Sheet08: Shapes

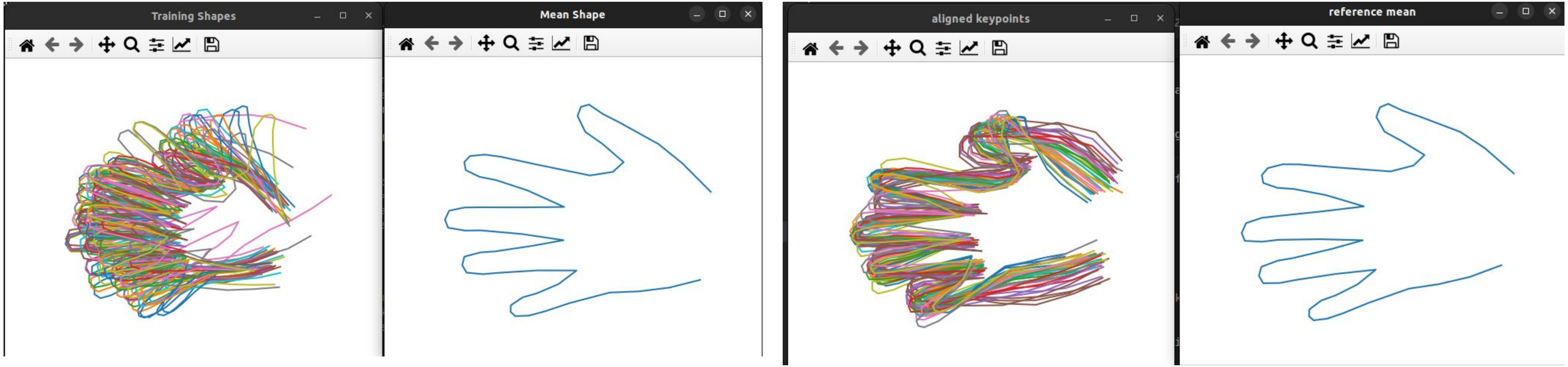
Task 1: Procruster Analysis

1. **Procrustres Analysis:** We are given *hands_orig_train.txt.new* which contains 56 landmark points on hand contours from 39 different subjects. The underlying structure of the given data is further explained in the *readme* file. The goal of this task is to align the data. The rough outline of the analysis is as follows:

    - Compute the mean shape (μ_s).
    - Align each shape to μ_s.
    - Compute the RMS error between aligned shapes and the new mean shape.
    - Repeat above steps until convergence (either small error or max number of steps).

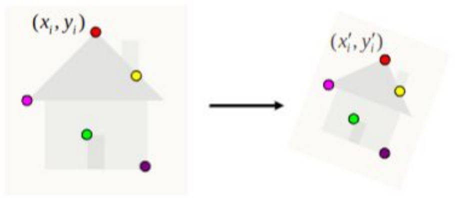
    Display the shapes and the mean shape before and after the alignment to verify your results.

Before and After Alignment



Estimating Affine transformation

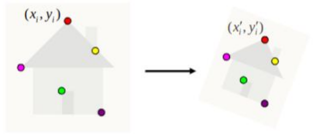
Fitting an affine transformation



$$\begin{bmatrix} x_i' \\ y_i' \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$\begin{bmatrix} u_i & v_i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_i & v_i & 1 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \tau_x \\ \phi_{21} \\ \phi_{22} \\ \tau_y \end{bmatrix}$$



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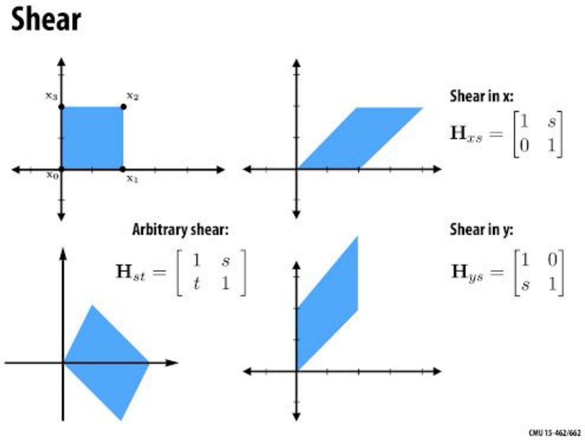
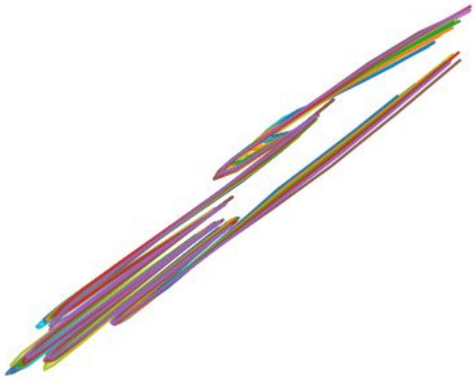
3D Geometric Transformations

