HW9. Problem 1.b.

Tuesday, 22. November 2022 11:05

Ornstein- Uhlenbeck Stochastic Process

$$dX_t = \mu(1 - c\ln X_t)X_t\,dt + \sigma X_t\,dW_t \quad \text{—— (A)}$$

with $\mu, c > 0$ and $0 \le c \le 1$.

Here, $f = \mu(1 - c\ln X_t)$ and
$g = \sigma X_t$.

Itô's Lemma to $F(X_t, t)$

$$dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} f + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} g^2\right] dt + g\frac{\partial F}{\partial x}\partial W \quad \text{—— (1)}$$

Portfolio: (from lecture slides)

$$\underbrace{\pi}_{\text{bonds}} = \underbrace{\alpha C}_{\text{option}} + \underbrace{\beta S}_{\text{stock}} \Rightarrow d\pi = \alpha\, dC + \beta\, dS \quad \text{—— (2)}$$

Comparing (1) & (2)

$$d\pi = \alpha\left[\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} f + \frac{1}{2}\frac{\partial^2 C}{\partial x^2} g^2\right] dt + \alpha g \frac{\partial C}{\partial x} dW + \beta f dt + \beta g dW \quad \text{—— (B)}$$

Here we only need to option part. So, to Remove the stochastic part, let $\beta = -\alpha \frac{\partial C}{\partial x}$

Putting this into (B).

$$d\pi = \alpha\left[\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial x^2} g^2\right] dt \quad \text{—— (3)}$$

Eq^n (3) grows with rate r.
i.e; $\pi = \pi \cdot 0 \cdot e^{rt} \Rightarrow d\pi = \pi r dt = \pi\left[C - \frac{\partial C}{\partial x} x\right] r\, dt$ —— (4)

From (3) & (4)

$$\alpha\left[\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial x^2} g^2\right] \cancel{dt} = \pi\left[C - \frac{\partial C}{\partial x} x\right] r\,\cancel{dt}$$

or, $$\alpha\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial x^2} g^2 \alpha = \pi r C - \pi r \frac{\partial C}{\partial x} x$$

or, $$\cancel{\alpha}\left[\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial x^2} g^2\right] = \cancel{\alpha}\left[rC^2 - Cr\frac{\partial C}{\partial x} x\right] \quad \left[\pi = \alpha C + \underbrace{\beta S}_{\text{this part is removed.}}\right]$$

or, $$\frac{\partial C}{\partial t} + \frac{1}{2} g^2 \frac{\partial^2 C}{\partial x^2} + Crx\frac{\partial C}{\partial x} = rC^2$$

$$\boxed{\text{or, } \frac{\partial C}{\partial t} + Crx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 C}{\partial x^2} = rC^2} \quad \text{—— (5)}$$

Eq^n (5) is the required SDE.

This required SDE is the Black - scholes PDE

Mean does not matter.
The result is same as before.