

Camera Calibration

Shree K. Nayar

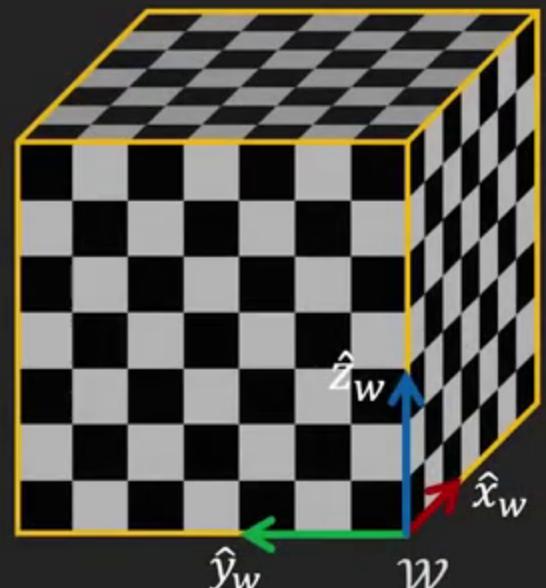
Columbia University

Topic: Camera Calibration, Module: Reconstruction II

First Principles of Computer Vision

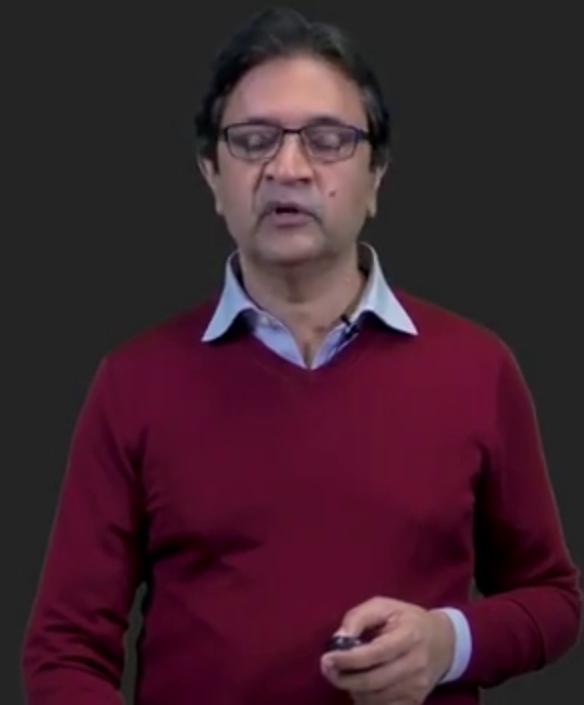
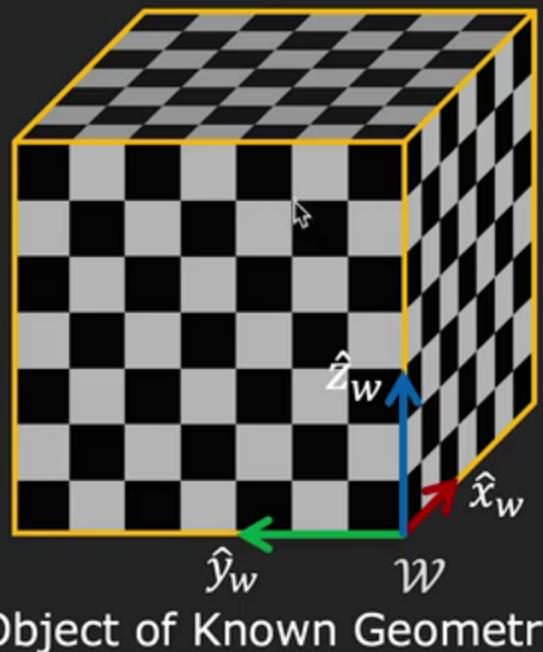
Camera Calibration Procedure

Step 1: Capture an image of an object with known geometry.



