Exercise Sheet 5 for MA-INF 2201 Computer Vision WS22/23

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1 Problem 1

Submodular functions:

$$P(\alpha, \beta) + P(\lambda, \tau) \geq P(\lambda, \beta) + P(\alpha, \tau) \tag{1}$$

for all α, β, λ, τ such that $\tau > \beta$ and $\alpha > \lambda$.

1.1

$$P(\omega_m, \omega_n) = \sin(|\omega_m - \omega_n|) \text{ where } |.| \text{ is the absolute function.} \tag{2}$$

Solution:
Substituting the given funtion $P(\omega_m, \omega_n)$ in the above inequality, we get:

$$\sin(|\alpha - \beta|) + \sin(|\lambda - \tau|) \geq \sin(|\lambda - \beta|) + \sin(|\alpha - \tau|) \tag{3}$$

The given function might not be submodular.
Consider the following counter example:
Let $\alpha = 2$, $\beta = 1$, $\lambda = 0$, $\tau = 3$
Then,

$$\sin(|\alpha - \beta|) + \sin(|\lambda - \tau|) = \sin(1) + \sin(3) = 0.841 + 0.141 = 0.982 \tag{4}$$

$$\sin(|\lambda - \beta|) + \sin(|\alpha - \tau|) = \sin(1) + \sin(1) = 0.841 + 0.841 = 1.682 \tag{5}$$

Here, $P(\alpha, \beta) + P(\lambda, \tau) < P(\lambda, \beta) + P(\alpha, \tau)$
Hence, we can conclude that the given function is not submodular.

1.2

$$P(\omega_m, \omega_n) = \begin{cases} (\omega_m - \omega_n)^2 & \text{if } |\omega_m - \omega_n| \leq 1 \\ 1 & \text{otherwise} \end{cases} \tag{6}$$

Solution:
Substituting the given funtion $P(\omega_m, \omega_n)$ in the above inequality, we get:

$$(\alpha - \beta)^2 + (\lambda - \tau)^2 \geq (\lambda - \beta)^2 + (\alpha - \tau)^2 \tag{7}$$

The given function might not be submodular.
Consider the following counter example:
Let $\alpha = 2$, $\beta = 1$, $\lambda = 0$, $\tau = 3$
Then,

$$(\alpha - \beta)^2 + (\lambda - \tau)^2 = (2 - 1)^2 + (0 - 3)^2 = 1 + 9 = 10 \tag{8}$$

$$(\lambda - \beta)^2 + (\alpha - \tau)^2 = (0 - 1)^2 + (2 - 3)^2 = 1 + 1 = 2 \tag{9}$$

Here, $P(\alpha, \beta) + P(\lambda, \tau) < P(\lambda, \beta) + P(\alpha, \tau)$
Hence, we can conclude that the given function is not submodular.

1.3

$$P(\omega_m, \omega_n) = c(\omega_m - \omega_n)^2 \text{ for } c \geq 0 \tag{10}$$

Solution:
Substituting the given funtion $P(\omega_m, \omega_n)$ in the above inequality, we get:

$$
\begin{aligned}
c(\alpha-\beta)^2 + c(\lambda-\tau)^2 &\geq c(\lambda-\beta)^2 + c(\alpha-\tau)^2 \quad (11)\\
\text{or, } c(\alpha^2+\beta^2-2\alpha\beta) + c(\lambda^2+\tau^2-2\lambda\tau) &\geq c(\lambda^2+\beta^2-2\lambda\beta) + c(\alpha^2+\tau^2-2\alpha\tau) \quad (12)\\
\text{or, } (\alpha^2+\beta^2-2\alpha\beta) + (\lambda^2+\tau^2-2\lambda\tau) &\geq (\lambda^2+\beta^2-2\lambda\beta) + (\alpha^2+\tau^2-2\alpha\tau) \quad (13)\\
\text{or, } \alpha^2+\beta^2+\lambda^2+\tau^2-2\alpha\beta-2\lambda\tau &\geq \lambda^2+\beta^2-2\lambda\beta+\alpha^2+\tau^2-2\alpha\tau \quad (14)\\
\text{or, } -2\alpha\beta-2\lambda\tau &\geq -2\lambda\beta-2\alpha\tau \quad (15)\\
\text{or, } 2\lambda(\beta-\tau)+2\alpha(\tau-\beta) &\geq 0 \quad (16)\\
\text{or, } -2\lambda(\tau-\beta)+2\alpha(\tau-\beta) &\geq 0 \quad (17)\\
\text{or, } (\tau-\beta)(\alpha-\lambda) &\geq 0 \quad (18)\\
& \quad (19)
\end{aligned}
$$

$$
\begin{aligned}
Either(\tau-\beta)or(\lambda-\alpha) \geq 0 \quad (20)\\
\textbf{Case 1:}(\tau-\beta) \geq 0 \implies \tau \geq \beta \implies \tau > \beta \quad (21)\\
\textbf{Case 2:}(\alpha-\lambda-) \geq 0 \implies \alpha \geq \lambda \implies \alpha > \lambda \quad (22)\\
(23)
\end{aligned}
$$

Hence, since the definition of the submodular function holds true for the given function, we can conclude that the given function is **submodular**.

2 Problem 2

6 nodes: a, b, c, d, e, f
Initial States: $\beta\beta\gamma\alpha\alpha\gamma$
Label α is expanded.
Solution:
The Graph structure using *alpha expansion* is shown in the figure below:

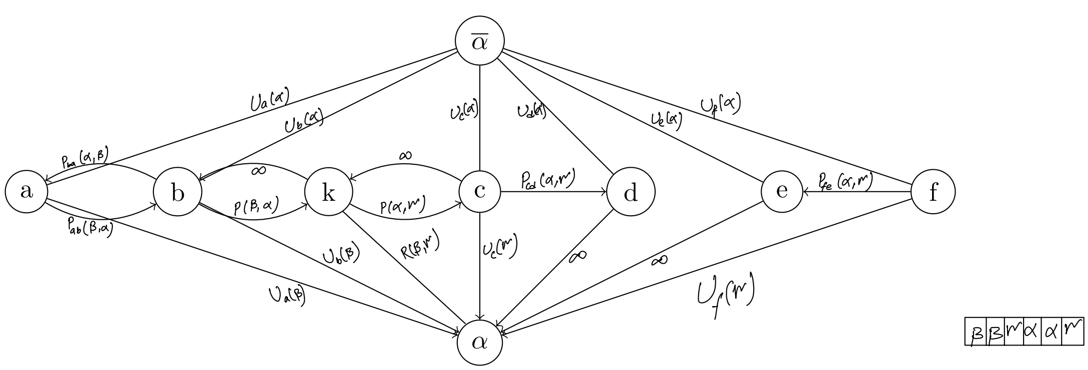

Figure 1: Graph structure using alpha expansion