Lucas-Kanade Method

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Columbia University

Topic: Motion and Optical Flow, Module: Reconstruction II

First Principles of Computer Vision

Lucas-Kanade Solution

Assumption: For each pixel, assume Motion Field, and hence Optical Flow (u, v), is constant within a small neighborhood W.

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Let the size of window W be $n \times n$

In matrix form:

$$\begin{bmatrix} I_x(1,1) & I_y(1,1) \\ I_x(k,l) & I_y(k,l) \\ \vdots & \vdots \\ I_x(n,n) & I_y(n,n) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} I_t(1,1) \\ I_t(k,l) \\ \vdots \\ I_t(n,n) \end{bmatrix}$$

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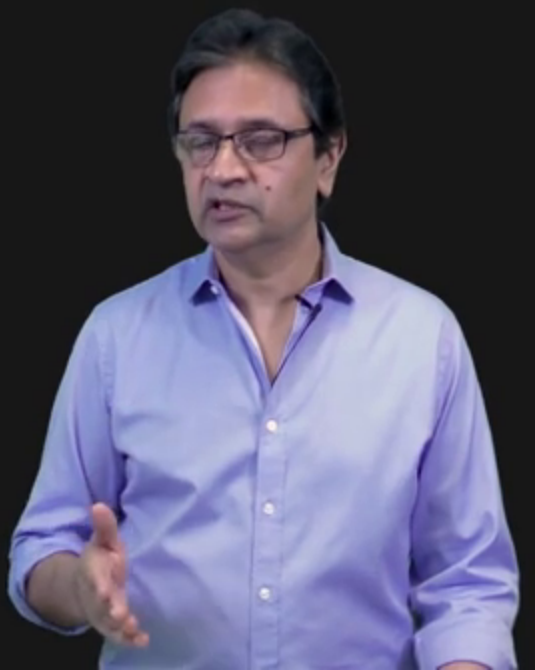

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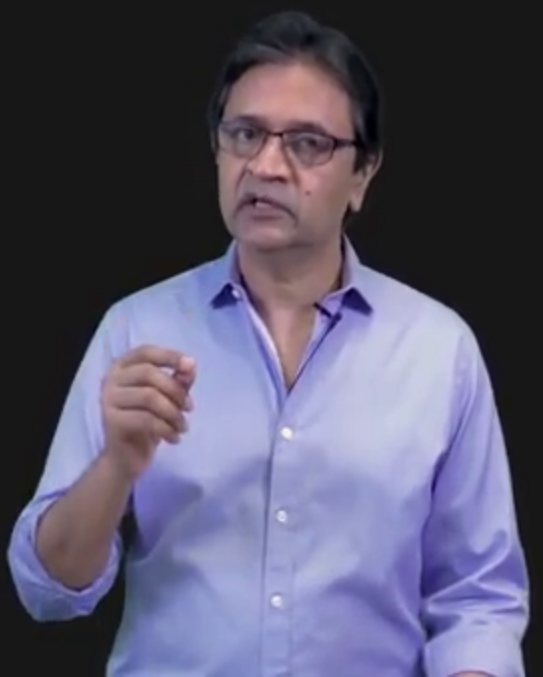

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A (Known) $n^2 \times 2$ | $\mathbf{u}$ (Unknown) 2×1 | B (Known) $n^2 \times 1$

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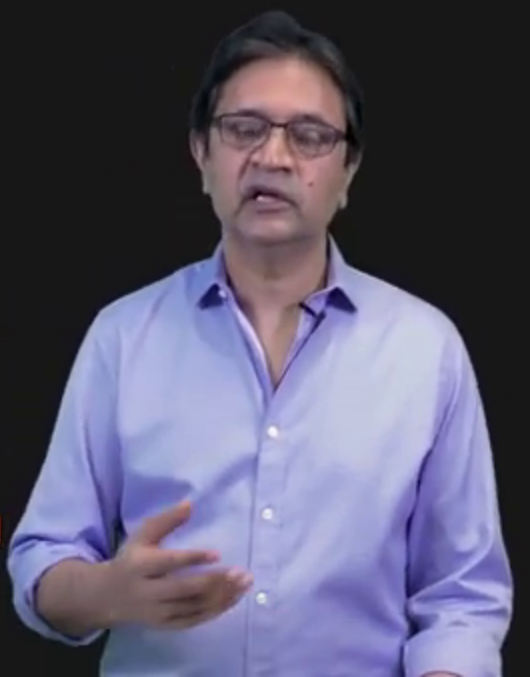
n^2 Equations, 2 Unknowns: Find Least Squares Solution

