Exercise for MA-INF 2201 Computer Vision WS22/23
Submission on 03.12.2023

1. A function is *submodular* when it satisfies the equation:

$$P(\alpha, \beta) + P(\lambda, \tau) \geq P(\lambda, \beta) + P(\alpha, \tau)$$

for all $\alpha, \beta, \tau, \lambda$ such that $\tau > \beta$ and $\alpha > \lambda$.

(a) A function is *submodular* when it satisfies the equation:

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Show that the following functions are *submodular* and if not present a counter-example.

i. $P(\omega_m, \omega_n) = sin(|\omega_m - \omega_n|)$, where $|.|$ is the absolute function

ii. $P(\omega_m, \omega_n) = \begin{cases} (\omega_m - \omega_n)^2 & \text{if } |\omega_m - \omega_n| \leq 1 \\ 1 & \text{else} \end{cases}$

iii. $P(\omega_m, \omega_n) = c(\omega_m - \omega_n)^2$ for $c > 0$

Solution:

i. Not submodular, counterexample:
$\alpha = \beta = \frac{5\pi}{6}(\approx 2.61)$ and $\tau = \pi(\approx 3.14), \lambda = \frac{\pi}{2}(\approx 1.57)$

ii. Not submodular, counterexample:
$\alpha = 3, \beta = 1, \tau = \lambda = 1.5,$
$P(3, 1) + P(1.5, 1.5) - P(1.5, 1) - P(3, 1.5) = -0.25$

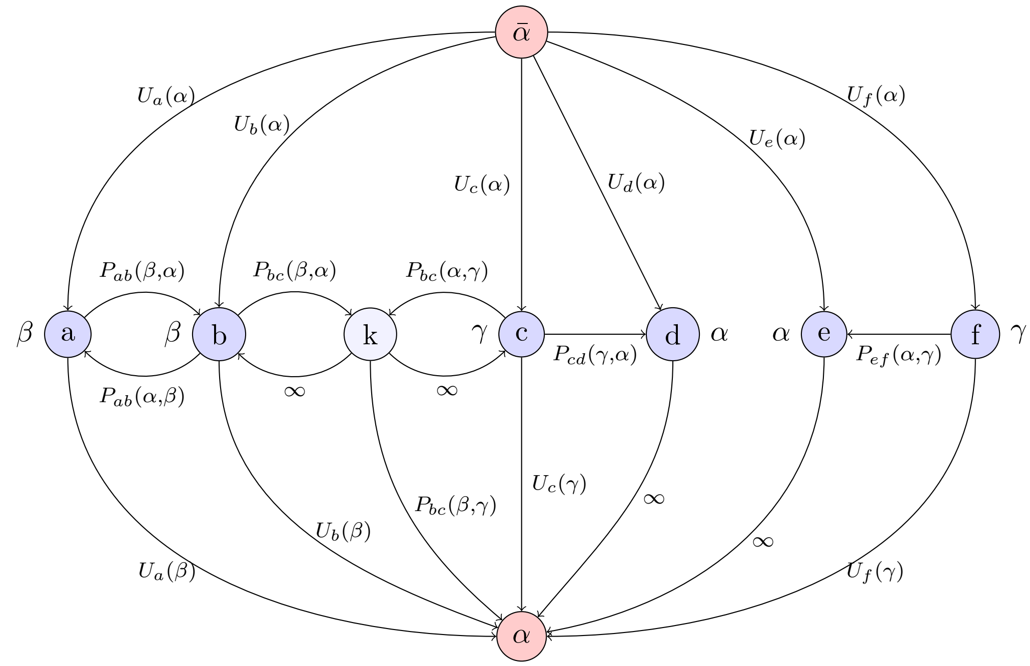
iii. Submodular, proof: $P(\omega_m, \omega_n) = c(\omega_m - \omega_n)^2$

$$\begin{aligned}
&P(\alpha, \beta) + P(\lambda, \tau) - P(\lambda, \beta) - P(\alpha, \tau) && = \\
&c\Big((\alpha - \beta)^2 + (\lambda - \tau)^2 - (\lambda - \beta)^2 - (\alpha - \tau)^2\Big) && = \\
&c\Big(\cancel{\alpha^2} - 2\alpha\beta + \cancel{\beta^2} + \cancel{\lambda^2} - 2\lambda\tau + \cancel{\tau^2} - \cancel{\lambda^2} + 2\lambda\beta - \cancel{\beta^2} - \alpha^2 + 2\alpha\tau - \cancel{\tau^2}\Big) && = \\
&2c\big(\alpha\tau + \lambda\beta - \lambda\tau - \alpha\beta\big) && = \\
&2c\Big(\tau(\alpha - \lambda) - \beta(\alpha - \lambda)\Big) && = \\
&2c\Big((\tau - \beta)(\alpha - \lambda)\Big)
\end{aligned}$$

from $c > 0$, $\tau - \beta > 0$ and $\alpha - \lambda > 0$ we derive

$$2c\Big((\tau - \beta)(\alpha - \lambda)\Big) > 0$$

2. Provide a graph structure using the *alpha expansion* method that encodes the initial state of 6 nodes (a,b,c,d,e,f) with initial states $\beta\beta\gamma\alpha\alpha\gamma$ for the case where the label α is expanded. ***(4 points)***
Solution:



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U_a(α)
U_b(α)
U_c(α)
U_d(α)
U_e(α)
U_f(α)
P_ab(β,α)
P_bc(β,α)
P_bc(α,γ)
β a
β b
k
γ c
d α
α e
f γ
P_cd(γ,α)
P_ef(α,γ)
P_ab(α,β)
∞
∞
U_c(γ)
∞
P_bc(β,γ)
U_b(β)
∞
U_a(β)
U_f(γ)
α