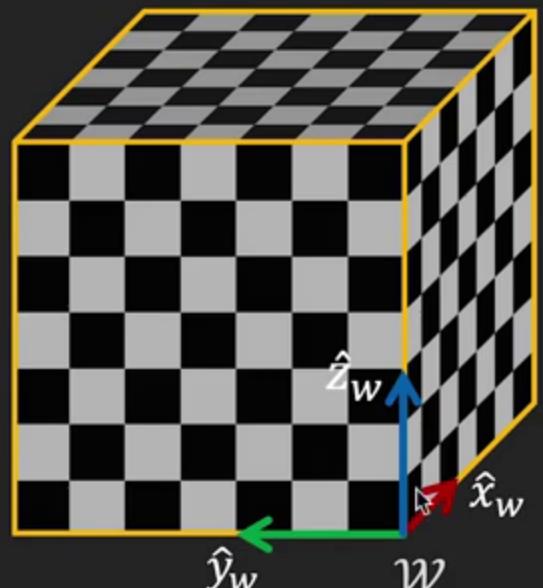
Camera Calibration Procedure

Step 1: Capture an image of an object with known geometry.



Camera Calibration Procedure

Step 1: Capture an image of an object with known geometry.

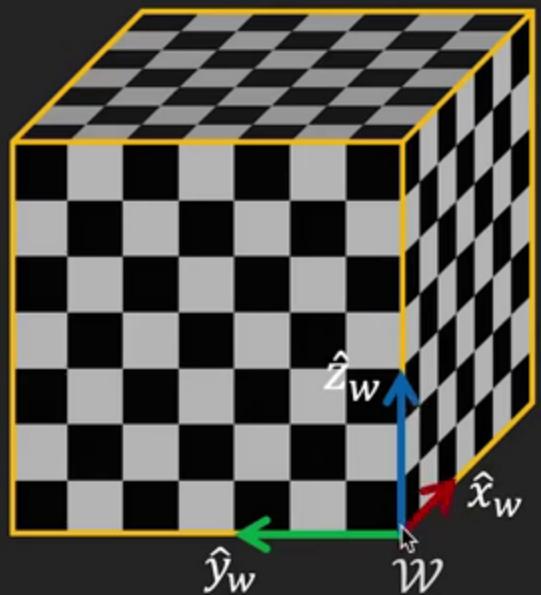


Object of Known Geometry

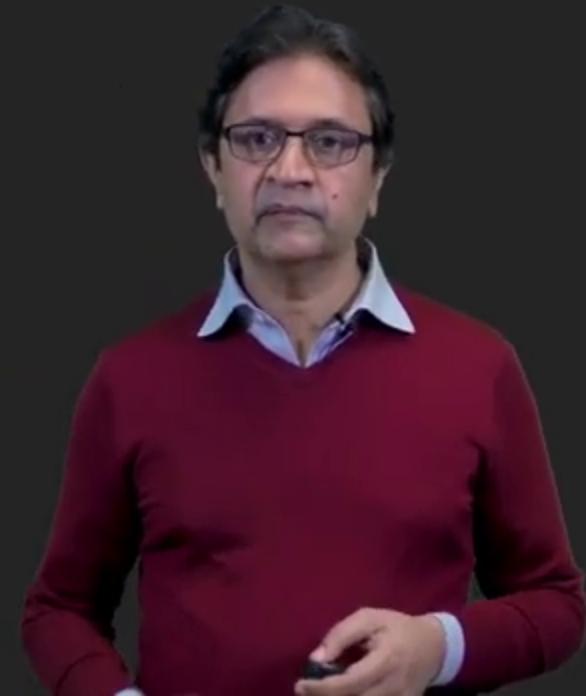


Camera Calibration Procedure

Step 1: Capture an image of an object with known geometry.