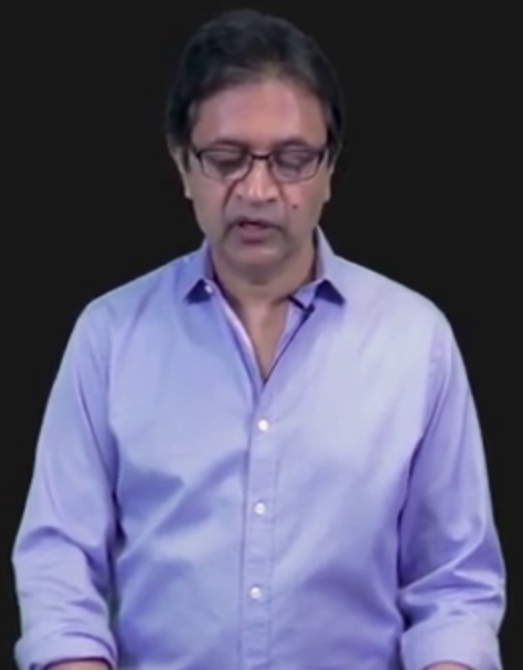


Least Squares Solution

Solve linear system: $A\mathbf{u} = B$

$$A^T A\mathbf{u} = A^T B$$

(Least-Squares using Pseudo-Inverse)



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$A^T A$	$\mathbf{u}$	$A^T B$
(Known)	(Unknown)	(Known)
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Indices (k, l) not written for simplicity

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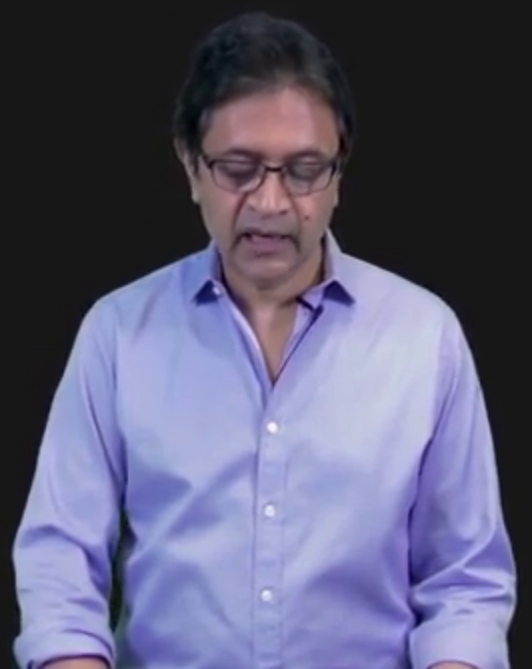

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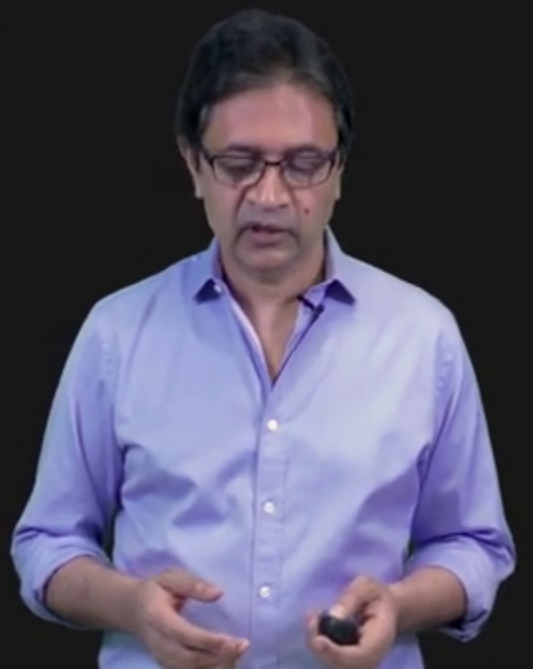

When Does Optical Flow Estimation Work?

$$A\mathbf{u} = B \qquad A^T A\mathbf{u} = A^T B$$

- $A^T A$ must be invertible. That is $\det(A^T A) \neq 0$
- $A^T A$ must be well-conditioned.

