Optical Flow Constraint Equation

Shree K. Nayar

Columbia University

Topic: Motion and Optical Flow, Module: Reconstruction II

First Principles of Computer Vision

Optical Flow

t

$t + \delta t$

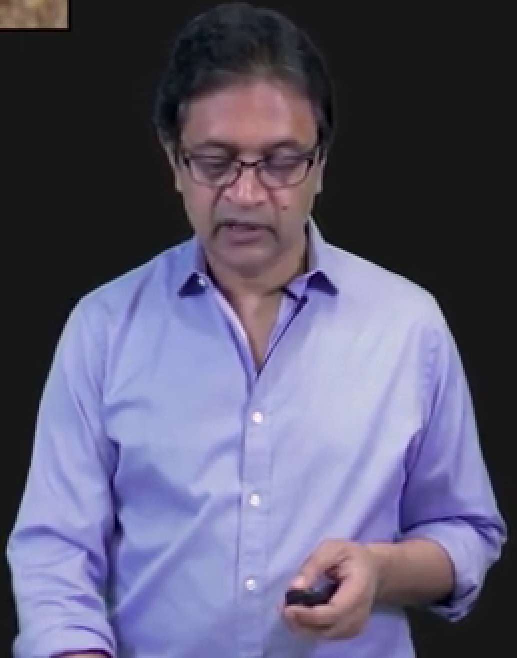
[Horn 1981]

Optical Flow

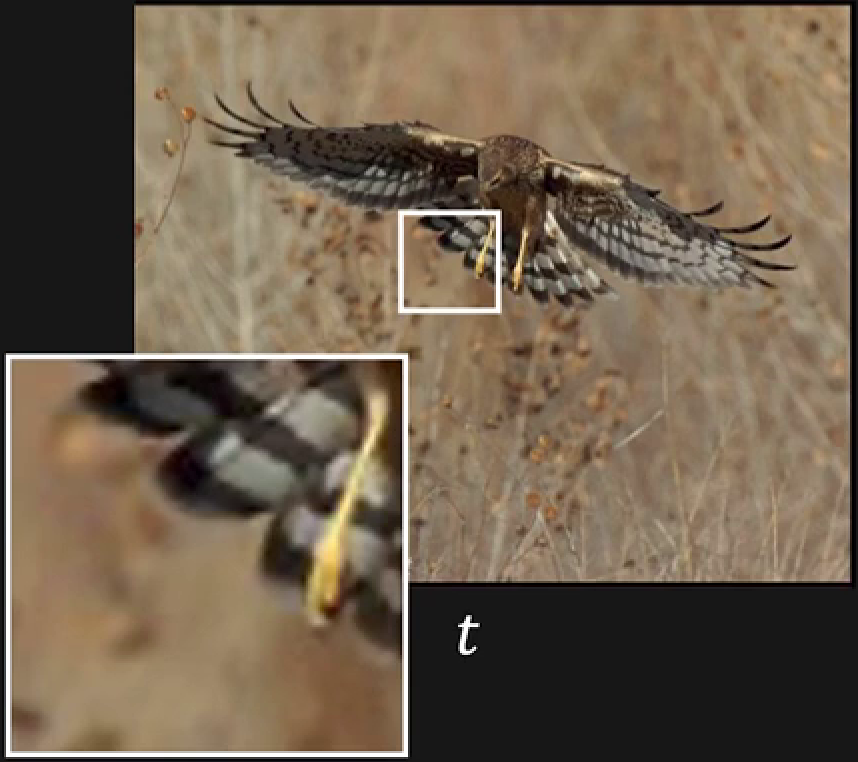
t

$t + \delta t$

[Horn 1981]

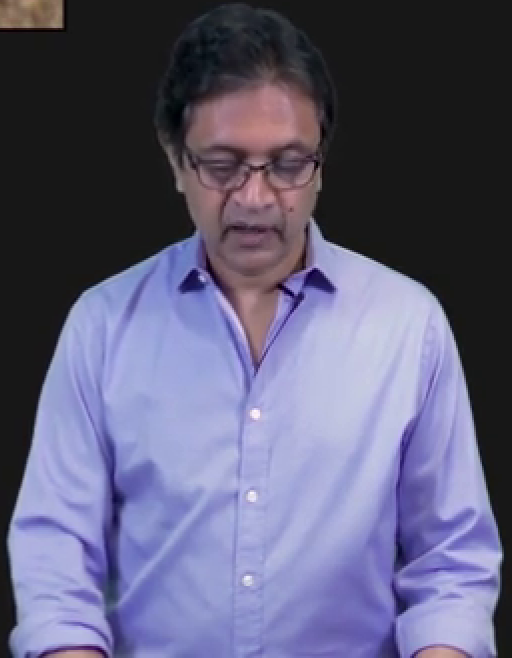


Optical Flow



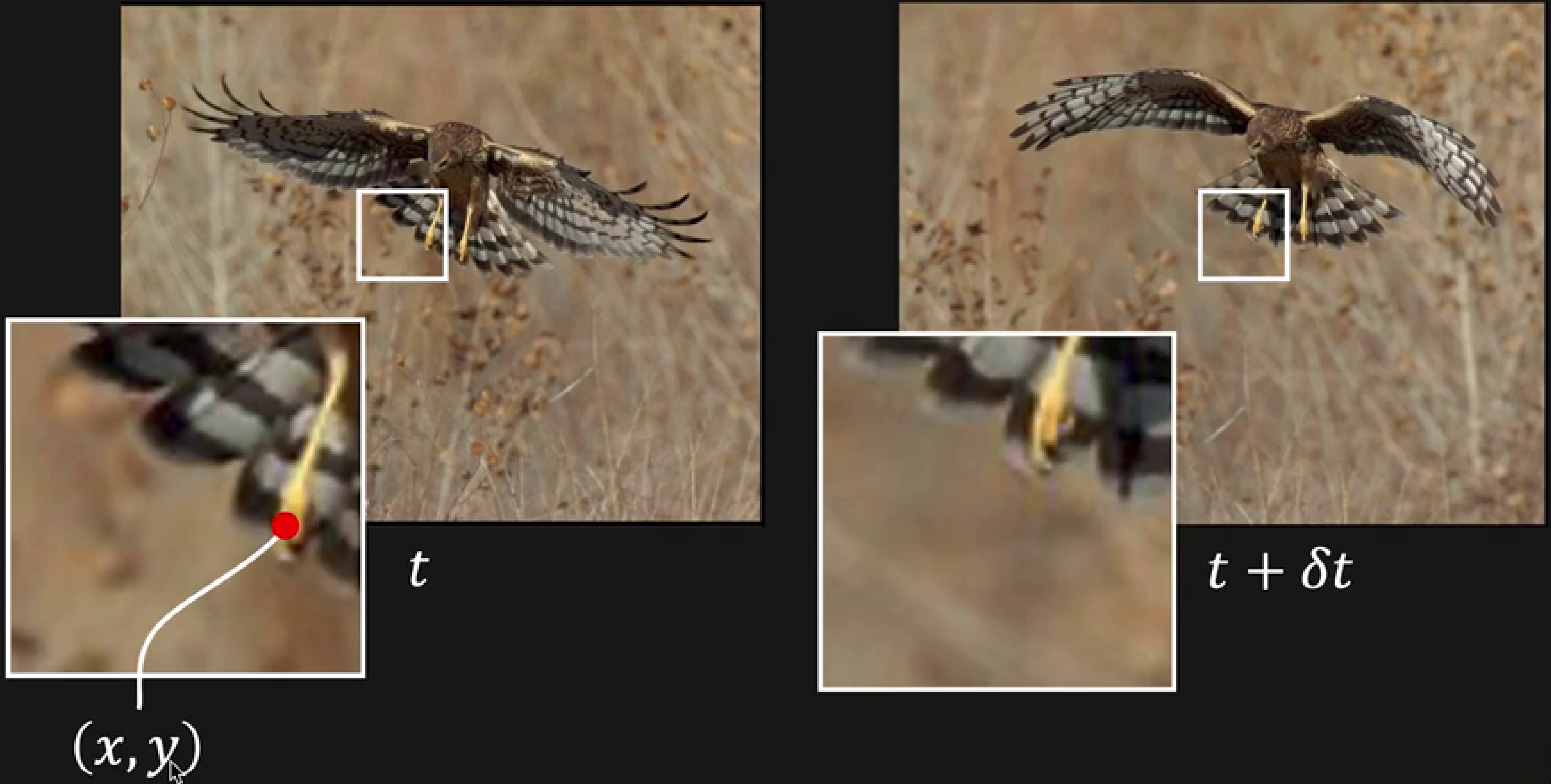
t

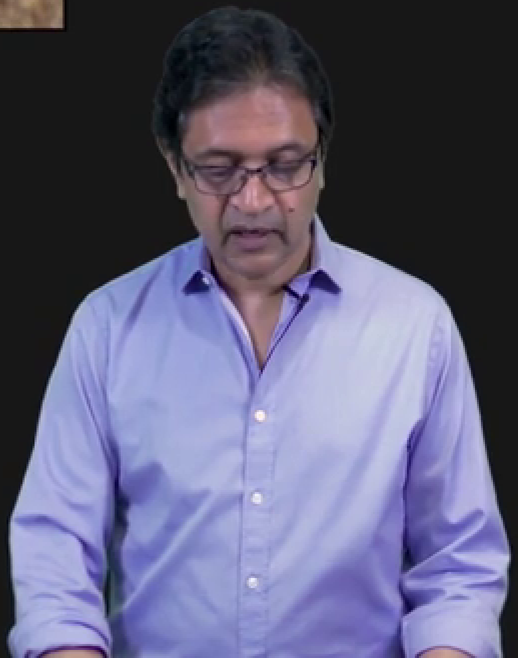
$t + \delta t$



[Horn 1981]

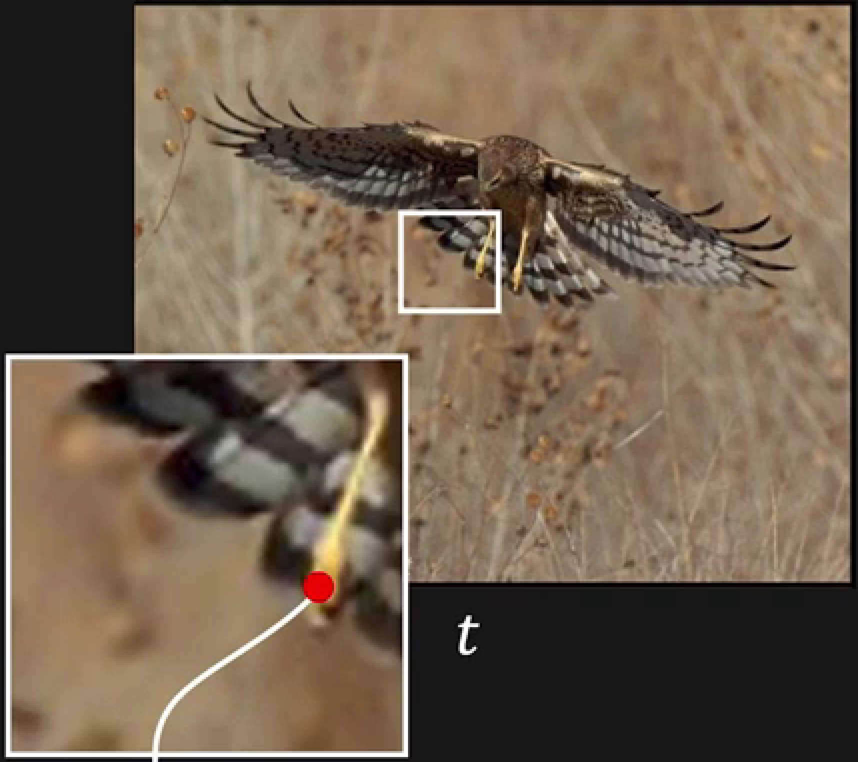
Optical Flow





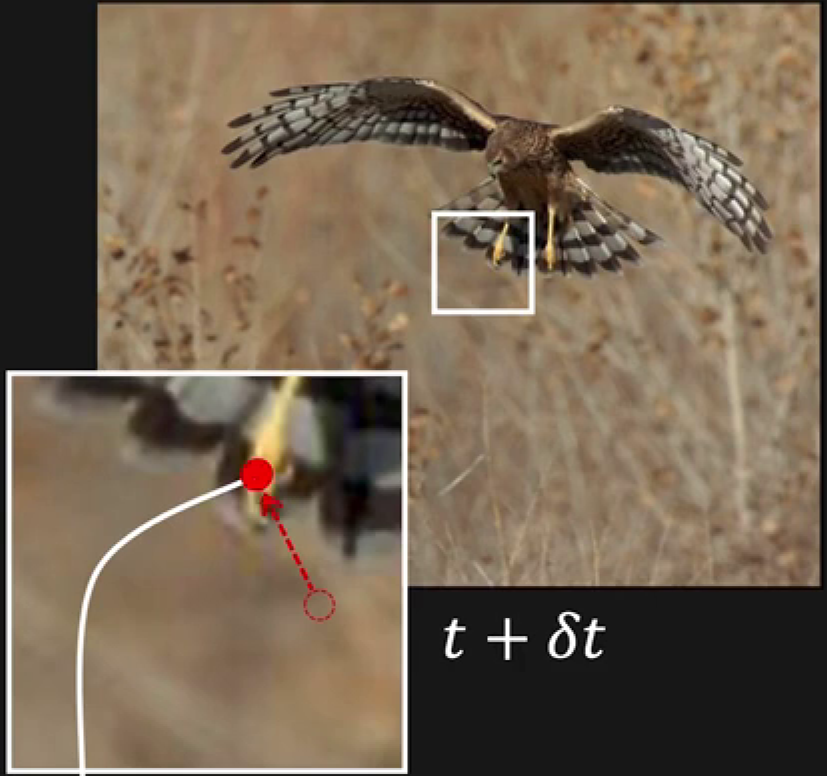
[Horn 1981]

Optical Flow



t

(x, y)



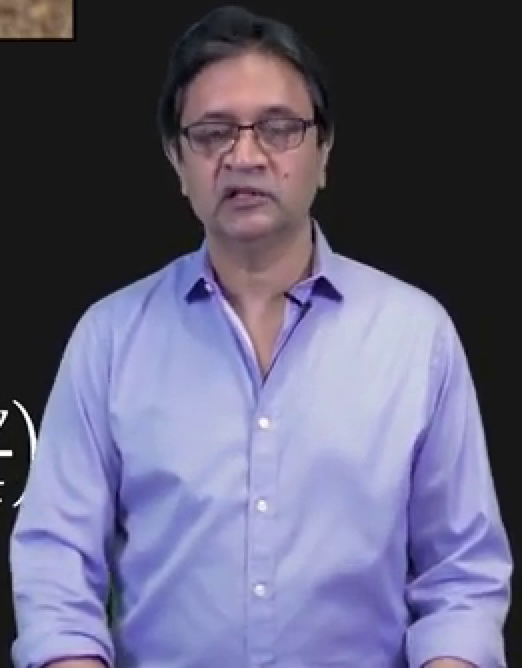
$t + \delta t$

$(x + \delta x, y + \delta y)$

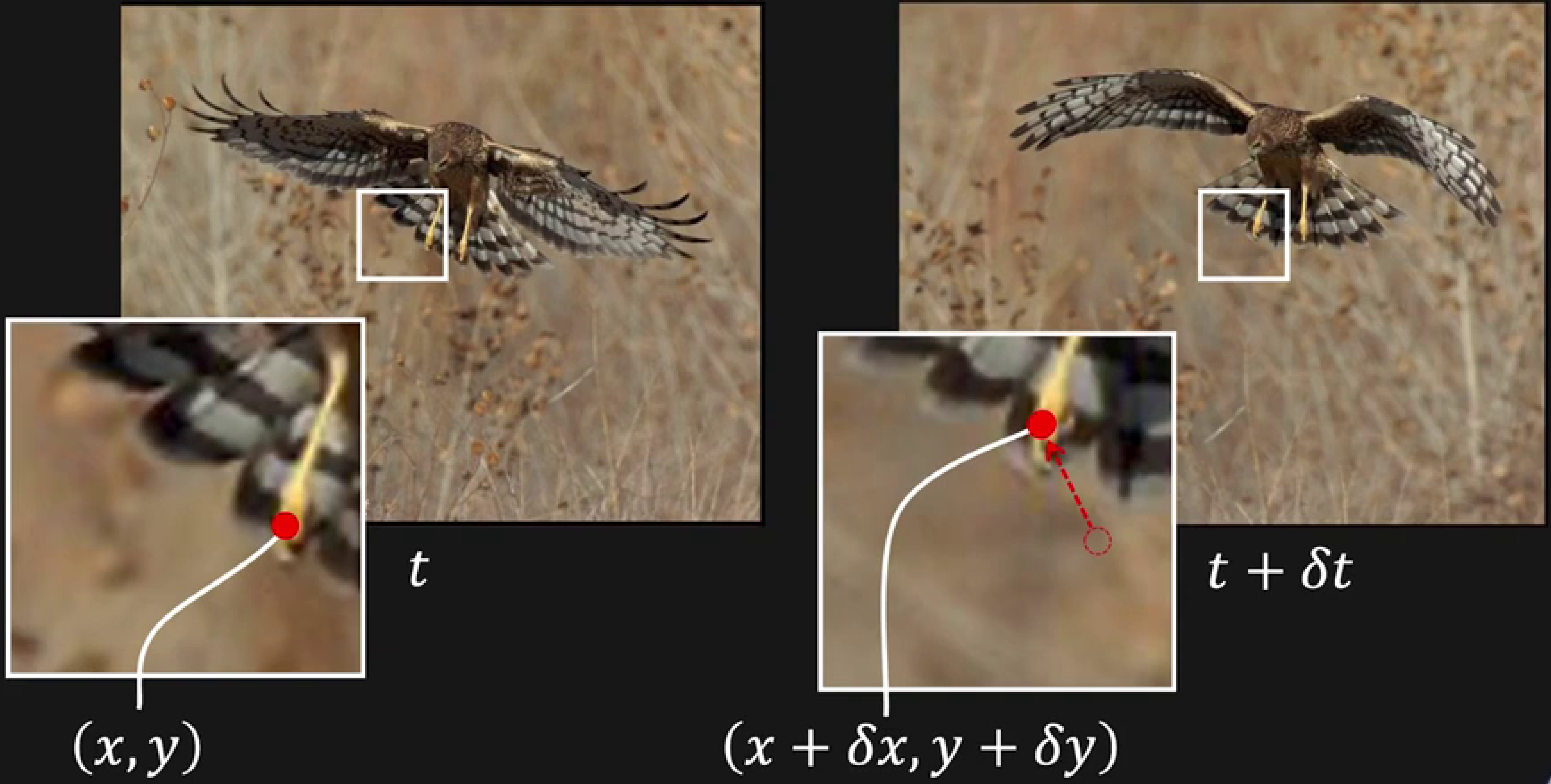
Displacement: $(\delta x, \delta y)$

Optical Flow: $(u, v) = \left(\frac{\delta x}{\delta t}, \frac{\delta y}{\delta t}\right)$

