7]$$

is valid along the curve,

$$\frac{dx}{dt} = V + c \qquad [6.6.8]$$

while, $$\frac{dV}{dt} - \frac{g}{c}\frac{dy}{dt} = g(S_0 - S_f) \qquad [6.6.9]$$

along, $$\frac{dx}{dt} = V - c \qquad [6.6.10]$$

Equation [6.6.7] is called positive characteristic equation and equation [6.6.9] is called negative characteristic equation. Equation [6.6.8] and [6.6.10] represent positive and negative characteristic lines respectively.

NOTE

Method of characteristics is generally not used now-a-days for open channel flow analysis.

WORKED OUT PROBLEMS

Problem 1

Prepare an algorithm to compute discharge and head based on the following form of finite difference equation for unsteady pipe flow problem using rectangular grid.

$$H_{Pi} - H_{i-1} = B(Q_{Pi} - Q_{i-1}) - RQ_{i-1}|Q_{i-1}|$$

$$H_{Pi} = H_{i+1} + B(Q_{Pi} - Q_{i+1}) + RQ_{i+1}|Q_{i+1}|$$

where, H is head, Q is discharge, H_{Pi} and Q_{Pi} is head and discharge at point of intersection of two characteristics, B and R is the coefficients. **[2070 Bhadra]**

Solution: See the definition part 6.5.3

Problem 2

Define characteristics curve and method of characteristics (MOC). Develop characteristics equations from partial differential form of the unsteady pipe flow equations. **[2071 Bhadra]**

Solution: See the definition part 6.1, 6.2 and 6.5

Problem 3

What is the significance of using method of characteristics? Develop the positive and negative characteristics equation used for analyzing a gradually varied flow in open channel. **[2071 Magh]**

Solution: See the definition part 6.2 and 6.6

Problem 4

Write an algorithm for simulation of water hammer process using method of characteristics. **[2072 Ashwin]**

Solution: See the definition part 6.5.3

Problem 5

The following is the characteristic form of equation for one dimensional unsteady pipe flow:

$$\frac{dV}{dt} + \frac{1}{\rho c}\frac{dP}{dt} + g\sin\theta + \frac{fV|V|}{2D} = 0;$$

where, V is the average velocity over a section,

ρ is the density of fluid,

P is the pressure at a point,

c is the celerity of wave,

g is the acceleration due to gravity,

sin θ is the slope of pipe,

f is the friction,

D is the diameter of pipe.

Develop finite difference equations for above equations using the concept of method of characteristics (MOC). Also, explain the criteria for the stability of the MOC solution. [2073 Bhadra]

Solution: See the definition part 6.5.1 and 6.5.4

Problem 6

Develop a finite difference solution of the characteristics form of unsteady flow equations to obtain solutions in terms of velocity and pressure. [2073 Magh]

Solution: See the definition part 6.5.1

NOTE

Make 3 × 3 matrix at last only.

CHAPTER 7

SIMULATION OF GROUND WATER FLOW

7.1 BASIC GOVERNING EQUATIONS FOR GROUNDWATER FLOW

i) Darcy's law

Darcy's law may be defined as:

$$v = -K\frac{dh}{dl} \quad [7.1.1]$$

where, v is the velocity of flow of water through porous media (Darcy flux).

$\frac{dh}{dl}$ is the hydraulic gradient.

K is the hydraulic conductivity.

Hydraulic conductivity (K) represents the volume of water flowing through 1 m × 1 m cross-section area of an aquifer under a hydraulic gradient of 1 m/1 m in a given amount of time (usually a day). Its unit is m/day.

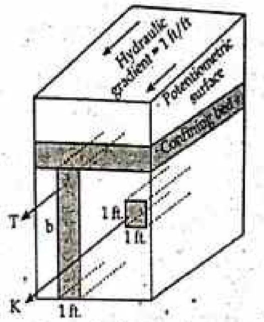


Figure: Concept of conductivity and transmissivity

Another term, transmissivity (T) is the volume of water flowing through a cross-section area of an aquifer that is 1 m × the aquifer thickness (b), under a hydraulic gradient of 1 m / 1 m in a given unit of time. It is given by;

$$T = Kb \quad [7.1.2]$$

Its unit is m^2/day.

From equation [7.1.1] and [7.1.2]; we have,

$$v = -\frac{T}{b}\frac{dh}{dl} \quad [7.1.3]$$

ii) Steady state continuity equation

Steady state continuity equation for groundwater flow is given by;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad [7.1.4]$$

where, u, v and w are the velocity of water in X, Y and Z direction respectively.

From equation [7.1.1]; we have,

$$u = -K_x\frac{\partial h}{\partial x}$$

$$v = -K_y \frac{\partial h}{\partial y}$$

$$w = -K_z \frac{\partial h}{\partial z}$$

where, K_x, K_y and K_z are the hydraulic conductivity in x, y and z direction respectively.

Assuming the soil mass to be homogenous and isotropic (i.e., $K_x = K_y = K_z$) and substituting velocity in equation [7.1.4]; we get,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \qquad [7.1.5]$$

Equation [7.1.5] is the Laplace equation of groundwater flow in 3-D in homogenous and isotropic medium.

Laplace equation for 2-D flow is:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad [7.1.6]$$

The general solution of equation [7.1.6] yields two sets of curves orthogonal to each other. One set of curves is known as the flow lines (stream lines) and the other set equipotential lines. Flow lines indicate the direction of flow and the equipotential lines are lines joining points where the total potential (head) is equal. The graphical representation of equation [7.1.6] is called flow net.

Velocity potential (ϕ) is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction i.e., for steady flow,

$$u = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y}$$

and, $$w = -\frac{\partial \phi}{\partial z}$$

Substituting these values into equation [7.1.4]; we get,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad [7.1.7]$$

Equation [7.1.7] is the Laplace equation for velocity potential.