  If λ_1 and λ_2 are eigen values of $A^T A$, then

  $\lambda_1 > \epsilon$ and $\lambda_2 > \epsilon$

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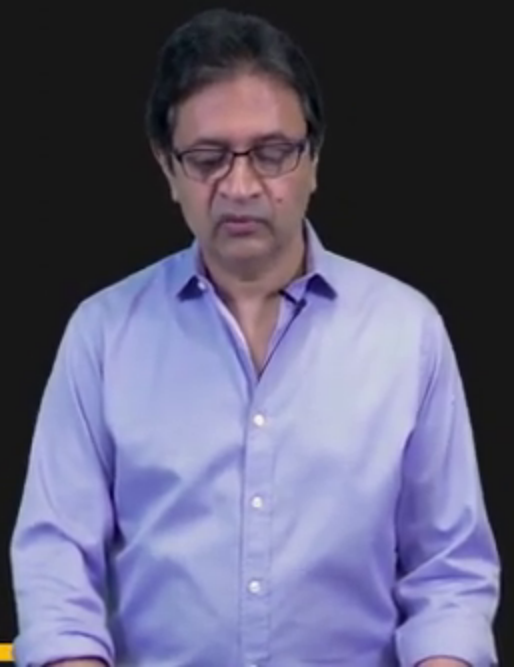

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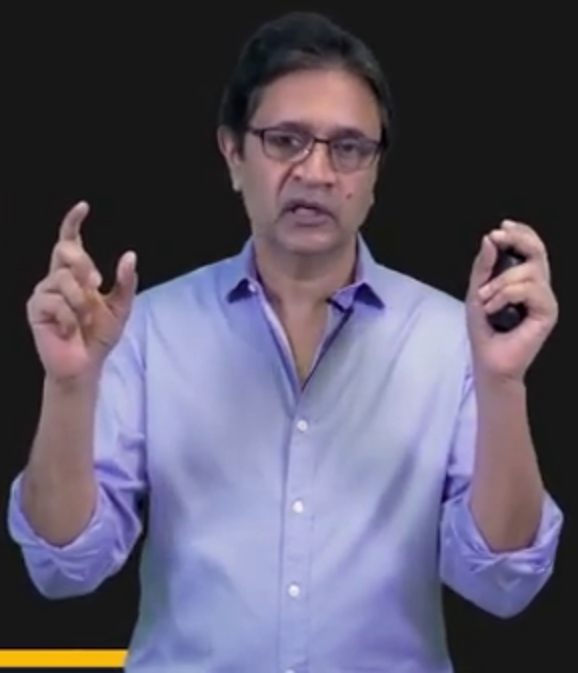

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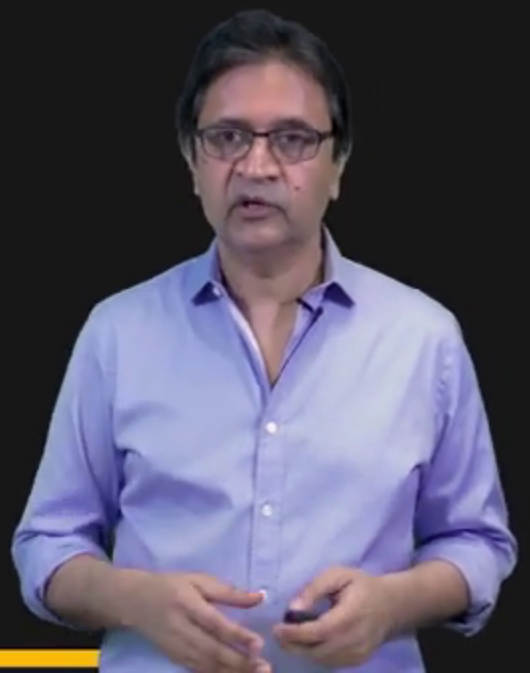

