SML HW 08

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1 Problem 1.a. Solution

Given Equation in question:

$$F(X,t) = (1+t)^2 cos(X) \tag{1}$$

From HW 07: *The SDE (Stochastic Differential Equation for a Brownian Motion is given by:*

$$dX = \mu X d(t) + \sigma X dW(t) \tag{2}$$

where $f = \mu X$ and $g = \sigma X$

From Slide 19: *Itô Lemma:*

$$dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X} f + \frac{1}{2}\frac{\partial^2 F}{\partial X^2} g^2\right] dt + g\frac{\partial F}{\partial X} dW \tag{3}$$

Applying *Itô Lemma* to eq 1 from eq 3

$$dF = \left[2cos(X)(1+t) - sin(X)(1+t)^2 \mu X - \frac{1}{2} cos(X)(1+t)^2 \sigma^2 X^2\right] dt$$
$$-\sigma X sin(X)(1+t)^2 dW$$

From eq. 1

$$cos(X) = \frac{F}{(1+t)^2}$$

$$\implies X = cos^{-1}\frac{F}{(1+t)^2}$$

and

$$sin^2(X) + cos^2(X) = 1$$

So,

$$sin(X) = \frac{\sqrt{(1+t)^4 - F^2}}{(1+t)^2}$$

Using these sin(X) and cos(X) in eq. for dF

$$dF = \left[2\frac{F}{(1+t)^2}(1+t) - (1+t)^2 \mu X \left(\frac{\sqrt{(1+t)^4 - F^2}}{(1+t)^2} \right) - \frac{1}{2}(1+t)^2 \sigma^2 X^2 \left(\frac{F}{(1+t)^2} \right) \right] dt$$
$$- \sigma X (1+t)^2 \left(\frac{\sqrt{(1+t)^4 - F^2}}{(1+t)^2} \right) dW$$

Changing X in terms of F

$$dF = \left[\frac{2F}{(1+t)} - \mu cos^{-1} \left(\frac{F}{(1+t)^2} \right) \left(\sqrt{(1+t)^4 - F^2} \right) - \frac{\sigma^2}{2} F \left(cos^{-1} \left(\frac{F}{(1+t)^2} \right) \right)^2 \right] dt$$
$$- \sigma cos^{-1} \left(\frac{F}{(1+t)^2} \right) \left(\sqrt{(1+t)^4 - F^2} \right) dW$$

This final equation for dF is our required SDE.