Chapter 1

Introduction to Numerical Method

APPROXIMATION

An approach designed or used to solve complex mathematical problem by using basic arithmetic operations. This can be achieved by using

- Computer based algorithmic mathematics.
- Most of numerical methods. [Usually iterative methods]

PROCESS OF NUMERICAL COMPUTING

The process of numerical computing can be divided into following four phases.

i) Formulation of a mathematical model.
ii) Construction of an appropriate numerical method.
iii) Implementation of method to obtain solution.
iv) Validation of the solution.

CHARACTERISTICS

i) Accuracy

It usually depends on number of iteration being carried out.

ii) Rate of convergence

This rate depends on how fast a desired result can be achieved.

iii) Numerical stability

A numerical Method is usually unstable if solution is unreachable despite of infinite number of iterations.

iv) Efficiency

Efficiency depends on the effort of human or computer to get desired solution.

IMPORTANCE OF NUMERICAL METHOD

Numerical computation plays a very important role in solving real life mathematical, physical and engineering problems. The development of digital computers has enhanced the speed and accuracy of numerical computations.

In the field of science and engineering complex mathematical problems are solved using the methods and algorithms of numerical methods. This has helped in both the field to tackle the complex numericals in a more relevant and accurate way to obtain the desired solution.

The numerical method involves a large number of arithmetic calculations, which includes the iterative methods. Hence, use of such algorithm in science and engineering provide greater accuracy.

Before the invention of computer, the calculation of such complex numerical was very difficult but the numerical method provided us with methods and the algorithms to solve such problems with great accuracy and efficiency. The computer also used their algorithm to compute such numerical, so on the whole the numerical method has provided a great help to the field of science and Engineering.

ERRORS

Errors are integral part of any numerical calculations. Different types of errors arise during the process of numerical computing. All these errors contribute to the total error in the final result. The figure below shows the "Taxonomy of Errors" which includes contribution of different errors to the total errors obtained in any numerical computation.

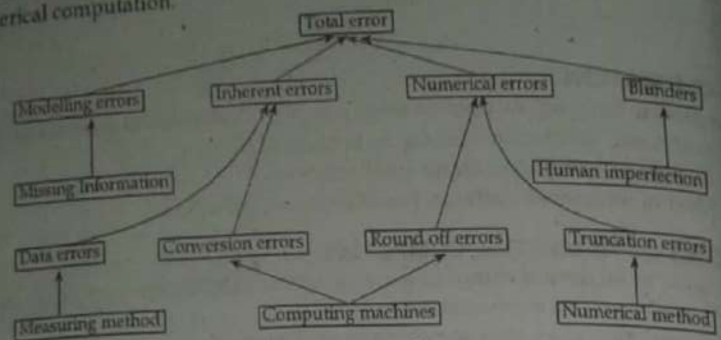


Figure: Taxanomy of errors

TYPES OF ERRORS

In any numerical computation, we come across the following types of errors.

i) Inherent errors

Inherent errors are those errors which are present in the data supplied to the model. Inherent errors (also known as input errors) contain two components normally, Data errors and Conversion errors.

ii) Data errors

Data errors (also known as **empirical error**) arise when data for a problem are obtained by some experimental means and are therefore of limited accuracy and precision.

iii) Conversion errors

Conversion errors (also known as representation errors) arise due to the limitation of computer to store the data exactly; we know that, the floating point representation retains only a specified number of digits.

iv) Round off errors

Round off errors occurs when a fixed number of digits are used to represent exact number. Since, the numbers are stored at every stage of computation; round off errors is introduced at the end of every arithmetic operation.

v) Truncation errors

Truncation errors arise from using an approximation in place of exact mathematical procedure. It is the error resulting from the truncation of the numerical process. We often use some finite number of terms to estimate the sum of an infinite series. *For example;*

$S = \sum_{i=0}^{\infty} a_i x^i$ is replaced by some finite sum which give rise to truncation errors.

ABSOLUTE AND RELATIVE ERRORS

There are many ways which can be used to describe error. The simplest is the absolute errors; this is the difference between the measured or calculated value and the true value.

$$\varepsilon = |\text{True value} - \text{Approximation}|.$$

The absolute error does not consider the order of magnitude of the value under consideration. For this we usually consider the relative error.

$$\varepsilon = \frac{|\text{True value} - \text{Approximation}|}{|\text{True value}|}$$

And the percentage error is;

$$\varepsilon_p = \frac{|\text{True value} - \text{Approximation}|}{|\text{True value}|} \times 100\%$$

Chapter 2

Solution of Nonlinear Equations

Different methods for the solutions of Non linear equations are explained below.

i) BISECTION METHOD

Bisection method is an iterative method used for the solution of non-linear equations, also known as binary chopping or half-interval method.

It is based on intermediate value property. It means if f(x) is continuous in the interval [a, b] and f(a) and f(b) have different sign then the equation f(x) = 0 has at least one root between x = a and x = b.

This method is most reliable and simplest iterative method for solution of non linear equation.

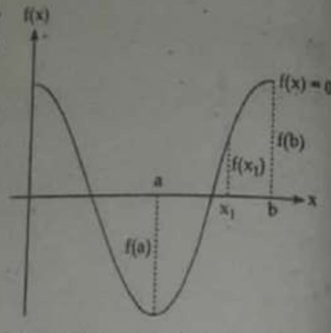


Figure: Illustration of bisection method

$$a < x < b$$

Where, x is not of non linear equation.

Let f(x) = 0 be continuous between 'a' and 'b'. By definition let f(a) be negative and f(b) be positive. The first approximation to the root is

$$x_1 = \frac{a+b}{2};$$

where, a and b is selected such that,

$$f(a) \times f(b) < 0.$$

Now, if $f(x_1) = 0$ then x_1 is the root of f(x) otherwise the root lies between 'a' and x_1 or x_1 and 'b' according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

ALGORITHM OF BISECTION METHOD

1. Define function f(x) and Error (E).
2. Take two initial value for root as x_1 and x_2.
3. Compute $f(x_1)$ and $f(x_2)$.
4. If $f(x_1) \times f(x_2) > 0$, then x_1 and x_2 do not converge and root does not lies in between x_1 and x_2 and go to step (9); otherwise

5. If $f(x_1) < 0$ then set $a = x_1$ and $b = x_2$
   Else,
   set $a = x_2$ and $b = x_1$
6. Calculate root, $x_n = \frac{a+b}{2}$ and also calculate $f(x_n)$.
7. If $f(x_n) > 0$ (i.e. positive) then,
   set $b = x_n$
   Else,
   set $a = x_n$
8. Repeat till step (7), until absolute value of $\frac{b-a}{a}$ is less than E, then print root.
   Root $= \left(\frac{a+x_n}{2}\right)$, go to step (9), else to step (6).
9. Stop.

SOLVED NUMERICALS

Problem 1

Find the root of the equation $e^x - 3x = 0$ correct up to three decimal place using bisection method. [T.U. 2068 Baishakh]

Solution:

Here,

$f(x) = e^x - 3x$

Now, let us follow following steps to find initial guess x_1 and x_2 using calculator.

Here, the initial guess be 1 and 2

$f(1) = -0.281$

$f(2) = 1.389$

$\because$ $f(1) = -0.281 < 0$

and, $f(2) = 1.389 > 0$

Therefore the root lies in between 1 and 2.

so, a = 1 and b = 2 and E = 0.0001

Now, let us calculate root by tabulation method.

n	X	Y	$A = \frac{X+Y}{2}$	$f(x_n) = C$
1	1	2	1.5	– 0.0183
2	1.5	2	1.75	0.5046
3	1.5	175	1.625	0.2034
4	1.5	1.625	1.5625	0.0832
5	1.5	1.5625	1.5313	0.0303
6	1.5	1.5313	1.5157	0.0055
7	1.5	1.5157	1.5079	– 0.0065
8	1.5079	1.5157	1.5118	– 0.0005

9	1.5118	1.5157	1.5138	0.0026
10	1.5118	1.5138	1.5128	0.0010
11	1.5118	1.5128	1.5123	0.0003
12	1.5118	1.5123	1.5121	− 0.0001
13	1.5121	1.5123	1.5122	END

Since, 'a' and 'b' values are equation three decimal place so the root of equation $e^x - 3x$ is 1.5122.

Problem 2

Find at least one root of $x^3 - 2x - 5 = 0$ with the accuracy of 0.08% using bisection method. [T.U. 2067 Ashadh]

Solution:

Given that;

$$x^3 - 2x - 5 = 0$$

Let the initial guess be 2 and 3

$f(2) = -1$

and, $f(3) = 16$

$f(2) = -1 < 0$

and, $f(3) = 16 > 0$

Therefore the root lies between 2 and 3, with error = 0.0008

Now let us calculate the root using tabulation method.

n	a	b	x_n	$f(x_n)$
1	2	3	2.5	5.625
2	2	2.5	2.25	1.891
3	2	2.25	2.215	0.346
4	2	2.125	2.0625	− 0.35132
5	2.0625	2.125	2.0938	− 0.0083
6	2.0938	2.125	2.1094	0.1671
7	2.0938	2.1094	2.1016	0.0789
8	2.0938	2.1016	2.0977	0.0352
9	2.0938	2.0977	2.0958	0.0139
10	2.0938	2.0958	2.0948	0.0027
				END

Since, 'a' and 'b' values are equal up to two decimal place as the root of given equation $f(x) = x^3 - 2x - 5$ is 2.0948.

Problem 3

Find at least one root of the equation $x^2 + \tan x + e^x = 0$ correct up to 3 significant digits using bisection method. [T.U. 2064 Falgun]

Solution:

Given that;

$$f(x) = x^2 + \tan x + e^x$$

NOTE

Set the calculator in radian for trigonometric function equation.

Let, the initial guess be (−1) and (−0.75)

For $f(-1) = -0.189$

$f(-0.75) = 0.103$

$\because$ For $f(-1) = -0.189 < 0$ and $f(-0.75) = 0.103 > 0$

Therefore, the root lies in between −1 and −0.75 with error 0.0001

Now, let us calculate root by tabulation method.

n	a	b	x_n	$f(x_n)$
1	−1	−0.75	−0.875	−0.0149
2	−0.875	−0.75	−0.8125	0.0482
3	−0.875	−0.8125	−0.84375	0.0179
4	−0.84375	−0.8125	−0.82813	0.0333
5	−0.82813	−0.8125	−0.82032	0.0408
6	−0.82032	−0.8125	−0.81641	0.0445
7	−0.81641	−0.8125	−0.81446	0.0463
8	−0.81446	−0.8125	−0.81348	0.04726
9	−0.81348	−0.8125	−0.8130	0.0477
10	−0.8130	−0.8125	−0.8128	0.0480
11	−0.8128	−0.8125	−0.8127	0.0480
				END

Since, 'a' and 'b' are same up to three decimal place, so the root of equation $f(x) = x^2 + \tan x + e^x$ is − 0.8127.

Problem 4

Find out the root of a non-linear equation $2x^3 + 4x^2 - 4x - 6 = 0$ using bisection method up to two decimal place. [T.U. 2063 Jestha]

Solution:

Given that;

$$f(x) = 2x^3 + 4x^2 - 4x - 6$$

Let the initial guess be 1 and 2

$f(1) = -4$ and $f(2) = 18$

$\because$ $f(1) = -4 < 0$ and $f(2) = 18 > 0$

$f(1) \times f(2) < 0$

$-4 \times 18 < 0$

Now, let us calculate root by tabulation method.

n	a	b	x_n	$f(x_n)$
1	1	2	1.5	3.75
2	1	1.5	1.25	−0.8438
3	1.25	1.5	1.375	1.2617
4	1.25	1.375	1.3125	0.1626

5	1.25	1.3125	1.2813	−0.3512
6	1.2813	1.3125	1.2969	−0.0972
7	1.2969	1.3125	1.3047	0.0320
8	1.2969	1.3047	1.3008	−0.0327
9	1.3008	1.3047	1.3028	0.00040

Since, 'a' and 'b' are same up to two decimal place so the root of equation $f(x) = 2x^3 + 4x^2 - 4x - 6$ is 1.3018.

Problem 5

Find out all the real roots of $x^3 - 6x^2 + 11x - 6 = 0$ using bisection method to two decimal places.

Solution:

Given that

$f(x) = x^3 - 6x^2 + 11x - 6$

Let the initial guess be 2.5 and 4 for first root

$f(2.5) = -0.375$ and $f(4) = 6$

$\because$ $f(2.5) = -0.370 < 0$ and $f(4) = 6 > 0$,

Therefore the root lies in between 2.5 and 4 with given E = 0.001.

Now, let us calculate root by tabulation method.

n	a	b	x_n	$f(x_n)$
1	2.5	4	3.25	0.70313
2	2.5	3.25	2.875	−0.2051
3	2.875	3.25	3.0625	0.1370
4	2.875	3.0625	2.9688	−0.0595
5	2.9688	3.0625	3.0157	0.03214
6	2.9688	3.0157	2.9923	−0.01522
7	2.9923	3.0157	3.004	0.00805
8	2.9923	3.004	2.9982	−0.00360
9	2.9982	3.004	3.0011	0.00220
10	2.9982	3.0011	2.9996	−0.00080
11	2.9996	3.001	3.0003	0.00060

Since, 'a' and 'b' are same up to three decimal place, so the root of equation $f(x) = x^3 - 6x^2 + 11x - 6$ is 3.0003.

For 2nd root let the initial guess be

$a = 0$ and $b = 1.5$

$f(a) = -6$ and $f(b) = 1.5$

so, the root lies in between 0 and 1.5.

n	a	b	x_n	$f(x_n)$
1	0	1.5	0.75	−0.703
2	0.75	1.5	1.125	0.205
3	0.75	1.125	0.937	−0.138
4	0.937	1.125	1.031	0.059
5	0.937	1.031	0.984	−0.033
6	0.984	1.031	1.007	0.014
7	0.984	1.007	0.995	−0.011
8	0.995	1.007	1.001	0.002
9	0.995	1.001	0.998	−0.004
10	0.998	1.001	0.999	−0.002

Hence, the root is 0.999

For 3rd root

Let initial guess be a = 1.4 and b = 2.4

n	a	b	x_n	$f(x_n)$
1	1.4	2.4	1.9	0.99
2	1.4	1.9	1.65	0.307
3	1.4	1.65	1.525	0.368
4	1.4	1.525	1.462	0.383
5	1.4	1.462	1.431	0.385
6	1.4	1.431	1.415	0.385
7	1.4	1.415	1.407	0.385
8	1.4	1.407	1.407	0.385

so, the 3rd root is 1.403.

Problem 6

Find the root of following equation using bisection method to four decimal places $\log x - \cos x = 0$. **[P.U. 2005 Fall]**

Solution:

Given that;

$f(x) = \log x - \cos x$

Let initial guess be 1 and 2

$f(1) = -0.54 < 0$ and $f(2) = 0.7171 > 0$,

Therefore, the root lies in between 1 and 2

Now, let us calculate root by tabular form.

n	a (−ve)	b (+ve)	x_n	$f(x_n)$
1	1	2	1.5	0.1054
2	1	1.5	1.25	−0.2184
3	1.25	1.5	1.375	−0.0562
4	1.375	1.5	1.4375	0.0247
5	1.375	1.4375	1.4063	−0.01568
6	1.4063	1.4375	1.4219	0.0045

7	1.4063	1.4219	1.4141	−0.0056
8	1.4141	1.4219	1.4180	−0.00053
9	1.4180	1.4219	1.4199	0.00193
10	1.4180	1.4199	1.4190	0.00077
11	1.4180	1.4190	1.4185	0.000121
12	1.4180	1.4185	1.4183	−0.00014
13	1.4180	1.4185	1.4184	−0.0000084
14	1.4184	1.4185	1.41845	0.000056

Since, 'a', 'b' and x_n are equal to 4 decimal place so the root of equation $f(x) = \log x - \cos x$ is 1.41841.

Problem 7

Calculate the root of given equation using bisection method, correct up to 3 decimal place $f(x) = \tan x + \tan hx$. [P.U. 2005 Spring]

Solution:

Given that

$f(x) = \tan x + \tan hx$

Let the initial guess be 2 and 3

$f(2) = -1.221$

$f(3) = 0.8525$

$f(2) \times f(3) < 0$

Now, let us calculate root using tabular form.

n	a (−ve)	b (+ve)	x_n	$f(x_n)$
1	2	3	2.5	0.2396
2	2	2.5	2.25	−0.2606
3	2.25	2.5	2.375	0.01976
4	2.25	2.375	2.3125	−0.11086
5	2.3125	2.375	2.3438	−0.4335
6	2.3438	2.375	2.3589	−0.01231
7	2.3589	2.375	2.3670	0.00395
8	2.3589	2.3670	2.3630	−0.00405
9	2.3630	2.3670	2.3650	−0.00004
10	2.3650	2.3670	2.3660	0.00196
11	2.3650	2.3660	2.3655	0.00096
12	2.3650	2.3655	2.3653	0.00056

Since, 'a', 'b' and x_n have same value up to 3 decimal place so the root of equation $f(x) = \tan x + \tan hx$ is 2.365.

Problem 8

Calculate the root of given equation using bisection method; correct up to three decimal place $f(x) = \sin x - 2x + 1$. [P.U. 2009 Fall]

Solution:

Given that;

$f(x) = \sin x - 2x + 1$

Let the initial guess be 0 and 1

Here,

$f(0) \times f(1) < 0$

Now, let us calculate root using tabular form.

n	a	b	x_n	$f(x_n)$
1	1	0	0.5	0.4794
2	1	0.5	0.75	0.1816
3	1	0.75	0.875	0.01754
4	1	0.875	0.9375	−0.06892
5	0.9375	0.875	0.90625	−0.02530
6	0.90625	0.875	0.89063	−0.00380
7	0.89063	0.875	0.88282	0.00689
8	0.89063	0.88282	0.8868	0.00145
9	0.89063	0.8868	0.88872	−0.00117
10	0.88872	0.8868	0.88776	0.000359
11	0.88872	0.88776	0.88824	−0.000517
12	0.88824	0.88776	0.888	−0.000188

Since, 'a', 'b' and x_n has same value up to 3 decimal place so root of equation $f(x) = \sin x - 2x + 1$ is 0.888.

Problem 9

Find a positive rot of the equation $f(x) = \cos x - 1.3x$ by using bisection method. [P.U. 2010 Spring]

Solution:

Given that;

$f(x) = \cos x - 1.3x$

Let the initial guess be 0 and 1

$f(0) = 1$

$f(1) = -0.759$

$f(0) \times f(1) < 0$

Now, let us calculate root using tabular form.

n	a	b	x_n	$f(x_n)$
1	1	0	0.5	0.2275
2	1	0.5	0.75	−0.2433
3	0.75	0.5	0.625	−0.00154
4	0.625	0.5	0.5625	0.11467
5	0.625	0.5625	0.5938	0.05688
6	0.625	0.5938	0.6094	0.02777
7	0.625	0.6094	0.6172	0.01314

8	0.625	0.6172	0.6211	0.00580
9	0.625	0.6211	0.6231	0.00204
10	0.625	0.6231	0.6241	0.00016
11	0.625	0.6241	0.6246	−0.000783

Since, 'a', 'b' and x_n has same value up to 3 decimal place so equation $f(x) = \cos x - 1.3x$ has root 0.624.

Problem 10

Find a root of the equation by using Bisection method up to three decimal place. $f(x) = x^2 - 10\log x$ [P.U. 2011 Spring]

Solution:

Given that;

$f(x) = x^2 - 10\log x$ [In this equation take *ln*]

$f(3.5) = -0.2777$

and, $f(4) = 2.137$

$f(3.5) \times f(4) < 0$

Now, let us calculate root using tabular form.

n	a	b	x_n	$f(x_n)$
1	3.5	4	3.75	0.844
2	3.5	3.75	3.625	0.262
3	3.5	3.625	3.5625	−0.013
4	3.5625	3.625	3.5937	0.12
5	3.5625	3.59375	3.5781	0.054
6	3.5625	3.5781	3.5703	0.0206
7	3.5625	3.5703	3.5664	−0.0036
8	3.5625	3.5664	3.5644	−0.0047
9	3.5644	3.5664	3.5654	−0.0005
10	3.5654	3.5664	3.5659	0.0015

Hence, root of equation $f(x) = x^2 - 10\log x$ is 3.565.

Problem 11

Calculate the root of the given equation using bisection method correct up to 3 decimal place $f(x) = \sin x - \frac{1}{x}$ [P.U. 2012 Fall]

Solution:

Given that;

$f(x) = \sin x - \frac{1}{x}$

$f(1) = -0.1585$

and, $f(2) = 0.4092$

$f(1) \times f(2) < 0$

Now, let us calculate root using tabular form.

n	a	b	x_n	$f(x_n)$
1	1	2	1.5	0.3308
2	1	1.5	1.25	0.1489
3	1	1.25	1.125	0.0134
4	1	1.125	1.0625	−0.0676
5	1.0625	1.125	1.09375	−0.0260
6	1.09375	1.125	1.1094	−0.00595
7	1.1094	1.125	1.1172	0.00378
8	1.1094	1.1172	1.1133	−0.001069
9	1.1133	1.1172	1.11525	0.00136
10	1.1133	1.11525	1.114275	0.000149
11	1.1133	1.114275	1.11379	−0.000458
12	1.11379	1.114275	1.11403	−0.000158
13	1.11403	1.114275	1.1142	0.000053
				END

Since, 'a' and 'b' are same up to three decimal place, so the root of equation $f(x) = \sin x - \frac{1}{x}$ is 1.1142.

Problem 12

Find a positive root of the equation $f(x) = \cos x - 3x + 1$ correct up to 3 decimal place by using bisection method. [P.U. 2013 Fall]

Solution:

Given that;

$f(x) = \cos x - 3x + 1$

Let the initial guess be 0 and 1.

$f(0) = 2 > 0$

and, $f(1) = -1.459 < 0$

Now, let us calculate root using tabular form.

n	a	b	x_n	$f(x_n)$
1	1	0	0.5	0.37758
2	1	0.5	0.75	−0.5183
3	0.75	0.5	.0625	−0.06404
4	0.625	0.5	0.5625	0.1584
5	0.625	0.5625	0.59375	0.04759
6	0.625	0.59375	0.6094	−0.00821
7	0.6094	0.59375	0.6016	0.01963
8	0.6094	0.6016	0.6055	0.00572
9	0.6055	0.6016	0.6035	0.01285
10	0.6035	0.6016	0.60255	0.01624
11	0.60255	0.6016	0.6021	0.0178

12	0.6021	0.6016	0.6019	0.01856
13	0.6019	0.6016	0.60175	0.0191
				END

Since, a, b and x_n have same value up to 3 decimal place so the root of equation (x) = cos x − 3x + 1 is 0.60175.

Problem 15

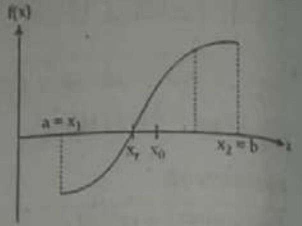
Write an algorithm to find a real root of a non-linear equation using bisection method. [IOE, 2069/2070 Chaitra, IOE, 2071 Shrawan]

Solution:

See Algorithm of bisection method on page no. 4

CONVERGENCE OF BISECTION METHOD

Here, x_0 is the midpoint of x_1 and x_2 so depending upon the sign of $f(x_0)$, $f(x_1)$ and $f(x_2)$, x_1 or x_2 is set equal to x_0.



i.e., $f(x_0) \times f(x_1) < 0 \Rightarrow x_2 = x_0$

$f(x_0) \times f(x_2) < 0 \Rightarrow x_1 = x_0$

If n is the number of iteration then,

$$\frac{x_2 - x_1}{2^n} = \frac{\Delta x}{2^n} = \varepsilon_n,$$

which is the error at n^{th} iteration.

$$\therefore \quad \varepsilon_{n+1} = \left|\frac{x_2 - x_1}{2^n + 1}\right|$$

$$= \frac{1}{2}\left|\frac{\Delta x}{2^n}\right|$$

$$\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n \qquad (1)$$

The error decreases linearly with each state by factor $\frac{1}{2}$. Therefore, the bisection method is linearly convergent, the convergence is slow to achieve high degree of accuracy and large number of iteration may be required.

ii) FALSE POSITION METHOD

In bisection method, the interval between x_1 and x_2 is divided into two equal halves, irrespective of their location of root. The root may be closer to one end or the other. Now let us join point x_1 and x_2 by a straight line. The point of intersection of this line with x axis gives an improved estimate of the root and is called false position of the root. This point then replaces one of the initial guess that has a function value of the same sign as $f(x_0)$.

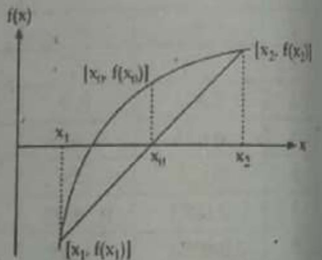


Figure: Illustration of false position method

This process is repeated with the new value of x_1 and x_2. Since, this method uses the false position of the root repeatedly so it is called **false position method**. It is also called **linear interpolation**.

False Position Method Algorithm (Regular False Method)

1. Define function f(x) and Error (E).
2. Take two initial point of interval for root as x_1 and x_2.
3. Calculate $f(x_1)$ and $f(x_2)$.
4. If $f(x_1) \times f(x_2) > 0$ then, root does not lie in between x_1 and x_2, go to step 9.
5. If $f(x_1) < 0$ then set $a = x_1$ and $b = x_2$

   Else,

   Set $a = x_2$ and $b = x_1$
6. Calculate root $x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$ and $f(x_n)$.
7. If $f(x_n) > 0$ (positive) then set $b = x_n$

   Else,

   Set $a = x_n$
8. Repeat step 7 till $\left(\text{abs}\frac{b-a}{b}\right) < E$ and print root.
9. Stop

NOTE

The bisection and regular false method has similar algorithm except the formula for calculation of x_n.

SOLVED NUMERICAL

Problem 1

Find a real root of the following equation correct to 4 decimal place using False position method. $e^{\cos x} - \sin x - 1 = 0$ [T.U. 2070 Bhadra]

Solution:

Given that;

$$f(x) = e^{\cos x} - \sin x - 1$$

Let the initial guess be 3 and 4

$\because \quad f(3) = -0.769 < 0$

and, $f(4) = 0.2769 > 0$,

So, root lies between 3 and 4.

Then according to false position method;

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{3 \times 0.2769 - 4 \times -0.769}{0.2769 + 0.769} = 3.73525$$

Let us calculate root from tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	3	4	−0.7695	0.2769	3.73537	−0.003945
2	3.73537	4	−0.003945	0.2769	3.73908	0.00003489
3	3.73537	3.73908	−0.003945	0.0003489	3.73905	0.000002707
4	3.73537	3.73905	−0.003945	0.000002707	3.73904	−0.00000802
						END

Since, the value of two consecutive x_n is similar up to 4 decimal place, so the root of equation $f(x) = e^{\cos x} - \sin x - 1$ is 3.73904.

Problem 2

Find the root of following equation using regula false method to four decimal place. $f(x) = \log x - \cos x$ **[P.U. 2005 Fall]**

Solution:

Given that;

$$f(x) = \log x - \cos x$$

Let the initial guess be 1 and 2

$\because$ $f(1) = -0.54 < 0$ and $f(2) = 0.7171 > 0$,

so, root lies between 1 and 2.

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Now, let us calculate root from tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	1	2	−0.54030	0.71717	1.42967	0.014577
2	1	1.42967	−0.54030	0.014577	1.41838	−0.0000343
3	1.41838	1.42967	−0.0000343	0.014577	1.41840	−0.000008397
4	1.41840	1.42967	−0.000008397	0.014577	1.41840	END

Since, the two consecutive values of x_n in 3rd and 4th iteration is same up to 4 decimal place so the root of equation $f(x) = \log x - \cos x$ is **1.41840**.

Problem 3

Find a root of equation $\sin x - x + 2 = 0$ by false position method correct up to three decimal place. **[P.U. 2007 Fall]**

Solution:

Given that;

$$f(x) = \sin x - x + 2$$

Let the initial guess be 2 and 3

$\because$ $f(2) = 0.9092 > 0$

and, $f(3) = -0.858 < 0$

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

So, root lies in between 2 and 3, where a = 3 and b = 2

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	3	2	−0.858	0.9092	2.5144	0.07247
2	3	2.5144	−0.858	0.07247	2.5522	0.003656
3	3	2.5522	−0.858	0.003656	2.5541	0.000175
4	3	2.5541	−0.858	0.000175	2.5541	END

Since, value of two consecutive root x_n is same up to 3 decimal place so root is 2.5541.

Problem 4

Use False-position method to find the root of the following equation, $x^3 - 4x^2 + x + 6 = 0$ correct up to two decimal place. **[P.U. 2009 Spring]**

Solution:

Here,

$$f(x) = x^3 - 4x^2 + x + 6$$

Let the initial guess be 2.5 and 3.5

$\because$ $f(2.5) = -0.875 < 0$

and, $f(3.5) = 3.375 > 0$

so, root lies between 2.5 and 3.5.

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

where, a = 2.5 and b = 3.5.

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	2.5	3.5	−0.875	3.375	2.7058	−0.7695
2	2.7058	3.5	−0.7695	3.375	2.8533	−0.48235
3	2.8533	3.5	−0.48235	3.375	2.9342	−0.24184
4	2.9342	3.5	−0.24184	3.375	2.9720	−0.10810
5	2.9720	3.5	−0.10810	3.375	2.9884	−0.04573
6	2.9884	3.5	−0.04573	3.375	2.9952	−0.01908
7	2.9952	3.5	−0.01908	3.375	2.9980	END

Here, the value of x_n in consecutive iteration 6th and 7th is same up to 2 decimal place so the root of equation $f(x) = x^3 - 4x^2 + x + 6$ is 2.9980.

Problem 5

Find the root of equation $x \sin x + \cos x = 0$ with accuracy 0.008, using false position method. **[P.U. 2011 Fall]**

Solution:

Given that;

$$f(x) = x \sin x + \cos x$$

Let the initial guess be 2 and 3

$\because$ $f(2) = 1.4024 > 0$

and, $f(3) = -0.566 < 0$

Therefore, the root lies between 2 and 3 but according to algorithm step 5, a = 3 and b = 2, Error = 0.008

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Now, let us calculate the root in tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	3	2	−0.5666	1.4020	2.7122	0.2199
2	3	2.7122	−0.5666	0.2199	2.7926	0.0152
3	3	2.7926	−0.5666	0.0152	2.7980	0.0010
4	3	2.7980	−0.5666	0.0010	2.7983	0.0002
5	3	2.7983	−0.5666	0.0002	2.7983	END

Hence, root of equation $x \sin x + \cos x$ is 2.7983.

Problem 6

Find the root of the equation $f(x) = x^3 + 4x - 1$ with accuracy 0.0001 using False position method. [P.U. 2011 Spring]

Solution:

Given that;

$$f(x) = x^3 + 4x - 1$$

Let the initial guess be 0 and 1

$\because$ $f(0) = -1 < 0$

and, $f(1) = 4 > 0$

Therefore, the root lies between 0 and 1.

Here, a = 0 and b = 1, Error = 0.0001

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Let us calculate root in tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	0	1	−1	4	0.20	−0.1920
2	0	0.20	−1	−0.1920	0.2475	0.0052
3	0.2475	0.20	0.0052	−0.1920	0.2462	−0.0003
4	0.2475	0.2462	0.0052	−0.0003	0.2462	END

Since, the value of x_n in 3rd and 4th iteration is same up to 4 decimal place, so the root of equation $f(x) = x^3 + 4x - 1$ is 0.2462.

Problem 7

Find the root of the equation $\cos x = xe^x$ using the regula-false method correct up to four decimal place. [P.U. 2012 Fall]

Solution:

Given that;

$$f(x) = \cos x - xe^x$$

Let the initial guess be 0 and 1

$\because$ $f(0) = 1 > 0$

and, $f(1) = -2.177 < 0$

So, root lies between 0 and 1, but according to algorithm step 5.

a = 1 and b = 0, Error = 0.00001

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Now, let us calculate using tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	1	0	−2.177	1	0.31476	0.51967
2	1	0.31476	−2.177	0.51967	0.44681	0.20332
3	1	0.44681	−2.177	0.20332	0.49408	0.07061
4	1	0.49408	−2.177	0.07061	0.50997	0.02354
5	1	0.50997	−2.177	0.02354	0.51521	0.007735
6	1	0.51521	−2.177	0.00773	0.51692	0.002545
7	1	0.51692	−2.177	0.002545	0.51748	0.0008436
8	1	0.51748	−2.177	0.0008436	0.51766	0.0002962
9	1	0.51766	−2.177	0.0002962	0.51772	0.00011366
10	1	0.51772	−2.177	0.000113366	0.51774	0.000053
						END

Since, the value of x_n in 9th and 10th iteration are similar up to 4 decimal place so the root of equation $f(x) = \cos x - xe^x$ is 0.51774.

Problem 8

Find root of the equation $x \log_{10} x = 1.2$ by false position method correct up to 4 decimal places.

Solution:

Given that;

$$f(x) = x \log_{10} x - 1.2$$

Let the initial guess be 2 and 3.

$\because$ $f(2) = -0.597 < 0$

and, $f(3) = 0.2313 > 0$

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

so, root lies between a = 2 and b = 3, with Error = 0.00001

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	2	3	−0.597	0.2313	2.72075	−0.0732
2	2.72075	3	−0.01732	0.2313	2.74020	−0.000389
3	2.74020	3	−0.000389	0.2313	2.74063	−0.000014
4	2.74063	3	−0.000014	0.2313	2.74064	−0.0000053

Since, the value of x_n in 3rd and 4th iteration are same up to 4 decimal place so the root of equation $f(x) = x \log_{10} x - 1.2$ is 2.74064.

Problem 9

Use the method of false position to find the fourth root of 32 correct up to three decimal place.

Solution:

Given that:

$x = (32)^{\frac{1}{4}}$

$f(x) = x^4 - 32 = 0$

Let the initial guess be 2 and 3

$\because$ $f(32) = -16 < 0$

and, $f(3) = 49 > 0$

Therefore, the root lies between 2 and 3.

Hence, a = 2 and b = 3

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Let us calculate the root using tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	2	3	−16	49	2.2461	−6.54833
2	2.2461	3	−6.54833	49	2.2349	−2.27833
3	2.3349	3	−2.27833	49	2.3644	−0.74757
4	2.3644	3	−0.74757	49	2.3739	−0.24225
5	2.3739	3	−0.24225	49	2.3769	−0.08141
6	2.3769	3	−0.08141	49	2.3779	−0.02766
7	2.3779	3	−0.02766	49	2.3782	−0.01153
8	2.3782	3	−0.01153	49	2.3783	−0.00614
						END

Since, the value of x_n in 7^{th} and 8^{th} iteration are similar up to 3 decimal place so the root of equation $f(x) = x^4 - 32$ is 2.3783.

Problem 10

Find a real root of following equation correct to four decimal place; using the false position.

$x^3 - 5x - \sin(x) - 6 = 0.$ [T.U., 2071 Shrawan]

Solution:

$f(x) = x^3 - 5x - \sin(x) - 6$

Let the initial guess be: 2 and 3

$\therefore$ $f(2) = -8.90929 < 0$

$f(3) = 5.85887 > 0$

$\therefore$ Root les between 2 and 3.

Now,

According to false position method,

$$x_n = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

Let us calculate the root using tabular form

n	a	b	f (a)	f (b)	x_n	$f(x_n)$
1	2	3	− 8.90929	5.85887	2.60327	− 1.88653
2	2.60327	3	− 1.88653	5.85887	2.69990	− 0.246051
3	2.69990	3	− 0.24602	5.85887	2.71200	− 0.02988
4	2.71200	3	− 0.29877	5.85887	2.71346	-3.599×10^{-3}
5	2.71346	3	-3.6325×10^{-3}	5.85887	2.71363	-4.367×10^{-4}
6	2.71363	3	-5.726×10^{-4}	5.85887	2.71365	-5.1972×10^{-5}

$\therefore$ Root = 2.71363

iii) SECANT METHOD

Secant method is like false position and bisection method; it uses two initial estimations but does not require that they must bracket the root. Let us consider the following figure.

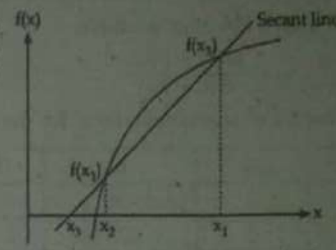


Figure: Illustration of secant method

The point x_1 and x_2 are starting point, they do not bracket the root. Slope of line (secant line) passing through x_1 and x_2 is given by.

$$\frac{f(x_1)}{x_1 - x_3} = \frac{f(x_2)}{x_2 - x_3}$$

$$f(x_1)(x_2 - x_3) = f(x_2)(x_1 - x_3)$$

$$x_3[f(x_2) - f(x_1)] = f(x_2) \cdot x_1 - f(x_1) \cdot x_2$$

$$x_3 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$

Now, by adding and subtracting $f(x_2)x_2$ to the numerator and rearranging the terms; we get,

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} \quad (1)$$

This equation is called secant formula. The secant formula in general form is

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Secant Method Algorithm

1. Define function f(x) and Error (E)
2. Take two initial guess of interval for root as x_1 and x_2
3. Calculate $f(x_1)$ and $f(x_2)$
4. If $f(x_1) \times f(x_2) > 0$ then root does not lie in between x_1 and x_2, so go to step 8.
5. If $f(x_1) < 0$ then set $a = x_1$ and $b = x_2$
   Else,
   Set $a = x_2$ and $b = x_1$
6. Calculate root by $x_n = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ and x_n
7. If $\left(\text{abs}\dfrac{(b-a)}{b}\right) < E$ then print root = x_n
   Else,
   go to step 5
8. Stop.

SOLVED NUMERICALS

Problem 1

Find an approximate root of $x\log_{10}x - 1.2 = 0$ using secant method up to three decimal place of accuracy. [T.U. 2067 Ashadh]

Solution:

Given that;

$f(x) = x\log_{10}x - 1.2$

Let the initial guess be 2 and 3

$\because$ $f(2) = -0.597 < 0$

and, $f(3) = 0.2313 > 0$

Therefore, the root lies in between 2 and 3, where a = 2 and b = 3, Error = 0.0001

By secant formula; we have,

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Now let us use tabular form for the calculation of root.

n	a	b	f(a)	f(b)	x_n
1	2	3	−0.597	0.2313	x_1 = 2.7207
2	3	x_1 = 2.7207	0.2313	−0.01736	x_2 = 2.7401
3	2.7207	x_2 = 2.7401	−0.01736	−0.000476	x_3 = 2.7406

The 2nd and 3rd iteration has similar value of x_n up to 3 decimal place, so the root of equation $f(x) = x\log_{10}x - 1.2$ is 2.7406.

Problem 2

Find at least one root of the equation $x^2 + \tan x + e^x = 0$ correct up to 3 decimal place with secant method. [T.U. 2064 Falgun]

Solution:

Given that;

$f(x) = x^2 + \tan x + e^x$

Let the initial guess be −1 and −0.75

$\because$ $f(-1) = -0.189 < 0$

and, $f(-0.75) = 0.1032 > 0$

Therefore, the root lies in between −1 and −0.75, where a = −1 and b = −0.75

Let us use secant method and find the root, by secant method formula

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Now let us use tabular form for the calculation of root.

n	a	b	f(a)	f(b)	x_n
1	−1	−0.75	−0.189	0.1032	−0.83829
2	−0.75	−0.83829	0.1032	0.023378	−0.868522
3	−0.83892	−0.868522	0.023378	0.0078748	−0.88387
4	−0.868522	−0.88387	0.0078748	−0.024835	−0.87221
5	−0.88387	−0.87221	−0.024835	−0.011876	−0.86152
6	−0.87221	−0.86152	−0.011876	−0.0003969	-0.86115

Since, the 5th and 6th iteration has similar value up to three decimal place so the root of equation $f(x) = x^2 + \tan x + e^x$ is −0.86115.

Problem 3

Calculate the root of non-linear equation $f(x) = \sin x - 2x + 1$ using secant method. The absolute error of functional value at or calculated root should be less than 10^{-3}. [T.U. 2065 Shrawan]

Solution:

Given that;

$f(x) = \sin x - 2x + 1$

Let the initial guess be 0 and 1

$\because$ $f(0) = 1 > 0$

and, $f(1) = -0.158 < 0$

Therefore, the root lies in between 0 and 1

Since, the value of a = 1 and b = 0

Now let us calculate the root of the given equation using tabular form.

n	a	b	f(a)	f(b)	x_n
1	1	0	−0.1585	1	0.863185
2	0	0.863185	1	0.03355	0.89315
3	0.863185	0.89315	0.03355	−0.00725	0.88782
4	0.89315	0.88782	−0.00725	0.000577	0.88786

Hence, the value of x_n in 3rd and 4th iteration is similar up to 3 decimal place so the root of equation $f(x) = \sin x - 2x + 1$ is 0.8878.

Problem 4

Calculate a root of non-linear equation $f(x) = 3x + \sin x - e^x$ using secant method. The absolute error of function value of your calculated root should be less than 10^{-5}. [T.U. 2063 Baishakh]

Solution:

Given that;

$f(x) = 3x + \sin x - e^x$

Let the initial guess be 0 and 1

$\because$ $f(0) = -1 < 0$

and, $f(1) = 1.1231 > 0$

Therefore, the root lies in between 0 and 1

Now let us calculate the root of the given equation using tabular form.

n	a	b	f(a)	f(b)	x_n
1	0	1	−1	1.1231	0.4710
2	1	0.4710	1.1231	0.26518	0.30748

3	0.4710	0.30748	0.26518	−0.13489	0.3626
4	0.30748	0.3626	−0.13489	0.00544	0.36046
5	0.3626	0.36046	0.00544	0.0000958	0.36042

Hence, the value of x_n is 4^{th} and 5^{th} iteration is similar up to 4 decimal place, the root of equation $f(x) = 3x + \sin x - e^x$ is 0.36042.

Problem 5

Find out all the roots of a non-linear equation $2x^3 + 4x^2 - 4x - 6 = 0$ using secant method. [T.U. 2063 Jestha]

Solution:

Given that;

$$f(x) = 2x^3 + 4x^2 - 4x - 6$$

Let the initial guess be 1 and 2

$\because$ $f(1) = -4 < 0$

and, $f(2) = 18 > 0$

So, the root lies in between 1 and 2

n	a	b	f(a)	f(b)	x_n
1	1	2	−4	18	1.1818
2	2	1.1818	18	−1.83947	1.3006
3	1.1818	1.3006	−1.83947	−0.03607	1.3029
4	1.3006	1.3029	−0.03607	0.00206	1.3027

Since, the value of x_n in 3^{rd} and 4^{th} iteration is similar up to 3 decimal place, so the root of equation $f(x) = 2x^3 + 4x^2 - 4x - 6$ is 1.3027.

Problem 6

Find the root of the following equation correct up to the three decimal place by the secant method (chord method) $x^3 + x^2 + x + 7 = 0$. [T.U. 2061 Baishakh]

Solution:

Given that;

$$f(x) = x^3 + x^2 + x + 7$$

Let the initial guess be −3 and −2

$\because$ $f(-3) = -14 < 0$

and, $f(-2) = 1 > 0$

So, the root lies in between −3 and −2, the error = 0.0001

Let us tabulate root from the tabular form

n	a	b	f(a)	f(b)	x_n
1	−3	−2	−14	1	−2.0666
2	−2	−2.0666	1	0.37813	−2.1070
3	−2.0666	−2.1070	0.37813	−0.02147	−2.1048
4	−2.1070	−2.1048	−0.02147	0.000733	−2.107
5	−2.1048	−2.107	0.000733	−0.02147	−2.104

Since, the value of x_n is same up to two decimal place so the root of equation $f(x) = x^3 + x^2 + x + 7$ is −2.104.

Problem 7

Find the real root of the function $\tan x + \tan hx = 0$ secant method with initial value 2.3 and 2.4. [P.U. 2005 Spring]

Solution:

Given that;

Function $f(x) = \tan x + \tan hx$

Then given initial value is 2.3 and 2.4

$\because$ $f(2.3) = -0.13912 < 0$

and, $f(2.4) = 0.06766 > 0$

Therefore, the root lie in between 2.3 and 2.4.

Here, a = 2.3 and b = 2.4

Let us tabulate root from the tabular form

n	a	b	f(a)	f(b)	x_n
1	2.3	2.4	−0.13912	0.06766	2.36728
2	2.4	2.36728	0.06766	0.00451	2.36494
3	2.36728	2.36494	0.00451	−0.00016	2.36502
4	2.36494	2.36502	−0.00016	−0.0000007	2.36502

Since, the x_n value in 3^{rd} and 4^{th} iteration is same up to 5 decimal place, so the root of equation $f(x) = \tan x + \tan hx$ is 2.36502.

Problem 8

Using secant method, find the root of the equation $x^6 - x^4 - x^3 - 1 = 0$ correct up to 2 significant digits. [P.U. 2006 Spring]

Solution:

Given that;

$$f(x) = x^6 - x^4 - x^3 - 1$$

Let the initial guess be 1 and 2

$\because$ $f(1) = -2 < 0$

and, $f(2) = 39 > 0$

So, the root lies in between 1 and 2

Here, a = 1 and b = 2

Now let us consider root by using tabulation method.

n	a	b	f(a)	f(b)	x_n
1	1	2	−2	39	1.0488
2	2	1.0488	39	−2.0327	1.09592
3	1.0488	1.09592	−2.0327	−2.02624	15.88
4	1.09592	15.88	−2.02624	15968658.83	1.09592
5	15.88	1.09592	15968658.83	1.09592	1.09592

Since, the value of x_n in 4^{th} and 5^{th} iteration is same up to 5 decimal place, so the root of equation $f(x) = x^6 - x^4 - x^3 - 1$ is 1.09592.

Problem 9

> By using secant method find a real root of $x^3 + x^4 - 3x - 3 = 0$ correct up to three decimal places. [P.U. 2006 Fall and 2008 Fall]

Solution:

Given that;

$$f(x) = x^3 + x^4 - 3x - 3$$

Let the initial guess be 1 and 2

$\because$ $f(1) = -4 < 0$

and, $f(2) = 3 > 0$

So, the root lies in between 1 and 2

n	a	b	f(a)	f(b)	x_n
1	1	2	−4	3	1.5714
2	2	1.5714	3	−1.3646	1.7054
3	1.5714	1.7054	−1.3646	−0.24784	1.73509
4	1.7054	1.73509	−0.24784	0.02882	1.73699
5	1.73509	1.73699	0.02882	0.04689	1.7321
6	1.73699	1.7321	0.000465	0.000465	1.7321

Since, the value of x_n in 5^{th} and 6^{th} iteration is same up to 3 decimal place, so the root of equation $f(x) = x^3 + x^4 - 3x - 3$ is 1.7321.

Problem 10

> Solve $f(x) = 3x + \sin x - e^x$ by secant method up to 5th iteration. [P.U. 2007 Fall, 2007 Spring, 2010 Spring]

Solution:

Given that;

$$f(x) = 3x + \sin x - e^x$$

Let the initial guess be 0 and 1

$\because$ $f(0) = -1 < 0$ and $f(1) = 1.1231 > 0$,

Therefore, the root lies in between 0 and 1

Here, $a = 0$ and $b = 1$

Let us calculate the root by using tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	0	1	−1	1.1231	0.4710	0.2651
2	1	0.4710	1.1231	0.2651	0.30755	−0.13471
3	0.4710	0.30755	0.2651	−0.13471	0.36262	0.00549
4	0.30755	0.36262	−0.13471	0.00549	0.36046	0.00095
5	0.36262	0.36046	0.00549	0.000095	0.36616	0.1432
						END

Hence, according to 5^{th} iteration the root of equation $f(x) = 3x + \sin x - e^x$ is 0.36616.

