Chapter-1.

Introduction, Approximation And errors of Computation

Introduction

- Numerical methods are capable of handling large system of equations, non-linearities and complicated geometrics.
- Numerical methods are extremely powerful problem solving tools.
- Generate reliable solutions to mathematical problems.

Why do we need numerical method?

- If no analytical solution exist and if it is difficult to obtain or not practical.

$$4x^2 + 2x + 3 = 0$$

quadratic eqn: $ax^2 + bx + c = 0$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\left.\begin{array}{l} x^9 + 2x^2 - 3 = 0 \\ x = e^{-x} \end{array}\right\} \text{no analytical solution.}$$

Approximation

The numbers: $\pi = 3.1415....$

$\sqrt{3} = 1.732050...$

$4/3 = 1.33333...$

cannot be expressed by finite number of digits.

These may be ~~approxima~~ approximate by such numbers which represent the given number to a certain degree of accuracy is called approximation.

So, $\pi = 3.1415$
$\sqrt{3} = 1.7320$
$4/3 = 1.3333$

Errors Of Computation

An error represents inaccuracy and imprecision (not exact) of a numerical calculation or computation.

Types of Error

explain
1. Inherent error — error in data before processing.
2. Rounding error — error by round off $\Rightarrow 8.5 \rightarrow 9$
3. Truncation error
4. Absolute, Relative and Percentage errors.

4. Absolute, Relative and Percentage errors

If x is the true value of a quantity and x' is its approximate value then $|x - x'|$ is called the absolute error.

- Denoted by E_a i.e $E_a = |x - x'|$
- Relative error is defined by, $E_r = \left|\frac{x - x'}{x}\right|$
- Percentage error is $E_p = 100 E_r$

$$= 100 * \left|\frac{x - x'}{x}\right|$$

Q. Round off the numbers 865250 and 37.46235 to 4 significant figures and compute ε_a, ε_r, ε_p in each case.

⇒

$$x = 865250$$
$$x' = 865200$$

$$\varepsilon_a = |x - x'|$$
$$= 50$$

$$\varepsilon_r = \left| \frac{x - x'}{x} \right|$$
$$= 5.778 \times 10^{-5}$$

$$\varepsilon_p = 100\ \varepsilon_r$$
$$= 5.778 \times 10^{-3}\ \%$$

⇒

$$x = 37.46235$$
$$x' = 37.46$$

$$\varepsilon_a = |x - x'|$$
$$= 0.00235$$

$$\varepsilon_r = \left| \frac{x - x'}{x} \right|$$
$$= 6.272 \times 10^{-5}$$

$$\varepsilon_p = 100\ \varepsilon_r$$
$$= 6.272 \times 10^{-3}\ \%$$

Date
Page

1. Inherent Error

It is an error found in a program that causes it to fail regardless of what user does and is commonly unavoidable This error requires the programmer to modify the code to correct the issue. This is usually caused by error in data before processing.

2. Rounding error

Roundoff error occurs because of the computing device's inability to deal with certain numbers. such numbers need to be rounded off to some near approximation which is dependent on the word size used to represent numbers of the device.

3. Truncation Error.

It refers to the error in method, which occurs because some series (infinite or finite) is truncated to a fewer number of terms. Such errors are essentially algorithmic errors and we can predict the extent of the error that will occur in the method.

Date 2074-02-02
Page

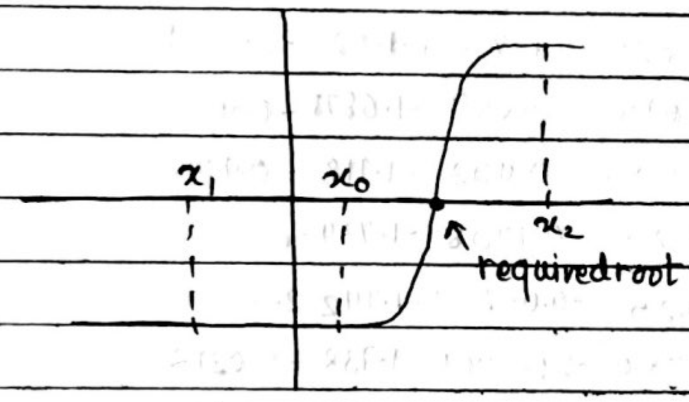
Chapter-2

Solution Of Non-linear Equations.

Non-linear Equation Solver

- Bracketing
  - Bisection method
  - Falsi method
- Graphical
- Open Method (Non-bracketing)
  - Newton Raphson
  - Secant

1) Bisection Method / Bolzano method



i) $f(x_0) = 0$ exact root

ii) $f(x_1) \cdot f(x_2) < 0$
root lies bet^n x_1 and x_2

iii) $f(x_0) \cdot f(x_2) < 0$
root lies bet^n x_0 and x_2

- Also called Bolzano method
- Most simplest and reliable method for finding solution of non-linear equations.
- Let x_1 and x_2 be two points between which root lies then $x_0 = \dfrac{x_1 + x_2}{2}$

- Now, there exist following three conditions.

1) If $f(x_0) = 0$ then x_0 is the exact root of given equation.
2) If $f(x_0) \cdot f(x_2) < 0$ then there is a root between x_0 and x_2.
3) If $f(x_0) \cdot f(x_1) < 0$ then there is a root between x_0 and x_1.

Here, two initial guess x_1 and x_2 must bracket the root.

Mode 7. table - eqⁿ → start -3 to 1 (range) ⇒ -ve and +ve betⁿ so, -2 to -1

$B = x_1$, $A = F(x_1)$, $C = F(x_2)$, $D = x_2$

$F(x_1)$: $A = B^2 - 4B - 10$: $F(x_2)$: $C = D^2 - 4D - 10$

$\underbrace{x_0}_{E} = \frac{B+D}{2}$, $\underbrace{F(x_0)}_{F} = E^2 - 4E - 10$

Q. Find a root of the equation $x^2 - 4x - 10 = 0$. Using bisection method. Correct upto 3 decimal place.

⇒ Solution,

let $F(x) = x^2 - 4x - 10$ -3 to 1

Iteration	x_1	x_2	$f(x_1)$	$F(x_2)$	x_0	$f(x_0)$	Remarks
1	-2	-1	2	-5	-1.5	-1.75	$f_0 \cdot f_1 < 0$ so, $x_1 = x_1$, $x_2 = x_0$
2	-2	-1.5	2	-1.75	-1.75	0.0625	
3	-1.75	-1.5	0.0625	-1.75	-1.625	-0.859	
4	-1.75	-1.625	0.0625	-0.859	-1.6875	-0.402	$F_0 \cdot F_{x_1} < 0$ $x_1 = x_1$ $x_2 = x_0$
5	-1.75	-1.6875	0.0625	-0.402	-1.718	-0.1708	
6	-1.75	-1.718	0.0625	-0.1708	-1.734	-0.057244	
7	-1.75	-1.734	0.0625	-0.057244	-1.742	2.564×10^{-3}	
8	-1.742	-1.734	2.564×10^{-3}	-0.057244	-1.738	-0.0273	
9	-1.742	-1.738	2.564×10^{-3}	-0.0273	-1.74	-0.0124	
10	-1.742	-1.74	2.564×10^{-3}	-0.0124	-1.741	-4.919×10^{-3}	
11	-1.742	-1.741	2.564×10^{-3}	-4.919×10^{-3}	-1.7415	-1.17775×10^{-3}	

$A = B^2 - 4B - 10$: $C = D^2 - 4D - 10$: $E = (B+D)/2$: $F = E^2 - 4E - 10$ calc

$B = -2$, $A = 2$, $C = -5$, $D = -1$

$E = x_0$

$F = F(x_0)$

$F_0 \cdot F_2 < 0$ (false)

So, root doesnot lie betⁿ x_0 to x_2

exp

Q. $x^3 - 2x - 5 = 0$

Q. $\sin x = 1/x$ (save in radian)

Q. $x^3 - 2x - 5 = 0$ $A = B^3 - 2B - 5 : C = D^3 - 2D - 5 : E = (B+D)/2$

let $F(x) = x^3 - 2x - 5$ $: F = E^3 - 2E - 5$

Iteration	x_1	x_2	$F(x_1)$	$F(x_2)$	x_0	$F(x_0)$
1	2	3	-1	16	2.5	5.625
2	2	2.5	-1	5.625	2.25	1.8906
3	2	2.25	-1	1.8906	2.125	0.3457
4	2	2.125	-1	0.3457	2.0625	-0.3513
5	2.0625	2.125	-0.3513	0.3457	2.0937	-8.94×10^{-3}
6	2.0937	2.125	-8.94×10^{-3}	0.3457	2.1093	0.1665
7	2.0937	2.1093	-8.94×10^{-3}	0.1665	2.1015	0.7785

Q. $\sin x = 1/x$

let $f(x) = \sin x - 1/x$

	B	D	A	C	E	F
Iteration	x_1	x_2	$f(x_1)$	$f(x_2)$	x_0	$f(x_0)$
1	-3	-2	0.1922	-0.409	-2.5	-0.1984
2	-3	-2.5	0.1922	-0.1984	-2.75	-0.0180
3	-3	-2.75	0.1922	-0.0180	-2.875	0.0843
4	-2.875	-2.75	0.0843	-0.0180	-2.8125	0.0323
5	-2.8125	-2.75	0.0323	-0.0180	-2.7812	6.955×10^{-3}
6	-2.7812	-2.75	6.955×10^{-3}	-0.0180	-2.7656	-5.61×10^{-3}
7	-2.7812	-2.7656	6.955×10^{-3}	-5.61×10^{-3}	-2.7734	6.384×10^{-4}
8	-2.7734	-2.7656	6.384×10^{-4}	-5.61×10^{-3}	-2.7695	-2.489×10^{-3}
	-2.7734	-2.7695	6.384×10^{-4}	-2.489×10^{-3}		

Date
Page

1Q. Find at least one root of $x^3 - 2x - 5 = 0$ with the accuracy of 0.08 % using bisection method. [8]

2Q. Using the bisection method, find an approximate root of the eqn. $\sin x = 1/x$, that lies between $x = 1$ and $x = 1.5$ (measured in radian) carry out computations upto the 7th stage.

3Q. Find the root of the eqn, $\cos x = xe^x$ using the bisection method correct to 4 decimal places.

4Q. Find a positive real root of $x\log_{10}x = 1.2$ using the bisection method.

Q2. $\sin x = 1/x$

	B	D	A	C	E	F
	x_1	x_2	$f(x_1)$	$f(x_2)$	x_0	$f(x_0)$
1	1	1.5	-0.1585	0.3308	1.25	0.1489
2	1	1.25	-0.1585	0.1489	1.125	0.0133
3	1	1.125	-0.1585	0.0133	1.0625	-0.0676
4	1.0625	1.125	-0.0676	0.0133	1.09375	-0.02593
5	1.09375	1.125	-0.02593	0.0133	1.10937	-5.98×10^{-3}
6	1.10937	1.125	-5.98×10^{-3}	0.0133	1.1171	3.7635×10^{-3}
7	1.10937	1.1171	-5.98×10^{-3}	3.7635×10^{-3}	1.1132	-1.1504×10^{-3}

The approximate root is 1.1132

Q.3 $\cos x = xe^x$ $\Rightarrow$ cosx - (x)*exp(x)

B	D	A	C	E	F
x_1	x_2	$f(x_1)$	$f(x_2)$	x_0	$f(x_0)$
0	1	1	-2.177	0.5	0.05322
0.5	1	0.05322	-2.177	0.75	-0.85606
0.5	0.75	0.05322	-0.85606	0.625	-0.35669
0.5	0.625	0.05322	-0.35669	0.5625	-0.14129

x_1	x_2	$F(x_1)$	$F(x_2)$	x_0	$f(x_0)$
0.5	0.5625	0.05322	-0.14129	0.53125	-0.04151
0.5	0.53125	0.05322	-0.04151	0.51562	6.475×10^{-3}
0.51562	0.53125	6.475×10^{-3}	-0.04151	0.52343	-0.01735
0.51562	0.52343	6.475×10^{-3}	-0.01735	0.519525	-5.385×10^{-3}
0.51562	0.519525	6.475×10^{-3}	-5.385×10^{-3}	0.517572	5.622×10^{-4}
0.517572	0.519525	5.622×10^{-4}	-5.385×10^{-3}	0.51854	-2.408×10^{-3}
0.517572	0.51854	5.622×10^{-4}	-2.408×10^{-3}	0.51805	-9.087×10^{-4}
0.517572	0.51805	5.622×10^{-4}	-9.087×10^{-4}	0.517811	-1.631×10^{-4}
0.517572	0.517811	5.622×10^{-4}	-1.631×10^{-4}	0.517691	2.003×10^{-4}
0.517691	0.517811	2.003×10^{-4}	-1.631×10^{-4}	0.517751	1.935×10^{-5}
0.517751	0.517811	1.935×10^{-5}	-1.631×10^{-4}	0.517781	-7.19×10^{-5}

∴ The approximate root is 0.517781 *

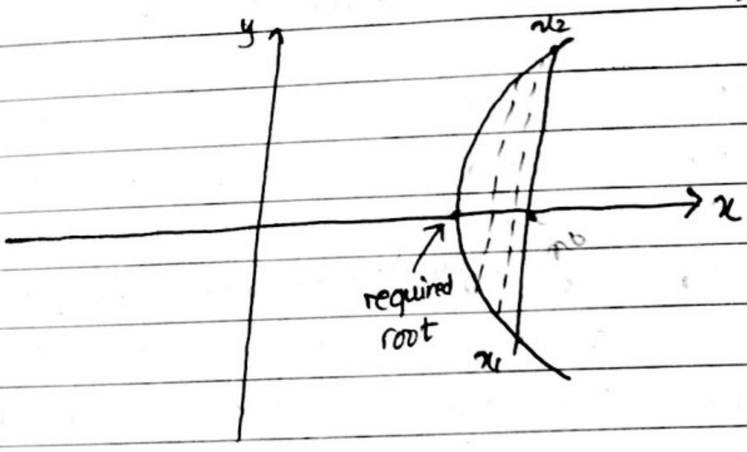
Q1. $x^3 - 2x - 5 = 0$

B	D	A	C	E	F
x_1	x_2	$F(x_1)$	$F(x_2)$	x_0	$F(x_0)$
2	3	-1	16	2.5	5.625
2	2.5	-1	5.625	2.25	1.8906
2	2.25	-1	1.8906	2.125	0.3457
2	2.125	-1	0.3457	2.0625	-0.3513
2.0625	2.125	-0.3513	0.3457	2.0937	-8.94×10^{-3}
2.0937	2.125	-8.94×10^{-3}	0.3457	2.1093	0.1665
2.0937	2.1093	-8.94×10^{-3}	0.1665	2.1015	0.7785
2.0937	2.1015	-8.94×10^{-3}	0.7785	2.0976	0.03408
2.0937	2.0976	-8.94×10^{-3}	0.03408	2.0956	0.01226
2.0937	2.0956	-8.94×10^{-3}	0.01226	2.09465	1.099×10^{-3}
2.0937	2.09465	-8.94×10^{-3}	1.099×10^{-3}	2.09417	-4.201×10^{-3}

∴ The approximate root is 2.09417 upto 3 decimal places.

