Stochastic Methods Lab.

H.W.3.

Problem 1.

Immunize liability at time D with two bonds of Macaulay Durations D_1 and D_2

To show: Fraction w_1 and w_2 of the two bonds in the initial portfolio needs to satisfy

$$w_1 + w_2 = 1$$
$$w_1 D_1 + w_2 D_2 = D$$

We know,

$$FV = (1+y)^m P \quad \text{—— (1)}$$

$$MD = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n}\right] \quad \text{—— (2)}$$

$$P = \sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n} \quad \text{—— (3)}$$

Then, $FV = FV_1 + FV_2$

or, $FV = P_1(1+y)^D + P_2(1+y)^D$ [P_1 and P_2 are bonds prices. $D \rightarrow$ time / horizon $= my$]

—— (4)

From (3) and (4)

$$FV = (1+y)^D\left[\sum_{i=1}^{n_1} \frac{C_1}{(1+y)^i} + \frac{F_1}{(1+y)^{n_1}}\right] + (1+y)^D\left[\sum_{i=1}^{n_2} \frac{C_2}{(1+y)^i} + \frac{F_2}{(1+y)^{n_2}}\right]$$

$$\cancel{= \sum_{i=1}^{n_1} \frac{C_1}{1+y}}$$

$$= \sum_{i=1}^{n} C_1(1+y)^{D-i} + F_1(1+y)^{D-n_1} + \sum_{i=1}^{n_2} C_2(1+y)^{D-i} + F_2(1+y)^{D-n_2}$$

$$\cancel{FV = \sum_{i=1}^{n} C_1(1+y)^{D-i} + nF_1(1+y)^{D-n}}$$

$$FV = \sum_{i=1}^{n_1} C_1(1+y)^{D-i} + F_1(1+y)^{D-n_1} + \sum_{i=1}^{n_2} C_2(1+y)^{D-i} + F_2(1+y)^{D-n_2}$$

For immunization,

$$\frac{\partial FV}{\partial y} = 0.$$

$$\Rightarrow \frac{\partial FV_1}{\partial y} + \frac{\partial F_2}{\partial y} = 0$$

$$\Rightarrow \sum_{i=1}^{n_1} C_1(D-i)(1+y)^{D-i-1} + F_1(D-n_1)(1+y)^{D-n_1-1}$$
$$+ \sum_{i=1}^{n_2} C_2(D-i)(1+y)^{D-i-1} + F_2(D-n_2)(1+y)^{D-n_2-1} = 0$$

$$\Rightarrow (1+y)^{D-1}\left[\sum_{i=1}^{n_1} \frac{C_1(D-i)}{(1+y)^i} + \frac{(D-n_1)F_1}{(1+y)^{n_1}}\right] +$$
$$(1+y)^{D-1}\left[\sum_{i=1}^{n_2} \frac{C_2(D-i)}{(1+y)^i} + \frac{(D-n_2)F_2}{(1+y)^{n_2}}\right] = 0$$

$$\Rightarrow \sum_{i=1}^{n_1} \frac{C_1(D-i)}{(1+y)^i} + \frac{(D-n_1)F_1}{(1+y)^{n_1}} + \sum_{i=1}^{n_2} \frac{C_2(D-i)}{(1+y)^i} + \frac{(D-n_2)F_2}{(1+y)^{n_2}} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{DC_1 - iC_1}{(1+y)^i} + \frac{DF_1}{(1+y)^{n_1}} - \frac{n_1F_1}{(1+y)^{n_1}} + \sum_{i=1}^{n_2} \frac{DC_2 - iC_2}{(1+y)^i} + \frac{DF_2 - n_2F_2}{(1+y)^{n_2}} = 0.$$

$$\Rightarrow \underline{\sum_{i=1}^{n} \frac{DC_1}{(1+y)^i}} - \underline{\sum_{i=1}^{n} \frac{iC_1}{(1+y)^i}} + \underline{\frac{DF_1}{(1+y)^{n_1}}} - \underline{\frac{n_1F_1}{(1+y)^{n_1}}} + \underline{\sum_{i=1}^{n_2} \frac{DC_2}{(1+y)^i}} - \underline{\sum_{i=1}^{n_2} \frac{iC_2}{(1+y)^i}} +$$
$$\underline{\frac{DF_2}{(1+y)^{n_2}}} - \underline{\frac{n_2F_2}{(1+y)^{n_2}}} = 0$$

$$\Rightarrow -\left[\sum_{i=1}^{n} \frac{iC_1}{(1+y)^i} + \frac{n_1F_1}{(1+y)^{n_1}}\right] + D\left[\sum_{i=1}^{n} \frac{C_1}{(1+y)^i} + \frac{F_1}{(1+y)^{n_1}}\right]$$
$$-\left[\sum_{i=1}^{n_2} \frac{iC_2}{(1+y)^i} + \frac{n_2F_2}{(1+y)^{n_2}}\right] + D\left[\sum_{i=1}^{n} \frac{C_2}{(1+y)^i} + \frac{F_2}{(1+y)^{n_2}}\right] = 0$$

$$\Rightarrow -P_1D_1 + DP_1 - P_2D_2 + DP_2 = 0$$

$$\Rightarrow D(P_1+P_2) = P_1D_1 + P_2D_2$$

$$\Rightarrow D = \frac{P_1D_1 + P_2D_2}{P_1+P_2}$$

$$\Rightarrow D = \frac{P_1D_1}{P_1+P_2} + \frac{P_2D_2}{P_1+P_2}$$

$$\Rightarrow \boxed{D = w_1D_1 + w_2D_2} \quad \left[\text{let } \frac{P_1}{P_1+P_2} = w_1 \ \& \ \frac{P_2}{P_1+P_2} = w_2\right]$$

$$\therefore \quad w_1D_1 + w_2D_2 = D$$

Also, from our recent supposition.

$$P_1 + P_2 = P_1 + P_2.$$

$$\Rightarrow \frac{P_1+P_2}{P_1+P_2} = 1$$

$$\Rightarrow \frac{P_1}{P_1+P_2} + \frac{P_2}{P_1+P_2} = 1$$

$$\Rightarrow \boxed{w_1 + w_2 = 1}$$