Problem 11

> Find out at least one real root of $x^3 + x^2 - x - 1 = 0$ using secant method to three decimal places. [P.U. 2008 Spring]

Solution:

Given that;

$$f(x) = x^3 + x^2 - x - 1$$

Let the initial guess be 0.5 and 1.5

$\because$ $f(0.5) = -1.125 < 0$ and $f(1.5) = 3.125 > 0$,

Therefore, the root lies in between $a = 0.5$ and $b = 1.5$

Let us calculate the root from tabular form.

n	a	b	f(a)	f(b)	x_n
1	0.5	1.5	−1.125	3.125	0.7647
2	1.5	0.7647	3.125	−0.73276	0.90436
3	0.7647	0.90436	−0.73276	−0.34685	1.0298
4	0.90436	1.0298	−0.34685	0.12277	0.99700
5	1.0298	0.99700	0.12277	−0.01196	0.9999
6	0.99700	0.9999	−0.01196	−0.000399	1.0000

Since, the root of equation $f(x) = x^3 + x^2 - x - 1$ is 1.0000. Therefore, x_n in 5^{th} and 6^{th} iteration is same.

Problem 12

> Find a real root of the equation $x^3 - 2x - 5 = 0$ correct up to 3 decimal places using secant method. [P.U. 2010 Fall]

Solution:

Given that;

$$f(x) = x^3 - 2x - 5$$

Let the initial guess be 2 and 3

$\because$ $f(2) = -1 < 0$

and, $f(3) = 16 > 0$

Therefore, the root lies in between 2 and 3.

Let us calculate the root from tabular form.

n	a	b	f(a)	f(b)	x_n
1	2	3	−1	16	2.05882
2	3	2.05882	16	−0.390837	2.08126
3	2.05882	2.08126	−0.390837	−0.14724	2.09482
4	2.08129	2.09482	−0.14724	0.002998	2.09460

Since, the value of x_n is same up to 3 decimal place in 3^{rd} and 4^{th} iteration so the root of equation $f(x) = x^3 - 2x - 5$ is 2.09460.

Problem 13

Using secant method find a root of the equation $x - e^x + 2 = 0$ correct up to three decimal place. [P.U. 2011 Fall]

Solution:

Given that;

$f(x) = x - e^x + 2$

Let the initial guess be 1 and 2

$\because \quad f(1) = 0.2817 > 0$

and, $f(2) = -3.389 < 0$

Therefore, the root lie in between 1 and 2, but according to step 5 of algorithm

$a = 2$ and $b = 1$

Let us tabulate the root of equation by using tabular form

n	a	b	f(a)	f(b)	x_n
1	2	1	−3.389	0.2817	1.07674
2	1	1.07674	0.2817	0.14164	1.15434
3	1.07674	1.15434	0.14164	−0.01756	1.14578
4	1.15434	1.14578	−0.01756	0.000887	1.1462
5	1.14578	1.1462	0.000887	−0.0000145	1.1462

Since, the 4th and 5th iteration has same value of x_n up to 3 decimal place, so the root of equation $f(x) = x - e^x + 2$ is 1.1462.

Problem 14

Find the root of the equation $xe^x = \cos x$ using the secant method, correct to 3 decimal places.

Solution:

Given that;

$f(x) = \cos x - xe^x$

Let the initial guess be 0 and 1

$\because \quad f(0) = 1 > 0$

and, $f(1) = -2.177 < 0$

Therefore, the root lie in between 0 and 1, but according to the step 5 of secant algorithm $a = 1$ and $b = 0$.

n	a	b	f(a)	f(b)	x_n
1	1	0	−2.177	1	0.314762
2	0	0.314762	1	0.519667	0.655299
3	0.314762	0.655299	0.519667	−0.469055	0.49375
4	0.655299	0.49375	−0.469055	0.071577	0.515138
5	0.49375	0.515138	0.071577	0.007951	0.517810
6	0.515138	0.517810	0.007951	−0.000160	0.51776

Since, the value of x_n is same up to 3 decimal place in 5th and 6th iteration, so $f(x) = \cos x - xe^x$ has root 0.51776.

Problem 15

Write an algorithm to find real root of a non-linear equation using secant method. [T.U., 2070 Ashadh, 2072 Kartik]

Solution:

See Secant method algorithm on page no. 21

Problem 16

Find an approximate root of $x\log_{10} x - 1.2 = 0$ using secant method up to 3 decimal places of accuracy. [T.U., 2067 Ashadh]

Solution:

$f(x) = x\log_{10} x - 1.2 = 0$

Let initial guess be: 2 and 3

$\therefore \quad f(2) = -0.59794 < 0$

$f(3) = 0.2313 > 0$

$\therefore$ Root lies between 2 and 3.

Now,

According to secant method,

$$x_n = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

n	a	b	f (a)	f (b)	x_n
1	2	3	− 0.59794	0.2313	2.7210
2	3	2.7210	5.85888	0.02917	2.7145
3	2.7210	2.7145	0.132812	-6.4011×10^{-4}	2.1366
4	2.7145	2.71366	0.01588	-5.316×10^{-6}	2.71366

$\therefore$ Root = 2.71366

Problem 17

Find a real root of following equation correct to three decimals using secant method, $e^{\cos x} - \sin(x)$. [T.U., 2069 Chaitra]

Solution:

$f(x) = e^{\cos x} - \sin(x)$

Let initial guess be 1 and 2.

$\therefore \quad f(1) = 0.87505 > 0$

$f(2) = -0.2437 < 0$

$\therefore$ Root lies between 1 and 2.

By secant method,

$$x_n = \frac{a\,f(b) - b\,f(a)}{b - a}$$

n	a	b	f (a)	f (b)	x_n
1	1	2	0.87505	− 0.2437	1.77798
2	2	1.77798	− 0.24971	− 0.16454	1.34907

3	1.7798	1.34907	− 0.16454	0.27045	1.61573
4	1.34907	1.61573	0.27045	− 0.0429	1.57921
5	1.61573	1.57921	− 0.0429	-8.342×10^{-3}	1.57039
6	1.57921	1.57039	-8.342×10^{-3}	4.0649×10^{-4}	1.57079

∴ Root = 1.57079

Problem 18

How do we obtain a real root of non-linear equation using secant method? Explain graphically and hence deduce the iteration formula. [T.U., 2070 Chaitra]

Solution:

See Secant method on page no. 21

iv) NEWTON-RAPHSON METHOD

Consider the graph below of f(x). Let us assume that x_1 is an approximate root of $f(x) = 0$. Draw a tangent at the curve f(x) at $x = x_1$ as shown in figure. The point of intersection of this tangent with the x-axis gives the second approximation to the root. Let the point of intersection be x_2. The slope of the tangent is given by;

$$\tan\alpha = \frac{f(x_1)}{x_1 - x_2} = f'(x_1)$$

where, $f(x_1)$ is the slope of $x = x_1$.

Solving for x_2; we obtain,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

This is called the Newton Raphson method.

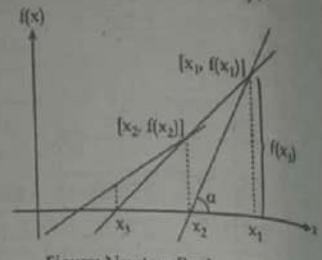


Figure: Newton-Raphson method

In general $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

This method of successive approximation is called Newton-Raphson method. The process is terminated when the difference between two successive value is within a prescribed limit.

Algorithm of Newton Raphson Method

1. Assign an initial value to 'x' say x_0.
2. Evaluate $f(x_0)$ and $f'(x_0)$.
3. Find the improved estimate of x_0

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

4. Check for accuracy of the latest estimate.
5. Compare relative error to a predefined value E.

If $\left|\frac{x_1 - x_0}{x_1}\right| \leq E$ stop;

Otherwise continue.

6. Replace x_0 by x_1 and repeat 3, 4 and 5.

SOLVED NUMERICALS

Problem 1

Find the roots of the following equation correct to three decimal places by Newton-Raphson Method. $x^3 + x^2 + x + 7 = 0$ [T.U. 2061 Baishakh]

Solution:

$f(x) = x^3 + x^2 + x + 7 = 0$

$f'(x) = 3x^2 + 2x + 1$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	−2.5	−4.875	14.75	−2.1694
2	−2.1694	−0.67294	107800	−2.10697
3	−2.10697	−0.021166	10.10402	−2.10487
4	−2.10487	2.8084×10^{5}	10.0816	−2.10487

∴ Root = −2.10487

Problem 2

Find a real root of $3x - e^x = 0$ using Newton Raphson method to three decimal place. [T.U. 2062 Baishakh]

Solution:

$f(x) = 3x - e^{-x}$

$f'(x) = 3 + e^{-x}$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	0.89346	3.6065	0.25226
2	0.25226	−0.02026	3.77704	0.257624
3	0.257624	−1.37825	3.77288	0.25762

∴ Root = 0.25762

Problem 3

Find out at least one root of the equation $x \cdot e^x - 2 = 0$ correct up to 3 decimal place using Newton-Raphson method. [T.U. 2063 Jestha]

Solution:

$f(x) = x \times e^x - 2$

$f'(x) = x \times e^x + e^x$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	−1.17563	2.473081	0.97537
2	0.97537	0.58682	5.238974	0.863358
3	0.863358	0.047116	4.41822	0.852693
4	0.852693	3.812700×10^{-4}	4.34633	0.85260

∴ Root = 0.85260

Problem 4

Calculate a real root of non-linear equation. $x\sin x + \cos x = 0$ using Newton Raphson Method. The absolute error of functional value to our calculator must should be less than 10^{-4}. [T.U. 2066 Bhadra, P.U. 2009 Spring and 2010 Fall]

Solution:

$f(x) = x\sin x + \cos x$

$f'(x) = x\cos x$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	2.5	−0.695036	−2.0028	2.847022
2	2.847022	−0.13035	−2.72439	2.7991750
3	2.7991750	-2.0796×10^{-3}	−2.636670	2.7983862
4	-2.0796×10^{-3}	−2.636670		2.7983862

∴ Root = 2.7983

Problem 5

Find a root of $e^x = 3x$ using Newton Raphson method correct up to 3 decimal places. [T.U. 2069 Bhadra]

Solution:

$f(x) = e^x - 3x$

$f'(x) = e^x - 3$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	0.14872	−1.351278	0.61005
2	0.61005	0.010373	−1.15947	0.61899
3	0.61273	8.1472×10^{-3}	−1.14294	0.61273
4	0.61273	3.5755×10^{-3}	−1.14287	0.61903
5	0.61903	3.575×10^{-5}	−1.14287	0.61906

∴ Root = 0.61906

Problem 6

Find the reciprocal of 3 using Newton Raphson method. [T.U. 2068 Baishakh]

Solution:

$x = \frac{1}{3}$

$f(x) = x - \frac{1}{3}$

$f'(x) = 1$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	0.1666	1	0.33333
2	0.3333	-3.333×10^{-5}	1	0.3333
3	0.3333	-3.333×10^{-5}	1	0.3333

∴ Root = 0.3333

Problem 7

Find a real root of the equation $x = e^{-x}$ using Newton Raphson Method to six decimal place. [P.U. 2005 Fall]

Solution:

$f(x) = x - e^{-x}$

$f'(x) = 1 + e^{-x}$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	2.5	2.41791	13.182	2.3165
2	2.3165	2.21788	11.14012	2.11741
3	2.11741	1.997067	9.3095	1.90289
4	1.90289	1.75375	7.7052	1.67528
5	1.67528	1.488024	6.34029	1.44058
6	1.44058	1.20378	5.22314	1.21010
7	1.21010	0.91193	4.35382	1.00064
8	1.00064	0.63299	3.72002	0.83048
9	0.83048	0.394639	3.29441	0.71068
10	0.71068	0.21937	3.03537	0.63840
11	0.63840	0.11026	2.03487	0.566311
12	0.566311	−1.19544	1.5676	0.567143
13	0.567143	−1.1960	1.5671	0.567143

∴ Root = 0.567143

Problem 8

Find the real root of the function $\tan x + \tan hx = 0$ by Newton Raphson method with initial value 2.2. [P.U. 2005 Spring]

Solution:

$f(x) = \tan x + \tan hx$

$f'(x) = \sec^2 x + \sec h^2 x$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	2.2	−0.39807	2.93531	2.33561
2	2.33561	−0.06058	2.12260	2.36415
3	2.36415	−1.742235	2.0034	2.36501
4	2.36501	-2.07450×10^{-5}	2.0000	2.3650

∴ Root = 2.3650

Problem 9

Using Newton-Raphson method. Solve $f(x) = e^{x-1} - 5x^3$ correct up to four decimal place. [P.U. 2006 Fall and 2011 Spring]

Solution:

$f(x) = e^{x-1} - 5x^3$

$f'(x) = e^{x-1} - 15x^2$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	0.01846	−3.1434	0.49412
2	0.49142	7.974×10^{-3}	−3.02105	0.49405
3	0.49405	-1.9448×10^{-5}	−3.05834	0.49404

∴ Root = 0.49404

Problem 10

Find the real root of the equation $x^3 - 4x + 1 = 0$ that lie in between 1 and 4, correct to three decimal place by using the Newton-Raphson method. [P.U. 2006 Spring]

Solution:

$f(x) = x^3 - 4x + 1$

$f'(x) = 3x^2 - 4$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	2.5	6.625	14.75	2.0508
2	2.0508	1.42201	8.617	1.88580
3	1.88580	0.16316	6.66872	1.86133
4	1.86133	3.349×10^{-3}	6.39364	1.86080
5	1.86080	-3.7388×10^{-5}	6.38772	1.86080

∴ Root = 1.86080

Problem 11

Find out all the possible real roots of the equation $x^3 - 2x - 5 = 0$ using Newton Raphson method to 3 decimal places. [P.U. 2007 Spring]

Solution:

$f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	2.5	5.625	16.75	2.1642
2	2.1642	0.80819	12.0512	2.0971
3	2.0971	0.02848	11.1934	2.0945
4	2.0945	-5.7459×10^{-4}	11.1607	2.0945

∴ Root = 2.0945

Problem 12

Find the real root of $x \log x = 1.2$, using Newton-Raphson method. [P.U. 2008 Fall and 2008 Spring]

Solution:

$f(x) = x \log x - 1.2$

$f'(x) = \log x + \log e$ $\left[\because \frac{d}{dx}\log x = \frac{\log_x e}{x}\right]$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	−1.35051	0.13326	2.74650
2	2.74650	5.1081×10^{-3}	0.87307	2.740649
3	2.740649	2.5327×10^{-6}	0.87214	2.740646

∴ Root = 2.740646

Problem 13

Find the root of equation $x^3 - 3x + 2 = 0$ correct to four decimal place using Newton Raphson method. [P.U. 2009 Fall]

Solution:

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	0.625	2.75	0.272727
2	0.272727	1.20210	2.22314	−0.267996
3	−0.267996	2.22314	2.21546	−1.52495
4	−1.52495	3.0286	8.97641	−1.86234
5	−1.86234	1.12784	12.4049	−1.95325
6	−1.95325	0.4077	13.445	−1.98357
7	−1.98357	0.14625	13.8036	−1.9816

∴ Root = 0.60710

Problem 14

Find the root of the equation $f(x) = 3x - \cos x - 1$ using Newton Raphson method, correct up to three decimal place. [P.U. 2012 Fall]

Solution:

$f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	−0.377758	3.47942	0.60851
2	0.60851	5.029×10^{-3}	3.571645	0.60710
3	0.60710	-5.884×10^{-6}	3.57048	0.60710

$\therefore$ Root = 0.60710

Problem 15

Find the cube root of 30, correct up to 3 decimal places using Newton-Raphson method. [T.U., 2070 Ashadh]

Solution:

Let, $f(x) = x^3 - 30$

$f'(x) = 3x^2$

Let the initial guess be 2.5

So, from Newton-Raphson method;

$$x_n = x_i - \frac{f(x_i)}{f'(x_i)}$$

n	x_i	$f(x_i)$	$f'(x_i)$	$x_n = x_i - \frac{f(x_i)}{f'(x_i)}$
1	2.5	14.375	18.75	3.2667
2	3.2667	4.8600	32.0139	3.11489
3	3.11489	0.22234	29.1076	3.10725
4	3.10725	5.0671×10^{-4}	28.9650	3.10723

$\therefore$ Root = 3.107

Problem 16

Derive iterative formula for Newton Raphson method using Taylor series. [T.U., 2068 Chaitra]

Solution:

Let, x_0 be the approximate root of $f(x) = 0$

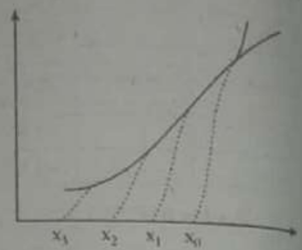
and, $x_1 = x_0 + h$ be correct root so that;

$f(x_1) = 0_1$;

where, h is a small interval.

i.e., $h = (x_1 - x_0)$

We know that;



Taylor series is expanded as;

$$f(x+h) = f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 + \dots\dots\dots + \frac{f^n(x)}{n!}h^n \quad (1)$$

$\because$ h is so small, higher power of h is also small,

Thus, neglecting higher power of x; we get;

$f(x+h) = f(x) + f'(x)h = 0$

or, $h = \frac{-f(x)}{f'(x)}$

or, $(x_1 - x_0) = \frac{-f(x_0)}{f'(x_0)}$ at $x = x_0$

$\therefore$ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Similarly;

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

Thus,

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

which is the required iterative formula for Newton Raphson method.

Problem 17

Find the positive real root of $\sin(x) + \cos(x) + e^x - 8 = 0$ correct up to 4 decimal place using Newton-Raphson method. [T.U., 2070 Chaitra]

Solution:

$f(x) = \sin x + \cos x + e^x - 8$

$f'(x) = \cos x - \sin x + e^x$

Let the initial guess be 1.5.

By, Newton-Raphson method;

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

n	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	1.5	−2.45007	3.55493	2.18920
2	2.18920	1.16363	7.53352	2.03480
3	2.03480	0.09745	6.30892	2.01935
4	2.01935	8.3913×10^{-4}	6.19868	2.01921
5	2.01921	6.717×10^{-8}	6.197699	2.01921

$\therefore$ Root = 2.01921

Problem 18

Derive analytically the iteration formula for Newton-Raphson method to find a real root of non-linear equation. [T.U., 2071 Shrawan]

Solution:

Refer page no. 30, Newton-Raphson method

Problem 19

Find a positive real root of the equation $xe^x + \sin x = 0.5$ with an accuracy of 6 decimal places using Newton-Raphson Method. [T.U., 2072 Kartik]

Solution:

$f(x) = xe^x + \sin x - 0.5$

$f'(x) = xe^x + e^x + \cos x$

Let the initial guess be 0.5.

By Newton-Raphson method;

$$x_n = x_i - \frac{f(x_i)}{f'(x_i)}$$

n	x_i	$f(x_i)$	$f'(x_i)$	$x_n = x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	0.803786	3.350664	0.260111
2	0.260111	0.094571	2.600818	0.223748
3	0.223748	1.7398×10^{-3}	2.505682	0.223053
4	0.223053	6.186×10^{-7}	2.5039044	0.223053

∴ Root = 0.223053

LIMITATION OF NEWTON RAPHSON METHOD

The Newton-Raphson method has certain limitation and pit falls. The method will fail in the following situations.

i) Division by zero may occur if $f'(x_1)$ is zero or very close to zero.

ii) If the initial guess is too far away from the required root, the process may converge to some other root.

iii) A particular value in the iteration sequence may repeat, resulting in an infinite loop. This occurs when the tangent to the curve $f(x)$ at $x = x_{i+1}$ cuts the x-axis again at $x = x_i$.

CONVERGENCE OF NEWTON-RAPHSON METHOD

Let x_n be an estimate of the root of the function f(x). If x_n and x_{n+1} are close to each other, then using Taylor's series expansion, we can state,

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(R)}{2}(x_{n+1} - x_n)^2 \quad (1)$$

where 'R' lie somewhere in the interval x_n to x_{n+1} and third and higher order have been dropped.

Let us assume that the exact root of f(x) is x_r. Then, $x_{n+1} = x_r$.

Therefore, $f(x_{n+1}) = 0$ and substituting then value in (1); we get,

$$0 = f(x_n) + f'(x_n)(x_r - x_n) + \frac{f''(R)}{2}(x_r - x_n)^2 \quad (2)$$

We know that the Newton's iterative formula is given by;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Rearranging the terms; we get,

$$f(x_n) = f'(x_n)(x_n - x_{n+1})$$

Substituting this for $f(x_n)$ in equation (2) yields;

$$0 = f'(x_n)(x_r - x_{n+1}) + \frac{f''(R)}{2}(x_r - x_n)^2 \quad (3)$$

We know the error in the estimate x_{n+1} is given by;

$$e_{n+1} = x_r - x_{n+1}$$

Similarly,

$$e_{n+1} = x_r - x_n$$

Now equation (3) can be expressed in terms of these errors as;

$$0 = f'(x_n)e_{n+1} + \frac{f''(R)}{2}e_n^2$$

Rearranging the terms; we get,

$$e_{n+1} = -\frac{f''(R)}{2f'(x_n)}e_n^2 \quad (4)$$

This equation (4) shows that the error is roughly propogational to the square of the error in the previous iteration. Therefore, the Newton-Raphson method is said to have quadratic convergence.

v) FIXED POINT ITERATION METHOD

Any function in the form of

$$f(x) = 0; \quad (1)$$

can be manipulated such that 'x' is on the left hand side of the equation

$$x = g(x) \quad (2)$$

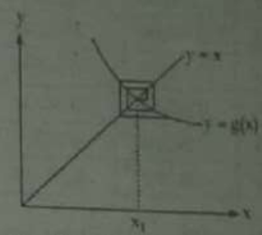


Figure: Illustration of fixed point method

Equation (1) and (2) are equivalent and therefore a root of equation (2) is root of equation (1). The root of equation (2) is given the point of interaction of the curve y = x and y = g(x). This intersection point is known as the fixed point of g(x).

The equation x = g(x) is known as the fixed point equation. It provides a convenient from for predicting the value of 'x' as a function of 'x'. If x_0 is the initial guess to a root, the next approximation is given by;

$$x_1 = g(x_0)$$

Further approximation is given by;

$$x_2 = g(x_1)$$

Thus iteration process can be expressed in general form as;

$$x_{i+1} = g(x_i)$$

where, i = 0, 1, 2, 3, ...; which is fixed point iteration formula.

This method of solution is known as the method of successive approximations or method of direct substitution. The iteration terminates if two successive approximations aggress within some specified error.

SOLVED NUMERICALS

Problem 1

Find a real root of the following equation, correct to three decimal place using the fixed point iteration method $\sin x + 3x - 2 = 0$. [T.U. 2070 Magh]

Solution:

Given that;

$$f(x) = \sin x + 3x - 2 \quad (1)$$

Now for fixed point iteration method, arranging equation (1) in terms of ∅

or, $3x = -\sin x + 2$

or, $x = -\frac{\sin x + 2}{3}$

$$\emptyset(x) = -\frac{\sin x + 2}{3}$$

By iteration formula; $x_{n+1} = \emptyset(x) = -\frac{\sin x + 2}{3}$

Calculating the root of equation in tabular form

n	x_n	x_{n+1}
1	0	0.6667
2	0.6667	0.4605
3	0.4605	0.5185
4	0.5185	0.50147
5	0.50147	0.50643
6	0.50643	0.50498
7	0.50498	0.50540
8	0.50540	0.50554
9	0.50554	0.50524
10	0.50524	0.50532
11	0.50532	0.50530

Since, in 10th and 11th iteration x_{n+1} value is same up to 3 decimal place so the root is 0.5053 for $f(x) = \sin x + 3x - 2$.

Problem 2

Find a real root of the equation $x^3 + x^2 - 1 = 0$ by the fixed point iteration method, correct up to six decimal place. [T.U. 2066 Bhadra]

Solution:

Given that;

$$f(x) = x^3 + x^2 - 1 \quad (1)$$

For fixed point iteration method, arranging equation (1) in terms of $\emptyset(x)$

$x^3 + x^2 - 1 = 0$

or, $x^3 + x^2 = 1$

or, $x^2(x + 1) = 1$

or, $x^2 = \frac{1}{x+1}$

or, $x = \sqrt{\frac{1}{x+1}}$

$$\emptyset(x) = \sqrt{\frac{1}{x_{n+1}}}$$

Now, calculating the root of equation in tabular form

n	x_n	x_{n+1}
1	0	1
2	1	0.707106
3	0.707106	0.765365
4	0.765365	0.7526316
5	0.7526316	0.755361
6	0.755361	0.754773
7	0.754773	0.754900
8	0.754900	0.754873
9	0.754873	0.754878
10	0.754878	0.754878

Since, the value of x_{n+1} in 9th and 10th iteration has similar value up to 6 decimal place so the root of equation $f(x) = x^3 + x^2 - 1$ is 0.754878.

Problem 3

Use fixed point iteration method to evaluate root of the equation $x^2 - x - 1 = 0$ correct up to 3 decimal place. [P.U. 2006 Spring]

Solution:

Given that;

$$f(x) = x^2 - x - 1 \quad (1)$$

For fixed point iteration method, arranging equation (1) in terms of $\emptyset(x)$

$x^2 - x - 1 = 0$

or, $x(x - 1) = 1$

or, $x = \frac{1}{x-1}$

or, $\emptyset(x) = \frac{1}{x-1}$

$$x_{n+1} = \emptyset(x) = \frac{1}{x_{n-1}}$$

Now calculating the root using tabular form

n	x_n	x_{n+1}
1	0	−1
2	−1	−0.5
3	−0.5	−0.6667
4	−0.6667	−0.59998
5	−0.59998	−0.62500
6	−0.62500	−0.61538
7	−0.61538	−0.61904
8	−0.61904	−0.61764
9	−0.61764	−0.61818
10	−0.61818	−0.6179
11	−0.6179	−0.61808
12	−0.61808	−0.61802

Since, the value of x_{n+1} in 11^{th} and 12^{th} iteration has same value up to 3 decimal place so $f(x) = x^2 - x - 1$ has root -0.61802

Problem 4

Find one root of equation $x^2 - 2x - 3 = 0$ using fixed point iteration method. [P.U. 2007 Fall]

Solution:

Given that;

$$f(x) = x^2 - 2x - 3 \quad (1)$$

For fixed point iteration method, arranging equation (1) in terms of $\emptyset(x) = x^2 - 2x - 3$

or, $x(x-2) = 3$

or, $x = \frac{1}{x-2}$

$$\emptyset(x_n) = \frac{1}{x_{n-2}}$$

Calculating the other value of root in tabular form

n	x_n	x_{n+1}
1	0	−1.5
2	−1.5	−0.85714
3	−0.85714	−1.05000
4	−1.05000	−0.983606
5	−0.983606	−1.00549
6	−1.00549	−0.99817
7	−0.99817	−1.0006
8	−1.0006	−0.99980
9	−0.99980	−1.00006
10	−1.00006	−0.99998
11	−0.99998	−1.00000
12	−1.00000	−1

Hence, root of equation $f(x) = x^2 - 2x - 3$ is -1.

Problem 5

Find root of equation $f(x) = x^2 - 3$ using fixed point iteration method. [P.U. 2010 Fall]

Solution:

Here,

$$f(x) = x^2 - 3 \quad (1)$$

For fixed point iteration method, arranging equation (1) in terms of $\emptyset(x) = x$

$x^2 - 3 = 0$

or, $x^2 = 3$

or, $x = \frac{3}{x}$

or, $x + x = \frac{3}{x} + x$

or $2x = \frac{3 + x^2}{x}$

or, $x = \frac{3 + x^2}{2x}$

$$\emptyset(x_n) = \frac{3 + x_n^2}{2x_n}$$

Let us calculate root by tabular form

n	x_n	x_{n+1}
1	0	∞
2	1	2
3	2	1.75
4	1.75	1.73214
5	1.73214	1.73205
6	1.73205	1.73205

Hence, root is 1.73205 for $f(x) = x^2 - 3$.

Problem 6

Find the square root of 17. Using fixed point iteration method. [P.U. 2009 Fall]

Solution:

Given that;

$x^2 = 17$

$f(x) = x^2 - 17$

or, $x = \frac{17}{x}$

or, $x + x = \frac{17}{x} + x$

or $2x = \frac{17 + x^2}{x}$

or, $x = \frac{17 + x^2}{2x}$

$$\emptyset(x_n) = \frac{17 + x_n^2}{2x_n}$$

Let calculate root by tabular form

n	x_n	x_{n+1}
1	1	9
2	9	5.444
3	5.444	4.28335
4	4.28335	4.1261
5	4.1261	4.1231
6	4.1231	4.1231

Since, the 5^{th} and 6^{th} iteration has similar value of x_{n+1} up to 4 decimal place. So, $f(x) = x^2 - 17$ has root 4.1231.

Chapter 3

Interpolation and Approximation

INTRODUCTION

The computing of value for a tabulated function at a point is called interpolation. Suppose we are given the following value of $y = f(x)$ for a set of values of 'x'.

x	x_0	x_1	x_2	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_n)$

Then the process of finding the value of 'y' corresponding to any value of 'x' is called interpolation. Thus, interpolation is the technique to estimate the value of a function for any intermediate value of the independent variable.

i) INTERPOLATION WITH UNEQUAL INTERVALS

There are basically two methods to find the interpolation with unequal intervals.

a) Lagrange's interpolation method

b) Newton's divided difference interpolation method

a) Lagrange's Interpolation Method

Suppose $y = f(x)$ be a function with $f(x_0), f(x_1), f(x_2)\ldots\ldots f(x_n)$ corresponding to the values $x_0, x_1, x_2\ldots\ldots x_n$, then;

$$y_0 = f(x_0), y_1 = f(x_1) \ldots\ldots y_n = f(x_n)$$

The Lagrange's interpolation formula is given by;

$$y = f(x)$$

$$= \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} \times y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} \times y_1 \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} \times y_n$$

b) Newton's Divided Difference Interpolation Method

If $f(x_1), f(x_2), f(x_3) \ldots\ldots$ be the value of $y = f(x)$ corresponding to $x_1, x_2, x_3 \ldots$ Then, Newton divided difference interpolation is given by;

$$y = y_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots (x - x_n)f[x_0, x_1, \ldots\ldots x_n]$$

Now, we can construct the divided difference table as follows;

x	f	1st divided difference Δf	2nd divided difference $\Delta^2 f$	3rd divided difference $\Delta^3 f$	4th divided difference $\Delta^4 f$
x_0	f_0				
		$\frac{f_1 - f_0}{x_1 - x_0} = \Delta f_0$			
x_1	f_1		$\frac{\Delta f_1 - \Delta f_0}{x_2 - x_0} = \Delta^2 f_0$		
		$\frac{f_2 - f_1}{x_2 - x_1} = \Delta f_1$		$\frac{\Delta^2 f_1 - \Delta^2 f_0}{x_3 - x_0} = \Delta^3 f_0$	
x_2	f_2		$\frac{\Delta f_2 - \Delta f_1}{x_3 - x_1} = \Delta^2 f_1$		$\frac{\Delta^3 f_1 - \Delta^3 f_0}{x_4 - x_0}$
		$\frac{f_3 - f_2}{x_3 - x_2} = \Delta f_2$		$\frac{\Delta^2 f_2 - \Delta^2 f_1}{x_4 - x_1} = \Delta^3 f_1$	
x_3	f_3		$\frac{\Delta f_3 - \Delta f_2}{x_4 - x_2} = \Delta^2 f_2$		
		$\frac{f_4 - f_3}{x_4 - x_3} = \Delta f_3$			
x_4	f_4				

ii) INTERPOLATION WITH EQUAL INTERVALS

There are two methods to find the interpolation with equal intervals.

a) Newton forward interpolation method/Newton-Gregory forward interpolation

b) Newton backward interpolation method/Newton-Gregory backward interpolation

a) Newton Forward Interpolation Method

We know that, Newton's forward interpolation formula as;

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0 + \ldots\ldots$$

where, $p = \frac{x_p - x_0}{h}$

x_p = Value at which interpolation is to be found

x_0 = initial value relative to x_p.

h = interval of 'x'.

b) Newton Backward Interpolation Method

We know that, Newton's backward interpolation formula as;

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{(p+1)(p+2)p}{3!}\nabla^3 y_n + \frac{(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \ldots\ldots$$

where, $p = \frac{x_p - x_n}{h}$

x_p = Value at which interpolation is to be found.

x_n = final value relative to x_p.

h = interval 'x'.

The difference table (Forward)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0		forward	
		$\Delta y_0 = y_1 - y_0$		
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y = \Delta^2 y_1 - \Delta^2 y_0$
		$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
x_2	y_2			
		$\Delta y_2 = y_3 - y_2$		
x_3	y_3			

The difference table (Backward)

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$\nabla y_0 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_0 = \nabla y_1 - \nabla y_0$	$\nabla^3 y = \nabla^2 y_1 - \nabla^2 y_0$
		$\nabla y_1 = y_2 - y_1$		
x_2	y_2		$\nabla^2 y_1 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_3 - y_2$		
x_3	y_3		backward	

SOLVED NUMERICALS

i) LAGRANG'S INTERPOLATION

Problem 1

Use Lagrange method to find f(2.5) from following data. [T.U. 2069 Bhadra]

x	1	2	4	5	7
f(x)	1	1.414	1.732	2.00	2.6

Solution:

Here,

$x_0 = 1$	$y_0 = 1 = f(4)$
$x_1 = 2$	$y_1 = 1.414$
$x_2 = 4$	$y_2 = 1.732$
$x_3 = 5$	$y_3 = 2.00$
$x_2 = 7$	$y_2 = 2.6$

By Lagrange's formula, we have,

$$y = f(x)$$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \times y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \times y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \times y_4$$

$$y = f(x)$$

$$= \frac{(x-2)(x-4)(x-5)(x-7)}{(1-2)(1-4)(1-5)(1-7)} \times 1 + \frac{(x-1)(x-4)(x-5)(x-7)}{(2-1)(2-4)(2-5)(2-7)} \times 1.414$$

$$+ \frac{(x-1)(x-2)(x-5)(x-7)}{(4-1)(4-2)(4-5)(4-7)} \times 1.732 + \frac{(x-1)(x-2)(x-4)(x-7)}{(5-1)(5-2)(5-4)(5-7)} \times 2$$

$$+ \frac{(x-1)(x-2)(x-4)(x-5)}{(7-1)(7-2)(7-4)(7-5)} \times 2.6$$

$$y = f(2.5)$$

$$= \frac{(2.5-2)(2.5-4)(2.5-5)(2.5-7)}{(1-2)(1-4)(1-5)(1-7)} \times 1$$

$$+ \frac{(2.5-1)(2.5-4)(2.5-5)(2.5-7)}{(2-1)(2-4)(2-5)(2-7)} \times 1.414 +$$

$$\frac{(2.5-1)(2.5-2)(2.5-5)(2.5-7)}{(4-1)(4-2)(4-5)(4-7)} \times 1.732$$

$$+ \frac{(2.5-1)(2.5-2)(2.5-4)(2.5-7)}{(5-1)(5-2)(5-4)(5-7)} \times 2$$

$$+ \frac{(2.5-1)(2.5-2)(2.5-4)(2.5-5)}{(7-1)(7-2)(7-4)(7-5)} \times 2.6$$

$$y = f(2.5) = -0.117188 + 1.193063 + 0.811875 - 0.421875 + 0.040625$$

$$y = f(2.5) = 1.5065$$

Problem 2

Use Lagrange interpolation and estimate the square of 3.25 if:

x	1	2	3	4	5
x^2	1	4	9	16	25

[T.U. 2066 Jestha]

Solution:

Here,

$x_0 = 1$	$y_0 = 1$
$x_1 = 2$	$y_1 = 4$
$x_2 = 3$	$y_2 = 9$
$x_3 = 4$	$y_3 = 16$
$x_2 = 5$	$y_2 = 25$

$$y = f(3.25)$$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}\times y_2+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}\times y_3$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}\times y_4$$

$$y=f(3.25)$$

$$=\frac{(3.25-2)(3.25-3)(3.25-4)(3.25-5)}{(1-2)(1-3)(1-4)(1-5)}\times 1$$

$$+\frac{(3.25-1)(3.25-3)(3.25-4)(3.25-5)}{(2-1)(2-3)(2-4)(2-5)}\times 4$$

$$+\frac{(3.25-1)(3.25-2)(3.25-4)(3.25-5)}{(3-1)(3-2)(3-4)(3-5)}\times 9$$

$$+\frac{(3.25-1)(3.25-2)(3.25-3)(3.25-5)}{(4-1)(4-2)(4-3)(4-5)}\times 16$$

$$+\frac{(3.25-1)(3.25-2)(3.25-3)(3.25-4)}{(5-1)(5-2)(5-4)(5-7)}\times 25$$

$$=0.17089-0.492188+8.30566+3.28125-0.54932$$

$$=10.5625$$

Problem 3

Find the missing value of collected water level using Lagrange's Interpolation

Time duration of rain fall (t) min	1	3	6.5	10
Collected water level (h) mm	23	61	?	203

[T.U. 2065 Shrawan]

Solution:

Here,

$x_0 = 1$

$x_1 = 3$

$x_2 = 10$

$y_0 = 23$

$y_1 = 61$

$y_2 = 203$

Now according to Lagrange interpolation; we have,

$$y=f(x)$$

$$=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\times y_0+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\times y_1+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\times y_2$$

$$y=f(6.5)$$

$$=\frac{(6.5-3)(6.5-10)}{(1-3)(1-10)}\times 1+\frac{(6.5-1)(6.5-10)}{(3-1)(3-10)}\times 61+\frac{(6.5-1)(6.5-3)}{(10-1)(10-3)}\times 203$$

$$=-0.68055+83.875+62.0277$$

$$y=f(6.5)=145.217$$

Thus above table can be completed as below.

Time duration of rainfall (t) min	1	3	6.5	10
Collected water level (h) mm	23	61	145.217	203

Problem 4

Find out the missing value in the following set of data using Lagrange interpolation method.

[T.U. 2063 Jestha]

x	−2	−1	0	1	6
f(x)	−20	?	2	?	70

Solution:

Here,

$x_0 = -2$

$x_1 = 0$

$x_2 = 6$

$y_0 = -20$

$y_1 = 2$

$y_2 = 70$

Now according to Lagrange interpolation; we have,

$$y=f(x)$$

$$=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\times y_0+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\times y_1+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\times y_2$$

$$y=f(-1)$$

$$=\frac{(-1+2)(-1-6)}{(-2-0)(-2-6)}\times -20+\frac{(-1+2)(-1-6)}{(0+2)(0-6)}\times 2+\frac{(-1+2)(-1-0)}{(6+2)(6-0)}\times 70$$

$$y=f(-1)=8.459$$

$$y=f(1)=\frac{(1-0)(1-0)}{(-2-0)(-2-6)}\times -20+\frac{(1+2)(1-6)}{(0+2)(0-6)}\times 2+\frac{(1+2)(1-0)}{(6+2)(6-0)}\times 70$$

$$y=f(1)=3.125$$

So the complete table is;

x	−2	−1	0	1	6
f(x)	−20	8.459	2	3.125	70

Problem 5

Use Lagrange's interpolation formula find the value of f(1.3) if

x	1	2	3	4
y	4.28	5.79	2.18	4.13

[T.U. 2062 Baishakh]

Solution:

Here,

$x_0 = 1$

$x_1 = 2$

$x_2 = 3$

$x_3 = 4$

$y_0 = 4.28$

$y_1 = 5.79$

$y_2 = 2.18$

$y_3 = 4.13$

Now according to Lagrange interpolation; we have,

$$y = f(x)$$
$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$y = f(1.3)$$
$$= \frac{(1.3-2)(1.3-3)(1.3-4)}{(1-2)(1-3)(1-4)} \times 4.28 + \frac{(1.3-1)(1.3-3)(1.3-4)}{(2-1)(2-3)(2-4)} \times 5.79 + \frac{(1.3-1)(1.3-2)(1.3-4)}{(3-1)(3-2)(3-4)} \times 2.18 + \frac{(1.3-1)(1.3-2)(1.3-3)}{(4-1)(4-2)(4-3)} \times 4.13$$

$$y = f(1.3) = 2.292 + 3.986415 - (0.61803) + 0.245735$$
$$y = f(1.3) = 5.90612$$

Problem 6

Find the polynomial using Lagrange's method also calculate absolute and relative error at $x = 2.7$. [P.U. 2009 Fall]

x	3.2	2.7	1.0	4.8
f(x)	22.0	17.8	14.2	38.3

Solution:

Here,

$x_0 = 3.2$
$x_1 = 1.0$
$x_2 = 4.8$
$y_0 = 22.0$
$y_1 = 14.8$
$y_2 = 38.3$

Here, the value of 'x' at which corresponding 'y' is to be calculated is taken according to Lagrange's interpolation.

$$y = f(x)$$
$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

Now, at $x = 2.7$

$$y = f(2.7)$$
$$= \frac{(2.7-1)(2.7-4.8)}{(3.2-1)(3.2-4.8)} \times 22 + \frac{(2.7-3.2)(2.7-4.8)}{(1-3.2)(1-4.8)} \times 14.8 + \frac{(2.7-3.2)(2.7-1)}{(4.8-3.2)(4.8-1)} \times 38.2$$
$$y = f(2.7) = 22.3125 + 1.7835 + 5.3405$$
$$y = f(2.7) = 18.7555$$

But from table; we have,

$$y = f(x) = 17.8$$

so Absolute error $= 18.7555 - 17.8 = 0.9555$

$$\text{Relative error} = \frac{18.7555 - 17.8}{18.7555} = 0.05095$$

Problem 7

Write an algorithm for Lagrange's interpolation method to interpolate any function value of 'x' from given set of data points.

Solution:

Algorithm of Lagrange's interpolation

i) Let $x_0, x_1, x_2, \ldots, x_n$ denote 'n' distinct real numbers and $f_0, f_1, f_2, \ldots, f_n$ be corresponding value of f(x).

ii) For (n + 1) points: $(x_0, f_0), (x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n)$ can be data value connected by a curve.

iii) Let $P_n(x)$ be a polynomial
if $P_n(x) = f_K$, $K = 0, 1, \ldots, n$ where, $P_n(x_K)$ is interpolating function, then

iv) Consider 2[nd] order polynomial

$$P_2(x) = (x_0, f_0), (x_1, f_1), (x_2, f_2)$$
$$(x_0, f_0) \Rightarrow P_2(x_0) = b_2(x_0 - x_1)(x_0 - x_2) = f_0$$
$$\therefore \quad b_2 = \frac{f_0}{(x_0 - x_1)(x_0 - x_2)}$$
$$(x_1, f_1) \Rightarrow P_2(x_1) = b_3(x_1 - x_2)(x_1 - x_0) = f_1$$
$$\therefore \quad b_3 = \frac{f_1}{(x_1 - x_2)(x_1 - x_0)}$$
$$(x_2, f_2) \Rightarrow P_2(x_2) = b_1(x_2 - x_0)(x_2 - x_1) = f_2$$
$$\therefore \quad b_1 = \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}$$
$$P_2(x) = f_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$
$$P_2(x) = \textstyle\sum_{i=0}^{2} f_i \cdot l_i(x)$$
$$\text{where, } l_i(x) = \prod_{j=0 \;\; j \neq 1}^{2} \frac{(x - x_j)}{(x_i - x_j)}$$

v) For n[th] order

$$P_n(x) = \textstyle\sum_{i=0}^{n} f_i \cdot l_i(x)$$
$$\text{where, } l_i(x) = \prod_{j=0 \;\; j \neq 1}^{2} \frac{(x - x_j)}{(x_i - x_j)}$$

vi) End.

Problem 8

Use suitable interpolation formula to find the value of y(10) from the following data.