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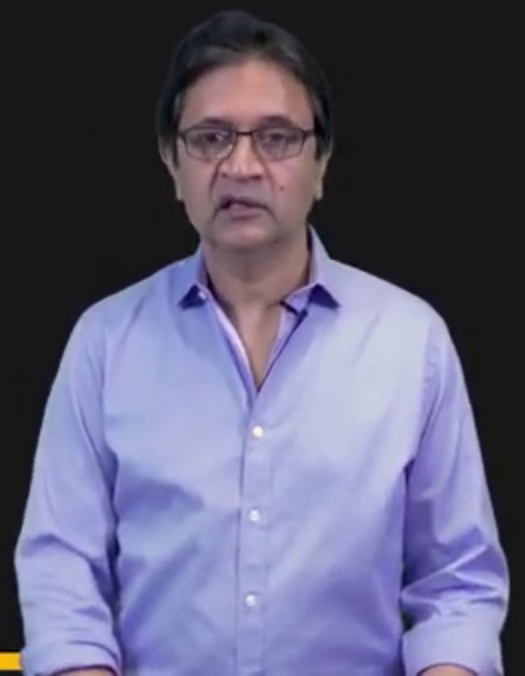

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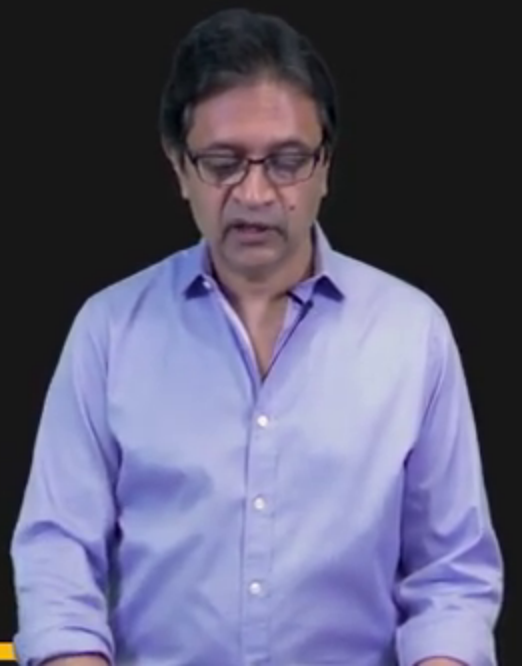

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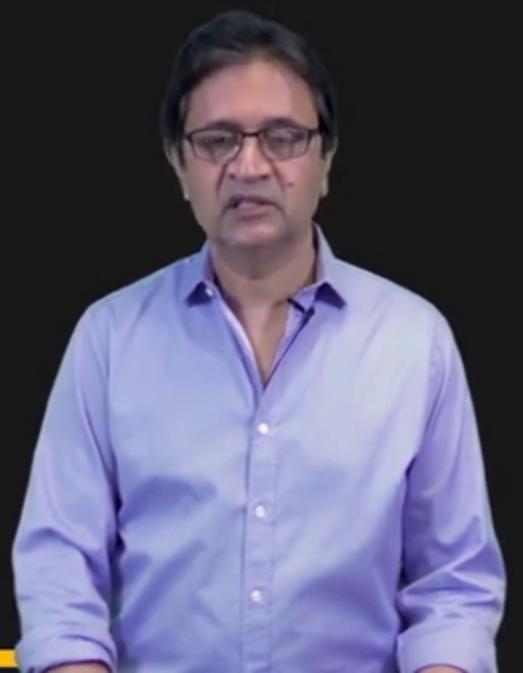