2) Falsi Method.

⇒ from one side
⇒ moves closer to actual root



Line joining the points
$(x_1, f(x_1))$ and $(x_2, f(x_2))$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $$y - y_1 = \frac{F(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

∴ The line cuts the x-axis at when $x = x_0$, $y = 0$, so,
we have,

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} = \frac{-F(x_1)}{x_0 - x_1}$$

or, $$x_0 - x_1 = -\frac{F(x_1)(x_2 - x_1)}{F(x_2) - F(x_1)}$$

$$\boxed{\therefore x_0 = x_1 - \frac{F(x_1) \times (x_2 - x_1)}{F(x_2) - F(x_1)}}$$

$A = B^3 - 2B - 5$: $C = D^3 - 2D - 5$: $E = B - \frac{A(D-B)}{C-A}$: $F = E^3 - 2E - 5$

Q. Find a real root of the eq^n $x^3 - 2x - 5 = 0$ by the method of false position. Correct upto 3 decimal places

B	D	A	C	E	F
x_1	x_2	$f(x_1)$	$f(x_2)$	x_0	$f(x_0)$
2	3	-1	16	2.0588	-0.3907
2.0588	3	-0.3907	16	2.0812	-0.1473
2.0812	3	-0.1473	16	2.0896	-0.0549
2.0896	3	-0.0549	16	2.0927	-0.02036
2.0927	3	-0.02036	16	2.0938	-7.613×10^{-3}
2.0938	3	-7.613×10^{-3}	16	2.0942	-3.0898×10^{-3}
2.0942	3	-3.0898×10^{-3}	16	2.0944	-1.4451×10^{-3}

∴ The approximate root is 2.0944

Q.4 Find a positive real root of $x \log_{10} x = 1.2$ using bisection method.

B	D	A	C	E	F
x_1	x_2	$f(x_1)$	$f(x_2)$	x_0	$f(x_0)$
2	3	-0.5979	0.2313	2.5	-0.2051
2.5	3	-0.2051	0.2313	2.75	8.164×10^{-3}
2.5	2.75	-0.2051	8.164×10^{-3}	2.625	-0.0997
2.625	2.75	-0.0997	8.164×10^{-3}	2.6875	-0.04612
2.6875	2.75	-0.04612	8.164×10^{-3}	2.71875	-0.01905
2.71875	2.75	-0.01905	8.164×10^{-3}	2.73437	-5.46×10^{-3}
2.73437	2.75	-5.46×10^{-3}	8.164×10^{-3}	2.74218	1.342×10^{-3}
2.73437	2.74218	-5.46×10^{-3}	1.342×10^{-3}	2.73827	-2.067×10^{-3}
2.73827	2.74218	-2.067×10^{-3}	1.342×10^{-3}	2.740225	-3.672×10^{-4}
2.740225	2.74218	-3.672×10^{-4}	1.342×10^{-3}	2.741202	4.852×10^{-4}
2.740225	2.741202	-3.672×10^{-4}	4.852×10^{-4}	2.74071	5.878×10^{-5}
2.740225	2.74071	-3.672×10^{-4}	5.878×10^{-5}	2.74046	-1.557×10^{-4}
2.74046	2.74071	-1.557×10^{-4}	5.878×10^{-5}	2.74058	-5.328×10^{-5}
2.74058	2.74071	-5.328×10^{-5}	5.878×10^{-5}	2.74064	-9.558×10^{-7}
2.74064	2.74071	-9.558×10^{-7}	5.878×10^{-5}	2.74067	2.520×10^{-5}

∴ The approximate root is 2.74067 upto 4 decimal plac.

Date
Page

Q. $x\log_{10}x = 1.2$ [Falsi method]

$$x_0 = x_1 - \frac{F(x_1)(x_2-x_1)}{F(x_2)-F(x_1)}$$

B	D	A	C	E	F
x_1	x_2	$F(x_1)$	$F(x_2)$	x_0	$F(x_0)$
2	3	-0.5979	0.2313	2.7210	-0.01709
2.720	3	-0.01709	0.2313	2.7402	-3.84×10^{-4}
2.7402	3	-3.84×10^{-4}	0.2313	2.74063	-8.692×10^{-6}
2.74063	3	-8.692×10^{-6}	0.2313	2.74064	-3.136×10^{-7}

Hence, the required root is 2.74064 correct upto 4 decimal places

Q. Find a real root of $x^5-3x^3-1=0$ correct upto 4 decimal places using the falsi method.

x_1	x_2	$F(x_1)$	$F(x_2)$	x_0	$F(x_0)$
1	2	-3	7	1.3	-3.87807
1.3	2	-3.87807	7	1.5495	-3.22824
1.5495	2	-3.22824	7	1.69169	-1.66886
1.69169	2	-1.66886	7	1.75104	-0.64475
1.75104	2	-0.64475	7	1.77204	-0.22025
1.77204	2	-0.22025	7	1.77899	-0.07201
1.77899	2	-0.07201	7	1.78124	-0.02323
1.78124	2	-0.02323	7	1.78196	-7.48×10^{-3}

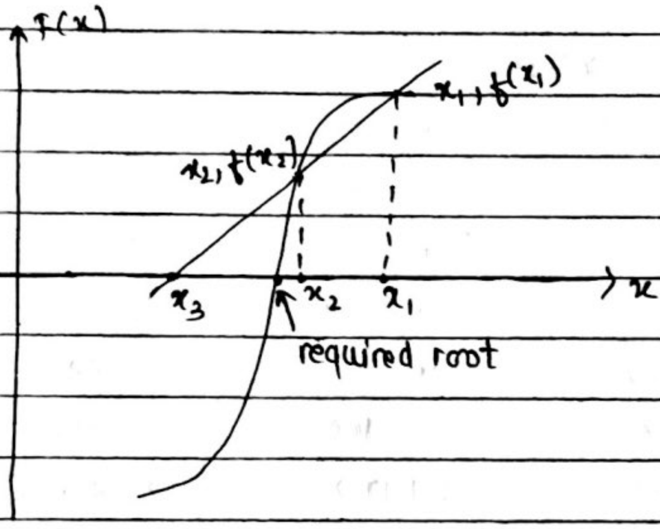
Hence, the required root is 1.7819 correct upto 3 decimal places

Q. $x = (32)^{1/4}$ correct upto 3 decimal place.

3) Open Method

i) Secant Method.

- condition check garnu pardaina
- Faster
- Iteration less.



$x_3 = x_0$

$\left(\begin{array}{l} x_1 = x_2 \\ x_2 = x_0 \end{array} \right)$

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$E = D - \frac{C(D-B)}{C-A}$$

Q. Find root of $x^5 - 3x^3 - 1$ using secant method.

B	D	A	C	E
x_1	x_2	$f(x_1)$	$f(x_2)$	x_0
1	2	-3	7	1.3
2	1.3	7	-3.87807	1.54955
1.3	1.54955	-3.87807	-3.22826	2.7893
1.54955	2.7893	-3.22826	102.736	1.5873
2.7893	1.5873	102.736	-2.9215	1.6205
1.5873	1.6205	-2.9215	-2.5914	1.8811
1.6205	1.8811	-2.5914	2.5846	1.7509
1.8811	1.7509	2.5846	-0.6476	1.7769
1.7509	1.7769	-0.6476	-0.1170	1.7826
1.7769	1.7826	-0.1170	6.37×10^{-3}	1.7823

The required root is 1.7823 correct upto 3 decimal places.

Q. Find the root of eqⁿ $x^3-2x-5=0$ using secant method correct upto 4 decimal places.

$$f(x) = x^3-2x-5$$

$$x_1 = 2 \quad \text{and} \quad x_2 = 3$$

x_1	x_2	$f(x_1)$	$f(x_2)$	x_0
2	3	-1	16	2.05882
3	2.05882	16	-0.3908	2.08126
2.05882	2.08126	-0.3908	-0.1472	2.09482
2.08126	2.09482	-0.1472	2.99×10^{-3}	2.09454
2.09482	2.09454	2.99×10^{-3}	-2.25×10^{-5}	2.09455

∴ The required root is 2.09455 correct upto 4 decimal places

Q. Find the root of eqⁿ $xe^x = \cos x$ using the secant method correct upto 4 decimal places.

x_1	x_2	$f(x_1)$	$f(x_2)$	x_0
1	2	1.718	13.77	0.85751
2	0.85751	13.77	1.0215	0.76602
0.85751	0.76602	1.0215	0.64797	0.60733
0.76602	0.60733	0.64797	0.11483	0.57315
0.60733	0.57315	0.11483	0.0167	0.567126
0.57315	0.56732	0.0167	5.39×10^{-4}	0.567126
0.56732	0.567126	5.39×10^{-4}	2.656×10^{-6}	0.567125

The required root is 0.567125 correct upto 5 decimal places

Q. Find the positive root of $x^4 - x = 10$ correct upto 3 decimal places, using secant method.

x_1	x_2	$f(x_1)$	$f(x_2)$	x_0
1	2	-10	4	1.7142
2	1.7142	4	-3.077	1.8385
1.7142	1.8385	-3.077	-0.412	1.8577
1.8385	1.8577	-0.412	+0.0539	1.85555
1.8577	1.85555	+0.0539	-7.77×10^{-4}	1.85558

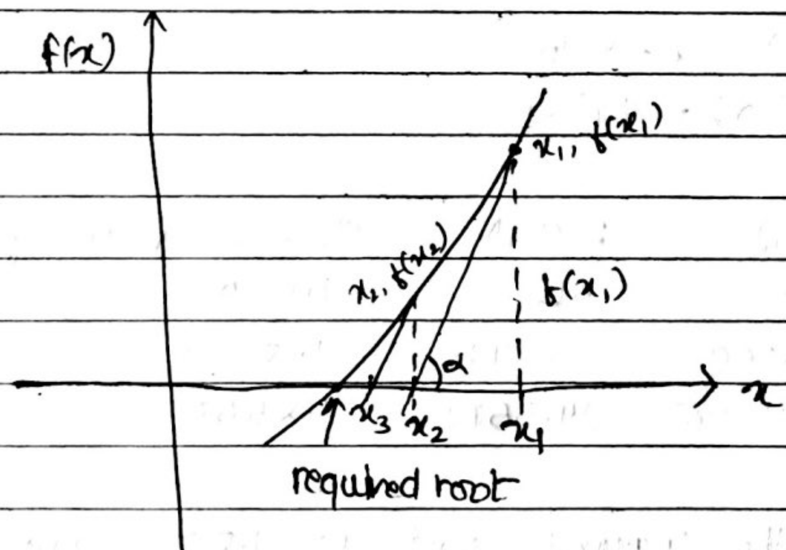
The required root is 1.85558 correct upto 4 decimal places.

Open Method.

3 ii) Newton Raphson Method

imp
Derivation
merits
Demerits

- long process if initial guess is taken away from root.



- Consider a graph as in figure.

We assume that x_1 is the approximate root of $f(x) = 0$. Draw tangent at $x = x_1$ as in fig.

The point of intersection gives the 2nd approximation. Then, the slope of tangent is given by:

$$\tan \alpha = \frac{f(x_1)}{x_1 - x_2} = f'(x_1)$$

where $f'(x_1)$ is the slope of $f(x)$ at $x = x_1$.

Solving for x_2 we obtain,

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

This is called the Newton - Raphson formula.
The next approximation would be.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In general,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Q. Find the positive root of $x^4 - x = 10$. Correct to 3 decimal places, using N-R method

eq^n type
Shift-Cal
$X = ?$
necessit[illegible]
suppose
[illegible]

Sol^n,

Let $f(x) = x^4 - x - 10$

$f'(x) = 4x^3 - 1$

$x_2 = x_1$

B	A	C	D
x_1	$f(x_1)$	$f'(x_1)$	$x_2 = x_1 - f(x_1) / f'(x_1)$
2	4	31	1.8709
1.8709	0.3809	25.1945	1.8557
1.8557	2.835×10^{-3}	24.5613	1.8555

Hence, the required root is 1.8555 correct upto 3 decimal places.

Q1. Find the root of the equation $f(x) = x^2 - 3x + 2$ of $x = 0$ using NR method.

Q2. Calculate a real root of non-linear eqn $x\sin x + \cos x = 0$ using NR method. Correct upto 4 decimal places. [8]

Q3. Find the reciprocal of 3 using NR method. [8]

→ Soln:

Reciprocal of 3

i.e. $x = 1/3$

or, $x^2 = 1/9$ [∵ squaring on both sides)

$$\therefore f(x) = x^2 - 1/9$$

$$f'(x) = 2x$$

Q4. Evaluate the required root of $f(x) = 4\sin x - e^x$, using NR method correct upto 4 decimal places. [8]

Q5. Using NR method, find the real root of $x\log_{10}x = 1.2$. Correct upto 5 decimal places.

Chapter - 3

Solution of System of linear algebraic equations

Linear Equations Solver

- Elimination method / Direct method
  - → Gauss elimination
  - → Gauss Jordan.
- Iterative method.
  - Jacobi's Iterative
  - Gauss Seidal

1) Elimination method:

a) Gauss elimination

Apply Gauss elimination method to solve the equations
$x+4y-z=-5$; $x+y-6z=-12$; $3x-y-z=4$

⇒ Solution,

$$A = \left[\begin{array}{ccc:c} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

Applying $R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - 3R_1$

$$[A/B] = \left[\begin{array}{ccc:c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & -13 & 2 & 19 \end{array}\right]$$

Applying $R_3 \rightarrow R_3 - \frac{(-13)}{(-3)} R_2$, we get

$$[A/B] = \left[\begin{array}{ccc:c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & 0 & 23.67 & 49.33 \end{array}\right]$$

Now,

$$x + 4y - z = -5$$
$$0.x - 3y - 5z = -7$$
$$0.x + 0.y + 23.67z = 49.33$$

By backward substitution, we get.

$$z = 2.084$$
$$y = -1.14$$
$$x = 1.645$$ Ans

Q. Solve:

$$10x - 7y + 3z + 5u = 6$$
$$-6x + 8y - z - 4u = 5$$
$$3x + y + 4z + 11u = 2$$
$$5x - 9y - 2z + 4u = 7$$ by gauss elimination method.