- 3D Transformation hierarchy

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} I \mid t \end{bmatrix}_{3\times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}_{3\times 4}$	6	lengths	
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}_{3\times 4}$	7	angles	
affine	$\begin{bmatrix} A \end{bmatrix}_{3\times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}_{4\times 4}$	15	straight lines	

Shear transformation





Task 2: Statistical Shape Modelling

2. **Statistical Shape Modeling:**

   - Build a PPCA based statistical shape model $\mathcal{M}$ using the data in *hands_align_train.txt.new*. The data is a set of 56 corresponding landmark points on hand-contours from 39 instances that have already been aligned using Procrustres Analysis. Refer to the *readme* file for details about data organization. Choose the number of basis functions N to be the minimum number of principal components preserving 90% of the energy. Visualize μ and the effect of varying positive and negative weights of each ϕ_k.
   - When does the computation of the eigenvalue decomposition for matrix WW^T become computationally prohibitive? What can you do in this case?

   Restriction: Implement PPCA by yourself. You can utilize *np.linalg.eig* or *np.lingalg.svd* for this task.

   (8 Points)

3. **Inference:** Express the test shape in *hands_align_test.txt* in terms of the generated model $\mathcal{M}$. Display the values of h_{ik}. Also, reconstruct the test shape as $\hat{w}_{test}$, visualize the original and the reconstructed shapes and calculate the RMS error between both shapes.

   (4 Points)

Statistical Shape Modelling

- Generate data from model:

$$\mathbf{w}_i = \boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{h}_i + \epsilon_i$$

  - $\boldsymbol{\mu}$ is the mean shape
  - the matrix $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_K]$ contains K basis functions in it columns
  - ϵ_i is normal noise with covariance $\sigma^2\mathbf{I}$
- Can alternatively write

$$\mathbf{w}_i = \boldsymbol{\mu} + \sum_{k=1}^{K} \boldsymbol{\phi}_k h_{ik} + \epsilon_i$$

Statistical Shape Modelling: PPCA

Probabilistic version:

$$Pr(\mathbf{w}_i|\mathbf{h}_i, \boldsymbol{\mu}, \boldsymbol{\Phi}, \sigma^2) = \text{Norm}_{\mathbf{w}_i}[\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{h}_i, \sigma^2\mathbf{I}]$$

Add prior: $Pr(\mathbf{h}_i) = \text{Norm}_{\mathbf{h}_i}[\mathbf{0}, \mathbf{I}]$

Density: $Pr(\mathbf{w}_i) = \int Pr(\mathbf{w}_i|\mathbf{h}_i)Pr(\mathbf{h}_i)d\mathbf{h}_i$

Learn parameters $\boldsymbol{\mu}, \boldsymbol{\Phi}$ and σ^2 from data $\{\mathbf{w}_i\}_{i=1}^I$
where $\mathbf{w}_i = [\mathbf{w}_{i1}^T, \mathbf{w}_{i2}^T \ldots, \mathbf{w}_{iN}^T]^T$.