7.2 NUMERICAL SCHEME FOR GROUNDWATER FLOW

The groundwater flow equation for transient flow in unisotropic medium is given by;

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{yy}\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_{zz}\frac{\partial h}{\partial z}\right) - W' = S_S \frac{\partial h}{\partial t}$$

or, $$K_{xx}\frac{\partial^2 h}{\partial x^2} + K_{yy}\frac{\partial^2 h}{\partial y^2} + K_{zz}\frac{\partial^2 h}{\partial z^2} - W' = S_s \frac{\partial h}{\partial t} \qquad [7.2.1]$$

where, first three terms represent Darcy's law and steady state continuity equation, 'W' represents source and sinks (withdrawal), and R.H.S. represents transient flow term (effect of change of storage with time.)

Above equation is valid for 3-D ground water flow. The general governing equation for 2-D ground water flow is:

$$K_{xx}\frac{\partial^2 h}{\partial x^2} + K_{yy}\frac{\partial^2 h}{\partial y^2} - W' = S_s \frac{\partial h}{\partial t} \qquad [7.2.2]$$

The equation can be transformed into another form with T = Kb and $S_s = \frac{S}{b}$

$$T_{xx}\frac{\partial^2 h}{\partial x^2} + T_{yy}\frac{\partial^2 h}{\partial y^2} = S\frac{\partial h}{\partial t} + W \qquad [7.2.3]$$

NOTE

- 'S' represents storage coefficient. It is a dimensionless quantity and measures the volume of water expelled (absorbed) per unit surface area per unit head change.
- 'S_S' represents specific storage and $S_S = \frac{S}{b}$. It measures the water volume per unit aquifer volume that is expelled (stored) per unit head change.
- 'W' represents net ground water withdrawal per unit area per unit time.

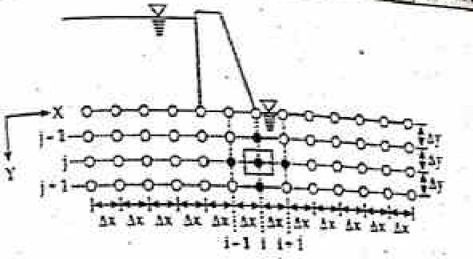


Figure: Discretization of aquifer

Considering point (i, j) and its neighbouring points.

Recall the equation [5.2.4],

$$f'' = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2}$$

∴ at point (i, j); we have,

$$\frac{\partial^2 h}{\partial x^2} = \frac{h^j_{i+1} - 2h^j_i + h^j_{i-1}}{\Delta x^2}$$

and, $$\frac{\partial^2 h}{\partial y^2} = \frac{h^{j+1}_i - 2h^j_i + h^{j-1}_i}{\Delta y^2}$$

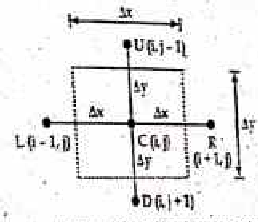


Figure: point (i, j) and its neighbouring points in domain

substituting in equation [7.2.3]; we get,

$$T_{xx}\left(\frac{h^j_{i+1} - 2h^j_i + h^j_{i-1}}{\Delta x^2}\right) + T_{yy}\left(\frac{h^{j+1}_i - 2h^j_i + h^{j-1}_i}{\Delta y^2}\right) = S\frac{\partial h}{\partial t} + W$$

or, $$\frac{T_{xx}}{\Delta x^2}h^j_{i-1} + \frac{T_{xx}}{\Delta x^2}h^j_{i+1} + \frac{T_{yy}}{\Delta y^2}h^{j-1}_i + \frac{T_{yy}}{\Delta y^2}h^{j+1}_i - 2\left(\frac{T_{xx}}{\Delta x^2} + \frac{T_{yy}}{\Delta y^2}\right)h^j_i = S\frac{\partial h}{\partial t} + W \qquad [7.2.4]$$

or, $$Lh^j_{i-1} + Rh^j_{i+1} + Uh^{j-1}_i + Uh^{j+1}_i - Ch^j_i = S\frac{\partial h}{\partial t} + W \qquad [7.2.5]$$

where, $L = R = \frac{T_{xx}}{\Delta x^2}$

$$U = D = \frac{T_{yy}}{\Delta y^2}$$

$$C = L + R + U + D = 2\left(\frac{T_{xx}}{\Delta x^2} + \frac{T_{yy}}{\Delta y^2}\right)$$

If the domain is homogeneous $(\Delta x = \Delta y)$ and isotropic $(T_{xx} = T_{yy})$, the equation [7.2.4] can be written as:

$$h_{i-1}^j + h_{i+1}^j + h_i^{j-1} + h_i^{j+1} - 4h_i^j = \frac{\Delta x^2}{T_{xx}}\left(S\frac{\partial h}{\partial t} + W\right)$$

$$h_{i-1}^j + h_{i+1}^j + h_i^{j-1} + h_i^{j+1} - 4h_i^j = \frac{\Delta x^2}{T_{xx}}\left(S\frac{\partial h}{\partial t} + W\right) \quad [7.2.6]$$

$$Lh_{i-1}^j + Rh_{i+1}^j + Uh_i^{j-1} + Dh_i^{j+1} - Ch_i^j = \frac{\Delta x^2}{T_{xx}}\left(S\frac{\partial h}{\partial t} + W\right)$$

where, $L = R = U = D = 1$

$$C = L + R + U + D = 4$$

For steady state condition with no withdrawal,

$$h_{i-1}^j + h_{i+1}^j + h_i^{j-1} + h_i^{j+1} - 4h_i^j = 0 \quad [7.2.7]$$

NOTE

Hydraulic conductivity or transmissivity for impervious layers would be '0'. So the coefficients for them *i.e.*, L, R, U, D for such part of domain would be '0'. This concept is necessary for the application of boundary condition.