Gauss Elimination with Pivoting:

1) Partial Pivoting

Q. Solve the following system of equations. Using partial pivoting technique.

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$
$$x_1 - x_2 + x_3 = 0$$

Original system:

$$\begin{array}{r c} & \left[\begin{array}{ccc:c} 2 & 2 & 1 & 6 \\ 4 & 2 & 3 & 4 \\ 1 & -1 & 1 & 0 \end{array}\right] \end{array} \text{ Interchange}$$

largest: 4

$$= \left[\begin{array}{ccc:c} 4 & 2 & 3 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$R_2 \to R_2 - \tfrac{2}{4} R_1$$

$$R_3 \to R_3 - R_1/4$$

$$= \left[\begin{array}{ccc:c} 4 & 2 & 3 & 4 \\ 0 & 1 & -\tfrac{1}{2} & 4 \\ 0 & -\tfrac{3}{2} & \tfrac{1}{4} & -1 \end{array}\right]$$

$$= \left[\begin{array}{ccc:c} 4 & 2 & 3 & 4 \\ 0 & -\tfrac{3}{2} & +\tfrac{1}{4} & -1 \\ 0 & 1 & -\tfrac{1}{2} & 4 \end{array}\right]$$

$$R_3 \to R_2 + \tfrac{2}{3} R_1$$

$$= \left[\begin{array}{ccc:c} 4 & 2 & 3 & 4 \\ 0 & -3/2 & 1/4 & -1 \\ 0 & 0 & -1/3 & 10/3 \end{array}\right]$$

$$4x_1 + 2x_2 + 3x_3 = 4$$

$$0 \cdot x_1 - \tfrac{3}{2} x_2 + \frac{x_3}{4} = -1$$

$$0 \cdot x_1 + 0 \cdot x_2 - \tfrac{1}{3} x_3 = \tfrac{10}{3}$$

$\therefore$ $x_3 = -10$

$x_2 = -1$

$x_1 = 9$

Q. $2x_1 + x_2 + x_3 - 2x_4 = -10$

$4x_1 + 2x_3 + x_4 = 8$

$3x_1 + 2x_2 + 2x_3 = 7$

$x_1 + 3x_2 + 2x_3 - x_4 = -5$

Complete Pivoting.

$$\left[\begin{array}{ccc:c} 5 & 8 & 1 & 3 \\ 6 & \boxed{9} & \boxed{2} & 4 \\ 7 & \boxed{10} & \boxed{3} & 5 \end{array}\right]$$

b) Gauss Jordan

Q. Apply Gauss Jordan method to solve the equation.

$$x + y + z = 9$$
$$2x - 3y + 4z = 13$$
$$3x + 4y + 5z = 40$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

$R_2 \to R_2 - \frac{2}{1} R_1$

$R_3 \to R_3 - 3 R_1$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

$R_3 \to R_3 - \frac{1}{-5} R_2$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 2.4 & 12 \end{array}\right]$$

$R_1 \to R_1 - \left(\frac{1}{2.4}\right) R_3$

$R_2 \to R_2 - \left(\frac{2}{2.4}\right) R_3$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & -5 & 0 & -15 \\ 0 & 0 & 2.4 & 12 \end{array}\right]$$

$R_2 \to \left(R_2/-5\right)$

$R_3 \to R_3/2.4$

Rule. - upper triangle
- Lower triangle
- Diagonal ~~zeroone~~.

- only row operation allowed
- But column and row interchange allowing

$$\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Hence, required solⁿ ie:

$x = 1$

$y = 3$

$z = 5$

Q. Solve by Gauss Jordan:

$10x - 7y + 3z + 5u = 6$

$-6x + 8y - z - 4u = 5$

$3x + y + 4z + 11u = 2$

$5x - 9y - 2z + 4u = 7.$

$$[A:B] = \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ -6 & 8 & -1 & -4 & : & 5 \\ 3 & 1 & 4 & 11 & : & 2 \\ 5 & -9 & -2 & 4 & : & 7 \end{bmatrix}$$

$R_2 \to R_2 + \frac{6}{10} R_1$, $R_3 \to R_3 - \frac{3}{10} R_1$, $R_4 \to R_4 - \frac{5}{10} R_1$

$$= \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 19/5 & 4/5 & -1 & : & 43/5 \\ 0 & 31/10 & 31/10 & 19/2 & : & 1/5 \\ 0 & -11/2 & -7/2 & 3/2 & : & 4 \end{bmatrix}$$

$R_3 \to R_3 - \frac{31}{10} \times \frac{5}{19} R_2$

$R_4 \to R_4 + \frac{11}{2} \times \frac{5}{19} R_2$

$$= \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 19/5 & 4/5 & -1 & : & 43/5 \\ 0 & 0 & 93/38 & \frac{196}{19} & : & \frac{-259}{38} \\ 0 & 0 & -89/38 & 1/19 & : & 625/38 \end{bmatrix}$$

$R_4 \to R_4 + \frac{89}{93} R_3$

$$= \left[\begin{array}{cccc:c} 10 & -7 & 3 & 5 & 6 \\ 0 & 19/5 & 4/5 & -1 & 43/5 \\ 0 & 0 & 93/38 & 196/19 & -259/38 \\ 0 & 0 & 0 & 9.9247 & 923/93 \end{array}\right]$$

$R_1 \to R_1 - \frac{93 \times 5}{923} R_4$, $R_2 \to R_2 + \frac{93}{923} R_4$

$R_3 \to R_3 - \frac{93}{923} \times \frac{196}{19} R_4$

$$= \left[\begin{array}{cccc:c} 10 & -7 & 3 & 0 & 1 \\ 0 & 19/5 & 4/5 & 0 & 48/5 \\ 0 & 0 & 93/38 & 0 & -651/38 \\ 0 & 0 & 0 & 923/93 & 923/93 \end{array}\right]$$

$R_1 \to R_1 - \frac{38}{93} \times 3 R_3$, $R_2 \to R_2 - \frac{38}{93} \times \frac{4}{5} R_3$

$$= \left[\begin{array}{cccc:c} 10 & -7 & 0 & 0 & 22 \\ 0 & 19/5 & 0 & 0 & 76/5 \\ 0 & 0 & 93/38 & 0 & -651/38 \\ 0 & 0 & 0 & 923/93 & 923/93 \end{array}\right]$$

$R_1 \to R_1 + \frac{5}{19} \times 7 R_2$

$$= \left[\begin{array}{cccc:c} 10 & 0 & 0 & 0 & 50 \\ 0 & 19/5 & 0 & 0 & 76/5 \\ 0 & 0 & 93/38 & 0 & -651/38 \\ 0 & 0 & 0 & 923/93 & 923/93 \end{array}\right]$$

$R_1 \to R_1/10$, $R_2 \to \frac{5}{19} R_2$, $R_3 \to \frac{38}{93} R_3$, $R_4 \to \frac{93}{923} R_4$

$$= \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 76/19 \\ 0 & 0 & 1 & 0 & -651/93 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

So, $\therefore x = 5$ $\therefore u = 1$

$\therefore y = 4$

$\therefore z = -7$

Lower Δ

LU factorization

Upper Δ

Q) Apply factorization method to solve the eq^n:

$$3x + 2y + 7z = 4$$
$$2x + 3y + z = 5$$
$$3x + 4y + z = 7$$

$$A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

or, $$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$\therefore$ The product of L and U doesnot produce unique solution. So, in order to produce unique ~~solution~~ factors, we assume diagonal elements L or U to be unity.

- The decomposition with L having unit diagonal values is called the Dolittle LU decomposition.
- While the other one with U having unit diagonal elements is called crout LU decomposition.

$\therefore$ It has no unique solution. So, we assume U_{ii} or $L_{ii} = 1$

ie $$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Now, equating with [A] we get.

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & U_{13}L_{21}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix}$$

Equating the matrices,

$\therefore U_{11} = 3$

$\therefore U_{12} = 2$

$\therefore U_{13} = 7$

$\Rightarrow L_{21} U_{11} = 2$

$\therefore L_{21} = 2/3$

$\Rightarrow U_{11} L_{31} = 3$

$\therefore L_{31} = 1$

$L_{21} U_{12} + U_{22} = 3$

$\Rightarrow \frac{2}{3} \times 2 + U_{22} = 3$

$\therefore U_{22} = 5/3$

$L_{21} U_{13} + U_{23} = 1$

or, $2/3 \times 7 + U_{23} = 1$

$\therefore U_{23} = -11/3$

$L_{31} U_{12} + L_{32} U_{22} = 4$

or, $1 \times 2 + L_{32} \times 5/3 = 4$

$\therefore L_{32} = +6/5$

$L_{31} U_{13} + U_{23} L_{32} + U_{33} = 1$

or, $1 \times 7 + \frac{-11}{3} \times 6/5 + U_{33} = 1$

$\therefore U_{33} = -8/5$

Thus,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

writing,

$[L][Y] = [B]$ the given system becomes,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

Solving these system, we have,

$\therefore\ y_1 = 4$

$\frac{2}{3} Y_1 + y_2 = 5$

$\therefore\ y_2 = 7/3$

$y_1 + \frac{6}{5} y_2 + y_3 = 7$

$\therefore\ y_3 = 1/5$

Now,

$[U][X] = [Y]$

or, $$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

or, $$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

So,

$\therefore\ z = -1/8$

$\frac{5}{3} y - \frac{11}{3} z = 7/3$

$\Rightarrow\ y = 9/8$

$3x + 2y + 7z = 4$

$\Rightarrow\ x = 7/8$ Ans

Inverse matrix using Gauss Jordan Elimination.

$$AI = IA^{-1}$$

Q. Find the inverse of the matrix.

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

Sol

Writing the given matrix side by side with the unit matrix of same order, we have,

$$\left[\begin{array}{ccc:ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + \frac{1}{2} R_1$

$$\left[\begin{array}{ccc:ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1/2 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1/2 \;;\; R_2 \to R_2/5 \;;\; R_3 \to R_3/3$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & -2/5 & -1/5 & 1/5 & 0 \\ 0 & 0 & 1 & 1/6 & 0 & 1/3 \end{array}\right]$$

$R_1 \to R_1 - 2R_3 \;;\; R_2 \to R_2 + 2/5\, R_3$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1/6 & 0 & -2/3 \\ 0 & 1 & 0 & -2/15 & 1/5 & 2/15 \\ 0 & 0 & 1 & 1/6 & 0 & 1/3 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/30 & 1/5 & -8/15 \\ 0 & 1 & 0 & -2/15 & 1/5 & 2/15 \\ 0 & 0 & 1 & 1/6 & 0 & 1/3 \end{array}\right]$$

Hence the inverse matrix is : $\begin{bmatrix} 1/30 & 1/5 & -8/15 \\ -2/15 & 1/5 & 2/15 \\ 1/6 & 0 & 1/3 \end{bmatrix}$ #

2. data
Mat
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Imp **Eigen Value and Eigen Vector using Power Method.**

Q. Determine the eigen value and corresponding eigen vector of the following matrix by power method. [8 marks]

$$\begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix}$$

⇒ Solution,

$$\underset{\substack{\uparrow \\ \text{eigen value}}}{AX = \lambda} X \leftarrow \text{Eigen Vector.}$$

$$\begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

highest value common.

$$= 4 \begin{bmatrix} 1/4 \\ 1 \\ 1/2 \end{bmatrix}$$

$$\text{or, } AX^1 = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5.75 \\ 6.5 \\ 9 \end{bmatrix}$$

$$= 9 \begin{bmatrix} 0.638 \\ 0.722 \\ 1 \end{bmatrix}$$

$$\text{or, } AX^2 = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 0.638 \\ 0.722 \\ 1 \end{bmatrix} = \begin{bmatrix} 6.526 \\ 10.996 \\ 10.608 \end{bmatrix}$$

$$= 10.996 \begin{bmatrix} 0.5934 \\ 1 \\ 0.9647 \end{bmatrix}$$

$$\text{or, } AX^3 = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 0.5934 \\ 1 \\ 0.9647 \end{bmatrix} = \begin{bmatrix} 7.4875 \\ 11.126 \\ 12.01 \end{bmatrix} = 12.01 \begin{bmatrix} 7.4875 \\ 11.126/12 \\ 1 \end{bmatrix}$$

or, $AX^4 = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 7.4875/12.01 \\ 11.126/12.01 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.329 \\ 11.346 \\ 11.805 \end{bmatrix}$

$= 11.805 \begin{bmatrix} 0.6208 \\ 0.9611 \\ 1 \end{bmatrix}$

or, $AX^5 = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 0.6208 \\ 0.9611 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.4652 \\ 11.405 \\ 12.008 \end{bmatrix}$

$= 12.008 \begin{bmatrix} 0.6216 \\ 0.9497 \\ 1 \end{bmatrix}$

or, $AX^6 = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 0.6216 \\ 0.9497 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.4204 \\ 11.385 \\ 11.941 \end{bmatrix}$

$= 11.941 \begin{bmatrix} 7.4204/11.941 \\ 11.385/11.941 \\ 1 \end{bmatrix}$

or, $AX^7 = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 0.6214 \\ 0.9534 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.4351 \\ 11.392 \\ 11.963 \end{bmatrix}$

$= 11.963 \begin{bmatrix} 0.6215 \\ 0.9522 \\ 1 \end{bmatrix}$

or, $AX^8 = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 0.6215 \\ 0.9522 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.4305 \\ 11.39 \\ 11.956 \end{bmatrix}$

$= 11.956 \begin{bmatrix} 0.6214 \\ 0.9526 \\ 1 \end{bmatrix}$

$$Ax^9 = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 7 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 0.6214 \\ 0.9526 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.4321 \\ 11.391 \\ 11.958 \end{bmatrix}$$

$$= 11.958 \begin{bmatrix} 0.6215 \\ 0.9525 \\ 1 \end{bmatrix}$$

The required eigen value $\lambda = 11.958$ and eigen vector is $\begin{bmatrix} 0.6215 \\ 0.9525 \\ 1 \end{bmatrix}$

Iterative Methods (Jacobi Method)

Q. Solve by Jacobi's method, the eq^{n}s:
$5x - y + z = 10$; $2x + 4y = 12$; $x + y + 5z = -1$, start with $(2, 3, 0)$ suppose $(0,0,0)$ if not given.

→ Solution,

$$x = \frac{1}{5}(10 + y - z) = \frac{10 + y - z}{5}$$

$$y = \frac{12 - 2x}{4}$$

$$z = \frac{(-1 - x - y)}{5}$$

Iteration	x	y	z
0	2	3	0
1	2.6	2	-1.2
2	2.64	1.7	-1.12
3	2.564	1.68	-1.068
4	2.5496	1.718	-1.0488
5	2.55336	1.7252	-1.05352
6	2.5557	1.7233	-1.0557

7. 2.5558 1.7221 -1.0558

$\therefore$ The value of 6th and 7th Iteration is practically same, so we can stop.

Hence, $x = 2.5558$, $y = 1.7221$, $z = -1.0558$ #

Q. Solve by Jacobi's Iterative method. eq^{n}s.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Gauss seidal Iteration method.

It is an improved version of Jacobi Iteration method or modification of Jacobi's method.

Q. Apply Gauss-seidal Iteration method to solve eq^n of

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

$\Rightarrow$ Sol^n,

$$x = \frac{17 - y + 2z}{20}$$

$$y = \frac{-18 - 3x + z}{20}$$

$$z = \frac{25 - 2x + 3y}{20}$$

Let us assume initial guess,

$x_0 = 0$, $y_0 = 0$, $z_0 = 0$

Iteration	x	y	z
0	0	0	0
1	0.85	-1.027	1.011
2	1.0024	-0.9998	0.9997
3	0.9999	-1	1.00001
4.	1.000001	-0.9999	1.00001
5.	0.9999	-0.9999	1.00002
6.	0.9999	-0.9999	1.00002

Since value of 5^{th} and 6^{th} are same so,

$x = 0.9999$

$y = -0.9999$

$z = 1.00002$

CHAPTER-4

Interpolation:

Suppose we are given the following values of $y=f(x)$ for a set of values x

x	x_0	x_1	x_2	....	x_n
y	y_0	y_1	y_2	....	y_n

Then, the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation.