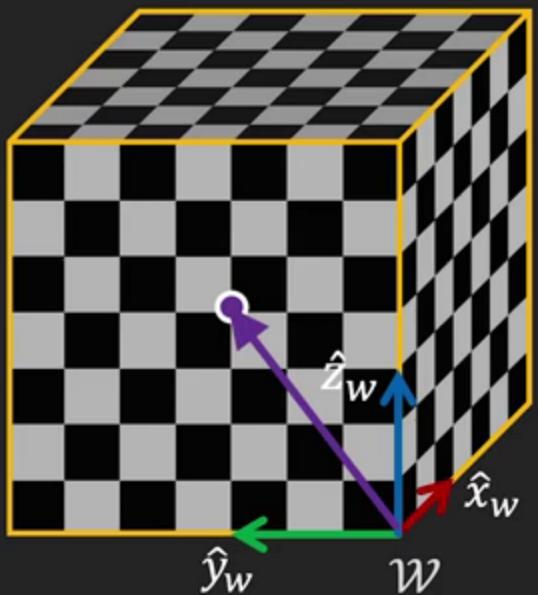


Object of Known Geometry

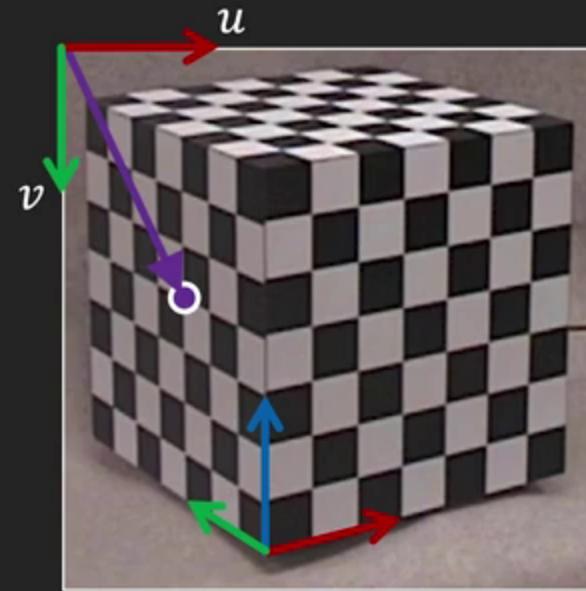


Camera Calibration Procedure

Step 2: Identify correspondences between 3D scene points and image points.



Object of Known Geometry



Captured Image

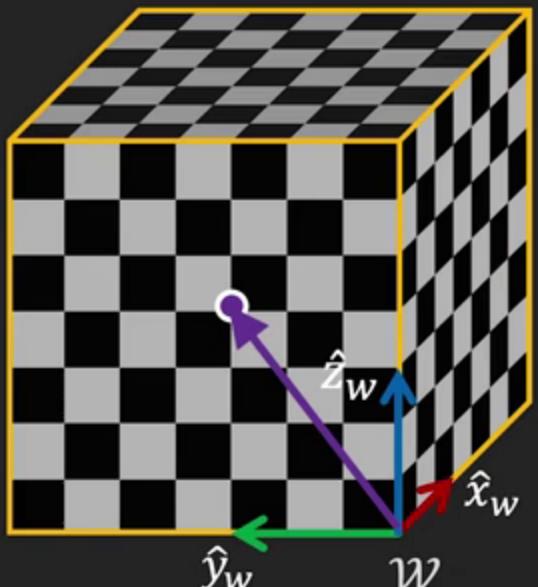
- $\mathbf{x}_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$
(inches)

- $\mathbf{u} = [u] = \begin{bmatrix} 56 \\ 115 \end{bmatrix}$
(pixels)

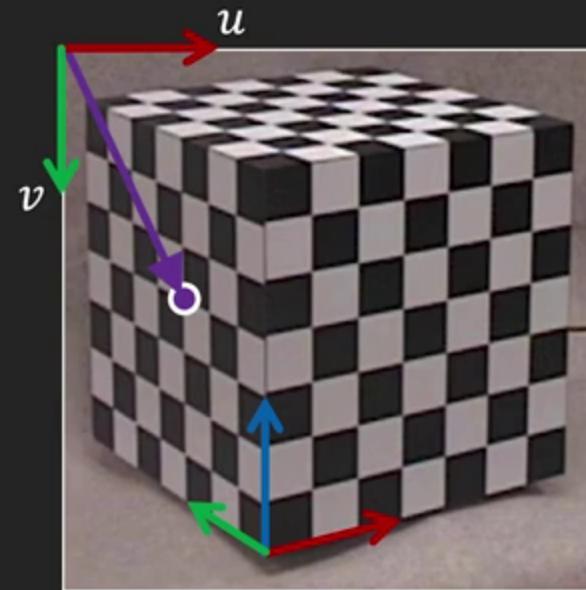


Camera Calibration Procedure

Step 2: Identify correspondences between 3D scene points and image points.



