Hw10_p1a

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$$\begin{bmatrix} b_1 & c_1 & \cdots & & 0 \\ a_2 & b_2 & c_2 & & \vdots \\ & a_2 & b_3 & & \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}$$

Perform Gaussian elimination.

$$\Rightarrow \left[\begin{array}{ccccc|c} b_1 & c_1 & \cdots & & 0 & d_1 \\ a_2 & b_2 & c_2 & & \vdots & d_2 \\ & a_2 & b_3 & & & d_3 \\ \vdots & & \ddots & \ddots & c_{n-1} & \vdots \\ 0 & \cdots & & a_n & b_n & d_n \end{array}\right]$$

$R_1 \to \frac{R_1}{b_1}$

$$\left[\begin{array}{ccccc|c} 1 & \frac{c_1}{b_1} & \cdots & & 0 & d_1/c_1 \\ a_2 & b_2 & c_2 & & \vdots & d_2 \\ & a_2 & b_3 & & & d_3 \\ \vdots & & \ddots & \ddots & c_{n-1} & \vdots \\ 0 & \cdots & & a_n & b_n & d_n \end{array}\right]$$

$R_2 \to R_2 - a_2 R_1$

$$\left[\begin{array}{ccccc|c} 1 & \frac{c_1}{b_1} & \cdots & & 0 & d_1/b_1 \\ 0 & b_2 - \frac{a_2 c_1}{b_1} & c_2 & & \vdots & d_2 - \frac{a_2 d_1}{b_1} \\ & a_2 & b_3 & & & d_3 \\ \vdots & & \ddots & \ddots & c_{n-1} & \vdots \\ 0 & \cdots & & a_n & b_n & d_n \end{array}\right]$$

$R_2 \to R_2 / \left(b_2 - \frac{a_2 c_1}{b_1}\right)$

$$\left[\begin{array}{ccccc|c} 1 & \frac{c_1}{b_1} & \cdots & & 0 & d_1/b_1 \\ 0 & 0 & c_2/\left(b_2 - \frac{a_2 c_1}{b_1}\right) & & \vdots & \frac{d_2 b_1 - a_2 d_1}{b_2 b_1 - a_2 c_1} \\ & a_2 & b_3 & & & d_3 \\ \vdots & & \ddots & \ddots & c_{n-1} & \vdots \\ 0 & \cdots & & a_n & b_n & d_n \end{array}\right] \quad \text{——— } (1)$$

from the starting question (equation), we can see.

$$\left.\begin{array}{ll} b_i x_i + c_i x_{i+1} = d_i & i = 1 \\ a_i x_{i-1} + b_i x_i + c_{i+1} = d_i & i = 2, 3, \ldots n-1 \\ a_n x_{n-1} + b_n x_n = d_n & \end{array}\right\} (A)$$

From (1), we can see that

$$x_{i+1} + x_{i+2}\left(\frac{c_{i+1}}{b_{i+1} - \frac{a_i c_i}{b_i}}\right) = \frac{d_{i+1} - \frac{a_{i+1} d_i}{b_i}}{b_{i+1} - \frac{a_i c_i}{b_i}}$$

$\Rightarrow$ Solving this, we get

$$(x_{i+1})\left(b_{i+1} b_i - a_{i+1} c_i\right) + b_i c_{i+1} x_{i+2} = d_{i+1} b_i - a_{i+1} d_i \quad \text{——— } (B)$$

Also, from (1) We get

$$\left(b_{i+2}\left(b_{i+1} b_i - c_i a_{i+1}\right) - c_{i+1} b_i a_{i+2}\right)\right) x_{i+2} + c_{i+2}\left(b_{i+1} b_i - c_i a_{i+1}\right) x_{i+3} = \left(d_{i+2}\left(b_{i+1} b_i - c_i a_{i+1}\right) - \left(d_{i+1} b_i - d_i a_{i+1}\right) a_{i+2}\right. \quad \text{——— } (C)$$

We can continue using Gaussian elimination for n-1 steps to eliminate all a_i, after that divide each row by its pivot element for n steps and then use recursion for n-1 steps.

$\therefore$ After these $3n-2$ steps. We will get:

$$c_i' = \begin{cases} \frac{c_i}{b_i} & \text{for } i = 1 \\ \frac{c_i}{b_i - a_i c_{i-1}'} & \text{for } i = 2, 3, \ldots n-1 \end{cases}$$

$$d_i' = \begin{cases} \frac{d_i}{b_i} & \text{for } i = 1 \\ \frac{d_i - a_i d_{i-1}'}{b_i - a_i c_{i-1}'} & \text{for } i = 2, 3, \ldots, n \end{cases}$$

and,

$$x_i = \begin{cases} d_i' & \text{for } i = 1 \\ d_i' - c_i' x_{i+1} & \text{for } i = 2, 3, \ldots n \end{cases}$$

Ans