Discharge from L to C (Q_L) = Cross section area normal to flow velocity × Flow velocity

$$= (\Delta y \times b) \times \left(-\frac{T_{xx}}{b}\right) \times \frac{(h_i^j - h_{i-1}^j)}{\Delta x}$$

$$= \frac{T_{xx}}{\Delta x^2} \times (\Delta x \Delta y) \times \left\{-\left(h_i^j - h_{i-1}^j\right)\right\}$$

$$= L\Delta x\Delta y\left(h_{i-1}^j - h_i^j\right)\ m^3/s \quad [7.2.8(a)]$$

Similarly; we can get,

$$(Q_R) = R\Delta x\Delta y\left(h_i^j - h_{i+1}^j\right)\ m^3/s \quad [7.2.8(b)]$$

$$(Q_U) = U\Delta x\Delta y\left(h_i^{j-1} - h_i^j\right)\ m^3/s \quad [7.2.8(c)]$$

$$(Q_L) = D\Delta x\Delta y\left(h_i^j - h_i^{j+1}\right)\ m^3/s \quad [7.2.8(d)]$$

where, $L = R = \frac{T_{xx}}{\Delta x^2}$

$$U = D = \frac{T_{yy}}{\Delta y^2}$$

Formation of FDM equation of ground water flow by alternative method

For grid point (i, j),

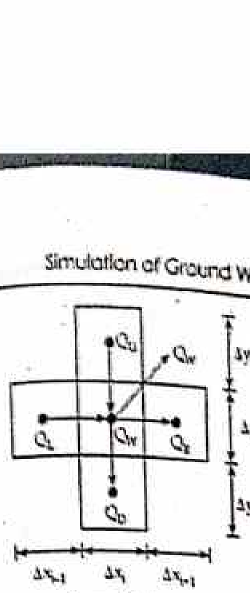


Figure: Discharge at (i, j)

Inflow discharge − Outflow discharge = Rate of change of storage

$$Q_L - Q_R + Q_U - Q_D - Q_W = S\frac{\partial h}{\partial t}\Delta x_i \Delta y_j \quad [7.2.9]$$

where Q_L, Q_R, Q_U, Q_D are grid to grid flow rate.

Q_W is net groundwater withdrawal rate from grid (i, j)

or, $$\frac{Q_L}{\Delta x_i \Delta y_j} - \frac{Q_R}{\Delta x_i \Delta y_j} + \frac{Q_U}{\Delta x_i \Delta y_j} - \frac{Q_D}{\Delta x_i \Delta y_j} - W = S\frac{\partial h}{\partial t} \quad [7.2.10]$$

Now,

$$Q_L = \text{Flow velocity} \times \text{Normal cross-section area}$$

$$= \left(-\frac{T_x}{b}\frac{h_i^j - h_{i-1}^j}{\frac{\Delta x_{i-1}}{2} + \frac{\Delta x_i}{2}}\right) \times \left(b\Delta y_j\right)$$

$$= -2T_x\left(\Delta y_j\right)\frac{h_i^j - h_{i-1}^j}{\Delta x_{i-1} + \Delta x_i} \quad [7.2.11]$$

or, $$\Delta h = -\frac{Q_L}{2T_x(\Delta y_j)}(\Delta x_{i-1} + \Delta x_i) \quad [7.2.12]$$

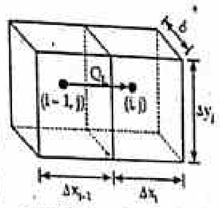


Figure: Discharge from (i-1, j) to (i, j)

where, $\Delta h = h_i^j - h_{i-1}^j$.

We can write,

$$\Delta h = \Delta h_{i-1} + \Delta h_i$$

or, $$-\frac{Q_L}{2T_x(\Delta y_j)}(\Delta x_{i-1} + \Delta x_i) = -\frac{Q_L}{2T_{x_{i-1,j}}(\Delta y_j)}(\Delta x_{i-1}) - \frac{Q_L}{2T_{x_{i,j}}(\Delta y_j)}(\Delta x_i)$$

or, $$T_x = \frac{T_{x_{i,j}}T_{x_{i-1,j}}}{\Delta x_{i-1}T_{x_{i,j}} + \Delta x_i T_{x_{i-1,j}}}(\Delta x_{i-1} + \Delta x_i) \quad [7.2.13]$$

Now from equations [7.1.11] and [7.1.13]; we can write,

$$\frac{Q_L}{\Delta x_i \Delta y_j} = -2\frac{T_{x_{i,j}}T_{x_{i-1,j}}}{\Delta x_{i-1}T_{x_{i,j}} + \Delta x_i T_{x_{i-1,j}}}(\Delta x_{i-1} + \Delta x_i)\frac{h_i^j - h_{i-1}^j}{\Delta x_i(\Delta x_{i-1} + \Delta x_i)}$$

or, $$\frac{Q_L}{\Delta x_i \Delta y_j} = -L_{i,j}\left(h_i^j - h_{i-1}^j\right) = L_{i,j}\left(h_{i-1}^j - h_i^j\right) \quad [7.2.14(a)]$$

where, $$L_{i,j} = \frac{2T_{x_{i,j}}T_{x_{i-1,j}}}{\Delta x_i\left(\Delta x_{i-1}T_{x_{i,j}} + \Delta x_i T_{x_{i-1,j}}\right)}$$