[Horn 1981]



Optical Flow

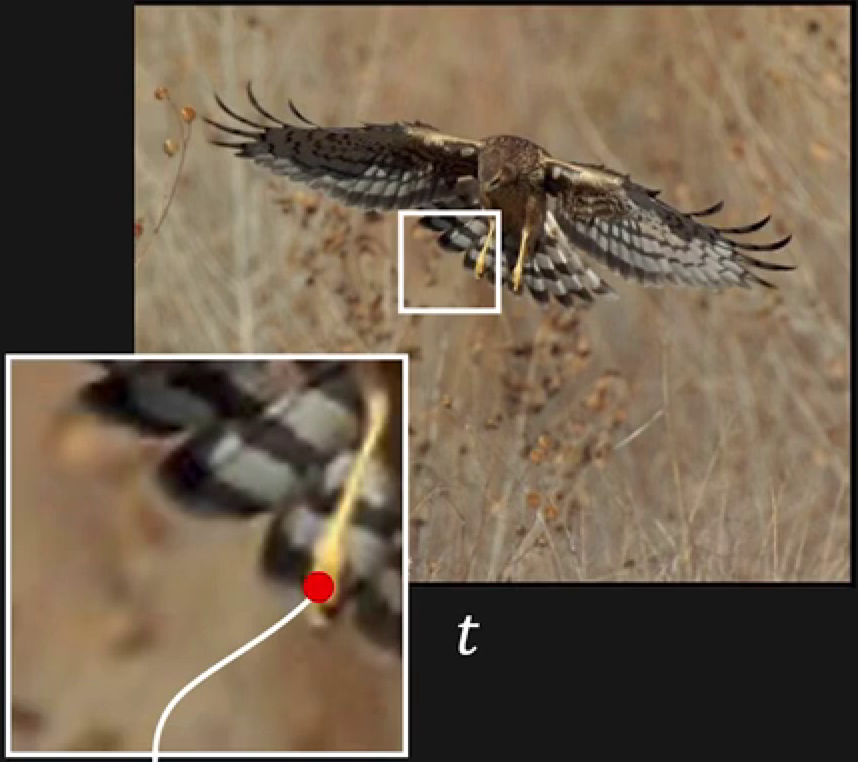


Displacement: $(\delta x, \delta y)$

Optical Flow: $(u, v) = \left(\frac{\delta x}{\delta t}, \frac{\delta y}{\delta t}\right)$

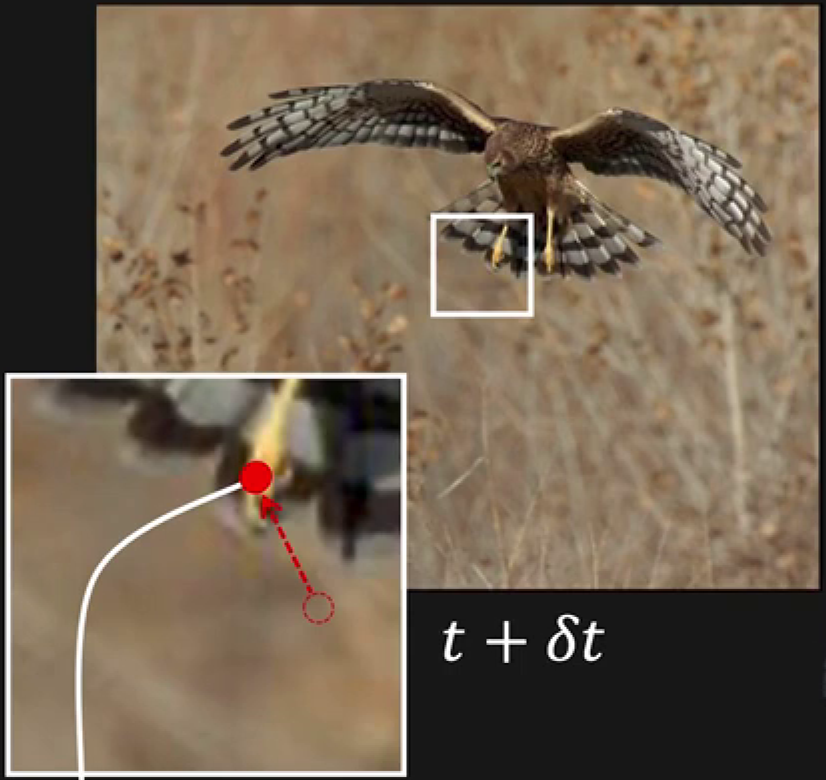
[Horn 1981]

Optical Flow



t

(x, y)



$t + \delta t$

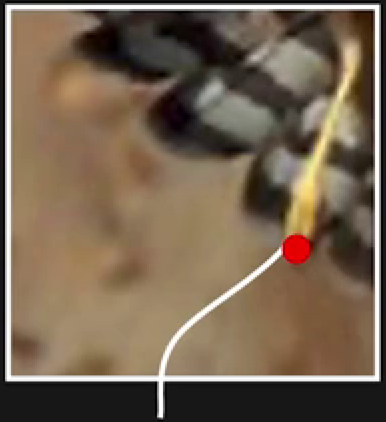
$(x + \delta x, y + \delta y)$

Displacement: $(\delta x, \delta y)$

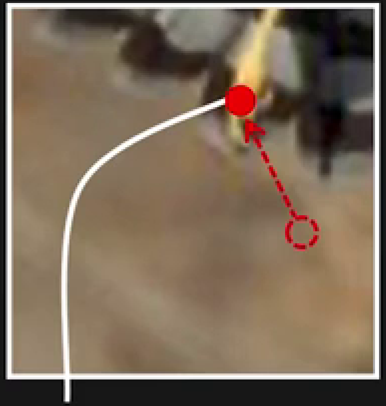
Optical Flow: $(u, v) = \left(\frac{\delta x}{\delta t}, \ldots\right.$

[Horn 1981]

Optical Flow Constraint Equation



$I(x, y, t)$

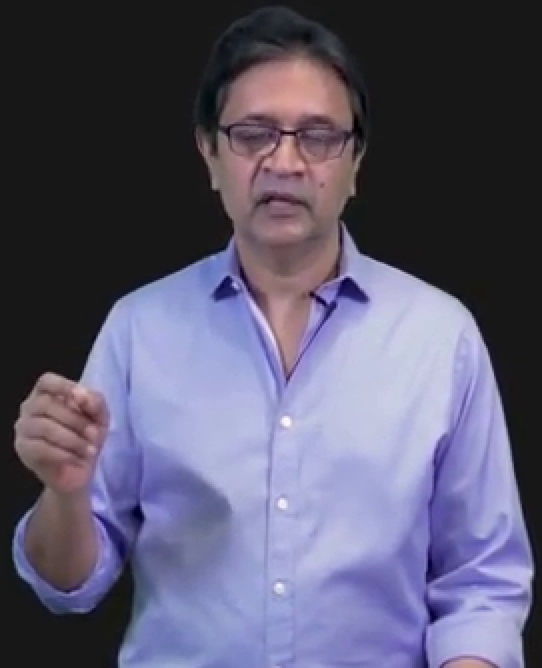


$I(x + \delta x, y + \delta y, t + \delta t)$

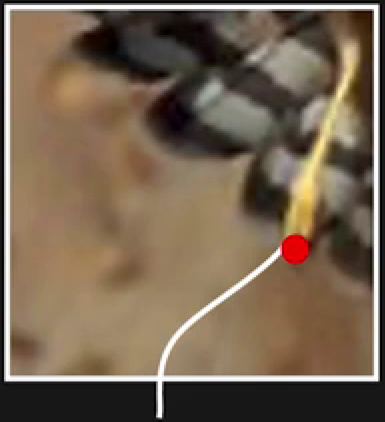
Assumption #1:

Brightness of image point remains constant over time

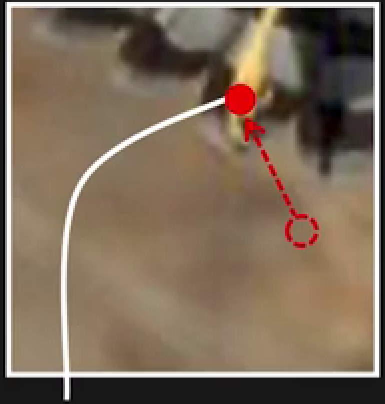
$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t)}$$



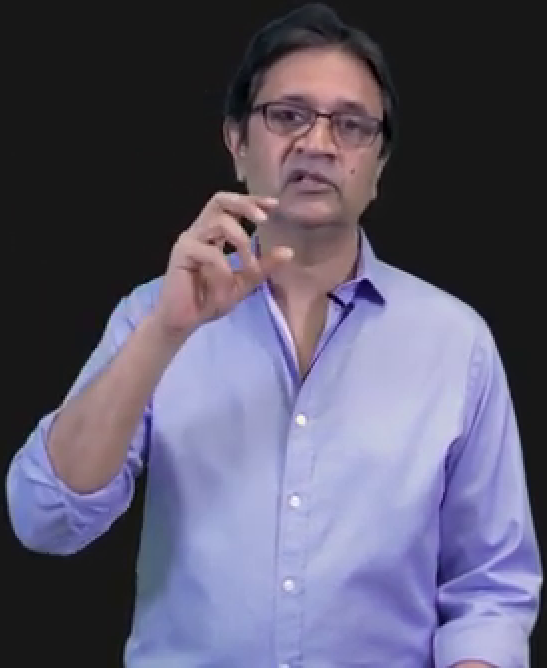
Optical Flow Constraint Equation



$I(x, y, t)$



$I(x + \delta x, y + \delta y, t + \delta t)$

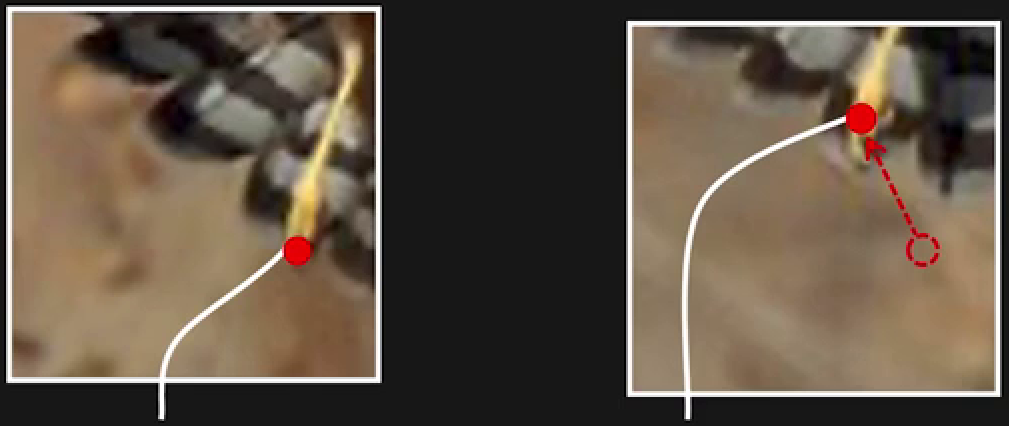


Assumption #1:

Brightness of image point remains constant over time

$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t)}$$

Optical Flow Constraint Equation

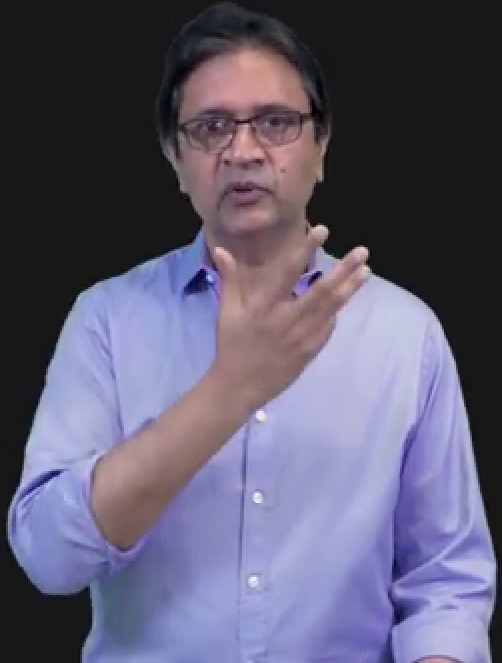


$I(x, y, t)$ $\qquad$ $I(x + \delta x, y + \delta y, t + \delta t)$

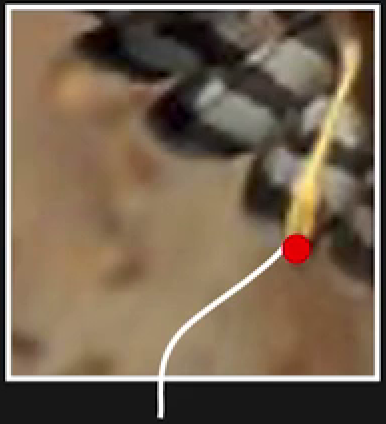
Assumption #1:

Brightness of image point remains constant over time

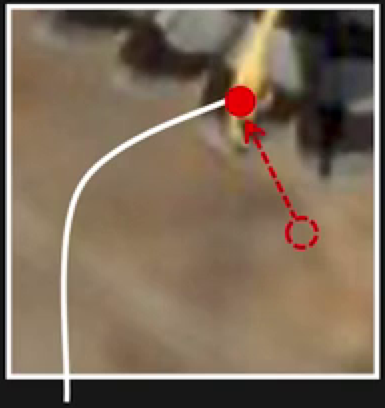
$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t)}$$



Optical Flow Constraint Equation



$I(x, y, t)$

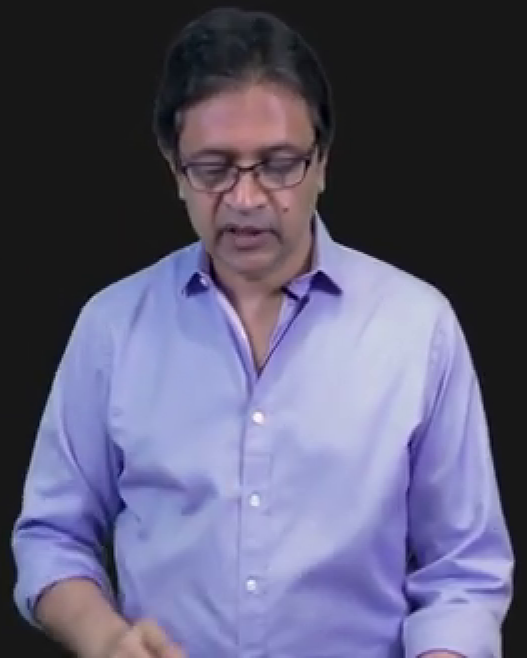


$I(x + \delta x, y + \delta y, t + \delta t)$

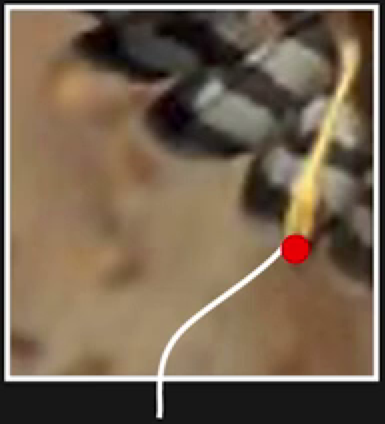
Assumption #1:

Brightness of image point remains constant over time

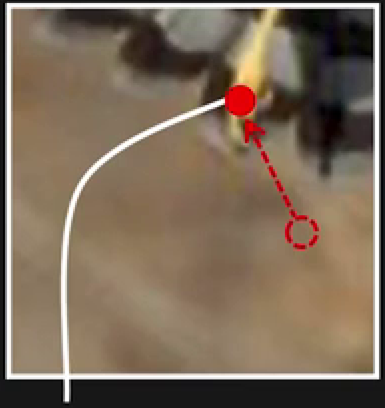
$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t)}$$



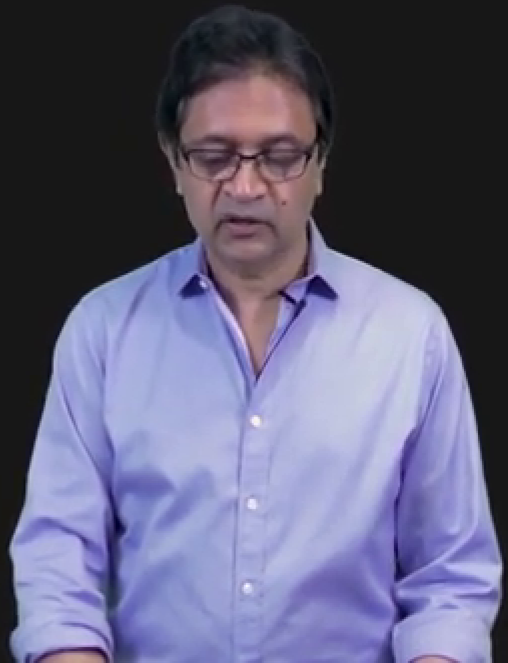
Optical Flow Constraint Equation



$I(x, y, t)$



$I(x + \delta x, y + \delta y, t + \delta t)$

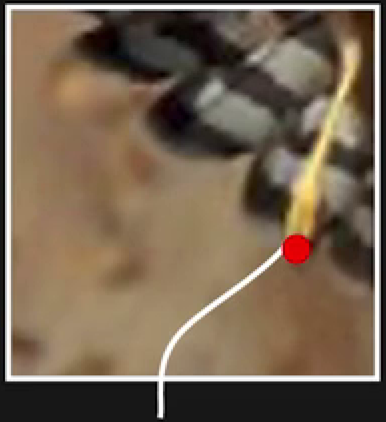


Assumption #1:

Brightness of image point remains constant over time

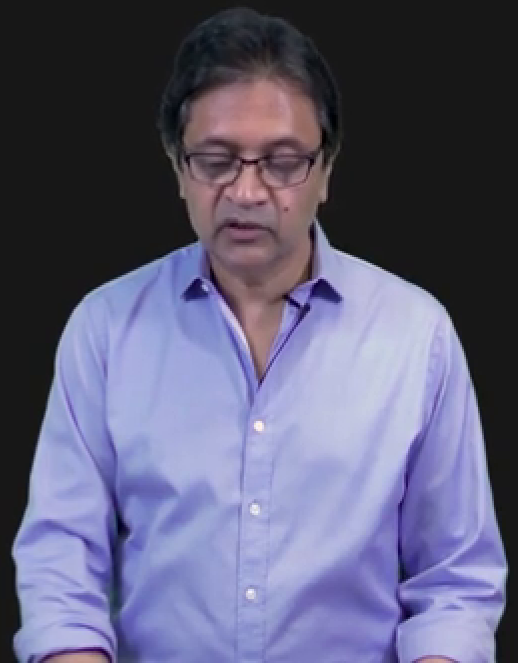
$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t)}$$

Optical Flow Constraint Equation



$I(x, y, t)$

$I(x + \delta x, y + \delta y, t + \delta t)$

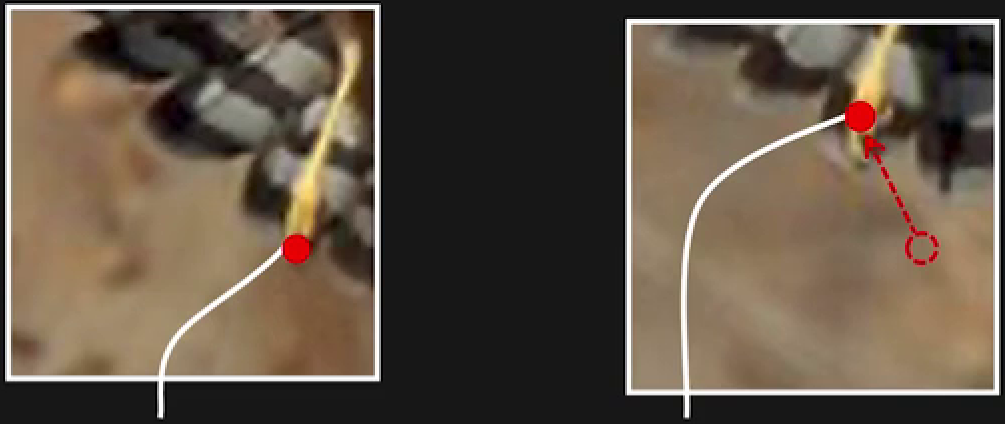


Assumption #1:

Brightness of image point remains constant over time

$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t)}$$

Optical Flow Constraint Equation

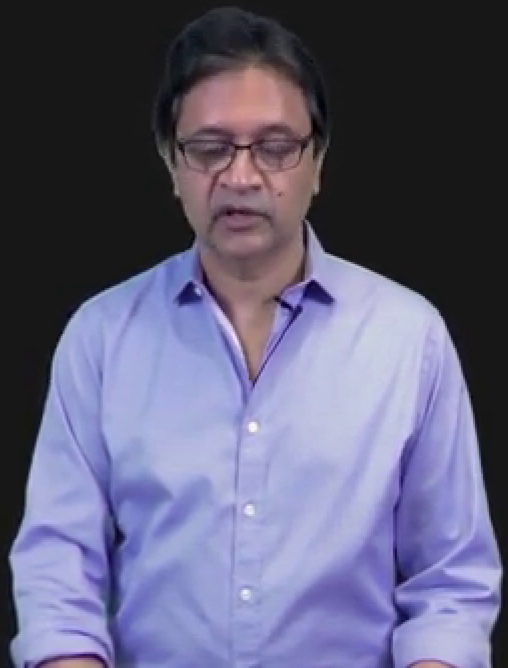


$I(x, y, t)$ $\quad$ $I(x + \delta x, y + \delta y, t + \delta t)$

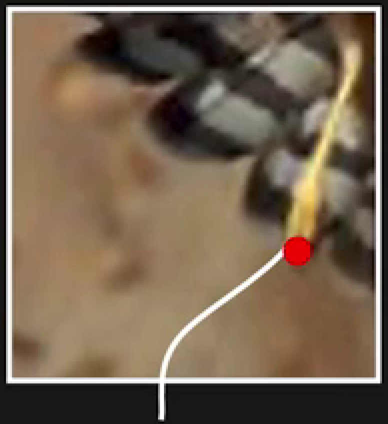
Assumption #1:

Brightness of image point remains constant over time

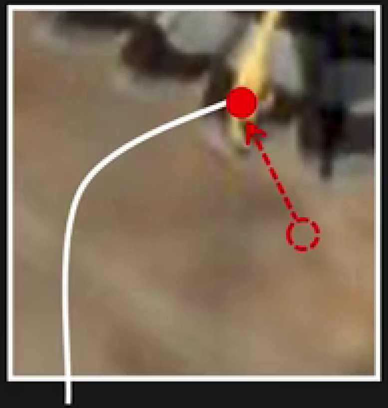
$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t)}$$



Optical Flow Constraint Equation



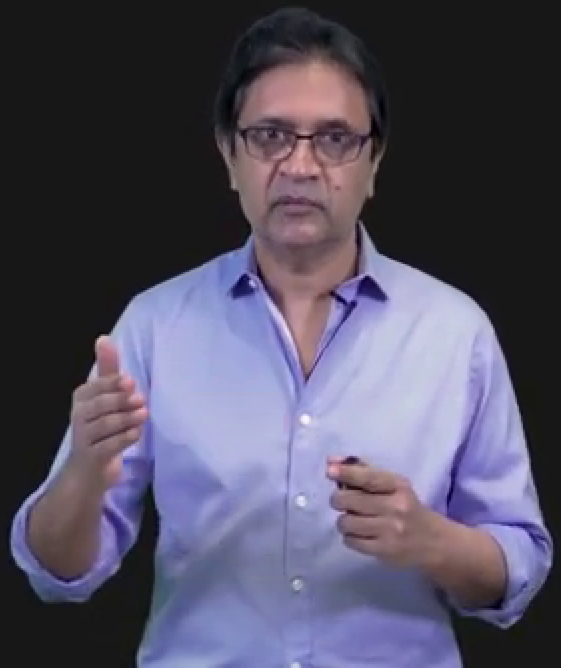
$I(x, y, t)$



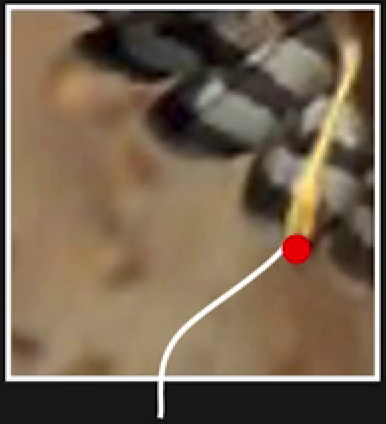
$I(x + \delta x, y + \delta y, t + \delta t)$

Assumption #2:

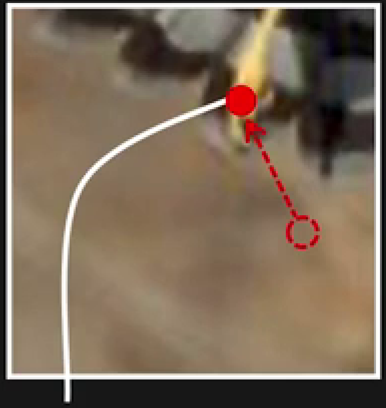
Displacement $(\delta x, \delta y)$ and time step δt are small



Optical Flow Constraint Equation



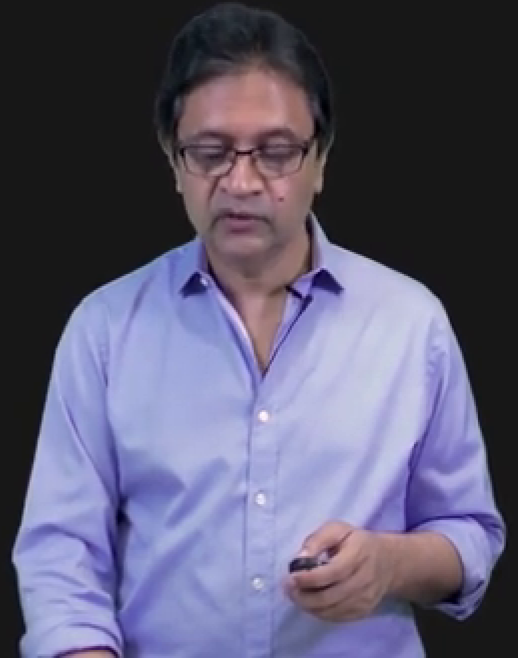
$I(x, y, t)$



$I(x + \delta x, y + \delta y, t + \delta t)$

Assumption #2:

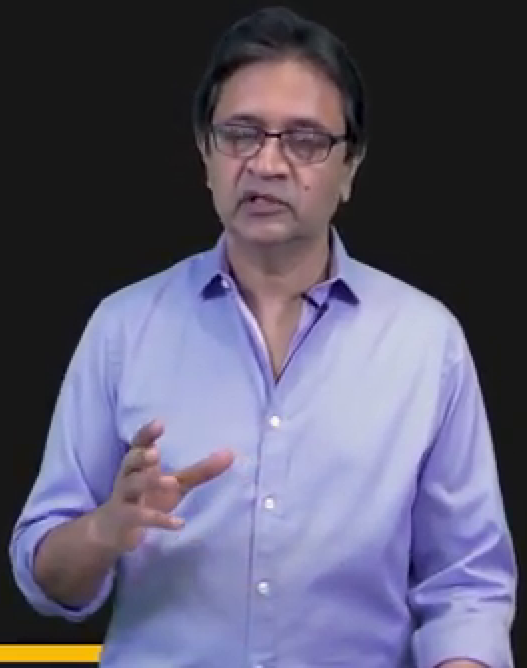
Displacement $(\delta x, \delta y)$ and time step δt are small



Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

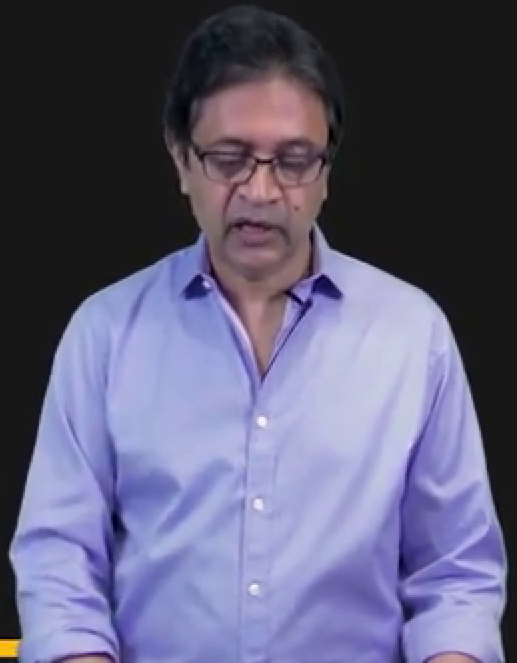
$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$



Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

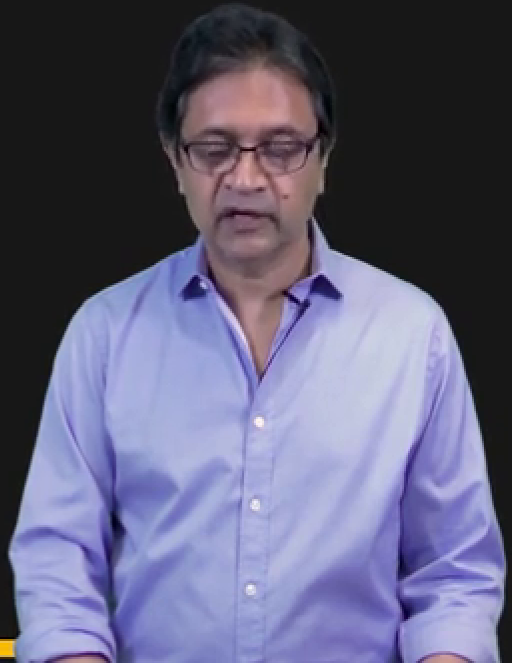
$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$



Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

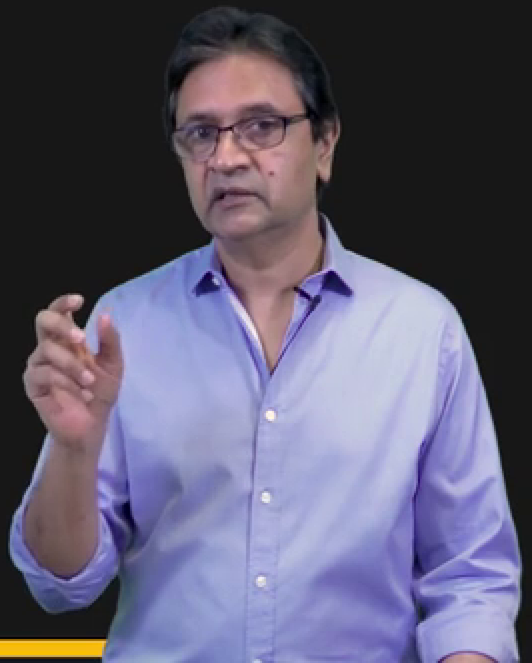
$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$



Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$



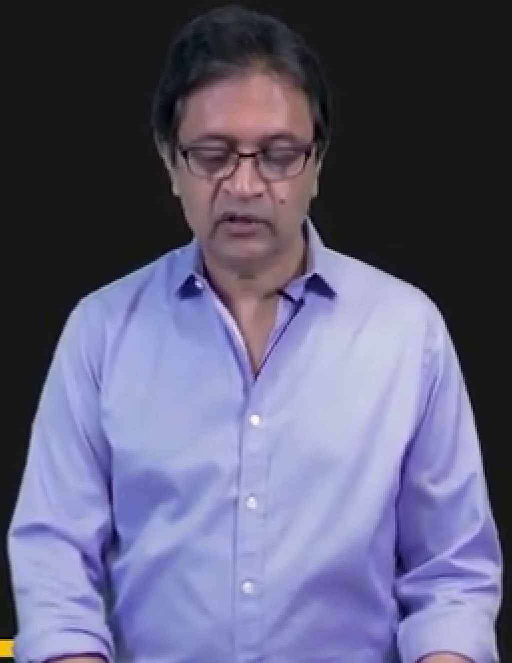
Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \longrightarrow \text{Almost Zero}$$



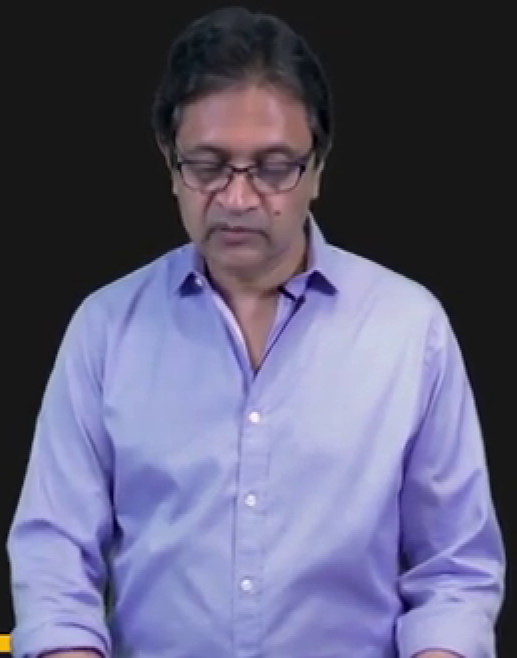
Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$



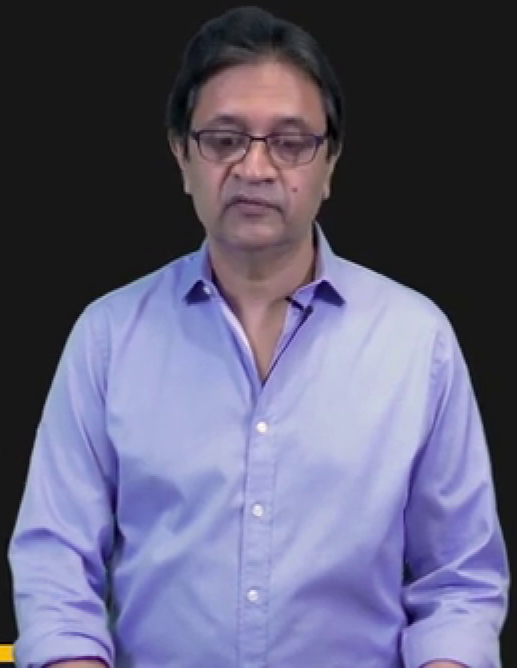
Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$



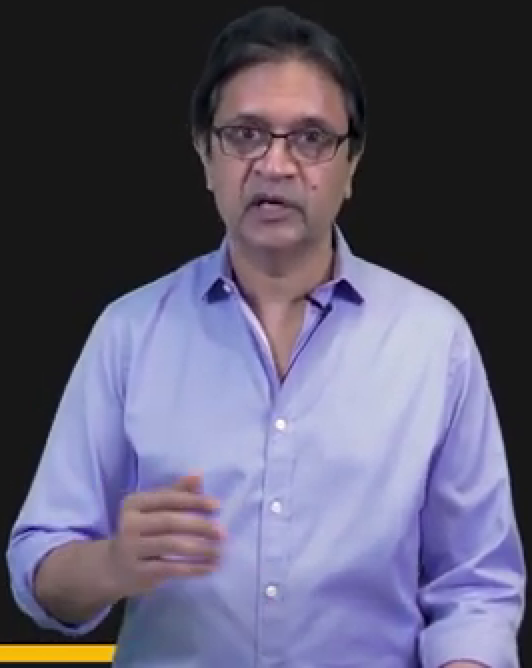
Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$



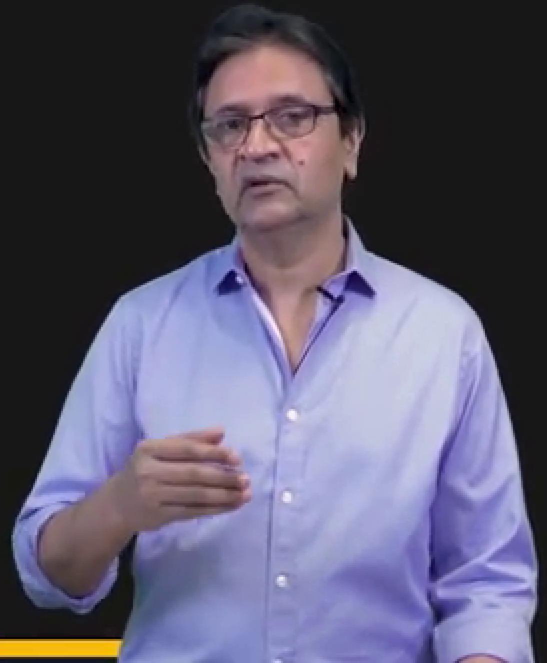
Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$



Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

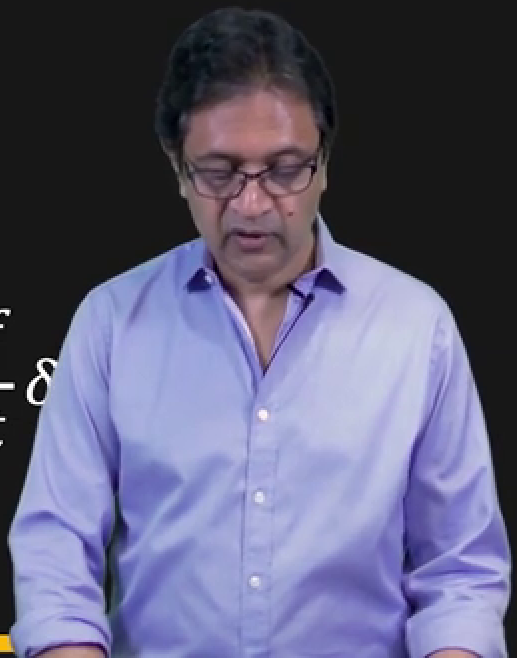
$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$

For a function of three variables with small $\delta x, \delta y, \delta t$:

$$f(x+\delta x, y+\delta y, t+\delta t) \approx f(x,y,t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial t}\delta$$



Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

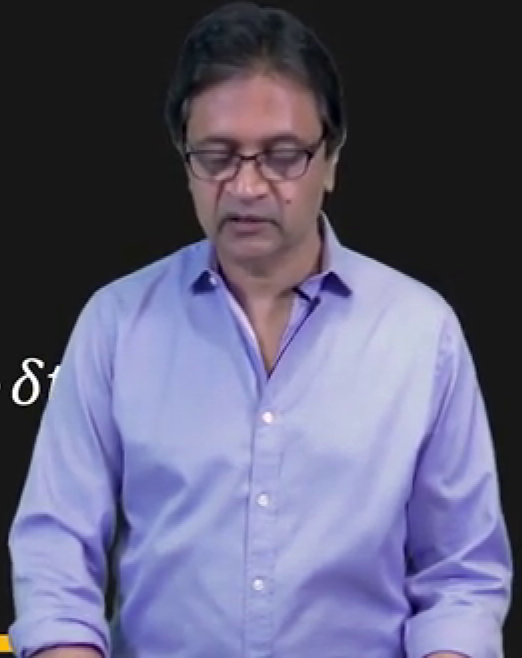
$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$

For a function of three variables with small $\delta x, \delta y, \delta t$:

$$f(x+\delta x, y+\delta y, t+\delta t) \approx f(x,y,t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial t}\delta t$$



Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

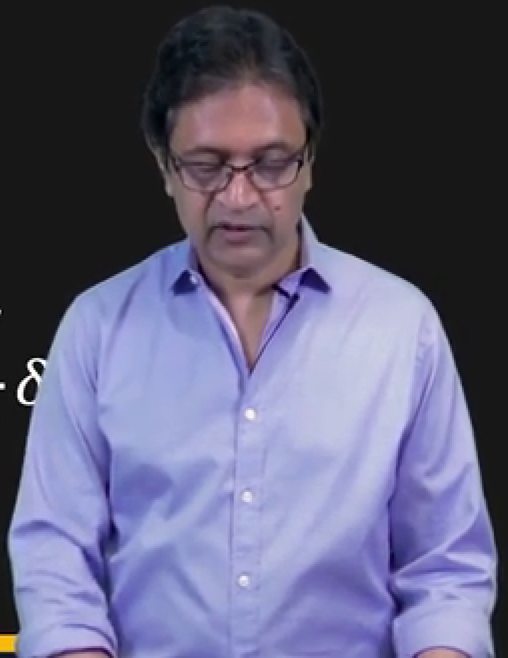
$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

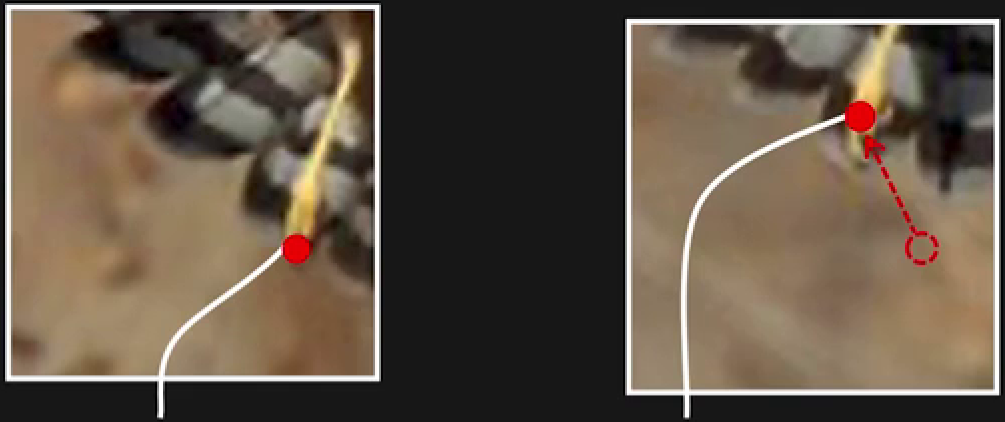
$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$

For a function of three variables with small $\delta x, \delta y, \delta t$:

$$f(x+\delta x, y+\delta y, t+\delta t) \approx f(x,y,t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial t}\delta$$



Optical Flow Constraint Equation

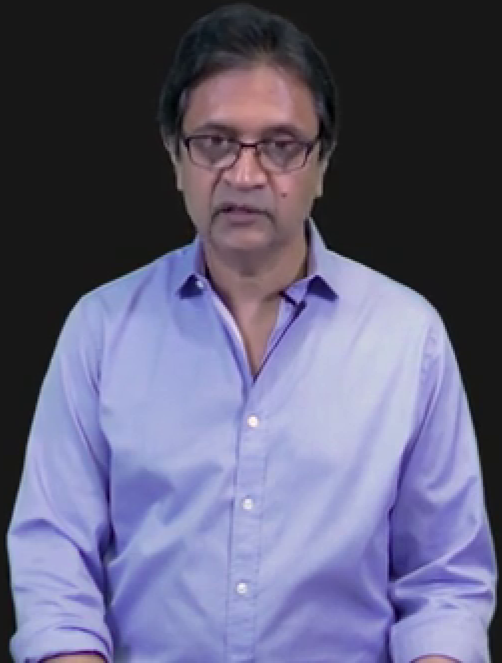


$I(x, y, t)$ $\qquad$ $I(x + \delta x, y + \delta y, t + \delta t)$

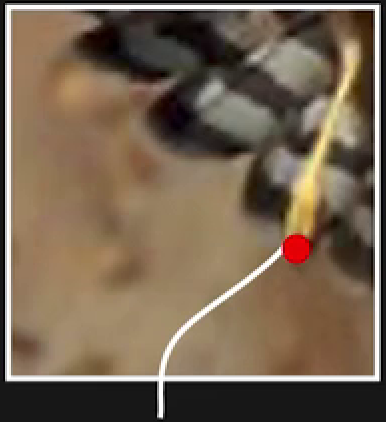
Assumption #2:

Displacement $(\delta x, \delta y)$ and time step δt are small

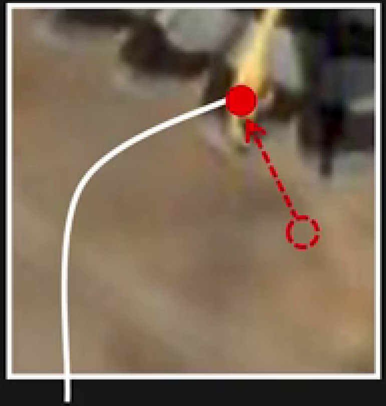
$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t$$



Optical Flow Constraint Equation



$I(x, y, t)$

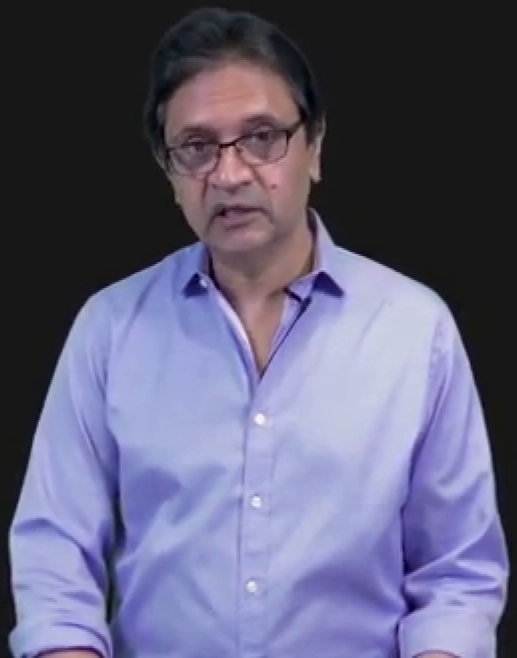


$I(x + \delta x, y + \delta y, t + \delta t)$

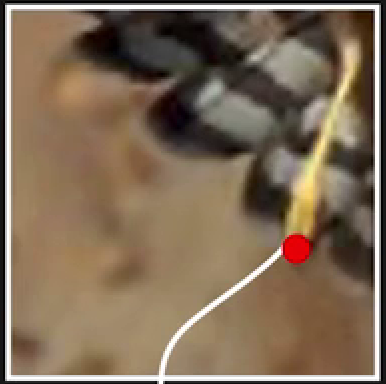
Assumption #2:

Displacement $(\delta x, \delta y)$ and time step δt are small

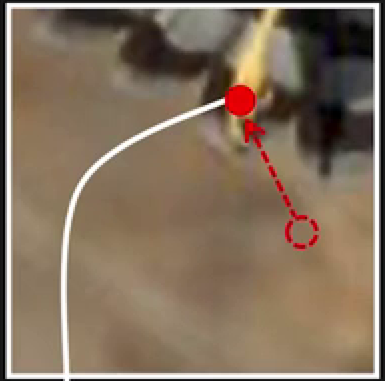
$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t$$



Optical Flow Constraint Equation



$$I(x, y, t)$$



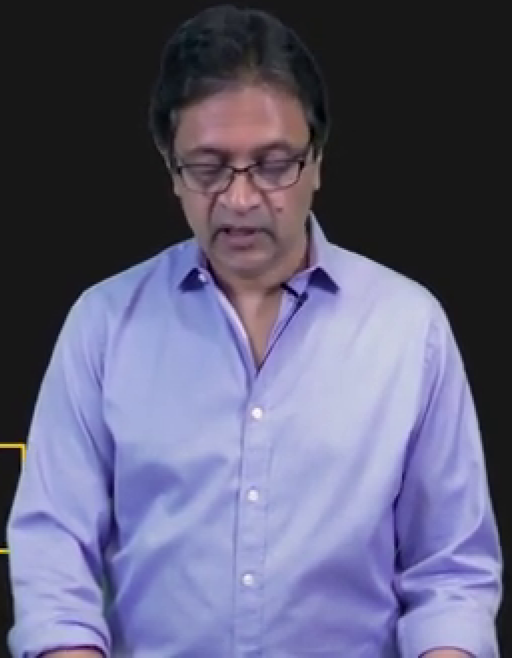
$$I(x + \delta x, y + \delta y, t + \delta t)$$