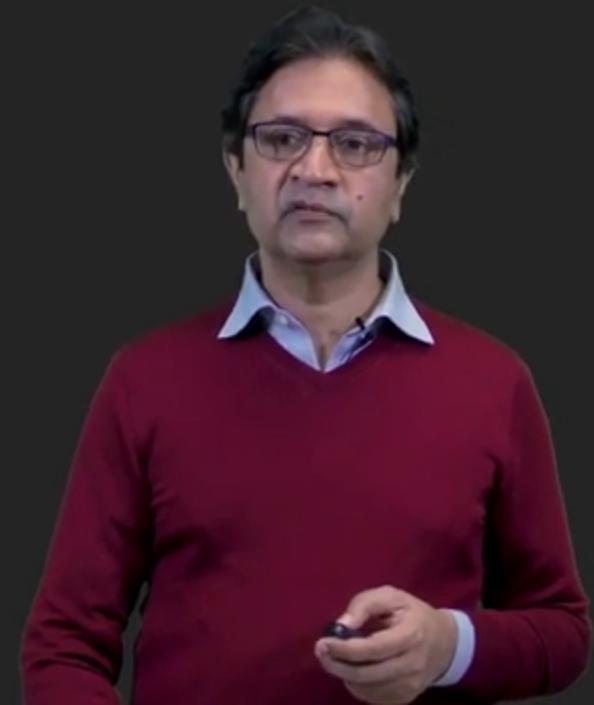
Object of Known Geometry



Captured Image

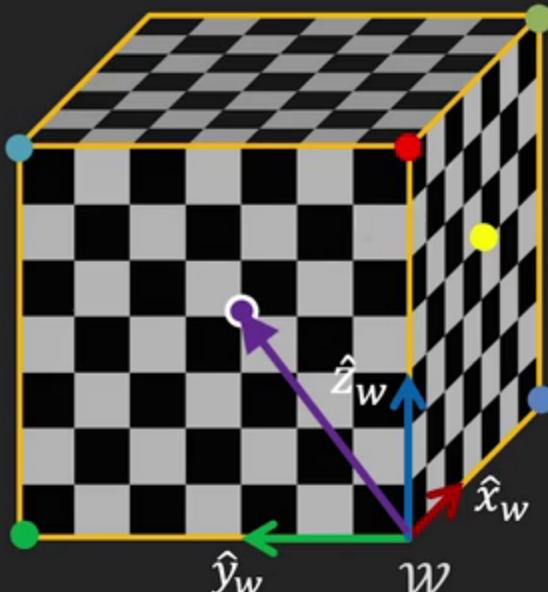
- $\mathbf{x}_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$
(inches)

- $\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 56 \\ 115 \end{bmatrix}$
(pixels)

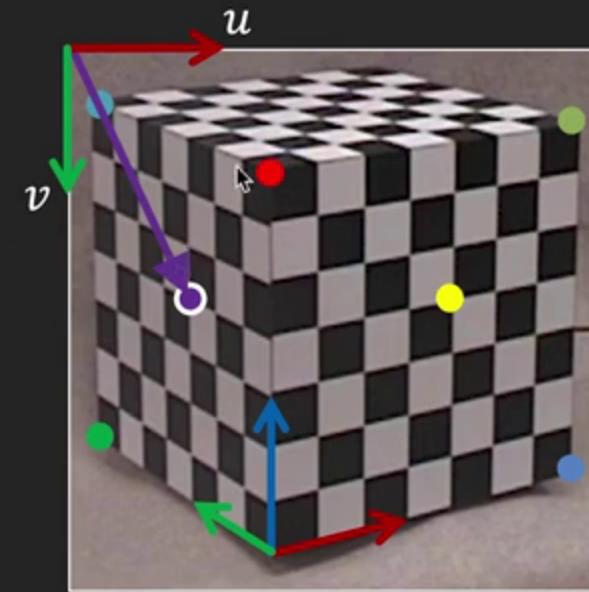


Camera Calibration Procedure

Step 2: Identify correspondences between 3D scene points and image points.