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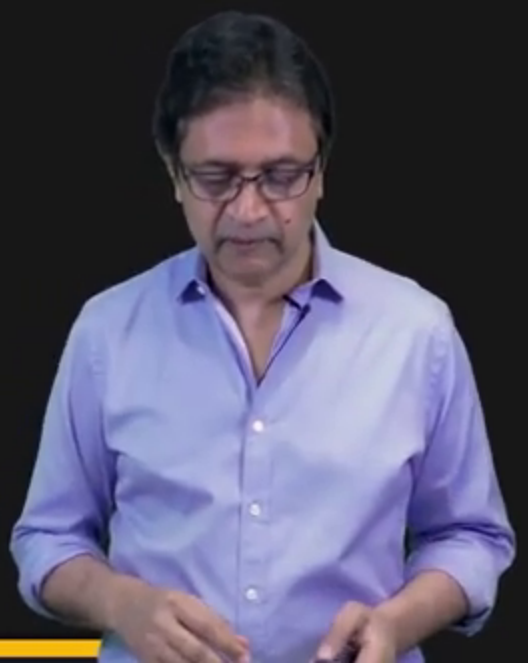
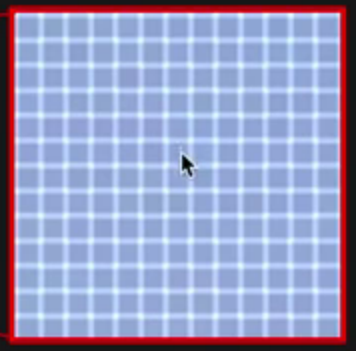
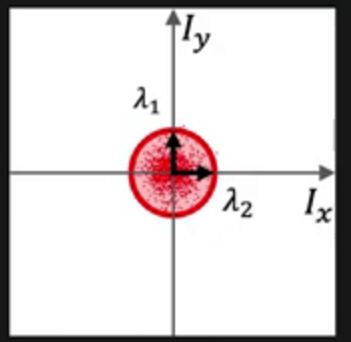


Smooth Regions (Bad)

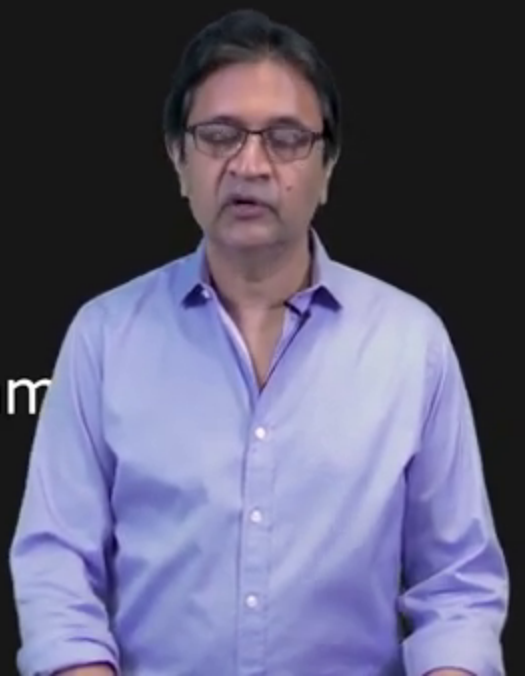




$\lambda_1 \sim \lambda_2$
Both are Small

Equations for all pixels in window are more or less the sam

Cannot reliably compute flow!



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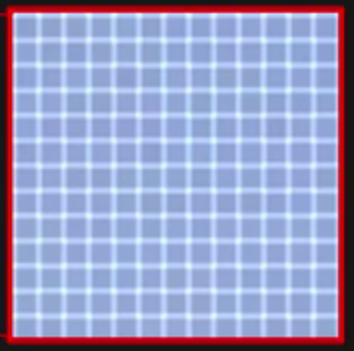

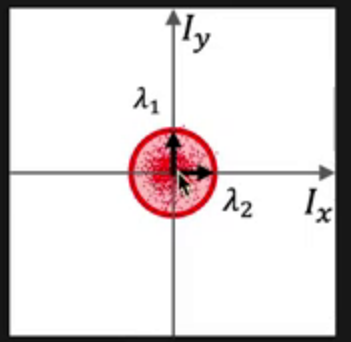

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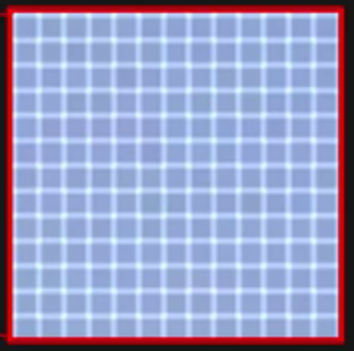

