Jacobs University
Fall 2022

November 2, 2022

Stochastic Methods Lab

Assignment Sheet 7

Due on November 9, 2022

Note: The work is to be submitted via `git`, as discussed in class. The coding language is Python. Please make sure that your code actually runs and produces the requested output. Please make your code readable for the instructor and TA, and include comments wherever necessary. Please submit .py source code, not jupyter notebooks. Theoretical questions may be submitted as a scan of handwritten notes or typed up (e.g., using LaTeX). The submission deadline is midnight of the stated due date.

Problem 1 [10 points]

It is known that the stochastic differential equation

$$\begin{aligned} dS(t) &= \mu\, S(t)\, dt + \sigma\, S(t)\, dW(t), \\ S(0) &= S_0, \end{aligned} \tag{1}$$

is solved by geometric Brownian motion

$$S(t) = S_0\, e^{(\mu - \sigma^2/2)\, t + \sigma\, W(t)}. \tag{2}$$

(a) Use the Euler-Maruyama method to solve (1) with $\mu = 1.5$, $\sigma = 0.8$, and $S_0 = 1$ up to final time $T = 1$. Compare the result in a plot pathwise against the exact solution (2).

(b) Find the *strong order of convergence*, i.e., an exponent p such that

$$\mathbb{E}\big[|S_N - S(T)|\big] \leq c\,(\Delta t)^p\,,$$

where $S(T)$ denotes true geometric Brownian motion and S_N its Euler-Maruyama approximation at the final time T.

(c) Find the *weak order of convergence*, i.e., an exponent q such that

$$\big|\mathbb{E}[S_N] - \mathbb{E}[S(T)]\big| \leq c\,(\Delta t)^q\,.$$

Problem 2 [3 points]

Confirm numerically with a python program that $\sum_{i=0}^{n-1} (\Delta W_i)^2$ converges to a constant in the limit $n \to \infty$. Here, $W(t)$ is a Brownian motion. What is the constant?

Problem 3 [4 points]

Use the Black-Scholes formula that we discussed in class (see also Problem 1 of Assignment Sheet 5) and plot the call price C against

(a) the stock price S,

(b) the interest rate r,

(c) the volatility σ.

For each plot, use reasonable parameters.

Problem 4 [3 points]

A theoretical exercise: Show that it is never optimal to exercise an American call option on a non-dividend-paying stock before expiration.