X	5	6	9	11
Y	12	13	14	16

Solution:

Here,

$x_0 = 5$
$x_1 = 6$
$x_2 = 9$
$x_3 = 11$
$y_0 = 12$
$y_1 = 13$
$y_2 = 14$
$y_3 = 16$

By Lagrange's interpolation; we have,

$$y = f(x)$$
$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$y = f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$y = f(10) = 2 - 4.3333 + 11.6666 + 5.3333$$
$$\therefore y = f(10) = 14.6666$$

Problem 9

The table given below shows the temperatures T (in degree census) and length L (in mm) of heated rod. Find the temperature at L = 800.8 mm, using Lagrange interpolation method **[P.U. 2006 Fall]**

T	30	40	50	60	70
L	800.4	800.6	800.7	800.9	801.0

Solution:

Here,

$x_0 = 30$
$x_1 = 40$
$x_2 = 50$
$x_3 = 60$
$x_4 = 70$
$y_0 = 800.4$
$y_1 = 800.6$
$y_2 = 800.7$
$y_3 = 800.9$
$y_4 = 801.0$
$x = f(y)$

$$= \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} \times x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)} \times x_1 + \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)} \times x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)} \times x_3 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)} \times x_4$$

At y = 800.8,

$$x = f(800.8)$$
$$= \frac{(800.8-800.6)(800.8-800.7)(800.8-800.9)(800.8-801.0)}{(800.4-800.6)(800.4-800.7)(800.4-800.9)(800.8-801.0)} \times 30 + \frac{(800.8-800.4)(800.8-800.7)(800.8-800.9)(800.8-801.0)}{(800.6-800.4)(800.6-800.7)(800.6-800.9)(800.6-801.0)} \times 40 + \frac{(800.8-800.4)(800.8-800.6)(800.8-800.9)(800.8-801.0)}{(800.7-800.4)(800.7-800.6)(800.7-800.9)(800.7-801.0)} \times 50 + \frac{(800.8-800.4)(800.8-800.6)(800.8-800.7)(800.8-801.0)}{(800.9-800.4)(800.9-800.6)(800.9-800.7)(800.9-801.0)} \times 60 + \frac{(800.8-800.4)(800.8-800.6)(800.8-800.7)(800.8-800.9)}{(801-800.4)(801-800.6)(801-800.7)(801-800.9)} \times 70$$

$$x = f(800.8) = 55.94$$

Problem 10

Using Lagrange interpolation formula, find value of f (1.3) from following data; **[T.U., 2068 Chaitra]**

X	1	3	4
Y	4.28	2.18	4.13

Solution:

Here,

$x_0 = 1$
$x_1 = 3$
$x_2 = 4$
$y_0 = 4.28$
$y_1 = 2.18$
$y_2 = 4.13$

From Lagrange's formula; we have,

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

Now,

$$f(1.3) = \left[\frac{(1.3-3)(1.3-4)}{(1-3)(1-4)} \times 4.28\right] + \left[\frac{(1.3-1)(1.3-4)}{(3-1)(3-4)} \times 2.18\right] + \left[\frac{(1.3-1)(1.3-3)}{(4-1)(4-3)} \times 4.13\right]$$
$$= \frac{(4.59 \times 4.28)}{6} + \frac{(-0.81 \times 2.18)}{(-2)} + \frac{(-0.51 \times 4.13)}{3}$$
$$= 3.2742 + 0.8829 - 0.7021$$

$= 3.455$

$\therefore \quad f(1.3) = 3.455$

Problem 11

Develop pseudocode to interpolate the given sets of data using Lagrange interpolation method. [T.U., 2070 Ashadh]

Solution:

i) Let $f = f(x)$ be the function.

ii) Let $y_0, y_1, y_2, \dots\dots, y_n$ be the corresponding to $x = x_0, x_1, x_2, \dots\dots, x_n$.

iii) Evaluate f(x) as,

$$f(x) = \left\{\frac{(x - x_1)(x - x_2)\dots\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots\dots(x_0 - x_n)} \times y_0\right\} + \left\{\frac{(x - x_0)(x - x_2)\dots\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots\dots(x_1 - x_n)} \times y_1\right\} + \dots\dots + \left\{\frac{(x - x_0)(x - x_1)\dots\dots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\dots\dots(x_n - x_{n-1})} \times y_n\right\}$$

ii) NEWTON DIVIDED DIFFERENCE METHOD

Problem 1

The following are the measurement of 't' (time) made on a curve recorded by an oscillograph a change in the condition of an electric current (I).

t (time)	1.2	2.0	2.5	3.0
I	1.36	0.58	0.34	0.20

Find the value of 'I' when t = 1.6 with appropriate Newton's Gregory Interpolation method. [T.U. 2067 Ashadh]

Solution:

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.2	1.36			
		$\frac{0.58 - 1.36}{2 - 1.2} = -0.975$		
2.0	0.58		$\frac{-0.48 + 0.975}{2.5 - 1.2} = 0.38077$	
		$\frac{0.34 - 0.58}{2.5 - 2.0} = -0.48$		$\frac{0.4 - 0.38077}{3.0 - 1.2} = 0.01068$
2.5	0.34		$\frac{-0.28 + 0.48}{3 - 2.0} = 0.4$	
		$\frac{0.20 - 0.34}{3 - 2.5} = -0.28$		
3.0	0.20			

$$y = y_0 + (x - x_0)\Delta y + (x - x_0)(x - x_1)\Delta^2 y + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y$$

$$y = 1.36 + (x - 1.2) \times -0.975 + (x - 1.2)(x - 2) \times 0.38077 + (x - 1.2)(x - 2)(x - 2.5) \times 0.01068$$

$$y = 1.36 + (1.6 - 1.2) \times -0.975 + (1.6 - 1.2)(1.6 - 2) \times 0.38077 + (1.6 - 1.2)(1.6 - 2)(1.6 - 2.5) \times 0.01068$$

$$y = f(1.6) = 0.9106$$

Problem 2

Use Newton's Divided Difference method interpolate the value of x = 1.75

Temperature	1.1	2.0	3.5	5	7.1
Viscosity	0.6981	1.4715	2.1287	2.0521	1.4480

Solution:

Let us first construct Newton divided difference table.

x	y = f(x)	1st difference	2nd difference	3rd difference	4th difference
1.1	0.6981				
		$\frac{1.4715 - 0.6981}{2.0 - 1.1} = 0.8593$			
2.0	1.4715		$\frac{0.4381 - 0.8593}{3.5 - 1.1} = -0.1755$		
		$\frac{2.1287 - 1.4715}{3.5 - 2.0} = 0.4381$		$\frac{-0.1630 + 0.1755}{5 - 1.1} = 0.0031$	
3.5	2.1287		$\frac{-0.0510 - 0.4381}{5 - 2.0} = -0.1630$		$\frac{0.0019 - 0.0031}{7.1 - 1.1} = 0.0026$
		$\frac{2.0521 - 2.1287}{5 - 3.5} = -0.0510$		$\frac{-0.0657 + 0.1630}{7.1 - 2.0} = 0.0190$	
5	2.0521		$\frac{-0.2876 + 0.0510}{7.1 - 3.5} = -0.0657$		
		$\frac{1.4480 - 2.0521}{7 - 5} = 0.2876$			
7.1	1.4480				

For x = 1.175; which falls in between 1.1 and 2.0 interval

$$y = f(1.175)$$

$$= y_0 + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)f(x_0, x_1, x_2, x_3, x_4)$$

$$= 0.6981 + (1.175 - 1.1)0.85933 + (1.175 - 1.1)(1.175 - 2.0) \times -0.1755 + (1.175 - 1.1)(1.175 - 2)(1.1756 - 3.5) \times 0.000318897 + (1.175 - 1.1)(1.175 - 2)(1.1756 - 3.5)(1.1756 - 5) \times 0.0026495$$

$$y = f(1.175) = 0.77199$$

Problem 3

Estimate f(1.732) and f(2.646) from following set of data using Newton's divided difference interpolation. [T.U. 2063 Baishakh]

X	−2	−1	0	1	2	3
f(x)	64	−5.5	−10	−9.5	56	366.5

Solution:

x	y = f(x)	1st difference	2nd difference	3rd difference	4th difference	5th difference
−2	64					
		$\frac{-5.5-64}{-1-2}$ = 23.167				
−1	−5.5		$\frac{-4.5-23.167}{0+2}$ = −13.834			
		$\frac{-10+5.5}{0+1}$ = −4.5		$\frac{2.5+13.834}{1+2}$ = 5.445		
0	−10		$\frac{0.5+4.5}{1+1}$ = 2.5		$\frac{6.833-5.445}{2+2}$ = 0.347	
		$\frac{-9.5+10}{1-0}$ = 0.5		$\frac{23-2.5}{2+1}$ = 6.833		$\frac{5.77175-0.347}{3+2}$ = 1.08495
1	−9.5		$\frac{65.5-0.5}{2-0}$ = 32.5		$\frac{29.92-6.833}{3+1}$ = 5.77175	
		$\frac{56+9.5}{2-1}$ = 65.5		$\frac{122.25-32.5}{3-0}$ = 29.92		
2	56		$\frac{310.5-65.5}{3-1}$ = 122.25			
		$\frac{366.5-56}{3-2}$ = 310.5				
3	366.5					

$$y = f(x) = y_0 + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2)$$
$$y = f(1.732) = -9.5 + (1.732 - 1)65.5 + (1.732 - 1)(1.732 - 2) \times 122$$
$$y = f(1.732) = 14.5123$$

Problem 4

Using the divided different table, show that following date satisfies a cubic polynomial; [T.U., 2069 Chaitra]

x	1	3	4	5	7	9
y	2.9	2.3	14.6	41.5	166.7	418.1

Solution:

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
1	2.9				
		$\frac{2.3-2.9}{3-1}$ = −0.3			
3	2.3		$\frac{12.3-0.3}{4-1}$ = 4.2		
		$\frac{14.6-2.6}{4-3}$ = 12.3		$\frac{7.3-4.2}{5-1}$ = 0.775	
4	14.6		$\frac{26.9-12.3}{5-3}$ = 7.3		$\frac{1.15-0.775}{7-1}$ = 0.0625
		$\frac{41.5-14.6}{5-4}$ = 7.3		$\frac{11.9-7.3}{7-3}$ = 1.15	
5	41.5		$\frac{62.6-26.9}{7-4}$ = 11.9		$\frac{0.775-1.15}{9-3}$ = −0.0625
		$\frac{166.7-41.5}{7-5}$ = 62.6		$\frac{15.775-11.9}{9-4}$ = 0.775	
7	166.7		$\frac{125.7-62.6}{9-5}$ = 15.775		
		$\frac{418.1-166.7}{9-7}$ = 125.7			
9	418.1				

Problem 5

Evaluate f(2.5) from the following Newton's divided difference interpolation.

x	1	2	3	4	5	6
f(x)	8.9	9.2	16.3	35.6	72.5	132.4

[T.U., 2071 Shrawan]

Solution:

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
1	8.9				
		$\frac{9.2-8.9}{2-1}$ = 0.3			
2	9.2		$\frac{2.218-0.3}{3-1}$ = 0.959		
		$\frac{16.3-9.2}{3-2}$ = 2.218		$\frac{8.541-0.959}{4-1}$ = 2.52	
3	16.3		$\frac{19.3-2.218}{4-2}$ = 8.541		$\frac{0.086-2.52}{5-1}$ = −0.6085
		$\frac{35.6-16.3}{4-3}$ = 19.3		$\frac{8.8-8.54}{5-2}$ = 1.15	

4	35.6		$\frac{36.9-19.3}{5-3} = 8.8$		$\frac{0.9-0.086}{6-2} = 0.2035$
		$\frac{72.5-35.6}{5-4} = 36.9$		$\frac{11.5-8.8}{6-3} = 0.9$	
5	72.5		$\frac{59.9-36.9}{6-4} = 11.5$		
		$\frac{132.4-72.5}{6-5} = 59.9$			
6	132.4				

INTERPOLATION WITH EQUIDISTANCE INTERVAL

Problem 1

Apply Newton's forward difference formula to find y(3.5) [T.U. 2068 Baishakh]

X	1	2	3	4	5	6	7	8
Y	1	8	27	64	125	216	343	512

Solution:

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	$\Delta^7 y$
1	1							
		$8-1=7$						
2	8		12					
		$27-8=19$		6				
3	27		18		0			
		$64-27=37$		6		0		
4	64		24		0		0	
		$125-64=61$		6		0		0
5	125		30		0		0	
		$216-125=91$		6		0		
6	216		36		0			
		$343-216=127$		6				
7	343		42					
		$512-3453=169$						
8	512							

$x_p = 3.5$ lie in between 3 and 4.

Therefore, $x_0 = 3$ and $y = 27$

$h = 1$

$$p = \frac{x_p - x_0}{h} = \frac{3.5-3}{1} = 0.5$$

Let y_p be the value of 'x' when x = 3.5.

Using Newton's forward difference formula; we have,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0$$

$$= 27 + 0.5 \times 37 + \frac{0.5(0.5-1)}{2!} \times 24 + \frac{(0.5-2)(0.5-1)0.5}{3!} \times 6 + 0$$

$$= 27 + 18.5 - 3 + 0.375$$

$$y_p = 42.875$$

Problem 2

Generate forward difference table from the following data [T.U. 2069 Bhadra]

X	1	2	3	4	5	6
f(x)	2	9	28	65	126	217

Solution:

X	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	2					
		$9-2=7$				
2	9		$19-7=12$			
		$28-9=19$		$18-12=6$		
3	28		$37-19=18$		$6-6=0$	
		$65-28=37$		$24-18=6$		0
4	65		$61-37=24$		$6-6=0$	
		$126-65=61$		$30-24=6$		
5	126		$91-61=30$			
		$217-126=91$				
6	217					

Problem 3

Compute the value of y(3) and y(7) from the following data using Newton's interpolation formula [T.U. 2070 Magh]

X	2	4	6	8	10	12
Y	5.1	4.2	3.1	3.5	6.2	7.3

Solution:

Newton's forward difference table is given below.

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2	5.1					
		−0.9				
4	4.2		−0.2			
		−1.1		1.7		
6	3.1		1.5		−0.9	
		0.4		0.8		−3.8
8	3.5		2.3		−4.7	
		2.7		−3.9		
10	62		−1.6			
		1.1				
12	7.3					

$$y(3) = y(p)$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0 + \frac{(p-4)(p-3)(p-2)(p-1)}{5!}\Delta^5 y_0$$

$$= 5.1 + 0.5 \times -0.9 + \frac{0.5(0.5-1)}{2!} \times 0.2 + \frac{(0.5-2)(0.5-1)0.5}{3!} \times 1.7 + \frac{(0.5-3)(0.5-2)(0.5-1)0.5}{4!}(-0.9) + \frac{(0.5-4)(0.5-3)(0.5-2)(0.5-1)0.5}{5!}(-3.8)$$

$$= 5.1 - 0.45 + 0.025 + 0.10625 + 0.03516 - 0.10391 = 47125$$

$$y(7) = 3.1 + 0.5 \times -0.4 + \frac{0.5(0.5-1)}{2!} \times 2.3 + \frac{(0.5-2)(0.5-1)0.5}{3!} \times (-3.9)$$

$$y(7) = 3.1 + 0.2 - 0.2875 - 0.24375$$

$$y(7) = 2.76875$$

Problem 4

Using Newton's forward difference formula or Lagrange interpolation estimate the square of 3.25, if **[T.U. 2066 Jestha]**

X	1	2	3	4	5
x^2	1	4	9	16	25

Solution:

The forward difference table

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		$4-1=3$			
2	4		$5-3=2$		
		$9-4=5$		$2-2=0$	
3	9		$7-5=2$		0
		$16-9=7$		$2-2=0$	
4	16		$9-7=2$		
		$25-16=9$			
5	25				

Now according to Newton's forward difference formula;

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

where, $p = \frac{x_p - x_0}{h}$

$x_p = 3.25$

$x_0 = 3$

$h = 3$

$$p = \frac{3.25 - 3}{1} = 0.25$$

$$y(3.15) = 9 + 0.25 \times 7 + \frac{0.25(0.25-1)}{2!} \times 2$$

$$y(3.15) = 9 + 1.75 - 0.1875$$

$$y(3.15) = 10.5625$$

Problem 5

Use appropriate method of interpolation to get f(0.675) from the given table.

X	0.125	0.25	0.375	0.5	0.625	0.75
f(x)	0.7916	0.7733	0.7437	0.7041	0.6532	0.6022

[T.U. 2066 Bhadra]

Solution:

The forward difference table;

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.125	0.7916					
		−0.00183				
0.25	0.7733		0.0479			
		0.0296		0.1171		
0.375	0.7437		−0.0692		0.1472	
		−0.0396		0.03002		−0.11026
0.5	0.7041		−0.03918		0.03694	
		−0.07878		0.06696		
0.625	0.6532		0.02778			
		−0.051				
0.75	0.6002					
		∇y	$\nabla^2 y$	$\nabla^3 y$		

From above forward difference table we calculate f(0.675)

Here,

$x_p = 0.675;$

which lie in between 0.75 and 0.625 so using backward interpolation

$$p = \frac{x_p - x_n}{h} = \frac{0.675 - 0.75}{0.125} = -0.6$$

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{(p+1)(p+2)p}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$

$$y(0.675) = 0.6022 + (-0.6 \times -0.051) + \left[\frac{-0.6(-0.6+1)}{2!}\right] \times 0.02778 + \left[\frac{-0.6(-0.6+1)(-0.6+2)}{3!}\right] \times 0.06696 + \left[\frac{-0.6(-0.6+1)(-0.6+2)(-0.6+5)}{4!}\right] \times 0.03694 + \left[\frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{5!}\right] \times -0.11026$$

$$y(0.675) = 0.6022 + 0.0306 - 0.003336 - 0.0012412 - 0.0037498 + 0.0025192$$

$$y(0.675) = 0.62699$$

Problem 6

The velocity distribution of the fluid near a flat surface is given below

X (cm)	0.1	0.3	0.5	0.7	0.9
V (cms)	0.72	1.81	2.73	3.47	3.98

'X' is the distance from the surface using a suitable interpolation formula obtain the velocity at x = 0.2. [P.U. 2006 Spring, 2009 Spring]

Solution:

Since the value x = 0.2 is near the beginning of the difference table. Hence, to get the corresponding V (cm/s), we use Newton forward interpolation formula.

The difference table is tabulated below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	0.72				
		1.09			
			−0.17		
0.3	1.81			−0.01	
		0.92			−0.04
			0.92		
0.5	2.73			−0.05	
		0.74			
			0.51		
0.7	3.47				
		0.51			
0.9	3.98				

Since, $x_p = 0.2$ lie between 0.1 and 0.3 therefore, $x_0 = 0.1$ and $V_0 = 0.72$

$$h = 0.3 - 0.1 = 0.2$$

$$p = \frac{x_p - x_0}{h} = \frac{0.2 - 0.1}{0.2} = 0.5$$

Let y_p be the value of 'x' when x = 0.2

Using Newton's forward interpolation formula; we have,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{p(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0$$

$$= 0.72 + 0.5 \times 1.09 + \frac{0.5(0.5-1)}{2!} \times -0.17 + \frac{(0.5-2)(0.5-1)0.5}{3!} \times -0.01 + \frac{(0.5-3)(0.5-2)(0.5-1)0.5}{4!} \times -0.04$$

$y_{0.2} = 1.28$ is the required value.

Problem 7

The following data gives the melting point of an alloy of lead zinc, where t is the temperature and P is the percentage of lead in the alloy. Using Newton Interpolation formula, find the melting point of an alloy containing 75% lead.

P	40	50	60	70	80
t	184	204	226	250	276

[P.U. 2008 Fall]

Solution:

Since, the value p = 75% is near the end of the difference table. Hence, to get corresponding 't', we use Newton backward interpolation formula.

The difference table is tabulated below.

x = p	y = t	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	184				
		20			
50	204		2		
		22		0	
60	226		2		0
		24		0	
70	250		2		
		26			
80	276				

$x_p = 75\%$ lie in between 70 and 80.

Therefore, $x_n = 80$ and $y_n = 276$, h = 10

$$p = \frac{x_p - x_n}{h} = -0.5$$

Using Newton's backward interpolation formula; we have,

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{(p+1)(p+2)p}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 276 + (-0.5 \times 26) + \frac{-0.5(-0.5-1)}{2!} \times 2 + 0 + 0$$

$\therefore \quad y_{75\%} = 262.875$

Problem 8

The following set of data represents the position of a car in a road at specified time, calculate the position of the car at T = 1.75 hours. [P.U. 2008 Spring]

Time (hr)	0	0.5	1.0	1.5	2.0
Position (Km)	0	0.25	1.0	2.25	40

Solution:

x = t	y = p	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	0				
		0.25			
0.5	0.25		0.5		
		0.75		0	
1	1		0.5		0
		1.25		0	
1.5	2.25		0.5		
		1.75			
2	4				

$x_p = 1.75$ lie in between 1.5 and 2 therefore, $x_n = 2$ and $y_n = 4$, h = 0.5

$$p = \frac{x_p - x_n}{h} = -0.5$$

By Newton's backward interpolation formula; we have,

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{(p+1)(p+2)p}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 4 + (-0.5 \times 1.75) + \frac{-0.5(-0.5-1)}{2!} \times 0.5 + 0 + 0$$

$\therefore \quad y_{1.75} = 3.0625$ km

Problem 9

Find Newton's forward difference interpolation polynomial for the following data. Use the obtained formula to estimate the value of f(0.24) [P.U. 2010 Fall]

X	0.1	0.2	0.3	0.4	0.5
f(x)	1.40	1.56	1.76	2.00	2.28

Solution:

The difference table is tabulated below.

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.2		0	
0.3	1.76		0.04		0
		0.24		0	
0.4	2.0		0.04		
		0.28			
0.5	2.28				

Here, $x_p = 0.24$ lie in between 0.3 and 0.2 therefore, $x_0 = 0.2$ and $y_0 = 1.56$

$$h = 0.3 - 0.2 = 0.1$$

$$p = \frac{x_p - x_0}{h} = \frac{0.24 - 0.2}{0.1} = 0.4$$

Let y_p be the value of 'x' when x = 0.24

Using Newton's forward interpolation formula; we have,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{p(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0$$

$$= 1.56 + 0.4 \times 0.2 + \frac{0.4(0.4-1)}{2!} \times 0.04 + 0 + 0$$

$\therefore \quad y_p = 1.6448$

Problem 10

Estimate the value of sin θ at θ = 25 using Newton-Gregory forward difference formula with the help of the following table.

θ	10	20	30	40	50
sin θ	0.1736	0.3420	0.5000	0.6428	0.7660

Solution:

The difference table is tabulated below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	0.1736				
		0.1684			
20	0.3420		−0.0104		
		0.158		−0.0048	
30	0.5		−0.0152		0.0004
		0.1428		−0.0044	
40	0.6428		−0.0196		
		0.1232			
50	0.7660				

$$x_p = 25$$
$$x_0 = 20$$
and, $y_0 = 0.3420$

$$p = \frac{x_p - x_0}{h} = \frac{25-20}{10} = 0.5$$

Let y_p be the value at x = 25

From Newton's forward interpolation formula; we have,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{p(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0$$

$$= 0.3420 + 0.5 \times 0.158 + \frac{0.5(0.5-1)}{2!}(-0.0152) + \frac{(0.5-2)(0.5-1)0.5}{3!}(-0.0044)$$

$\therefore \quad y_{25} = 0.4226$

Problem 11

Evaluate y(1.6), y(7.8) and y(4.2) from the following data using appropriate polynomial interpolation technique used for equally spaced intervals. [T.U., 2072 Kartik]

x	1	2	3	4	5	6	7	8
y	2.3	1.8	2.0	3.0	4.4	5.0	3.9	1.7

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	$\Delta^7 y$
1	2.3							
		− 0.5						
2	1.8		0.7					
		0.2		0.1				
3	2.0		0.8		− 0.5			
		1		− 0.4		− 0.3		
4	3.0		0.4		− 0.8		1.4	
		1.4		− 1.2		1.1		− 1.3

5	4.4		− 0.8		0.3		0.1	
		0.6		− 0.9		1.2		
6	5.0		− 1.7		1.5			
		− 1.1		0.6				
7	3.9		− 1.1					
		− 2.2						
8	1.7							

Taking $x_0 = 2$, $h = 1$ we have to find y (1.6).
Newton's forward interpolation formula is,

$$y_P = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0 + \frac{P(P-1)(P-2)(P-3)(P-4)}{5!}\Delta^5 y_0 + \frac{P(P-1)(P-2)(P-3)(P-4)(P-5)}{6!}\Delta^6 y_0 + \frac{P(P-1)(P-2)(P-3)(P-4)(P-5)(P-6)}{7!}\Delta^7 y_0$$

Taking $x_0 = 1.6$, $x = 2$, $y_0 = 1.8$

$$P = \frac{x_0 - x}{h} = -0.4$$

$$\therefore \quad y_P = 1.8 + (-0.4)(0.2) + (-0.4)\frac{(-0.4-1)}{2!} \times 0.8 + \frac{(-0.4)(-0.4-1)(-0.4-2)}{3!} \times (-0.4) + \frac{(-0.4)(-0.4-1)(-0.4-2)(-0.4-3)}{4!} \times (-0.8) + \frac{(-0.4)(-0.4-1)(-0.4-2)(-0.4-3)(-0.4-4)}{5!} \times (1.1) + \frac{(-0.4)(-0.4-1)(-0.4-2)(-0.4-3)(-0.4-4)(-0.4-5)}{6!} \times (0.1)$$

$$= 1.8 - 0.08 + 0.28 + 0.0896 - 0.15232 - 0.1859$$
$$= 1.768$$

iii) CUBIC SPLINE INTERPOLATION

The concept behind cubic spline interpolation is from the mechanical drafting tool called 'spline' used for drawing smooth curves. These curve resemble cubic curve and hence the name cubic spline has been given to the piecewise cubic interpolating polynomial. We consider the construction of the cubic spline function which would interpolate the point $(x_0, f_0), (x_1, f_1), \ldots\ldots (x_n, f_n)$. The cubic spline s(x) consists of (n − 1) cubic corresponding to (n − 1) sub interval.

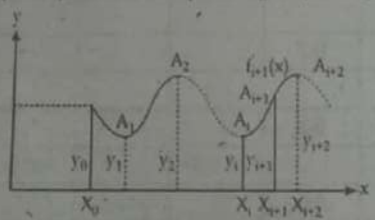


Figure: Curve for cubic spline

Formula

1. $h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$

   where, a_0 and $a_n = 0$

2. $S_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1} - U_i)$

Important Points

1. This method is applicable for uniformly and non-uniformly distributed for any value of 'x'.
2. $h_i = x_i = x_{i-1}$ and $v_i = x - x_i$
3. Number of coefficient is equal to number of points.
4. Evaluate equation (1) by starting i = 1 and onward a numbers of time equal to number of unknown coefficients.
5. Now evaluate equation (2) at i = a value given by position of interval.

Example 1

Given the data points

x =	4	9	16
f =	2	3	4

Estimate the functional value of 'f' at x = 7 using cubic splines.

Solution:

Here,

$$h_i = x_i - x_{i-1}$$

and, $v_i = x - x_i$

In this question there are 3 points

$$a_0 = a_2 = 0$$

For formula 1

$$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$$

For i = 1

$$\cancel{h_1 a_0}^{0} + 2a_1(h_1 + h_2) + \cancel{h_2 a_2}^{0} = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

$$h_1 = x_1 - h_0 = 9 - 4 = 5$$

$$2a_1 \times (5 + 7) = 6\left[\frac{4-3}{7} - \frac{3-2}{5}\right]$$

$$24a_1 = 6\left[\frac{1}{7} - \frac{1}{5}\right]$$

$$a_1 = \frac{6}{24}\left[\frac{1}{7} - \frac{1}{5}\right]$$

$$\therefore \quad a_1 = -0.0143$$

From second formula; we have,

$$S_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1} U_i)$$

$$S_1(x) = \frac{a_0}{6h_1}(h_1^2U_1 - U_1^3) + \frac{a_1}{6h_1}(U_0^3 - h_1^2 - U_0) + \frac{1}{h_1}(f_1U_0 - f_0U_1)$$

$U_0 = x - x_0 = x - 4$

$U_1 = x - x_1 = x - 9$

$$S_1(7) = \frac{-0.0143}{6 \times 5}[(x-4)^3 - 5^2(x-4)] + \frac{1}{5}[3(x-4) - 2(x-9)]$$

$$= \frac{-0.0143}{30}[(7-4)^3 - 5^2(7-4)] + \frac{1}{5}[3(7-4) - 2(7-9)]$$

$= 0.0228 + 2.6$

$S_1(7) = 2.6228$

Example 2

Find 'y' at x = 8 from following data using Natural cubic spline interpolation

X	3	5	7	9
Y	3	2	3	1

[T.U. 2070 Bhadra]

Solution:

In this question there are 4 points

$a_0, a_1, a_2,$ and a_3

$a_0 = a_3 = 0$

We have three interval and three cubics and therefore, only a_1 and a_2 are to be determined.

When i = 1

$$h_1 \overset{0}{\cancel{a_0}} + 2a_1(h_1 + h_2) + h_2a_2 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

$h_1 = x_1 - x_0 = 5 - 3 = 2$

$h_2 = x_2 - x_1 = 7 - 5 = 2$

$$\begin{bmatrix} 2(2+2) & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$d_1 = \frac{6}{h^2}[f_2 - 2f_1 + f_0] = \frac{6}{4}[3 - 2 \times 2 + 3] = 3$$

$$d_2 = \frac{6}{h^2}[f_3 - 2f_2 + f_1] = \frac{6}{4}[1 - 6 + 3] = -3$$

$$a_1 = \frac{d_1 \times 8 - d_2 \times 1}{60} = \frac{3 \times 8 - (-3 \times 1)}{60} = 0.45$$

$$a_2 = \frac{d_2 \times 8 - d_1 \times 1}{60} = \frac{-3 \times 8 - 3 \times 1}{60} = -0.45$$

The target point x = 8

$$S_2(x) = \frac{a_1}{6}(U_2 - U_2^3) + \frac{a_2}{6}(U_1^3 - U_1) + (f_2U_1 - f_1U_2)$$

$$S_2(x) = \frac{a_1}{6}[(x - x_2) - (x - x_2)^3] + \frac{a_2}{6}[(x - x_1)^3 - (x - x_1)] + [f_2(x - x_1) - f_1(x - x_2)]$$

$$S_2(8) = \frac{0.45}{6}[(8-7) - (8-7)^3] + \frac{(-0.45)}{6}[(8-5)^3 - (8-5)] + [3(8-5) - 2(8-7)]$$

$S_2(8) = 5.2$

Example 3

Given the table of value

i	0	1	2	3
x_i	1	2	3	4
$8(x_i)$	0.5	0.3333	0.25	0.20

Estimate the value of f(2.5) using cubic spline functions.

Solution:

The point are equally spaced and therefore

$h_1 = h_2 = h_3 = 1$

Since, n = 4 we have intervals and three cubics and therefore, only a_1 and a_2 are to be determined, we have

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \qquad \left(\because \begin{bmatrix} 2(h_1 + h_2) & 1 \\ 1 & 2(h_2 + h_3) \end{bmatrix}\right)$$

$$d_1 = \frac{6}{h^2}[f_2 - 2f_1 + f_0] = 6[0.25 - 2 \times 0.3333 + 0.5] = 0.5004$$

$$d_2 = \frac{6}{h^2}[f_3 - 2f_2 + f_1] = 6[0.2 - 2 \times 0.25 + 0.3333] = 0.1998$$

Solving a_1 and a_2; we get,

$$a_1 = \frac{d_1 \times 4 - d_2 \times 1}{15} = \frac{0.5004 \times 4 - 0.1998}{15} = 0.1201$$

$$a_2 = \frac{d_2 \times 4 - d_1 \times 1}{15} = \frac{0.1998 \times 4 - 0.5004}{15} = 0.0199$$

The target point x = 2.5

$$S_2(x) = \frac{a_1}{6}(U_2 - U_2^3) + \frac{a_2}{6}(U_1^3 - U_1) + (f_2U_1 - f_1U_2)$$

$$S_2(x) = \frac{a_1}{6}[(x - x_2) - (x - x_3)^3] + \frac{a_2}{6}[(x - x_1)^3 - (x - x_1)] + [f_2(x - x_1) - f_1(x - x_2)]$$

$$S_2(2.5) = \frac{0.1201}{6}[(2.5-3) - (2.5-3)^3] + \frac{0.0199}{6}[(2.5-2)^3 - (2.5-2)] + [0.25(2.5-2) - 0.3333(2.5-3)]$$

$= -0.0075 - 0.0012 + 0.125 + 0.1667$

$= 0.2829$

CURVE FITTING

Least square method/Fitting the linear equation (Straight line)

The equation y = a + b ; (1)

which is an equation of first degree in 'x' and 'y'. It represents a straight line. Usually fitting a straight line means fitting the values of the parameters 'a' and 'b' of the straight line given by equation (1).

Method

Let, y = a + b; (1)

be the equation of straight line having

$$\sum Y = nA + b\sum X \quad (2)$$

$$\sum XY = A\sum X + b\sum X^2 \quad (3)$$

Equation (2) and (3) are called normal equation.

Problem 1

Obtain a relation of the form $y = ae^{bx}$ for the following data by the method of least square. [T.U. 2068 Baishakh]

X	0.0	0.5	1.0	1.5	2.0	2.5
Y	0.10	0.45	2.15	9.15	40.35	180.75

Solution:

The given curve is $y = ae^{bx}$ (1)

Taking log on both sides; we have,

$\log y = \log a + bx \log e$; which is of the formal.

$Y = A + BX$ (2)

$Y = \log y$

$A = \log a$

$B = b \log e$

The normal equation of (2) is;

$$\sum Y = nA + B\sum X \quad (3)$$

$$\sum XY = A\sum X + B\sum X^2 \quad (4)$$

To solve using calculator to find value of $\sum X, \sum Y, \sum XY, \sum X^2$ etc

→ Press ⟹ (Mode) and 3

→ Select 3, put the value of 'x' and 'y' and press $\boxed{AC}$

→ Press ⟹ (Shift) and (Stat)

→ Now select (4) : SUM and take the required submation vale.

X	Y	Y = log y	X^2	XY
0.0	0.10	−1	0.0	0
0.5	0.45	−0.34679	0.25	−0.1734
1.0	2.15	0.33244	1	0.33244
1.5	9.15	0.96142	2.25	1.44213
2.0	40.35	1.60584	4	3.21168
2.5	180.75	2.25708	6.25	5.6427
$\sum X = 7.5$		$\sum Y = 3.80999$	$\sum X^2 = 13.75$	$\sum XY = 10.4556$

$3.80999 = 6A + B(7.5)$

$10.4556 = A(7.5) + B(13.75)$

On solving; we get,

$A = -0.99161$

$B = 1.3013$

$\therefore$ $a = \text{antilog}(-0.99161) = 0.101951$

$$b = \frac{B}{\log e} = \frac{1.3013}{\log e} = 2.99635$$

Hence, the required curve is $y = 0.101951e^{2.99635x}$.

Problem 2

Fit the following set of data to a curve of from $y = ae^{bx}$ from the following observation by least square method. [T.U. 2069 Bhadra]

X	1	2	3	4	5	6
Y	5.5	6.5	9.4	15.2	30.6	49.8

Solution:

The given curve is $y = ae^{bx}$ (1)

Taking log on both sides; we have,

$\log y = \log a + bx \log x$ (2)

Equation (2) is form of equation given below.

$Y = A + BX$ (3)

$Y = \log y$

$A = \log a$

$B = b \log e$

Now the normal equation (3) is;

$$\sum Y = nA + B\sum X \quad (4)$$

$$\sum XY = A\sum X + B\sum X^2 \quad (5)$$

X	Y	Y = log y	X^2	XY
1	5.5	0.740363	1	0.740363
2	6.5	0.81291	4	1.62582
3	9.4	0.97313	9	2.91939
4	15.2	1.18184	16	4.72736
5	30.2	1.48000	25	7.40000
6	49.8	1.69723	36	10.18338
$\sum X = 21$		$\sum Y = 6.885473$	$\sum X^2 = 91$	$\sum XY = 27.596313$

Put these values in equation (4) and (5); we have,

$6.885473 = 6A + 21B$

$27.596313 = 21A + 91B$

$A = 0.44815$

$B = 0.19964$

$a = \text{antilog}(A) = \text{antilog}(0.44815) = 2.806403$

$$b = \frac{B}{\log e} = \frac{0.19984}{\log e} = 0.46015$$

The equation (1) becomes;

$y = 2.806403e^{0.46015x}$

Problem 3

Fit the following set of data to a curve of the form $y = ab^x$. Also evaluate at y(7). [T.U. 2070 Bh

X	2	4	6	8	10	12
Y	16.0	17.1	8.7	6.4	4.7	2.6

Solution:

Given that;

$$y = ab^x \quad (1)$$

Taking log on both sides; we have,

$$\log y = \log a + x \log b \quad (2)$$

Comparing equation (2) with

$$Y = A + BX \quad (2)$$

$$Y = \log y$$

$$B = \log b$$

and, $A = \log a$

The normal equation to (3) is;

$$\sum Y = nA + B\sum X \quad (4)$$

$$\sum XY = A\sum X + B\sum X^2 \quad (5)$$

Calculate $\sum Y, \sum X, \sum XY$ and $\sum X^2$ using calculator

X	Y	Y = log y
2	16	1.20412
4	11.1	1.04532
6	8.7	0.93952
8	6.4	0.806179
10	4.7	0.672098
12	2.6	0.414973

$$\sum X = 42$$

$$\sum Y = 5.126642$$

$$\sum XY = 30.64332$$

$$\sum X^2 = 364$$

Put the value in equation (4) and (5); we get,

$$5.126642 = 6A + 42B$$

$$30.64332 = 42A + 364B$$

$$A = 1.378758$$

$$B = -0.074902$$

$$a = \text{antilog}(1.378758) = 23.91983$$

$$b = \text{antilog}(-0.074902) = 0.841585$$

so, equation (1) becomes;

$$y = 23.91983 \times 0.841585^x$$

For y(7) *i.e.*, x = 7

$$y = 23.91983 \times 0.841585^7$$

$$y = 7.1523$$

Hence, the required value of 'y' is 7.1523.

Problem 4

Fit the following data to the curve $y = \log_e(ax + b)$ [T.U. 2070 Magh]

X	0	1	2	3	4	5	6
Y	0.9	1	1.5	1.9	2.1	2.4	2.5

Solution:

Given that;

$$y = x \log_e a + \log_e b \quad (1)$$

Comparing with Y = A + BX (2)

$$Y = y$$

$$A = \log_e a$$

$$B = \log_e b$$

x	y
0	0.9
1	1
2	1.5
3	1.9
4	2.1
5	2.4
6	2.5

$$\sum X = 21$$

$$\sum Y = 12.3$$

$$\sum XY = 45.1$$

$$\sum X^2 = 91$$

The normal equation to Y = A + BX is;

$$\sum Y = nA + B\sum X \quad (3)$$

$$\sum XY = A\sum X + B\sum X^2 \quad (4)$$

Now, substituting value in (3) and (4); we get,

$$12.3 = 7A + 21B$$

$$45.1 = 21A + 91B$$

$$A = 0.878571$$

$$B = 0.292857$$

$\therefore$ $A = \log_e a$

$$a = e^A$$

$$a = e^{0.878571}$$

$$a = 2.407457$$

$\therefore$ $B = \log_e b$

$$b = e^B$$

$$b = e^{0.292857}$$

$$b = 1.34025$$

$$y = \log_e(2.4075x + 1.34025)$$

Problem 5

Fit the following data to the function $y = a(ax + b)$ using least square

X	−2.0	−1.0	0.5	2.0	3.0	5.5
Y	−0.4	1.2	3.5	6.0	7.4	11.0

[T.U. 2063 Jestha]

Solution: Proceed as Q. no. 4.

Problem 6

Use suitable method to fit quadratic curve $y = ax^2 + bx + c$ for the following data

X	−3	−2	−1	0	1	2	3
Y	4.63	2.11	0.67	0.09	0.63	2.15	4.56

Solution:

The quadratic equation $y = ax^2 + bx + c$; then the normal equation is;

$$nc + b\sum X + a\sum X^2 = \sum Y$$
$$c\sum X + b\sum X^2 + a\sum X^3 = \sum XY$$
$$c\sum X^2 + b\sum X^3 + a\sum X^4 = \sum X^2Y$$

X	Y	X^2	X^3	X^4	XY	X^2Y
−3	4.63	9	−27	81	−13.89	49.67
−2	2.11	4	−8	16	−4.22	8.44
−1	0.67	1	−1	1	−0.67	0.67
0	0.09	0	0	0	0	0
1	0.63	1	1	1	0.63	0.63
2	2.15	4	8	16	4.3	8.6
3	4.56	9	27	81	13.68	41.04
$\sum X = 0$	$\sum Y = 14.84$	$\sum X^2 = 28$	$\sum X^3 = 0$	$\sum X^4 = 196$	$\sum XY = 0.17$	$\sum X^2Y = 109.05$

so the equation (1), (2) and (3), we get,

$$7c + 28a = 14.84$$
$$28b = -0.17$$
$$28c + 196a = 109.05$$
$$b = -\frac{0.17}{28} = -0.006071 = -0.0061$$
$$7c + 28a = 14.84$$
$$28c + 196a = 109.05$$
$$a = 0.591547 = 0.6$$
$$c = -0.24619 = -0.25$$
$$y = 0.6x^2 - 0.0061x - 0.25$$

Problem 7

Use the suitable method and determine the exponential fit of $y = Ce^{Ax}$ for the following data [T.U. 2063 Shrawan]

X	0	1	2	3	4
Y	1.5	2.5	3.5	5.0	7.5

Solution:

Given that:

$$y = Ce^{Ax}$$

Take log on both sides; we have,

$$\log y = \log C + Ax \log e$$

Comparing with $y = V + ax$

$$Y = \log y$$

$$V = \log C$$
$$a = A \log e$$

The normal equation of (2) is;

$$\sum Y = nV + a\sum X \quad (3)$$
$$\sum XY = V\sum X + a\sum X^2 \quad (4)$$

X	Y	Y = log y
0	1.5	0.1761
1	2.5	0.39794
2	3.5	0.544068
3	5.0	0.69897
4	7.5	0.87506

$$\sum X = 10$$
$$\sum X^2 = 30$$
$$\sum Y = 2.69214$$
$$\sum XY = 7.08323$$
$$2.69214 = 5V + 10a$$
$$7.08323 = 10V + 30a$$
$$a = 0.169895$$
$$b = 0.198638$$
$$A = \frac{a}{\log e} = \frac{0.169895}{\log e} = 2.99635$$
$$C = \text{antilog}(0.198638) = 1.57993$$
$$y = 1.57993e^{0.39119x}$$

Problem 8

The temperature of metal strip was measured at various time interval during heating and the values are given in table below

Time (t) mm	1	2	3	4
Temperature T (C°)	70	83	100	125

If the relationship between the temperature 'T' and time 't' is of the form $T = be^{\frac{t}{4}} + a$ estimate the temperature at t = 6 min [T.U. 2064 Falgun]

Solution:

We can write the temperature equation in the form $y = bf(x) + a$

This is similar to the linear equation except that the variable 'x' is replaced by function f(x).

Therefore, we can solve the parameter 'a' and 'b' by replacing x_i by $f(x_i)$.

$$\sum x_i = \sum f(x_i)$$
$$\sum x_i^2 = \sum f(x_i)^2$$

Thus,

$$b = \frac{\left(\sum f(x_i)\sum y_i\right) - \sum f(x_i)\sum y_i}{n\sum\left[f(x_i)^2\right] - \left[\sum f(x_i)\right]^2}$$

$$a = \frac{\sum y_i - b\sum f(x)}{n}$$

We can set the following table to obtain the various any we have $f(x) = e^{0.5x}$

x	y	f(x)	yf(x)	$[f(x)]^2$
1	70	1.28	89.6	1.64
2	83	1.65	136.95	2.73
3	100	2.12	212	4.49
4	124	2.72	337.28	7.40
	$\sum Y = 377$	$\sum f(x) = 7.77$	$\sum yf(x) = 775.83$	$\sum f(x)^2 = 16.26$

$$b = \frac{4 \times 775.83 - 7.77 \times 377}{4 \times 16.26 - (7.77)^2} = 37.29$$

$$a = \frac{377 - (37.29)(7.77)}{4} = 21.82$$

The equation is $T = 37.29e^{0.25\times t} + 21.82$

The temperature at $t = 6$ is;

$$T = 37.29e^{0.25\times 6} + 21.82$$

Problem 9

Fit the following set of data to estimate the coefficient 'a' and 'b' for the function. **[T.U. 2069 Bhadra]**

$$y = \frac{1}{ax - b}$$

X	−5	−4	−3	0	2	3	6	8
Y	0.15	0.18	0.23	2	−0.5	−0.31	−0.143	−0.1

Solution:

Here,

$$y = \frac{1}{ax - b}$$

$$Z = \frac{1}{y}$$

$$Z = ax + b$$

X	Y	$Z = \frac{1}{y}$	X^2	XZ
−5	0.15	6.67	25	−33.35
−4	0.18	5.56	16	−22.24
−3	0.23	4.35	9	−13.05
0	2	0.5	0	0
2	−0.5	−2	4	−4
3	−0.31	−3.23	9	−9.69
6	−0.143	−6.993	36	−41.958
8	−0.1	−10	64	−80
$\sum X = 7$	$\sum Y = 1.507$	$\sum Z = -5.143$	$\sum X^2 = 163$	$\sum XZ = -204.288$

$$b = \frac{n\sum XZ - \sum X \sum Z}{n\sum X^2 - (\sum X)^2} = \frac{8(-204.288) - 7\times -5.143}{8 \times 163 - (7)^2} = 0.0156637$$

$$a = \frac{\sum Z}{n} - \frac{b\sum f(x)}{n} = \frac{-5.143}{8} - (0.0156637)\frac{7}{8} = -0.65658$$

$$y = \frac{1}{Z} = \frac{1}{-0.65658x - 0.0156637}$$

Problem 10

Find the best fit curve in the form of $y = ax^2 + bx + C$ using least square approximation form the following discrete data. **[T.U. 2063 Jestha]**

X	−1.5	−1	0	1	2.5	4	5.5
f(x)	−0.75	0	0.72	1.12	1.5	1.8	2

Solution:

The normal to quadratic equation $y = ax^2 + bx + C$

$nC + b\sum X + a\sum X^2 = \sum Y$ (1)

$C\sum X + b\sum X^2 + a\sum X^3 = \sum XY$ (2)

$C\sum X^2 + b\sum X^3 + a\sum X^4 = \sum X^2Y$ (2)

X	Y	X^2	X^3	X^4	XY	X^2Y
−1.5	−0.75	2.25	−3.375	5.0625	1.125	−1.6875
−1	0	1	−1	1	0	0
0	0.72	0	0	0	0	0
1	1.12	1	1	1	1.12	1.12
2.5	1.5	6.25	15.625	39.0625	3.75	9.375
4	1.8	16	64	256	72	28.8
5.5	2	30.25	166.375	915.0625	11	60.5
$\sum X = 10.5$	$\sum Y = 6.39$	$\sum X^2 = 56.75$	$\sum X^3 = 242.625$	$\sum X^4 = 1217.168$	$\sum XY = 16.995$	$\sum X^2Y = 98.1075$

$76 + 10.56 + 56.75 = 6.39$

$10C + 56.75b + 242.625a = 16.995$

$56.756 + 242.625b + 1217.188a = 98.1075$

$a = 0.11261$

$b = -0.247333$

$C = 0.37091$

$y = 0.1126x^2 + 0.24733x + 0.37091$

Problem 11

Estimate coefficient $y = ax + b$ for the following data using least square method. **[T.U. 2062 Baishakh]**

X	−2.0	−1.0	0.5	2.0	3.0	5.5
Y	−0.4	1.2	3.7	6.0	7.4	11.0

Solution:

The given equation is $y = ax + b$ (1)

The normal equation of (1) are;

$$\sum Y = nb + a\sum X$$
$$\sum XY = b\sum X + a\sum X^2$$

X	Y
−2.0	−0.4
−1.0	1.2
0.5	3.5
2.0	6.0
3.0	7.4
5.5	11.0

$\sum X = 8$

$\sum Y = 28.7$

$\sum XY = 96.05$

$\sum X^2 = 48.5$

$28.7 = 6b + 8a$

$96.05 = 8b + 48.5a$

$a = 1.5273$

$b = 2.7469$

$y = 1.5273x + 2.7469$

Problem 12

From the following data

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

Fit a power function of the form $y = ax^b$ [P.U. 2006 Fall and 2008 Fall]

Solution:

The given curve is $y = ax^b$ (1)

Taking log on both sides; we get,

$$10 \log y = \log a + b \log x \quad (2)$$

Comparing it with;

$y = A + bx$ (3)

$Y = \log y$

$A = \log a$

$X = \log x$

The normal equation of (3) are;

$$\sum Y = nA + b\sum X \quad (4)$$
$$\sum XY = A\sum X + b\sum X^2 \quad (5)$$

x	y	$X = \log x$	$Y = \log y$	X^2	XY
1	0.5	0	−0.3010	0	0
2	2.0	0.3010	0.3010	0.0906	0.0906
3	4.5	0.4771	0.6532	0.2276	0.3116
4	8.0	0.6021	0.9031	0.3625	0.5438
5	12.5	0.6990	1.0969	0.4886	0.7667
		$\sum X = 2.0792$	$\sum Y = 2.6532$	$\sum X^2 = 1.1693$	$\sum XY = 1.7127$

Substituting in (4) and (5); we have,

$2.532 = 2.0792b + 5A$ (A)

$1.7127 = 1.1693b + 2.0792A$ (B)

On solving (A) and (B); we have,

$b = 2$

$A = -0.3010$

$\therefore \quad a = \text{antilog}(-0.3010) = 0.5$

Hence, $y = 0.5x^2$

Problem 13

Use the data linearization method and determine the exponential fit of $y = Ce^{Ax}$ for the following data [P.U. 2006 Spring]

X	0	1	2	3	4
Y	1.5	2.5	3.5	5.0	7.5

Solution:

The given curve is $y = Ce^{Ax}$ (1)

Taking log on both sides; we get,

$\log y = \log C + Ax \log e$ (2)

$Y = \log y$

$V = \log C$

$a = \log e$

The normal equation of (2) are;

$$\sum Y = nV + a\sum X \quad (3)$$
$$\sum XY = V\sum X + a\sum X^2 \quad (4)$$

X	Y	$Y = \log y$	X^2	XY
0	1.5	0.1760	0	0.2640
1	2.5	0.3979	1	0.9947
2	3.5	0.5440	4	1.9040
3	5.0	0.6989	9	3.4945
4	7.5	0.8750	16	3.5629
$\sum X = 10$		$\sum Y = 2.6918$	$\sum X^2 = 30$	$\sum XY = 7.0828$

Now, equation (3) and (4) becomes;

$2.6918 = 5V + 10a$

$7.0824 = 10V + 30a$

$V = 0.1985$

$a = -0.1699$

$\therefore \quad A = \dfrac{a}{\log e} = 0.3911$

$\therefore$ $C = \text{antilog}(0.1985) = 1.5794$

Hence, the required curve is $y = 1.5794e^{0.3911x}$.

Problem 14

Find the missing value in the following table of a chemical dissolved in water using least square approximation to straight line fitting [P.U. 2005 Fall]

Temperature	10°	15°	20°	25°	30°	35°
Solubility	20	22	23	?	26	27

Solution:

The given equation of straight line is;

$$y = mx + C \quad (1)$$

The normal equation of (1) is;

$$\sum Y = nC + m\sum X \quad (2)$$

$$\sum XY = C\sum X + m\sum X^2 \quad (3)$$

X	Y	X^2	XY
10	20	100	200
15	22	225	330
20	23	400	460
30	26	900	780
35	27	1225	945
$\sum X = 110$	$\sum Y = 118$	$\sum X^2 = 2850$	$\sum XY = 2715$

The equation (2) and (3) now becomes;

$$118 = 110m + 5C \quad (4)$$

$$2715 = 285m + 110C \quad (5)$$

On solving (4) and (5); we get,

$$M = 0.2767$$

$$C = 17.5116$$

The required equation curve is $y = 0.2767x + 17.5116$ at $x = 25$

$$y = 0.2767 \times 25 + 17.5116 = 24.429$$

Problem 15

Use the suitable method to fit a quadratic curve for the following data

X	−5	−4	−3	−2	−1	0
Y	12.96	6.94	4.63	2.11	0.67	0.09

[P.U. 2010 Spring]

Solution:

The given curve is $y = e^{a+bx}$ (1)

Taking log on both sides; we get,

$$\log y = a \log e + bx \log e$$

Comparing with $Y = A + BX$ (2)

$$Y = \log y$$

$$A = \log e$$

$$B = b \log e$$

The normal equation of (2) is;

$$\sum Y = nA + b\sum X \quad (3)$$

$$\sum XY = A\sum X + b\sum X^2 \quad (4)$$

X	Y	Y = log y	X^2	XY
−5	12.96	1.1126	25	−5.563
−4	6.94	0.8413	16	−3.3652
−3	4.63	0.6655	9	−1.9965
−2	0.67	−0.1739	1	0.17392
0	0.09	−1.0457	0	0
$\sum X = 15$		$\sum X = 1.7240$	$\sum X^2 = 55$	$\sum XY = -11.3994$

$$1.7240 = 6A - 15B \quad (5)$$

$$-11.3994 = -15A + 55B \quad (6)$$

On solving; we get,

$$A = -0.7254$$

$$B = -0.4051$$

$$\therefore \quad b = \frac{A}{\log e} = -1.67029$$

$$b = \frac{B}{\log e} = -0.9327$$

Hence, the required curve $y = e^{-1.6702-0.9327x}$.