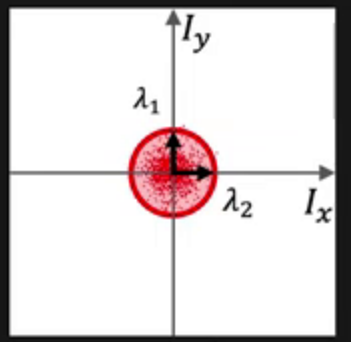

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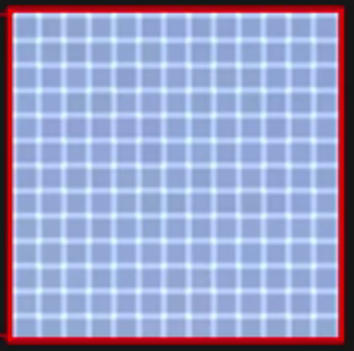

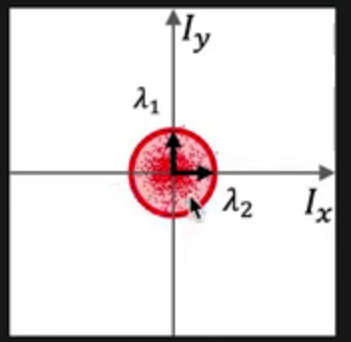

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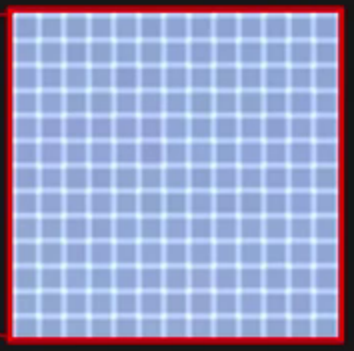

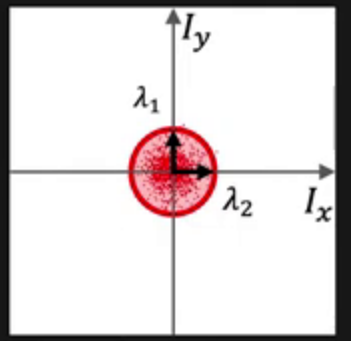

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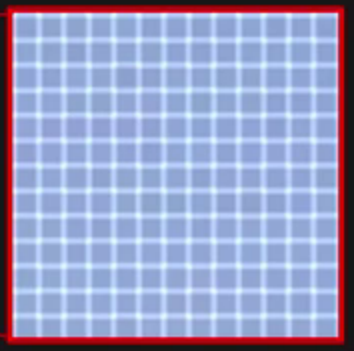

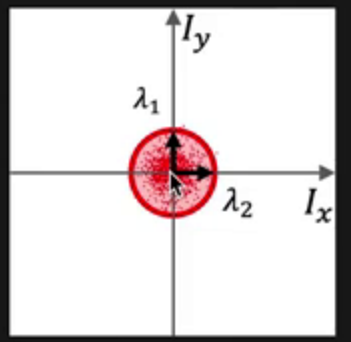

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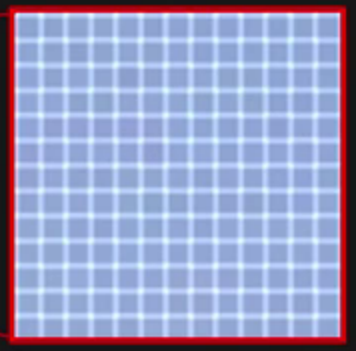
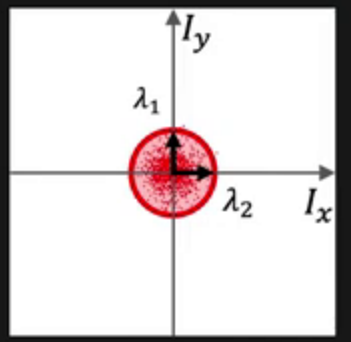


