Exercise Sheet 5 for MA-INF 2201 Computer Vision WS22/23

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1 Theory Question

Prove the following property of k dimensional Gaussian distributions $Norm_x[\mu, \Sigma]$:

$$\int Norm_x[a, A] Norm_x[b, B]\, dx = Norm_a[b, A + B] \int Norm_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*]\, dx$$

where $\Sigma_* = (A^{-1} + B^{-1})^{-1}$.

Solution:

LHS:

$$\int Norm_x[a, A] Norm_x[b, B]\, dx$$

$$= \int \frac{1}{\sqrt{(2\pi)\det(A)}} \exp\left(-\frac{1}{2}(x-a)^T A^{-1}(x-a)\right) \frac{1}{\sqrt{(2\pi)\det(B)}} \exp\left(-\frac{1}{2}(x-b)^T B^{-1}(x-b)\right) dx$$

$$= \frac{1}{(2\pi)\sqrt{\det(A)\det(B)}} \int \exp\left(-\frac{1}{2}(x-a)^T A^{-1}(x-a)\right) \exp\left(-\frac{1}{2}(x-b)^T B^{-1}(x-b)\right) dx$$

$$= \frac{1}{(2\pi)\sqrt{\det(A)\det(B)}} \int \exp\left(-\frac{1}{2}(x-a)^T A^{-1}(x-a) - \frac{1}{2}(x-b)^T B^{-1}(x-b)\right) dx$$

$$= \frac{1}{(2\pi)\sqrt{\det(A)\det(B)}} \int \exp -\frac{1}{2}((x-a)^T A^{-1}(x-a) + (x-b)^T B^{-1}(x-b))\, dx$$

Solving the exponent term:

$$(x-a)^T A^{-1}(x-a) + (x-b)^T B^{-1}(x-b)$$

$$= x^T A^{-1}x - x^T A^{-1}a - a^T A^{-1}x + a^T A^{-1}a + x^T B^{-1}x - x^T B^{-1}b - b^T B^{-1}x + b^T B^{-1}b$$

$$= x^T A^{-1}x + x^T B^{-1}x - a^T A^{-1}x - x^T A^{-1}a - b^T B^{-1}x - x^T B^{-1}b + a^T A^{-1}a + b^T B^{-1}b$$

$$= x^T(A^{-1} + B^{-1})x - x(a^T A^{-1} + b^T B^{-1}) - a^T A^{-1}x - b^T B^{-1}x + a^T A^{-1}a + b^T B^{-1}b$$

$= x^T(A^{-1}+B^{-1})x - x(A^{-1}+B^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}a^T + B^{-1}b^T) - (aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})x^T + aA^{-1}(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})a^T + bB^{-1}(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})b^T$

Solving the above term will yield the following:

$$(x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})(x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1})^T)+(a-b)(A+B)^{-1}(a-b)^T \quad (1)$$

Subsituting the equation (1) in the LHS:

$$\int Norm_x[a, A] Norm_x[b, B]\, dx$$

$$= \frac{1}{(2\pi)\sqrt{\det(A)\det(B)}} \int \exp -\frac{1}{2}((x-a)^T A^{-1}(x-a) + (x-b)^T B^{-1}(x-b))\,dx$$

$$= \frac{1}{\sqrt{(2\pi)\det(A)\det(B)}} \int \frac{1}{\sqrt{2\pi}} \exp\Bigg(-\frac{1}{2}\Bigg[\Bigg(x - (aA^{-1} + bB^{-1})(A^{-1}+B^{-1})^{-1}\Bigg)(A^{-1}+B^{-1})$$

$$(x - (aA^{-1} + bB^{-1})(A^{-1}+B^{-1})^{-1})^T + (a-b)(A+B)^{-1}(a-b)^T\Bigg]\Bigg)\,dx$$

Let $\mu = (aA^{-1} + bB^{-1})(A^{-1}+B^{-1})^{-1}$ and $\Sigma_* = (A^{-1}+B^{-1})^{-1}$, then we have:

$$\int Norm_x[a,A] Norm_x[b,B]\,dx$$

$$= \frac{\sqrt{|(A^{-1}+B^{-1})^{-1}|}}{\sqrt{(2\pi)\det(A)\det(B)}} \int \frac{1}{\sqrt{2\pi}\sqrt{|(A^{-1}+B^{-1})^{-1}|}} \exp\Bigg(-\frac{1}{2}\Bigg((x-\mu)(\Sigma_*)^{-1}(x-\mu)^T +$$

$$(a-b)(A+B)^{-1}(a-b)^T\Bigg)\Bigg)\,dx$$

$$= \frac{1}{\sqrt{(2\pi)\det(A)A^{-1} + \det(B)B^{-1}}} \exp\Bigg(-\frac{1}{2}(a-b)(A+B)^{-1}(a-b)^T\Bigg) \int Norm_x[\mu, \Sigma_*]\,dx$$

$$= \frac{1}{\sqrt{(2\pi)\det(A+B)}} \exp\Bigg(-\frac{1}{2}(a-b)(A+B)^{-1}(a-b)^T\Bigg) \int Norm_x[\mu, \Sigma_*]\,dx$$

$$= Norm_a[b, A+B] \int Norm_x[\mu, \Sigma_*]\,dx$$

$$= Norm_a[b, A+B] \int Norm_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*]\,dx$$

because $\mu = (aA^{-1} + bB^{-1})(A^{-1}+B^{-1})^{-1} = \Sigma_*(A^{-1}a + B^{-1}b)$ and $\Sigma = (A^{-1}+B^{-1})^{-1} = \Sigma_*$ from above.

$$\int Norm_x[a,A] Norm_x[b,B]\,dx = Norm_a[b, A+B] \int Norm_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*]\,dx$$