Newton's interpolation (forward, backward)

1) Newton Gregory forward interpolation.

x	x_0	x_1	x_2	---	x_n
$f(x)$	y_0	y_1	y_2	---	y_n

Here, the f^n are tabulated at equal intervals i.e $x_2 - x_1 = x_3 - x_2 = \dots = x_n - x_{n-1} = h$, a constant with tabulation at equal intervals, a difference table for n-point is expressed as:

x	$f(x)$	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$	$\Delta^4 f_0$
x_0	f_0				
		$\Delta f_0 = f_1 - f_0$			
x_0+h	f_1		$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$		
		$\Delta f_1 = f_2 - f_1$		$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$	
x_0+2h	f_2		$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$		$\Delta^4 f_0 = \Delta^3 f_1 - \Delta^3 f_0$
		$\Delta f_2 = f_3 - f_2$		$\Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1$	
x_0+3h	f_3		$\Delta^2 f_2 = \Delta f_3 - \Delta f_2$		
		$\Delta f_3 = f_4 - f_3$			
x_0+4h	f_4				

$\therefore$ h = Uniform difference in the value of x

Here, $\Delta f_0 = f_1 - f_0$, $\Delta f_1 = f_2 - f_1$, $\Delta f_2 = f_3 - f_2$

$$\boxed{\Delta f_i = f_{i+1} - f_i}$$ " first forward difference"

Similarly,

$$\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = f_2 - f_1 - (f_1 - f_0)$$
$$= f_2 - 2f_1 + f_0$$

$$\boxed{\therefore \Delta^2 f_i = f_{i+2} - 2f_{i+1} + f_i}$$ " second forward Difference"

$$\therefore f(x) = f_0 + P\Delta f_0 + \frac{P(P-1)}{2!}\Delta^2 f_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 f_0 + \dots + \frac{P(P-1)(P-2)\dots(P-(n-1))}{n!}\Delta^n f_0$$

Q. Estimate the value of function at $x = 0.16$ from the following tabulated f^n.

x	0.1	0.2	0.3	0.4
$f(x)$	1.005	1.020	1.045	1.081

$$p = \frac{x - x_0}{h} = \frac{0.16 - 0.1}{0.1} = 0.6$$

x	$f(x)$	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$
		0.015	0.01	0.001
0.1	1.005			
		0.025	0.011	
0.2	1.020			
		0.036		
0.3	1.045			
0.4	1.081			

$$f(x) = f_0 + P\Delta f_0 + \frac{P(P-1)}{2!}\Delta^2 f_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 f_0$$

$$= 1.005 + 0.6 \times 0.015 + \frac{0.6(0.6-1)}{2!} \times 0.01 + \frac{0.6(0.6-1)(0.6-2)}{3!} \times 0.001$$

$$= 1.012856 \text{ \#}$$

2. Newton's Backward Interpolation.

If the table is too long and if the required point is closed to the end of the table, we can use Newton-Gregory Backward difference formula. Hence, the ~~requir~~ reference point is x_n instead of x_0

$$y_x = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$
$$+ \dots + \frac{P(P+1)\dots(P+(n-1))}{n!}\nabla^2 y_n$$

Q. The table gives the distance in miles of the visible horizon for the given heights in feet above the earth's surface.

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find value of y when $x = 410$

⇒ Given $x = 410$, and $x_n = 400$

$$p = \frac{x - x_n}{h} = \frac{410 - 400}{\cancel{4}50} = 0.20$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
100	10.63						
150	13.03	2.4					
200	15.04	2.01	-0.39				
250	16.81	1.73	-0.24	0.15			
300	18.42	1.61	-0.16	0.08	-0.07		
350	19.90	1.48	-0.13	0.03	-0.05	0.02	
400	21.27	1.37	-0.11	0.02	-0.01	0.04	0.02

$$y_x = 21.27 + 0.20 \times 1.37 + \frac{0.20(0.20+1)}{2!} \times (-0.11)$$

$$+ \frac{0.20(0.20+1)(0.20+2)}{3!} \times (0.02) + \frac{0.20(0.20+1)(0.20+2)(0.2+3)}{4!} \times -0.0$$

$$+ \frac{0.20(0.20+1)(0.20+2)(0.20+3)(0.20+4)}{5!} \times (0.04)$$

$$+ \frac{0.20(0.20+1)(0.20+2)(0.20+3)(0.20+4)(0.20+5)}{6!} \times 0.02$$

$= 21.53$ feet. #

Central Difference Interpolation:

1) Stirling's Formula:

$$y_p = y_0 + p\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4_{y-2} + \ldots\ldots$$

where $P = \frac{x - x_m}{h}$

* For P lying between ($-1/4$ to $1/4$) or (-0.25 to 0.25) prefer stirling formula.
* Calulation upto Δ^4 only needed.

2) Bessel's formula:

$$y_p = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{(P-\frac{1}{2})P(P-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{P(P+1)P(p-1)(P-2)}{4!}\frac{\Delta^4_{y-2} + \Delta^4_{y-1}}{2} + \ldots\ldots$$

* For P lying between $1/4$ and $3/4$, use Bessel's formula
* calculation upto Δ^4 only needed

Q. Find the functional value at 0.644 from the table given below

x	0.62	0.63	0.64	0.65	0.66	0.67
y	1.858928	1.877610	1.896481	1.915541	1.934792	1.95437

$\Rightarrow$ Sol^n,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.62	1.858928 y_{-2}	0.018682 y_{-2}	1.89×10^{-4} y_{-2}	0 y_{-2}	0.02×10^{-4} y_{-2}
0.63	1.877610 y_{-1}	0.018871 y_{-1}	1.89×10^{-4} y_{-1}	0.02×10^{-4} y_{-1}	1.34×10^{-4} y_{-1}
x_0 0.64	1.896481 y_0	0.01906 y_0	1.91×10^{-4} y_0	1.36×10^{-4} y_0	
0.65	1.915541 y_{+1}	0.019251 y_{+1}	3.27×10^{-4} y_{+1}		
0.66	1.934792 y_{+2}	0.019578 y_{+2}			
0.67	1.95437 y_{+3}				

$x = 0.644$

x_0 or $x_m = 0.64$ (select just less than x)

$h = 0.01$

$$P = \frac{x - x_m}{h}$$

$= 0.4$

Since P lies bet^n $1/4$ and $3/4$ using Bessel's formula,

$$y_p = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{(P-\frac{1}{2})P(P-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{P(P+1)P(P-1)(P-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}$$

$$= 1.896481 + 0.4 \times (0.01906) + \frac{0.4(0.4-1)}{2!}\left(\frac{1.89\times10^{-4} + 1.91\times10^{-4}}{2}\right)$$

$$+ \frac{(0.4-\frac{1}{2})\,0.4\,(0.4-1)}{3!}(0.02\times10^{-4})$$

$$+ \frac{0.4\,(0.4+1)\,0.4\,(0.4-1)\,(0.4-2)}{4!}\left(\frac{0.02\times10^{-4} + 1.34\times10^{-4}}{2}\right)$$

$= 1.904082817$

⇒ If nothing said do this.

Divided Difference : (Use this if interval not equal)

x	$f(x)$	1st DD	2nd DD
x_0	F_0	$f_0[x_0, x_1] = \frac{F_1 - F_0}{x_1 - x_0}$	$F_0[x_0, x_1, x_2] = \frac{F_1[x_1,x_2] - F_0[x_0,x_1]}{x_2 - x_0}$
x_1	F_1	$F_1[x_1, x_2] = \frac{F_2 - F_1}{x_2 - x_1}$	$F_1[x_1, x_2, x_3] = \frac{F_2[x_2, x_3] - F_1[x_1, x_2]}{x_3 - x_1}$
x_2	F_2	$F_2[x_2, x_3] = \frac{F_3 - F_2}{x_3 - x_2}$	$F_2[x_2, x_3, x_4] = \frac{F_3[x_3, x_4] - F_2[x_2, x_3]}{x_4 - x_2}$
x_3	F_3	$F_3[x_3, x_4] = \frac{F_4 - F_3}{x_4 - x_3}$	
x_4	F_4		

$$f(x) = F[x_0] + F[x_0, x_1](x - x_0) + F[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots + F[x_0, x_1, x_2 \dots x_n](x - x_0)(x - x_1)\dots(x, x_{n-1})$$

Q. Given the values

x	5	7	11	13	17
f	150	392	1452	2366	5202

Evaluate F [9] using Divided Difference table:

x	f	1st DD	2nd DD	3rd DD	4th DD
5	150	$\frac{392 - 150}{7 - 5} = 121$	$\frac{265 - 121}{11 - 5} = 24$	$\frac{32 - 24}{13 - 5} = 1$	0
7	392	265	$\frac{457 - 265}{13 - 7} = 32$	$\frac{42 - 32}{17 - 7} = 1$	
11	1452	457	$\frac{709 - 457}{11} = 42$		
13	2366	709			
17	5202				

$x = 9$

$$f(x) = 150 + 121(9-5) + 24(9-5)(9-7) + 1(9-5)(9-7)(9-11) + 0$$

$\therefore$ $F(9) = 810$

3rd DD   4th DD

$F[x_0, x_1, x_2, x_3]$

$F[x_0, x_1, x_2, x_3, x_4]$

$3]$ $F[x_1, x_2, x_3, x_4]$

Lagrange Interpolation.

$$F(x) = l_0^{(x)}F_0(x) + l_1^{(x)}F_1(x) + l_2^{(x)}F_2(x) + l_3^{(x)}F_3(x) + l_4^{(x)}F_4(x) + \cdots$$

where,

$$l_0 = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$$

$$l_1 = \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$\vdots$

Given the values:

x	5	7	11	13	17
f	150	392	1452	2366	5202

Estimate F[9] using Lagrange Interpolation.

$$l_0(x) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}$$

$$= -\frac{1}{9}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$= \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}$$

$$= \frac{8}{15}$$

$$l_2(x) = \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}$$

$$= \frac{8}{9}$$

$$l_3(x) = \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}$$

$$= -\frac{1}{3}$$

$$l_4(x) = \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}$$

$$= \frac{1}{45}$$

$$F(9) = F_0(x)\, l_0(x) + F_1(x)\, l_1(x) + F_2(x)\, l_2(x) + f_3(x)\, l_3(x) + F_4(x)\, l_4(x)$$

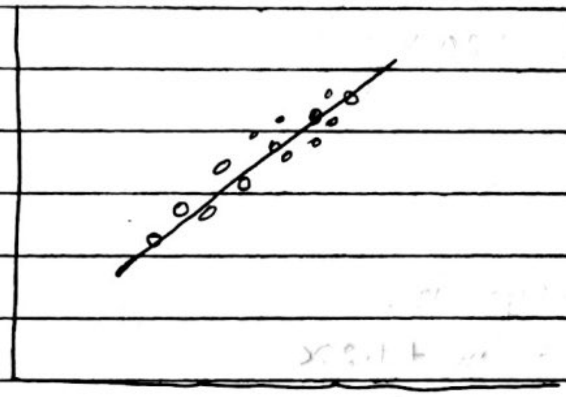
$$= -\frac{1}{9} \times 150 + \frac{8}{15} \times 392 + \frac{8}{9} \times 1452 + \left(\frac{-1}{3}\right) \times 2366 + \frac{1}{45} \times 5202$$

$$= 810$$ \#

Demerits

- If another Interpolation value were inserted, then the interpolation co-efficient were required to be recalculated.

Least Square method of fitting lineas and non-linear curve for data and continuous function.



$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

$$a = \frac{\sum y_i}{n} - b\frac{\sum x_i}{n}$$

⇒ The process of establishing a relationship between two

Q. Fit a str. line to the following set of data

x	1	2	3	4	5
y	3	4	5	6	8

Sol^n,

x_i	y_i	x_i^2	$x_i y_i$
1	3	1	3
2	4	4	8
3	5	9	15
4	6	16	24
5	8	25	40
15	26	55	90

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$= \frac{5 \times 90 - 15 \times 26}{5 \times 55 - (15)^2}$$

$$= 1.20$$

$$a = \frac{\sum y_i}{n} - b \frac{\sum x_i}{n}$$

$$= \frac{26}{5} - 1.20 \times \frac{15}{5}$$

$$= 1.60$$

$\therefore$ The linear eq^n is :

$$\therefore y = 1.6 + 1.2x$$

Fitting transactional eq^n :

→ The non-linear eq^n bet^n the variables is transactional eq^n. It may be Power function expressed as $y = ax^b$ or any other non-linear.

Q. Fit the given data using power method eq^n.

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

⇒ Sol^n,

The power model eq^n is:

$$y = ax^b$$

Taking natural log (ln) on both sides.

$$y = ax^b$$

i.e. $\ln y = \ln(ax^b)$

$$= \ln a + \ln x^b$$

or, $\ln y = \ln a + b \ln x$

$$Y = A + bX$$

where $Y = \ln y$

$A = \ln a$

$X = \ln x$

This is now in linear form.

Now, expressing in table for different summation.