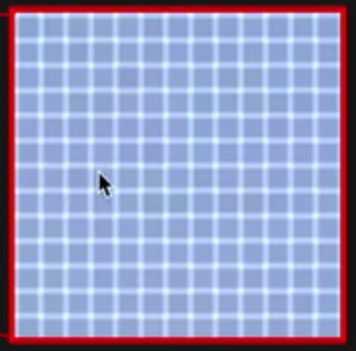


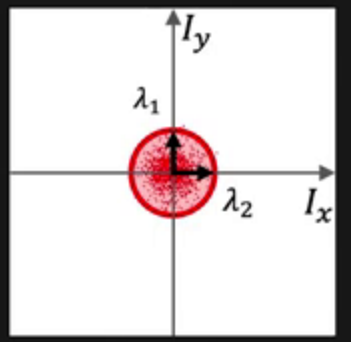
$\lambda_1 \sim \lambda_2$
Both are Small

Equations for all pixels in window are more or less the sam

Cannot reliably compute flow!

Smooth Regions (Bad)





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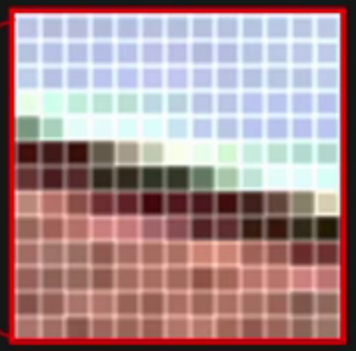
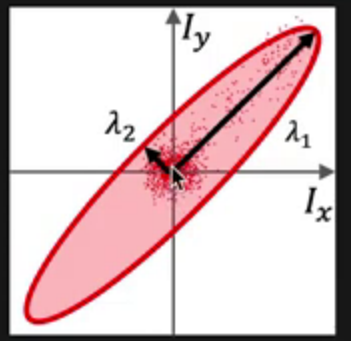
Edges (Bad)





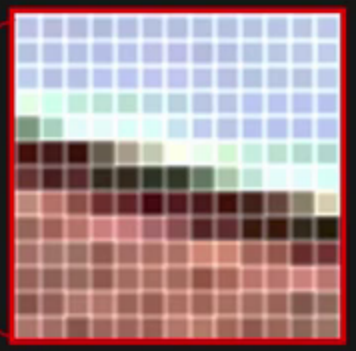
$\lambda_1 \ggg \lambda_2$

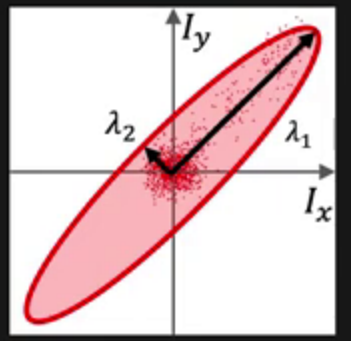
Badly conditioned. Prominent gradient in one direction.

Cannot reliably compute flow!

Same as Aperture Problem.

Edges (Bad)





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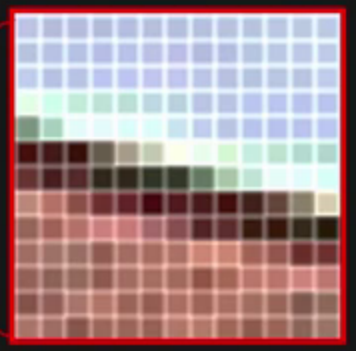