Assumption #2:

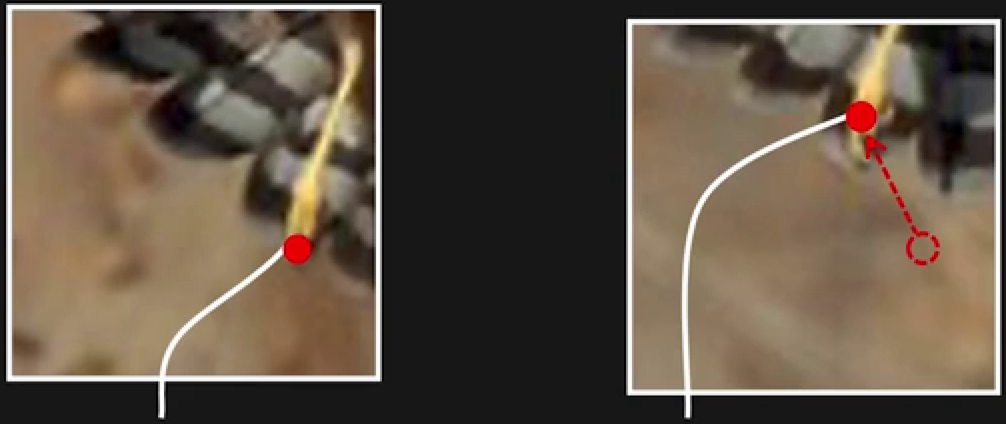
Displacement $(\delta x, \delta y)$ and time step δt are small

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t$$

$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t}$$



Optical Flow Constraint Equation



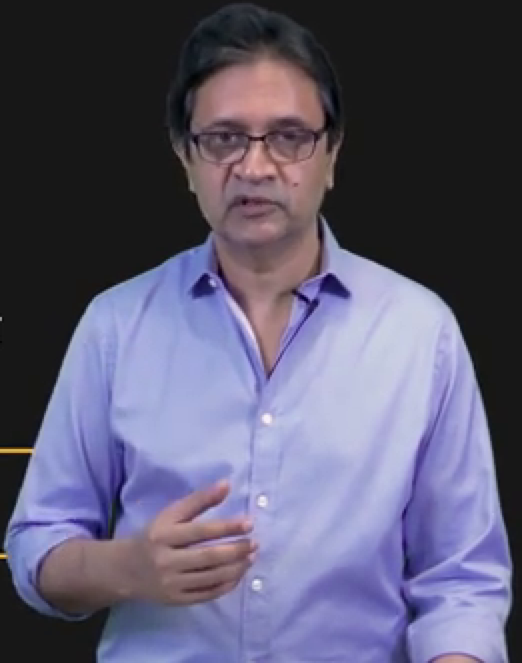
$I(x, y, t)$ $\quad$ $I(x + \delta x, y + \delta y, t + \delta t)$

Assumption #2:

Displacement $(\delta x, \delta y)$ and time step δt are small

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t$$

$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t}$$

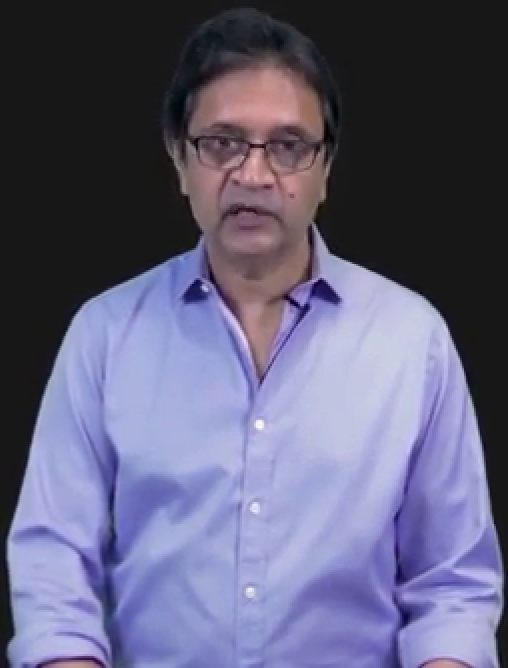


Optical Flow Constraint Equation

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) \quad \text{------------ (1)}$$

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t \quad \text{------------ (2)}$$

Subtract (1) from (2): $I_x \delta x + I_y \delta y + I_t \delta t = 0$



Optical Flow Constraint Equation

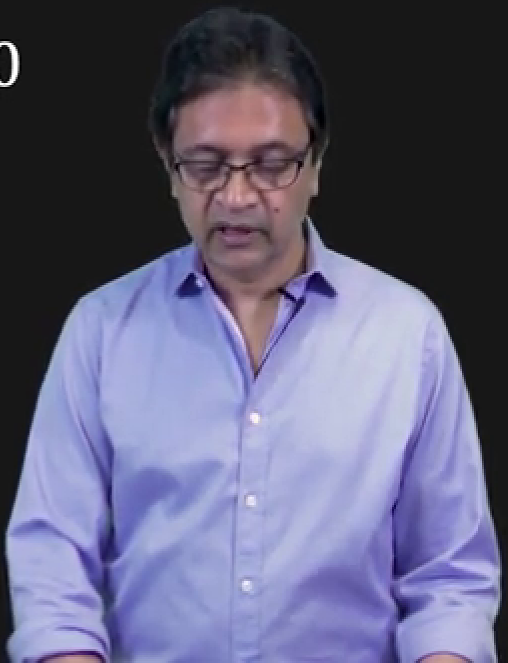
$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) \quad \text{------------ (1)}$$

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t$$

------------ (2)

Subtract (1) from (2): $I_x \delta x + I_y \delta y + I_t \delta t = 0$

Divide by δt and take limit as $\delta t \to 0$: $I_x \dfrac{\partial x}{\partial t} + I_y \dfrac{\partial y}{\partial t} + I_t = 0$



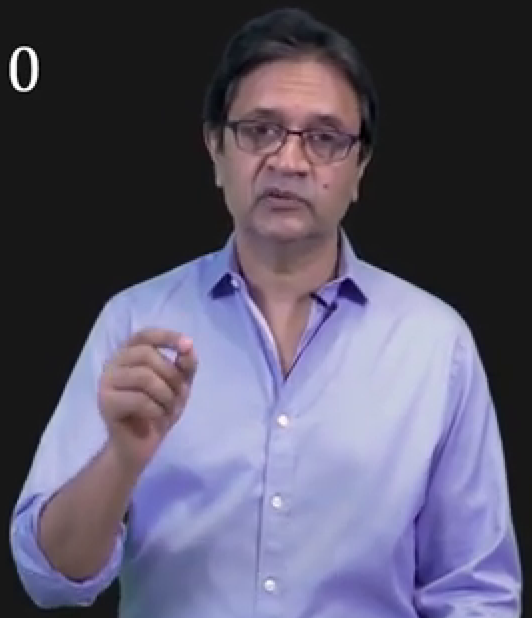
Optical Flow Constraint Equation

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) \qquad (1)$$

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t \qquad (2)$$

Subtract (1) from (2): $I_x \delta x + I_y \delta y + I_t \delta t = 0$

Divide by δt and take limit as $\delta t \to 0$: $I_x \dfrac{\partial x}{\partial t} + I_y \dfrac{\partial y}{\partial t} + I_t = 0$



Optical Flow Constraint Equation

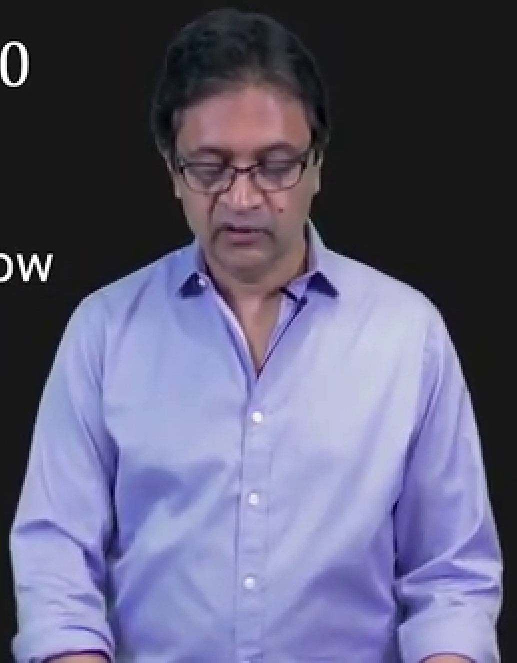
$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) \qquad \text{------------ (1)}$$

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t \qquad \text{------------ (2)}$$

Subtract (1) from (2): $I_x \delta x + I_y \delta y + I_t \delta t = 0$

Divide by δt and take limit as $\delta t \to 0$: $I_x \dfrac{\partial x}{\partial t} + I_y \dfrac{\partial y}{\partial t} + I_t = 0$

Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$ (u, v): Optical Flow



Optical Flow Constraint Equation

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t) \quad \text{------------ (1)}$$

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t) + I_x \delta x + I_y \delta y + I_t \delta t$$

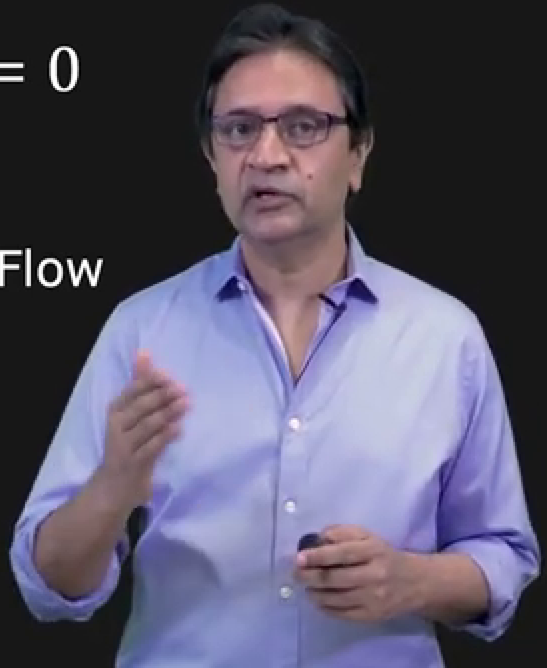
------------ (2)

Subtract (1) from (2): $I_x \delta x + I_y \delta y + I_t \delta t = 0$

Divide by δt and take limit as $\delta t \to 0$: $I_x \frac{\partial x}{\partial t} + I_y \frac{\partial y}{\partial t} + I_t = 0$

Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$ (u, v): Optical Flow

(I_x, I_y, I_t) can be easily computed from two frames



Optical Flow Constraint Equation

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t) \quad \text{------------ (1)}$$

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t) + I_x \delta x + I_y \delta y + I_t \delta t$$

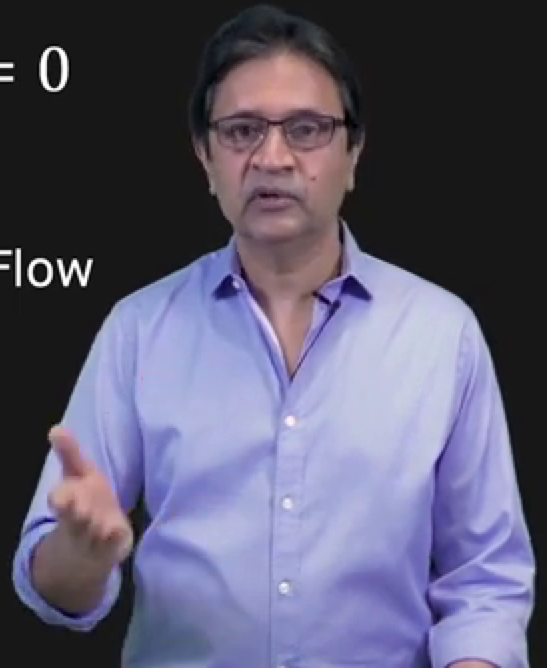
$$\text{------------ (2)}$$

Subtract (1) from (2): $I_x \delta x + I_y \delta y + I_t \delta t = 0$

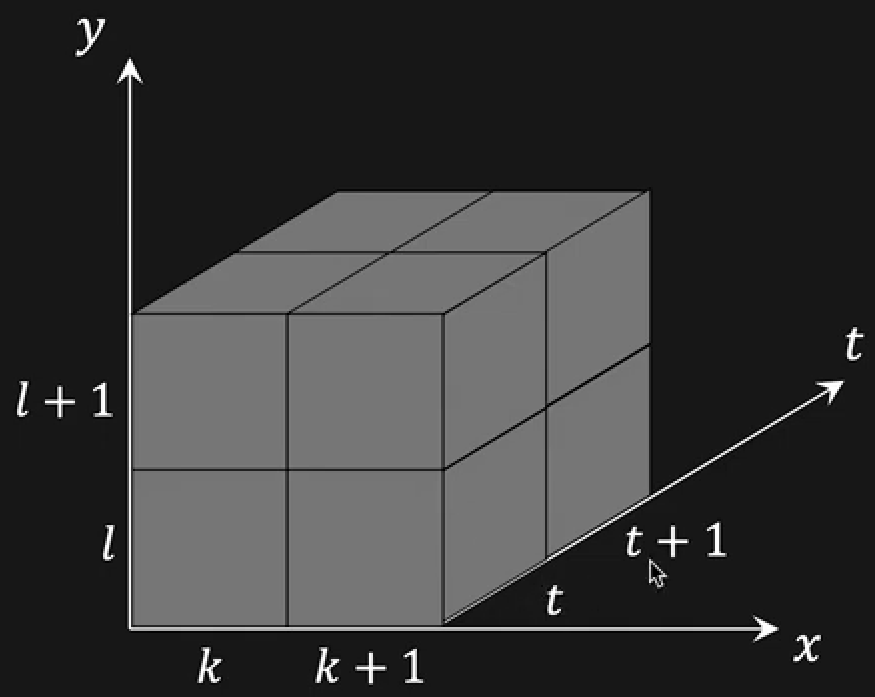
Divide by δt and take limit as $\delta t \to 0$: $I_x \dfrac{\partial x}{\partial t} + I_y \dfrac{\partial y}{\partial t} + I_t = 0$

Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$ (u, v): Optical Flow

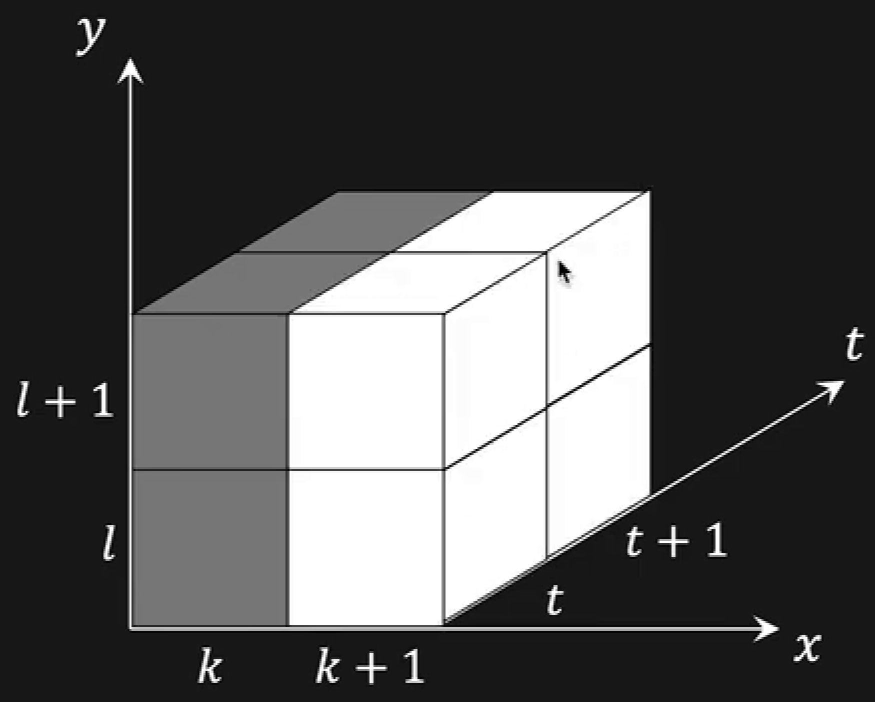
(I_x, I_y, I_t) can be easily computed from two frames