~~Q. fit the given data.~~

$$y = a + bx$$

x_i	y_i	$\ln x_i$ (X)	$\ln y_i$ (Y)	X^2	XY
1	0.5	0	−0.6931	0	0
2	2	0.6931	0.6931	0.4803	0.4803
3	4.5	1.0986	1.5040	1.2069	1.6522
4	8	1.3862	2.0794	1.9215	2.8827
5	12.5	1.6094	2.5257	2.5901	4.0648
		4.7873	6.1091	6.1988	9.08

$$y = a + bx$$

$$b = \frac{n \sum XY - \sum X \sum Y}{n \sum X^2 - (\sum X)^2}$$

$$= \frac{5 \times 9.08 - \cancel{9.08}\, 4.7873 \times 6.1091}{5 \times 6.1988 - (4.7873)^2}$$

$$= 2.00029$$

Date
Page

$$A = \frac{\sum Y_i}{n} - b\frac{\sum X_i}{n}$$

$$= \frac{6.1091}{5} - 2.00029 \times \frac{4.7873}{5}$$

$$= -0.69337$$

$\therefore$ $A = \ln a$

$a = \exp(A)$

$= 0.49988$

$\therefore$ We obtain power function eq^n as:

$$y = 0.49988\, x^{2.00029}$$

$$= 0.5\, x^2 \quad \#$$

$0.49988 \approx 0.5$

$2.00029 \approx 2$

Q. Obtain a relation of form $y = a e^{bx}$ for the following data by the method of least square.

x	0.0	0.5	1.0	1.5	2.0	2.5
y	0.10	0.45	2.15	9.15	40.35	180.75

The given curve is $y = a e^{bx}$

Taking natural log (ln) on both sides;

$\ln y = \ln a + \ln e^{bx}$

or, $\ln y = \ln a + bx \ln e$;

which is of the form

$$Y = A + Bx$$

where,

$Y = \ln y$

$A = \ln a$

$B = b \ln e$

x	$\ln y\,(Y)$	x^2	xY
0.0	-2.3025	0.0	0
0.5	-0.7985	0.25	-0.3992
1.0	0.7654	1.0	0.7654
1.5	2.2137	2.25	3.3205
2.0	3.6975	4.0	7.395
2.5	5.1971	6.25	12.9927
7.5	8.7727	13.75	24.0744

$$B = \frac{n\sum xY - \sum x \sum Y}{n\sum X^2 - (\sum X)^2}$$

$$= \frac{6 \times 24.0744 - 7.5 \times 8.7727}{6 \times 13.75 - (7.5)^2}$$

$$= 2.99623$$

$$b = \frac{B}{\ln e}$$

$$= 2.99623$$

$$A = \frac{\sum Y_i}{n} - \frac{B \sum x_i}{n}$$

$$= \frac{8.7727}{6} - \frac{2.99623 \times 7.5}{6}$$

$$= -2.28317$$

$$A = \ln a$$

$$\therefore a = 0.10196$$

The required curve is,

$$y = ae^{bx}$$

$$\therefore y = 0.10196\, e^{2.99623x}$$

Q. fit the following set of data into a curve.

$$y = \frac{ax}{b+x}$$

x	1	2	3	4	5
y	0.5	0.667	0.75	0.2	0.833

given,

$$y = \frac{ax}{b+x}$$

or, $xy = ax - by$

or, $y = a - \left(\frac{y}{x}\right) b$

or, $Y = a + BX$ form, $y = a + bx$.

where,

~~$Y = y$~~

$B = -b$

$X = y/x$

x	y	X	X^2	XY
1	0.5	0.5	0.25	0.25
2	0.667	0.3335	0.1112	0.2224
3	0.75	0.25	0.0625	0.1875
4	0.2	0.05	2.5×10^{-3}	0.01
5	0.833	0.1666	0.02757	0.13877
15	2.95	1.3001	0.4539	~~2.0576~~ 0.808

$$B = \frac{n\Sigma Xy - \Sigma X \Sigma y}{n\Sigma X^2 - (\Sigma X)^2}$$

$$= \frac{5 \times 0.8086 - 1.3001 \times 2.95}{5 \times 0.4539 - (1.3001)^2}$$

$$= 0.35858$$

$b = -B = -0.3585$

$$a = \frac{\Sigma y}{n} - B\frac{\Sigma X_i}{n}$$

$$= \frac{2.95}{5} - 0.3585 \times \frac{1.3001}{5}$$

$$= 0.4967$$

Hence, the required curve is:

$$y = \frac{ax}{b+x}$$

$$= \frac{0.4967x}{-0.3585+x} \quad \#$$

Q. Temperature of metal strip was measured at Various time. Interval with following data.

time (t) min	1	2	3	4
Temp^r (T)	70	83	100	124.

if the relation T and t is in form $T = be^{t/4} + a$ Find the temperature (T) at t=6.

Sol^n,

the given eq^n is:

$$T = be^{t/4} + a$$

$$\text{or, } T = bX + a$$

$$\text{where } X = e^{t/4}$$

$t(x)$	$T(y)$	$(e^{t/4}) = X$	X^2	Xy
1	70	1.2840	1.6486	89.88
2	83	1.6487	2.7182	136.8421
3	100	2.1170	4.4816	211.7
4	124	2.7182	7.3886	337.0568
	377	7.7679	16.237	775.4789

$$b = \frac{n \Sigma Xy - \Sigma X \Sigma y}{n \Sigma X^2 - (\Sigma X)^2}$$

$$= \frac{4 \times 775.4789 - 7.7679 \times 377}{4 \times 16.237 - (7.7679)^2}$$

$$= 37.6361$$

$$a = \frac{\Sigma y}{n} - b \frac{\Sigma X}{n}$$

$$= \frac{377}{4} - 37.6361 \times \frac{7.7679}{4}$$

$$= 21.1616$$

The given relation is:

$$T = b e^{t/4} + a$$

$$\therefore T = 37.6361 e^{t/4} + 21.1616$$ #

At $t = 6$,

$$T = 37.6361 e^{6/4} + 21.1616$$

$$= 189.8348 \text{ °c}$$ #

Q Fit the 2^{nd} order polynomial with data.

x	1	2	3	4
y	6	11	18	27

→ Sol^n,

Second order polynomial is given as:

$$y = a + bx + cx^2 \quad \text{— (A)}$$

While taking Σ for different data,

$$\Sigma y = n.a + b\Sigma x + c\Sigma x^2 \quad \text{— (1)}$$

Now, multiplying eq^n (1) by x,

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad \text{— (2)}$$

Again, multiplying eq^n (2) by x,

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad \text{— (3)}$$

Expressing in tabular form for different summations.

x	y	x^2	x^3	x^4	xy	x^2y
1	6	1	1	1	6	6
2	11	4	8	16	22	44
3	18	9	27	81	54	162
4	27	16	64	256	108	432
10	62	30	100	354	190	644

Now, eq^n (1), (2), (3) becomes,

$$62 = 4a + 10b + 30c$$

$$190 = 10a + 30b + 100c$$

$$644 = 30a + 100b + 354c$$

On solving,

$$a = 3$$

$$b = 2$$

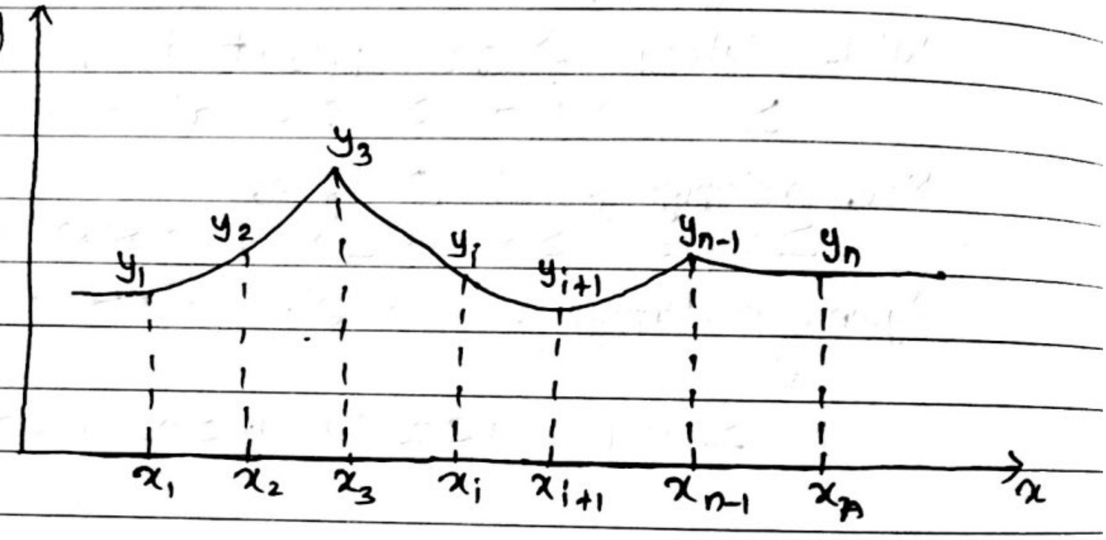
$$c = 1$$

So, eq^2 (A) becomes,

The 2nd order polynomial Eq^2 is,

$$y = 3 + 2\cancel{x}x + 1x^2$$
$$= x^2 + 2x + 3$$

Curve Fitting: Cubic Spline:



Since, we have explained about the different methods of Interpolation. And it seems that the Interpolation technique is not practicable for more number of datas.

Thus, we lifted towards the curve fitting technique where single polynomial equation has been fitted to the given sets of tabulated data

This method gives us the rough approximation of the data only

Thus, to minimize this error cubic spline is developed where every consecutive points shows a unique quodractive curve (or simply cubic curve).

Conditions for cubic spline:

1) $F(x)$ is linear outside the (x_1, x_n) and called natural cubic spline.

2) $F(x)$ is cubic at each sub-interval

3) $F'(x)$ and $F''(x)$ are continuous at each point.

4) If $F'(x)$ is of 2nd degree and $F''(x)$ is linear at each interval and if $F'(x)$ is constant then $F''(x)$ is zero for 1st and last point.

Formula:

Functional value:

$$f_{i,i+1}(x) = \frac{K_i}{6}\left[\frac{(x-x_{i+1})^3}{(x_i-x_{i+1})} - (x-x_{i+1}) * (x_i - x_{i+1})\right]$$
$$-\frac{K_{i+1}}{6}\left[\frac{(x-x_i)^3}{(x_i-x_{i+1})} - (x-x_i)(x_i-x_{i+1})\right]$$
$$+ \frac{y_i(x-x_{i+1}) - y_{i+1}(x-x_i)}{(x_i-x_{i+1})}$$

where, K represents the $f''(x)$.

1) If interval is equal:

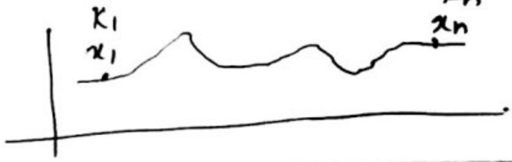
$$K_{i-1} + 4K_i + K_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$$

where, $i = 2, 3, 4, \ldots (n-1)$

2) If interval is not equal:

$$K_{i-1}(x_{i-1} - x_i) + 2K_i(x_{i-1} - x_{i+1}) + K_{i+1}(x_i - x_{i+1})$$
$$= 6\left[\frac{y_{i-1} - y_i}{x_{i-1} - x_i} - \frac{y_i - y_{i+1}}{x_i - x_{i+1}}\right]$$

where $i = 2, 3, 4, \ldots (n-1)$



x_1, x_2 linear

Date
Page

Q) Find $f(1.5)$ for following data

x	1	2	3	4
y	1	2	5	11

where x is at equal interval then,
we have,

when $i=2$,

$$K_1 + 4K_2 + K_3 = \frac{6}{1}\left[y_1 - 2y_2 + y_3\right]$$

or, $\cancel{K_1}^{0} + 4K_2 + K_3 = 6[1-4+5]$

or, $4K_2 + K_3 = 12$ —①

When $i=3$,

$$K_2 + 4K_3 + \cancel{K_4}^{0} = \frac{6}{1}\left[y_2 - 2y_3 + y_4\right]$$

$\therefore K_2 + 4K_3 = 18$ —②

Solving ①, ②, we get,

$K_2 = 2$

$K_3 = 4$

$$f_{1,2}(x) = \frac{K_1}{6}\left[\frac{(x-x_2)^3}{(x_1-x_2)} - (x-x_2)(x_1-x_2)\right] - \frac{K_2}{6}\left[\frac{(x-x_1)^3}{x_1-x_2} - (x-x_1)(x_1-x_2)\right] + \frac{y_1(x-x_2) - y_2(x-x_1)}{x_1-x_2}$$

$i, i+1$

$$f_{1,2}(x) = -\frac{1}{3}\left[-(x-1)^3 - (x-1)(1-x)\right] + \frac{(x-2) - 2(x-1)}{-1}$$

$$= \frac{1}{3}\left(x^3 - 3x^2 + 3x - 1 + (-x) + 1\right) - (x + 2 + 2x - 2)$$

$$f_{1,2}(x) = \frac{1}{3}(x^3 - 3x^2 + 5x) \quad \text{for } 1 \le x \le 2$$

$$\therefore f_{1,2}(1.5) = \frac{1}{3}\left[1.5^3 - 3\times(1.5)^2 + 5 * (1.5)\right]$$

$$= \frac{11}{8}$$

$$f'_{1,2}(x) = \frac{1}{3}(3x^2 - 6x + 5)$$

$$\therefore f'_{1,2}(1.5) = \frac{1}{3}(3\times 1.5^2 - 6\times 1.5 + 5)$$

$$= 0.9166$$

H/W

Q. Find $f'(1.5)$ and $f''(3)$

x	1	2	3	4
y	1	2	5	11

CHAPTER-5

Numerical Differentiation and Integration.

- For equally spaced data.

1) Newton's forward Interpolation formula:

- The formula is given by:

$$f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$+ \dots + \frac{P(P-1)(P-2)\dots(P-(n-1))}{n!}\Delta^n y_0 \quad \text{—} ①$$

Also, we have,

$$P = \frac{x-x_0}{h}$$

$$\text{or, } x = x_0 + ph \quad \text{—} ②$$

$$\therefore P = \frac{x-x_0}{h}$$

$$\therefore \frac{dP}{dx} = \frac{1}{h}$$

Now, differentiating eq^n ① wrt 'P' we get,

$$\frac{dy}{dP} = \Delta y_0 + \frac{(2P-1)}{2!}\Delta^2 y_0 + \frac{3P^2-6P-2}{3!}\Delta^3 y_0$$

$$+ \frac{4P^3-18P^2+22P-6}{4!}\Delta^4 y_0 + \dots \quad \text{—} ③$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dP} \times \frac{dP}{dx}$$

At $x = x_0$, $P = 0$, Hence putting $p=0$ we get,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right]$$

2) Derivatives using backward difference formula :

$$\therefore y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla_n^3 y_n + \cdots$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right]$$

3) Derivatives using central difference formula stirling formula.

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots\right]$$

Q. Given that:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $h = 0.1$

(a) $x = 1.1$

(b) $x = 1.6$

$\Rightarrow$ The difference table is:

(a) $x = 1.1$ [Forward Substitution]

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989	0.414	-0.036	6×10^{-3}	-2×10^{-3}	1×10^{-3}	$+2\times10^{-3}$
→ 1.1	8.403	0.378	-0.03	4×10^{-3}	-1×10^{-3}	3×10^{-3}	
1.2	8.781	0.348	-0.026	3×10^{-3}	2×10^{-3}		
1.3	9.129	0.322	-0.023	5×10^{-3}			
1.4	9.451	0.299	-0.018				
1.5	9.750	0.281					
1.6	10.031						

(a) For $x = 1.1$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0\right]$$

$$= \frac{1}{0.1}\left[0.378 + \frac{0.03}{2} + \frac{4\times10^{-3}}{3} + \frac{10^{-3}}{4} + \frac{3\times10^{-3}}{5} \cancel{+2}\right]$$

$$= 3.9518$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.03 - 4\times10^{-3} - \frac{11\times10^{-3}}{12} - \frac{5\times3\times10^{-3}}{6}\right]$$

$$= -3.7416$$

(b) $x = 1.6$ [Backward substitution]

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
1.0	7.989						
1.1	8.403	0.414					
1.2	8.781	0.378	-0.036				
1.3	9.129	0.348	-0.03	6×10^{-3}			
1.4	9.451	0.322	-0.026	4×10^{-3}	-2×10^{-3}		
1.5	9.750	0.299	-0.023	3×10^{-3}	-1×10^{-3}	1×10^{-3}	
→ 1.6	10.031	0.281	-0.018	5×10^{-3}	2×10^{-3}	3×10^{-3}	-2×10^{-3}

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n\right]$$

$$= \frac{1}{0.1}\left[0.281 - \frac{0.018}{2} + \frac{5\times10^{-3}}{3} + \frac{2\times10^{-3}}{4} + \frac{3\times10^{-3}}{5} + \frac{2\times10^{-3}}{6}\right]$$

$$= 2.751$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.018 + 5\times10^{-3} + \frac{11\times2\times10^{-3}}{12} + \frac{5\times3\times10^{-3}}{6} + \frac{137\times2\times10^{-3}}{180}\right]$$

$$= -0.7144$$

Q. The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data:

Time (t) sec:	0	5	10	15	20
Velocity v (m/s)	0	3	14	69	228

The difference table is:

t	v	Δ	Δ^2	Δ^3	Δ^4
→ 0	0	3	8	36	24
5	3	11	44	60	
10	14	55	104		
15	69	159			
20	228				

An initial acc. $\left(\text{ie } \frac{dv}{dt}\right)$ at $t=0$ is required we use Newton's forward formula:

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \dots\right]$$

$$= \frac{1}{5}\left[3 - \frac{8}{2} + \frac{36}{3} - \frac{24}{4}\right]$$

$$= 1$$

Hence, the initial acceleration is $1 \, m/s^2$.