Learn mean: $\boldsymbol{\mu} = \frac{\sum_{i=1}^I \mathbf{w}_i}{I}$

Then set $\mathbf{W} = [\mathbf{w}_1 - \boldsymbol{\mu}, \mathbf{w}_2 - \boldsymbol{\mu}, \ldots, \mathbf{w}_I - \boldsymbol{\mu}]$
and compute eigen-decomposition

$$\mathbf{W}\mathbf{W}^T = \mathbf{U}\mathbf{L}^2\mathbf{U}^T$$

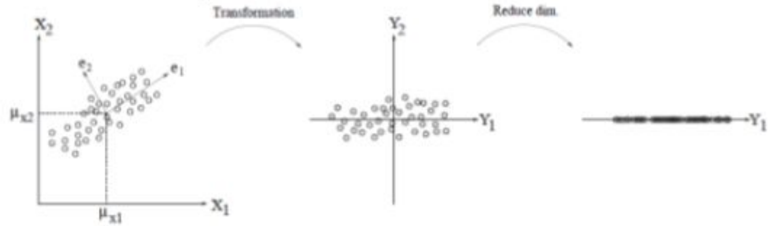
Choose parameters

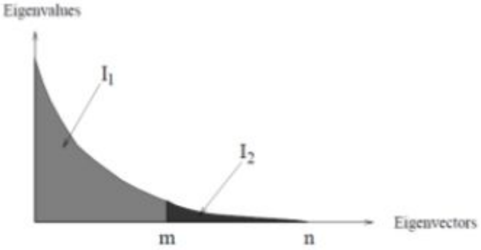
$$\hat{\sigma}^2 = \frac{1}{D-K}\sum_{j=K+1}^{D} L_{jj}^2$$

$$\hat{\boldsymbol{\Phi}} = \mathbf{U}_K(\mathbf{L}_K^2 - \hat{\sigma}^2\mathbf{I})^{1/2}$$

How many basis functions?

- Dimensionality reduction





$$\frac{\sum_{i=1}^{i=K} L_{ii}^2}{\sum_{i=1}^{i=D} L_{ii}^2} > 1 - \epsilon$$

Source: T. Moeslund

Eigenvalue decomposition

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Eigenvalue decomposition

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If the dimensionality D of the data is very high, then the eigenvalue decomposition of the $D \times D$ matrix $\mathbf{W}\mathbf{W}^T$ will be computationally expensive. If the number of training examples I is less than the dimensionality D, then a more efficient approach is to compute the singular value decomposition of the $I \times I$ scatter matrix $\mathbf{W}^T\mathbf{W}$

$$\mathbf{W}^T\mathbf{W} = \mathbf{V}\mathbf{L}^2\mathbf{V}^T, \tag{17.29}$$

and then re-arrange the SVD relation $\mathbf{W} = \mathbf{U}\mathbf{L}\mathbf{V}^T$ to compute $\mathbf{U}$.

Visualization

We could visualize the PPCA model by drawing samples from the marginal density (equation 17.25). However, the properties of the basis functions permit a more systematic way to examine the model. In figure 17.11 we visualize the PPCA model by manipulating the hidden variable $\mathbf{h}_i$ and then illustrating the vector $\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{h}_i$. We can choose $\mathbf{h}_i$ to elucidate each basis function $\{\boldsymbol{\phi}_k\}_{k=1}^{K}$ in turn. For example, by setting $\mathbf{h}_i = \pm[1, 0, 0, 0, ..., 0]$ we investigate the first basis function.

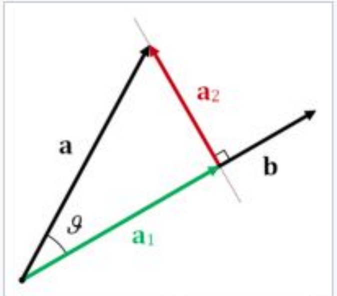
Projecting onto the subspace

Given a vector x that you want to project onto the subspace spanned by the eigenvectors, the projection P is given by:

$$P = \sum_{i=1}^{n} (x \cdot v_i) \cdot v_i$$

Here:

- $x \cdot v_i$ is the dot product between the vector x and the i-th eigenvector v_i.
- $\sum$ denotes the sum over all eigenvectors.



Projection of **a** on **b** ($\mathbf{a}_1$), and rejection of **a** from **b** ($\mathbf{a}_2$).

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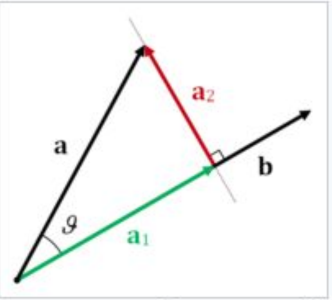

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