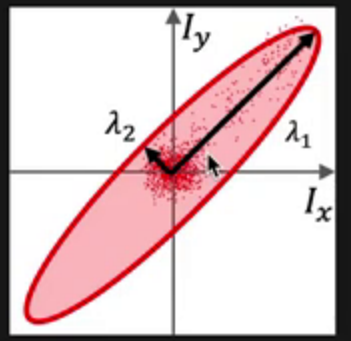

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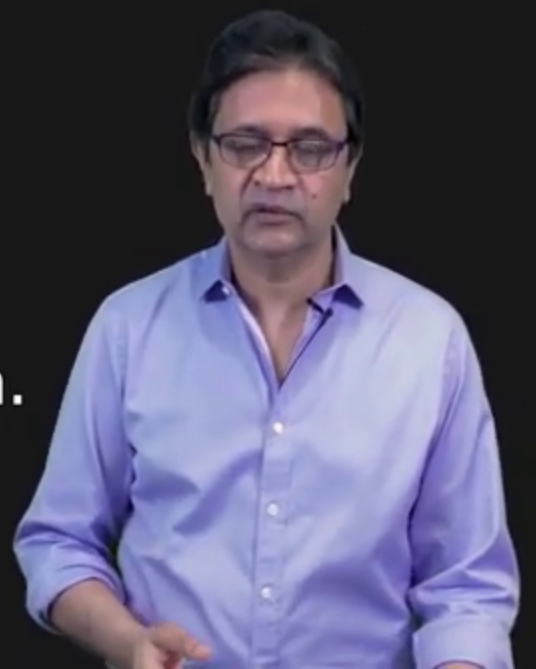
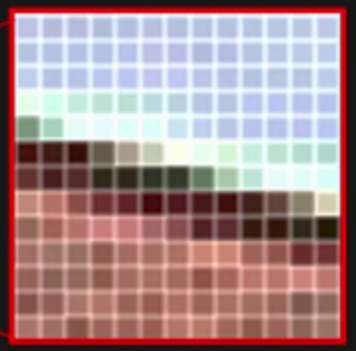
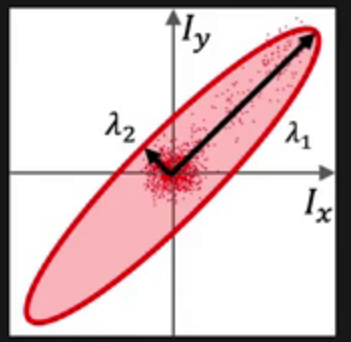


Edges (Bad)





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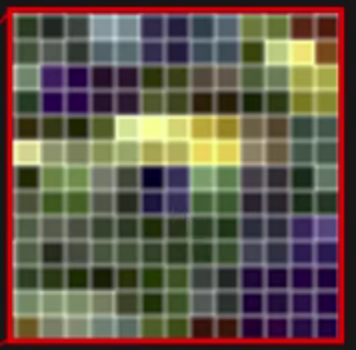
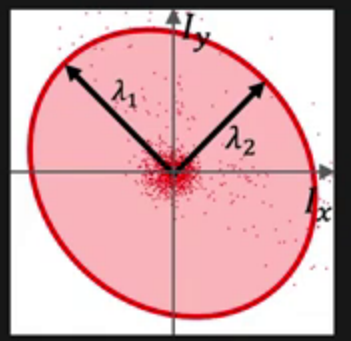
Same as Aperture Problem.

Textured Regions (Good)



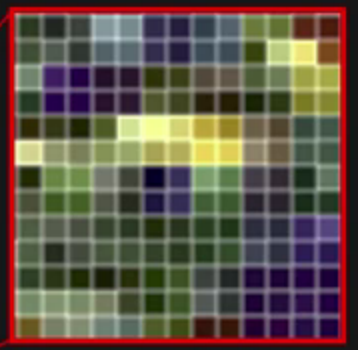


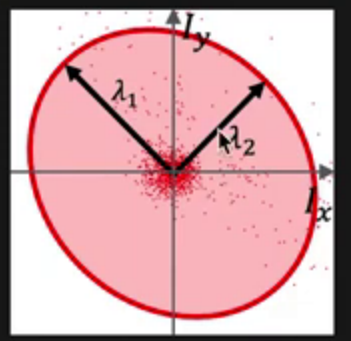
$\lambda_1 \sim \lambda_2$
Both are Large

Well conditioned. Large and diverse gradient magnitudes.

Can reliably compute optical flow.

Textured Regions (Good)





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