Object of Known Geometry



Captured Image

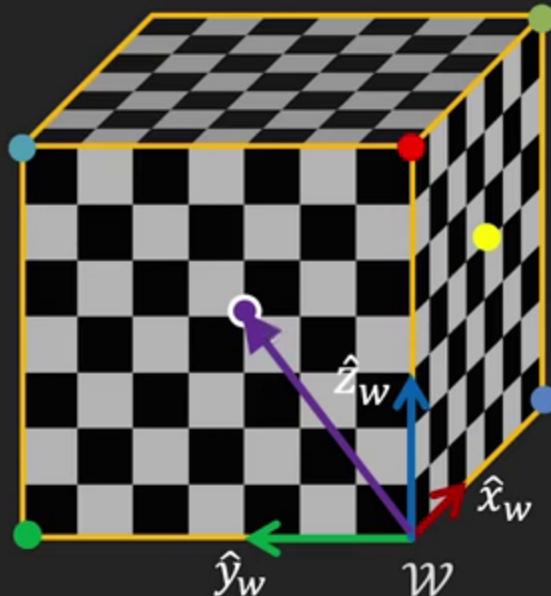
- $\mathbf{x}_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$
(inches)

- $\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 56 \\ 115 \end{bmatrix}$
(pixels)

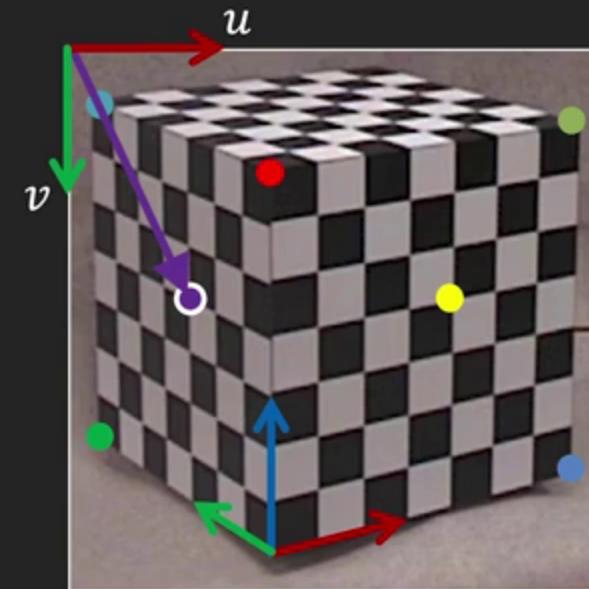


Camera Calibration Procedure

Step 2: Identify correspondences between 3D scene points and image points.



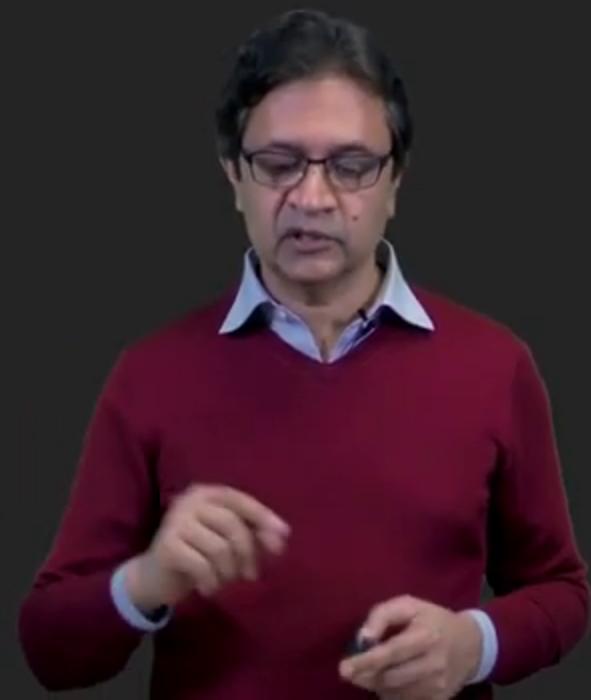
Object of Known Geometry



Captured Image

- $\mathbf{x}_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$
(inches)