Q. A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when $t = 0.3$ s.

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.13	31.62	32.87	33.64	33.94	33.81	33.24

t	x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	$30.13_{x_{-3}}$	1.49	-0.24	-0.24	0.26	-0.27	0.29
0.1	$31.62_{x_{-2}}$	1.25	-0.48	0.02	-0.01	0.02	
0.2	$32.87_{x_{-1}}$	0.77	-0.46	0.01	0.01		
t_0 0.3	33.64_{x_0}	0.31	-0.45	0.02			
0.4	$33.94_{x_{+1}}$	-0.14	-0.43				
0.5	$33.81_{x_{+2}}$	-0.57					
0.6	$33.24_{x_{+3}}$						

As the derivatives are required near the middle of the table, we use stirling's formula,

$h = 0.1$, $t_0 = 0.3$, $\Delta x_0 = 0.31$

$$\left(\frac{dx}{dt}\right)_{0.3} = \frac{1}{h}\left[\frac{\Delta x_0 + \Delta x_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right]$$

$$= \frac{1}{0.1}\left[\frac{0.31 + 0.77}{2} - \frac{1}{6}\,\frac{0.01 + 0.02}{2} + \frac{1}{30}\,\frac{0.02 - 0.27}{2}\right]$$

$$= 5.33$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.3} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3}\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.46 + \frac{0.01}{12} + \frac{0.29}{90}\right]$$

$$= -45.59$$

Hence, the required velocity is 5.33 cm/sec and acceleration is $-45.59\ cm/s^2$ #

H/w Q. The elevation above a datum line of seven points of a road are given below:

x	0	300	600	900	1200	1500	1800
y	135	149	157	183	201	205	193

Find the gradient of the road at the middle point.

Q. Using Bessel's formula, find $f'(7.5)$ from the following table:

x	7.47	7.48	7.49	7.50	7.51	7.52	7.53
$f(x)$	0.193	0.195	0.198	0.201	0.203	0.208	0.208

Soln,

$$\Rightarrow \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})\right]$$

Q. Find $F'(10)$ from the following data:

x	3	5	11	27	34
$f(x)$	-13	23	899	17315	35606

$\Rightarrow$ Solution,

Here the values of x are not equispaced, we shall use Newton's divided difference formula. The divided difference table is:

x	$f(x)$	1st DD	2nd DD	3rd DD	4th DD
3	-13	18	16	0.998	0.0002
5	23	146	39.96	1.003	
11	899	1025	69.04		
27	17315	2613			
34	35606				

$$f'(x) = f(x_0,x_1) + (2x-x_0-x_1)f(x_0,x_1,x_2) + [3x^2-2x(x_0+x_1+x_2) + x_0x_1 + x_1x_2 + x_2x_0)] * f(x_0,x_1,x_2,x_3)$$
$$+ [4x^3 - 3x^2(x_0+x_1+x_2+x_3) + 2x(x_0x_1+x_1x_2+x_2x_3+x_3x_0 + x_1x_3+x_0x_2) - (x_0x_1x_2 + x_1x_2x_3 + \cancel{x_0x_2}) x_2x_3x_0 + x_0x_1x_3)] f(x_0,x_1,x_2,x_3,x_4)$$

Putting $x_0=3$, $x_1=5$, $x_2=11$, $x_3=27$ and $x=10$
we obtain,

$f'(10) = 18 + 12\times16 + 23 * 0.998 - 426 * 0.0002$

$= 232.869$ #

Maxima and Minima.

Newton's forward Interpolation formula is:

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

Differentiating it wrt 'p' we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \dots$$

for maxima or minima $\frac{dy}{dp} = 0$

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 = 0$$

$\therefore\ p = \frac{x-x_0}{h}$

Substituting the value of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ from the difference table, we solve this quadratic for p. Then the corresponding values of x are given by $x = x_0 + ph$ at which y is maximum or minimum.

Q. From the table below, for what value of x, y is minimum. Also, find this value of y.

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.262	0.250	0.224

⇒ Solution,

The difference table is:

x	y	Δy	Δ^2	Δ^3
3	0.205	0.035	-0.016	0
4	0.240	0.019	-0.016	1×10^{-3}
5	0.259	0.003	-0.015	1×10^{-3}
6	0.262	-0.012	-0.014	
7	0.250	-0.026		
8	0.224			

$h = 1$ $\qquad p = \frac{x - x_0}{h}$

Taking $x_0 = 3$, Newton's forward difference formula.

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 = 0$$

or, $0.035 + \frac{2\left(\frac{x-3}{1}\right)-1}{2} \times (-0.016) + \left[\frac{1}{2}\left(\frac{x-3}{1}\right)^2 - (x-3) + \frac{1}{3}\right] \times 0 = 0$

or, $0.035 - 0.016x + 0.048 - 8\times10^{-3} = 0$

$\therefore x = 5.6875$

$p = 2.6875$

Now,

$$y = 0.205 + p(0.035) + \frac{p(p-1)}{2} * (-0.016) \quad \text{—①}$$

$$= 0.205 + 2.687 \times 0.035 + \frac{2.687(2.687-1)}{2}(-0.016)$$

$\therefore y = 0.2628.$

$\therefore$ Minimum $x = 5.6875$

$y = 0.2628.$ #

Numerical Integration:

Newton's Cote Formula:

let $I = \int_a^b F(x)\,dx$ where $f(x)$ is determined by Interpolation technique.

As we know from Newton's forward Interpolation formula,

$$y(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \dots$$

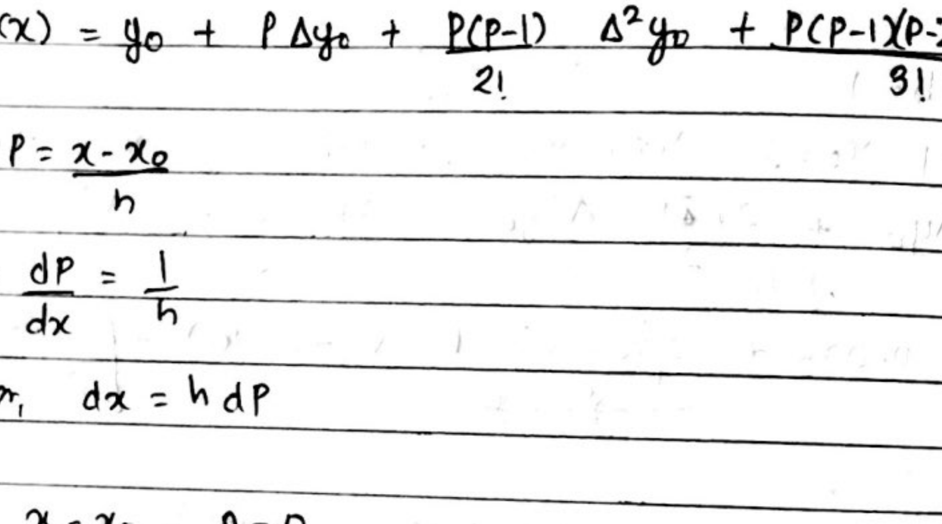
and, $P = \frac{x - x_0}{h}$

$$\therefore \frac{dP}{dx} = \frac{1}{h}$$

or, $dx = h\,dP$

When $x = x_0$, $P = 0$

When $x = x_0 + nh$, $P = n$



$$\therefore I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_{x_0}^{x_0+nh} \left[y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P+1)(P-2)}{3!}\Delta^3 y_0 \right. dx$$

$$= h\int_0^n \left[y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \dots \right] dP$$

$$\therefore \; I = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \dots \right] \quad \text{--- (2)}$$

which is the general expression for numerical integration and is called "General Newton - Cotes Formula".

Imp Trapezoidal Rule (2 point rule):

As we know,

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

n even

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \dots \right] \quad \text{--- (1)}$$

Now putting $n = 1$, in eq^n (1) and neglecting 2^{nd} and higher order difference, we get,

$$I = \int_{x_0}^{x_1} f(x)\,dx$$

$$= h\left(y_0 + \frac{\Delta y_0}{2} \right)$$

$$= h\left(y_0 + \frac{y_1 - y_0}{2} \right)$$

$$= h\left(\frac{2y_0 + y_1 - y_0}{2} \right)$$

$$= h\left(\frac{y_0 + y_1}{2} \right)$$

where $h = \frac{b-a}{n}$

Similarly,

$$I = \int_{x_1}^{x_2} f(x)\, dx$$

$$= h\left(\frac{y_1+y_2}{2}\right)$$

And so on.

This is simple trapezoidal rule.

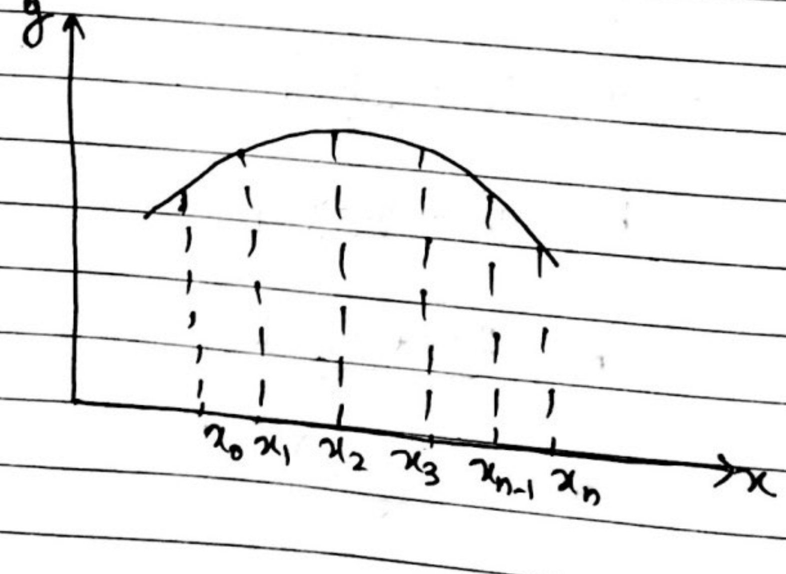
Composite trapezoidal rule

- Hence, the given interval $[a,b]$ is further sub-divided into 'n' equal parts.
- This is done to improve accuracy and non-linear functions can be better explained by this method than simple one.

Then the integral can be given by:

$$I = \int_{a}^{b} f(x)\, dx.$$

$$= \int_{x_0}^{x_1} f(x)\, dx + \int_{x_1}^{x_2} f(x)\, dx + \dots + \int_{x_{n-1}}^{x_n} f(x)\, dx$$



$$I = \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \dots + \frac{h}{2}[y_{n-1} + y_n]$$

$$\boxed{\therefore I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})\right]}$$

This is composite trapezoidal rule.

Simpson's 1/3 rule (3 point)

As we know,

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} - \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \dots\right]$$

Putting $n=2$ and neglecting 3rd and higher order difference.

$$I = \int_{x_0}^{x_2} f(x)\,dx$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \left(\frac{8}{3} - \frac{4}{2}\right)\left(\frac{y_2 - 2y_1 - y_0}{2}\right)\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}y_2 - \frac{2}{3}y_1 + \frac{1}{3}y_0\right]$$

$$\therefore I = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

This is Simple Simpson's 1/3 rule.

Composite Simpson's $\frac{1}{3}$ rule :

Here, the total Interval is further divided into 'n' ... 'interval and n must be even.

$$I = \int_{x_0}^{x_2} f(x)\, dx + \int_{x_2}^{x_4} f(x)\, dx + \dots + \int_{x_{n-2}}^{x_n} f(x)\, dx$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \dots + \frac{h}{3}[y_{n-2} \dots$$

$$\therefore I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots\right.$$

This is composite simpson's $\frac{1}{3}$ formula.

Simpson's $3/8$ rule (4-point)

As we know from Newton's cote formula

- - - -

Putting $n=3$ and neglecting 4^{th} and higher order,

$$I = \int_{x_0}^{x_3} f(x)\, dx$$

$$\therefore I = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

This is simple Simpson's $3/8$ rule.

Composite 3/8 rule

- 'n' must be multiple of 3.

$$\therefore I = \int_a^b f(x)\,dx$$

$$= \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \dots\dots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$\therefore I = \frac{3h}{8}\Big[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+y_8+\dots+y_{n-1})$$

$$+ 2(y_3+y_6+y_9+\dots+y_{n-3})\Big]$$

This is composite Simpson's 3/8 formula.

Q. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

1) Trapezoidal rule
2) Simpson's 1/3 rule
3) Simpson's 3/8 rule.

$$\therefore h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

Suppose $n=6$

Divide the interval $(0,6)$ into 6 parts each of width $h=1$.

The values of $f(x) = \frac{1}{1+x^2}$

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.0385	0.027

1) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.05+0.0385)\right]$$

$$\therefore I = 1.4108$$ #

2) By Simpson's 1/3 rule

$$I = \int_0^6 \frac{dx}{1+x^2}$$

$$= \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right]$$

$$= 1.3662$$ #

3) By Simpson's 3/8 rule,

$$I = \int_0^6 \frac{dx}{1+x^2}$$

$$= \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3}{8}\left[(1+0.027)+3(0.5+0.2+0.0588+0.0385)+2(0.1)\right]$$

$$= 1.3571$$ #

Q. Evaluate the Integral $\int_0^1 \frac{x^2}{1+x^3} dx$, using Simpson's $1/3^{rd}$ rule compare the error with the exact value.