Similarly; we can achieve,

$$\frac{Q_R}{\Delta x_i \Delta y_j} = -R_{i,j}\left(h^j_{i+1} - h^j_i\right) \quad [7.2.14(b)]$$

$$\frac{Q_U}{\Delta x_i \Delta y_j} = -U_{i,j}\left(h^j_{i+1} - h^{j-1}_i\right) = U_{i,j}\left(h^{j-1}_i - h^j_i\right) \quad [7.2.14(c)]$$

$$\frac{Q_D}{\Delta x_i \Delta y_j} = -D_{i,j}\left(h^{j+1}_i - h^j_i\right) \quad [7.2.14(d)]$$

where, $R_{i,j} = \dfrac{2T_{x_{i+1,j}} T_{x_{i,j}}}{\Delta x_i\left(\Delta x_i T_{x_{i+1,j}} + \Delta x_{i+1} T_{x_{i,j}}\right)}$

$U_{i,j} = \dfrac{2T_{x_{i,j-1}} T_{x_{i,j}}}{\Delta y_j\left(\Delta y_j T_{x_{i,j-1}} + \Delta y_{j-1} T_{x_{i,j}}\right)}$

and, $D_{i,j} = \dfrac{2T_{x_{i,j}} T_{x_{i,j+1}}}{\Delta y_j\left(\Delta y_{j+1} T_{x_{i,j}} + \Delta y_j T_{x_{i,j+1}}\right)}$

Substituting these values into [7.2.10]; we get,

$$L_{ij}\left(h^j_{i-1} - h^j_i\right) + R_{ij}\left(h^j_{i+1} - h^j_i\right) + U_{ij}\left(h^{j-1}_i - h^j_i\right) + D_{ij}\left(h^{j+1}_i - h^j_i\right) = S\frac{\partial h}{\partial t} + W$$

or, $$L_{ij}\left(h^j_{i-1}\right) + R_{ij}\left(h^j_{i+1}\right) + U_{ij}\left(h^{j-1}_i\right) + D_{ij}\left(h^{j+1}_i\right) - C_{ij}\left(h^j_i\right) = S\frac{\partial h}{\partial t} + W \quad [7.2.15]$$

where, $C_{ij} = L_{ij} + R_{ij} + U_{ij} + D_{ij}$

If $\Delta x_{i-1} = \Delta x_i = \Delta x_{i+1} = \Delta x;$

$\Delta y_{j-1} = \Delta y_j = \Delta y_{j+1} = \Delta y$

and, $T_{x_{i-1,j}} = T_{x_{i,j}} = T_{x_{i+1,j}} = T_x;$

$T_{y_{i,j-1}} = T_{y_{i,j}} = T_{y_{i,j+1}} = T_y$

Then, $L = R = \dfrac{T_x}{\Delta x^2}$

$U = D = \dfrac{T_y}{\Delta y^2}$

$C = 2\left(\dfrac{T_x}{\Delta x^2} + \dfrac{T_y}{\Delta y^2}\right)$; which we had previously obtained for equation [7.2.5].

7.3 SIMULATION OF SEEPAGE UNDER A DAM

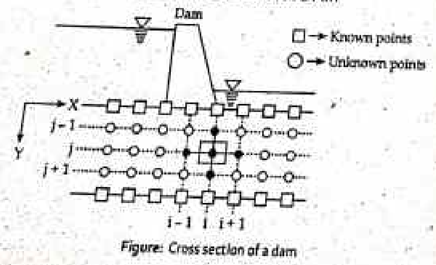


Figure: Cross section of a dam

Seepage under a dam can be simulated considering 2-D equation of groundwater flow; recall equation [7.2.5]

$$Lh^j_{i-1} + Rh^j_{i+1} + Uh^{j-1}_i + Dh^{j+1}_i - Ch^j_i = S\frac{\partial h}{\partial t} + W$$

Consider steady state condition with W = 0;

$$Lh^j_{i-1} + Rh^j_{i+1} + Uh^{j-1}_i + Dh^{j+1}_i - Ch^j_i = 0 \quad [7.3.1]$$

For iteration, re-arrange the equation as:

$$\left(h^j_i\right)^* = \frac{1}{C}\left(Lh^j_{i-1} + Rh^j_{i+1} + Uh^{j-1}_i + Uh^{j+1}_i\right) \quad [7.3.2]$$

Algorithm

1. Form the equation of form [7.3.2] for every grid point.
2. Employ boundary conditions.
3. Assign initial values for h^j_i.
4. Compute $\left(h^j_i\right)^*$ for every grid point using equation [7.3.2].
5. Evaluate error $e = h^* - h$
6. If error is within allowable limit, go to step 7. Else go to step 4.
7. Evaluate seepage per width

$$\left(Q^j_i\right)_{Horizontal} = L\Delta x\Delta y\left(h^j_{i-1} - h^j_i\right) \text{ cumec}$$

$$\left(Q^j_i\right)_{Vertical} = U\Delta x\Delta y\left(h^{j-1}_i - h^j_i\right) \text{ cumec}$$

EXAMPLE 7.1

The value of potentials at four points in a domain of seepage under dam are given as $\phi^j_{i+1} = 2.1$, $\phi^{j-1}_i = 2.12$, $\phi^j_{i-1} = 2.09$ and $\phi^{j+1}_i = 2.1$. Compute the value of potential head at grid point (i, j) considering aquifer as homogenous, isotropic and steady flow with no withdrawal.