Computing Partial Derivatives I_x, I_y, I_t



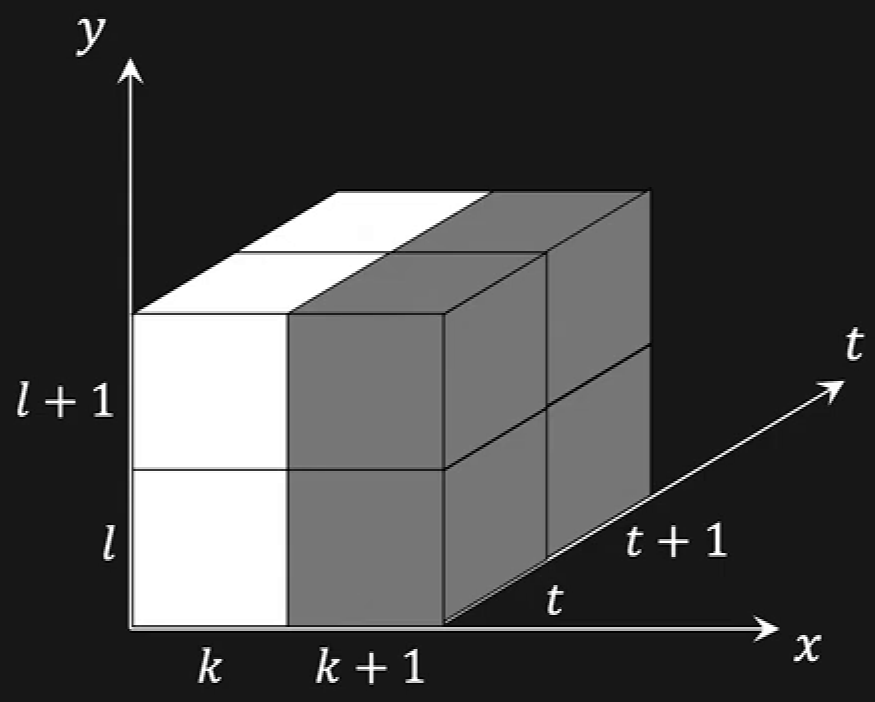
Computing Partial Derivatives I_x, I_y, I_t



$I_x(k,l,t) =$

$$\frac{1}{4}[I(k+1,l,t) + I(k+1,l,t+1) + I(k+1,l+1,t) + I(k+1,l+1,t+1)]$$

Computing Partial Derivatives I_x, I_y, I_t

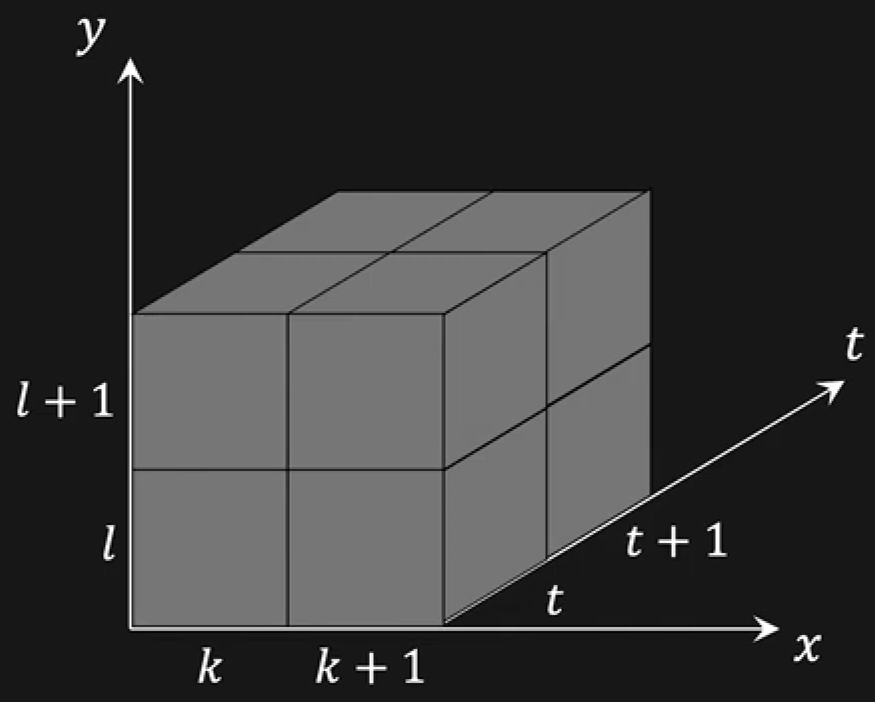


$I_x(k, l, t) =$

$$\tfrac{1}{4}[I(k+1, l, t) + I(k+1, l, t+1) + I(k+1, l+1, t) + I(k+1, l+1, t+1)]$$

$$-\tfrac{1}{4}[I(k, l, t) + I(k, l, t+1) + I(k, l+1, t) + I(k, l+1, t+1)]$$

Computing Partial Derivatives I_x, I_y, I_t

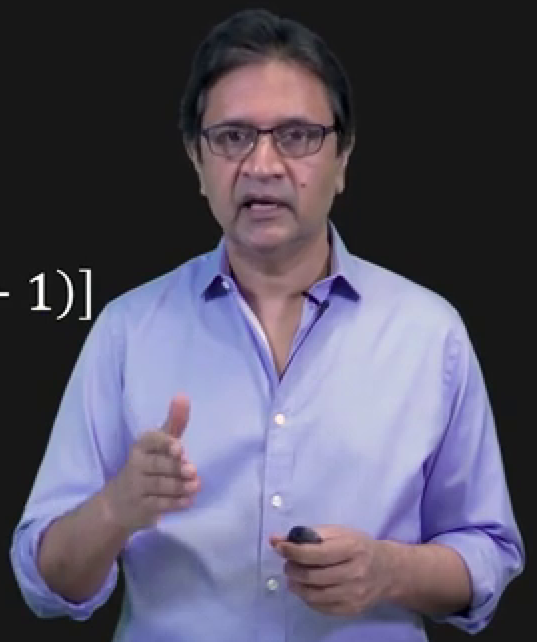


$I_x(k,l,t) =$

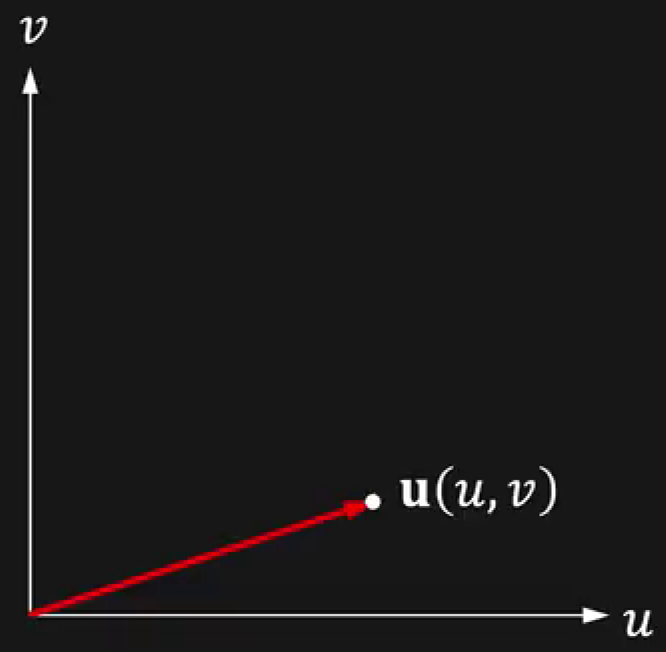
$$\tfrac{1}{4}[I(k+1,l,t) + I(k+1,l,t+1) + I(k+1,l+1,t) + I(k+1,l+1,t+1)]$$

$$-\tfrac{1}{4}[I(k,l,t) + I(k,l,t+1) + I(k,l+1,t) + I(k,l+1,t+1)]$$

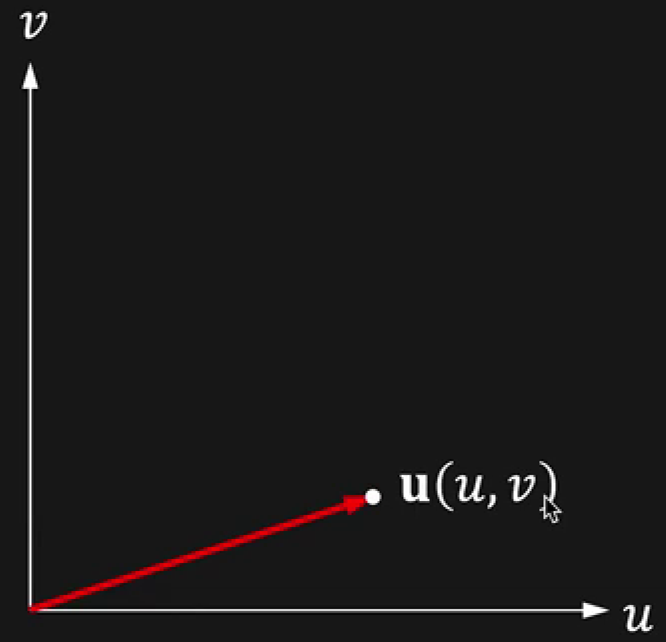
Similarly find $I_y(k,l,t)$ and $I_t(k,l,t)$



Geometric Interpretation



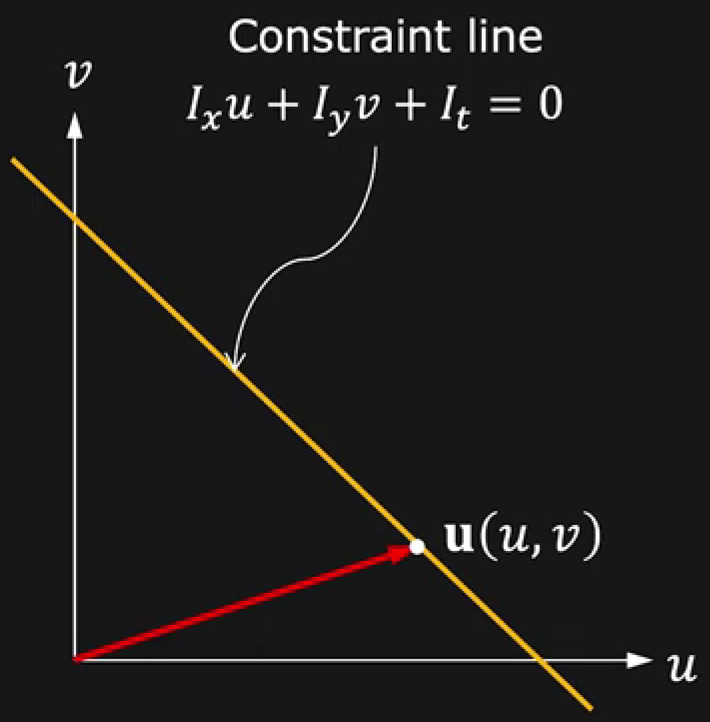
Geometric Interpretation



Geometric Interpretation

For any point (x, y) in the image, its optical flow (u, v) lies on the line:

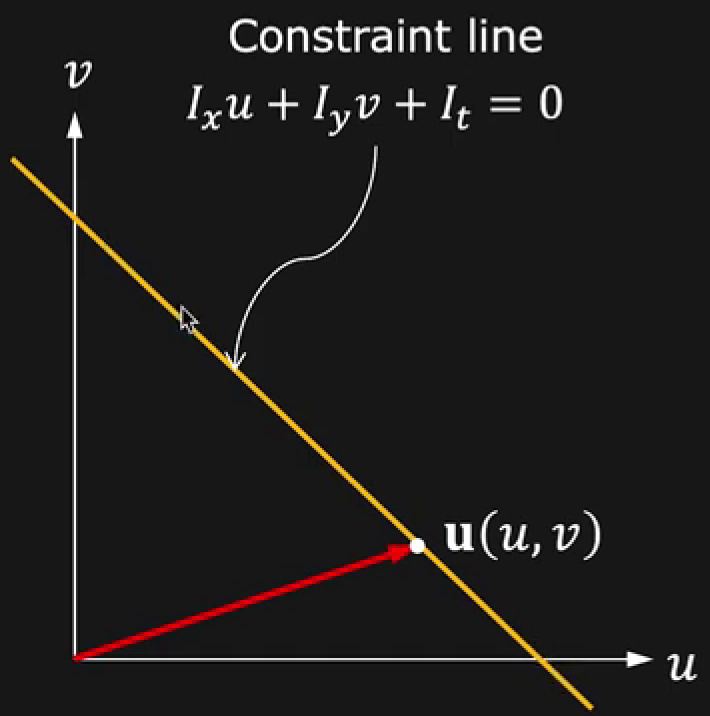
$$I_x u + I_y v + I_t = 0$$



Geometric Interpretation

For any point (x, y) in the image, its optical flow (u, v) lies on the line:

$$I_x u + I_y v + I_t = 0$$



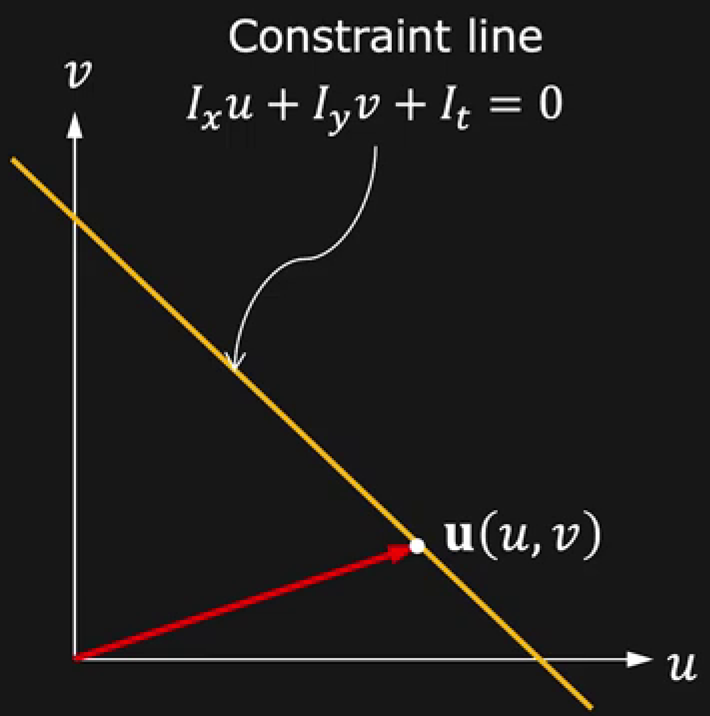
Geometric Interpretation

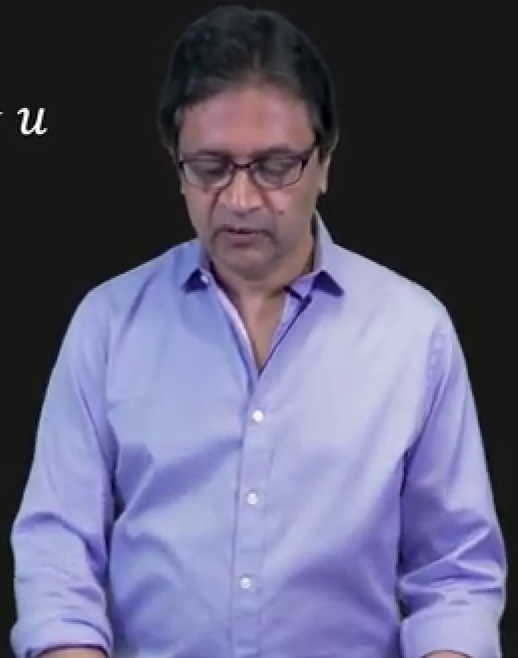
For any point (x, y) in the image, its optical flow (u, v) lies on the line:

$$I_x u + I_y v + I_t = 0$$

Optical Flow can be split into two components.

$$\mathbf{u} = \mathbf{u}_n + \mathbf{u}_p$$

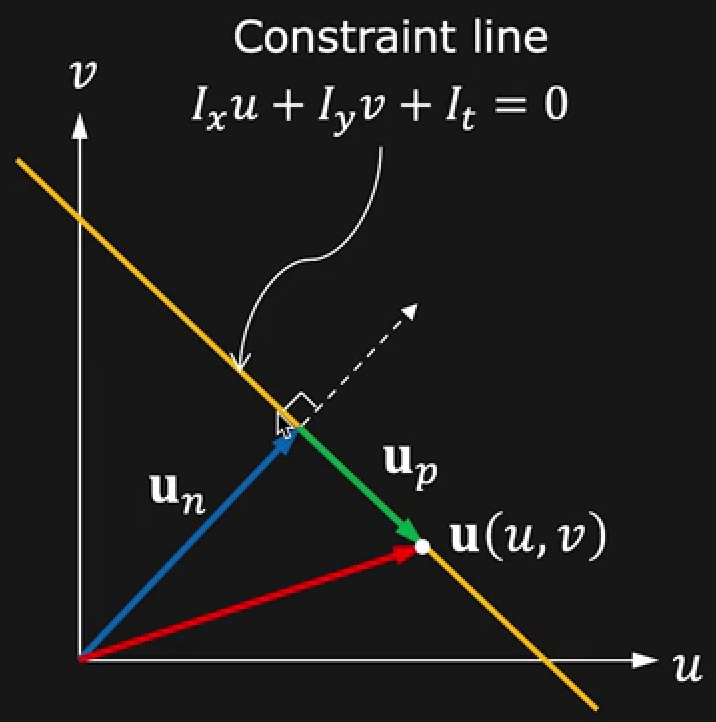




Geometric Interpretation

For any point (x, y) in the image, its optical flow (u, v) lies on the line:

$$I_x u + I_y v + I_t = 0$$



Optical Flow can be split into two components.