⇒ Solution,

Let us divide the interval $(0, 1)$ into 4 equal parts so, that

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x	0	0.25	0.50	0.75	1.00
y	0	0.06153	0.22222	0.39560	0.5

$$\therefore I = \int_0^1 \frac{x^2}{1+x^3} dx$$

$$= \frac{h}{3}\left[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)\right]$$

$$= \frac{0.25}{3}\left[0.5 + 0.44444 + 1.82852\right]$$

$$= 0.23108 \quad \#$$

Also, $\int_0^1 \frac{x^2}{1+x^3} dx$

$$= \frac{1}{3}\left|\log(1+x^2)\right|_0^1$$

$$= \frac{1}{3}\log_e 2$$

$$= 0.23105$$

Thus, the error $= 0.23108 - 0.23105$

$= +0.00003 \quad \#$

Q. Use the Trapezoidal rule to estimate the Integral $\int_0^2 e^{x^2}\,dx$ taking the numbers of 10. intervals.

⇒ Sol^n:

Here, $f(x) = e^{x^2}$ and $n = 10$, $h = \frac{b-a}{n} = 0.2$

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1	1.0408	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.935[illegible]	25.5337	54.5981

Ans: 17.0621

Q. Use Simpson's $\frac{1}{3}^{rd}$ rule to find $\int_0^{0.6} e^{-x^2}\,dx$ by taking seven ordinates.

⇒ Sol^n,

Divide the interval $(0, 0.6)$ into 6 parts each of width $h = 0.1$

$\because f(x) = e^{-x^2}$

$n = 6$

Romberg Integration:

From the Newton's Cote Quadrature formula it is observed that its integration can be improved by:

i) Increasing the no. of sub-intervals (making h small)
ii) Increasing the order of Integration polynomial.

Formula

$$I_1^* = I_2 + \frac{1}{3}(I_2 - I_1)$$

$$I_2^* = I_3 + \frac{1}{3}(I_3 - I_2)$$

$$I_1^{**} = I_2^* + \frac{1}{3}(I_2^* - I_1^*)$$

$\therefore$ Best Estimate = More accurate + $\frac{1}{3}$ (More accurate - less accurate)

Q. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Romberg Integration

Take $h = 0.5, 0.25, 0.125$

Sol^n,

$h = 0.5$

$\therefore f(x) = \frac{1}{1+x^2}$

x	0	0.5	1
y	1	0.8	0.5

Using Trapezoidal rule we have:

$$I_1 = \frac{h}{2}\left[(1+0.5) + 2 * 0.8\right]$$

$$= 0.775 \#$$

Taking $h = 0.25$

x	0	0.25	0.5	0.75	1
y	1	0.9911	0.8	0.64	0.5

$$\therefore I_2 = \frac{h}{2}\left[(1+0.5) + 2 * (0.9411 + 0.8 + 0.64)\right]$$

$$= 0.782775$$

Taking $h = 0.125$

x	0	0.125	0.25.	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9911	0.8767	0.8	0.7191	0.64	0.5663	0.

$$I_3 = \frac{h}{2}\Big[(1+0.5) + 2 * (0.9845 + 0.9411 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5663)$$

$$= 0.7847$$

$$I_1^* = I_2 + \frac{1}{3}(I_2 - I_1)$$

$$= 0.7852$$

$$I_2^* = I_3 + \frac{1}{3}(I_3 - I_2)$$

$$= 0.78536$$

$$I_1^{**} = I_2^* - \frac{1}{3}(I_2^* - I_1^*)$$

$$= 0.785413$$

Gaussian Integration: → limit -1 to 1 otherwise change the range into -1 to 1

- This method is used for non-equal interval of scope.
- Gaussian formula is expressed as:

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + \dots + w_n f(x_n)$$

$$= \sum_{i=1}^{n} w_i f(x_i) \quad \text{— (1)}$$

where, w_i = weight and x_i = abscissae

Table

n	Weight (w_i)	x_i
2	$W_1 = W_2 = 1$	$x_1 = \frac{-1}{\sqrt{3}}$ and $x_2 = \frac{1}{\sqrt{3}}$
3	$W_1 = 5/9$	$x_1 = -\sqrt{3/5}$
	$W_2 = 8/9$	$x_2 = 0$
	$W_3 = 5/9$	$x_3 = \sqrt{3/5}$

Q. Evaluate : using 2 point formula:

$$\int_{-1}^{1} e^x\,dx$$

⇒ Solution;

Using Gaussian two point formula,

We have,

$f(x) = e^x$

$I = w_1 f(x_1) + w_2 f(x_2)$

$= 1 * f\left(-\frac{1}{\sqrt{3}}\right) + 1 * f\left(\frac{1}{\sqrt{3}}\right)$

$= e^{-1/\sqrt{3}} + e^{1/\sqrt{3}}$

$= 0.5613 + 1.7813$

$\therefore I = 2.3426$ #

Changing the limit of integration:

Q. Evaluate $\int_0^1 \frac{dx}{1+x}$ using 3-point Gaussian quadrature formula

⇒ Solution,

$$\because x = \left(\frac{b-a}{2}\right)u + \left(\frac{b+a}{2}\right)$$ ⇒ If interval not in -1 to 1.

$$= \left(\frac{1-0}{2}\right)u + \left(\frac{1+0}{2}\right)$$

$$= \frac{1u}{2} + \frac{1}{2}$$

$$\therefore x = \frac{1}{2}(u+1)$$

$$\therefore \frac{dx}{du} = \frac{1}{2}$$

When $x=0$ in ①,

$$0 = \frac{u}{2} + \frac{1}{2}$$

$$\therefore u = -1$$

When $x=1$ in eq^n ①

$$1 = \frac{u}{2} + \frac{1}{2}$$

$$\therefore u = 1$$

$$\therefore I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{du/2}{1+\frac{1}{2}(u+1)}$$

$$= \int_{-1}^{1} \frac{du}{3+u}$$

$$= w_1 g(u_1) + w_2 g(u_2) + w_2 g(u_3)$$

$$= \frac{5}{9}\left(\frac{1}{3+(-\sqrt{3/5})}\right) + \frac{8}{9}\left(\frac{1}{3+0}\right) + \frac{5}{9}\left(\frac{1}{3+\sqrt{3/5}}\right)$$

$$= 0.2496 + 0.29629 + 0.14718$$

$$= 0.69307$$

CHAPTER-7

Sol^n of Partial Differential Eq^{n}s.

Classification of partial differential eq^n (Elliptic parabolic and Hyperbolic)

The general linear ~~pde~~ partial differential eq^n of the second order in two independent variables is of the form.

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0 \quad \text{——(1)}$$

where A, B, C are the co-efficients may be constants or functions of x and y.

- Depending on the values of these co-efficients may be classified into one of the 3-types of eq^{n}s namely,

Elliptic, if $B^2 - 4AC < 0$
Parabolic, if $B^2 - 4AC = 0$
Hyperbolic, if $B^2 - 4AC > 0$

Eg:

Classify the following eq^n:

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

Solution,

comparing this eq^n with general linear partial differential eq^n

We find;

$A = 1$, $B = 4$, $C = 4$

$\therefore B^2 - 4AC = 4^2 - 4\times 1\times 4$

$= 16 - 16$

$= 0$

$B^2 - 4AC = 0$

So, the eq^n is parabolic..

Elliptic Eqns:

1) Laplace eqn^s.

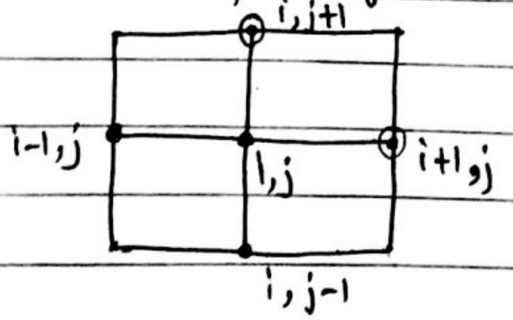
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2) Poisson's eq^n:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

3) Laplace Eq^n.

- Standard 5 point formula

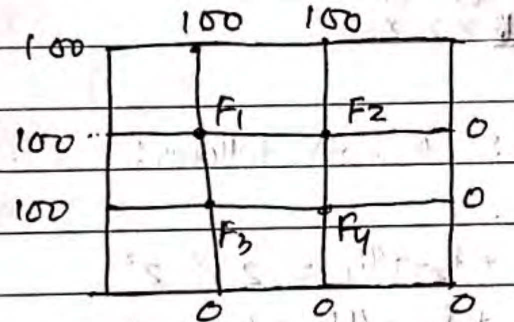


$$F_{i-1,j} + F_{i,j+1} + F_{i+1,j} + F_{i,j-1} - 4F_{i,j} = 0$$

$$\therefore \; F_{i,j} = \frac{1}{4}\left(F_{i+1,j} + F_{i-1,j} + F_{i,j+1} + F_{i,j-1}\right)$$

Q. Consider a steel plate of size 15 cm x 15 cm. If two of the sides are held at 100°c and the other two sides are held at 0°c what are the steady state temperature at interior points assuming a grid size of 5cm x 5cm

⇒ Solution:



The system of equation is as follows:

At point 1 : $F_2 + F_3 - 4F_1 + 100 + 100 = 0$

At point 2 : $F_1 + F_4 - 4F_2 + 100 + 0 = 0$

At point 3 : $F_1 + F_4 - 4F_3 + 100 + 0 = 0$

At point 4 : $F_2 + F_3 - 4F_4 + 0 + 0 = 0$

i.e

$$-4F_1 + F_2 + F_3 + 0 = -200 \quad \text{—①}$$

$$F_1 - 4F_2 + 0 + F_4 = -100 \quad \text{—②}$$

$$F_1 + 0 - 4F_3 + F_4 = -100 \quad \text{—③}$$

$$0 + F_2 + F_3 - 4F_4 = 0 \quad \text{—④}$$

Solving ①, ②, ③, ④ we get

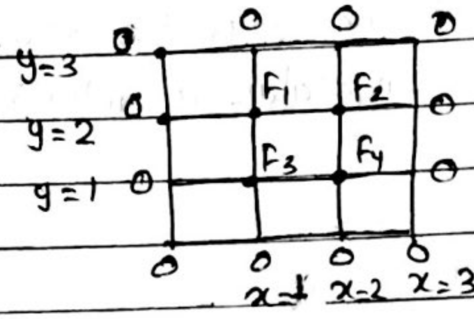
$F_1 = 75$, $F_2 = 50$

$F_3 = 50$, $F_4 = 25$

Poisson's Eq^n.

Solve the Poisson's eq^n $\nabla^2 F = 2x^2y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $F = 0$ on the boundary and $h = 1$

⇒ solution:



The system of eq^{n}s is as follows:

At point 1: $0 + 0 + F_2 + F_3 - 4F_1 = 2 \cdot 1^2 \cdot 2^2$

i.e $F_2 + F_3 - 4F_1 = 8$ — (1)

At point 2: $0 + 0 + F_1 + F_4 - 4F_2 = 2\,2^2 2^2$

i.e $F_1 - 4F_2 + F_4 = 32$ — (2)

At point 3: $0 + 0 + F_1 + F_4 - 4F_3 = 2 \cdot 1^2 1^2$

i.e $F_1 + F_4 - 4F_3 = 2$ — (2)

At point 4: $0 + 0 + F_2 + F_3 - 4F_4 = 2\,2^2\,1^2$

i.e $F_2 + F_3 - 4F_4 = 8$ — (4)

Rearranging the eq^n (1) to (4) we get.

$$-4F_1 + F_2 + F_3 = 8$$
$$F_1 - 4F_2 + F_4 = 32$$
$$F_1 - 4F_3 + F_4 = 2$$
$$F_2 + F_2 - 4F_4 = 8$$

Solving these eq² by elimination method, we get

$F_1 = -\frac{22}{4}$, $F_2 = -\frac{43}{4}$, $F_3 = -\frac{13}{4}$, $F_4 = -\frac{22}{4}$

Q. Solving the above problem by Gauss-Seidal Iteration method.

By rearranging the eq²s, we have,

$F_1 = \frac{1}{4}(F_2 + F_3 - 8)$

$F_2 = \frac{1}{4}(F_1 + F_4 - 32)$

$F_3 = \frac{1}{4}(F_1 + F_4 - 2)$

$F_4 = \frac{1}{4}(F_2 + F_3 - 8)$

$\because F_1 = F_4$

$F_1 = \frac{1}{4}(F_2 + F_3 - 8)$

$F_2 = \frac{1}{4}(2F_1 - 32)$

$F_3 = \frac{1}{4}(2F_1 - 2)$

Assuming starting values as

$f_2 = 0 = f_3$

Iteration 1:

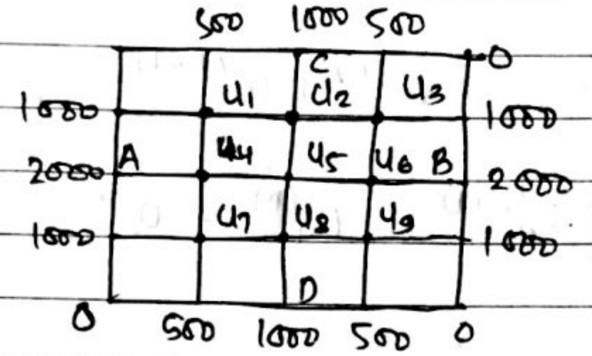
$f_1 = -2$, $f_2 = -9$, $f_3 = -1.5$

Iteration 2:

$f_1 = -\frac{37}{8}$, $f_2 = \frac{-165}{16}$, $f_3 = -\frac{45}{16}$

Iteration
3

Q. Solve the elliptic eq^n $u_{xx} + u_{yy} = 0$. for the following square mesh with boundary values:



⇒ Sol^n,

Let $u_1, u_2, \ldots u_9$ be the values of u at the interior mesh points. Since the boundary values of u are symmetrical about AB.

$$\therefore u_7 = u_1, \quad u_8 = u_2, \quad u_3 = u_9$$

Also, the values of u being symmetrical about CD

$$\therefore u_3 = u_1, \quad u_6 = u_4, \quad u_9 = u_7$$

Thus, it is sufficient to find the values u_1, u_2, u_4, u_5.