Solution:

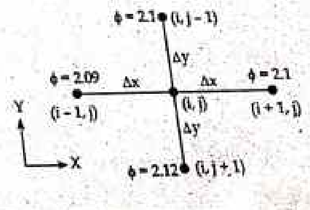
> **NOTE**
> In groundwater problems under steady flow, you can treat velocity potential 'ϕ' and static head 'h' equivalent in expressions.

$$\phi^j_{i-1} = \left(L\phi^j_{i-1} + R\phi^j_{i+1} + U\phi^{j-1}_i + D\phi^{j+1}_i\right)$$

For isotropic, homogenous medium

$L = R = U = D = 1$

and, $C = L + R + U + D = 4$



$$\therefore \quad \phi^{j+1}_i = \frac{1}{4}\left(\phi^j_{i-1} + \phi^j_{i+1} + \phi^{j-1}_i + \phi^{j+1}_i\right)$$

$$= \frac{1}{4}(2.09 + 2.1 + 2.1 + 2.12)$$

$$= 2.10$$

$\therefore$ Velocity potential at $(i, j) = 2.10$

EXAMPLE 7.2

The figure shows the 2 dimensional grid for simulating seepage under a dam. The value of potential function (ϕ) are shown in the grid. Compute vertical and horizontal seepage to grid A. Take transmissivity in x-direction = 2900 m^2/day for all grids. Transmissivity in y-direction = 2400 m^2/day for all grids, Δx = 100 m and Δy = 75 m.

[2072 Magh]

0	0	0	0
0	2	2	0
0	2.1	2.1	0
0	2.12	2.09 Grid A	0
0	2.12	2.12	0
0	0	0	0

Solution:

Here, $L = R = \frac{T_{xx}}{\Delta x^2} = \frac{2900}{100^2} = 0.29\ m^2/day$

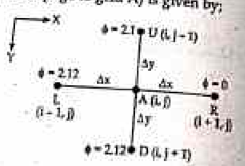
$U = D = \frac{T_{YY}}{\Delta y^2} = \frac{2400}{75^2} = 0.43\ m^2/day$

Now, Seepage of water from left to A (horizontal seepage to grid A) is given by;

$(Q_L) = L\Delta x\Delta y\left(\phi_{i-1}^{j} - \phi_i^{j}\right)$

$= 0.29 \times 100 \times 75 \times (2.12 - 2.09)$

$= 65.25\ m^3/day$



Seepage of water from below grid to A (vertical seepage to grid A) is given by;

$(Q_D) = D\Delta x\Delta y\left(\phi_i^{j} - \phi_i^{j+1}\right) m^3/s$

$= 0.43 \times 100 \times 75 \times (2.12 - 2.09)$

$= 96.75\ m^3/day$

Seepage of water from above grid to A (vertical seepage to grid A) is given by;

$(Q_U) = U\Delta x\Delta y\left(\phi_i^{j-1} - \phi_i^{j}\right) m^3/s$

$= 0.43 \times 100 \times 75 \times (2.1 - 2.09)$

$= 32.25\ m^3/day$

From data, it seems that there occurs an impermeable layer as boundary of domain, hence horizontal seepage to the right = 0

$\therefore$ Total horizontal seepage = $65.25\ m^3/day$

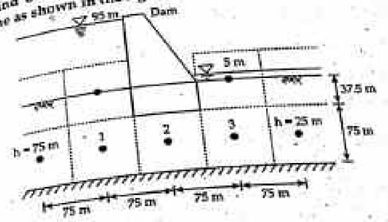
$\therefore$ Total vertical seepage = $96.75 + 32.25\ m^3/day$

NOTE

The question is unrealistic.

EXAMPLE 7.3

Considering homogenous and isotropic domain, steady flow with no withdrawal, find out the head at points 1, 2 and 3 in the under dam water seepage scheme as shown in the figure.

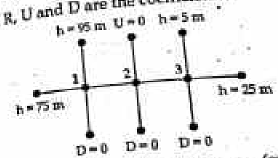


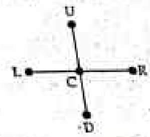
Solution:

For point C,

$$(h_C)^+ = \frac{1}{L + R + U + D}(Lh_L + Rh_R + Uh_U + Dh_D)$$

where, L, R, U and D are the coefficients.





For homogenous, $(\Delta x = \Delta y)$; isotropic $(T_{xx} = T_{yy})$ medium and steady flow without withdrawal, we can take L = R = U = D = 1 for every point except the direction of barriers where we put coefficient = 0.

Now,

$$h_1 = \frac{1}{1+1+1+0}(1 \times 75 + 1 \times h_2 + 1 \times 95 + 0)$$

or, $h_1 = \frac{1}{3}(h_2 + 170)$

$$h_2 = \frac{1}{1+1+0+0}(1 \times h_1 + 1 \times h_3 + 0 + 0)$$

or, $h_2 = \frac{1}{2}(h_1 + h_3)$

$$h_3 = \frac{1}{1+1+1+0}(1 \times h_2 + 1 \times 25 + 1 \times 5 + 0)$$

or, $h_3 = \frac{1}{3}(h_2 + 30)$

Let initial trial values be $h_1 = h_2 = h_3 = 0$.

Trial 1

$$h_1 = \frac{1}{3}(0 + 170) = 56.67\ m$$

$h_2 = \frac{1}{3}(0 + 0) = 0$ m

and, $h_3 = \frac{1}{3}(0 + 30) = 10$ m

Trial 2

$h_1 = \frac{1}{3}(0 + 170) = 56.67$ m

$h_2 = \frac{1}{3}(56.67 + 10) = 33.33$ m

and, $h_3 = \frac{1}{3}(0 + 30) = 10$ m

Carry out more trials with trial value of pervious trials:

	Trial 3	Trial 4	Trial 5	Trial 6	Trial 7	Trial 8	Trial 9	Trial 10	Trial 11	Trial 12	Trial 13	Trial 14
h_1	67.77	67.77	71.48	71.48	72.71	72.71	73.13	73.12	73.26	73.26	73.31	73.31
h_2	33.33	44.44	44.44	48.14	48.14	49.38	49.38	49.79	49.79	49.93	49.93	49.98
h_3	21.11	21.11	24.81	24.81	26.05	26.05	26.46	26.46	26.60	26.59	26.64	26.64

Trial 14 values ≈ Trial 13 values

Hence, we stop the iteration. Thus, the result is:

$h_1 = 73.31$ m

$h_2 = 49.98$ m

and, $h_3 = 26.64$ m

NOTE

Above solution uses Gauss-Jacobi method. You may use Gauss-Seidal method for more convergence.