$$\mathbf{u} = \mathbf{u}_n + \mathbf{u}_p$$

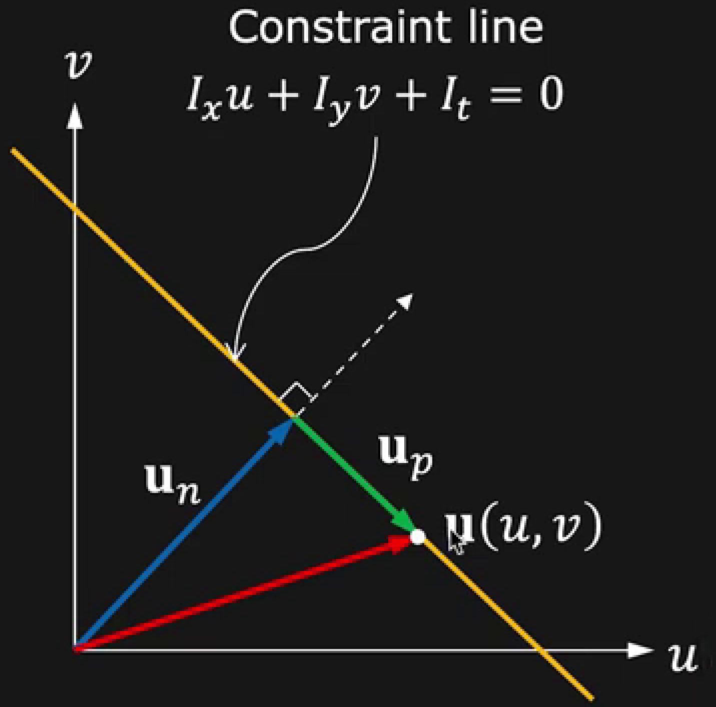
$\mathbf{u}_n$: Normal Flow

$\mathbf{u}_p$: Parallel Flow

Geometric Interpretation

For any point (x, y) in the image, its optical flow (u, v) lies on the line:

$$I_x u + I_y v + I_t = 0$$

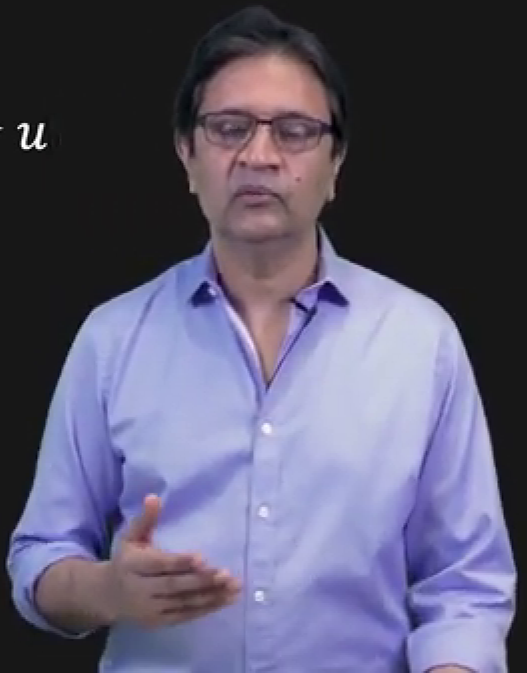


Optical Flow can be split into two components.

$$\mathbf{u} = \mathbf{u}_n + \mathbf{u}_p$$

$\mathbf{u}_n$: Normal Flow

$\mathbf{u}_p$: Parallel Flow

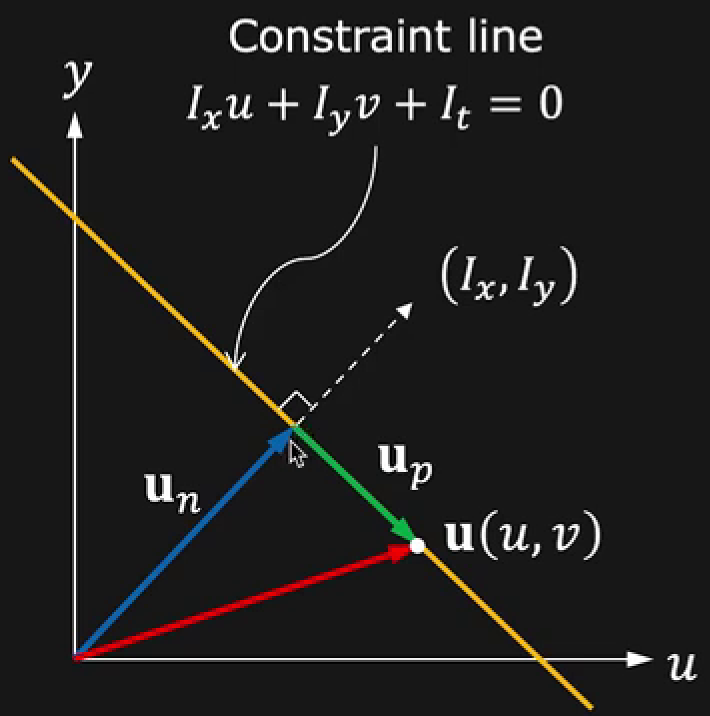


Normal Flow

Direction of Normal Flow:

Unit vector perpendicular to the constraint line:

$$\hat{\mathbf{u}}_n = \frac{(I_x, I_y)}{\sqrt{I_x^2 + I_y^2}}$$

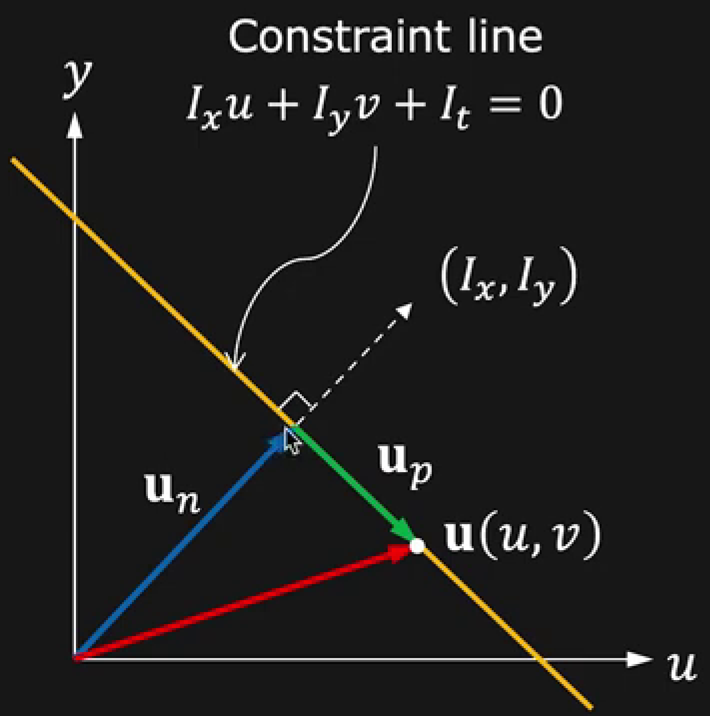


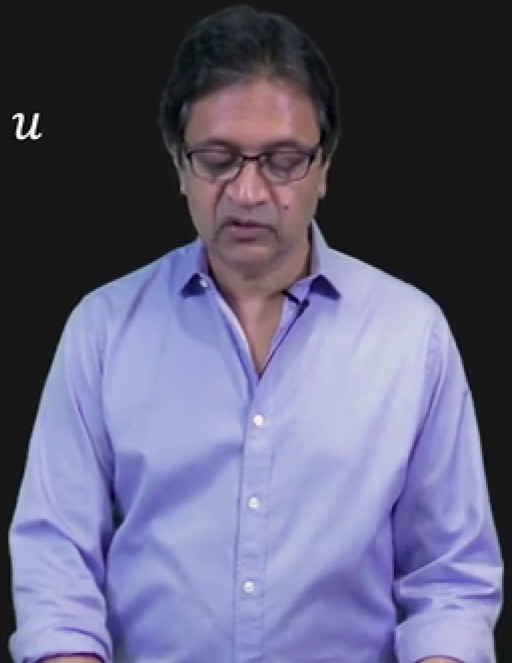
Normal Flow

Direction of Normal Flow:

Unit vector perpendicular to the constraint line:

$$\hat{\mathbf{u}}_n = \frac{(I_x, I_y)}{\sqrt{I_x^2 + I_y^2}}$$



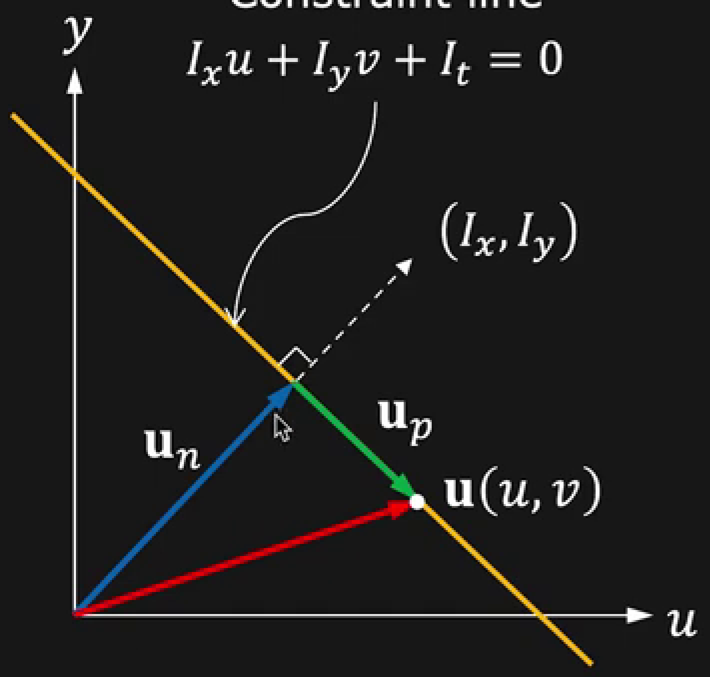


Normal Flow

Direction of Normal Flow:

Unit vector perpendicular to the constraint line:

$$\hat{\mathbf{u}}_n = \frac{(I_x, I_y)}{\sqrt{I_x^2 + I_y^2}}$$

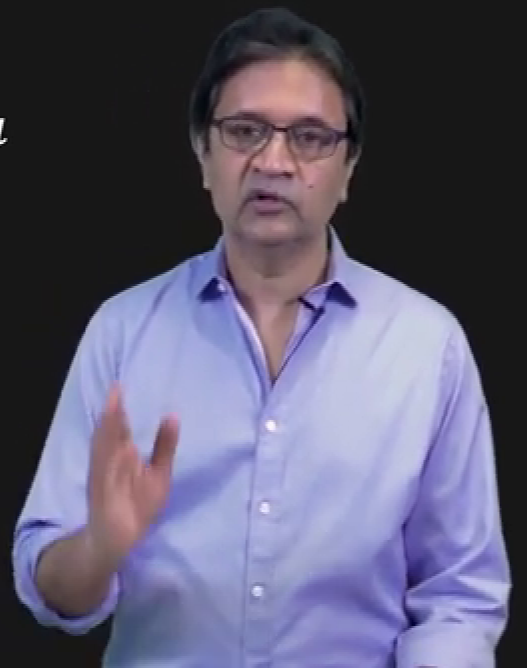


Magnitude of Normal Flow:

Distance of origin from the constraint line:

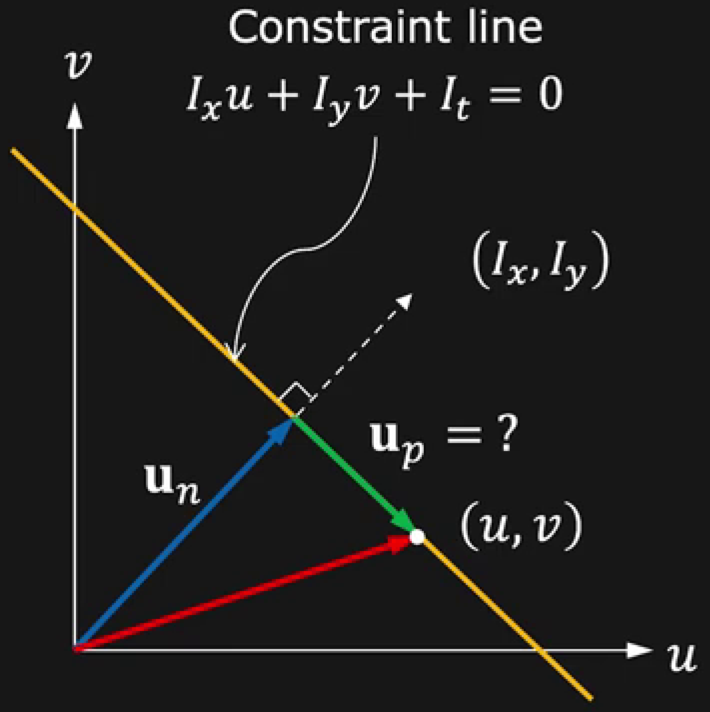
$$|\mathbf{u}_n| = \frac{|I_t|}{\sqrt{I_x^2 + I_y^2}}$$

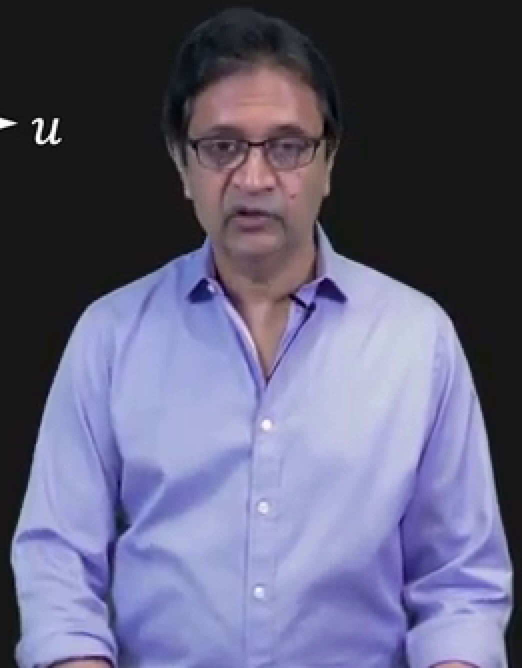
$$\boxed{\mathbf{u}_n = \frac{|I_t|}{(I_x^2 + I_y^2)}(I_x, I_y)}$$



Parallel Flow

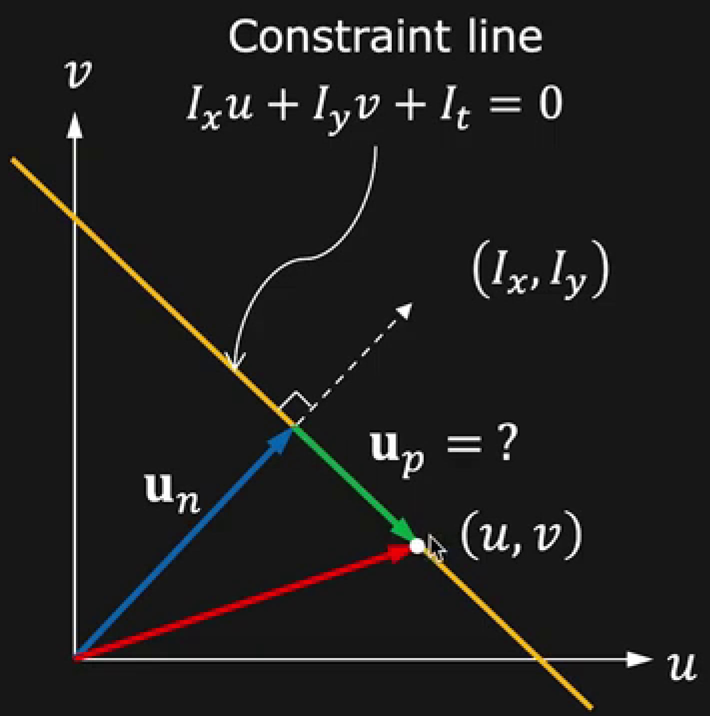
We cannot determine $\mathbf{u}_p$, the optical flow component parallel to the constraint line.



