Sheet 01 P3

Saturday, 21. October 2023 14:48

Team Members: Suraj Giri and Suyash Thapa.

Problem 3: Proof that convolutions are in the continuous case associative.

Proof:

Definition of Convolution:

$$(f*g)(x) = \int_0^t f(x-u)\, g(u)\, du = \int f(u)\, g(x-u)\, du.$$

Let another convolution exists such that

$$((f*g)*h)(x) = \int_0^t (f*g)(\delta)\, h(x-\delta)\, d\delta$$

$$= \int_\delta^t \left(\int_{u=0}^{\delta} f(u)\, g(\delta-u)\, du \right) h(x-\delta)\, d\delta$$

$$= \int_{\delta=0}^{t} \int_{u=0}^{\delta} f(u)\, g(\delta-u)\, h(x-\delta)\, du\, d\delta$$

$$= \int_{u=0}^{t} \int_{\delta=u}^{t} f(u)\, g(\delta-u)\, h(x-\delta)\, d\delta\, du$$

$$= \int_{u=0}^{t} \int_{\delta=0}^{t-u} f(u)\, g(\delta-u)\, h(x-\delta)\, d\delta\, du$$

$$= \int_{u=0}^{t} f(u) \left(\int_{\delta=0}^{t-u} g(\delta-u)\, h(x-\delta-u)\, d\delta \right) du$$

$$= \int_{u=0}^{t} f(u) \left((g*h)(x-u) \right) du$$

$$= f*(g*h)(x)$$

proved

Sheet 01 P6

Sunday, 22. October 2023 16:50

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Problem 6: Proof that convolution two times with a Gaussian kernel with standard deviation σ is the same as convolution once with standard deviation $\sqrt{2}\,\sigma$.

Proof:

Firstly, we have the following Gaussian kernel with std. deviation σ.

$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}\right)\left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2}}\right)$$

Let the original image is I

Applying convolution to image I with Gaussian kernel G_σ, we have.

$$I * G_\sigma \quad \text{——} \quad (1)$$

Applying convolution again to eq^n (1) we have

$(I * G_\sigma) * G_\sigma$ · which according to associative property of convolution is as follows:

$$(I * G_\sigma) * G_\sigma = I * (G_\sigma * G_\sigma)$$

i.e; convolving pixel x twice resulted in:

$$I * \int_0^u \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-u)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{u^2}{2\sigma^2}}\, du \qquad g(x) = \int f(x-u)\, h(u)\, du$$

$$= I * \frac{1}{2\pi\sigma^2} \int_0^u e^{-\frac{(x-u)^2}{2\sigma^2} + \frac{(-u^2)}{2\sigma^2}}\, du \qquad = I * \frac{1}{2\pi\sigma^2} \int_0^u e^{\frac{-x^2-u^2-u^2}{2\sigma^2}}\, du$$

$$= I * \frac{1}{2\pi\sigma^2} \int_0^u e^{\frac{-x^2+2xu-u^2-u^2}{2\sigma^2}}\, du$$

$$= I * \frac{1}{2\pi\sigma^2} \int_0^u e^{\frac{-x^2-2u^2+2ux}{2\sigma^2}}\, du$$

$$= I * \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{x^2}{2\sigma^2}} \cdot \int_0^u e^{\frac{2u(x-u)}{2\sigma^2}}\, du$$

$$= I * \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{x^2}{2\sigma^2}} \int_0^u e^{\frac{u(x-u)}{\sigma^2}}\, du$$

$$= I * \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{x^2}{4\sigma^2}}$$

$$= I * \frac{1}{\sqrt{2\pi}\,(\sqrt{2}\,\sigma)^2} \cdot e^{-\frac{x^2}{2\cdot(\sqrt{2}\,\sigma)^2}}$$

$$= I * G'$$

In G' new value of std. deviation after convolving twice becomes $\sqrt{2}\,\sigma$.

$\therefore$ Convolution with Gaussian kernel with std. dev. σ two times is equal to convolution with Gaussian kernel with std. dev. $\sqrt{2}\,\sigma$.

proved