7.4 RIVER STAGE WATER TABLE INTERACTION

This is the case of river recharging groundwater aquifer or groundwater aquifer supplying water to river.

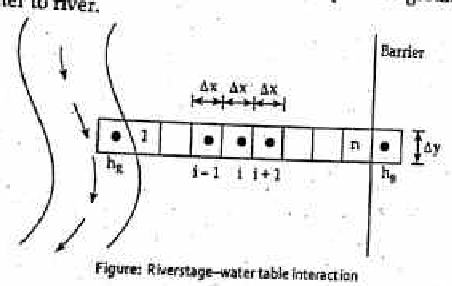


Figure: Riverstage–water table interaction

In this scheme,

h_R is the head at riverside is known.

h_B is the head at barrier is known.

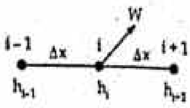
Our objective is to find head at intermediate points using finite difference method.

Governing equation for 1-D unsteady groundwater flow is:

$$\frac{\partial}{\partial x}\left(K\frac{\partial h}{\partial x}\right) = S_S\frac{\partial h}{\partial t} + W' \qquad [7.4.1]$$

In terms of T;

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) = S\frac{\partial h}{\partial t} + W \qquad [7.4.2]$$



Consider node i and its neighbouring nodes,

From equation [5.2.4]; we get,

$$f''(x) = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{2\Delta x^2}$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i+1} - 2h_i + h_{i-1}}{2\Delta x^2}$$

Substituting this value into equation [7.4.2]; we get,

$$T\left\{\frac{h_{i+1}^+ - 2h_i^+ + h_{i-1}^+}{\Delta x^2}\right\} = \frac{S_i(h_i^+ - h_i)}{\Delta t} + W_i;$$

where, h^+ is the value at present iteration.

h is the value at previous iteration.

$$\frac{T}{\Delta x^2}h_{i+1}^+ - \frac{2T}{\Delta x^2}h_i^+ + \frac{T}{\Delta x^2}h_{i-1}^+ = \frac{S_i(h_i^+ - h_i)}{\Delta t} + W_i$$

or, $$\frac{T}{\Delta x^2}h_{i+1}^+ + \left(-\frac{2T}{\Delta x^2} - \frac{S_i}{\Delta t}\right)h_i^+ + \frac{T}{\Delta x^2}h_{i-1}^+ = -\frac{S_i h_i}{\Delta t} + W_i$$

or, $$A_i h_{i-1}^+ + B_i h_i^+ + C_i h_{i+1}^+ = E_i \qquad [7.4.3]$$

where, $A_i = C = \frac{T}{\Delta x^2}$

$$B_i = -\left(\frac{2T}{\Delta x^2} + \frac{S_i}{\Delta t}\right)$$

$$E_i = -\frac{S_i h_i}{\Delta t} + W_i$$

Equation [7.4.3] for i = 1 to n can be represented in the matrix form as:

$$\begin{bmatrix} B_1 & C_1 & 0 & 0 & \cdots & 0 & 0 \\ A_2 & B_2 & C_2 & 0 & \cdots & 0 & 0 \\ 0 & A_3 & B_3 & C_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & \cdots & & & A_n & B_n \end{bmatrix} \begin{bmatrix} h_1^+ \\ h_2^+ \\ \vdots \\ \vdots \\ h_n^+ \end{bmatrix} = \begin{bmatrix} E_1 - A_1 h_R \\ E_2 \\ E_3 \\ \vdots \\ E_n - C_n h_B \end{bmatrix}$$

The seepage rate between grid points can be computed using:

$$\text{Discharge} = \frac{T}{\Delta x}(h_{i+1}^+ - h_i^+)\ m^3/sec./m$$

EXAMPLE 7.4

A schematic for simulating river stage-water table fluction is shown in the figure below:

The following data are given for the simulation homogenous and isotropic aquifer, river stage (h_L) = 203 m, length of aquifer = 500 m, Δx = 100 m, Δt = 1 day, transmissivity of aquifer = 0.03 m/sec., storage coefficient = 0.01. The initial value of water table at 5 grids is given as 200.1, 200.11, 200.2, 200.3,

200.4 m respectively. Compute coefficients of the 1D implicit finite difference model and display the matrix.
[2074 Magh]

Solution:

Since the aquifer is homogeneous and isotropic,

River stage (h_R) = 205 m

Length of aquifer = 500 m

Grid size, Δx = 100 m

Δt = 1 day = 24 × 60 × 60 = 86400 sec.

Transmissivity (T) = 0.03 m²/sec.