Problem 16

The heat of water Hf(t) and the quantity of water Q (t^3) flowing per second are related by the law $Q = CH^{\alpha}$. Find out the best fir value for coefficient C and α for the following data [P.U. 2009 Fall]

H	5	10	25	20	25	30
Q	20	150	360	800	1500	2200

Solution:

The given equation is $Q = CH^{\alpha}$ (1)

Let, $Q = y$ and $C = a$, $H = x$ and $\alpha = b$

We get,

$$y = ax^b$$

Taking log on both sides; we get,

$$\log y = \log a + bx \log x \quad (2)$$

Comparing (2) with

$$Y = A + BX \quad (3)$$

We get,

$$Y = \log y$$

$$A = \log a$$

$$X = \log x$$

The normal equation of (3) is;

$$\sum Y = nA + b\sum X \quad (4)$$

$$\sum XY = A\sum X + b\sum X^2 \quad (5)$$

X	y	x = log H	Y = log y	X^2	XY
5	20	0.699	1.301	0.489	0.909
10	150	1.0	2.556	1	2.176
15	360	1.176	23.556	1.383	3.006
20	800	1.301	2.903	1.693	3.777
35	1500	1.398	3.176	1.954	9.44
30	2200	1.477	3.342	2.182	4.936
		$\sum X = 7.051$	$\sum Y = 15.454$	$\sum X^2 = 8.7$	$\sum XY = 19.244$

Now equation (4) and (5) can be written as;

$15.454 = 6A + 7.051x$

$19.244 = 7.051A + 8.7x$

On solving; we get,

$X = 2.617$

$A = -0.499$

$a = \text{antilog}(A) = 0.317$

Hence, $y = 0.317b^{2.617}$

Problem 17

From the following data

X	1	2	3	4	5
Y	0.5	2	4.5	8	12.5

Fit a power function model of the form $y = ax^b$ [P.U. 2006 Fall and 2007 Fall]

Solution:

Given that;

$y = ax^b$

Taking log on both sides; we have,

$\log y = \log a + b \log x$

Comparing (2) to $y = A + bx$; we get,

$Y = \log y$

$A = \log a$

$X = \log x$

The normal equation is;

$\sum Y = nA + b\sum X$

$\sum XY = A\sum X + b\sum X^2$

x	y	X = log x	Y = log y	X^2	XY
1	0.5	0	−0.3010	0	0
2	2	0.3010	0.3010	0.090601	0.090601
3	4.5	0.4771	0.6532	0.2276	0.3116
4	8	0.6020	0.9030	0.3624	0.5436
5	12.5	0.6989	1.0969	0.48846	0.7667
		$\sum X = 2.0890$	$\sum Y = 2.6532$	$\sum X^2 = 1.1752$	$\sum XY = 1.127$

Now equation (3) and (4) can be written as,

$2.6532 = 2.0890 \times b + 5A$

$1.7127 = 1.1752b + 2.0890A$

On solving; we get,

$b = 2$

$A = -0.301$

$a = \text{antilog}(-0.301) = 0.5$

Hence, the required curved is $y = 0.5x^2$

Problem 18

For the following set of data fir a linear curve using least square method and find (2)

X	0.5	1.5	4.5	7.5
f(x)	2.5	3.5	6.5	9.5

Solution:

The given curve is $y = ax + b$ (1)

The normal equation is;

$\sum Y = nb + a\sum X$ (2)

$\sum XY = b\sum X + a\sum X^2$ (3)

Now we have to find missing value 2.

X	Y	X^2	XY
0.5	2.5	0.25	1.25
1.5	3.5	2.25	5.25
4.5	6.5	20.25	29.25
7.5	9.5	56.25	71.25
$\sum X = 14$	$\sum Y = 22$	$\sum X^2 = 79$	$\sum XY = 107$

Equation (2) and (3) can be written as;

$107 = 79a + 14b$

$22 = 14a + 4b$

On solving; we get,

$a = 1$

$b = 2$

Hence the required curve is $y = x + 2$.

Problem 19

Fit the exponential equation of the form $y = ae^{bx}$ using the least square method from the following data. [P.U. 2010 Fall]

X	1	2	3	4	5	6
Y	1.65	2.70	4.50	7.35	12.2	15

Solution:

Given that;

$y = ae^{bx}$ (1)

Taking log on both sides; we have,

$\log y = \log a + bx \log e$

Comparing (1) with $Y = A + BX$ (2)

We get,

$Y = \log y$

$A = \log a$

$B = b \log e$

The normal equation of (2) is;

$\sum Y = nA + b\sum X$ (3)

$\sum XY = A\sum X + b\sum X^2$ (4)

X	Y	Y = log y	X^2	XY
1	1.65	0.217	1	0.217
2	2.7	0.43	4	0.86
3	4.5	0.65	9	1.95
4	7.35	0.866	16	3.464
5	12.2	1.08	25	5.4
6	15	1.176	36	7.056
$\sum X = 21$		$\sum Y = 4.419$	$\sum X^2 = 91$	$\sum XY = 18.94$

Now the equation (3) and (4) can be written as;

$4.419 = 6A + 21B$

$18.94 = 21A + 91B$

On solving; we get,

$A = 0.0418$

$B = -0.19848$

$\therefore$ $a = \text{antilog}(0.0418) = 1.1010$

$b = \frac{B}{\log e} = 0.4571$

Hence, the required curve $y = 1.1010b^{0.4571x}$.

Problem 20

Determine the constant 'a' and 'b' the method of least square such that $y = ae^{bx}$ fit the following data [P.U. 2011 Fall]

X	2	4	6	8	10
Y	4.077	11.084	30.128	81.897	222.62

Solution:

The given curve is $y = ae^{bx}$ (1)

Taking log on both sides; we have,

$\log y = \log a + bx \log e$

Comparing it with $Y = A + BX$ (2)

We get,

$Y = \log y$

$A = \log a$

$B = b \log e$

The normal equation of (2) are;

$\sum Y = nA + b\sum X$ (3)

$\sum XY = A\sum X + b\sum X^2$ (4)

X	Y	Y = log y	X^2	XY
2	4.077	0.610	4	1.22
4	11.084	1.045	16	4.18
6	30.128	1.479	36	8.874
8	81.897	1.913	64	15.304
10	222.62	2.3475	100	23.475
$\sum X = 30$		$\sum Y = 7.3945$	$\sum X^2 = 220$	$\sum XY = 53.053$

Now equation (3) and (4) becomes,

$7.3945 = 5A + 30B$

$53.053 = 30A + 220B$

On solving, we get,

$A = 0.1794$

$B = 0.21657$

$\therefore$ $a = \text{antilog}(0.1794) = 1.5114$

$b = \frac{B}{\log e} = 0.4586$

Hence, the required curve is $y = 1.5114e^{0.4586x}$.

Problem 21

Fit the following data using least square method to estimate 'a' and 'b' for the relation y = a + bx. Also find f(15) from the fitted equation [P.U. 2007 Fall]

X	10	12	18	22	30
f(x)	0	2	4	8	10

Solution:

The given curve is $y = a + bx$ (1)

The normal equation of (1) is;

$\sum Y = na + b\sum X$ (2)

$\sum XY = a\sum X + b\sum X^2$ (3)

X	Y	X^2	XY
10	0	100	0
12	2	144	24
18	4	324	72
22	8	484	176
30	10	900	300
$\sum X = 92$	$\sum Y = 24$	$\sum X^2 = 1952$	$\sum XY = 572$

Now equation (2) and (3) can be written as;

$24 = 5a + 92b$

$572 = 92a + 1952b$

$a = -4.4568$

$b = 0.503086$

Hence, the required curve is $y = -4.4568 + 0.503086x$.

For $x = 15$

$y = -4.4568 + 0.503086 \times 15$

$y = 3.08949$

Hence, the value of y at $x = 15$ is 3.08949.

Problem 22

Use regression method to fit the geometrical curve $y = ab^x$ to the data given below and obtain the value of 'y' at $x = 5.5$

X	0	1	2	3	4	5	6	7
Y	10	21	35	59	92	200	400	610

Solution:

The curve $y = ab^x$

$\log y = \log a + x \log b$

Comparing with $Y = A + BX$

The normal equation of (2) is;

$\sum Y = nA + b\sum X$

$\sum XY = A\sum X + b\sum X^2$

X	Y	Y = log y	X^2	XY
0	10	1	0	0
1	21	1.32	1	1.32
2	35	1.54	4	3.08
3	59	1.77	9	5.31
4	92	1.963	16	7.852
5	200	2.30	25	11.5
6	400	2.60	36	15.6
7	610	2.785	49	19.495
$\sum X = 28$		$\sum Y = 15.27$	$\sum X^2 = 140$	$\sum XY = 64.11$

Now equation (3) and (4) can be written as;

$15.27 = 8A + 28B$

$64.11 = 28A + 140B$

On solving; we have,

$B = 0.25$

and, $A = 1.02$

$a = \text{antilog}(A) = 10.47$

$b = \text{antilog}(B) = 1.778$

so, the curve is $y = 10.47 \times 1.778^x$

Now,

$x = 5.5$

$y = 10.47 \times 1.778^{5.5} = 248.068$

Hence, the value of 'y' at $x = 5.5$ is $y = 248.068$.

Problem 23

Estimate the coefficient of $y = ax + b$ the following data using least square method.

[T.U., 2068 Chaitra]

x	− 2.0	− 1.0	0.5	2.0	3.0	5.5
y	− 0.4	1.2	3.5	6.0	7.4	11.0

Solution:

x	y	x^2	xy
− 2	− 0.4	4	0.8
− 1	1.2	1	1.2
0.5	3.5	0.25	1.75
2.0	6.0	4	12
3.0	7.4	9	22.2
5.5	11.0	30.25	60.5
$\sum x = 8$	$\sum y = 28.7$	$\sum x^2 = 48.5$	$\sum xy = 98.45$

$n = 6$

$\therefore \quad b = \dfrac{n\sum xy - \Sigma x \sum y}{n\sum x^2 - (\Sigma x)^2} = \dfrac{(6 \times 98.45) - (8 \times 28.7)}{(6 \times 48.5) - (8)^2}$

$= 1.59074$

$\therefore \quad a = \dfrac{\sum y}{n} - b\dfrac{\Sigma x}{n}$

$= \dfrac{28.7}{6} - 1.590 \times \dfrac{8}{6}$

$= 4.7833 - 2.1209$

$= 2.6624$

$\therefore \quad y = ax + b = 1.59074\,x + 2.6624$

Problem 24

Fit the following set of data to a curve of the form $y = ae^{bx}$.

x	2	3	4	5	6	7
y	15.1	10.2	7.8	5.5	3.8	1.7

[T.U., 2069 Chaitra]

Solution:

x	y	ln y	x^2	x · ln y
2	15.1	2.7146	4	10.8584
3	10.2	2.3223	9	20.9007
4	7.8	2.0541	16	32.8656
5	5.5	1.7047	25	42.6175
6	3.8	1.3350	36	48.06
7	1.7	0.5306	49	48.06
$\sum x = 27$		$\sum \ln y = 10.6713$	$\sum x^2 = 139$	$\sum x \cdot \ln y = 181.3016$

$$b = \frac{n\sum x \cdot \ln y - \Sigma x \sum \ln y}{n\sum x^2 - (\Sigma x)^2}$$

$$= \frac{(6 \times 181.3016) - (27 \times 10.6713)}{(6 \times 139) - (27)^2}$$

$= 7.61604$

$$a = e^{\left(\frac{\sum \ln y}{n} - b\frac{\Sigma x}{n}\right)}$$

$$= e^{\left(\frac{10.6713}{6} - 7.61604 \times \frac{27}{6}\right)}$$

$= e^{1.77855 - 34.27218}$

$\therefore$ $a = 7.73029 \times 10^{-15}$

$\therefore$ $y = 029 \times 10^{-15} e^{7.61604x}$ is the required curve.

Problem 25

Fit the following data to a curve $y = ae^{bx}$. [T.U., 2070 Chaitra]

x	1	2	3	4	5	6	7	8
y	2	3	4	5	7	10	15	30

Solution:

x	y	$\ln y$	x^2	$x \cdot \ln y$
1	2	0.69314	1	0.69314
2	3	1.09861	4	2.19722
3	4	1.38629	9	4.15887
4	5	1.60943	16	6.43772
5	7	1.9459	25	9.7295
6	10	2.30258	36	13.81548
7	15	2.70805	49	18.95635
8	30	3.40119	64	27.20952
$\Sigma x = 36$		$\sum \ln y = 15.14$	$\sum x^2 = 204$	$\sum x \cdot \ln y = 83.19811$

$n = 8$

$$\therefore \quad b = \frac{n\sum x \cdot \ln y - \Sigma x \sum \ln y}{n\sum x^2 - (\Sigma x)^2}$$

$$= \frac{(8 \times 83.19811) - (36 \times 15.14)}{(8 \times 204) - (36)^2} = \frac{665.584 - 545.04}{1632 - 1296}$$

$= 0.3587$

$$\therefore \quad a = e^{\left(\frac{\sum \ln y}{n} - b\frac{\Sigma x}{n}\right)}$$

$$= e^{\left(\frac{15.14}{8} - 0.3587 \times \frac{36}{8}\right)} = e^{(1.8925 - 1.6144)} = 1.32061$$

$\therefore$ $y = 132061e^{0.3587x}$

Problem 26

Fit the following data to an exponential curve of the form $y = ab^x$.

x	2	4	6	8	10
y	2	6	25	115	300

[T.U., 2071 Shrawan]

Solution:

$y = ab^x$

$\log y = \log(ab^x)$

$\log y = \log a + x \log b$

x	y	$Y = \log y$	x^2	xY
2	2	0.3010	4	0.602
4	6	0.77781	16	3.1124
6	25	1.3979	36	8.3874
8	115	2.0606	64	16.4848
10	300	2.4771	100	24.771
$\Sigma x = 30$		$\sum y = 7.0147$	$\sum x^2 = 220$	$\Sigma xY = 53.35$

$$\therefore \quad b = \frac{n\sum xY - \Sigma x \sum Y}{n\sum x^2 - (\Sigma x)^2} = \frac{(5 \times 53.35) - (30 \times 7.0147)}{(5 \times 220) - (30)^2} = \frac{266.788 - 210.441}{1100 - 900}$$

$= 0.2817$

$$\therefore \quad a = \frac{\sum Y}{n} - b\frac{\Sigma x}{n} = \frac{7.0147}{5} - 0.2817 \times \frac{30}{5} = 1.40294 - 1.6902 = -0.28726$$

$\therefore$ $y = -0.0809^x$

Problem 27

Fit the following set of data to a curve of the form $y = a\log_e x + b$.

x	2	4	6	8	10	12	14
y	4.7	7.2	8.3	9.6	10.4	10.7	10.9

[T.U., 2072 Kartik]

Solution:

Comparing with $y = a + bx$,

x	y	x^2	xy
2	4.7	4	9.4
4	7.2	16	28.8
6	8.3	36	49.8
8	9.6	64	76.8
10	10.4	100	104
12	10.7	144	128.4
14	10.9	196	152.6
$\Sigma x = 56$	$\sum y = 61.8$	$\sum x^2 = 560$	$\sum xy = 549.8$

$$b = \frac{n\sum xy - \Sigma x \sum y}{n\sum x^2 - (\Sigma x)^2} = \frac{(7 \times 549.8) - (56 \times 61.8)}{(7 \times 560) - (56)^2} = 0.49464$$

$$a = \frac{\sum y}{n} - b\frac{\Sigma x}{n} = \frac{61.8}{7} + 0.49464 \times \frac{56}{7} = 4.87142$$

Hence, required best line to fit given data is;

$$y = a + bx = 0.49464 + 4.87142 \log_e x$$

Chapter 4

Numerical Differentiation and Integration

NUMERICAL DIFFERENTIATION

We have,

Taylor series is;

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots\dots \quad (1)$$

Neglecting higher order of derivative,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

Again,

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \dots\dots \quad (3)$$

Neglecting higher order derivative,

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Adding equation (1) and (3) by neglecting higher derivative; we get,

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + 0$$

$$\therefore \quad f''(x) = \frac{f(x+h) - 2f(x) - f(x-h)}{h^2}$$

DERIVATIVE USING NEWTON FORWARD DIFFERENCE FORMULA

Newton's forward difference formula is,

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \dots\dots$$

Differentiating with respect to p; we get,

$$\frac{dy}{dP} = y_0 + \frac{2P-1}{2!}\Delta^2 y_0 + \frac{3P^2 - 6P + 2}{3!}\Delta^3 y_0 + \dots\dots$$

$$\because \quad P = \left(\frac{x - x_0}{h}\right) \Rightarrow \frac{dP}{dx} = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dP} \times \frac{dP}{dx} = \frac{1}{h}\left[y_0 + \frac{2P-1}{2!}\Delta^2 y_0 + \frac{3P^2 - 6P + 2}{3!}\Delta^3 y_0 + \frac{4P^3 - 18P^2 - 22P - 6}{4!}\Delta^4 y_0 + \dots\dots\right] \quad (1)$$

At $x = x_0$, $P = 0$. Hence, putting $P = 0$.

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \dots\dots\right]$$

Differentiating equation (I) with respect to x; we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dP}\left(\frac{dy}{dP}\right)\frac{dP}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6P-6}{3!}\Delta^3 y_0 + \frac{12P^2 - 36P - 22}{4!}\Delta^4 y_0 + \dots\dots\right]\frac{1}{h}$$

$$\therefore \quad \left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \dots\dots\right]$$

Similarly;

$$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \dots\dots\right]$$

Similarly, using Newton's Backward difference;

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2P+1}{2!}\nabla^2 y_n + \frac{3P^3 + 6P + 2}{3!}\nabla^3 y_n + \dots\dots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6P+6}{3!}\nabla^3 y_n + \frac{6P^2 + 18P + 11}{12!}\nabla^4 y_n + \dots\dots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \dots\dots\right].$$

Problem 1

A slider in a machine moves along a fixed straight rod. Its displacement x cm, along the rod is given below at different instant of time t seconds. Find velocity of slider and its acceleration at t = 0.2 sec. [T.U., 2070 Chaitra]

t	0.0	0.1	0.2	0.3	0.4
x	30.13	31.62	32.87	33.64	33.95

Solution:

The difference table is;

t	x	Δ	Δ^2	Δ^3	Δ^4
0	30.13				
		1.49			
0.1	31.62		−0.24		
		1.25		−0.24	
0.2	32.87		−0.48		0.26
		0.77		0.02	
0.3	33.64		−0.46		
		0.31			
0.4	33.95				

As derivative is required near the middle of table we use Stirling's formula

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left[\frac{\Delta x_0 + \Delta x_{-1}}{2}\right] - \frac{1}{6}\left[\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right] + \frac{1}{30}\left[\frac{\Delta^5 x_{-1} + \Delta^5 x_{-3}}{2}\right] + \dots\dots$$

$h = 0.1$

$t_0 = 0.2$

$\Delta x_0 = 0.77$

$\Delta x_{-1} = 1.25$

$\Delta^2 x_{-1} = -0.48$

$$\left(\frac{dx}{dt}\right)_{0.2} = \frac{1}{0.1}\left[\frac{0.77 + 1.25}{2}\right] - \frac{1}{6}\left[\frac{0.002 + 0.24}{2}\right] = 10.1 + 0.0198 = 10.080 \text{ m/sec.}$$

and, $$\frac{d^2x}{dt^2} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} + \dots\dots\right]$$

or, $$\left(\frac{d^2x}{dt^2}\right)_{0.2} = \frac{1}{0.1^2}\left[-0.46 - \frac{1}{12}\times 0.26\right] = -48.16 \text{ m/sec}^2$$

Problem 2

The distance y covered in time t by an object moving in a straight line is given below; approximate velocity at t = 1 sec.

t (sec)	0	1	2	3	4	5
y (meter)	0	15	71	143	245	367

Solution:

The difference table is;

t	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0					
		15				
1	15		41			
		56		−25		
2	71		16		39	
		72		14		−63
3	143		30		−24	
		102		−10		
4	245		20			
		122				
5	367					

$$\left(\frac{dy}{dt}\right)_{y_0} = \frac{1}{h}\left[y_0 - \frac{1}{2}\Delta y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0\right]$$

$h = 1$

$y_0 = 1$

$$\therefore \quad \left(\frac{dy}{dt}\right) = 1\left[15 - \left(\frac{1}{2}\times 41\right) + \frac{1}{3}(-25) - \frac{1}{4}(39) + \frac{1}{5}(-63)\right]$$

$$= 15 - 20.5 - 8.33 - 9.75 - 12.6$$

$$= -33.18 \text{ m/sec.}$$

Problem 3

Find y′(0.2) and y″(0.2) from the following data: [T.U., 2071 Shrawan]

x	0.1	0.2	0.3	0.4	0.5
y	2.6	8.2	15.4	25.6	37.8

Solution:
The difference table is;

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	2.6				
		5.6			
0.2	8.2		1.6		
		7.2		1.4	
0.3	15.4		3		− 2.4
		10.2		− 1	
0.4	25.6		2		
		12.2			
0.5	37.8				

$h = 0.1$

$$y'(0.2) = \frac{1}{h}\left[y_0 - \frac{1}{2}\Delta y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right] = \frac{1}{0.1}\left[7.2 - \frac{1}{2}\times 3 - \frac{1}{3}\times 1\right]$$

$$= 53.667$$

$$y''(0.2) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \dots\dots\right] = \frac{1}{0.1^2}[3+1]$$

$$= 400$$

$\therefore \quad y'(0.2) = 53.667$

$\therefore \quad y''(0.2) = 400$

Problem 4

A rod is rotating in a plane. The following table gives the angle θ (radian) through which the rod is turned for various values of the time t seconds, calculate the angular velocity and angular acceleration of the rod when t = 0.2 seconds. [T.U., 2067 Ashadh, 2069 Chaitra]

t	0.0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.20	4.67

Solution:
The forward table is;

t	θ	Δθ	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$	$\Delta^6\theta$
0	0						
		0.12					
0.2	0.12		0.25				
		0.37		0.01			
0.4	0.49		0.26		0		
		0.63		0.01		0	
0.6	1.12		0.27		0		0
		0.9		0.01		0	
0.8	2.02		0.28		0		
		1.18		0.01			
1.0	3.20		0.29				
		1.47					
1.2	4.67						

$$\text{Angular velocity (W)} = \left(\frac{d\theta}{dt}\right)_{0.2}$$

$$= \frac{1}{h}\left[y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\dots\right]$$

$$= \frac{1}{0.2}\left[0.37 - \frac{1}{2}\times 0.26 + \frac{1}{3}\times 0.01\right] = \frac{0.2433}{0.2}$$

$$= 1.2167 \text{ rad/sec.}$$

$$\text{Angular acceleration } (\alpha) = \frac{d^2\theta}{dt^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \dots\dots\right]$$

$$= \frac{1}{0.2^2}[0.26 - 0.01] = \frac{0.16}{0.2^2} = 4 \text{ rad/sec}^2$$

$\therefore$ Angular velocity = 1.2167 rad/sec

$\therefore$ Angular acceleration = 4 rad/sec^2

Problem 5

Derive formula for first derivative using Newton forward interpolation formula. [T.U., 2072 Kartik]

Solution:
For theory derivative using Newton's forward difference formula
Refer page no. 91

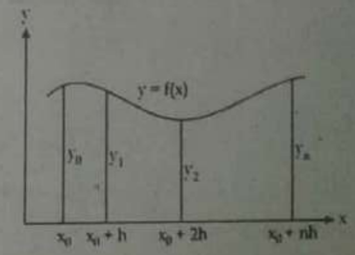
NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated value of the integral f(x) is called numerical integration.

$$I = \int_a^b f(x)dx$$

NEWTON–COTES FORMULA

Let $I = \int_a^b f(x)\,dx$, where f(x) takes values $y_0, y_1, y_2, \dots\dots, y_n$ for $x = x_0, x_1, x_2, \dots\dots, x_n$.



Let us divide the interval (a, b) into n sub-intervals of width h so that,

$$x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots\ldots, x_0 + 2h = b.$$

$$I = \int_{x_0}^{x_0+2h} f(x)\,dx = h\int_0^n f(x_0 + rh)\,dr \qquad [\text{Put } x = x_0 + rh,\ dx = h\,dr]$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\ldots\right] dr$$

[Using Newton's forward interpolation]

on integrating term by term we get,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2(r-2)}{24}\Delta^3 y_0 + \ldots\ldots\right] \quad (1)$$

This is called Newton's-Cotes formula.

This gives the general quadration formula for equidistant ordinate and is also known as Newton's cote formula.

There are following to solve integration.

i) Trapezoidal rule

ii) Simpson's $\frac{1}{3}$ rule

iii) Simpson's $\frac{3}{8}$ rule

i) TRAPEZOIDAL RULE

Take n = 1 and neglecting the higher order differences more than one in Newton's cote quadrature formula; we get,

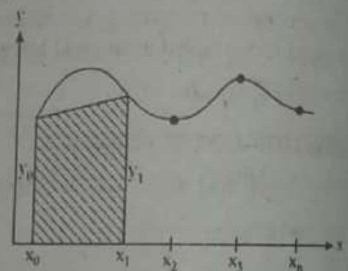
$$\int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = \frac{h}{2}[2y_0 + \Delta y_0] = \frac{h}{2}[y_0 + y_0 + \Delta y_0]$$

$$= \frac{h}{2}[y_0 + y_1] \qquad [\because y_0 + \Delta y_0 = y_1]$$

In the interval $(x_0 + h, x_0 + 2h)$; we get,

$$\int_{x_0+h}^{x_0+2h} f(x)dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right]$$

$$= \frac{h}{2}[y_1 + y_2]$$



Similarly;

$$\int_{x_0+2h}^{x_0+3h} f(x)dx = \frac{h}{2}[y_2 + y_3]$$

In general;

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}[y_{n-1} + y_n]$$

Adding all; we get,

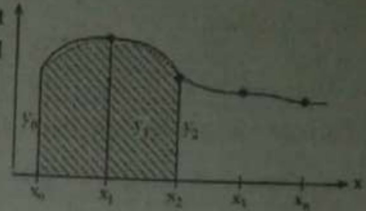
$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[y_0 + y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1} + y_n]$$

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})]$$

This is called Trapezoidal formula.

ii) SIMPSON'S $\frac{1}{3}$ RULE

Putting n = 2 in Newton's-cotes formula and taking curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as parabola as in figure, a polynomial of second order so that difference of order higher than second vanish, we get,



$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly;

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

....

....

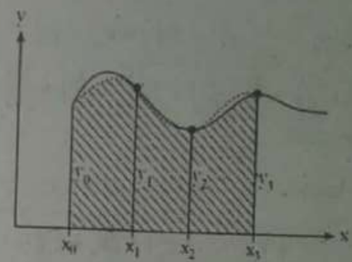
$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n);$$

where, n being even.

Adding all these integrals,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2})]$$

iii) SIMPSON'S $\frac{3}{8}$ RULE



Putting n = 3 in Newton's-cotes formula and taking the curve through (x_i, y_i): i = 0, 1, 2, 3 as polynomial of third order, the difference above third order vanishes.

$$\therefore \int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly;

$$\int_{x_0+3h}^{x_0+5h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all integrals;

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{n-3})]$$

SOLVED NUMERICALS

Problem 1

The velocity 'V' of particle at a distance 'S' from a point on its path is given in the table below.

S (f(t))	0	10	20	30	40	50	60
V (f(t)/sec)	47	58	64	65	61	52	38

Estimate the time taken to travel a distance of 60 feet by using Simpson's $\frac{1}{3}$ rule. Compare the result with Simpson's $\frac{3}{8}$ rule. [T.U. 2068 Baishakh]

Solution:

If 't' second be the time taken to travel a distance s(f(t)); then,

$$\frac{ds}{dt} = v$$

or, $\frac{dt}{ds} = \frac{1}{v} = y \text{ (say)}$

S (ft)	0	10	20	30	40	50	60
$\frac{1}{v}$	$\frac{1}{47}$	$\frac{1}{58}$	$\frac{1}{64}$	$\frac{1}{65}$	$\frac{1}{61}$	$\frac{1}{52}$	$\frac{1}{38}$

0 1 2 3 4 . 5 6

Then,

$$|t|_{s=0}^{s=60} = \int_0^{60} y\, ds$$

where, h = 10

By Simpson's $\frac{1}{3}$ rule; we know,

$$\int_0^{60} y\, ds = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6]$$
$$= \frac{10}{3}\left[\frac{1}{47} + 4\left(\frac{1}{58} + \frac{1}{65} + \frac{1}{52}\right) + 2\left(\frac{1}{64} + \frac{1}{61}\right) + \frac{1}{58}\right]$$
$$= 1.0635 \text{ seconds}$$

By Simpson's $\frac{3}{8}$ rule; we know,

$$\int_0^{60} y\, ds = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3 + y_6)]$$
$$= \frac{30}{8}\left[\frac{1}{47} + 3\left(\frac{1}{58} + \frac{1}{64} + \frac{1}{61} + \frac{1}{52}\right) + 2\times\left(\frac{1}{65} + \frac{1}{38}\right)\right]$$
$$= 1.0644 \text{ seconds}$$

Problem 2

Derive the expression of Simpson's $\frac{1}{3}$ rule for integration. [T.U. 2069 Bhadra]

Solution: Refer to the definition part on page no. 97

Problem 3

Evaluate the following using Simpson's $\frac{1}{3}$ rule (take h = 0.2). [T.U. 2070 Magh]

$$\int_0^2 \frac{4e^x}{1+x^3} dx$$

Solution:

Given that;

$$\int_0^2 \frac{4e^x}{1+x^3} dx$$

h = 0.2

We have,

$$f(x) = y = \frac{4e^x}{1+x^3}$$

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y = f(x)	4	4.84684	5.6084	5.9938	5.88767	5.73656	4.86820	4.3325	3.88778	3.54195	3.28402
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_n

By Simpson's $\frac{1}{3}$ rule; we have,

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \dots) + 4(y_1 + y_3 + y_5 + y_7 + \dots)]$$

$$\int_0^2 \frac{4e^x}{1+x^3} dx = \frac{0.2}{3}[(4 + 3.28402) + 2(5.6084 + 5.88767 + 4.86820 + 3.88778)$$
$$+4(4.84684 + 5.9938 + 5.43656 + 4.3325 + 3.54195)]$$

$$\int_0^2 \frac{4e^x}{1+x^3} dx = 9.62630$$

Problem 4

Integrate $\int_0^{\pi} (1 + 3\cos^2 x)dx$

i) Trapezoidal Rule

ii) Simpson's $\frac{3}{8}$ Rule, taking number of interval n = 6 [P.U. 2005 Spring]

Solution:

We have,

$$nh = b - a$$

or, $h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$

$$f(x_i) = y_i = 1 + 3\cos^2 x$$

X_i	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
Y_i	4	3.9997	3.9989	3.9977	3.9959	3.9937	3.9909
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By trapezoidal rule; we have,

$$i = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{12}[4 + 3.9909 + 2(3.9997 + 3.9989 + 3.9977 + 3.9959 + 3.9937)]$$
$$= 12.494$$

By Simpson's $\frac{3}{8}$ rule; we have,

$$i = \frac{3h}{8}[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$
$$= \frac{3\pi}{48}[4 + 3.9909 + (2 \times 3.9977) + 3(3.9997 + 3.9989 + 3.9959)]$$
$$i = 10.204$$

Problem 5

Evaluate the following integral $\int_0^{\frac{\pi}{2}} \sin x\, dx$. Using

i) Trapezoidal Rule

ii) Simpson's $\frac{1}{3}$ Rule

iii) Simpson's $\frac{3}{8}$ comment on the result.

[P.U. 2006 Spring, 2007 Spring, 2010 Fall, 2011 Fall, 2011 Spring]

Solution:

We have,

$$nh = b - a$$

or, $h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$

Let us define a function $f(x_i) = y_i = \sin x$

X_i	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
Y_i	0	0.00456	0.0091	0.0137	0.0182	0.0228	0.0274
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By trapezoidal rule; we have,

$$i = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$
$$= \frac{\pi}{24}[0 + 0.274 + 2(0.00456 + 0.0091 + 0.0137 + 0.0182 + 0.0228)]$$
$$i = \int_0^{\frac{\pi}{2}} \sin x\, dx = 0.02148$$

ii) By Simpson's $\frac{1}{3}$ rule; we have,

$$i = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$
$$= \frac{\pi}{36}[(0 + 0.0274) + 2(0.0091 + 0.0182) + 4(0.00456 + 0.0137 + 0.0228)]$$
$$= \int_0^{\frac{\pi}{2}} \sin x\, dx = 0.02147$$

iii) By Simpson's $\frac{3}{8}$ rule; we have,

$$i = \frac{3h}{8}[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$
$$= \frac{3\pi}{96}[0 + 0.274 + (2 \times 0.0137) + 3(0.00456 + 0.0091 + 0.0182 + 0.0228)]$$
$$\therefore \int_0^{\frac{\pi}{2}} \sin x\, dx = 0.023010$$

Here, value obtained from Trapezoidal and Simpson's $\frac{1}{3}$ rule is near the exact value (0.02153). But the value obtained by Simpson's $\frac{3}{8}$ rule is largely deviated from exact value.

Problem 6

Integrate the following function by Simpson's $\frac{3}{8}$ rule. Take n = 5

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx$$

[P.U. 2007 Fall]

Solution:

We have,

$$nh = b - a$$

or, $h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{5} = \frac{\pi}{10}$

$$f(x) = y_i = \frac{\sin x}{x}$$

X_i	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{\pi}{2}$
Y_i	0	0.017453	0.017452	0.017451	0.017451	0.17451
	y_0	y_1	y_2	y_3	y_4	y_5

By Simpson's $\frac{3}{8}$ rule; we have,

$$\int_0^{\frac{\pi}{2}} f(x)dx = i = \frac{3h}{8}[(y_0 + y_5) + 2y_3 + 3(y_1 + y_2 + y_4)]$$
$$= \frac{3\pi}{80}[0 + 0.017451 + (2 \times 0.01745) + 3(0.017453 + 0.017452 + 0.017451)]$$
$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx = 0.02467$$

Problem 7

Integrate the given integral $\int_1^3 \cos x\, dx$;

using Trapezoidal, Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule [P.U. 2008 Spring, 2009 Fall]

Solution:

We have,

$nh = b - a$

or, $h = \frac{b-a}{n} = \frac{3-1}{5} = \frac{2}{5}$

X_i	1	$\frac{7}{5}$	$\frac{9}{5}$	$\frac{11}{5}$	$\frac{13}{5}$	3
$Y_i = f(x) = \cos x$	0.9998	0.9997	0.9995	0.9992	0.9989	0.9986
	y_0	y_1	y_2	y_3	y_4	y_5

By trapezoidal rule; we have,

$$i = \frac{h}{2}[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{2}{10}[0.9998 + 0.9986 + 2(0.9997 + 0.9995 + 0.9992 + 0.9989)]$$

$$i = 1.9986$$

$\therefore$ $\int_1^3 \cos x\, dx = 1.9986$

By Simpson's $\frac{1}{3}$ rule; we have,

$$i = \frac{h}{3}[y_0 + y_5 + 2(y_2 + y_4) + 4(y_1 + y_3)]$$

$$= \frac{2}{15}[0.9998 + 0.9986 + 2(0.9995 + 0.9989) + 4(0.9997 + 0.9992)]$$

$\therefore$ $\int_1^3 \cos x\, dx = 1.81$

By Simpson's $\frac{3}{8}$ rule; we have,

$$i = \frac{3h}{8}[y_0 + y_5 + 2y_3 + 3(y_1 + y_2 + y_4)]$$

$$= \frac{3\pi}{80}[0.9998 + 0.9986 + 2 \times 0.9992 + 3(0.9997 + 0.9995 + 0.9989)]$$

$\therefore$ $\int_1^3 \cos x\, dx = 1.9484$

Problem 8

Evaluate the following integral using Simpson's $\frac{3}{8}$ rule.

$\int_1^2 (x^3 + 1)dx$; using $n = 3$ [P.U. 2009 Spring]

Solution:

We have,

$nh = b - a$

or, $h = \frac{b-a}{n} = \frac{2-1}{3} = \frac{1}{3}$

$f(x) = x^3 + 1$

X_i	1	$\frac{4}{3}$	$\frac{5}{3}$	2
Y_i	2	3.37	5.62	9
	y_0	y_1	y_2	y_3

By Simpson's $\frac{3}{8}$ rule; we have,

$$i = \frac{3h}{8}[y_0 + y_3 + 3(y_1 + y_2)] = \frac{3}{24}[2 + 9 + 3(3.37 + 5.62)] = 4.74$$

$\therefore$ $\int_1^2 (x^3 + 1)dx = 4.74$

Problem 9

Integrate the given integral $\int_0^1 \frac{1}{1+x^2}dx$ using Trapezoidal, Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule. [P.U. 20012 Fall]

Solution:

We have,

$nh = b - a$

or, $h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$

$f(x) = \frac{1}{1+x^2}$

X_i	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
Y_i	1	0.9615	0.8620	0.7352	0.6097	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

By trapezoidal rule; we have,

$$i = \frac{h}{2}[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{1}{10}[1 + 0.5 + 2(0.9615 + 0.8620 + 0.7352 + 0.6097)]$$

$\therefore$ $\int_0^1 \frac{1}{1+x^2}dx = 0.78368$

By Simpson's $\frac{1}{3}$ rule; we have,

$$i = \frac{h}{3}[y_0 + y_5 + 2(y_2 + y_4) + 4(y_1 + y_3)]$$

$$= \frac{1}{15}[1 + 0.5 + 2(0.8620 + 0.6097) + 4(0.9615 + 0.7352)]$$

$\therefore$ $\int_0^1 \frac{1}{1+x^2}dx = 0.74863$

By Simpson's $\frac{3}{8}$ rule; we have,

$$i = \frac{3h}{8}[y_0 + y_5 + 2y_3 + 3(y_1 + y_2 + y_4)]$$

$$= \frac{3}{40}[1 + 0.5 + 2 \times 0.7352 + 3(0.9615 + 0.8620 + 0.6097)]$$

$\therefore$ $\int_0^1 \frac{1}{1+x^2}dx = 0.77025$

Problem 10

Derive Simpson's $\frac{1}{3}$ rule for integration. Evaluate the following integral using Simpson's $\frac{1}{3}$ rule, taking h = 0.25.

$$\int_0^1 \frac{e^x}{x+1} dx$$

[T.U., 2069 Chaitra]

Solution:

Simpson's $\frac{1}{3}$ rule for integration

Refer to page no. 97

Given that;

$$\int_0^1 \frac{e^x}{x+1} dx$$

h = 0.25

We have,

$$f(x) = y = \frac{e^x}{x+1}$$

x	0	0.25	0.5	0.75	1
y = f(x)	1	1.0272	1.0991	1.2097	1.3591
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\frac{1}{3}$ rule;

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \dots\dots) + 4(y_1 + y_3 + y_5 + \dots\dots)]$$

$$= \frac{0.25}{3}[(1 + 1.3591) + 2(1.0991) + 4(1.0272 + 1.2097)]$$

(0.7456)(2.3591 + 2.1982 + 8.9476)

$$= 10.069$$

Problem 11

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Simpson's $\frac{1}{3}$ rule taking unit interval size.

Solution:

Given that;

$$\int_0^6 \frac{dx}{1+x^2}$$

h = 1

We have,

$$f(x) = y = \frac{dx}{1+x^2}$$

x	0	1	2	3	4	5	6
f(x) = y	1	0.5	0.2	0.1	0.0588	0.0384	0.0270
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rule;

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{1}{3}[(1 + 0.0270) + 2(0.2 + 0.0588) + 4(0.5 + 0.1 + 0.0384)]$$

$$= \frac{1}{3}[1.027 + 0.5176 + (0.6384)4]$$

$$= 1.3660$$

GAUSS INTEGRATION/GAUSS LEGENDRE INTEGRATION

Gauss Integration is based on the concept that the accuracy of numerical integration can be improved by choosing the sampling point rather than on the basis of equal spacing.

Let us consider a curve, a straight line has been moved up such that the area B = A + C, the sampling points are moved away from the end point. The problem is to compute the value of x_1 and x_2 and to choose appropriate weights w_1 and w_2.

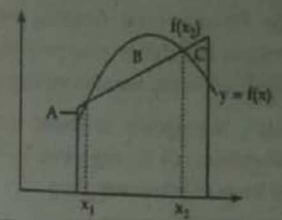


Figure: Illustration of Gauss integration

The method of implementing the strategy of obtaining the value of x_i and w_i and that of the integral of f(x) is called Gauss Legendre integration.

Gauss integration assumes an approximation of the form.

$$I_g = \int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Formula

i) For two point

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

where, $w_1 = w_2 = 1$

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

ii) For three point

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

Here,

$$w_1 = \frac{5}{9} = 0.555$$

$$w_2 = \frac{8}{9} = 0.8889$$

$$w_3 = \frac{5}{9} = 0.555$$

$$x_1 = -\sqrt{\frac{3}{5}} = -0.77459$$

$$x_2 = 0$$

$$x_1 = \sqrt{\frac{3}{5}} = 0.77459$$

$$\int_{-1}^{1} f(x)dx = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

COMPARISON BETWEEN SIMPSON'S METHOD AND GAUSSIAN INTEGRATION

Simpson's method for calculation of integration is one of Newton's cotes formula. The Newton-cotes formula was derived by integrating the Newton-Gregory forward difference interpolating polynomial. Consequently, all the rules were based on evenly faced sampling points within the range of integral.

Gauss integration is based on the concept that the accuracy of numerical integration can be improved by choosing the sampling point wisely rather than on the basis of equal sampling.

SOLVED NUMERICALS

Problem 1

Evaluate the integral $I = \int_0^1 e^{-x^2} dx$ and compare the result in both the condition of Simpson's $\frac{1}{3}$ rule and 3 point gauss Legendre method. [T.U. 2069 Bhadra]

Solution:

For Simpson's method find, the integral for n = 3 as,

Given that;

$$I = \int_0^1 e^{-x^2} dx$$

We have,

$$nh = b - a$$

or, $h = \frac{b-a}{n} = \frac{1-0}{3} = \frac{1}{3} = 0.33$

Here,

$$f(x) = e^{-x^2}$$

X	0	0.33	0.66	0.99
Y	1	0.89682	0.646876	0.37527
	y_0	y_1	y_2	y_3

By Simpson's $\frac{1}{3}$ rule; we have,

$$I = \frac{0.33}{3}[1 + 0.37527 + 2(0.646876) + 4(0.646876)] = 0.68819$$

According to Gauss Legendre; we have,

We need to change the limit of integral;

$I = \int_0^1 e^{-x^2} dx$; from (0, 1) to (−1, 1) by transformation

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

$$x = \frac{1}{2}(1-0)u + \frac{1}{2}(1+0)$$

$$x = \frac{u+1}{2}$$

$$dx = \frac{1}{2}du$$

Now,

$$I = \int_0^1 e^{-x^2} dx = \int_{-1}^{1} e^{-\left(\frac{u+1}{2}\right)^2} \frac{1}{2} du$$

$$\therefore \int_{-1}^{1} f(u) = \frac{1}{2} e^{-\left(\frac{u+1}{2}\right)^2} du$$

$$f(0) = \frac{1}{2} e^{-\left(\frac{u+1}{2}\right)^2} = \frac{1}{2} \times e^{-\frac{1}{4}} = 0.3894$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = 0.49368$$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.2275$$

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] = 0.7467$$

Problem 2

Evaluate the integral $I = \int_0^{1.5} \sin x\, dx$, compare the absolute error in both condition for Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule.

Solution:

Given that;

$$I = \int_0^{1.5} \sin x\, dx$$

$$nh = b - a$$

or, $h = \frac{b-a}{n}$

For n = 6,

$$h = \frac{1.5 - 0}{6} = 0.25$$

Here,

$$f(x) = \sin x$$

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X	0	0.25	0.5	0.75	1	1.25	1.5
f(x)	0	0.24740	0.47943	0.68164	0.84147	0.94898	0.997495
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rule; we have,

$$I = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{0.25}{3}[(0 + 0.997495) + 2(0.04794 + 0.84147) + 4(0.24740 + 0.681647 + 0.94898)]$$

$$I = 0.929763$$

By Simpson's $\frac{3}{8}$ rule; we have,

$$I = \frac{3h}{8}[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3 \times 0.25}{8}[(0 + 0.997495) + (2 \times 0.68164) + 3(0.24740 + 0.47943 + 0.84147 + 0.94898)]$$

$$I = 0.9293077$$

So absolute error by using both rules is;

$$E = |0.9292763 - 0.9293077|$$

$$E = 0.0000314$$

Problem 3

Evaluate $\int_2^3 \frac{\cos 2x}{1 + \sin x}$ by using Gauss quadrature three point formula.

Solution:

First change the limit (2, 3) to (−1, 1) by transformation

$$x = \frac{1}{2}(b - a)u + \frac{1}{2}(b + a) = \frac{1}{2}(3 - 2)U + \frac{1}{2}(3 + 2) = \frac{u}{2} + \frac{5}{2} = \frac{u + 5}{2}$$

$$dx = \frac{1}{2}du$$

Now,

$$I = \int_{-1}^{1} \frac{\cos 2\left(\frac{u+5}{2}\right)}{1 + \sin\left(\frac{u+5}{2}\right)} \frac{1}{2} du = \int_{-1}^{1} \frac{1}{2} \frac{\cos(u + 5)}{1 + \sin\left(\frac{u+5}{2}\right)} du$$

$$I = \int_{-1}^{1} \frac{\cos 2\left(\frac{u+5}{2}\right)}{1 + \sin\left(\frac{u+5}{2}\right)} \frac{1}{2} du = \int_{-1}^{1} \frac{\cos(u + 5)}{1 + \sin\left(\frac{u+5}{2}\right)} du$$

We know,

$$\int_{-1}^{1} f(x)\, du = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] \quad (1)$$

$$f(0) = \frac{1}{2} \frac{\cos 5}{1 + \sin\frac{5}{2}} = 0.08872$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = \frac{1}{2} \frac{\cos 2\left(\frac{-\sqrt{\frac{3}{5}} + 5}{2}\right)}{1 + \sin\left(\frac{-\sqrt{\frac{3}{5}} + 5}{2}\right)} = -0.12601$$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.348936$$

$$\therefore \quad I = \left(\frac{8}{9} \times 0.08872\right) + \frac{5}{9}(-0.12601 + 0.348936) = 0.20135$$

Problem 4

Evaluate $I = \int_0^2 \frac{x^2 + 2x + 1}{1 + (x + 1)^4} dx$ using Gauss two point and three point formula.