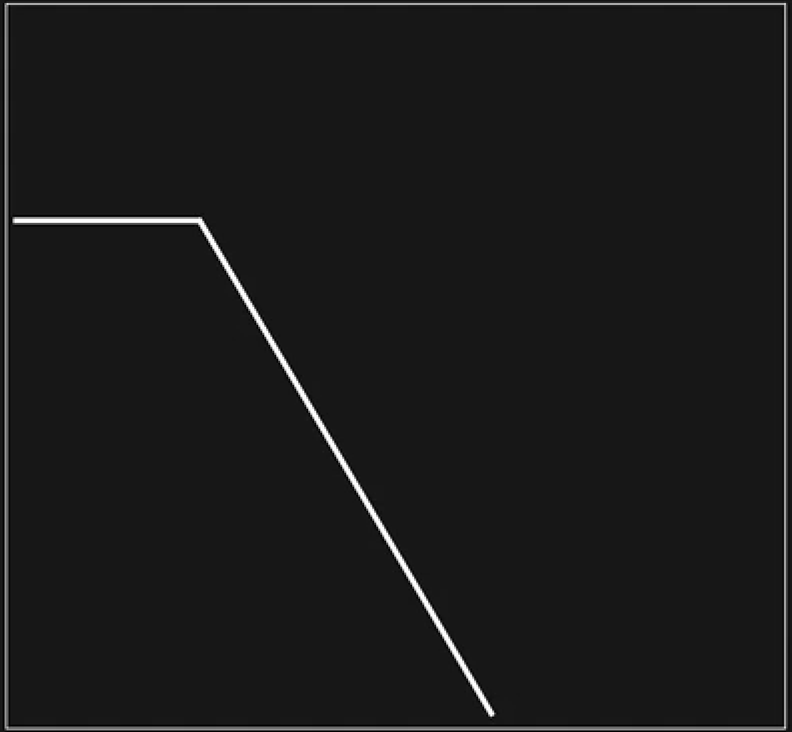


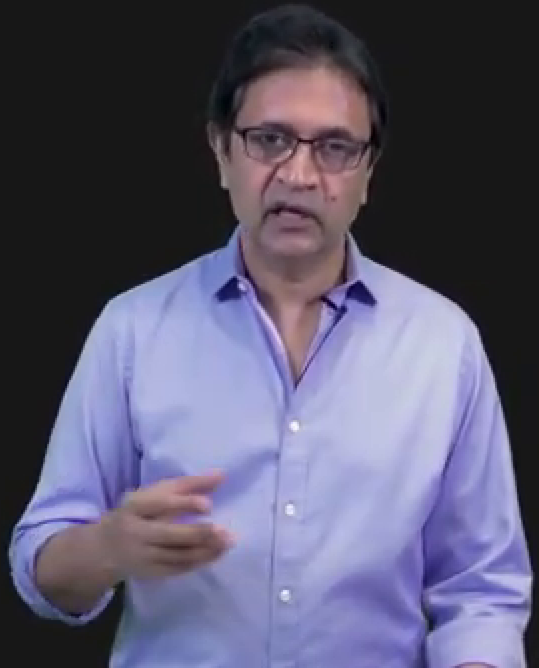
Parallel Flow

We cannot determine $\mathbf{u}_p$, the optical flow component parallel to the constraint line.

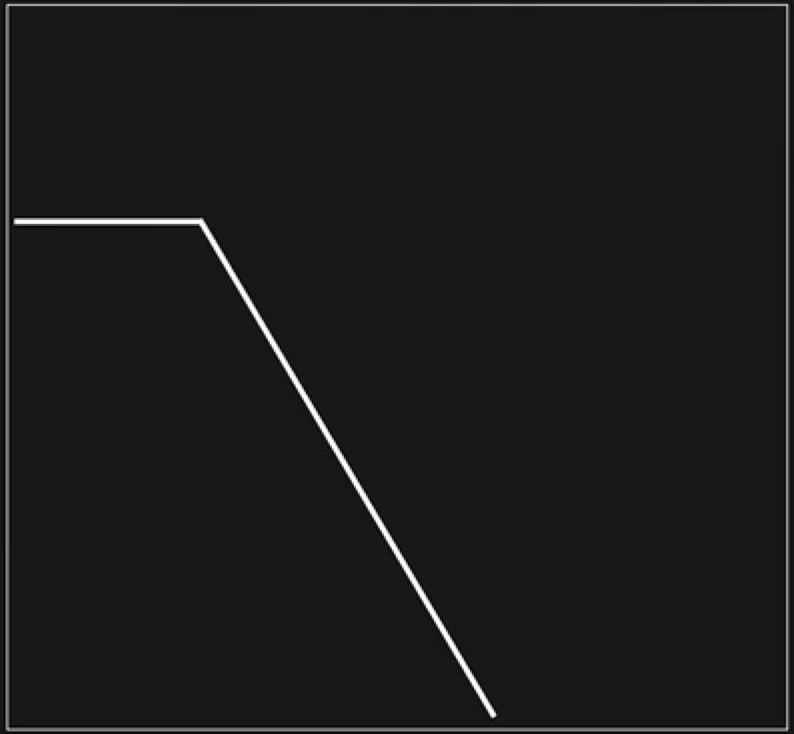


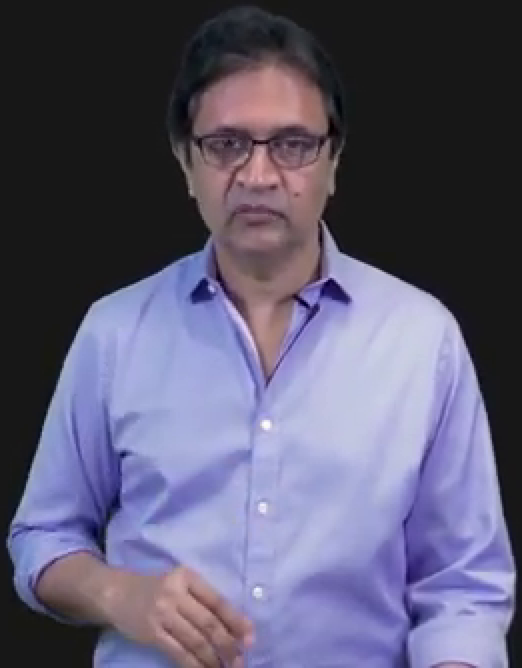
Aperture Problem



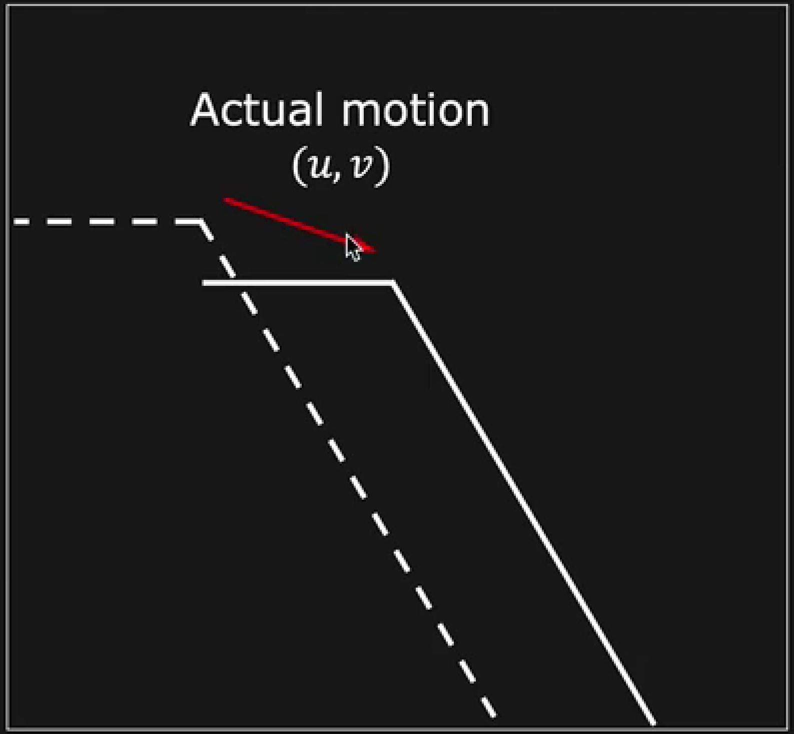


Aperture Problem

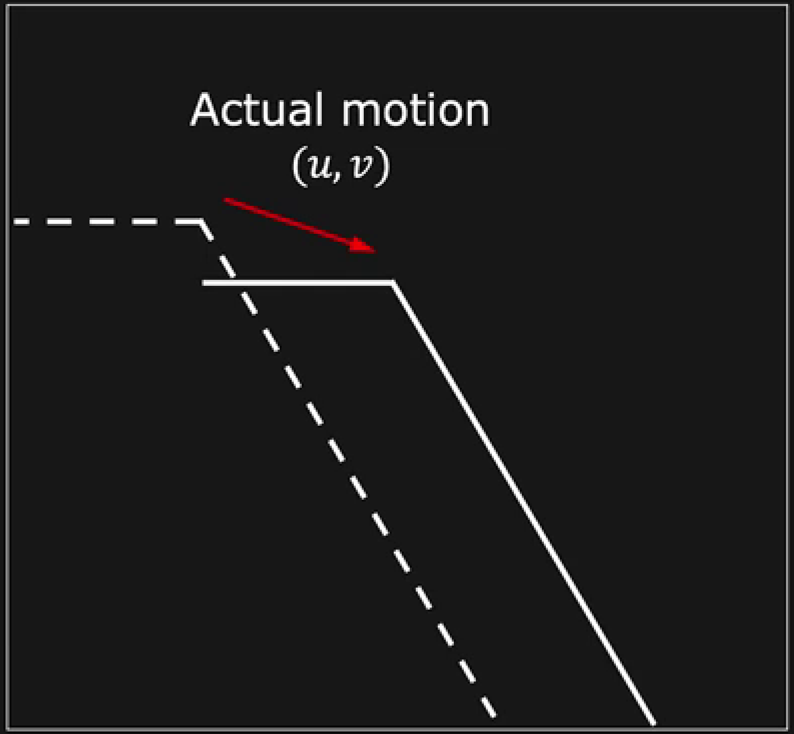




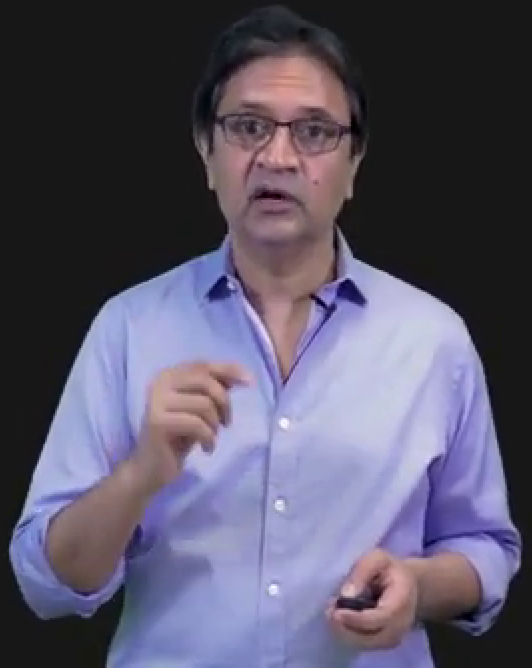
Aperture Problem



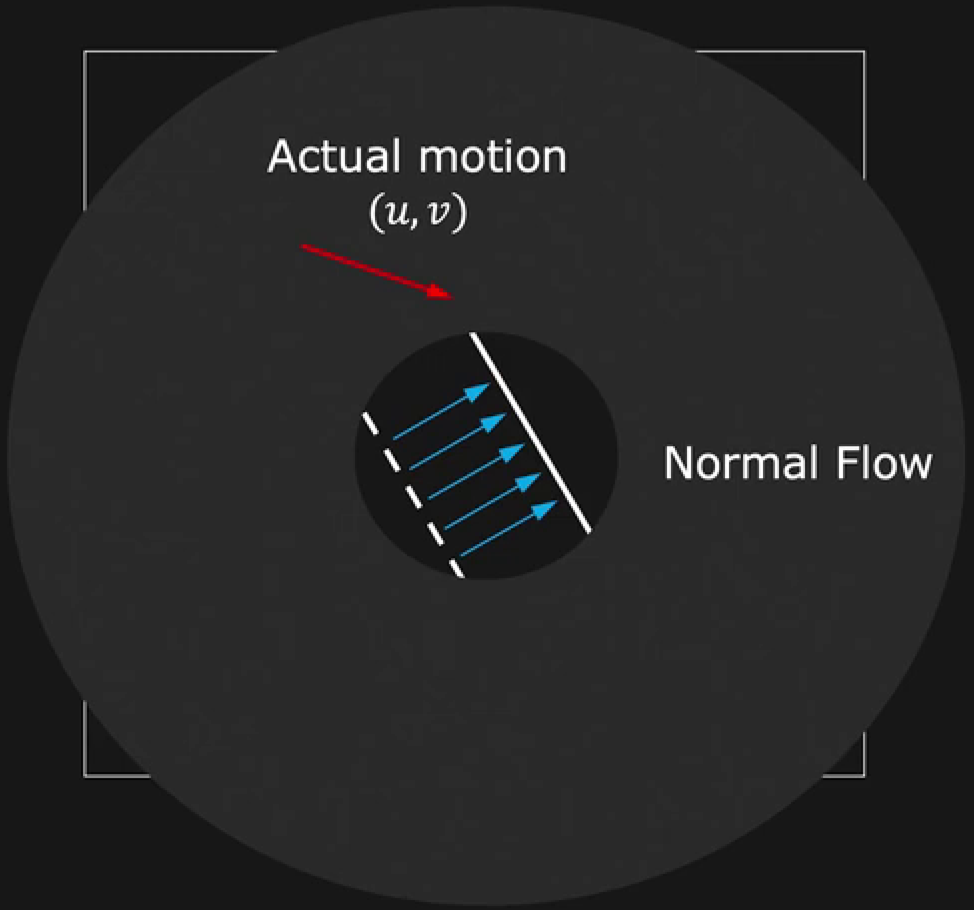
Aperture Problem



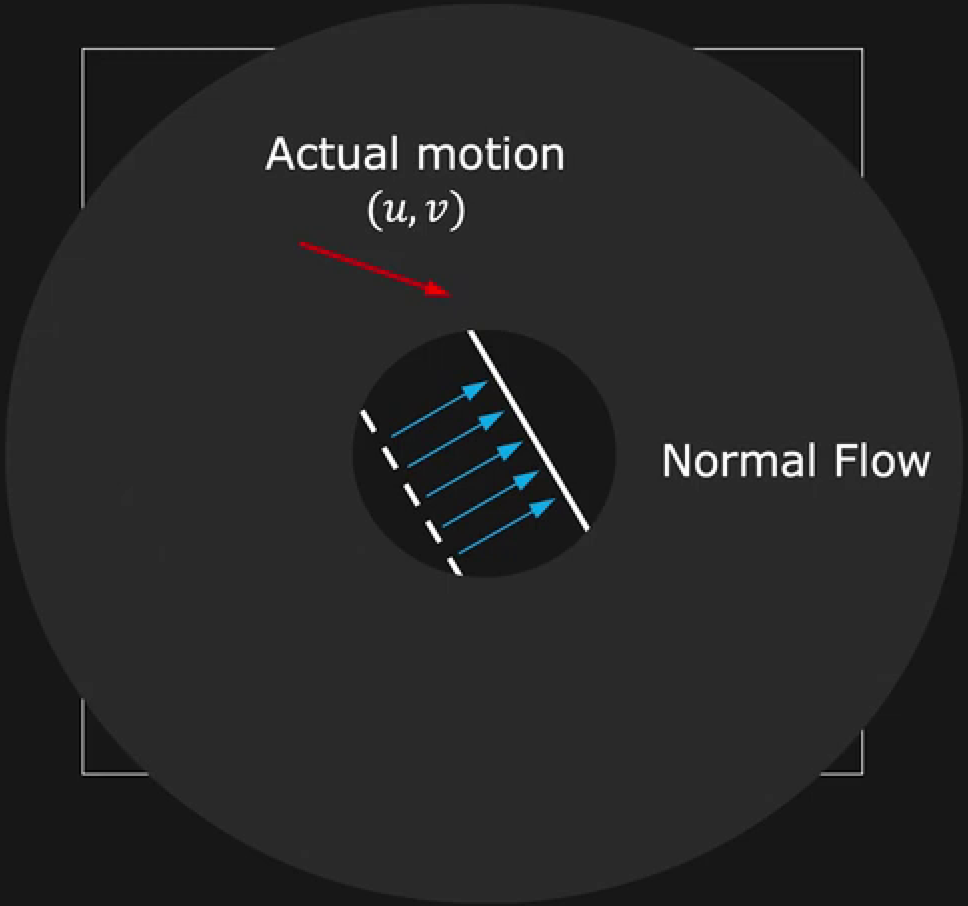
Aperture Problem



Aperture Problem



Aperture Problem



Locally, we can only determine normal flow!

Optical Flow is Under Constrained

Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$

2 unknowns, 1 equation.