Storage coefficient = 0.01

Initial value of grids,

$h_1 = 200.1$ m

$h_2 = 200.11$ m

$h_3 = 200.2$ m

$h_4 = 200.3$ m

$h_5 = 200.4$ m

$C_n = 0$; as there is a barrier on the right.

$$A_i = C_i = \frac{T}{\Delta x^2} = \frac{0.03}{(100)^2} = 3 \times 10^{-6}$$

$$B_i = -\left(\frac{2T}{\Delta x^2} + \frac{S}{\Delta t}\right) = -\left(\frac{2 \times 0.03}{(100)^2} + \frac{0.01}{86400}\right) = -6.115 \times 10^{-6}$$

and, $E_i = -\frac{S h_i}{\Delta t} + W = -\frac{0.01 h_i}{86400} + 0 = -1.157 \times 10^{-7} h_i$

Then,

$$E_1 = -1.157 \times 10^{-7} h_i = -1.157 \times 10^{-7} \times 200.1 = -2.315 \times 10^{-5}$$

$$E_2 = -1.157 \times 10^{-7} h_i = -1.157 \times 10^{-7} \times 200.1 = -2.315 \times 10^{-5}$$

$$E_3 = -1.157 \times 10^{-7} h_i = -1.157 \times 10^{-7} \times 200.2 = -2.316 \times 10^{-5}$$

$$E_4 = -1.157 \times 10^{-7} h_i = -1.157 \times 10^{-7} \times 200.3 = -2.317 \times 10^{-5}$$

$$E_5 = -1.157 \times 10^{-7} h_i = -1.157 \times 10^{-7} \times 200.4 = -2.318 \times 10^{-5}$$

Again,

$$E_1 - A_1 h_R = -2.315 \times 10^{-5} - (3 \times 10^{-6} \times 205) = -6.38 \times 10^{-4}$$

$$E_5 - A_n h_B = E_5 = -2.318 \times 10^{-5}$$

Thus, the matrix can be written as:

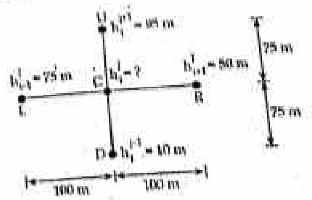
$$\begin{bmatrix} B_1 & C_1 & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & 0 & 0 \\ 0 & A_3 & B_3 & C_3 & 0 \\ 0 & 0 & A_4 & B_4 & C_4 \\ 0 & 0 & 0 & A_5 & B_5 \end{bmatrix} \begin{Bmatrix} h_1^j \\ h_2^j \\ h_3^j \\ h_4^j \\ h_5^j \end{Bmatrix} = \begin{bmatrix} E_1 - A_1 h_R \\ E_2 \\ E_3 \\ E_4 \\ E_5 - C_n h_B \end{bmatrix}$$

or, $$10^{-6} \begin{bmatrix} -6.115 & 3 & 0 & 0 & 0 \\ 3 & -6.115 & 3 & 0 & 0 \\ 0 & 3 & -6.115 & 3 & 0 \\ 0 & 0 & 3 & -6.115 & 3 \\ 0 & 0 & 0 & 3 & -6.115 \end{bmatrix} \begin{Bmatrix} h_1^j \\ h_2^j \\ h_3^j \\ h_4^j \\ h_5^j \end{Bmatrix} = 10^{-5} \begin{bmatrix} -63.8 \\ -2.315 \\ -2.316 \\ -2.317 \\ -2.318 \end{bmatrix}$$

EXAMPLE 7.6

The final value of head at four grid points are: $h_{i+1}^j = 50$ m, $h_i^{j-1} = 10$ m, $h_i^{j+1} = 95$ m and $h_{i-1}^j = 75$ m. The conductivity are: $K_{xx} = 0.9 \times 10^{-5}$ m/sec. and $K_{yy} = 1 \times 10^{-5}$ m/sec. Evaluate head at h_i^j and discharges. Take: $\Delta x = 100$ m and $\Delta y = 75$ m. Consider steady flow with no withdrawal.

Solution:



Coefficients

$$L = R = \frac{K_{xx}}{\Delta x^2} = \frac{0.9 \times 10^{-5}}{(100)^2} = 9 \times 10^{-10}$$

$$U = D = \frac{K_{yy}}{\Delta y^2} = \frac{1 \times 10^{-5}}{(75)^2} = 17.78 \times 10^{-10}$$

$$C = L + R + U + D = 53.56 \times 10^{-10}$$

Now,

$$h_i^j = \frac{1}{C}\left(Lh_{i-1}^j + Rh_{i+1}^j + Dh_i^{j+1} + Uh_i^{j-1}\right)$$

$$= \frac{1}{53.56 \times 10^{-10}}(9 \times 75 + 9 \times 50 + 17.78 \times 95 + 17.78 \times 10) \times 10^{-10}$$

$$= 55.86 \text{ m}$$

Again,

$$\text{Flow from L to C} = L\Delta x\Delta y\left(h_{i-1}^j - h_i^j\right)$$

$$= 9 \times 10^{-10} \times 100 \times 75 \times (75 - 55.86)$$

$= 1.29 \times 10^{-4}\ m^3/sec./m$

Flow from C to R $= R\Delta x \Delta y \left(h_i^j - h_{i+1}^j\right)$

$= 9 \times 10^{-10} \times 100 \times 75 \times (55.86 - 50)$

$= 0.39 \times 10^{-4}\ m^3/sec./m$

Flow from U to C $= U\Delta x \Delta y \left(h_i^{j+1} - h_i^j\right)$

$= 17.78 \times 10^{-10} \times 100 \times 75 \times (95 - 55.86)$

$= 5.21 \times 10^{-4}\ m^3/sec./m$

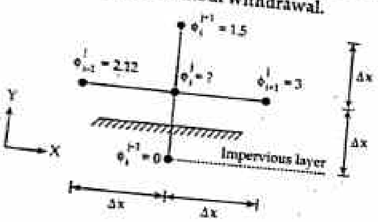
Flow from C to D $= D\Delta x \Delta y \left(h_i^j - h_i^{j-1}\right)$

$= 17.78 \times 10^{-10} \times 100 \times 75 \times (55.86 - 10)$

$= 6.11 \times 10^{-4}\ m^3/sec./m$

EXAMPLE 7.6

For the scheme shown in the figure, determine ϕ_i^j. Consider homogenous isotropic domain with steady flow without withdrawal.