Solution:

Given that;

$$I = \int_0^2 \frac{x^2 + 2x + 1}{1 + (x + 1)^4} dx$$

Changing limit (0, 2) to by (−1, 1) transformation,

$$x = \frac{1}{2}(b - a)u + \frac{1}{2}(b + a) = \frac{1}{2}(2 - 0)u + \frac{1}{2}(2 + 0) = u + 1$$

$$dx = du$$

$$I = \int_{-1}^{1} \frac{(u + 1)^2 + 2(u + 1) + 1}{1 + ((u + 1) + 1)^4} du = \int_{-1}^{1} \left(\frac{u^2 + 2u + 1 + 2u + 2 + 1}{1 + (u + 2)^4}\right) du$$

$$= \int_{-1}^{1} \frac{u^2 + 4u + 4}{1 + (u + 2)^4} du$$

Here,

$$f(u) = \frac{u^2 + 4u + 4}{1 + (u + 2)^4}$$

$$f(0) = \frac{4}{1 + 2^4} = \frac{4}{17}$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = \frac{\left(-\sqrt{\frac{3}{5}}\right)^2 + 4 \times \left(-\sqrt{\frac{3}{5}}\right) + 4}{1 + \left(-\sqrt{\frac{3}{5}} + 2\right)^4} = 0.4614$$

$$f\left(\sqrt{\tfrac{3}{5}}\right) = \frac{\left(\sqrt{\frac{3}{5}}\right)^2 + 4\times\sqrt{\frac{3}{5}}+4}{1+\left(\sqrt{\frac{3}{5}}+2\right)^4} = 0.1277$$

We know,

Gauss-3 point formula is;

$$\int_{-1}^{1} f(x)\,du = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\tfrac{3}{5}}\right) + f\left(\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \left(\frac{8}{9}\times\frac{4}{17}\right) + \frac{5}{9}(0.4614 + 0.1277)$$

$$= 0.5365$$

Problem 5

Evaluate $\int_3^5 \frac{1}{x^6+5}dx$ by using Gauss Legendre two and three point formula.

[P.U. 2006 Fall]

Solution:

By changing the limit (3, 5) to (−1, 1) by transformation,

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(5-3)u + \frac{1}{2}(5+3) = u+4$$

$$dx = du$$

$$I = \int_3^5 \frac{1}{x^6+5}dx = \int_{-1}^{1}\frac{1}{(u+4)^6+5}du$$

$$f(u) = \frac{1}{(u+4)^6+5}$$

i) Gauss Legendre (Gauss-quadrature) (n = 2) two point formula is;

$$f(0) = \frac{1}{4^6+5} = \frac{1}{4101}$$

$$f\left(-\sqrt{\tfrac{3}{5}}\right) = \frac{1}{\left(-\sqrt{\frac{3}{5}}+4\right)^6+5} = 8.8424\times10^{-4}$$

$$f\left(\sqrt{\tfrac{3}{5}}\right) = 8.4371\times10^{-5}$$

By Gauss three points formula;

$$\int_{-1}^{1} f(x)\,du = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\tfrac{3}{5}}\right) + f\left(\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \left(\frac{8}{9}\times\frac{1}{4101}\right) + \frac{5}{9}(8.8424\times10^{-4} + 8.4371\times10^{-5})$$

$$= 7.5486\times10^{-4}$$

Problem 6

Using Gauss-quadrature formula for n = 3, evaluate

$$\int_0^1 \frac{dx}{1+x^2}$$

[P.U. 2007 Spring]

Solution:

By changing limit (0, 1) to (−1, 1) by transformation

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(1-0)u + \frac{1}{2}(1+0) = \frac{u}{2}+\frac{1}{2} = \frac{u+1}{2}$$

$$dx = \frac{1}{2}du$$

Now,

$$I = \int_0^1 \frac{dx}{1+x^2} = \int_{-1}^{1}\frac{1}{1+\left(\frac{u+1}{2}\right)^2}\frac{1}{2}du$$

$$\therefore\quad f(u) = \frac{2}{u^2+2u+5}$$

From Gauss Legendre three point formula; we have,

$$f(u) = \frac{1}{u+3}$$

$$f(0) = \frac{1}{3}$$

$$f\left(-\sqrt{\tfrac{3}{5}}\right) = \frac{1}{-\sqrt{\frac{3}{5}}+3}$$

$$f\left(\sqrt{\tfrac{3}{5}}\right) = \frac{1}{\sqrt{\frac{3}{5}}+3}$$

$$\int_{-1}^{1} f(x)\,du = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\tfrac{3}{5}}\right) + f\left(\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \left(\frac{8}{9}\times\frac{1}{3}\right) + \frac{5}{9}\left(\frac{1}{-\sqrt{\frac{3}{5}}+3} + \frac{1}{\sqrt{\frac{3}{5}}+3}\right) = \frac{8}{27}+\frac{25}{63}$$

$$= 0.6931$$

Problem 7

Using two and three point Gauss Legendre formula. Evaluate

$$\int_{0.2}^{1.5} e^{-x^2}dx$$

[P.U. 2008 Spring, 2009 Fall]

Solution:

Changing limit from (0.2, 1.5) to (−1, 1)

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(1.5-0.2)u + \frac{1}{2}(1.5+0.2)$$

$$= (0.654 + 0.85)$$

Now,

$$dx = 0.65$$

$$I = \int_{0.2}^{1.5} e^{-x^2} dx = \int_{-1}^{1} e^{-(0.65u+0.85)^2} \times 0.65$$

$$\therefore \quad f(u) = e^{-(0.65u+0.85)^2}$$

$$f(0) = e^{-(0.65u+0.85)^2} = 0.4855$$

$$f\left(-\sqrt{\tfrac{3}{5}}\right) = f(-0.7746) = e^{-[0.65(-0.7746)+0.85]^2} = 0.8869$$

$$f\left(\sqrt{\tfrac{3}{5}}\right) = f(0.7746) = 0.1601$$

Using Gauss 3 point formula;

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\tfrac{3}{5}}\right) + f\left(\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \left(\frac{8}{9} \times 0.4855\right) + \frac{5}{9}(0.8869 + 0.1601)$$

$$= 0.65865$$

Problem 8

Evaluate the following integral using Gaussian three point formula.

$$\int_0^2 \frac{x\sin(\cos x)}{2} dx$$ [T.U., 2071 Shrawan]

Solution:

Given that;

$$I = \int_0^2 \frac{x\sin(\cos x)}{2}$$

Changing limit (0, 2) to (−1, 1),

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

$$= \frac{1}{2}(2-0)u + \frac{1}{2}(2+0)$$

$$= 4 + 1$$

$$\therefore \quad dx = du$$

$$\therefore \quad I = \int_0^2 \frac{x\sin(\cos x)}{2} = \int_{-1}^{1} \frac{(u+1)\sin[\cos(u+1)]}{2} du$$

$$\therefore \quad f(u) = \frac{1}{2}(u+1)\sin[\cos(u+1)]$$

$$f(0) = \sin(\cos 1) = 0.51439$$

$$f\left(-\sqrt{\tfrac{3}{5}}\right) = \frac{1}{2}\left[\left\{\left(-\sqrt{\tfrac{3}{5}}\right) + 1\right\}\sin\left\{\cos\left(-\sqrt{\tfrac{3}{5}} + 1\right)\right\}\right] = 0.09326$$

$$f\left(\sqrt{\tfrac{3}{5}}\right) = -0.17835$$

$$\therefore \quad \int_{-1}^{1} f(x)\, du = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\tfrac{3}{5}}\right) + f\left(\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \left(\frac{8}{9} \times 0.51439\right) + \frac{5}{9}(0.09326 - 0.17835)$$

$$= 0.40995$$

Problem 9

Evaluate the following using Gaussian three point formula;

$$\int_0^2 x\sin(\cos x) + 2\, dx$$ [T.U., 2071 Shrawan]

Solution:

Given that;

$$I = \int_0^2 x\sin(\cos x) + 2\, dx$$

Changing limit (0, 2) to (−1, 1); we have,

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(2-0)u + \frac{1}{2}(2+0) = u + 1$$

$$dx = du$$

Now,

$$I = \int_{-1}^{1} (u+1)\sin[\cos(u+1)] + 2\, du$$

$$\therefore \quad f(u) = (u+1)\sin[\cos(u+1)] + 2$$

$$f(0) = 2.5143$$

$$f\left(-\sqrt{\tfrac{3}{5}}\right) = \left(-\sqrt{\tfrac{3}{5}} + 1\right)\sin\left[\cos\left(-\sqrt{\tfrac{3}{5}} + 1\right)\right] + 2$$

$$= (0.2254 \times 0.8275) + 2 = 0.1865 + 2$$

$$= 2.1865$$

$$f\left(\sqrt{\tfrac{3}{5}}\right) = \left(\sqrt{\tfrac{3}{5}} + 1\right)\sin\left[\cos\left(\sqrt{\tfrac{3}{5}} + 1\right)\right] + 2 = [1.7745 \times (-0.2010)] + 2$$

$$= 1.6433$$

By Gauss 3–pointy formula;

$$\int_{-1}^{1} f(x)\, du = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\tfrac{3}{5}}\right) + f\left(\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \left(\frac{8}{9} \times 2.5143\right) + \frac{5}{9}(2.1865 + 1.6433) = 2.2349 + \frac{3.8298 \times 5}{9}$$

$$= 4.3625$$

Problem 10

Evaluate $\int_0^{\pi} x\sin x\, dx$ using Gauss 3–point Legendre formula. [T.U., 2072 Kartik]

Solution:

Given that;

$$I = \int_0^{\pi} x\sin x\, dx$$

Changing limit (0, π) to (−1, 1); we have,

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$
$$= \frac{1}{2}(\pi - 0)u + \frac{1}{2}(\pi + 0)$$
$$= \frac{\pi}{2}u + \frac{\pi}{2}$$
$$= \frac{\pi}{2}(1+u)$$
$$dx = \frac{\pi}{2}$$

Now,

$$I = \int_{-1}^{1} x \sin x \, dx = \int_{-1}^{1} \frac{\pi}{2}(1+u)\sin\left(\frac{\pi}{2}(1+u)\right)\frac{\pi}{2}$$

$$f(0) = \frac{\pi}{2}\sin\left\{\frac{\pi}{2}(1+0)\right\}\frac{\pi}{2} = 2.4674$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = \frac{\pi}{2}\left(1-\sqrt{\frac{3}{5}}\right)\sin\left[\frac{\pi}{2}\left(1-\sqrt{\frac{3}{5}}\right)\right]\frac{\pi}{2}$$
$$= \left(\frac{\pi}{2}\times 0.2254\right)\sin\left[\frac{\pi}{2}\times 0.2254\right]\times\frac{\pi}{2}$$
$$= 0.3540 \times 0.54460$$
$$= 0.19279$$

$$f\left(\sqrt{\frac{3}{5}}\right) = \frac{\pi}{2}\left(1+\sqrt{\frac{3}{5}}\right)\sin\left[\frac{\pi}{2}\left(1+\sqrt{\frac{3}{5}}\right)\right]\frac{\pi}{2}$$
$$= \left(\frac{\pi}{2}\times 1.7745\right)\sin\left[\frac{\pi}{2}\times 1.7745\right]\times\frac{\pi}{2}$$
$$= 1.5180$$

∴ By Gauss 3-point formula;

$$\int_{-1}^{1} f(x)\,du = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$
$$= \left(\frac{8}{9}\times 2.4674\right) + \frac{5}{9}(1.5180 + 0.1927)$$
$$= 2.1932 + 0.9504$$
$$= 3.1436$$

ROMBERG INTEGRATION

The accuracy of numerical integration can be improved in two ways;

i) By changing the number of subinterval (*i.e.*, by decreasing 'h') this decreases the magnitude of error terms.
ii) By using higher order methods, this eliminates the lower order errors term, here the order of the method is varied and therefore, this method is known as variable order approach.

The Romberg integration method uses the trapezoidal rule. It use trapezoidal rule in iterative way.

TRAPEZOIDAL RULE

$$I = \int_a^b f(x)dx = \frac{h}{2}[f(a) + f(b)]$$

Now,

Find 'I' with 'h', which will give I_1 and E_1 similarly, 'I' is found with h_2 which will give I_2.

$$E_1 = -\frac{(b-a)h_1^2}{12}y''(x)$$
$$E_2 = -\frac{(b-a)h_2^2}{12}y''(x)$$

$$\therefore \quad \frac{E_1}{E_2} = \frac{h_1^2}{h_2^2}$$

$$\text{or,} \quad \frac{E_1}{E_2 - E_1} = \frac{h_2^2}{h_1^2 - h_2^2}$$
$$I = I_1 + E_1 = I_2 + E_2$$
$$E_2 - E_1 = I_1 - I_2$$

Now,

$$\frac{E_1}{E_2 - E_1} = \frac{h_2^2}{h_1^2 - h_2^2}$$
$$E_1 = \frac{h_2^2}{h_1^2 - h_2^2}(I_1 - I_2)$$
$$I = I_1 + \frac{h_2^2}{h_1^2 - h_2^2}(I_1 - I_2) = \frac{I_1h_2^2 - I_2h_1^2}{h_1^2 - h_2^2}$$

If $h_1 = h$

$$h_2 = \frac{h}{2}$$

$$\therefore \quad I = \frac{I_1\left(\frac{h}{2}\right)^2 - I_2(h)^2}{(h)^2 - \left(\frac{h}{2}\right)^2} = \frac{4I_2 - I_1}{3}$$

$$\therefore \quad I\left(h, \frac{h}{2}\right) = \frac{4I\left(\frac{h}{2}\right) - I(h)}{3} \qquad (1)$$

We use trapezoidal rule several times successively having 'h' and apply in equation (1) to each pair of values as per the following scheme.

$I(h)$			
	$I\left(h, \frac{h}{2}\right)$		
$I\left(\frac{h}{2}\right)$		$I\left(h, \frac{h}{2}, \frac{h}{4}\right)$	
	$I\left(\frac{h}{2}, \frac{h}{4}\right)$		$I\left(h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right)$
$I\left(\frac{h}{4}\right)$		$I\left(\frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right)$	
	$I\left(\frac{h}{4}, \frac{h}{8}\right)$		
$I\left(\frac{h}{8}\right)$			

Procedure to Find Romberg Integration

Use trapezoidal value to find

I_1: Divide the interval into two parts

I_2: Divide interval into four parts

I_3: Divide interval into eight parts

Then, calculate

$$I_4 = I_2 + \frac{I_2 - I_1}{3}$$

and, $I_5 = I_3 + \frac{I_3 - I_2}{3}$

Now, calculate

$$I = I_5 + \frac{1}{3}(I_5 - I_4)$$

SOLVED NUMERICALS

Problem 1

Use Romberg integration method to evaluate the integral $\int_1^2 \frac{dx}{x}$ correct up to place taking size as $h = \frac{b-a}{2}$

Solution:

We have,

$$h = \frac{b-a}{2} = \frac{2-1}{2} = 0.5$$

Therefore, taking $h = 0.5$, $\frac{0.5}{2} = 0.25$ and $\frac{0.25}{2} = 0.125$

Let us calculate the given integral using trapezoidal rule

$$f(x) = y_i = \frac{1}{x}$$

i) When h = 0.5

$X_i =$	1	1.5	2
$Y_i =$	1	0.6667	0.5

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{0.5}{2}[1 + 0.5 + 2 \times 0.6667] = 0.70835$$

ii) When h = 0.25

X_i	1	1.25	1.5	1.75	2
Y_i	1	0.8	0.6667	0.57143	0.5

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = \frac{0.25}{2}[1 + 0.5 + 2(0.8 + 0.6667 + 0.5714$$

$I_2 = 0.697033$

iii) When h = 0.125

X_i	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
Y_i	1	0.8889	0.8	0.72727	0.6667	0.6154	0.57143	0.533	0.5

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$
$$= \frac{0.125}{2}[1 + 0.5 + 2(0.8889 + 0.8 + 0.72727 + 0.6667 + 0.6154 + 0.57143 + 0.533)]$$

$I_3 = 0.69408$

$\therefore$ $I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.697033 + \frac{0.697033 - 0.70835}{3}$

$I_4 = 0.69326$

$\therefore$ $I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.69408 + \frac{0.69408 - 0.697033}{3}$

$I_5 = 0.693095$

$\therefore$ $I = I_5 + \frac{1}{3}(I_5 - I_4) = 0.693095 + \frac{0.693095 - 0.69326}{3}$

$I = 0.69304$

$\therefore$ $\int_1^2 \frac{dx}{x} = 0.69304$

Problem 2

Evaluate $\int_1^2 e^{-x^2}$ using Romberg method correct up to 3 decimal place.

Solution:

We have,

$$h = \frac{b-a}{n}$$

Taking n = 2

$$h = \frac{1.5 - 0.2}{2} = 0.65$$

Taking h = 0.65, 0.325, 0.162 respectively

$$f(x) = y_i = e^{-x^2}$$

i) When h = 0.65

X	0.2	0.85	1.5
Y	0.9607	0.4855	0.1053

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{0.65}{2}[0.960 + 0.1053 + 2 \times 0.4855]$$

$\therefore$ $I_1 = 0.6617$

ii) When h = 0.325

X	0.2	0.525	0.850	1.175	1.5
Y	0.9607	0.759	0.4855	0.251	0.1053

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$
$$= \frac{0.325}{2}[0.9607 + 0.1053 + 2(0.759 + 0.4855 + 0.251)]$$

$\therefore$ $I_2 = 0.6589$

iii) When h = 0.1625

X	0.2	0.363	0.525	0.688	0.850	1.013	1.175	1.338	1.5
Y	0.9607	0.876	0.759	0.6233	0.4855	0.35691	0.251	0.167	1.1053

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.1625}{2}[0.9607 + 1.1053 + 2(0.876 + 0.759 + 0.6233 + 0.4855 + 0.35691 + 0.251 + 0.167)]$$

$\therefore$ $I_3 = 0.6565$

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.6589 + \frac{1}{3}(0.6589 - 0.6617) = 0.657$$

Similarly,

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.6505 + \frac{0.6565 - 0.6589}{3} = 0.655$$

$\therefore$ $I = I_5 + \frac{1}{3}(I_5 - I_4) = 0.655 + \frac{0.655 - 0.657}{3} = 0.654$

$\therefore$ $\int_{0.2}^{1.5} e^{-x^2} = 0.654$

Problem 3

Evaluate the following integral using Romberg method. $\int_0^2 \frac{e^x + \sin x}{1 + x^2}dx$

Solution:

Here,

$$h = \frac{b-a}{n}$$

Taking n = 2

$\therefore$ $h = \frac{b-a}{n} = \frac{2-0}{2} = 1$

Taking h = 1, 0.5, 0.25

Here,

$$f(x) = y = \frac{e^x + \sin x}{1 + x^2}$$

i) When h = 1

X	0	1	2
Y	1	1.777988	1.659671

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[1 + 1.659671 + 2 \times 1.77988] = 3.10972$$

ii) When h = 0.5

X_i	0	0.5	1	1.5	2
Y_i	1	1.70252	1.77988	1.68590	1.659671

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.5}{2}[1 + 1.659671 + 2(1.70252 + 1.77988 + 1.68590)]$$

$\therefore$ $I_2 = 3.24906$

iii) When h = 0.25

X_i	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
Y_i	1	1.44134	1.70252	1.79112	1.77988	1.73242	1.68590	1.6587	1.659671

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.25}{2}[1 + 1.659671 + 2(1.44134 + 1.70252 + 1.79112 + 1.77988 + 1.73242 + 1.68590 + 1.6587)]$$

$\therefore$ $I_3 = 3.28043$

Now,

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 3.24906 + \frac{3.24906 - 3.10792}{3} = 3.2955$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 3.28043 + \frac{3.28043 - 3.24906}{3} = 3.29088$$

$$I = I_5 + \frac{1}{3}(I_5 - I_4) = 3.29088 + \frac{3.29088 - 3.2955}{3} = 3.28934$$

$\therefore$ $\int_0^2 \frac{e^x + \sin x}{1 + x^2}dx = 3.28934$

Problem 4

Use Romberg integration method $I = \int_0^2 \frac{e^x + e^{-x}}{2}dx$ **correct up to three decimal place.**

Solution:

Here,

$$h = \frac{b-a}{n}$$

Taking n = 2

$$h = \frac{2-0}{2}$$

Taking h = 1, 0.5, 0.25

$$y(i) = f(x_i) = \frac{e^x + e^{-x}}{2}$$

i) When h = 1

X_i	0	1	2
Y_i	1	1.54308	3.76219

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[1 + 3.76219 + 2 \times 1.54308] = 3.9242$$

ii) When h = 0.5

X_i	0	0.5	1	1.5	2
Y_i	1	1.12763	1.54308	2.35241	3.76219

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$
$$= \frac{0.5}{2}[1 + 3.76219 + 2(1.12763 + 1.54308 + 2.35241)]$$
$\therefore \quad I_2 = 3.70211$

iii) When h = 0.25

X_i	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
Y_i	1	1.03141	1.12763	1.59468	1.54308	1.878842	2.35241	2.964188	3.7621

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$
$$= \frac{0.25}{2}[1 + 3.76219 + 2(1.03141 + 1.12763 + 1.59468 + 1.56308 + 1.878842$$
$$+2.35241 + 2.964188$$

$\therefore \quad I_3 = 3.64573$

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 3.70211 + \frac{3.70211 - 3.9242}{3} = 3.62808$$
$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 3.64573 + \frac{3.64573 - 3.70211}{3} = 3.63694$$

so, $I = I_5 + \frac{1}{3}(I_5 - I_4) = 3.6394 + \frac{3.63694 - 3.62808}{3} = 3.6399$

Problem 5

Use Romberg Integration find the integral of $e^x \sin x$ between the limit −1 and 1

Solution:

Here,

$$\int_{-1}^{1} e^x \sin x \, dx$$
$$h = \frac{b-a}{n}$$

Taking n = 2

$$h = \frac{1+1}{2} = 1$$

Taking h = 1, 0.5, 0.25

$$f(x) = e^x \sin x$$

i) When h = 1

X_i	−1	0	1
Y_i	−0.30959	0	2.2874

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[-0.30959 + 2.2874 + 2 \times 0] = 0.98892$$

ii) When h = 0.5

X_i	−1	−0.5	0	0.5	1
Y_i	−0.30956	−0.29079	0	0.79044	2.2874

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.5}{2}[-0.30956 + 2.2874 + 2(-0.29079 + 0 + 0.79044)]$$
$\therefore \quad I_2 = 0.744285$

iii) When h = 0.25

X_i	−1	−0.75	−0.5	−0.25	0	0.25	0.5	0.75	1
Y_i	−0.30956	−0.32198	−0.29079	−0.192978	0	0.317673	0.79044	1.44303	2.2874

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$
$$= \frac{0.25}{2}[-0.30956 + 2.2874 + 2(-0.32198 - 0.2979 - 0.192978 + 0$$
$$+0.317673 + 0.79044 + 1.44303)]$$

$\therefore \quad I_3 = 0.683654$

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.744285 + \frac{0.744285 - 0.98892}{3} = 0.66274$$
$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.683654 + \frac{0.683654 - 0.744285}{3} = 0.66344$$

$\therefore \quad I = I_5 + \frac{1}{3}(I_5 - I_4) = 0.66344 + \frac{0.66344 - 0.66274}{3} = 0.66367$

Problem 6

Use Romberg integration method evaluate $\int_1^2 e^x \, dx$ with $h = \frac{b-a}{2}$.

Solution:

We have,

$$h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$$

Taking h = 0.5, 0.25, 0.125

i) When h = 0.5

X_i	0	0.5	1
Y_i	1	1.28403	2.7183

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{0.5}{2}[1 + 2.7183 + 2 \times 1.28403] = 1.57159$$

ii) When h = 0.25

X_i	0	0.25	0.5	0.75	1
Y_i	1	1.0645	1.28403	1.7551	2.7183

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$
$$= \frac{0.25}{2}[1 + 2.7183 + 2(1.0645 + 1.28403 + 1.7551)]$$

$\therefore \quad I_2 = 1.4907$

iii) When h = 0.125

X_i	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
Y_i	1	1.01575	1.0645	1.1510	1.28403	1.47790	1.7551	2.15034	2.7183

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.125}{2}[1 + 2.7183 + 2(1.01575 + 1.0645 + 1.1510 + 1.28403 + 1.47790 + 1.7551 + 2.1503$$

$\therefore$ $I_3 = 1.46986$

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 1.4907 + \frac{1.4907 - 1.57159}{3} = 1.46374$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 1.46374 + \frac{1.46374 - 1.4907}{3} = 1.46291$$

$\therefore$ $I = I_5 + \frac{1}{3}(I_5 - I_4) = 1.46291 + \frac{1.46291 - 1.46374}{3} = 1.4629$

so, $\int_1^2 e^x\,dx = 1.4629$

Problem 7

Use Romberg integration to evaluate $\int_0^{\frac{3\pi}{2}} e^x \sin x$ [P.U. 2005 Fall]

Solution: Refer the solution of Q. no. 5

Problem 8

Use Romberg integration to evaluate $\int_{0.2}^{1.5} e^{-x}dx$ correct up to three decimal place [P.U. 2005 Spring]

Solution: Refer the solution of Q. no. 2

Problem 9

Apply Romberg method to evaluate $\int_4^{5.2} \log x\,dx$. [P.U. 2006 Fall]

Solution:

Here,

$$h = \frac{b-a}{n}$$

For n = 2

$$h = \frac{5.2 - 4}{2} = 0.6$$

Taking h = 0.6, 0.3, 0.15 respectively

Here,

$$f(x) = \log x_i$$

i) When h = 0.6

X_i	4	4.6	5.2
Y_i	0.60	0.66	0.71

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{0.6}{2}[0.60206 + 0.662758 + 2 \times 0.716003]$$

$$= 0.789$$

ii) When h = 0.3

X_i	4	4.3	4.6	4.9	5.2
Y_i	0.60	0.63	0.66	0.69	0.71

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = \frac{0.3}{2}[0.60 + 0.71 + 2(0.63 + 0.66 + 0.69)]$$

$\therefore$ $I_2 = 0.7905$

iii) When h = 0.15

X_i	4	4.15	4.3	4.45	4.6	4.75	4.9	5.05	5.2
Y_i	0.60	0.61	0.63	0.64	0.66	0.67	0.69	0.70	0.71

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.15}{2}[0.60 + 0.71 + 2(0.61 + 0.63 + 0.64 + 0.66 + 0.67 + 0.69 + 0.7)]$$

$\therefore$ $I_3 = 0.78825$

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.7905 + \frac{0.7905 - 0.789}{3} = 0.791$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.78825 + \frac{0.78825 - 0.7905}{3} = 0.7875$$

$\therefore$ $I = I_5 + \frac{1}{3}(I_5 - I_4) = 0.7875 + \frac{0.7875 - 0.791}{3} = 0.7863$

Chapter 5

Numerical Solution of Ordinary Differential Equation

Solution of ordinary differential equation mean finding an explicit expression for 'y' in terms of a finite numbers of elementary function of 'x'.

Suppose the first order differential

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

All numerical technique for selling differential equation involves a series of estimate of y(x) starting from the given conditions. There are two basic approaches to estimate the values of y(x). They are known as one-step method and multistep methods.

In step by step method, we start from $y(x_0) = y_0$ and proceed with the approximate value of 'y' of the solution of 'y' of equation (1) at;

$$x = x_1 = x_0 + h$$

In second step, we compute an approximate value y_2 of the solution at $x = x_2 = x_0 + 2h$ etc.

Hence, there are 5 methods for solving the differential equation, numerically.

i) Taylor series method
ii) Picard's method
iii) Euler's Method
iv) Runga-kutta method
v) Heun's methods.

i) TAYLOR SERIES METHOD

Suppose y = f(x) be a solution of the equation $\frac{dy}{dx} = f(x, y)$; (1)

with an initial approximation $y(x_0) = y_0$. Expanding by Taylor's series about the point x_0; we get,

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!} \times f'(x) + \frac{(x - x_0)^2}{2!} \times f''(x_0) + \frac{(x - x_0)^3}{3!} \times f'''(x) + \ldots\ldots$$

This equation can be written as;

$$y(x) = y_0 + \frac{(x - x_0)}{1!} \times y_0' + \frac{(x - x_0)^2}{2!} \times y_0'' + \frac{(x - x_0)^3}{3!} \times y_0''' + \ldots\ldots \quad (2)$$

When x_0 and y_0 denote the initial values of 'x' and 'y' put $x = x_1 = x_0 + h$

Then equation (2) becomes;

$$y(x_0 + h) = y_0 + \frac{h}{1!} \times y_0' + \frac{h^2}{2!} \times y_0'' + \frac{h^3}{3!} \times y_0''' + \frac{h^4}{4!} \times y_0^{iv} + \ldots\ldots$$

Similarly;

$$y(x_0 + 2h) = y_1 + \frac{h}{1!} \times y_1' + \frac{h^2}{2!} \times y_1'' + \frac{h^3}{3!} \times y_1''' + \frac{h^4}{4!} \times y_1^{iv} + \ldots\ldots$$

This formula is known as Taylor's series formula.

Problem 1

Given that $\frac{dx}{dy} = x + y, y(0) = 1$ using Taylor series method compute y(0.1)

Solution:

Here,

$$y' = \frac{dx}{dy} = x + y$$

$$y(0) = 1$$

i.e., $x_0 = 0$

and, $y_0 = 1$

Taking h = 0.1 then,

$$x_1 = x_0 + h,$$

We know, Taylor's series expansion,

$$y(x_0 + h) = y_0 + \frac{h}{1!} \times y_0' + \frac{h^2}{2!} \times y_0'' + \frac{h^3}{3!} \times y_0''' + \frac{h^4}{4!} \times y_0^{iv} + \ldots\ldots \quad (1)$$

Now,

$y' = x + y$

or, $y_0' = x_0 + y_0 = 0 + 1 = 1$

$y'' = 1 + y'$

or, $y_0'' = 1 + y_0' = 1 + 1 = 2$

$y''' = y''$

or, $y_0''' = y_0'' = 2$

$y^{iv} = y'''$

or, $y_0^{iiv} = y_0''' = 2$

Substituting these values in equation (1); we get,

$$y(0 + 0.1) = 1 + \frac{0.1}{1!} \times 1 + \frac{(0.1)^2}{2!} \times 2 + \frac{(0.1)^3}{3!} \times 2 + \frac{(0.1)^4}{4!} \times 2 + \ldots\ldots$$

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12}$$

$$y(0.1) = 1.11034$$

Problem 2

Given that $y^i = 3x + \frac{y}{2}$ and y(0) = 1. Find y(0.1) and y(0.2) taking h = 0.1 using Taylor series method.

Solution:

Here,

$$y' = 3x + \frac{y}{2}$$

$$y(0) = 1$$

Taking h = 0.1 then,

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

We know that; Taylor's series expansion,

$$y(x_0 + h) = y_0 + \frac{h}{1!} \times y_0' + \frac{h^2}{2!} \times y_0'' + \frac{h^3}{3!} \times y_0''' + \frac{h^4}{4!} \times y_0^{iv} + \dots\dots$$

$$y' = 3x + \frac{y}{2}$$

or, $y_0' = 3x_0 + \frac{y_0}{2} = 3 \times 0 + \frac{1}{2} = 0.5$

$$y'' = 3 + \frac{y'}{2}$$

or, $y_0'' = 3 + \frac{y_0'}{2} = 3 + \frac{0.5}{2} = 3.25$

$$y''' = \frac{y''}{2}$$

or, $y_0''' = \frac{3.25}{2} = 1.625$

$$y^{iv} = \frac{y'''}{2}$$

or, $y_0^{iiv} = \frac{y_0'''}{2} = \frac{1.625}{2} = 0.8125$

Substituting there values in (2); we get,

$$y(0 + 0.1) = 1 + \frac{0.1}{1} \times 0.5 + \frac{(0.1)^2}{2} \times 3.25 + \frac{(0.1)^3}{6} \times 1.625 + \frac{(0.1)^4}{24} \times 0.8125$$

$$y(0.1) = 1.0665$$

Again for $y(0.2)$; we have,

$$x_2 = 0.2$$

$$x_2 = x_1 + h$$

$$0.2 = 0.1 + h$$

$$h = 0.1$$

$$y' = 3x + \frac{y}{2}$$

or, $y_1' = 3x_1 + \frac{y_1}{2} = 3 \times 0.1 + \frac{1.0665}{2} = 0.833$

$$y'' = 3 + \frac{y'}{2}$$

or, $y_1'' = 3 + \frac{y_1'}{2} = 3 + \frac{0.833}{2} = 3.14166$

$$y''' = \frac{y''}{2}$$

or, $y_1''' = \frac{y_1''}{2} \frac{3.14166}{2} = 1.7083$

$$y^{iv} = \frac{y'''}{2}$$

or, $y_1^{iiv} = \frac{y_1'''}{2} = \frac{1.7083}{2} = 0.8542$

Now,

$$y(0.1 + 0.1) = y_1 + \frac{h}{1!} \times y_1' + \frac{h^2}{2!} \times y_1'' + \frac{h^3}{3!} \times y_1''' + \frac{h^4}{4!} \times y_1^{iv} + \dots\dots$$

$$y(0.2) = 1.0665 + 0.1 \times 0.8333 + \frac{(0.1)^2}{2} \times 3.4166 + \frac{(0.1)^3}{6} \times 1.7083 + \frac{(0.1)^4}{24} \times 0.8542$$

$$y(0.2) = 1.16720$$

ii) PICARD'S METHOD

Suppose the given differential equation is of the form;

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

Let us solve equation (1) with the given initial condition $y(x_0) = y_0$

The given differential equation can be written as

$$dy = f(x, y)\,dx \qquad (2)$$

Integrating equation (2) form $x_0 + x$ with respect to 'x'

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx$$

$$[y]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x, y)\,dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx \qquad (3)$$

For first approximation we replace 'y' by y_0 in equation (3); we get,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

For second approximation, we replace 'y' by y_1; we get,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

Similarly, for other approximation we make a general form

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$$

We continue this process until we get two successive approximation value equal.

Problem 1

Use Picard's Method, find 'y' when x = 0.2. Given that y(0) = 1 and $\frac{dx}{dy} = x - y$

Solution:

Here,

$$\frac{dx}{dy} = x - y = f(x, y)$$

$$y(0) = 1$$

i.e., $x_0 = 0$

and, $y_0 = 1$

Using Picard's formula; we have,

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

First approximation;

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_1 = y_0 + \int_0^x f(x - y_0)\, dx$$

$$y_1 = y_0 + \int_0^x f(x - 1)\, dx$$

$$y_1 = y_0 + \frac{x^2}{2} - x$$

$$y_1 = y_0 + \frac{(0.2)^2}{2} - 0.2 \quad (\text{at } x = 0.2)$$

$$y_1 = 0.82$$

Second approximation;

$$y_2 = y_0 + \int_0^x f(x, y_1)dx$$

$$y_2 = 1 + \int_0^x f(x, y_1)dx$$

$$y_2 = 1 + \int_0^x \left(x - 1 - \frac{x^2}{2} + x\right) dx$$

$$y_2 = 1 + \frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{2}$$

$$y_2 = 1 - x + x^2 - \frac{x^3}{6}$$

At $x = 0.6$;

$$y_2 = 1 - 0.2 + 0.5^2 - (0.2)^3 = 0.8387$$

Third approximation;

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

$$y_3 = 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{6}\right) dx$$

$$y_3 = 1 + \frac{x^2}{2} - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{24}$$

$$y_3 = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$$

at $x = 0.2$

$$y_3 = 1 - 0.2 + 0.2^2 - \frac{0.2^3}{3} + \frac{0.2^4}{24} = 0.8374$$

Fourth approximation;

$$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3)dx$$

$$y_4 = 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24}\right) dx$$

$$y_4 = 1 + \frac{x^2}{2} - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

At $x = 0.2$;

$$y_4 = 1 + \frac{0.2^2}{2} - 0.2 + \frac{0.2^2}{2} - \frac{0.2^3}{3} + \frac{0.2^4}{12} - \frac{0.2^5}{120}$$

$$y_4 = 0.8375$$

$\therefore$ $y_3 = y_4$ is equal up to 3 decimal place

$\therefore$ $y(0.2) = 0.837$

iii) EULER'S METHOD: RK–1

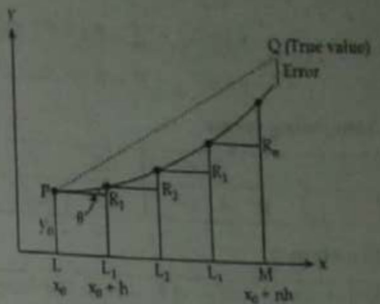
Consider the equation,

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

Given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in the figure.

We have to find the ordinate of any other point Q on this curve.

Let us divide LM into n-sub-intervals each of width h at $L_1, L_2, \ldots\ldots$ so that h is quite small.



Let the interval LL_1, we approximate the curve by tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$ then,

$$y_1 = L_1P_1 + R_1P_1 = y_0 + PR_1 \tan\theta.$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P = y_0 + h\, f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then,

$$y_2 = y_1 + h\, f(x_0 + 2h, y_2)$$

Repeating n–times;

$$y_n = y_{n-1} + h\, f(x_0 + \overline{n-1}h, y_{n-1})$$

Problem 1

Solve the equation $\frac{dx}{dy} = 1 - y$ with initial condition $y(0) = 0$ using Euler's method and tabulate the solution at $x = 0.1, 0.2$

Solution:

Given that;

$$\frac{dx}{dy} = f(x, y) = 1 - y$$

$y(0) = 0$

i.e., $x_0 = 0$

and $y_0 = 0$

Taking h = 0.1

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = ?$

$x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$

$y_2 = ?$

$x_3 = x_0 + 3h = 0 + 3(0.1) = 0.3$

$y_3 = ?$

From Euler's method, we have,

$$y_1 = y_0 + h[f(x_0, y_0)]$$

$$y_1 = 0 + 0.1(1 - y_0) = 0.1 \times (1 - 0) = 0.1$$

$$y_2 = y_1 + h[f(x_1, y_1)] = 0.1 + 0.1 \times (1 - 0.1) = 0.19$$

$$y_3 = y_2 + h[f(x_2, y_2)] = 0.19 + 0.1 \times (1 - 0.19) = 0.271$$

Tabulating result

x	0	0.1	0.2	0.3
y	0	0.1	0.19	0.271

Problem 2

Solve $\sin x + \cos y$, $y(0) = \pi$ in the range $0 \le x \le 2$ by dividing the interval into 5 sub intervals using Euler's method. [T.U., 2072 Kartik]

Solution:

x	y	$\frac{dy}{dx} = \sin x + \cos y$	Old $y_0^{+}0.4\left(\frac{dy}{dx}\right)$ = New y
0	π	$(\sin 0 + \cos \pi) = -1$	$\pi + 0.4 \times (-1) = 2.7415$
0.4	2.7415	$\sin(0.4) + \cos(2.7415) = -0.5316$	$2.7415 + 0.4(-0.5316) = 2.5288$
0.8	2.5288	$\sin(0.8) + \cos(2.5288) = -0.10068$	$2.5288 + 0.4(-0.10068) = 2.4885$
1.2	2.4885	$\sin(1.2) + \cos(2.4885) = 0.137815$	$2.4885 + 0.4(0.137815) = 2.5436$
1.6	2.5436	$\sin(1.6) + \cos(2.5436) = 0.173091$	$2.5436 + 0.4(0.173091) = 2.61283$
2.0	2.61283	$\sin(2.0) + \cos(2.612863) = 0.04586$	$2.61283 + 0.4(0.04586) = 2.61413$

$\therefore$ Required approximate value of $y = 2.61493$.

iv) RUNGE–KUTTA METHOD

This method refers to a family of one-step method used for numerical solution of initial value problems. They all based on the general form of extrapolation equation.

a) First order R–K method

$y_1 = y_0 + h f(x_0, y_0)$ which is Euler's method.

b) Second order R–K2 method

This method is a **Heun's method or modified Euler's method** / Predictor corrector method

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where, $k_1 = h f(x_0, y_0)$

$$k_2 = h f(x_0 + h, y_0 + k_1)$$

c) Third order R–K method (R–K3)

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where, $k_1 = h f(x_0, y_0)$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k}{2}\right)$$

$$k_3 = h f(x_0 + h, y_0 + 2k_2 - k_1)$$

d) Fourth order R–K method

This method is commonly used and is often referred to as **Runga-Kutta method classically forth order R–K method**.

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where, $k_1 = h f(x_0, y_0)$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

Similarly, for second interval

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where, $k_1 = h f(x_1, y_1)$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$

Problem 1

Apply R–K forth method to find an approximate value of 'y' when x = 0.2.

Solution:

Given that;

$$\frac{dx}{dy} = x + y$$

$$y(0) = 1$$

i.e., At $x_0 = 0$;

$$y_0 = 1$$

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$h = 0.2$

$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = ?$

From R-K forth order method; we have,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h\,f(x_0, y_0) = 0.2 \times (0 + 1) = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\,f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$

$$= h\,f(0.1, 1.1)$$

$$= 0.2 \times (0.1, 1.1) = 0.24$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\,f\left(0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right)$$

$$= h\,f(0.1, 1.12)$$

$$= 0.2 \times (0.1, 1.12)$$

$$= 0.244$$

and, $k_4 = h\,f(x_0 + h, y_0 + k_3)$

$$= h\,f(0 + 0.2, 1 + 0.244)$$

$$= h\,f(0.2, 1.244)$$

$$= 0.2 \times (0.2 + 1.244)$$

$$= 0.2888$$

Problem 2

> Solve the differential equation for $y(0.25)$ using R-K fourth order method
> $10\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 6x = 0$, with $y(0) = 1$ and $y'(0) = 0$ take $h = 0.25$.
>
> [P.U. 2011 Spring, 2011 Fall, 2010 Fall]

Solution:

Here, the given equation is of second order.

For this form;

$$\frac{d^2y}{dx^2} = \emptyset\left(x, y, \frac{dy}{dx}\right) \quad (1)$$

We put

$$\frac{dy}{dx} = z \quad (2)$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} \quad (3)$$

From (1), (2) and (3); we get,

$$\frac{dz}{dx} = \emptyset(x, y, z) \quad (4)$$

and, $\frac{dy}{dx} = f(x, y, z) \quad (5)$

Equation (4) and (5) gives a system of simultaneous equation which can be solved easily.

Here,

$$10y'' + (y')^2 + 6x = 0$$

$$y(0) = 1$$

$$y'(0) = 0$$

$$h = 0.5$$

$$y' = z = f(x, y, z),$$

$$z(0) = 0$$

Then,

$$10z' + z^2 + 6x = 0$$

$$z' = \frac{6x + z^2}{10} = \emptyset(x, y, z)$$

Now, using Rungee-Kutta method;

$$k_1 = hf(x_0, y_0, z_0) = 0.5 \times (f(0, 0, 0)) = 0$$

$$l_1 = h\emptyset(x_0, y_0, z_0) = 0.5 \times \emptyset(0, 0, 0) = 0$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.5 \times f(0.25, 0, 0) = 0$$

$$l_2 = h\emptyset\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.5 \times \emptyset(0.25, 1, 0) = -0.075$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.5 \times f(0.25, 0, -0.0375) = -0.01875$$

$$l_3 = h\emptyset\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.5 \times \emptyset(0.25, 0, -0.0375) = -0.0749$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.5 \times f(0.5, -0.01875, -0.07949) = -0.03746$$

$$l_4 = h\emptyset(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.5 \times \emptyset(0.5, -0.01875, -0.07949) = -0.299$$

Then,

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0124$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = -0.0998$$

$$y(0.5) = y_0 + k = 0.9876$$

$$z(0.5) = z_0 + l = 0.0998$$

Problem 3

> Solve the following initial value problem using the modified method for $0 < x < 0.6$ with an interval of 0.2.
> $\frac{dy}{dx} = \sin x + \cos y;\ y(0) = 3$ [T.U., 2071 Shrawan]

Solution:

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x	$y' = \sin x + \cos y$	Mean slope	New y = Old y (mean slope)
0.0	1	-	$1 + 0.2(1) = 1$
0.2	$\sin(0.2) + \cos(1) = 0.73897$	$\frac{1}{2}(1 + 0.73897) = 0.8694$	$1 + 0.2(0.8694) = 1.173897$
0.2	$\sin(0.2) + \cos(1.173897) = 0.58522$	$\frac{1}{2}(1 + 0.58522) = 0.79261$	$1 + 0.2(0.79261) = 1.62823$
0.2	$\sin(0.2) + \cos(1.62823) = 1.00308$	$\frac{1}{2}(1 + 1.00308) = 1.00163$	$1 + 0.2(1.00163) = 1.2003$
0.2	1.003271	-	$1.2003 + 0.2(1.003271) = 1.40094$
0.4	$\sin(0.4) + \cos(1.40094) = 1.00668$	$\frac{1}{2}(1 + 1.00668) = 1.006680$	$1.2003 + 0.2(1.006680) = 1.40429$
0.4	$\sin(0.4) + \cos(1.40094) = 1.006680$	$\frac{1}{2}(1 + 1.006680) = 1.00497$	$1.2003 + 0.2(1.006680) = 1.40429$
0.4	1.006680	-	$1.40429 + 0.2(1.006680) = 1.605626$
0.6	$\sin(0.6) + \cos(1.605626) = 1.010079$	$\frac{1}{2}(1 + 1.010079) = 1.008379$	$1.40429 + 0.2(1.008379) = 1.6059658$
0.6	$\sin(0.6) + \cos(1.606037) = 1.010078$	$\frac{1}{2}(1 + 1.010078) = 1.008373$	$1.40429 + 0.2(1.008373) = 1.605964$

$\therefore \quad y(0.6) = 1.6059$ (Approximately)

SHOOTING METHOD

Solving Boundary Value Problem Using Shooting Method

Shooting method is used for solving the boundary value problem. In this method the given boundary value problem is first converted into an equivalent initial value problem and then solved.

Consider the equation

$$y'' = f(x, y, y')$$
$$y(a) = A$$
and, $$y(b) = B$$
$$y' = z$$

Then,

$$y' = z$$
$$z' = f(x, y, z)$$

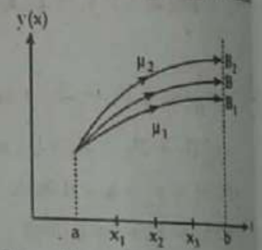


Figure: Illustration of shooting method

In order to solve this set as an initial value problem. We need two test conditions at x = a and therefore, required another condition for 'z' at x = a

Let us assume that $z(a) = \mu_1$ where, μ_1 is a queue.

Thus,

$$y^i = z$$

$$y(a) = A$$
$$z^i = f(x, y, z)$$
$$z(a) = \mu_1$$

Equation (1) can be solved by Heun's or R-K fourth method. (1)

Problem 1

Using shooting method solve the following equation $\frac{d^2y}{dx^2} = 6x$, $y(1) = 2$ and $y(2) = 9$ in the interval 1 and 2.

Solution:

Here, by transformation

$$\frac{dz}{dx} = z$$
$$y(1) = 2$$

and, $$\frac{dz}{dx} = 6x$$

Let us assume that $z(2) = 2(m_1)$

Applying Heun's method; we obtain,

First Iteration

$$x_0 = 1$$
$$y(1) = y_0 = 2$$
$$z(1) = z_0 = 2$$
$$m_1(1) = z_0 = 2$$
$$m_1(2) = 6 \times 1 = 6$$
$$m_2(1) = z_0 + hm_1(2) = 2 + 0.5(6) = 5$$
$$m_2(2) = 6(x_0 + h) = 6(1.5) = 9$$
$$m(1) = \frac{m_1(1) + m_2(1)}{2} = \frac{2 + 5}{2} = 3.5$$
$$m(2) = \frac{m_1(2) + m_2(2)}{2} = \frac{6 + 9}{2} = 7.5$$
$$y(x_1) = y(1.5) = y(1) + m(1)h = 2 + 3.5 \times 0.5 = 3.75$$
$$z(x_1) = z(1.5) = z(1) + m(2)h = 2 + 7.5 \times 0.5 = 5.75$$

Second Iteration

$$h = 0.5$$
$$x_1 = 1.5$$
$$y(1) = 3.75$$
$$z(1) = 5.75$$
$$m_1(1) = z_1 = 5.75$$
$$m_1(2) = 6 \times 1 = 6$$
$$m_2(1) = z_1 + hm_1(2) = 5.75 + 0.5(6) = 8.75$$
$$m_2(2) = 6(x_1 + h) = 12$$
$$m(1) = \frac{m_1(1) + m_2(1)}{2} = 7.25$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2} = 9$$

$$y(x_2) = y(1) + m(1)h = 3.75 + 7.25 \times 0.5 = 7.375$$

This gives $B_1 = 7.75$ which is less than $B = 9$.

