Jacobs University
Fall 2022

November 30, 2022

Stochastic Methods Lab

Assignment Sheet 11

Due on December 7, 2022

Note: The work is to be submitted via `git`, as discussed in class. The coding language is Python. Please make sure that your code actually runs and produces the requested output. Please make your code readable for the instructor and TA, and include comments wherever necessary. Please submit .py source code, not jupyter notebooks. Theoretical questions may be submitted as a scan of handwritten notes or typed up (e.g., using LaTeX). The submission deadline is midnight of the stated due date.

Problem 1 [20 points]

Suppose S_i for $i = 0, \ldots, N$ denotes time series data which we believe behaves like geometric Brownian motion with parameters μ and σ. Then estimates $\hat{\mu}$ and $\hat{\sigma}$ for μ and σ can be obtained by considering the log-returns

$$r_i = \ln S_{i+1} - \ln S_i \quad \text{for} \quad i = 0, \ldots, N-1,$$

as discussed in class. *For the following exercises no loops are allowed, please implement all operations vectorized.*

(a) [6 points] Generate a sample geometric Brownian path on the time interval $[0, 1]$ with fixed $\mu = 0.3$ and $\sigma = 0.5$, where the number of sub-intervals is $N = 2^k$.

Estimate $\hat{\sigma}$ and $\hat{\mu}$ on a coarsened data set where you use only every 2^i-th data point, where $i = 0, \ldots, k$.

Plot $\hat{\sigma}$ and $\hat{\mu}$ vs. the log of the number of sample points (`semilogx`). Do the estimated values converge to the true values of the model?

(b) [4 points] Now repeat the computation on a large ensemble of geometric Brownian paths and estimate the mean and the standard deviation of the estimate for σ and μ as a function of the number of sub-intervals N using a doubly logarithmic graph.

Note that it is known that the variance of the estimate for σ is approximately

$$\mathrm{Var}[\hat{\sigma}] = \frac{\mathbb{E}(\hat{\sigma})^2}{2N} \,.$$

Does your statistics reproduce this result?

How does the variance of the estimate for μ behave as N increases?

(c) [4 points] Use the single geometric Brownian path from Question (a), estimate its parameters, and use the estimated values $\hat{\mu}$ and $\hat{\sigma}$ to generate an ensemble of geometric Brownian paths as in Question (b). Plot a histogram (`hist`) of the estimates for μ and σ from this ensemble, compute the standard deviation, and visualize the standard deviation and the true value of the original model in this histogram.

(This procedure is a simple example of so-called *parametric bootstrapping.*)

(d) [2 points] How does the result of Question (a) change if

(1) you add Gaussian noise to the geometric Brownian motion;

(2) you add a high frequency periodic perturbation?

(e) [2 points] Perform a QQ-plot vs. the normal distribution for the distribution of the log-returns, and the distribution of the two noisy log-returns from Question (d), all into one graph. Briefly discuss the result.

(f) [2 points] Plot the autocorrelation function for the time series of log-returns, and the two noisy versions, all into one graph. Briefly discuss the result.