Now,

We carry out the Iteration process using the standard formulae.

$$u_1^{(n+1)} = \frac{1}{4}\left[1000 + u_2^{(n)} + 500 + u_4^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + u_3^{(n)} + 1000 + u_5^{(n)}\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[2000 + u_5^{(n)} + u_1^{(n+1)} + u_7^{(n)}\right]$$

$$u_5^{(n+1)} = \frac{1}{4}\left[u_4^{(n+1)} + u_6^{(n)} + u_2^{(n+1)} + u_8^{(n)}\right]$$

$$U_1^{(n+1)} = \frac{1}{4}\left[1000 + U_2^{(n)} + 500 + U_4^{(n)}\right]$$

$$U_2^{(n+1)} = \frac{1}{4}\left[U_1^{(n+1)} + U_1^{(n)} + 1000 + U_5^{(n)}\right]$$

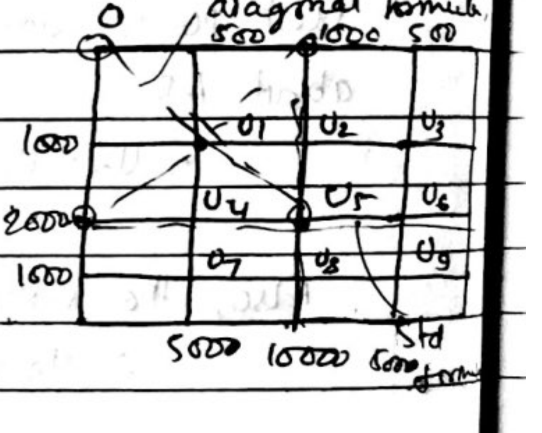
$$U_4^{(n+1)} = \frac{1}{4}\left[2000 + U_5^{(n)} + U_1^{(n+1)} + U_1^{(n)}\right]$$

$$U_5^{(n+1)} = \frac{1}{4}\left[U_4^{(n+1)} + U_4^{(n)} + U_2^{(n+1)} + U_2^{(n)}\right]$$

Now, we find their initial value in the following order:

$$U_5 = \frac{1}{4}\left[2000 + 2000 + 1000 + 1000\right]$$

$= 1500$ (std formula.)



$$U_1 = \frac{1}{4}\left[0 + 1500 + 1000 + 2000\right]$$

$= 1125$ (Diagonal formula)

$$U_2 = \frac{1}{4}\left[1000 + 1125 + 1500 + 1125\right]$$

$= 1188$

$$U_4 = \frac{1}{4}\left[1125 + 2000 + 1125 + 1500\right]$$

$= 1438$

Now,

1st iteration, (at $n=0$)

$$U_1^{1} = \frac{1}{4}\left[1000 + 1188 + 500 + 1438\right] = 1032$$

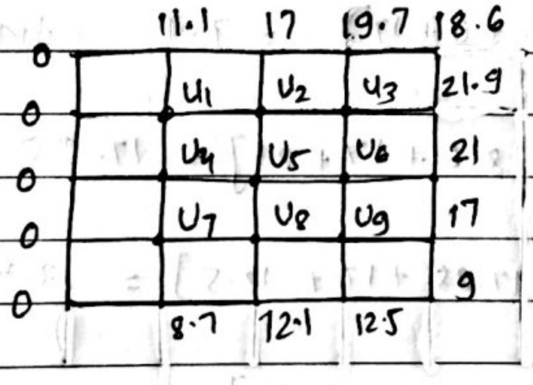
$$U_2^{1} = \frac{1}{4}\left[1032 + 1125 + 1000 + 1500\right] = 1164$$

Iteration	U_1	U_2	U_4	U_5
0	1022	1164	1414	1301
1	1020	1088	1338	1251
2	982	1063	1313	1201

Q. Solve the laplace eq^n.

$$U_{xx} + U_{yy} = 0$$

given that :



$$U_1^{(n+1)} = \frac{1}{4}\left[0 + U_2^{(n)} + 11.1 + U_4^{(n)}\right]$$

$$U_2^{(n+1)} = \frac{1}{4}\left[U_1^{(n+1)} + U_3^{(n)} + 17 + U_5^{(n)}\right]$$

$$U_3^{(n+1)} = \frac{1}{4}\left[U_2^{(n+1)} + 21.9 + 19.7 + U_6^{(n)}\right]$$

$$U_4^{(n+1)} = \frac{1}{4}\left[0 + U_5^{(n)} + U_1^{(n+1)} + U_7^{(n)}\right]$$

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Page

$$U_5^{(n+1)} = \frac{1}{4}\left[U_4^{(n+1)} + U_6^{(n)} + U_2^{(n+1)} + U_8^{(n)}\right]$$

$$U_6^{(n+1)} = \frac{1}{4}\left[U_5^{(n+1)} + 21 + U_3^{(n+1)} + U_9^{(n)}\right]$$

$$U_7^{(n+1)} = \frac{1}{4}\left[0 + U_8^{(n)} + U_4^{(n+1)} + 8.7\right]$$

$$U_8^{(n+1)} = \frac{1}{4}\left[U_7^{(n+1)} + U_9^{(n)} + U_5^{(n+1)} + 12.1\right]$$

$$U_9^{(n+1)} = \frac{1}{4}\left[U_8^{(n+1)} + 17 + U_5^{(n+1)} + 12.5\right]$$

Now, find the initial values,

$$U_5 = \frac{1}{4}\left[0+17+21+12.1\right]$$

$= 12.5$ (Standard formula)

$U_1 = \frac{1}{4}\left[0 + 12.5 + 0 + 17\right] = 7.4$ (diagonal formula)

$U_3 = \frac{1}{4}\left[12.5 + 18.6 + 17 + 21\right] = 17.28$ (D.F)

$U_2 = \frac{1}{4}\left[7.4 + 17.28 + 17 + 12.5\right] = 13.55$ (S.F)

$$U_7 = \frac{1}{4}\left[0 + 12.1 + 0 + 12.5\right]$$

$= 6.15$ (D.F)

$$U_4 = \frac{1}{4}\left[0 + 12.5 + 7.4 + 6.15\right]$$

$= 6.52$ (D.F)

$$U_9 = \frac{1}{4}[12.5 + 9 + 12.1 + 21] = 13.65 \quad (D.F)$$

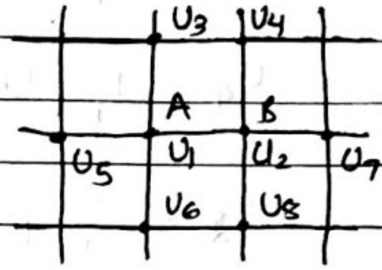
$$U_8 = \frac{1}{4}[6.15 + 12.5 + 12.1 + 13.65] = 11.12 \quad (S.F)$$

$$U_6 = \frac{1}{4}[12.5 + 21 + 17.28 + 13.65] = 16.12 \quad (S.F)$$

Calculation in tabular form:

Iteration	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
1	7.79	13.64	12.84	6.61	11.88	16.09	6.61	11.06	12.238
2	7.84	16.64	17.83	6.58	11.84	16.23	6.58	11.19	14.30

Solution of Elliptic Equation by Relaxation Method:



$\nabla^2 u = 0$

$$U_1 = \frac{1}{4}\left[U_5 + U_3 + U_2 + U_6\right]$$

$$\therefore\ U_5 + U_3 + U_2 + U_6 - 4U_1 = 0$$

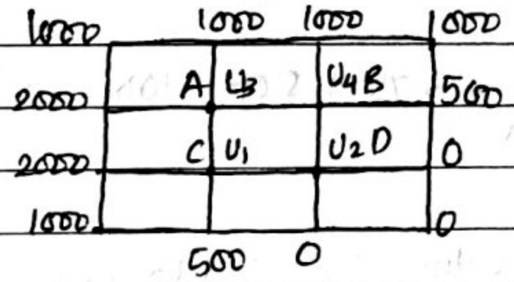
If r_1 be the residual at the mesh point A, then

$$r_1 = U_5 + U_3 + U_2 + U_6 - 4U_1 \quad \text{—①}$$

Similarly residual at the point B, is given by

$$r_2 = U_1 + U_4 + U_7 + U_8 - 4U_2 \quad \text{and so on.}$$

Q. Solve by Relaxation method. The square region with square mesh as shown in figure.



Standard 5-point formula for the given eq^n is:

$$U_1 = \frac{1}{4}\left[2000 + U_2 + U_3 + 500\right]$$

$$U_2 = \frac{1}{4}\left[U_1 + 0 + U_4 + 0\right]$$

$$U_4 = \frac{1}{4}\left[U_3 + 500 + 1000 + U_2\right]$$

Now, find initial value.

$$U_3 = \frac{1}{4}\left[1000 + 2000 + 1000 + U_2\right] \quad \Rightarrow \text{Diagonal formula.}$$

$$= 1000 \quad [\because \text{Assume } U_2 = 0]$$

$U_1 = \frac{1}{4}[2000 + 0 + 1000 + 500] = 875$ std formula.

$U_4 = \frac{1}{4}[1000 + 500 + 1000 + 0] = 625$ std formula

$U_2 = \frac{1}{4}[625 + 0 + 875 + 0] = 375$ std formula.

Now,

Hence, the calculation of residual at U_1, U_2, U_3, U_4 are

$r_1 = 2000 + 375 + 1000 + 500 - 4 \times 875$
$= 375$

$r_2 = 875 + 625 + 0 + 0 - 4 \times 375$
$= 0$

$r_3 = 1000 + 875 + 2000 + 625 - 4 \times 1000$
$= 500$

$r_4 = 1000 + 375 + 500 + 1000 - 4 \times 625$
$= 375$.

Now, the modified values of U_1, U_2, U_3 and U_4 are:

round off

$U_1 = \frac{1}{4}[2000 + 375 + 1000 + 500]$

$= 969$

$= 875 + \underbrace{94}_{\text{residue}}$

$U_2 = \frac{1}{4}[0 + 0 + 625 + 875]$

$= 375 + 0$

$$U_3 = \frac{1}{4}[1000 + 875 + 625 + 2000]$$

$= 1125$

$= 1000 + \underline{125}$

$$U_4 = \frac{1}{4}[1000 + 1000 + 500 + 375]$$

$= 719$

$= 625 + 94$

Again modified value of U_1, U_2, U_3, U_4 are:

$$U_1 = \frac{1}{4}[2000 + 500 + 375 + 1125]$$

$= 1000$

$= 969 + 31$

$$U_2 = \frac{1}{4}[0 + 0 + 719 + 969]$$

$= 422$

$= 375 + 47$

$$U_3 = \frac{1}{4}[1000 + 2000 + 969 + 719]$$

$= 1172$

$= 1125 + 47$

$$U_4 = \frac{1}{4}[1000 + 500 + 1125 + 375]$$

$= 750$

$= 719 + 31$

classmate

Date
Page

$r_1 = 2000 + 422 + 1172 + 500 - 4 \times 1000 = 94$

$r_2 = 1000 + 0 + 750 + 0 - 4 \times 422 = 62$

$r_3 = 1000 + 1000 + 2000 + 750 - 4 \times 1172 = 62$

$r_4 = 1172 + 500 + 1000 + 422 - 4 \times 750 = 94$

Modified values of U_1, U_2, U_3, U_4 are

$U_1 = 1023.5$

$= 1000 + 23.5$

$U_2 = 437.5$

$= 422 + 15.5$

$U_3 = 1187.5 = 1172 + 15.5$

$U_7 = 773.5 \doteq 750 + 23.5$

The residue is.

$r_1 =$

$r_2 =$

$r_3 =$

$r_4 =$

Solution of 1-Dimensional Heat eq^n by Schmidt method (Parabolic eq^n)

→ One-dimensional heat eq^n is given by:

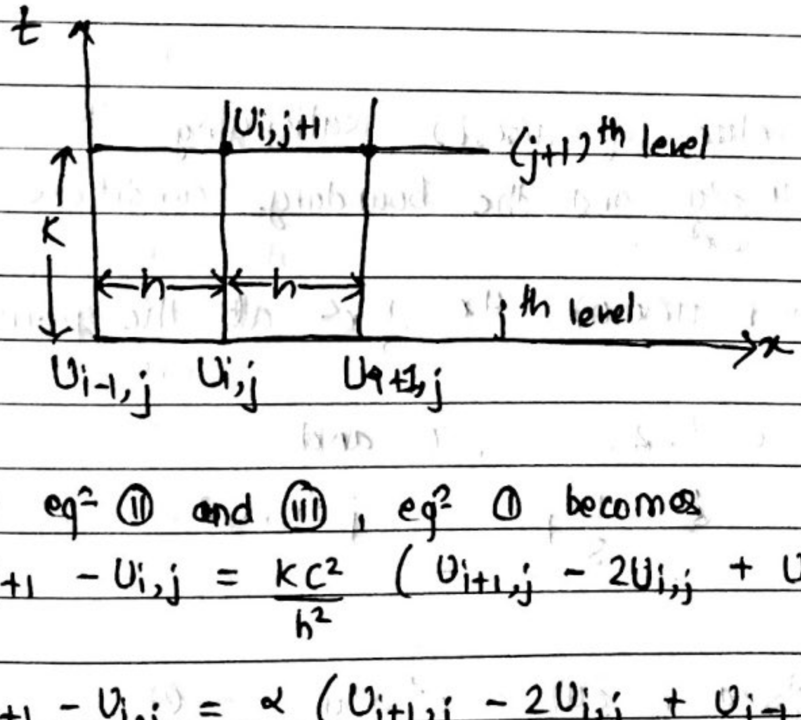
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial t^2} \quad \text{—} ①$$

where $c^2 = \frac{k}{\rho p}$ is the diffusivity of the substance

Consider a rectangular mesh in the x-t plane with spacing h along x direction and k along time t direction. denoting $(x,t) = (ih, jk)$ as simply i,j we get

$$\frac{\partial u}{\partial t} = \frac{U_{i,j+1} - U_{i,j}}{k} \quad \text{—} ⑪$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} \quad \text{—} ⑪⑪$$



With eq^n ⑪ and ⑪⑪, eq^n ① becomes

$$U_{i,j+1} - U_{i,j} = \frac{kc^2}{h^2}\left(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right)$$

$$U_{i,j+1} - U_{i,j} = \alpha\left(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right)$$

$$U_{i,j+1} - U_{i,j} = \alpha U_{i+1,j} - 2\alpha U_{i,j} + \alpha U_{i-1,j}$$

$$\boxed{U_{i,j+1} = \alpha U_{i+1,j} + (1-2\alpha) U_{i,j} + \alpha U_{i-1,j}} \quad \text{--- (IV)}$$

This formula enables us to determine the values of u the $(i+j+1)^{th}$ mesh point.

- It is a relation betⁿ the function values at the two time levels $j+1$ and j and therefore is called a 2-level formul
- Eqⁿ (IV) is called Schmid explicit formula which is valid only for $0 < \alpha \leq 1/2$

- In particular case when $\alpha = \frac{1}{2}$, eqⁿ (IV) becomes,

$$U_{i,j+1} = \frac{U_{i+1,j} + U_{i-1,j}}{2} \quad \text{--- (V)}$$

known as Bendre - Schmidt recurrence relation.

Q. Find the value of $u(x,t)$ satisfying the parabolic eqⁿ $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t) = 0 =$ $u(8,t)$ and $u(x,0) = 4x - \frac{1}{2}x^2$ at the point $x = i$

$\therefore i = 0, 1, 2, \ldots, 7$ and

$t = \frac{1}{8} j \qquad j = 0, 1, 2, \ldots, 5$

⇒ Solution,

Head eqⁿ, $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ --- (1)

$\therefore \frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$

Comparing with heat eq^n.

$$c^2 = 4 \quad , \quad h = 1 \quad , \quad K = \frac{1}{8}$$

then, $\alpha = \dfrac{Kc^2}{h^2} = \dfrac{1}{2}$

for $\alpha = \frac{1}{2}$

We have,

$$U_{i,j+1} = \frac{U_{i+1,j} + U_{i-1,j}}{2} \quad \text{— (A)}$$

$\therefore u(0,t) = 0 = u(8,t)$

$\because u(x,0) = 4x - \frac{1}{2}x^2$

$$U_{i,0} = 4i - \frac{1}{2}i^2$$

Now putting $i = 0, 1, \ldots, 7$, we have,

0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5

$\longrightarrow x$

t $\downarrow$

j\i	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	$\frac{3.5+7.5}{2}$ 5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0