Iteration 1

$h = 0.5$

$x_0 = 1$

$y(1) = y_0 = 2$

$z_0 = 4$

$m_1(1) = z_0 = 4$

$m_1(2) = 6 \times 1 = 6$

$m_2(1) = z_0 + hm_1(2) = 4 + 0.5(6) = 7$

$m_2(2) = 6(x_0 + h) = 6(1.5) = 9$

$$m(1) = \frac{m_1(1) + m_2(1)}{2} = \frac{4+7}{2} = 5.5$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2} = \frac{6+9}{2} = 7.5$$

$y(x_1) = y(1.5) = y(1) + m(1)h = 2 + 5.5 \times 0.5 = 4.75$

$z(x_1) = z(1.5) = z(1) + m(2)h = 4 + 7.5 \times 0.5 = 7.75$

Iteration 2

$h = 0.5$

$x_1 = 1.5$

$y_1 = 4.75$

$z_1 = 7.75$

$m_1(1) = z_1 = 7.75$

$m_1(2) = 6 \times 1 = 9$

$m_2(1) = z_1 + hm_1(2) = 7.75 + 0.5(9) = 12.25$

$m_2(2) = 6(x_1 + h) = 12$

$$m(1) = \frac{m_1(1) + m_2(1)}{2} = 12$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2} = 10.5$$

This gives $B_2 = 9.7$ which is greater than $B = 9$; hence, required answer.

Chapter 6

Classification of Partial Differential Equation

We classify partial differential equation of second order. The general lines partial equation of second order in two independent variable in the form;

$$A\frac{d^2U}{dx^2} + B\frac{d^2U}{dx\,dy} + C\frac{d^2U}{dy^2} + D\frac{dU}{dx} + \frac{dU}{dy} + FU = 0$$

or, $AU_{xx} + BU_{yy} + CU_{yy} + DU_x + EU_y + FU = 0$ (1)

where, A, B, C, D and E are function of 'x' and 'y'.

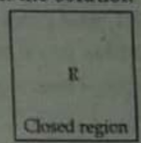
i) Elliptical at point (x, y) if $B^2 - 4AC < 0$

ii) Parabolic if $B^2 - 4AC = 0$

iii) Hyperbolic if $B^2 - 4AC > 0$

ELLIPTICAL EQUATION

A differential equation is elliptical in a region 'R' if $b^2 - 4AC < 0$ at all points of the region. The boundary condition of this type of equation specify the function 'u' at every point of the closed boundary of the region 'R' within which the solution $f(x, y)$ is to be determined as shown in the figure. Boundary conditions are given at every or some point of the boundary.



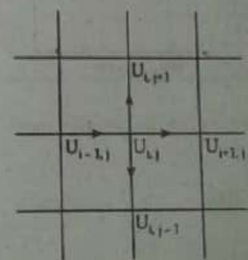
i) LAPLACE'S EQUATION

An important equation of the elliptic type is;

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = 0$$

i.e., $U_{xx} + U_{yy} = 0$ (1)

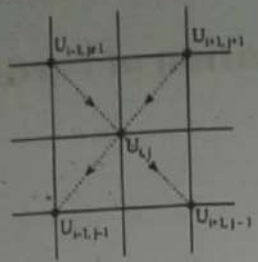
This is called Laplace's equation.



Taking square mesh and putting (1), we get,

$$U_{ij} = \frac{1}{4}[U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}]$$

That the value of 'u' at any interior meshes point is the arithmetic mean of its values at the four neighboring mesh point. This is called standard five point formula.
Instead of equation (2) we can also write the formula as;



$$U_{i,j} = \frac{1}{4}[U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j-1} + U_{i+1,j+1}]$$

This is called diagonal five point formula (DFPE)

ii) **POISSON EQUATION**

The partial differential equation is;

$$\nabla^2 U = f(x, y)$$

or, $$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = f(x, y)$$

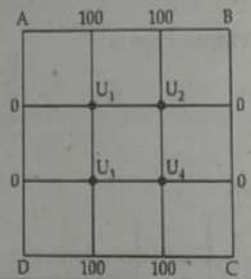
or, $$U_{xx} + U_{yy} = f(x, y)$$

where, f(x, y) is given function of 'x' and 'y' is called poisons equation.

$$U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij} = h^2 f(ih, jh)$$

Problem 1

A Steel plate of size of 15 cm × 15 cm. If two of the sides are held at 100°C and other two sides are held at 0°C. What is the steady state temperature at interior point assuming a grid of size 5 cm × 5 cm? [P.U. 2006 Fall]



Solution:
Let U_1, U_2, U_3, U_4 be the value of 'x' at the interior mesh points
From SFPF

$$U_1 = \frac{1}{4}(100 + U_1 + U_3 + 100)$$

$$U_2 = \frac{1}{4}(U_1 + 100 + U_4) \quad (2)$$

$$U_3 = \frac{1}{4}(U_1 + 100U_4) \quad (3)$$

$$U_4 = \frac{1}{4}(U_2 + U_3) \quad (4)$$

Here,

$$4U_1 - 2U_2 = 200 \quad (a)$$
$$4U_2 - U_1 - U_4 = 100 \quad (b)$$
$$4U_4 - 2U_2 = 0 \quad (c)$$

On solving; we have,

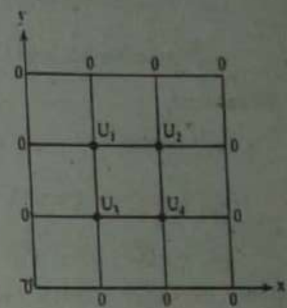
$$U_1 = 75$$
$$U_2 = U_3 = 50$$
$$U_4 = 25$$

Problem 2

Solve the Poisson's partial differential equation $u_{xx} + u_{yy} = -10(x^2 + y^2 + 10)$ over the region $0 \le x \le 3$ and $0 \le y \le 3$ with boundary condition $u(0, y) = 0$, $u(3, y) = 0$, $u(x, 0) = 0$ and $u(x, 3) = 0$. Assuming mess length 1. [T.U., 2070 Chaitra]

Solution:
Here,
$h = 1$
The standard 5-point formula is,

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \quad (1)$$



For $u_1(i = 1, j = 2)$,

$$0 + u_2 + u_3 - 4u_1 = -10(1 + 4 + 10)$$

or, $$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \quad (2)$$

For $u_2(i = 2, j = 2)$,

$$u_2 = \frac{1}{4}(u_1 + u_4 + 180) \quad (3)$$

For $u_3(i = 1, j = 1)$,

$$u_3 = \frac{1}{4}(u_1 + u_4 + 120) \quad (4)$$

For $u_4(i = 2, j = 1)$,

$$u_4 = \frac{1}{4}(u_2 + u_3 + 150) \quad (5)$$

From equation (2) and (5); we get,

$$u_4 = u_1$$

$\therefore$ $$u_1 = \frac{1}{4}(u_2 + u_3 + 150)$$

$\therefore$ $$u_2 = \frac{1}{2}(u_2 + 90)$$

$\therefore$ $$u_3 = \frac{1}{2}(u_1 + 60)$$

Solving equations by G-S method,

Iteration 1

Let $u_2 = u_3 = 0$,

$$u_1^{(1)} = 37.5$$

$$u_2^{(1)} = \frac{1}{2}(37.5 + 90) \approx 64$$

$$u_3^{(1)} = \frac{1}{2}(37.5 + 60) \approx 49$$

Iteration 2

$$u_1^{(2)} = \frac{1}{4}(64 + 49 + 150) \approx 66$$

$$u_2^{(2)} = \frac{1}{2}(66 + 90) \approx 78$$

$$u_3^{(2)} = \frac{1}{2}(66 + 60) \approx 63$$

Iteration 3

$$u_1^{(3)} = \frac{1}{4}(78 + 63 + 150) \approx 73$$

$$u_2^{(3)} = \frac{1}{2}(73 + 90) \approx 82$$

$$u_3^{(3)} = \frac{1}{2}(73 + 60) \approx 67$$

Iteration 4

$$u_1^{(4)} = \frac{1}{2}(82 + 67 + 150) \approx 75$$

$$u_2^{(4)} = \frac{1}{2}(75 + 90) \approx 82.5$$

$$u_3^{(4)} = \frac{1}{2}(75 + 60) \approx 67.5$$

Iteration 5

$$u_1^{(5)} = \frac{1}{2}(82 + 67 + 150) \approx 75$$

$$u_2^{(5)} = \frac{1}{2}(75 + 90) \approx 82.5$$

$$u_3^{(5)} = \frac{1}{2}(75 + 60) \approx 67.5$$

Hence, the values of iteration 4 and 5 are same.

$\therefore \quad u_1 = 75 = u_4$

$\therefore \quad u_2 = 82.5$

$\therefore \quad u_3 = 67.5$

Problem 3

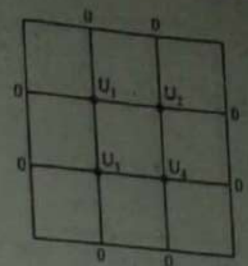
Solve the Poisson equation $\nabla^2 f = 2x^2 + y$ over the square domain $1 \le x \le 4$, $1 \le y \le 4$ with $f = 0$ on the boundary Take $h = k = 1$

[P.U. 2005 Spring, 2007 Spring. 2010 Fall]

Solution:

The given equation is;

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = 2x^2 + y^2 \quad (1)$$



Here,

$$U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j} = 2x^2 + y^2 \quad (2)$$

Put $i = 2, j = 2$

$$U_2 = \frac{1}{4}(U_1 + U_4 - 10) \quad (a)$$

Put $i = 1, j = 2$

$$U_1 = \frac{1}{4}(U_2 + U_3 - 4) \quad (b)$$

Put $i = 1, j = 1$

$$U_3 = \frac{1}{4}(U_1 + U_4 - 3) \quad (c)$$

Put $i = 2, j = 1$

$$U_4 = \frac{1}{4}(U_2 + U_3 - 9) \quad (d)$$

On solving (a), (b), (c) and (d) by Gauss seidel method

$U_1 = -2.6223$

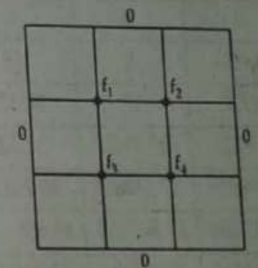
$U_3 = -4.1203$

$U_3 = -2.3736$

$U_3 = -3.874$

Problem 4

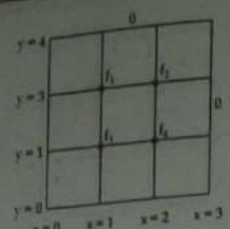
Solve the Poisson equation $\nabla^2 f = 2x^2y^2$ over the square domain $0 \le x \le 3$, $0 \le y \le 3$ with $f = 0$ on the boundary. Take $h = 1$



[T.U. 2067 Ashadh, 2063 Baishakh, P.U. 2005 Spring, 2007 Spring, 2009 Spring, 2010 fall]

Solution:

The domain is divided into square of 1 unit size as shown.



Now, applying standard 5 point formula at each grid point; we get,

At $f_1(x = 2, y = 3)$;

$$0 + 0 + f_2 + f_3 - 4f_1 = 2 \times 2^2 \times 3^2$$

or, $f_2 + f_3 - 4f_1 = 72$

or, $f_1 = \frac{1}{4}(f_2 + f_3 - 72)$ (1)

At $f_2(x = 3, y = 3)$;

$$0 + 0 + f_1 + f_4 - 4f_2 = 2 \times 2^2 \times 3^2$$

or, $f_2 = \frac{1}{4}(f_1 + f_4 - 162)$ (2)

At $f_3(x = 2, y = 2)$;

$$0 + 0 + f_1 + f_4 - 4f_3 = 2 \times 2^2 \times 2^2$$

or, $f_3 = \frac{1}{4}(f_1 + f_4 - 32)$ (3)

At $f_4(x = 3, y = 2)$;

$$0 + 0 + f_2 + f_3 - 4f_4 = 2 \times 3^2 \times 2^2$$

or, $f_4 = \frac{1}{4}(f_2 + f_3 - 72)$ (4)

For initial approximation;

Let $f_1 = f_2 = f_3 = f_4 = 0$

Now, using successive iteration method; we have,

Iteration	f_1	f_2	f_3	f_4
0	0	0	0	0
1	18	40.5	8	18
2	30.125	49.5	17	30.125
3	34.625	57.8125	25.3125	34.625
4	38.78125	59.390625	27.390625	38.78125
5	39.6953125	60.347656	27.847656	39.6953125
6	40.048828	60.524414	28.024414	40.048828
7	40.137207	60.568604	28.068604	40.137207
8	40.159302	60.579651	28.079651	40.159302
9	40.164826	60.579651	28.079651	40.164826
10	40.164826	60.579651	28.079651	40.164826

Here, values of f_1, f_2, f_3 and f_4 are correct up to 5 decimal points.

$\therefore$ $f_1 = -40.164826$

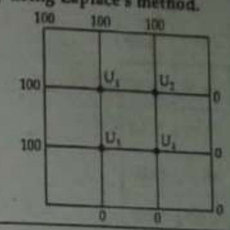
$f_2 = -60.579651$

$f_3 = -28.079651$

and, $f_4 = -40.164826$

Problem 5

Solve the elliptical equation $U_{xx} + U_{yy} = 0$ from the following square mesh with boundary values by using Laplace's method.



Solution:

Here, using standard 5-point formula;

$$U_1 = \frac{1}{4}(100 + 100 + U_2 + U_3) \quad (1)$$

or, $U_1 = \frac{1}{4}(200 + U_2 + U_3)$

Similarly;

$$U_2 = \frac{1}{4}(100 + U_1 + U_4 + 0)$$

$\therefore$ $U_2 = \frac{1}{4}(U_1 + U_4 + 100)$ (2)

$$U_3 = \frac{1}{4}(U_1 + 100 + 0 + U_4)]$$

$\therefore$ $U_3 = \frac{1}{4}(U_1 + U_4 + 100)$ (3)

and, $U_4 = \frac{1}{4}(0 + 0 + U_2 + U_3)$

$$U_4 = \frac{1}{4}(U_2 + U_3) \quad (4)$$

Again, using 5-point formula; we have,

$$U_1 = \frac{1}{4}(100 + 100 + 100 + U_4)$$

For initial condition;

Let $U_4 = 0$

$\therefore$ $U_1 = \frac{1}{4} \times 300 = 75$

Then,

$U_2 = 4.75$

$U_3 = 43.75$

$U_4 = 21.88$

Tabulating the successive approximation

Iteration	U_1	U_2	U_3	U_4
Initial	75	43.75	43.75	21.88
1	71.88	48.44	48.44	24.22
2	74.22	49.61	49.61	24.81
3	74.81	49.91	49.91	24.99
4	74.96	49.98	49.98	24.99
5	74.99	50.00	50.00	25.00
6	75.00	50.00	50.00	25.00
7	75.00	50.00	50.00	25.00

From sixth and seventh iteration; we have,

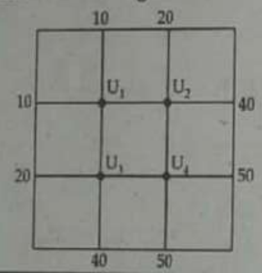
$U_1 = 75$

$U_2 = U_3 = 4.75$

and, $U_4 = 25$

Problem 6

Solve the elliptical equation $U_{xx} + U_{yy} = 0$ for the following square mesh with boundary conditions as shown in the figure below. [T.U. 2069 Bhadra]



Solution:

Using standard 5-point formula;

$$U_1 = \frac{1}{4}(10 + 10 + U_2 + U_3) = \frac{1}{4}(30 + U_2 + U_3) \quad (1)$$

Similarly;

$$U_2 = \frac{1}{4}(60 + U_1 + U_4) = \frac{1}{4}(60 + U_1 + U_4) \quad (2)$$

$$U_3 = \frac{1}{4}(60 + U_1 + U_4) \quad (3)$$

and, $$U_4 = \frac{1}{4}(100 + U_2 + U_3) \quad (4)$$

For initial approximation;

Let $U_1 = U_2 = U_3 = U_4 = 0$

Now, tabulating the successive approximation by using Gauss seidal method;

No. of Iteration	U_1	U_2	U_3	U_4
0	7.5	15	15	25
1	15	23.125	23.125	32.5
2	19.0625	26.875	26.875	36.5625
3	20.9375	28.90625	28.90625	38.4375
4	21.953125	29.84375	29.84375	39.453125
5	22.421875	30.351563	30.351563	39.921875
6	22.67578	30.585938	30.585938	40.175782
7	22.792969	30.71289	30.71289	40.292969
8	22.856445	30.771485	30.771485	40.356445
9	22.885743	30.83223	30.83223	40.385743
10	22.901612	30.817871	30.817871	40.401612
11	22.908936	30.825806	30.825806	40.408936
12	22.91903	30.829468	30.829468	40.412903
13	22.914374	30.831452	30.831452	40.414734
14	22.915726	30.832277	30.832277	40.415726
15	22.916169	30.832863	30.832863	40.416139
16	22.916432	30.833216	30.833216	40.416432

From the above table; values are correct up to 3 decimal points

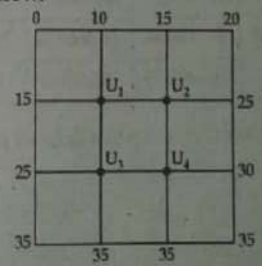
$\therefore$ $U_1 = 22.916432$

$U_2 = U_3 = 30.833216$

and, $U_4 = 40.416432$

Problem 7

Solve $U_{xx} + U_{yy} = 0$ for the following square mesh with boundary conditions as shown in the figure below. [T.U. 2070 Bhadra]



Solution:

Applying standard 5-point formula at each grid point; we get,

$$U_1 = \frac{1}{4}(15 + 10 + U_2 + U_3)$$

$$\therefore \quad U_1 = \frac{1}{4}(25 + U_2 + U_3)$$

Similarly;

$$U_2 = \frac{1}{4}(40 + U_1 + U_4)$$

$$U_3 = \frac{1}{4}(60 + U_1 + U_4)$$

and, $U_4 = \frac{1}{4}(65 + U_2 + U_3)$

Again, using 5 point formula at U_1; we get,

$$U_1 = \frac{1}{4}(40 + U_4)$$

For initial approximation;

Let $U_4 = 0$

$\therefore \quad U_1 = 10$

$U_2 = 12.5$

and, $U_3 = 18.75$

Now, tabulating the successive approximation by using Gauss Jordan's method;

No. of Iteration	U_1	U_2	U_3	U_4
0	10	12.5	18.75	0
1	14.0625	19.53125	24.53125	24.0625
2	17.265625	21.132813	26.132813	27.265625
3	18.066406	21.533203	26.533203	28.066406
4	18.266602	21.633301	26.633301	28.266602
5	18.316650	21.658325	21.658325	28.316650
6	18.319163	21.664581	21.664581	28.329163
7	18.332291	21.666145	26.666145	28,332291
8	18.333073	21.666536	26.666536	28.333073
9	18.333268	21.666634	21.666634	28.333268
10	18.333317	21.666659	26.666659	28.333317

From the above table; values are correct up to 4 decimal points

$\therefore \quad U_1 = 18.333317$

$U_2 = 21.666659$

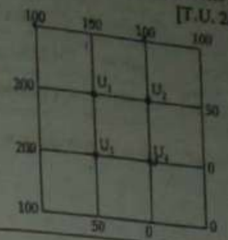
$U_3 = 26.666659$

and, $U_4 = 28.333317$

Problem 8

Find the value of $u(x, y)$ satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of the square region with boundary conditions as shown below.

[T.U. 2070 Magh, P.U. 2012 Fall]



Solution:

Applying standard 5-point formula; we get,

$$U_1 = \frac{1}{4}(200 + 100 + U_2 + U_3)$$

$$\therefore \quad U_1 = \frac{1}{4}(300 + U_2 + U_3) \qquad (1)$$

$$U_2 = \frac{1}{4}(150 + U_1 + U_4) \qquad (2)$$

$$U_3 = \frac{1}{4}(250 + U_1 + U_4) \qquad (3)$$

and, $U_4 = \frac{1}{4}(U_2 + U_3) \qquad (4)$

Again, using 5 point formula at U_1 (diagonal); we get,

$$U_1 = \frac{1}{4}(100 + 100 + U_4 + 200) = \frac{1}{4}(400 + U_4)$$

For initial approximation;

Let $U_4 = 0$

$\therefore \quad U_1 = 100$

$U_2 = 62.5$

and, $U_3 = 87.5$

Now, tabulating the successive approximation by using Gauss Jordan's method;

No. of Iteration	U_1	U_2	U_3	U_4
0	100	62.5	87.5	0
1	112.5	75.0	100	37.5
2	118.75	78.125	103.125	43.75
3	120.3125	78.90625	103.90625	45.3125
4	120.703125	79.101563	104.101563	45.703125
5	120.800781	79.150391	104.150391	45.800781
6	120.825195	79.162598	104.162598	45.825195
7	120.831299	79.165650	104.165650	45.831299

8	120.832825	79.166412	104.166412	45.832825
9	120.833206	79.166603	104.166603	45.833206
10	120.833302	79.166651	104.1666501	45.833302
11	120.833326	79.166663	104.16663	45.833326
12	120.833331	79.166666	104.166666	45.833331
13	120.833333	79.166666	104.166666	45.833333

From the above table; values are correct up to 6 decimal points

$\therefore \quad U_1 = 120.83333$

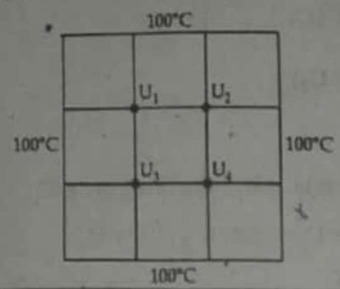
$U_2 = 79.166666$

$U_3 = 104.166666$

and, $U_4 = 45.833333$

Problem 9

Consider a metal plate of size 30 cm × 30 cm, the boundaries of which are held at 100°C. Calculate the temperature at interior points of the plate. Assume the grid size of 10 cm × 10 cm. [T.U. 2068 Baishakh]



Solution:

Let, U_1, U_2, U_3 and U_4 are the values of interior grid (mesh) point, then applying standard 5-point formula; we get,

$$U_1 = \frac{1}{4}(100 + 100 + U_2 + U_3)$$

$$\therefore \quad U_1 = \frac{1}{4}(200 + U_2 + U_3)$$

$$U_2 = \frac{1}{4}(200 + U_1 + U_4)$$

$$U_3 = \frac{1}{4}(200 + U_1 + U_4)$$

$$\text{and, } U_4 = \frac{1}{4}(200 + U_2 + U_3)$$

For initial approximation;

Let $U_1 = U_2 = U_3 = U_4$; then, tabulating the values of approximation by using Gauss Jordan's method;

No. of Iteration	U_1	U_2	U_3	U_4
0	0	0	0	0
1	50	75	75	50
2	87.5	93.75	93.75	87.5
3	96.875	98.4375	98.4375	96.875
4	99.21875	99.609375	99.609375	99.21875
5	99.804688	99.902344	99.902344	99.804688
6	99.951172	99.987793	99.987793	99.975586
7	99.993896	99.996948	99.996948	99.993896
8	99.9984474	99.999237	99.999237	99.998474
9	99.999619	99.999809	99.999809	99.999619
10	99.999905	99.999952	99.999952	99.999905
11	99.999976	99.999988	99.999988	99.999976
12	99.999994	99.999997	99.999997	99.999994
13	99.999999	99.999999	99.999999	99.999999
14	100.00	100.00	100.00	100.00
15	100.00	100.00	100.00	100.00

From the above table; values are exactly equal to preceding values.

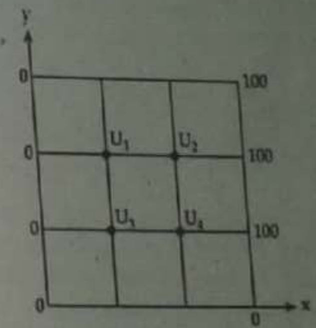
$\therefore \quad U_1 = U_2 = U_3 = U_4 = 100.00$

Problem 10

ih, ih, h²

Solve the Poisson's equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ with boundary condition $u(0, y) = u(x, 0) = 0$ and $u(1, y) = u(x, 1) = 100$ and $h = 100$, taking $h = \frac{1}{3}$.

Solution:



Here,

$$h = \frac{1}{3}$$

The standard 5-point formula for given equation is;

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2(ih, jh) = h^2[-81(ih \cdot jh)]$$

$$= h^4(-81)ij = -ij \quad (1)$$

For $u_1(i = 1, j = 2)$,

$$0 + u_2 + u_3 + 100 - 4u_1 = -2$$

For $u_2(i = 2, j = 2)$,

$$u_1 + 100 + u_4 + 100 - 4u_2 = -4$$

For $u_3(i = 1, j = 1)$,

$$0 + u_4 + 100 + u_1 - 4u_3 = -1$$

For $u_4(i = 2, j = 1)$,

$$u_3 + u_2 - 4u_4 = -102$$

Subtracting (5) from (2); we get,

$$-4u_1 + 4u_4 = 0$$

or, $u_1 = u_4$

Equation (2) becomes,

$$2u_1 - 4u_2 = -204$$

Now,

Multiply equation (2) by 4 and add equation (6); we get,

$$-14u_1 + 4u_3 = -612$$

Add equation (5) and (7) gives,

$$-12u_1 = -613$$

Thus,

$$u_1 = \frac{613}{12} = 51.0833 = \frac{1}{4}$$

$$u_2 = \frac{1}{2}(u_1 + 102) = 76.5477$$

$$u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916$$

Chapter 7

System of Linear Equations

INTRODUCTION

The techniques for solving system of linear algebraic equation are;

- Eliminative technique
- Iterative technique

ELIMINATIVE TECHNIQUE

In this technique, the system is reduces to a form, from which the solution can be obtained by direct and simple substitution. This method do not contains any truncation error. It is also called direct method. Some eliminative techniques used are as follows;

- Method of determinant
- Matrix inversion method
- Gauss elimination method
- Gauss Jordan method
- LU decomposition/Factorization method

ITERATIVE TECHNIQUE

Iterative technique requires some assumption on initial values. These initial assumption are refined until they reach to least error *i.e.,* negligible error. Some iterative techniques are as follows;

- Jacobi iteration method
- Gauss seidal iteration method

ILL CONDITIONED SYSTEM

The solution of the given system of equation depends upon the condition under which the system of equation is given. That is the system of equations will have unique solution infinitely many solution and no solution depending upon the nature of the system of equations. But, small change in the coefficient of variable in the system of equation shows large deviation in the solution then the system is known as ILL conditioned system.

Mathematically; let the two given equations be;

$$a_{11}x_1 + a_{12}x_2 = b_1$$

and, $a_{21}x_1 + a_{22}x_2 = b_2$

If two straight lines represented by the above two equations be almost parallel then the slopes must be nearly equal.

i.e., $\frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}}$

or, $a_{11}a_{22} = a_{12}a_{21}$

or, $a_{11}a_{22} - a_{12}a_{21} = 0$

This expression is the determinant of the coefficient of matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Thus, the condition for ILL defined system of the equation is that, the determinant of the coefficient matrix is small enough or approximately equal to zero.

METHOD OF DETERMINANT

Let us consider the system of linear equation;

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then, the augmented matrix of the given system is;

$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{pmatrix}$$

Now, determinant of the coefficient are;

$$\Delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix}$$

$$\Delta_2 = \begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix}$$

and, $$\Delta_3 = \begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix}$$

Now, by Cramer's rule, solutions of these equations are;

$$x = \frac{\Delta_1}{\Delta}$$
$$y = \frac{\Delta_2}{\Delta}$$

and, $$z = \frac{\Delta_3}{\Delta}$$

MATRIX INVERSION METHOD

Let us consider the system of linear equation;

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then, this system can be written as;

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

where, $$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Hence,

$$AX = B$$

or, $$X = A^{-1}B$$

where, A^{-1} is the inverse matrix of A.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

and, 'A', 'B' and 'C' are cofactor of 'a', 'b' and 'c' respectively and such matrix is called matrix of co-factor.

GAUSS ELIMINATION METHOD

In this method, the unknown values are eliminated successively and system is reduced to a triangular system from which these unknown values are determined using substitution. Gauss elimination can be done using two methods.

i) **Forward elimination**

Given system is reduces to upper triangular matrix.

i.e., $$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3'' & : & d_3'' \end{pmatrix}$$

ii) **Backward elimination**

Given system is reduces to lower triangular matrix.

i.e., $$\begin{pmatrix} a_1'' & 0 & 0 & : & d_1'' \\ a_2' & b_2' & 0 & : & d_2' \\ a_3 & b_3 & c_3 & : & d_3 \end{pmatrix}$$

Consider a system of linear equation;

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then,

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{pmatrix}$$

Step 2: Elimination of 'x' from second and third row;

Applying $R_2 \to R_2 - \frac{a_2}{a_1}R_1$, $R_3 \to R_3 - \frac{a_3}{a_1}R_1$

$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & b_3' & c_3' & : & d_3' \end{pmatrix}$$

where, $b_2' = b_2 - \frac{a_2}{a_1}b_1$, $c_2' = c_2 - \frac{a_2}{a_1}c_1$ and $d_2' = d_2 - \frac{a_2}{a_1}d_1$.

Step 3: Elimination of 'y' from third row

i.e., $R_3 \to R_3 - \frac{b_3'}{b_2'}R_2$

$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3'' & : & d_3'' \end{pmatrix}$$

where, $c_3'' = c_3 - \frac{b_3'}{b_2'}c_2$ and $d_3'' = d_3 - \frac{b_3'}{b_2'}d_2$.

Step 4: Evaluate the unknown values using backward substitution

i.e., $c_3''z = d_3''$

or, $z = \frac{d_3''}{c_3''}$

$b_2'y + c_2'z = d_2'$

or, $y = \frac{d_2' - c_2'z}{b_2'}$

$a_1x + b_1y + c_1z = d_1$

or, $x = \frac{d_1 - b_1y - c_1z}{a_1}$

Note

Similarly; in backward elimination technique, the given augmented matrix is reduced into lower triangular matrix and values are determined using forward substitution.

Problem 1

Solve the system of linear equations by using Gauss elimination method.

$$2x_1 + 2x_2 + 1 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$
$$x_1 - x_2 + x_3 = 0$$

Solution:

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} 2 & 2 & 1 & : & 6 \\ 4 & 2 & 3 & : & 4 \\ 1 & -1 & 1 & : & 0 \end{pmatrix}$$

Step 2: Elimination of 'x' from second and third row;

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - \frac{1}{2}R_1$

$$\begin{pmatrix} 2 & 2 & 1 & : & 6 \\ 0 & -2 & 1 & : & -8 \\ 0 & -2 & 0.5 & : & -3 \end{pmatrix}$$

Step 3: Elimination of 'y' from third row

Applying $R_3 \to R_3 - R_2$

$$\begin{pmatrix} 2 & 2 & 1 & : & 6 \\ 0 & -2 & 1 & : & -8 \\ 0 & 0 & -\frac{1}{2} & : & 5 \end{pmatrix}$$

Step 4: Using backward substitution

From R_3;

$-\frac{1}{2}x_3 = 5$;

or, $x_3 = -10$

From R_2;

$-2x_2 + x_3 = -8$

or, $x_2 = \frac{8-10}{2} = -1$

From R_1;

$2x_1 + 2x_2 + x_3 = 6$

or, $2x_1 = 6 + 2 + 10$

or, $x_1 = 9$

Hence,

$x_1 = 9$
$x_2 = -1$; which is the required solution of the system.
$x_3 = -10$

Problem 2

Solve the system of linear equations by using Gauss elimination method. [T.U. 2069 Bhadra]

$$x + 2y + 3z = 6$$
$$2x + 3y + 5z = 10$$
$$2x - y + 3z = 4$$

Solution:

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 2 & 3 & 5 & : & 10 \\ 2 & -1 & 3 & : & 4 \end{pmatrix}$$

Step 2: Elimination of 'x' from second and third row;

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -1 & : & -2 \\ 0 & -5 & -3 & : & -8 \end{pmatrix}$$

Step 3: Elimination of 'y' from third row

Applying $R_3 \rightarrow R_3 - 5R_2$

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -1 & : & -2 \\ 0 & 0 & 2 & : & 2 \end{pmatrix}$$

Step 4: Using backward substitution

From R_3;

$2z = 2$

or, $z = 1$

From R_2;

$-y - z = -2$

or, $z = 1$

From R_1;

$x + 2y + 3z = 6$

or, $x = 1$

Hence,

$x = 1$

$y = 1$; which is the required solution of the system.

$z = 1$

Problem 3

Solve the system of linear equations by using Gauss elimination method.	
$10x - 7y + 3z + 5u = 6$ $-6x + 8y - z - 4u = 5$ $3x + y + 4z + 11u = 2$ $5x - 9y - 2z + 4u = 7$	[Po.U. 2009 fall]

Solution:

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} 10 & -7 & 3 & 5 & : & 6 \\ -6 & 8 & -1 & -4 & : & 5 \\ 2 & 1 & 4 & 11 & : & 2 \\ 5 & -9 & -2 & 4 & : & 7 \end{pmatrix}$$

Step 2: Elimination of 'x'

Applying $R_2 \rightarrow R_2 + \frac{6}{10}R_1$, $R_3 \rightarrow R_3 - \frac{3}{10}R_1$, $R_4 \rightarrow R_4 - \frac{5}{10}R_1$

$$\begin{pmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & \frac{19}{5} & \frac{4}{5} & -1 & : & \frac{43}{5} \\ 0 & \frac{31}{10} & \frac{31}{10} & \frac{19}{2} & : & \frac{1}{5} \\ 0 & -\frac{11}{2} & -\frac{7}{2} & \frac{3}{2} & : & 4 \end{pmatrix}$$

or, $$\begin{pmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 19 & 4 & -5 & : & 43 \\ 0 & 31 & 31 & 95 & : & 2 \\ 0 & -11 & -7 & 3 & : & 8 \end{pmatrix}$$

Step 3: Elimination of 'y'

Applying $R_3 \rightarrow R_3 - \frac{31}{19}R_2$, $R_4 \rightarrow R_4 + \frac{11}{19}R_2$

$$\begin{pmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 19 & 4 & -5 & : & 43 \\ 0 & 0 & 465 & 1960 & : & -1295 \\ 0 & -11 & -89 & 2 & : & 625 \end{pmatrix}$$

Step 4: Elimination of 'z'

Applying $R_4 \rightarrow R_4 + \frac{89}{465}R_3$

$$\begin{pmatrix} 10 & 7 & 3 & 5 & : & 6 \\ 0 & 19 & 4 & -5 & : & 43 \\ 0 & 0 & 465 & 1960 & : & -1295 \\ 0 & 0 & 0 & 1 & : & 1 \end{pmatrix}$$

Step 5: Determination of unknowns using backward substitution

From R_4;

$u = 1$

From R_3;

$465z + 160u = -1295$

or, $z = 7$

From R_2;

$19y + 4z - 5u = 43$

or, $y = 1.05$

From R_1;

$10x - 7y + 3z + 5u = 6$

or, $x = -1.5$

Hence,

$x = -1.5$

$y = 1.05$; which is the required solution of the system.

$z = 7$

$u = 1$

Problem 4

Solve the system of linear equations by using Gauss elimination method.	
$5x_1 + x_2 + x_3 + x_4 = 4$ $x_1 + 7x_2 + x_3 + 4x_4 = 6$ $x_1 + x_2 + 6x_3 + x_4 = -5$ $x_1 + x_2 + x_3 + x_4 = 0$	[T.U. 2066 Bhadra]

Solution:

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 1 & 7 & 1 & 4 & : & 6 \\ 1 & 1 & 6 & 1 & : & -5 \\ 1 & 1 & 1 & 1 & : & 0 \end{pmatrix}$$

Step 2: Elimination of 'z'

Applying $R_2 \to R_2 - \frac{1}{5}R_1$, $R_3 \to R_3 - \frac{1}{5}R_1$, $R_4 \to R_4 - \frac{1}{5}R_1$

$$\begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 19 & : & 26 \\ 0 & 4 & 29 & 4 & : & -29 \\ 0 & 1 & 1 & 1 & : & -1 \end{pmatrix}$$

Step 3: Elimination of x_2

Applying $R_3 \to R_3 - \frac{4}{34}R_2$, $R_4 \to R_4 - \frac{1}{34}R_1$

$$\begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 19 & : & 26 \\ 0 & 0 & 97 & 6 & : & -109 \\ 0 & 0 & 2 & 1 & : & -4 \end{pmatrix}$$

Step 4: Elimination of x_3

Applying $R_4 \to R_4 - \frac{2}{97}R_3$

$$\begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 19 & : & 26 \\ 0 & 0 & 97 & 6 & : & 109 \\ 0 & 0 & 0 & \frac{85}{97} & : & \frac{170}{97} \end{pmatrix}$$

Step 5: Using backward substitution

From R_4;

$$x_4 = -2$$

From R_3;

$$97x_3 + 6x_4 = -109$$

or, $x_3 = -1.25$

From R_2;

$$34x_2 + 4x_3 + 19x_4 = 26$$

or, $x_2 = 2.03$

From R_1;

$$5x_1 + x_2 + x_3 + x_4 = 4$$

or, $x_1 = 1.04$

Hence,

$x_1 = 1.04$
$x_2 = 2.03$
$x_3 = -1.25$; which is the required solution of the system.
$x_4 = -2$

Problem 5

Using backward elimination, solve;

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - 4 - z = 4$$

Solution:

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} 1 & 4 & -1 & : & -5 \\ 1 & 1 & -6 & : & -12 \\ 3 & -4 & -1 & : & 4 \end{pmatrix}$$

Step 2: Elimination of 'z'

Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 6R_3$

$$\begin{pmatrix} -2 & 5 & 0 & : & -9 \\ -17 & 7 & -6 & : & -36 \\ 3 & -1 & -1 & : & 4 \end{pmatrix}$$

Step 3: Elimination of 'x'

Applying $R_1 \to R_1 - \frac{5}{7}R_2$

$$\begin{pmatrix} \frac{71}{7} & 0 & 0 & : & \frac{117}{7} \\ -17 & 7 & 0 & : & -36 \\ 3 & -1 & -1 & : & 4 \end{pmatrix}$$

Step 4: Using forward substitution

From R_1;

$$\frac{71}{7}x = \frac{117}{7}$$

or, $x = 1.6479$

From R_2;

$$-17x + 7y = -36$$

or, $y = -1.140$

From R_3;

$$3x - y - z = 4$$

or, $z = 2.0845$

Hence,

$x = 1.6479$
$y = -1.140$;
$z = 2.0845$

which is the required solution of the system.

GAUSS ELIMINATION USING PIVOTING

Pivoting is the process of recording the equation in order to get better accuracy. Pivoting can be done only if the pivot element is non-zero. The main diagonal element of augmented matrix is called pivot element. The pivoting is necessary if the main diagonal element of augmented matrix is zero. Pivoting can be applied by two ways;

i) Partial pivoting
ii) Complete pivoting

The partial pivoting involves the following steps.

- Search and locate largest absolute value among coefficient in first column.
- Exchange the first row with row containing that element.
- Eliminate the first variable in the second equation.

- When second row becomes pivot row, search for the coefficient in the second column from second row to n^{th} row and locate the largest coefficient.
- Exchange the second row with the row containing largest coefficient.
- Continue the process until $(n-1)^{th}$ unknown are eliminated.

Process for Complete Pivoting

- Check for the largest coefficient of unknown.
- Interchange the row and column, so that the largest coefficient of unknown appears as first element of augmented matrix.

Problem 1

Solve the given system of linear equations using partial pivoting.

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$
$$x_1 + x_2 + x_3 = 0$$

Solution:

The augmented matrix of given system is;

$$\begin{pmatrix} 2 & 2 & 1 & : & 6 \\ 4 & 2 & 3 & : & 4 \\ 1 & -1 & 1 & : & 0 \end{pmatrix}$$

Check the first column for largest coefficient. Here, R_2 is pivot equation. [Since, 4 is the largest coefficient.]

Interchanging R_1 and R_2; we have,

$$\begin{pmatrix} \boxed{4} & 2 & 3 & : & 4 \\ 2 & 2 & 1 & : & 6 \\ 1 & -1 & 1 & : & 0 \end{pmatrix} \text{Pivot}$$

Applying $R_2 \to R_2 - \frac{1}{2}R_1$, $R_3 \to R_3 - \frac{1}{4}R_1$

$$\begin{pmatrix} 4 & 2 & 3 & : & 4 \\ 0 & 1 & -0.5 & : & 4 \\ 1 & -1.5 & 0.25 & : & -1 \end{pmatrix}$$

Now, check the largest value for second column, except first row. (Leave first row)

−1.5 is the largest coefficient.

∴ R_3 is pivot equation.

Interchanging R_2 and R_3; we have,

$$\begin{pmatrix} 4 & 2 & 3 & : & 4 \\ \boxed{0 & -1.5 & 0.25 & : & -1} \\ 1 & 1 & 0.5 & : & 4 \end{pmatrix} \text{Pivot}$$

Applying $R_3 \to R_2 + \frac{2}{3}R_2$

$$\begin{pmatrix} 4 & 2 & 3 & : & 4 \\ 0 & -1.5 & 0.25 & : & -1 \\ 0 & 0 & -0.33 & : & 3.33 \end{pmatrix}$$

Using backward substitution;

From R_3;

$$x_3 = -10$$

From R_2;

$$-1.5x_2 + 0.25x_3 = -1$$

or, $x_2 = -1$

From R_1;

$$4x_1 + 2x_2 + 3x_3 = 4$$

or, $x_1 = 9$

Hence,

$$x_1 = 9$$
$$x_2 = -1$$
$$x_3 = -10$$

Problem 2

Solve the given system of linear equations using partial pivoting.

$$x_1 + x_2 - x_3 = 2$$
$$2x_1 + 3x_2 + 5x_3 = -3$$
$$3x_1 + 2x_2 - 3x_3 = 6$$

[Po.U. 2005 spring]

Solution:

The augmented matrix of given system is;

$$\begin{pmatrix} 1 & 1 & -1 & : & 2 \\ 2 & 3 & 5 & : & -3 \\ \boxed{3} & 2 & -3 & : & 6 \end{pmatrix}$$

Check the first column.

R_3 is the pivot equation.

Interchanging $R_1 \leftrightarrow R_3$

$$\begin{pmatrix} 3 & 2 & -3 & : & 6 \\ 2 & 3 & 5 & : & -3 \\ 1 & 2 & -1 & : & 2 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{2}{3}R_1$, $R_3 \to R_3 - \frac{1}{3}R_1$

$$\begin{pmatrix} 3 & 2 & -3 & : & 6 \\ 0 & \boxed{\frac{5}{3}} & 7 & : & -7 \\ 0 & \frac{1}{3} & 0 & : & 0 \end{pmatrix}$$

R_2 is pivot equation.

Applying $R_3 \to R_3 - \frac{1}{5}R_2$

$$\begin{pmatrix} 3 & 2 & -3 & : & 6 \\ 0 & \frac{5}{3} & 7 & : & -7 \\ 0 & 0 & -\frac{7}{5} & : & \frac{7}{5} \end{pmatrix}$$

Now, using backward substitution;

From R_3;

$$x_3 = -1$$

From R_2;

$$\frac{5}{3}x_2 + 7x_3 = -7$$

or, $x_2 = 0$

From R_1;

$$3x_1 + 2x_2 - 3x_3 = 6$$

or, $x_1 = 1$

Hence,

$x_1 = 1$
$x_2 = 0$
$x_3 = 1$

Problem 3

> Solve the given system of linear equations using partial pivoting.
> $$2x_1 + x_2 + x_3 - 2x_4 = 10$$
> $$4x_1 + 2x_3 + x_4 = 8$$
> $$3x_1 + 2x_2 + 2x_3 = 7$$
> $$x_1 + 3x_2 + 2x_3 - x_4 = -5$$
> [T.U. 2067, Ashadh]

Solution:

The augmented matrix of given system is;

$$\begin{pmatrix} 2 & 1 & 1 & -2 & : & -10 \\ \boxed{4} & 0 & 2 & 1 & : & 8 \\ 3 & 2 & 2 & 0 & : & 7 \\ 1 & 3 & 2 & -4 & : & -5 \end{pmatrix}$$

In the first column, R_2 is pivot equation.

Interchanging R_1 and R_2; we get,

$$\begin{pmatrix} 4 & 0 & 2 & 1 & : & 8 \\ 2 & 1 & 1 & -2 & : & -10 \\ 3 & 2 & 2 & 0 & : & 7 \\ 1 & 3 & 2 & -4 & : & -5 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - \frac{1}{2}R_1$, $R_3 \rightarrow R_3 - \frac{3}{4}R_1$ and $R_4 \rightarrow \frac{1}{4}R_1$

$$\begin{pmatrix} 4 & 0 & 2 & 1 & : & 8 \\ 0 & 1 & 0 & -\frac{5}{2} & : & -14 \\ 0 & 2 & \frac{1}{2} & -\frac{3}{4} & : & 1 \\ 0 & \boxed{3} & \frac{3}{2} & -\frac{17}{4} & : & -7 \end{pmatrix}$$

Checking column 2, R_4 is pivot equation.

Interchanging R_4 and R_2; we get,

$$\begin{pmatrix} 4 & 0 & 2 & 1 & : & 8 \\ 0 & 3 & \frac{3}{2} & -\frac{17}{4} & : & 7 \\ 0 & 2 & \frac{1}{2} & -\frac{3}{4} & : & 1 \\ 0 & 1 & 0 & -\frac{5}{2} & : & -14 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - \frac{2}{3}R_3$, $R_4 \rightarrow R_4 - \frac{1}{3}R_1$

$$\begin{pmatrix} 4 & 0 & 2 & 1 & : & 8 \\ 0 & 3 & \frac{3}{2} & -\frac{17}{4} & : & 7 \\ 0 & 0 & -\frac{1}{2} & \frac{25}{12} & : & -\frac{11}{3} \\ 0 & 0 & -\frac{1}{2} & -\frac{13}{12} & : & -\frac{49}{3} \end{pmatrix}$$

Now, applying $R_4 \rightarrow R_4 + R_3$

$$\begin{pmatrix} 4 & 0 & 2 & 1 & : & 8 \\ 0 & 3 & \frac{3}{2} & -\frac{17}{4} & : & 7 \\ 0 & 0 & -\frac{1}{2} & \frac{25}{12} & : & -\frac{11}{3} \\ 0 & 0 & 0 & 1 & : & -20 \end{pmatrix}$$

Using backward substitution

From R_4;

$$x_4 = -20$$

From R_3;

$$-\frac{1}{2}x_3 + \frac{25}{12}x_4 = -\frac{11}{3}$$

or, $x_3 = -76$

From R_2;

$$3x_2 + \frac{3}{2}x_3 - \frac{17}{4}x_1 = 7$$

or, $x_2 = 12$

From R_1;

$$4x_1 + 0 + 2x_3 + x_4 = 8$$

or, $x_1 = 45$

Hence,

$x_1 = 45$
$x_2 = 12$
$x_3 = -76$
$x_4 = -20$

Problem 4

> Solve the given system of linear equations using partial pivoting.
> $$b + 3c + 29 = 19$$
> $$3b + 2c + 2d = 20$$
> $$a + 4b + 2d = 17$$
> $$-2a + 2b + c + d = 9$$
> [P.U. 2010 fall, T.U. 2062, Baishakh]

Solution:

The augmented matrix of given system is;

$$\begin{pmatrix} 0 & 1 & 3 & 2 & : & 19 \\ 0 & 3 & 2 & 2 & : & 20 \\ 1 & 4 & 0 & 2 & : & 17 \\ \boxed{-2} & 2 & 1 & 1 & : & 9 \end{pmatrix}$$

Here, R_4 is pivot equation. Hence, pivoting,

$$\begin{pmatrix} -2 & 2 & 1 & 1 & : & 9 \\ 0 & 3 & 2 & 2 & : & 20 \\ 1 & 4 & 0 & 2 & : & 17 \\ 0 & 1 & 3 & 2 & : & 19 \end{pmatrix}$$

Applying $R_3 \to R_3 + 2R_1$

$$\begin{pmatrix} -2 & 2 & 1 & 1 & : & 9 \\ 0 & 3 & 2 & 2 & : & 20 \\ 0 & \boxed{10} & 1 & 5 & : & 43 \\ 0 & 1 & 3 & 2 & : & 19 \end{pmatrix}$$

Here, R_3 is pivot equation.
Interchanging R_2 and R_3; we get,

$$\begin{pmatrix} -2 & 2 & 1 & 1 & : & 9 \\ 0 & 10 & 1 & 5 & : & 43 \\ 0 & 3 & 2 & 2 & : & 20 \\ 0 & 1 & 3 & 2 & : & 19 \end{pmatrix}$$

Applying $R_3 \to R_3 - \frac{3}{10}R_2$, $R_4 \to R_4 - \frac{1}{10}R_2$

$$\begin{pmatrix} -2 & 2 & 1 & 1 & : & 9 \\ 0 & 10 & 1 & 5 & : & 43 \\ 0 & 0 & 17 & 5 & : & 71 \\ 0 & 0 & \boxed{29} & 15 & : & 147 \end{pmatrix}$$

Now,

Applying $R_4 \to R_4 - \frac{29}{17}R_3$

$$\begin{pmatrix} -2 & 2 & 1 & 1 & : & 9 \\ 0 & 10 & 1 & 5 & : & 43 \\ 0 & 0 & 17 & 5 & : & 71 \\ 0 & 0 & 0 & 110 & : & 440 \end{pmatrix}$$

From backward substitution; we have,

From R_4;

$$110d = 440$$

or, $d = 4$

From R_3;

$$17c + 5d = 71$$

or, $c = 3$

From R_2;

$$10b + c + 5d = 43$$

or, $b = 2$

From R_1;

$$-2a + 2b + c + d = 9$$

or, $a = 1$

Hence,

$a = 1$
$b = 2$
$c = 3$
$d = 4$

Problem 5

Solve the following system using complete pivoting.

$$2x + 7y + 3z = 5$$
$$3x + 4y + 9z = 15$$
$$2x + 3y + z = 9$$

Solution:

The augmented matrix of given system is;

$$\begin{array}{cccccc} x & y & z & & & \text{const.} \end{array}$$
$$\begin{pmatrix} 2 & 7 & 3 & : & 5 \\ 3 & 4 & \boxed{9} & : & 15 \\ 2 & 3 & 1 & : & 9 \end{pmatrix}$$

Here, 9 is the largest coefficient in R_2.