Solution:

Here,

$L = R = U = 1$

$D = 0$

$C = L + R + U + D = 3$

We know that;

$$\phi_i^j = \frac{1}{C}\left(L\phi_{i-1}^j + R\phi_{i+1}^j + U\phi_i^{j+1}\right) = \frac{1}{3}(2.12 + 3 + 1.5) = 2.21$$

WORKED OUT PROBLEMS

Problem 1

Explain 1-D implicit model to evaluate river stage water table interaction. **[2070 Bhadra]**

Solution: See the definition part 7.4

Problem 2

Finite difference equation for simulating river stage-water interactions considering one-dimensional flow. **[2070 Magh]**

Solution: See the definition part 7.4

Problem 3

Explain the continuity equation used in groundwater flow analysis. Write down the algorithm for simulation of seepage under a dam. **[2071 Bhadra]**

Solution: See the definition part 7.2 and 7.3

Problem 4

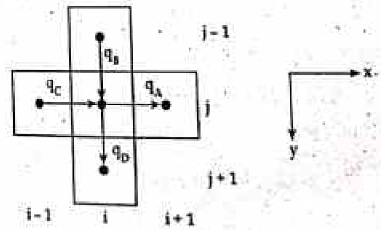
The figure shows a 2 dimensional grid with the value of potential function (ϕ) for simulating seepage. Calculate vertical and horizontal seepage into and out of grid A i.e., Q_A, Q_B, Q_C and Q_D. Transmissivity in x-direction and y-direction are 2200 m^2/day and 2400 m^2/day respectively, for all grids; $\Delta X = 80$ m and $\Delta Y = 90$ m. **[2073 Bhadra]**

0	0	0	0
2.3	2.2	2.1	2.08
2.27	2.23	2.06 Grid A	2.01
2.22	2.12	2.03	2
0	0	0	0

Solution: See the example 7.2

Problem 5

The figure below shows a central grid surrounded by four grids for simulating two dimensional groundwater flow under steady state condition. Values of potential function (ϕ) are given below:



$\phi_{i-1,j} = 12$, $\phi_{i+1,j} = 14$, $\phi_{i,j} = 13$, $\phi_{i,j-1} = 13.5$, $\phi_{i,j+1} = 11$

Transmissivity in x-direction = 0.013 m²/sec. for all grids, transmissivity in y-direction = 0.015 m²/sec. for all grids. Taking Δx = 20 m and Δy = 25 m, compute Darcy fluxes q_A, q_B, q_C and q_D from the finite difference equation in terms of φ. [2073 Magh]

Solution:

Given that;

$\phi_{i-1,j} = 12$

$\phi_{i+1,j} = 14$

$\phi_{i,j} = 13$

$\phi_{i,j-1} = 13.5$

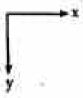
$\phi_{i,j+1} = 11$

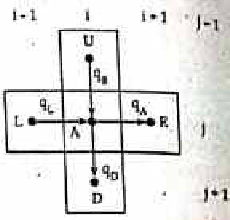
$T_{xx} = 0.013\ m^2/sec.$

$T_{yy} = 0.015\ m^2/sec.$

$\Delta x = 20\ m$

$\Delta y = 25\ m$





$$L = R = \frac{T_{xx}}{\Delta x^2} = \frac{0.013}{(20)^2} = 3.25 \times 10^{-5} \text{ per sec.}$$

$$U = D = \frac{T_{yy}}{\Delta y^2} = \frac{0.015}{(25)^2} = 2.4 \times 10^{-5} \text{ per sec.}$$

Now, seepage of water from left to A is given by;

$$Q_C = L \times \Delta x \times \Delta y \left(\phi^j_{i-1} - \phi^j_i\right)$$
$$= 3.25 \times 10^{-5} \times 20 \times 25 \times (12 - 13)$$
$$= -0.01625\ m^3/sec.$$

$$\therefore \quad q_C = \frac{Q_c}{\Delta x} = \frac{-0.01625}{20} = -8.125 \times 10^{-4}\ m^2/sec.$$

Similarly;

$$Q_A = L \times \Delta x \times \Delta y \left(\phi^j_i - \phi^j_{i+1}\right)$$
$$= 3.25 \times 10^{-5} \times 20 \times 25 \times (13 - 14)$$
$$= -0.01625\ m^3/sec.$$

$$\therefore \quad q_A = \frac{Q_c}{\Delta x} = \frac{-0.01625}{20} = -8.125 \times 10^{-4}\ m^2/sec.$$

Again,

$$Q_B = U \times \Delta x \times \Delta y \left(\phi^{j-1}_i - \phi^j_i\right)$$
$$= 2.4 \times 10^{-5} \times 20 \times 25 \times (13.5 - 13)$$
$$= 6 \times 10^{-3}\ m^3/sec.$$

$$\therefore \quad q_B = \frac{Q_c}{\Delta x} = \frac{6 \times 10^{-3}}{25} = 2.4 \times 10^{-4}\ m^2/sec.$$

Similarly;

$$Q_D = U \times \Delta x \times \Delta y \left(\phi^j_i - \phi^{j+1}_i\right)$$
$$= 2.4 \times 10^{-5} \times 20 \times 25 \times (13 - 11)$$
$$= 0.024\ m^3/sec.$$
$$q_D = \frac{Q_c}{\Delta x} = \frac{0.024}{25} = 9.6 \times 10^{-4}\ m^2/sec.$$

Problem 6

Develop tri-diagonal coefficient matrix to evaluate river stage water table interactions for an aquifer along a river. [2073 Magh]

Solution: See the definition part 7.4

NOTE

Make 3 × 3 matrix as final expression.