Solution to exercise 06 for MA-INF 2201 Computer Vision WS23/24
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1 Theory Question

NOTE: The derivation below is much nicer to follow in 1-D case and then work upto multivariate case.

Writing down the product of normal distributions

$$\begin{aligned}
Norm_x[a, A]Norm_x[b, B] &= \kappa_1 \cdot exp\left[-0.5(x-a)^T A^{-1}(x-a) - 0.5(x-b)^T B^{-1}(x-b)\right] \\
&= \kappa_2 \cdot exp\left[-0.5(x^T(A^{-1}+B^{-1})x - 2(A^{-1}a + B^{-1}b)x)\right] \\
&= \kappa_3 \cdot exp[-0.5(x^T(A^{-1}+B^{-1})x - 2(A^{-1}a + B^{-1}b)x \\
&\quad + (A^{-1}a + B^{-1}b)^T(A^{-1}+B^{-1})^{-1}(A^{-1}a + B^{-1}b))] \\
&= \kappa_3 \cdot exp[-0.5\left(x - (A^{-1}+B^{-1})^{-1}(A^{-1}a + B^{-1}b)\right)^T \left(A^{-1}+B^{-1}\right) \\
&\quad + \left(x - (A^{-1}+B^{-1})^{-1}(A^{-1}a + B^{-1}b)\right)] \\
&= \kappa \cdot Norm_x\left[\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}a + B^{-1}b\right), \left(A^{-1}+B^{-1}\right)^{-1}\right]
\end{aligned} \tag{1}$$

What is left is to write down the expression for κ:

$$\begin{aligned}
\kappa_1 &= \frac{1}{((2\pi)^D |A|^{0.5}|B|^{0.5})} \\
\kappa_2 &= \kappa_1 \cdot exp\left[-0.5a^T A^{-1}a - 0.5b^T B^{-1}b\right] \\
\kappa_3 &= \kappa_2 \cdot exp\left[0.5\left(A^{-1}a + B^{-1}b\right)^T \left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}a + B^{-1}b\right)\right] \\
\kappa &= \kappa_3 \cdot (2\pi)^{\frac{D}{2}} \left| \left(A^{-1}+B^{-1}\right)^{-1} \right|^{\frac{1}{2}}
\end{aligned} \tag{2}$$

unrolling the constant

$$\begin{aligned}
\kappa_1 &= \frac{1}{(2\pi)^D |A|^{0.5}|B|^{0.5}} \cdot (2\pi)^{\frac{D}{2}} \left| \left(A^{-1}+B^{-1}\right)^{-1} \right|^{0.5} \cdot exp[*] \\
\kappa &= \frac{1}{(2\pi)^{\frac{D}{2}} |A|^{0.5}|A^{-1}+B^{-1}|^{0.5}|B|^{0.5}} \cdot exp[*] \\
\kappa &= \frac{1}{(2\pi)^{\frac{D}{2}} |A+B|^{0.5}} \cdot exp[*]
\end{aligned} \tag{3}$$

working on the exponential part

$$
\begin{aligned}
\kappa &\propto exp\left[-0.5\left(a^T A^{-1}a + b^T B^{-1}b - (A^{-1}a + B^{-1}B)^T(A^{-1}+B^{-1})^{-1}(A^{-1}a + B^{-1}B)\right)\right] \\
&\propto exp[-0.5(a^T A^{-1}a + b^T B^{-1}b - a^T A^{-1}\left(A^{-1}+B^{-1}\right)^{-1} A^{-1}a \\
&\qquad - b^T B^{-1}\left(A^{-1}+B^{-1}\right)^{-1} B^{-1}b - 2a^T A^{-1}\left(A^{-1}+B^{-1}\right)^{-1} B^{-1}b)] \\
&\propto exp[-0.5(a^T A^{-1}a + b^T B^{-1}b - a^T A^{-1}\left(A^{-1}+B^{-1}\right)^{-1} A^{-1}a \\
&\qquad - b^T B^{-1}\left(A^{-1}+B^{-1}\right)^{-1} B^{-1}b - 2a^T\left(A+B\right)^{-1} b)] \\
&\propto exp[-0.5(a^T A^{-1}a + b^T B^{-1}b - a^T(A+B)^{-1}BA^{-1}a \\
&\qquad - b^T(A+B)^{-1}AB^{-1}b - 2a^T(A+B)^{-1}b)]
\end{aligned} \tag{4}
$$

Now using the identities below

$$
\begin{aligned}
a^T A^{-1}a &= a^T(A+B)^{-1}(A+B)A^{-1}a \\
&= a^T(A+B)^{-1}a + a^T(A+B)^{-1}BA^{-1}a
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
b^T B^{-1}b &= b^T(A+B)^{-1}(A+B)B^{-1}b \\
&= b^T(A+B)^{-1}b + b^T(A+B)^{-1}AB^{-1}b
\end{aligned} \tag{6}
$$

using equations 5, 6 in equation 4

$$
\begin{aligned}
\kappa &\propto exp[-0.5(a^T A^{-1}a + b^T B^{-1}b - a^T\left(A+B\right)^{-1} BA^{-1}a \\
&\qquad - b^T\left(A+B\right)^{-1} AB^{-1}b - 2a^T\left(A+B\right)^{-1} b)] \\
&\propto exp\left[-0.5\left(a^T(A+B)^{-1}a + b^T(A+B)^{-1}b - 2a^T(A+B)^{-1}b\right)\right] \\
&\propto exp\left[-0.5(a-b)^T(A+B)^{-1}(a-b)\right]
\end{aligned} \tag{7}
$$

Putting results of equation 7 in equation 3 we have

$$
\kappa = Norm_a[b, A+B] \tag{8}
$$