Interchanging R_2 and R_1; we have,

$$\begin{array}{cccccc} x & y & z & & & \text{const.} \end{array}$$
$$\begin{pmatrix} 3 & 4 & 9 & : & 15 \\ 2 & 7 & 3 & : & 5 \\ 2 & 3 & 1 & : & 9 \end{pmatrix}$$

Again, interchanging C_1 and C_3; we have,

$$\begin{array}{cccccc} z & y & x & & & \text{const.} \end{array}$$
$$\begin{pmatrix} 9 & 4 & 3 & : & 15 \\ 3 & 7 & 2 & : & 5 \\ 1 & 3 & 2 & : & 9 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{1}{3}R_1$, $R_3 \to R_3 - \frac{1}{9}R_1$

$$\begin{array}{cccccc} z & y & x & & & \text{const.} \end{array}$$
$$\begin{pmatrix} 9 & 4 & 3 & : & 15 \\ 0 & \frac{17}{3} & 1 & : & 0 \\ 0 & \frac{23}{9} & \frac{5}{3} & : & \frac{22}{3} \end{pmatrix}$$

Applying $R_3 \to R_3 - \frac{23}{51}R_1$

$$\begin{array}{cccccc} z & y & x & & & \text{const.} \end{array}$$
$$\begin{pmatrix} 9 & 4 & 3 & : & 15 \\ 0 & \frac{17}{3} & 1 & : & 0 \\ 0 & 0 & \frac{62}{51} & : & \frac{22}{3} \end{pmatrix}$$

Using backward substitution

From R_3;

$$x = \frac{187}{31}$$

From R_2;

$$y = -\frac{33}{31}$$

From R_1;

$$z = \frac{4}{31}$$

Hence,

$$x = \frac{187}{31}$$

$$y = -\frac{33}{31}$$

$$z = \frac{4}{31}$$

Problem 6

Solve the following system using complete pivoting. [T.U. 2065 Shrawan]

$$2x + 3y + 2z = 2$$
$$10x + 3y + 4z = 16$$
$$3x + 6y + z = 6$$

Solution:

The augmented matrix of given system is;

$$\begin{pmatrix} 2 & 3 & 2 & : & 2 \\ \boxed{10} & 3 & 4 & : & 16 \\ 3 & 6 & 1 & : & 6 \end{pmatrix}$$

Here, 10 is the largest coefficient in R_2.

Interchanging R_1 and R_2; we have,

$$\begin{pmatrix} 10 & 3 & 4 & : & 16 \\ 2 & 3 & 2 & : & 2 \\ 3 & 6 & 1 & : & 6 \end{pmatrix}$$

Applying $R_2 \rightarrow R - \frac{2}{10}R_1$, $R_3 \rightarrow \frac{3}{10}R_1$

$$\begin{pmatrix} 10 & 3 & 4 & : & 16 \\ 0 & \frac{12}{5} & \frac{6}{5} & : & -\frac{6}{5} \\ 0 & \frac{27}{5} & \frac{2}{5} & : & \frac{14}{5} \end{pmatrix}$$

Again, interchanging R_2 and R_3; we have,

$$\begin{pmatrix} 10 & 3 & 4 & : & 16 \\ 0 & \frac{27}{5} & \frac{2}{5} & : & \frac{14}{5} \\ 0 & \frac{12}{5} & \frac{6}{5} & : & -\frac{6}{5} \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - \frac{12}{27}R_2$

$$\begin{pmatrix} 10 & 3 & 4 & : & 16 \\ 0 & \frac{27}{5} & \frac{1}{5} & : & \frac{14}{5} \\ 0 & 0 & \frac{10}{9} & : & -\frac{22}{9} \end{pmatrix}$$

Using backward substitution

From R_3;

$$z = -\frac{22}{9}$$

From R_2;

$$y = \frac{6}{10}$$

From R_1;

$$x = \frac{23}{10}$$

Hence,

$$x = \frac{23}{10}$$

$$y = \frac{6}{10}$$

$$z = -\frac{22}{9}$$

Problem 7

Find determinant of matrix using Gauss elimination method.

$$\begin{pmatrix} 1.2 & -2.1 & 3.2 \\ -1.4 & -2.6 & 3.0 \\ 1.1 & 3.6 & 5.0 \end{pmatrix}$$

[Po.U. 2010 Spring]

Solution:

Using complete pivoting;

C_3 is the pivot equation. Hence, interchanging C_1 and C_3; we have,

$$\begin{pmatrix} 3.2 & -2.1 & 1.2 \\ 3.0 & -2.6 & -1.4 \\ 5.0 & 3.6 & 1.1 \end{pmatrix}$$

Again, interchanging R_1 and R_3; we have,

$$\begin{pmatrix} 5.0 & 3.6 & 1.1 \\ 3.0 & -2.6 & -1.4 \\ 3.2 & -2.1 & 1.2 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - \frac{3}{5}R_1$, $R_3 \rightarrow R_3 - \frac{3.2}{5}R_1$

$$\begin{pmatrix} 5.0 & 3.6 & 1.1 \\ 3.0 & -\frac{119}{25} & -\frac{103}{50} \\ 3.2 & -\frac{1101}{250} & \frac{262}{152} \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - \frac{1101}{1190}R_2$

$$\begin{pmatrix} 5.0 & 3.6 & 1.1 \\ 0 & -\frac{119}{25} & -\frac{103}{50} \\ 0 & 0 & 4.0 \end{pmatrix}$$

Now,

$$\text{Determinant of matrix} = 5.0\left|4 \times \left(-\frac{119}{25}\right) - 0\right|$$

$$= 95.240; \text{ which is the required answer.}$$

Problem 8

Find the determinant of the following matrix using Gauss elimination method with partial pivoting.

$$\begin{pmatrix} 1 & -2 & 4 & 3 \\ 2 & 0 & 2 & 4 \\ 3 & -1 & 0 & 2 \\ -1 & 2 & 2 & -3 \end{pmatrix}$$

Solution:

Here, R_3 is the pivot equation.

$\therefore$ Interchanging R_1 and R_3; we have,

$$\begin{pmatrix} 3 & -1 & 0 & 0 \\ 2 & 0 & 2 & 4 \\ 1 & -2 & 4 & 3 \\ -1 & 2 & 2 & -3 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{2}{3}R_1$, $R_3 \to R_3 - \frac{1}{3}R_1$, $R_4 \to R_3 + \frac{1}{3}R_1$

$$\begin{pmatrix} 3 & -1 & 0 & 2 \\ 0 & \frac{2}{3} & 2 & \frac{8}{3} \\ 0 & -\frac{5}{3} & 4 & \frac{7}{3} \\ 0 & \frac{5}{3} & 2 & -\frac{7}{3} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 & 0 & 2 \\ 0 & 2 & 6 & 8 \\ 0 & -5 & 12 & 7 \\ 0 & 5 & 2 & -7 \end{pmatrix}$$

R_4 is the pivot equation.

$\therefore$ Interchanging R_2 and R_4; we have,

$$\begin{pmatrix} 3 & -1 & 0 & 2 \\ 0 & 5 & 6 & -7 \\ 0 & -5 & 12 & 7 \\ 0 & 2 & 6 & 8 \end{pmatrix}$$

Applying $R_3 \to R_3 + R_2$, $R_4 \to R_4 - \frac{2}{5}R_2$

$$\begin{pmatrix} 3 & -1 & 0 & 2 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 18 & 0 \\ 0 & 0 & \frac{18}{5} & \frac{54}{5} \end{pmatrix}$$

Applying $R_4 \to R_4 - \frac{1}{5}R_3$

$$\begin{pmatrix} 3 & -1 & 0 & 2 \\ 0 & 5 & 6 & -7 \\ 0 & 0 & 18 & 0 \\ 0 & 0 & 0 & \frac{54}{5} \end{pmatrix}$$

Then,

$$\text{Determinant of matrix} = 3\begin{vmatrix} 5 & 6 & 7 \\ 0 & 18 & 0 \\ 0 & 0 & \frac{54}{5} \end{vmatrix} = 3 \times 5\begin{vmatrix} 18 & 0 \\ 0 & \frac{54}{5} \end{vmatrix} = 2916$$

$\therefore$ Det of A = 2916.

GAUSS JORDAN METHOD

Gauss Jordan method is the modified form of Gauss elimination method. In this method, the augmented matrix is reduced to unit diagonal matrix. Here, we get the solution without using back substitution.

Consider a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_2 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_3 + a_{32}x_2 + a_{33}x_3 = b_3$$

The augmented matrix is the form of;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{pmatrix}$$

After the completion of Gauss Jordan method the equation becomes;

$$\begin{pmatrix} 1 & 0 & 0 & : & B_1 \\ 0 & 1 & 0 & : & B_2 \\ 0 & 0 & 1 & : & B_3 \end{pmatrix}$$

Hence, the required solution is;

$x_1 = B_1$
$x_2 = B_2$
$x_3 = B_3$

Problem 1

Solve the following system of linear equation using Gauss Jordan elimination method. [Po.U. 2008 fall]

$$x + 2y + z = 8$$
$$2x + 3y + 4z = 20$$
$$4x + 3y + 2z = 16$$

Solution:

Arranging

$$4x + 3y + 2z = 16$$
$$2x + 3y + 4z = 20$$
$$x + 2y + z = 8$$

Step 1

The augmented matrix of the given system is;

$$\begin{pmatrix} 4 & 3 & 2 & : & 16 \\ 2 & 3 & 4 & : & 20 \\ 1 & 2 & 1 & : & 8 \end{pmatrix}$$

Step 2

Applying $R_2 \to 2R_2 - R_1$, $R_3 \to 4R_3 - R_1$

$$\begin{pmatrix} 4 & 3 & 2 & : & 16 \\ 0 & 3 & 6 & : & 24 \\ 0 & 5 & 2 & : & 16 \end{pmatrix}$$

Step 3

Applying $R_1 \to R_1 - R_2$, $R_3 \to 3R_3 - 5R_2$

$$\begin{pmatrix} 4 & 0 & 4 & : & 8 \\ 0 & 3 & 6 & : & 24 \\ 0 & 0 & -24 & : & -72 \end{pmatrix}$$

Applying $R_1 \to 6R_1 + 3R_3$, $R_2 \to 4R_2 + R_3$

$$\begin{pmatrix} 24 & 0 & 0 & : & 4 \\ 0 & -12 & 0 & : & 24 \\ 0 & 0 & 24 & : & -72 \end{pmatrix}$$

Hence,

From R_3;

$z = 3$

From R_2;

$y = 2$

From R_1;

$x = 1$

Hence,

$x = 1$

$y = 2$

$z = 3$

Problem 2

Solve the following system of linear equation using Gauss Jordan elimination method. [Po.U. 2007 fall]

$$4x_1 + 3x_2 - 4x_4 = 7$$
$$3x_2 - 4x_3 + 3x_4 = 5$$
$$x_1 + x_2 - 2x_3 + 3x_4 = -2$$
$$x_1 - 3x_3 + x_4 = -5$$

Solution:

The augmented matrix of the given system is;

$$\begin{pmatrix} 4 & 3 & 0 & -4 & : & 7 \\ 0 & 3 & -4 & 3 & : & 5 \\ 1 & 1 & -2 & 3 & : & -2 \\ 1 & 0 & -3 & 1 & : & -5 \end{pmatrix}$$

By pivoting;

$$\begin{pmatrix} 1 & 0 & -3 & 1 & : & 5 \\ 0 & 3 & -4 & 3 & : & 5 \\ 1 & 1 & -2 & 3 & : & -2 \\ 4 & 3 & 0 & -4 & : & 7 \end{pmatrix}$$

Applying $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 4R_1$, $R_2 \to \frac{R_2}{R_3}$

$$\begin{pmatrix} 1 & 0 & -3 & 1 & : & -5 \\ 0 & 1 & -\frac{4}{3} & 1 & : & \frac{5}{3} \\ 0 & 1 & 1 & 2 & : & 3 \\ 0 & 3 & 0.75 & -4.25 & : & 8.25 \end{pmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_4 \to R_4 - 3R_2$

$$\begin{pmatrix} 1 & 0 & -3 & 1 & : & -5 \\ 0 & 1 & -\frac{4}{3} & 1 & : & \frac{5}{3} \\ 0 & 0 & 2.33 & \frac{1}{2.333} & : & \frac{1.333}{2.333} \\ 0 & 0 & 4.75 & -7.25 & : & 3.25 \end{pmatrix}$$

Applying $R_1 \to R_1 + 3R_3$, $R_2 \to R_2 + \frac{4}{3}R_3$, $R_4 \to R_4 - 4.75$

$$\begin{pmatrix} 1 & 0 & 0 & 2.2859 & : & -3.285 \\ 0 & 1 & 0 & 1.5714 & : & 2.4285 \\ 0 & 0 & 1 & 0.4285 & : & 0.5714 \\ 0 & 0 & 0 & 1 & : & -0.0576 \end{pmatrix}$$

Applying $R_1 \to R_1 - 2.2859R_4$, $R_2 \to R - 1.57R_4$, $R_3 \to R_3 - 0.4285R_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & : & -3.4175 \\ 0 & 1 & 0 & 0 & : & 2.3378 \\ 0 & 0 & 1 & 0 & : & 0.5466 \\ 0 & 0 & 0 & 1 & : & -0.0576 \end{pmatrix}$$

Solving; we get,

$x_1 = -3.1538$

$x_2 = 2.519154$

$x_3 = 0.5961$

$x_4 = -0.05768$

Problem 3

Solve the following system of linear equation using Gauss Jordan elimination method. [T.U. 2063 Jestha]

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ -2 & 1 & 4 & 0 \\ -1 & 3 & 0 & 2 \\ 2 & 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 0 \\ -7 \end{bmatrix}$$

Solution:

The augmented matrix of the given system is;

$$\begin{pmatrix} 0 & 2 & 4 & 2 & : & 4 \\ -2 & 1 & 4 & 0 & : & -7 \\ -1 & 3 & 0 & 2 & : & 0 \\ \boxed{2} & 1 & 2 & -3 & : & -7 \end{pmatrix}$$

Pivoting (partial)

$$\left(\begin{array}{cccccc} 2 & 1 & 2 & -3 & : & 7 \\ -2 & 1 & 4 & 0 & : & -7 \\ -1 & 3 & 0 & 2 & : & 0 \\ 0 & 2 & 4 & 2 & : & 4 \end{array}\right)$$

Applying $R_2 \to R_2 + R_1, R_3 \to R_3 + \frac{1}{2}R_1$

$$\left(\begin{array}{cccccc} 2 & 1 & 2 & -3 & : & 7 \\ 0 & 2 & 6 & -3 & : & 0 \\ 0 & \frac{7}{2} & 1 & \frac{1}{2} & : & \frac{7}{2} \\ 0 & 2 & 2 & 4 & : & 4 \end{array}\right)$$

Applying $R_3 \to R_3 - \frac{7}{4}R_2, R_4 \to R_4 - R_2$

$$\left(\begin{array}{cccccc} 2 & 1 & 2 & -3 & : & 7 \\ 0 & 2 & 6 & -3 & : & 0 \\ 0 & 0 & -\frac{19}{2} & \frac{23}{4} & : & \frac{7}{2} \\ 0 & 0 & -4 & 1 & : & 4 \end{array}\right)$$

Applying $R_4 \to R_4 - \frac{8}{19}R_3$

$$\left(\begin{array}{cccccc} 2 & 1 & 2 & -3 & : & 7 \\ 0 & 2 & 6 & -3 & : & 0 \\ 0 & 0 & -\frac{19}{2} & \frac{23}{4} & : & \frac{7}{2} \\ 0 & 0 & 0 & -\frac{27}{19} & : & \frac{48}{19} \end{array}\right)$$

Applying $R_1 \to R_1 - \frac{19}{9}R_4, R_2 \to R_2 - \frac{19}{9}R_4, R_3 \to R_3 + \frac{23 \times 19}{4 \times 27}R_4$

$$\left(\begin{array}{cccccc} 2 & 1 & 2 & 0 & : & \frac{5}{3} \\ 0 & 2 & 6 & 0 & : & -\frac{16}{3} \\ 0 & 0 & -\frac{19}{2} & 0 & : & \frac{247}{18} \\ 0 & 0 & 0 & -\frac{27}{19} & : & \frac{48}{19} \end{array}\right)$$

Applying $R_1 \to R_1 - \frac{4}{19}R_3, R_2 \to R_2 + \frac{12}{19}R_3$

$$\left(\begin{array}{cccccc} 2 & 1 & 0 & 0 & : & \frac{41}{9} \\ 0 & 2 & 0 & 0 & : & \frac{10}{3} \\ 0 & 0 & -\frac{19}{2} & 0 & : & \frac{247}{18} \\ 0 & 0 & 0 & -\frac{27}{19} & : & \frac{48}{19} \end{array}\right)$$

Applying $R_1 \to R_1 - \frac{R_2}{2}$

$$\left(\begin{array}{cccccc} 2 & 1 & 0 & 0 & : & \frac{26}{9} \\ 0 & 2 & 0 & 0 & : & \frac{10}{3} \\ 0 & 0 & -\frac{19}{2} & 0 & : & \frac{247}{18} \\ 0 & 0 & 0 & -\frac{27}{19} & : & \frac{48}{19} \end{array}\right)$$

Solving; we get,

$x_1 = \frac{13}{9}$

$x_2 = \frac{10}{6}$

$x_3 = -\frac{13}{9}$

$x_4 = -\frac{48}{27}$

INVERSE OF MATRIX

We know that 'X' will be the inverse of matrix A, if

$$AX = I$$

where, 'I' is the unit matrix of same order as 'A'.

It is required to determine the element of 'X' such that $AX = I$ is satisfied.

Consider the system of inverse in matrix form $AX = I$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This equation is equivalent to the following three equations which are;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (2)$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (3)$$

We can therefore apply the Gauss elimination method of each of these systems and the resulting in each case will be corresponding column of A^{-1}.

Problem 1

Find the inverse of matrix.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

Solution:

Let, $x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A.

Then,

$AX = I$

The augmented matrix is;

$$\begin{pmatrix} 2 & 1 & 1 & : & 1 & 0 & 0 \\ 3 & 2 & 3 & : & 0 & 1 & 0 \\ 1 & 4 & 9 & : & 0 & 0 & 1 \end{pmatrix}$$

Applying $R_1 \to \frac{R_1}{2}$;

$$\begin{pmatrix} 1 & 0.5 & 0.5 & : & 0.5 & 0 & 0 \\ 3 & 2 & 3 & : & 0 & 1 & 0 \\ 1 & 4 & 9 & : & 0 & 0 & 1 \end{pmatrix}$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 0.5 & 0.5 & : & 0.5 & 0 & 0 \\ 0 & 0.5 & 1.5 & : & -1.5 & 1 & 0 \\ 0 & 3.5 & 8.5 & : & -0.5 & 0 & 1 \end{pmatrix}$$

Applying $R_2 \to \frac{R_2}{0.5}$

$$\begin{pmatrix} 1 & 0.5 & 0.5 & : & 0.5 & 0 & 0 \\ 0 & 1 & 3 & : & -3 & 2 & 0 \\ 0 & 3.5 & 8.5 & : & -0.5 & 0 & 1 \end{pmatrix}$$

Applying $R_3 \to R_3 - 3.5R_1$

$$\begin{pmatrix} 1 & 0.5 & 0.5 & : & 0.5 & 0 & 0 \\ 0 & 1 & 3 & : & -3 & 2 & 0 \\ 0 & 0 & -2 & : & 10 & -7 & 1 \end{pmatrix}$$

This is equivalent to three systems.

$$\begin{pmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0.5 \\ -3 \\ 10 \end{pmatrix}$$

By using backward substitution; we get,

From R_1;

$x_{31} = -5$

From R_2;

$x_{21} = 12$

From R_3;

$x_{11} = -3$

Second System

$$\begin{pmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -7 \end{pmatrix}$$

From R_1;

$x_{32} = 3.5$

From R_2;

$x_{22} = -8.5$

From R_3;

$x_{12} = 2.5$

Third System

$$\begin{pmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From R_1;

$x_{33} = -0.5$

From R_2;

$x_{23} = 1.5$

From R_3;

$x_{13} = -0.5$

Hence,

$$A^{-1} = \begin{pmatrix} -3 & 2.5 & 0.5 \\ 12 & 8.5 & 1.5 \\ -5 & 3.5 & -0.5 \end{pmatrix}$$

Problem 2

Find the inverse of matrix. [T.U. 2068 Bhadra]

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

Solution:

Let, $x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A.

i.e., $AX = I$

Then augmented matrix is;

$$\begin{pmatrix} 2 & 2 & 1 & : & 1 & 0 & 0 \\ 1 & 3 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 2 & : & 0 & 0 & 1 \end{pmatrix}$$

Applying $R_1 \to \frac{R_1}{2}$;

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 1 & 3 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 2 & : & 0 & 0 & 1 \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 2 & \frac{1}{2} & : & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & \frac{3}{2} & : & -\frac{1}{2} & 0 & 1 \end{pmatrix}$$

Applying $R_2 \to \frac{R_2}{2}$

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$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & : & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & : & -\frac{1}{2} & 0 & 1 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & : & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{5}{4} & : & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

Applying $R_3 \rightarrow \frac{R_5}{\frac{5}{4}}$

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & : & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{4} & : & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & : & -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

Hence from the first system; we have,

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & : & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} & : & -\frac{1}{4} \\ 0 & 0 & 1 & : & -\frac{1}{5} \end{pmatrix}$$

$x_{31} = -0.2$

$\therefore \quad x_{21} = -0.2$

$x_{11} = 0.8$

From the second system; we have,

$x_{32} = -0.4$

$x_{22} = 0.9$

$x_{12} = 0.8$

and, from the third system; we have,

$x_{31} = 0.8$

$x_{21} = -0.2$

$x_{11} = -0.2$

Hence, the required inverse matrix is;

$$\begin{pmatrix} 0.8 & 0.8 & -0.2 \\ -0.2 & 0.5 & -0.2 \\ -0.2 & -0.4 & -0.8 \end{pmatrix}$$

Problem 3

Find the inverse of matrix.

$$A = \begin{pmatrix} 2 & -2 & 4 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

[T.U. 2065 Shrawan]

Solution:

Let $x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A.

i.e. $AX = I$

Then augmented matrix is;

$$\begin{pmatrix} 2 & -2 & 4 & : & 1 & 0 & 0 \\ 2 & 3 & 2 & : & 0 & 1 & 0 \\ -1 & 1 & 1 & : & 0 & 0 & 1 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 + \frac{1}{2}R_2$

$$\begin{pmatrix} 2 & -2 & 4 & : & 1 & 0 & 0 \\ 0 & 5 & -2 & : & -1 & 1 & 0 \\ 0 & 0 & 3 & : & 0.5 & 0 & 1 \end{pmatrix}$$

Applying $R_1 \rightarrow \frac{R_1}{2}, R_2 \rightarrow \frac{R_2}{5}, R_3 \rightarrow \frac{R_3}{3}$

$$\begin{pmatrix} 1 & -1 & 2 & : & 0.5 & 0 & 0 \\ 0 & 1 & 0.4 & : & -0.2 & 0.2 & 0 \\ 0 & 0 & 1 & : & 0.167 & 0 & 0.33 \end{pmatrix}$$

We have, three systems which are;

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0.4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.167 \end{pmatrix}$$

Using backward substitution; we have,

$x_{31} = 0.167$

$x_{21} = 0.2332$

$x_{11} = 0.3992$

and, second system

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0.4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0.2 \\ 0 \end{pmatrix}$$

Using backward substitution; we have,

$x_{32} = 0$

$x_{22} = 0.2$

$x_{12} = 0$

and third system,

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0.4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.33 \end{pmatrix}$$

Solving; we get,

$x_{33} = 0.33$
$x_{32} = -0.132$
$x_{31} = -0.528$

Hence, inverse of matrix is;

$$\begin{pmatrix} 0.3992 & -0.2 & -0.528 \\ 0.2332 & 0.2 & -0.132 \\ 0.167 & 0 & 0.33 \end{pmatrix}$$

METHOD OF FACTORIZATION

This method is based on the fact that every square matrix 'A' can be expressed as the product of a lower triangular matrix and upper triangular matrix.

Let us consider the system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_2 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_3 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of equation can be written in the matrix form as;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

or, $AX = B$

where, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Let, $A = LU$ (1)

where, $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$ and $L = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$.

Now, equation (1) becomes;

$LUX = B$ (2)

or, $LZ = B$ (3)

where, $Z = UX$.

Now, we can solve equation in two stages.

i) Solve the equation $LZ = B$ for 'z' by forward substitution.

ii) Solve the equation $UX = Z$ for 'x' using 'z' by backward substitution.

DOOLITTLE AND CHOLESKY FACTORIZATION

The factorization is done by assuming diagonal element of 'L' and 'U' to be unity. The factorization with 'L' having unit diagonal values is called Doolittle factorization while other one with 'U' having uni-diagonal element called Cholesky's factorization.

i) Doolittle Factorization

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix} = A$$

ii) Cholesky's Factorization

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}\begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix} = A$$

Problem 1

Solve the following linear equation by using Doolittle LU method.

$3x_1 + 2x_2 + x_3 = 10$
$2x_1 + 3x_2 + 2x_3 = 14$
$x_1 + 2x_2 + 3x_3 = 14$

[P.U. 2007 Spring]

Solution:

The given system of linear equation can be written in matrix form as;

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}$$

or, $AX = B$

where, $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and, $B = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}$.

Let, $LU = A$

where, $L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$ and, $U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

or, $$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} & l_{21}U_{13} + U_{23} \\ l_{31}U_{11} & l_{31}U_{12} + l_{32}U_{22} & l_{31}U_{13} + l_{32}U_{23} + U_{33} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Equating corresponding elements; we get,

$U_{11} = 3$
$U_{12} = 2$
$U_{13} = 1$
$l_{21}U_{11} = 2$

or, $l_{21} = \frac{2}{3}$

$l_{21}U_{12} + U_{22} = 3$

or, $U_{22} = \frac{5}{3}$

$l_{21}U_{13} + U_{23} = 2$

or, $U_{23} = \frac{4}{3}$

$l_{31}U_{11} = 1$

or, $l_{31} = \frac{1}{3}$

$l_{31}U_{12} + l_{32}U_{22} = 2$

or, $l_{32} = \frac{4}{5}$

$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 3$

or, $U_{33} = \frac{8}{5}$

Hence,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{pmatrix}$$

and, $$U = \begin{pmatrix} 3 & 2 & 1 \\ 2 & \frac{5}{3} & \frac{4}{3} \\ 1 & 2 & \frac{8}{5} \end{pmatrix}$$

We know that;

$$LZ = B$$

i.e., $$\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}$$

Using forward substitution; we have,

$$z_1 = 10$$

$$z_2 = \frac{22}{3}$$

$$z_3 = \frac{24}{5}$$

Again, we have,

$$UX = Z$$

or, $$\begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ \frac{22}{3} \\ \frac{24}{5} \end{pmatrix}$$

Using backward substitution; we have,

From R_3;

$$x_3 = 3$$

From R_2;

$$x_2 = 2$$

From R_1;

$$x_1 = 1$$

Problem 2

Solve the following linear equation by using Doolittle LU method.

$x + y + z = 1$

$3x + 7 - 3z = 5$

$x - 2y - 5z = 10$ [P.U. 2006 Spring]

Solution:

The given system of linear equation can be written in matrix form as;

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}$$

or, $AX = B$

We have,

$$LU = A$$

or, $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{pmatrix}$$

or, $$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} & l_{21} + U_{23} \\ l_{31}U_{11} & l_{31}U_{12} + l_{32}U_{22} & l_{31}U_{13} + l_{32}U_{23} + U_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{pmatrix}$$

Equating; we get,

$U_{11} = 1$

$U_{12} = 2$

$U_{13} = 1$

$l_{21}U_{11} = 3$

or, $l_{21} = 3$

$l_{21}U_{12} + U_{22} = 1$

or, $U_{22} = -2$

$l_{21}U_{13} + U_{23} = -3$

or, $U_{23} = -6$

$l_{31}U_{11} = 1$

or, $l_{31} = 1$

$l_{31}U_{12} + l_{32}U_{22} = -2$

or, $l_{32} = \frac{3}{2}$

$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 0$

or, $U_{33} = 3$

Hence,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{3}{2} & 1 \end{pmatrix}$$

and, $$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{pmatrix}$$

We know,

$$LZ = B$$

or, $$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}$$

Using forward substituting; we have,

$$z_1 = 1$$
$$z_2 = 2$$
$$z_3 = 6$$

Again, we have,

$$UX = Z$$

or, $$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$

Using backward substitution; we have,

$$z = 2$$
$$y = -7$$
$$x = 6$$

ITERATIVE SOLUTION OF LINEAR EQUATION

Gauss Jacobi Method

Consider the system of the linear equations;

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad (1)$$

Then,

$$\left.\begin{aligned} x &= \frac{d_1 - b_1y - c_1z}{a_1} \\ y &= \frac{d_2 - b_2y - c_2z}{a_2} \\ z &= \frac{d_3 - b_3y - c_3z}{a_3} \end{aligned}\right\} \quad (2)$$

Here, a_1, b_2 and c_3 must be larger than other coefficient in each equation.

Suppose the initial approximation be;

$$x = x_0$$
$$y = y_0$$
and, $$z = z_0$$

Substituting these values in equation (2); we have,

$$x_1 = \frac{1}{a_1}[d_1 - b_1y_0 - c_1z_0]$$

$$y_1 = \frac{1}{b_2}[d_2 - a_1x_0 - c_2z_0]$$

$$z_1 = \frac{1}{c_3}[d_3 - a_3x_0 - b_3y_0]$$

Substituting these values of x_1, y_1 and z_1 in equation (2); we get the second approximation x_2, y_2 and z_2. This process is continued till the difference between two consecutive approximations is negligible.

Problem 1

Solve the following system using Gauss Jacobi iteration method.

$$3x + 4y + 15z = 54.8$$
$$x + 12y + 3z = 39.66$$
$$10x + y - 2z = 7.74$$

Solution:

Arranging the given system in larger in order.

$$10x + y - 2z = 7.74$$
$$x + 12y + 3z = 39.66$$
$$3x + 4y + 15z = 54.8$$

Now, the given system can be written as;

$$x = \frac{1}{10}[7.74 + 2z - y]$$

$$y = \frac{1}{12}[39.66 - x - 3z]$$

and, $$z = \frac{1}{15}[54.8 - 3x - 4y]$$

Let initial approximation be;

$$x = 0$$
$$y = 0$$
and, $$z = 0$$

Then, successive iteration is given as;

n	x	y	z
Initial	0	0	0
1	0.7740	3.3050	3.6533
2	1.1742	2.3272	2.6172
3	1.0647	2.5529	2.7979
4	1.0783	2.5168	2.7596
5	1.0748	2.5252	2.7665
6	1.0748	2.5239	2.7651
7	1.0746	2.5242	2.7653
8	1.0746	2.5241	2.7653

The iteration at 7th and 8th are similar up to 3 decimal points.

i.e., $x = 1.074$

$y = 2.524$

and, $z = 2.765$

GAUSS SEIDAL ITERATION METHOD

It is the modified version of Gauss Jordan method. In Gauss seidal, the latest approximation on Gauss Jordan method is used. i.e., the process is repeated till the values of 'x', 'y' and 'z' obtained to desired accuracy.

From Gauss Jordan approximation; we have,

$$x_1 = \frac{1}{a_1}[d_1 - b_1y_0 - c_1z_0]$$

$$y_1 = \frac{1}{b_2}[d_2 - a_2x_0 - c_2z_0]$$

$$z_1 = \frac{1}{c_3}[d_3 - a_3x_0 - b_3y_0]$$

and, Gauss seidal approximation;

$$x_1 = \frac{1}{a_1}[d_1 - b_1y_0 - c_1z_0]$$

$$y_1 = \frac{1}{b_2}[d_2 - a_2x_1 - c_2z_0]$$

and, $z_1 = \frac{1}{c_3}[d_3 - a_3x_1 - b_3y_1]$

Problem 1

Solve the following equation by an iterative method.

$$10x - 2y - w = 3$$
$$-2x + 10y - z - w = 15$$
$$-x - y + 10z - 2w = 27$$
$$-x - y - 2z + 10w = -9$$

Solution:

Here, the given system is diagonally dominant. It can be written as;

$$x = \frac{1}{10}[3 + 2y + z + w]$$

$$y = \frac{1}{10}[15 + 2x + z + w]$$

$$z = \frac{1}{10}[27 + 2w + x + y]$$

and, $w = \frac{1}{10}[x + y + 2z - 9]$

Let initial approximation be $y = z = w = 0$.

Now, tabulating the successive iterations; we have,

n	x	y	z	w
1	0.3	1.56	2.886	−0.1368
2	0.8869	1.9523	2.9566	−0.0248
3	0.9836	1.9899	2.9924	−0.0042
4	0.9968	1.9984	2.9987	−0.0008
5	0.9994	1.9997	2.9998	−0.0001
6	0.9999	2	3	0
7	1	2	3	0
8	1	2	3	0

The iteration at seventh and eighth positions is similar (or exact).

Hence,

$x = 1$

$y = 2$

and, $z = 3$

Problem 2

Solve the following system of linear equation by Gauss-seidal iteration method.

$$9x_1 + 2x_2 - 3x_3 = 10$$
$$5x_1 + 11x_3 - 24x_4 = 30$$
$$2x_1 + 8x_2 - 2x_4 = 15$$
$$x_2 + x_3 + 7x_4 = 25$$

[T.U., 2071 Shrawan]

Solution:

$$x_1 = \frac{1}{9}(10 - 2x_2 + 3x_3) \quad (1)$$

$$x_2 = \frac{1}{8}(15 - 2x_1 + 2x_4) \quad (2)$$

$$x_3 = \frac{1}{11}(30 - 5x_1 + 24x_4) \quad (3)$$

$$x_4 = \frac{1}{7}(25 - x_2 + x_3) \quad (4)$$

Iteration 1

Let $x_2 = x_3 = x_4 = 0$ (initial guess)

$$x_1^{(1)} = 1.111$$

$$x_2^{(1)} = \frac{1}{8}[15 - (2 \times 1.111) + 0] = 1.59725$$

$$x_3^{(1)} = \frac{1}{11}[30 - (5 \times 1.59725) + 0] = 2.00125$$

$$x_4^{(1)} = \frac{1}{7}(25 - 1.59725 - 2.00125) = 3.05735$$

Iteration 2

$$x_1^{(2)} = \frac{1}{9}[10 - (2 \times 1.59725) + (3 \times 2.00125)] = 0.08908$$

$$x_2^{(2)} = \frac{1}{8}[15 - (2 \times 0.08908) + (2 \times 3.05735)]$$

$$= \frac{1}{8}(15 - 0.17816 + 7.147)$$

$$= 1.0038$$

$$x_3^{(2)} = \frac{1}{11}[30 - (5 \times 0.08908) + (24 \times 3.05735)] = -3.9838$$

$$x_4^{(2)} = \frac{1}{7}(25 - 1.0038 + 3.9838) = 4.28394$$

Iteration 3

$$x_1^{(3)} = \frac{1}{9}[10 - (2 \times 1.0038) + 3 \times (-3.9838)] = 2.6622$$

$$x_2^{(3)} = \frac{1}{8}[15 - (2 \times 2.6622) + (2 \times 4.28394)] = 1.4695$$

$$x_3^{(3)} = \frac{1}{11}[30 - (5 \times 2.6622) + (24 \times 4.28394)] = -5.4094$$

$$x_4^{(3)} = \frac{1}{7}(25 - 1.4695 + 5.4094) = 4.1342$$

Solving on similar process; we get,

$x_1 = 2.6622$

$x_2 = 1.4321$

$x_3 = -4.3721$

$x_4 = 4.0134$

EIGEN VALUE AND EIGEN VECTOR USING POWER METHOD

If a 'A' is any square matrix of order n × n with elements a_{ij} we can find a column matrix 'X' and a constant λ.

Such that;

$$AX = \lambda X$$

where, λ is called eigen value and 'X' is called corresponding eigen vector.

We start with the column vector $X^{(0)}$ which is as near the solution as possible and evaluate $AX^{(0)}$ which is written as $\lambda^{(1)}X^{(1)}$ after normalization. This gives the first approximation $\lambda^{(1)}$ to the engine value and $X^{(1)}$ to the eigen vector.

Similarly $X^{(1)}$ to the eigen vector, and we evaluate;

$$AX^{(1)} = A^{(2)}X^{(2)};$$

which gives us second approximation.

Repeat the process till $[X^{(n)} - X^{(n-n)}]$ becomes negligible, then $\lambda^{(n)}$ will be the largest eigen value and $X^{(n)}$ be the corresponding eigen vector.

NOTE

For initial approximation of eigen vector take 1 for the rows not having zero and zero for other. *For example;*

$$\text{If } A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 5 & 3 \\ 8 & 0 & 2 \end{bmatrix}, X^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

In first row, there is no zero element.

∴ First element of $X^{(0)}$ is 1.

In second row, there is zero element.

∴ Second element of $X^{(0)}$ is 0.

Similarly; third element of $X^{(0)}$ is 0.

$$\text{For } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 6 & 9 \end{bmatrix}, X^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

If all row contains '0'

$$X^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and for all non zero element;

$$X^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Take largest value as common and divide the whole element of the vector with largest value.

Problem 1

Using power method, find smallest eigen value of the following matrix.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

[Po.U. 2010 Spring, 2009 fall]

Solution:

Here, initial eigen vector is,

$$X^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

First Iteration

$$AX^{(0)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \lambda^{(1)}X^{(1)}$$

Second Iteration

$$AX^{(1)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -4 \end{pmatrix} = -4\begin{pmatrix} 1 \\ 1.5 \\ 1 \end{pmatrix} = \lambda^{(2)}X^{(2)}$$

Third Iteration

$$AX^{(2)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1.5 \\ -1 \end{pmatrix} = \begin{pmatrix} -3.5 \\ 5 \\ -3.5 \end{pmatrix} = 3.5\begin{pmatrix} -1 \\ 1.43 \\ -1 \end{pmatrix} = \lambda^{(3)}X^{(3)}$$

Fourth Iteration

$$AX^{(3)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1.43 \\ -1 \end{pmatrix} = \begin{pmatrix} -3.43 \\ 4.86 \\ -3.43 \end{pmatrix} = 3.43\begin{pmatrix} -1 \\ 1.42 \\ -1 \end{pmatrix} = \lambda^{(4)}X^{(4)}$$

Fifth Iteration

$$AX^{(4)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1.42 \\ -1 \end{pmatrix} = \begin{pmatrix} -3.42 \\ 4.84 \\ -3.42 \end{pmatrix} = 3.42\begin{pmatrix} -1 \\ 1.42 \\ -1 \end{pmatrix} = \lambda^{(5)}X^{(5)}$$

∴ Smallest eigen value = 3.42

$$\text{and, Eigen vector} = \begin{pmatrix} -1 \\ 1.42 \\ -1 \end{pmatrix}$$

Problem 2

Find the largest eigen value and corresponding vector of the following matrix using power method.

$$A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution:

Here, row 2 and row 3 contains zero element.

$\therefore$ Initial eigen vector $= \begin{pmatrix}1\\0\\0\end{pmatrix}$

First Iteration

$$\begin{pmatrix}1&6&1\\1&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} = \lambda^{(1)}X^{(1)}$$

Second Iteration

$$AX^{(1)} = \begin{pmatrix}1&6&1\\1&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}7\\3\\0\end{pmatrix} = 7\begin{pmatrix}1\\0.429\\0\end{pmatrix} = \lambda^{(2)}X^{(2)}$$

Third Iteration

$$AX^{(2)} = \begin{pmatrix}1&6&1\\1&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1\\0.429\\0\end{pmatrix} = \begin{pmatrix}3.571\\1.858\\0\end{pmatrix} = 3.571\begin{pmatrix}1\\0.520\\0\end{pmatrix} = \lambda^{(3)}X^{(3)}$$

Fourth Iteration

$$AX^{(3)} = \begin{pmatrix}1&6&1\\1&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1\\0.520\\0\end{pmatrix} = \begin{pmatrix}4.120\\2.040\\0\end{pmatrix} = 4.120\begin{pmatrix}1\\0.495\\0\end{pmatrix} = \lambda^{(4)}X^{(4)}$$

Fifth Iteration

$$AX^{(4)} = \begin{pmatrix}1&6&1\\1&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1\\0.495\\0\end{pmatrix} = \begin{pmatrix}3.971\\1.990\\0\end{pmatrix} = 3.971\begin{pmatrix}1\\0.501\\0\end{pmatrix} = \lambda^{(5)}X^{(5)}$$

Sixth Iteration

$$AX^{(5)} = \begin{pmatrix}1&6&1\\1&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1\\0.501\\0\end{pmatrix} = \begin{pmatrix}4.006\\2.002\\0\end{pmatrix} = 4.006\begin{pmatrix}1\\0.500\\0\end{pmatrix} = \lambda^{(6)}X^{(6)}$$

Seventh Iteration

$$AX^{(6)} = \begin{pmatrix}1&6&1\\1&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1\\0.500\\0\end{pmatrix} = \begin{pmatrix}4\\2\\0\end{pmatrix} = 4\begin{pmatrix}1\\0.5\\0\end{pmatrix} = \lambda^{(7)}X^{(7)}$$

$\therefore$ Largest eigen value = 4

and, Eigen vector $= \begin{pmatrix}1\\0.500\\0\end{pmatrix}$

Problem 3

Find the largest eigen value and corresponding vector of the following matrix using power method.

$$A = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}$$ [T.U. 2070 Magh]

Solution:

Here, all row contains non-zero element,

$\therefore$ Initial eigen vector; $X^{(0)} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

First Iteration

$$AX^{(0)} = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}8\\6\\14\end{pmatrix} = 14\begin{pmatrix}0.571\\0.429\\1\end{pmatrix} = \lambda^{(1)}X^{(1)}$$

Second Iteration

$$AX^{(1)} = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}\begin{pmatrix}0.571\\0.429\\1\end{pmatrix} = \begin{pmatrix}4.287\\4.997\\11.858\end{pmatrix} = 11.858\begin{pmatrix}0.362\\0.421\\1\end{pmatrix} = \lambda^{(2)}X^{(2)}$$

Third Iteration

$$AX^{(2)} = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}\begin{pmatrix}0.362\\0.421\\1\end{pmatrix} = \begin{pmatrix}3.829\\3.698\\11.625\end{pmatrix} = 11.625\begin{pmatrix}0.329\\0.341\\1\end{pmatrix} = \lambda^{(3)}X^{(3)}$$

Fourth Iteration

$$AX^{(3)} = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}\begin{pmatrix}0.329\\0.341\\1\end{pmatrix} = \begin{pmatrix}3.363\\3.963\\10.337\end{pmatrix} = 10.337\begin{pmatrix}0.325\\0.383\\1\end{pmatrix} = \lambda^{(4)}X^{(4)}$$

Fifth Iteration

$$AX^{(4)} = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}\begin{pmatrix}0.325\\0.383\\1\end{pmatrix} = \begin{pmatrix}3.565\\3.859\\11.474\end{pmatrix} = 11.474\begin{pmatrix}0.311\\0.336\\1\end{pmatrix} = \lambda^{(5)}X^{(5)}$$

Sixth Iteration

$$AX^{(5)} = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}\begin{pmatrix}0.311\\0.336\\1\end{pmatrix} = \begin{pmatrix}3.302\\3.883\\11.319\end{pmatrix} = 11.319\begin{pmatrix}0.292\\0.343\\1\end{pmatrix} = \lambda^{(6)}X^{(6)}$$

Seventh Iteration

$$AX^{(6)} = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}\begin{pmatrix}0.292\\0.343\\1\end{pmatrix} = \begin{pmatrix}3.299\\3.774\\11.321\end{pmatrix} = 11.321\begin{pmatrix}0.291\\0.333\\1\end{pmatrix} = \lambda^{(7)}X^{(7)}$$

Eighth Iteration

$$AX^{(7)} = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}\begin{pmatrix}0.291\\0.333\\1\end{pmatrix} = \begin{pmatrix}3.247\\3.789\\11.290\end{pmatrix} = 11.290\begin{pmatrix}0.288\\0.336\\1\end{pmatrix} = \lambda^{(8)}X^{(8)}$$

Ninth Iteration

$$AX^{(8)} = \begin{pmatrix}2&5&1\\5&-2&3\\1&3&10\end{pmatrix}\begin{pmatrix}0.288\\0.336\\1\end{pmatrix} = \begin{pmatrix}3.256\\3.768\\11.296\end{pmatrix} = 11.296\begin{pmatrix}0.288\\0.334\\1\end{pmatrix} = \lambda^{(9)}X^{(9)}$$

$\therefore$ Largest eigen value = 11.296

and, Eigen vector $= \begin{pmatrix}0.288\\0.334\\1\end{pmatrix}$

Problem 4

Find the largest eigen value and corresponding vector of the following matrix.

$$A = \begin{pmatrix}1&2&[illegible]\\2&1&[illegible]\\0&0&[illegible]\end{pmatrix}$$ [Po.U. 2009 Spring]

Solution:

Here, all row contains zero element,

$\therefore$ Eigen vector of the matrix; $X^{(0)} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

First Iteration

$$AX^{(0)} = \begin{pmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} = 2\begin{pmatrix}1\\0.5\\0\end{pmatrix} = \lambda^{(1)}X^{(1)}$$

Second Iteration

$$AX^{(1)} = \begin{pmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\0.5\\0\end{pmatrix} = \begin{pmatrix}2\\2.5\\0\end{pmatrix} = 2.5\begin{pmatrix}0.8\\1\\0\end{pmatrix} = \lambda^{(2)}X^{(2)}$$

Third Iteration

$$AX^{(2)} = \begin{pmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}0.8\\1\\0\end{pmatrix} = \begin{pmatrix}2.8\\2.6\\0\end{pmatrix} = 2.8\begin{pmatrix}1\\0.929\\0\end{pmatrix} = \lambda^{(3)}X^{(3)}$$

Fourth Iteration

$$AX^{(3)} = \begin{pmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\0.929\\0\end{pmatrix} = \begin{pmatrix}2.857\\2.929\\0\end{pmatrix} = 2.929\begin{pmatrix}0.975\\1\\0\end{pmatrix} = \lambda^{(4)}X^{(4)}$$

Fifth Iteration

$$AX^{(4)} = \begin{pmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}0.975\\1\\0\end{pmatrix} = \begin{pmatrix}2.975\\2.95\\0\end{pmatrix} = 2.975\begin{pmatrix}1\\0.992\\0\end{pmatrix} = \lambda^{(5)}X^{(5)}$$

Sixth Iteration

$$AX^{(5)} = \begin{pmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\0.992\\0\end{pmatrix} = \begin{pmatrix}2.983\\2.992\\0\end{pmatrix} = 2.992\begin{pmatrix}0.997\\1\\0\end{pmatrix} = \lambda^{(6)}X^{(6)}$$

Seventh Iteration

$$AX^{(6)} = \begin{pmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}0.997\\1\\0\end{pmatrix} = \begin{pmatrix}2.997\\2.994\\0\end{pmatrix} = 2.997\begin{pmatrix}1\\0.999\\0\end{pmatrix} = \lambda^{(7)}X^{(7)}$$

$\therefore$ Largest eigen value = 2.997

and, Eigen vector $= \begin{pmatrix}1\\0.999\\0\end{pmatrix} \approx \begin{pmatrix}1\\1\\0\end{pmatrix}$

Problem 5

Find the largest eigen value and corresponding eigen vector of the following matrix using power method. [T.U. 2068 Baishakh]

$$A = \begin{pmatrix}1 & 2 & 3\\2 & -1 & 4\\3 & 4 & 5\end{pmatrix}$$

Solution: Refer the solution of Q. no. 3

Problem 6

Find the largest eigen value and corresponding eigen vector of the following matrix using power method. [T.U. 2066 Baishakh]

$$A = \begin{pmatrix}1 & 2 & 3\\-2 & 4 & 6\\-1 & -2 & 3\end{pmatrix}$$

Solution: Refer the solution of Q. no. 3

Problem 7

Find the largest eigen value and corresponding eigen vector of the following matrix using power method. [T.U. 2067 Ashadh, Po.U. 2011 Spring]

$$A = \begin{pmatrix}2 & -2 & 4\\2 & 3 & 2\\-1 & 1 & 1\end{pmatrix}$$

Solution: Refer the solution of Q. no. 3

Problem 8

Find the largest eigen value and corresponding eigen vector of the following matrix using power method. [T.U. 2062 Baishakh]

$$A = \begin{pmatrix}2 & 2 & 1\\1 & 2 & 0\\0 & 3 & 1\end{pmatrix}$$

Solution: Refer the solution of Q. no. 2

Problem 9

Find the largest eigen value and corresponding eigen vector of the following matrix using power method. [Po.U. 2008 Fall]

$$A = \begin{pmatrix}-1 & 0 & 0\\1 & -2 & 3\\0 & 2 & -3\end{pmatrix}$$

Solution: Refer the solution of Q. no. 1

Problem 10

Find the largest eigen value and corresponding eigen vector of the following matrix using power method. [Po.U. 2005 Fall]

$$A = \begin{pmatrix}2 & 3 & 0\\0 & 2 & -1\\-1 & 0 & -1\end{pmatrix}$$

Solution: Refer the solution of Q. no. 4

Problem 11

Find the largest eigen value and corresponding eigen vector of the following matrix using power method. [Po.U. 2006 Fall]

$$A = \begin{pmatrix}3 & -1 & 0\\-2 & 4 & -3\\0 & -1 & 1\end{pmatrix}$$

Solution: Refer the solution of Q. no. 1

Problem 12

Find the largest Eigen value and corresponding vector of following matrix using power method. [T.U., 2072 Kartik]

$$\begin{bmatrix}1.4 & 1.3 & 2.2\\1.3 & 3.5 & 1.5\\2.2 & 1.5 & 3.2\end{bmatrix}$$

Solution:

Let, Initial vector = $\begin{bmatrix}1\\1\\0\end{bmatrix}$

A	X	(A × X)	Largest value	X_{i+1}
$\begin{bmatrix}1.4 & 1.3 & 2.2\\1.3 & 3.5 & 1.5\\2.2 & 1.5 & 3.2\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}2.7\\4.8\\3.7\end{bmatrix}$	4.8	$\begin{bmatrix}0.5625\\1\\0.7708\end{bmatrix}$
$\begin{bmatrix}1.4 & 1.3 & 2.2\\1.3 & 3.5 & 1.5\\2.2 & 1.5 & 3.2\end{bmatrix}$	$\begin{bmatrix}0.5625\\1\\0.7708\end{bmatrix}$	$\begin{bmatrix}3.7833\\5.3875\\5.2041\end{bmatrix}$	5.3875	$\begin{bmatrix}0.7022\\1\\0.96[illegible]\end{bmatrix}$
$\begin{bmatrix}1.4 & 1.3 & 2.2\\1.3 & 3.5 & 1.5\\2.2 & 1.5 & 3.2\end{bmatrix}$	$\begin{bmatrix}0.7022\\1\\0.9659\end{bmatrix}$	$\begin{bmatrix}4.4082\\5.8618\\6.136\end{bmatrix}$	6.136	$\begin{bmatrix}0.71[illegible]\\0.95[illegible]\\1\end{bmatrix}$
$\begin{bmatrix}1.4 & 1.3 & 2.2\\1.3 & 3.5 & 1.5\\2.2 & 1.5 & 3.2\end{bmatrix}$	$\begin{bmatrix}0.7184\\0.9553\\1\end{bmatrix}$	$\begin{bmatrix}4.4477\\5.7776\\6.2135\end{bmatrix}$	6.2135	$\begin{bmatrix}0.71[illegible]\\0.929[illegible]\\1\end{bmatrix}$
$\begin{bmatrix}1.4 & 1.3 & 2.2\\1.3 & 3.5 & 1.5\\2.2 & 1.5 & 3.2\end{bmatrix}$	$\begin{bmatrix}0.7158\\0.9298\\1\end{bmatrix}$	$\begin{bmatrix}4.41096\\5.685\\6.1695\end{bmatrix}$	6.1695	$\begin{bmatrix}0.71496\\0.09214\\1\end{bmatrix}$
$\begin{bmatrix}1.4 & 1.3 & 2.2\\1.3 & 3.5 & 1.5\\2.2 & 1.5 & 3.2\end{bmatrix}$	$\begin{bmatrix}0.71496\\0.09214\\1\end{bmatrix}$	$\begin{bmatrix}4.3988\\5.6546\\6.155\end{bmatrix}$	6.150	$\begin{bmatrix}0.71468\\0.918[illegible]\\1\end{bmatrix}$
$\begin{bmatrix}1.4 & 1.3 & 2.2\\1.3 & 3.5 & 1.5\\2.2 & 1.5 & 3.2\end{bmatrix}$	$\begin{bmatrix}0.71468\\0.9187\\1\end{bmatrix}$	$\begin{bmatrix}4.3949\\5.6446\\6.1504\end{bmatrix}$	6.150	$\begin{bmatrix}0.7146\\0.9178\\1\end{bmatrix}$

∴ Largest Eigen value = 6.150

Corresponding vector = $\begin{bmatrix}0.7146\\0.9178\\1\end{bmatrix}$

Problem 1 (Chapter 4)

Construct divided difference table from the following data: [2073 Shrawan]

x	1	2	4	5	6
y	14	15	5	6	19

Solution:

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
1	14				
		1			
2	15		−2		
		−5		1	
4	5		2		0
		1		1	
5	6		6		
		13			
6	19				

Problem 2 (Chapter 2)

Find an approximation of the root of the equation $x^3 - x - 11 = 0$ by using bisection method correct to three decimal places. [2073 Shrawan]

Solution:

Here,

$f(x) = x^3 - x - 11$

Let, initial guess be 2 and 2.5.

$f(2) = -5 < 0$

$f(2.5) = 2.125 > 0$

Therefore, the root lies in between 2 and 2.5.

Now, let us calculate root by the tabular form.

n	a	b	x_n	$f(x_n)$
1	1	2.5	2.25	− 1.859
2	2.25	2.5	2.375	0.02148
3	2.25	2.375	2.3125	− 0.94604
4	2.3125	2.375	2.34375	− 0.46915
5	2.34375	2.375	2.35938	− 0.22556
6	2.35938	2.375	2.36719	− 0.10243
7	2.36719	2.375	2.37110	− 0.04058
8	2.37110	2.375	2.37305	− 0.00954
9	2.37305	2.375	2.37403	0.00597
10	2.37305	2.37403	2.37354	− 0.00175

x = 2.374

Problem 3 (Chapter 2)

Write an algorithm for finding a real root of non-linear equation using Newton Raphson method. [2073 Shrawan]

Solution:

1. Start
2. Read x, e, n, d
   * x is the initial guess
   e is the absolute error
   n is the operating loop
   d is for checking slope *
3. Do for i = 1 to n in step of 2
4. f = f(x)
5. f1 = f'(x)
6. If ([f1] < d), then display too small slope and go to 11
   [ ] is called modulus sign
7. $x1 = x - \frac{f}{f1}$
8. If $\left(\left[\frac{x1 - x}{x1}\right] < e\right)$, the display the root as x1 and go to 11
9. x = x1 and end loop
10. Display method doe not coverage due to oscillation.
11. Stop

Problem 4 (Chapter 7)

Solve the following system of linear equations using Gauss-Seidal iteration method. [2073 Shrawan]

$6x_1 + x_2 - x_3 + 2x_4 = 4$
$2x_1 + 5x_2 - 4x_3 + 6x_4 = -5$
$x_1 + 4x_2 + 3x_3 - x_4 = 2$
$x_1 + x_2 + 2x_3 + x_4 = 5$

Solution:

Arranging the given equations in the following order;

$$x_1 = -\frac{x_2 + x_3 - 2x_4 + 4}{6} \quad (1)$$

$$x_2 = -\frac{2x_1 + 4x_3 - 6x_4 - 5}{5} \quad (2)$$

$$x_3 = -\frac{x_1 - 4x_2 + x_4 + 2}{3} \quad (3)$$

$$x_4 = -\frac{x_1 - x_2 + 2x_4 + 5}{1} \quad (4)$$

However, the given equation is not diagonal dominant.

i.e., $|5| \ngtr |-5| + |4| + |-6|$
$|3| \ngtr |-1| + |-4| + |1|$
$|1| \ngtr |-1| + |-1| + |-2|$
$\because 6 > |1| + 1 + |-2|$

Hence, Gauss Seidal method does not yield an answer.

Problem 5 (Chapter 7)

Find the largest Eigen value and corresponding Eigen vector of the following matrix using power method. [2073 Shrawan]

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution:

Here,

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Let, Initial Eigen vector $X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

First iteration

$$AX^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda^{(0)}X^{(1)}$$

Second iteration

$$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

Third iteration

$$AX^{(2)} = 4\begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \lambda^{(3)}X^{(3)}$$

Fourth iteration

$$AX^{(3)} = 3.5\begin{bmatrix} 0.7142 \\ -1 \\ 0.7142 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

Fifth iteration

$$AX^{(4)} = 3.8257\begin{bmatrix} 0.7084 \\ -1 \\ 0.7084 \end{bmatrix} = \lambda^{(5)}X^{(5)}$$

Sixth iteration

$$AX^{(5)} = 3.417\begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix} = \lambda^{(6)}X^{(6)}$$

Seventh iteration

$$AX^{(6)} = 3.414\begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix}$$

$\therefore$ Largest Eigen value = 3.414

and, Eigen vector = $\begin{bmatrix} 0.7073 \\ -1.000 \\ 0.7073 \end{bmatrix}$

Problem 6 (Chapter 3)

Evaluate y(10) by using Lagrange's interpolation formula from the follow data: [2073 Shraw

x	5	6	9	11
y	12	13	14	16

Solution:

Here,

$x_0 = 5 \quad y_0 = 12$

$x_1 = 6 \quad y_1 = 13$

$x_2 = 9 \quad y_2 = 14$

$x_3 = 11 \quad y_3 = 16$

From Lagrange's formula; we have,

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$= 14.667$

Problem 7 (Chapter 3)

Using least square method, fit a curve $y = ae^{bx}$ to the following data: [2073 Shrawan]

x	4	5.5	7	8	10
y	18.47	39.11	82.79	136.5	371.03

Solution:

The given curve is;

$y = ae^{bx}$

Taking $\log_e$ on the both sides; we have,

$\log_e y = \log_e a + bx \log_e x$

Equation (2) is the form of the equation given below.

$Y = A + BX$

$Y = \log_e y$

$A = \log_e a$

$B = b \log_e e = b$

Now, the normal equation (3) is;

$\sum Y = nA + B\sum X$

$\sum XY = A\sum X + B\sum X^2$

x	y	Y = ln y	X^2	XY
4	18.47	2.91615	16	11.6646
5.5	39.11	3.66638	30.25	20.16509
7	82.79	4.41631	49	30.91417
8	136.5	4.91632	64	39.33056
10	371.03	5.91628	100	59.16280
$\sum X = 34.5$		$\sum Y = 21.83141$	$\sum X^2 = 259.25$	$\sum XY = 161.23722$

Putting these values in the equation (4) and (5); we get,

$21.83141 = 5A + 34.5B$

or, $161.23722 = 34.5A + 259.25B$

Using calculator; we get,

$A = 0.91620$

$B = 0.50001$

Then,

$a = e^{0.91620} = 2.5$

$b = 0.5$

$\boxed{y = 2.5e^{0.5x}}$

Problem 8 (Chapter 4)

Find the value of cos(1.74) from the following table. [2073 Shrawan]

x	1.7	1.74	1.78	1.82	1.86
sin x	0.9916	0.9857	0.9781	0.9691	0.9584

Solution:

Let, $y = f(x) = \sin x$;

so that;

$f'(x) = \cos x$

The difference table is given by;

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.7	0.9916				
		−0.0059			
1.74	0.9857		−0.0017		
		−0.0076		0.0003	
1.78	0.9781				−0.0006
		−0.0090		−0.0003	
182	0.9691		−0.0017		
1.84	0.9584	−0.0017			

Since, we require f′(1.74); we use Newton's forward formula;

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right] \quad (1)$$

Here, $h = 0.04$, $x_0 = 1.7$, $\Delta y_0 = -0.0059$ and $\Delta^2 y_0 = -0.0017$, etc.

$$\left(\frac{dy}{dx}\right)_{1.74} = \frac{1}{0.04}\left[0.0059 - \frac{1}{2}(-0.0017) + \frac{1}{3}(0.003) - \frac{1}{4}(-0.0006)\right]$$

$$= \frac{1}{0.04} \times (0.007) = 0.175$$

$\therefore \quad \cos(1.74) = 0.175$

Problem 9 (Chapter 4)

Derive composite Simpson's three-eight formula for the integration. [2073 Shrawan]

Solution:

See the definition part on page no. 97

Problem 10 (Chapter 5)

Write pseudo-code to solve a first order differential equation using R-K 4 method. [2073 Shrawan]

Solution:

Pseudo-code for the first order differential equation using RK 4 method

1. Define function F(x, y)
2. Define variable y0, x0, y1, n, h, f, f1, k1, k2, k3, k4;
3. Input the value of x0, y0, h
4. Input the value of last point 'n'
5. For x0 < n

```
        Repeat
            k1 = h * F(x0, y0)
            k2 = h * F(x0 + h/2, y0 + k1/2)
            k3 = h * F(x0 + h/2, y0 + k2/2)
            k4 = h * F(x0 + h, y0 + k3)
            x0 = x0 + h
            y1 = y0 + (k1 + 2k2 + 2k3 + k4/6)
            Print y1
            y0 = y1
```

6. Stop

Problem 11 (Chapter 5)

Solve the boundary value problem $y'' + xy' + y = 3x^2 + 2$, $y(0) = 0$, $y(1) = 1$ [2073 Shrawan]

Solution:

Let, $\dfrac{dy}{dx} = z$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = 3x^2 - xy' - y + 2$$

Now, boundary condition;

$y(0) = 0 \quad y(1) = 1$

or, $x_0 = 0 \quad x_1 = 1$

$y_0 = 0 \quad y_1 = 1$

Assuming $y'(0) = -0.5$

$Y_{i+1} = Y_i + f_1(x_i, y_i, z_i)$

$Z_{i+1} = Z_i + f_2(x_i, y_i, z_i)$

Taking $h = 0.5$;

For $i = 0$;

$Y_1 = 0 + (-0.5) \times 0.5 = -0.25$

$Z_1 = -0.5 + [3 \times (0)^2 - 0 \times 0 - 0 + 2] \times 0.5 = 0.5$

For $i = 1$;

$Y_2 = -0.25 + 0.5 \times 0.5 = 0$; which is not equal to $y(1) = 1$.

So, taking $y'(0) = 0$ and $h = 0.5$

For $i = 0$;

$Y_1 = 0 + 0 \times 0.5 = 0$

$Z_1 = 0 + [3 \times (0)^2 - 0 \times 0 - 0 + 2] \times 0.5 = 1$

For $i = 1$;

$Y_2 = 0 + 1 \times 0.5 = 0.5$

Taking $y'(0) = 1$

For $i = 0$;

$Y_1 = 0 + 1 \times 0.5 = 0.5$

$Z_1 = 1 + [3 \times (0)^2 - 0 \times 0 - 0 + 2] \times 0.5 = 2$

For $i = 1$;

$Y_2 = 0.5 + 2 \times 0.5 = 1.5$

Taking $y'(0) = 0.5$

For $i = 0$;

$Y_1 = 0 + 0.5 \times 0.5 = 0.25$

$Z_1 = 0.5 + [3 \times (0)^2 - 0 \times 0 - 0 + 2] \times 0.5 = 1.5$

For $i = 1$;

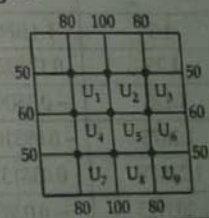
$Y_2 = 0.25 + 1.5 \times 0.5 = 1$

Here, $Y_2 = 1$ which is equal to $y(1) = 1$.

Hence, the required result is for $y'(0) = 0.5$.

Problem 12 (Chapter 6)

Solve the Laplace equation $U_{xx} + U_{yy} = 0$ over the square grid with boundary condition as shown in the figure. [2073 Shrawan]



Solution:

From the figure, we can see that it is symmetrical.

so, $U_1 = U_3 = U_7 = U_9$

$U_2 = U_8$

$U_4 = U_6$

Then, we have the following equations;

$$U_1 = \frac{50 + 80 + U_2 + U_4}{4} \quad (1)$$

$$U_2 = \frac{100 + 2U_1 + U_5}{4} \quad (2)$$

$$U_4 = \frac{60 + 2U_1 + U_5}{4} \quad (3)$$

$$U_5 = \frac{2U_2 + 2U_4}{4} \quad (4)$$

Using Gauss Seidal method in the equations (1), (2), (3) and (4); we get,

$U_1 = U_3 = U_7 = U_9 = 68.75$

$U_2 = U_8 = 77.5$

$U_4 = U_6 = 67.5$

$U_5 = 72.5$

Problem 13 (Chapter 1)

What are the applications of numerical methods in the field of science and engineering? Discuss briefly. [2073 Chaitra]

Solution:

See the definition part on page no. 1

Problem 14 (Chapter 2)

Find a real root of $e^x - \cos x = 3$ correct to three places of decimal using the bisection method. [2073 Chaitra]

Solution:

Given that;

$$f(x) = e^x - \cos x - 3 = 0$$

Let, the initial guess be 1 and 2.

$f(1) = -0.822$

$f(2) = 4.805$

Therefore, the root lies in between 1 and 2.

Now, let us calculate root by the tabular form.

n	a	b	$x_n = \frac{a+b}{2}$	$f(x_n)$
1	1	2	1.5	1.410951
2	1	1.5	1.25	0.1750206
3	1	1.25	1.125	− 0.350959
4	1.125	1.25	1.1875	− 0.0951058
5	1.1875	1.25	1.21875	0.038137
6	1.1875	1.21875	1.203125	− 0.0289
7	1.203125	1.21875	1.2109375	0.00449
8	1.203125	1.2109375	1.20703	− 0.01225
9	1.20703	1.2109375	1.20898	− 0.00389
10	1.20898	1.2109375	1.20996	0.00029
11	1.20898	1.20996	1.20947	− 0.00181
12	1.20947	1.20996	1.20972	− 0.00076
13	1.20972	1.20996	1.20984	− 0.0022

∴ $x = 1.210$

Problem 15 (Chapter 2)

What are the drawbacks of Newton-Raphson method? Discuss. [2073 Chaitra]

Solution:

The drawbacks of Newton-Raphson method may include the followings:

i) Calculating the required derivative for every iteration may be a costly tasks for some functions.

ii) Newton-Raphson method may not produce a root unless the starting value is close to the actual root of the function.

iii) It may not produce a root if for instance, the iterations get to a point x such that $f(x) = 0$. Then, the method fails.

iv) If the derivative of the function changes signs near a tested point, the Newton-Raphson method may oscillate around a point, now here near the nearest root.

v) The Newton-Raphson method is not guaranteed to find a root. For example; if the starting point x_1 is sufficiently far away from the root for the function $f(x) = \tan^{-1} x$, the function's small slope tends to derive the x guesses further and further away from the root.

vi) If the derivative of the function at any tested point x_i is sufficiently close to zero, the next point x_{i+1} will be very far away. The root may still be found, but the process is delayed.

Problem 16 (Chapter 7)

Solve the following system of linear equations using LU factorization method.

$x_1 + 2x_2 - x_3 = 2$

$x_1 - 3x_2 + 2x_3 = 1$ [2073 Chaitra]

$2x_1 + 4x_2 + 3x_3 = 19$

Solution:

The given system of linear equation can be written in the matrix form as;

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -3 & 2 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 19 \end{bmatrix}$$

or, $AX = B$

where,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ 1 \\ 19 \end{bmatrix}$$

Let, LU = A;

$$\text{where, } L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

$$\text{and, } U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + U_{32}U_{22} & L_{31}U_{13} + U_{32}U_{23} + U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

Equating corresponding elements; we get,

$U_{11} = 1$ $\quad U_{12} = 2$ $\quad U_{13} = -1$

$L_{21} = 1$ $\quad U_{22} = -5$ $\quad U_{23} = 3$

$L_{31} = 2$ $\quad L_{32} = 0$ $\quad U_{33} = 5$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

We know that;

$$LZ = B$$

$$\text{i.e., } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 19 \end{bmatrix}$$

Using forward substitution; we have,

$z_1 = 2$

$z_2 = -1$

$z_3 = 15$

Again; we have,

$$UX = Z$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 15 \end{bmatrix}$$

Using backward substitution; we get,

$x_3 = 3$

$x_2 = 2$

$x_1 = 1$

Problem 17 (Chapter 7)

Apply power method to find the largest Eigen value of the following matrix.

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

[2073 Chaitra]

Solution:

$$\text{Let, } A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

Here, all the row contains non-zero element.

$$\therefore \quad \text{Initial Eigen vector; } X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

First iteration

$$AX^{(0)} = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

Second iteration

$$AX^{(1)} = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = 4.5\begin{bmatrix} 1 \\ -0.44 \\ 0.55 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

Third iteration

$$AX^{(2)} = 5\begin{bmatrix} 1 \\ -0.688 \\ 0.7111 \end{bmatrix} = \lambda^{(3)}X^{(3)}$$

Fourth iteration

$$AX^{(3)} = 5.4\begin{bmatrix} 1 \\ -0.831 \\ 0.8353 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

Fifth iteration

$$AX^{(4)} = 5.67\begin{bmatrix} 1 \\ -0.911 \\ 0.9121 \end{bmatrix} = \lambda^{(5)}X^{(5)}$$

Sixth iteration

$$AX^{(5)} = 5.824\begin{bmatrix}1\\-0.954\\0.955\end{bmatrix} = \lambda^{(6)}X^{(6)}$$

Seventh iteration

$$AX^{(6)} = 5.908\begin{bmatrix}1\\-0.976\\0.977\end{bmatrix}$$

Here, stopping the iteration; we get,

Largest Eigen value = 5.91

and, Eigen vector = $\begin{bmatrix}1\\-0.98\\0.98\end{bmatrix}$

Problem 18 (Chapter 3)

Write the pseudo-code to fit a polynomial to a given set of data by Lagrange's interpolation method. [2073 Chaitra]

Solution:

1. Define variables n, i, j, x[max], f, f_p, f[max], sum, x_p;
2. Input number of data n
3. Print 'Input data points xi, fi and xp'
4. For i = 1; i <= n
   read x[i] and f[i]
   i = i + 1
5. Sum = 0
6. For i = 1; i <= n
   f = 1;
   a) for j = 1; j <= n
      if (i! = j)
         f! = f × (xp − x[j])/x[i] − x[j]
   b) sum = sum + f × f[i];
7. f = sum;
8. Print Lagrangian interpolation at 'xp' is 'sum'.

Problem 19 (Chapter 3)

Estimate y(3) from the following data using cubic spline interpolation technique. [2073 Chaitra]

x	2	4	6	8	10
y	2	3	6	3	2

Solution:

When i = 1;

$$h_1a_0 + 2a_1(h_1 + h_2) + h_2a_2 = 6\left(\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right)$$

$h_1 = x_1 - x_0 = 4 - 2 = 2$

$h_2 = x_2 - x_1 = 6 - 4 = 2$

$$\begin{bmatrix}2(h_1 + h_2) & 1\\ 1 & 2(h_2 + h_3)\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}d_1\\d_2\end{bmatrix} \quad (1)$$

$2(h_1 + h_2) = 2(2 + 2) = 8$

$2(h_2 + h_3) = 2(2 + 2) = 8$

$d_1 = \frac{6}{h^2}[f_2 - 2f_1 + f_0] = \frac{6}{4}[6 - 2\times3 + 2] = 3$

$d_2 = \frac{6}{h^2}[f_3 - 2f_2 + f_1] = \frac{6}{4}[3 - 2\times6 + 3] = -9$

Now, solving equation (1); we have,

$a_1 = 0.524$

and, $a_2 = -1.190$

The target point is x = 3;

$$S_2(x) = \frac{a_1}{6}[U_2 - U_2^3] + \frac{a_2}{6}[U_1^3 - U_1] + (f_2U_1 - f_1U_2)$$

$$S_2(x) = \frac{a_1}{6}[(x - x_2) - (x - x_2)^3] + \frac{a_2}{6}[(x - x_1)^3 - (x - x_1)] + [f_2(x - x_1) - f_1(x - x_1)]$$

$$S_2(3) = \frac{0.524}{6}[(3 - 6) - (3 - 6)^3] + \left(\frac{-1.190}{6}\right)[(3 - 4)^3 - (3 - 4)] + [6(3 - 4)^3 - (3 - 6)]$$

$= 5.096$

Problem 20 (Chapter 4)

Derive Newton-Cotes quadrature formula for integration. [2073 Chaitra]

Solution:

Let, $I = \int_a^b f(x)\,dx$

where, f(x) takes values $y_0, y_1, y_2, \ldots\ldots, y_n$ for $x = x_0, x_1, x_2, \ldots\ldots, x_n$.

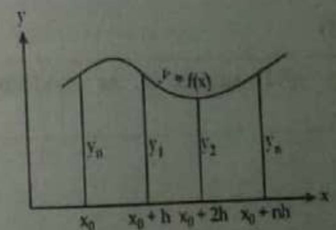
Let us divide the interval (a, b) into n-sub intervals of width h; so that,

$x_0 = a,\ x_1 = x_0, h_1,\ x_2 = x_0 + 2h_1, \ldots\ldots, x_0 + 2h = b$

$I = \int_{x_0}^{x_0+2h} f(x)\,dx = \int_0^n f(x_0 + rh)\,dr$ [Put $x = x_0 + rh$, $dx = h\,dr\phi$]

or, $I = h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\ldots\right] dz$

On integrating term by term; we get,



$$\int_{x_0}^{x_0+2h} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2(r-2)}{24}\Delta^3 y_0 + \dots\right]$$

This is called Newton's-Cotes formula.

Problem 21 (Chapter 4)

Evaluate $\int_{-1}^{2} e^{-x^2}$ using 3-point Gaussian quadrature formula. [2073 Chaitra]

Solution:

Here,

$$I = \int_{-1}^{2} e^{-x^2}$$

Changing limit (−1, 2) to (−1, 1); we have,

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}[2-(-1)]u + \frac{1}{2}(2-1) = \frac{3}{2}u + \frac{1}{2}$$

or, $dx = \frac{3}{2}du$

Now,

$$I = \int_{-1}^{1} e^{-\frac{(3u+1)^2}{4}}\frac{3}{2}du = \frac{3}{2}\int_{-1}^{1} e^{-\frac{(3u+1)^2}{4}}du$$

$$f(u) = e^{-\frac{(3u+1)^2}{4}}$$

$$f(0) = 0.77880$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = e^{-\frac{\left(3\sqrt{\frac{3}{5}}+1\right)^2}{4}} = 0.64526$$

$$f\left(\sqrt{\frac{3}{5}}\right) = e^{-\frac{\left(3\sqrt{\frac{3}{5}}+1\right)^2}{4}} = 0.06317$$

By Gauss 3-point formula; we have,

$$\int_{-1}^{1} f(x)\,du = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\times 0.7788 + \frac{5}{9}[0.64526 + 0.06317] = 1.08584$$

so, $I = \frac{3}{2}\times 1.08584 = 1.62876$

Problem 22 (Chapter 5)

Solve $y' = 2x + \sin y$ for $y(0.2)$ subject to the condition $y(0) = 1$ using modified Euler's method. [2073 Chaitra]

Solution:

Here,

$$y' = 2x + \sin y$$

$$y(0) = 1$$

i.e., at $x_0 = 0$;

$$y_0 = 1$$

$$h = 0.2$$

$$x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$y_1 = ?$$

From modified Euler's method; we have,

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where, $k_1 = hf(x_0, y_0)$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

or, $k_1 = 0.2 \times 0.84147 = 0.168294$

and, $k_2 = 0.26401$

Now,

$$y_1 = 0 + \frac{1}{2}(0.168294 + 0.26401) = 0.21616$$

Problem 23 (Chapter 5)

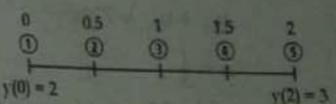
Solve the following boundary value problem using finite difference method by dividing the interval into four sub intervals. [2073 Chaitra]

$y'' + 3y' - y = \cos x$, $y(0) = 2$ and $y(2) = 3$

Solution:

Here,

$$h = \frac{2-0}{4} = 0.5$$



$$y'' \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$y' \approx \frac{y_{i+1} - y_i}{h}$$

so, $$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{3(y_{i+1} - y_i)}{h} - y_i = \cos x_i$$

For x = 0, y(0) = 2

At node i = 1;

$$y_1 = y(0) = 0$$

At node 2;

$$\frac{y_3 - 2y_2 + y_1}{(0.5)^2} + \frac{3(y_3 - y_1)}{0.5} - y_2 = \cos x_2$$

or, $$\frac{y_3 - 2y_2 + 0}{(0.5)^2} + \frac{3(y_3 - 0)}{0.5} - y_2 = 0.87758$$

or, $4y_3 - 8y_2 + 6y_3 - y_2 = 0.87758$

or, $-9y_2 + 10y_3 = 0.87758$ (1)

At node 3;

$$\frac{y_3 - 2y_2 + y_2}{(0.5)^2} + \frac{3(y_4 - y_2)}{0.5} - y_3 = \cos x_3$$

or, $4y_4 - 8y_3 + 4y_2 + 6y_4 - 6y_2 - y_3 = 0.5403$

or, $-2y_2 - 9y_3 + 10y_4 = 0.5403$

At node 4;

$$\frac{y_5 - 2y_4 + y_3}{(0.5)^2} + \frac{3(y_5 - y_3)}{0.5} - y_4 = \cos x_4$$

or, $-2y_3 - 9y_4 + 10y_5 = 0.07074$

At node 5;

$y_5 = 3$

On solving; we get,

$y_2 = 2.62228$

$y_3 = 2.44781$

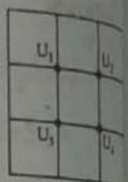
and, $y_4 = 2.78152$

Problem 24 (Chapter 6)

Solve Poisson's equation $u_{xx} + u_{yy} = 729x^2y^2$ on a square grid with $0 \le x \le 1$, $0 \le y \le 1$ with $u = 0$ on boundary. Take step size $n = \frac{1}{3}$ [2073 Chaitra]

Solution:

Here,



$$u_{xx} + u_{yy} = 729x^2y^2$$

$$n = \frac{1}{3}$$

The standard 5-point formula for the given equation is;

$$U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} = h^2(ih, jh) = h^2[729(ih \cdot jh)]$$

For $U_1(i = 1, j = 2)$

$0 + U_2 + U_3 + 0 + 4U_1 = 18$

For $U_2(i = 2, j = 2)$

$U_1 + 0 + 0 + U_4 - 4U_2 = 36$

For $U_3(i = 1, j = 1)$

$0 + U_4 + U_1 + 0 - 4U_3 = 9$

For $U_4(i = 2, j = 1)$

$U_3 + 0 + U_2 + 0 - 4U_4 = 18$

Solving the equation (1), (2), (3) and (4) by Gauss Siedal method; we get,

$U_1 = -9.75$

$U_2 = -13.875$

$U_3 = -7.125$

$U_4 = -9.75$

Problem 25 (Chapter 1)

Discuss the significance of numerical methods in the field of science and engineering. [2074 Ashwin]

Solution:

The significance of numerical methods is the field of science and engineering includes:

i) complex mathematical problems can be solved by numerical method easily.
ii) saves time for complex calculations.
iii) numerical method involves a large number of arithmetic calculations, which includes the iterative methods. Hence, use of such algorithm in science and engineering provides greater accuracy.
iv) In many cases, analytical method fails to give an answer, in which case numerical methods works fine.

Problem 26 (Chapter 2)

Find a real root of the equation $\cos x - xe^x = 0$, correct to four decimal places, using regula-falsi method. [2074 Ashwin]

Solution:

Here,

$f(x) = \cos x - xe^x$

Let, the initial guess be 0.5 and 1.

Since, $f(0.5) = 0.053 > 0$ and $f(1) = -2.177 < 0$; so, root lies between 0.5 and 1.

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	0.5	2	0.5322	− 2.17798	0.50524	0.03769
2	0.5	0.50524	0.5322	0.03768	0.51795	− 0.00057
3	0.50524	0.51795	0.03768	− 0.0059	0.51776	0.00001
4	0.51795	0.51776	− 0.0059	− 0.00001	0.51776	END

So, the value of x up to four decimal places.

$x = 0.5178$

Problem 27 (Chapter 2)

Write pseudo-code for finding a real root of a non-linear equation using the secant method. [2074 Ashwin]

Solution:

Pseudo-code

i) Function fun (x)
ii) Input: Acceptable error E
iii) Input: Two initial guess interval x and y
iv) Calculate:

$a = fun(x)$

$b = fun(y)$

v) If $a \times b > 0$

Print: The root does not lie in between x_1 and x_2

END

else

if $a < 0$ the set $c = x$ and $d = y$

else,

set $c = y$ and $d = x$

vi) Calculate

$$m = \frac{(c \times b - d \times a)}{(b - a)}$$

vii) if $abs(c - d) < E$,

Print: root = m

else

go to step (v)

viii) Stop

Problem 28 (Chapter 7)

Solve the following system of linear equations using the Gauss-elimination method.

$3x_1 - 2x_2 + 3x_3 + 2x_4 = 16$

$2x_1 - 3x_2 + 2x_3 + 3x_4 = 9$

$5x_1 + 3x_2 - 5x_3 + 4x_4 = 7$ [2074 Ashwin]

$4x_1 + 2x_2 + 2x_3 - 3x_4 = 16$

Solution:

Step 1: The augmented matrix of the given system is;

$$\begin{pmatrix} 3 & -2 & 3 & 2 & : & 16 \\ 2 & -3 & 2 & 3 & : & 9 \\ 5 & 3 & -5 & 4 & : & 7 \\ 4 & 2 & 2 & -3 & : & 16 \end{pmatrix}$$

Step 2: Elimination of x_1

Applying $R_2 \to R_2 - \frac{2}{3}R_1$, $R_3 \to R_3 - \frac{5}{3}R_1$, $R_4 \to R_4 - \frac{4}{3}R_1$

$$\begin{pmatrix} 3 & -2 & 3 & 2 & : & 16 \\ 0 & -\frac{5}{3} & 0 & \frac{5}{3} & : & -\frac{5}{3} \\ 0 & \frac{19}{3} & -10 & \frac{2}{3} & : & -\frac{59}{3} \\ 0 & \frac{14}{3} & -2 & -\frac{17}{3} & : & -\frac{16}{3} \end{pmatrix}$$

or, $$\begin{pmatrix} 3 & -2 & 3 & 2 & : & 16 \\ 0 & -5 & 0 & 5 & : & -5 \\ 0 & 19 & -30 & 2 & : & -59 \\ 0 & 14 & -6 & -17 & : & -16 \end{pmatrix}$$

Step 3: Elimination of x_2

Applying $R_3 \to R_3 + \frac{19}{5}R_2$, $R_4 \to R_4 + \frac{14}{5}R_2$

$$\begin{pmatrix} 3 & -2 & 3 & 2 & : & 16 \\ 0 & -5 & 0 & 5 & : & -5 \\ 0 & 0 & -30 & 21 & : & -78 \\ 0 & 0 & -6 & -3 & : & -30 \end{pmatrix}$$

Step 4: Elimination of x_3

Applying $R_4 \to R_4 - \frac{6}{30}R_3$

$$\begin{pmatrix} 3 & -2 & 3 & 2 & : & 16 \\ 0 & -5 & 0 & 5 & : & -5 \\ 0 & 0 & -30 & 21 & : & -78 \\ 0 & 0 & 0 & -\frac{36}{5} & : & -\frac{72}{5} \end{pmatrix}$$

Step 5: Determination of unknowns using backward substitution

From $R_4 \to x_4 = 2$

From $R_3 \to x_3 = 4$ (using $x_4 = 2$)

From $R_2 \to x_2 = 3$ (using $x_4 = 2$ and $x_3 = 4$)

From $R_1 \to x_2 = 2$ (using $x_4 = 2$, $x_3 = 4$ and $x_2 = 3$)

$\therefore$ $(x_1, x_2, x_3, x_4) = (2, 3, 4, 2)$

Problem 29 (Chapter 7)

Find the dominant Eigen value and corresponding vector of the following matrix using the power method. [2074 Ashwin]

$$\begin{bmatrix} 5 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

Solution:

Let, $A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix}$

Here, all row contains non-zero element.

Considering,

Initial Eigen vector, $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

First iteration

$$AX^{(0)} = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

Second iteration

$$AX^{(1)} = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 0.67 \\ 1 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

Third iteration

$$AX^{(2)} = \begin{bmatrix}5 & 2 & 3\\2 & 4 & 2\\3 & 2 & 5\end{bmatrix}\begin{bmatrix}1\\0.67\\1\end{bmatrix} = 9.333\begin{bmatrix}1\\0.7142\\1\end{bmatrix} = \lambda^{(3)}X^{(3)}$$

Fourth iteration

$$AX^{(3)} = 9.42857\begin{bmatrix}1\\0.7272\\1\end{bmatrix} = \lambda^{(4)}X^{(4)}$$

Fifth iteration

$$AX^{(4)} = 9.45455\begin{bmatrix}1\\0.73\\1\end{bmatrix} = \lambda^{(5)}X^{(5)}$$

Sixth iteration

$$AX^{(5)} = 9.46154\begin{bmatrix}1\\0.732\\1\end{bmatrix} = \lambda^{(6)}X^{(6)}$$

$\therefore$ Eigen vector = $\begin{bmatrix}1\\0.73\\1\end{bmatrix}$

Eigen value = 9.46

Problem 30 (Chapter 3)

Write the pseudocode to fix a given set of data to a second degree polynomial $y = a + bx + cx^2$ using the least square method. [2074 Ashwin]

Solution:

Pseudocode

```
1.  Define variable x[max], y[max], n, i, j, x2[max], x3[max], x4[max],xy, x2y
    A[3][4], c, K, z[3], Sum;
2.  Print 'Enter the value of n'
3.  Read 'n'
4.  Print 'Input the value of x and y'
5.  For i = 1 to n
        Read x[i], y[i]
        x2[i] = Power(x[i], 2)
        x3[i] = Power(x[i], 3)
        x4[i] = Power(x[i], 4)
        xy = x[i] * y[i]
        x2y = x2[i] * y[i]
        i = i + 1
6.  For j = 1 to 3
        For i = 1 to 3
            if (i > j),
                c = A[i][j]/A[i][j]
                For k = 1 to 4
                    A[i][k] = A[i][k] - C * A[j][k]
                    k = k + 1
            i = i + 1
        j = j + 1
7.  z[3] = A[3][4]/A[3][3]
    For i = 2, i >= 1; i - -
8.      Sum = 0
        For j = j + 1; i <= n; j + +
            Sum = Sum + A[i][j] * x[j];
        z[i] = (A[i][4] - Sum)/A[i][i]
9.  Print y = z[3] x x^2 + z[2] x x + z[3];
```


Problem 31 (Chapter 3)

Fit the following data to the curve $y = ax^b$ using least square method. [2074 Ashwin]

x	350	400	500	600
y	61	26	7	2.6

Solution:

Given that;

$y = ax^b$

or, $\log y = \log a + b \log x$

or, $Y = \log y$

$A = \log a$

$X = \log x$

The normal equation is;

$$\sum X = nA + b\sum X \quad (1)$$

$$\sum XY = A\sum X + b\sum X^2 \quad (2)$$

x	y	X = log x	Y = log y	X^2	XY
350	61	2.544	1.785	6.472	4.541
400	26	2.602	1.415	6.770	3.682
500	7	2.699	0.845	7.285	2.281
600	2.6	2.778	0.415	7.717	1.153
		$\sum X = 10.623$	$\sum Y = 4.460$	$\sum X^2 = 10.623$	$\sum XY = 11.657$

Now, equation (1) and (2) can be written as;

$4.460 = 4 \times A + b \times 10.623$

or, $11.657 = 10.623 \times A + b \times 28.244$

or, $A = 16.7038$

and, $B = -5.86982$

$a = \text{antilog}\ (A) = 5.0559 \times 10^{16}$

Here, the required curve is $y = 5.0559 \times 10^{16} x^{-5.86982}$.

Problem 32 (Chapter 4)

Evaluate $\int_0^2 (\sin x + \cos x)\, dx$ using Gaussian three point formula. [2074 Ashwin]

x	350	400	500	600
y	61	26	7	2.6

Solution:

Given that;

Changing limit (0, 2) to (−1, 1); we have,

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(2-0)u + \frac{1}{2}(2+0)$$

or, $x = u + 1$

Now,

$$dx = du$$

$$I = \int_{-1}^{1} [\sin(u+1) + \cos(u+1)]\, dx$$

$$f(u) = \sin(u+1) + \cos(u+1)$$

$$f(0) = 1.38177$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = 1.1982$$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.77691$$

By Gauss three point formula; we have,

$$\int_{-1}^{1} f(x)\, dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9} \times 1.38177 + \frac{5}{9}[1.1982 + 0.77691]$$

$$= 2.32552$$

Problem 33 (Chapter 3)

Derive the formula for computing first and second derivative using Newton's forward difference interpolation formula. [2074 Ashwin]

Solution:

See the definition part on page no. 91

Problem 34 (Chapter 5)

Solve the following boundary value problem using shooting method employing Euler's formula taking a step-size of 0.25. [2074 Ashwin]
$y'' = x - y + y'$ subject to boundary conditions $y(0.2)$ and $y(1) = 3$

Solution:

Let, $\frac{dy}{dx} = z$

so, $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}$

or, $\frac{dz}{dx} = x - y + y'$

∴ $\frac{dz}{dx} = x - y + z$

Boundary conditions;

$y(0.2) = 2$

and, $y(1) = 3$

Use Euler's formula taking step size of 0.25.

Assuming $y'(0) = 1$

and, given $y(0) = 2$

$$\frac{dy}{dx} = z \quad (1)$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} \quad (2)$$

$$Y_{i+1} = Y_i + f_1(x_i, y_i, z_i)h$$

$$Z_{i+1} = Z_i + f_2(x_i, y_i, z_i)h$$

For i = 0;

$$Y_1 = Y_0 + f_1(x_0, y_0, z_0)h$$

$$Z_1 = Z_0 + f_2(x_0, y_0, z_0)h$$

$$x_0 = 0$$

$$y_0 = 2$$

$$z_0 = 1$$

$$Y_1 = Y_i + f_1(x_i, y_i, z_i)h$$

$$Z_1 = 1 + f_2(0, 2, 1) \times 0.25 = 1 + (0 - 2 + 1) \times 0.25 = 0.75$$

so, $y(0.25) = 2.25$

and, $z(0.25) = 0.75$

For i = 1;

$$Y_2 = Y_1 + f_1(x_1, y_1, z_1)h$$

$$Z_2 = Z_1 + f_2(x_1, y_1, z_1)h$$

$$x_1 = 0.25$$

$$y_1 = 2.25$$

$$z_1 = 0.75$$

so, $Y_2 = 2.25 + 0.75 \times 0.25 = 2.4375$

and, $Z_2 = 0.75 + (0.25 - 0.25 + 0.75) \times 0.25 = 0.4375$

so, $x_2 = 0.5$

$$y_2 = 2.4375$$

$$z_2 = 0.4375$$

For i = 2;

$$x_2 = 0.5$$

$$y_2 = 2.4375$$

$$z_2 = 0.4375$$

so, $Y_3 = 2.4375 + 0.4375 \times 0.25 = 2.546875$

and, $Z_3 = 0.4375 + (0.5 - 2.4375 + 0.4375) \times 0.25 = 0.0625$

$\therefore \quad x_3 = 0.75$

$\quad\quad y_3 = 2.546875$

and, $z_3 = 0.0625$

For i = 3;

$$Y_4 = 2.546875 + 0.0625 \times 0.25 = 2.5625$$

$$Z_4 = 0.0625 + (0.75 - 2.546875 + 0.0625) \times 0.25 = -0.3711$$

Here, $Y_4 = 2.5625$; which is greater than 2.

Problem 35 (Chapter 6)

Solve the elliptic equation (Laplace) $\mu_{xx} + \mu_{yy} = 0$ for the square mesh $0 \le x \le 1, 0 \le y \le 1$ where, $h = \Delta x = 0.25$ and $k = \Delta y = 0.25$ with the following boundary conditions: [2074 Ashwin]

u(0, 0) = 0	u(0.25, 0) = 500	u(0.5, 0) = 1000	u(0.75, 0) = 500	u(1, 0) = 0
u(0, 0.25) = 1000				u(1, 0.25) = 1000
u(0, 0.50) = 2000				u(1, 0.50) = 2000
u(0, 0.75) = 1000				u(1, 0.75) = 1000
u(0, 1) = 0	u(0.25, 1) = 500	u(0.5, 1) = 1000	u(0.75, 1) = 500	u(1, 1) = 0

Solution:

From the figure above, it is clear that the figure is symmetrical; so,

$$U_1 = U_7 = U_3 = U_9$$

$$U_2 = U_8$$

and, $U_4 = U_6$

Then,

$$U_1 = \frac{U_2 + U_4 + 1000 + 500}{4} = \frac{U_2 + U_4 + 1500}{4} \quad (1)$$

$$U_2 = \frac{U_1 + U_3 + U_5 + 1000}{4} = \frac{2U_1 + U_5 + 1000}{4} \quad (2)$$

$$U_4 = \frac{U_1 + U_7 + U_5 + 2000}{4} = \frac{2U_1 + U_5 + 2000}{4} \quad (3)$$

$$U_5 = \frac{U_2 + U_8 + U_6 + U_4}{4} = \frac{2U_2 + 2U_4}{4} \quad (4)$$

Placing equations (1), (2), (3) and (4) in the calculator using Gauss Siedal method; we obtain,

$$U_1 = U_3 = U_7 = U_9 = 937.5$$

$$U_2 = U_8 = 1000$$

$$U_4 = U_6 = 1250$$

$$U_5 = 1125$$