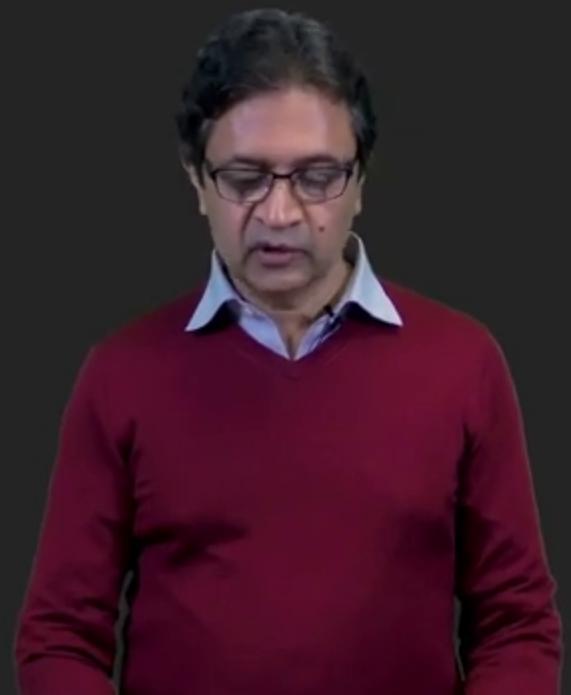
- $\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 56 \\ 115 \end{bmatrix}$
(pixels)



Camera Calibration Procedure

Step 3: For each corresponding point i in scene and image:

$$\begin{bmatrix} u^{(i)} \\ v^{(i)} \\ 1 \end{bmatrix} \equiv \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w^{(i)} \\ y_w^{(i)} \\ z_w^{(i)} \\ 1 \end{bmatrix}$$



Camera Calibration Procedure

Step 3: For each corresponding point i in scene and image:

$$\begin{bmatrix} u^{(i)} \\ v^{(i)} \\ 1 \end{bmatrix} \equiv \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w^{(i)} \\ y_w^{(i)} \\ z_w^{(i)} \\ 1 \end{bmatrix}$$



Camera Calibration Procedure

Step 3: For each corresponding point i in scene and image:

$$\begin{bmatrix} u^{(i)} \\ v^{(i)} \\ 1 \end{bmatrix} \equiv \frac{\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}}{\begin{bmatrix} x_w^{(i)} \\ y_w^{(i)} \\ z_w^{(i)} \\ 1 \end{bmatrix}}$$

Known Unknown Known

Expanding the matrix as linear equations:

$$u^{(i)} = \frac{p_{11}x_w^{(i)} + p_{12}y_w^{(i)} + p_{13}z_w^{(i)} + p_{14}}{p_{31}x_w^{(i)} + p_{32}y_w^{(i)} + p_{33}z_w^{(i)} + p_{34}}$$

$$v^{(i)} = \frac{p_{21}x_w^{(i)} + p_{22}y_w^{(i)} + p_{23}z_w^{(i)} + p_{24}}{p_{31}x_w^{(i)} + p_{32}y_w^{(i)} + p_{33}z_w^{(i)} + p_{34}}$$



Camera Calibration Procedure

Step 3: For each corresponding point i in scene and image:

$$\begin{bmatrix} u^{(i)} \\ v^{(i)} \\ 1 \end{bmatrix} \equiv \frac{\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}}{\begin{bmatrix} x_w^{(i)} \\ y_w^{(i)} \\ z_w^{(i)} \\ 1 \end{bmatrix}}$$

Known Unknown Known

Expanding the matrix as linear equations:

$$u^{(i)} = \frac{p_{11}x_w^{(i)} + p_{12}y_w^{(i)} + p_{13}z_w^{(i)} + p_{14}}{p_{31}x_w^{(i)} + p_{32}y_w^{(i)} + p_{33}z_w^{(i)} + p_{34}}$$

$$v^{(i)} = \frac{p_{21}x_w^{(i)} + p_{22}y_w^{(i)} + p_{23}z_w^{(i)} + p_{24}}{p_{31}x_w^{(i)} + p_{32}y_w^{(i)} + p_{33}z_w^{(i)} + p_{34}}$$



Camera Calibration Procedure

Step 4: Rearranging the terms

$$\begin{bmatrix} x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & 0 & 0 & 0 & -u_1 x_w^{(1)} & -u_1 y_w^{(1)} & -u_1 z_w^{(1)} & -u_1 \\ 0 & 0 & 0 & 0 & x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & -v_1 x_w^{(1)} & -v_1 y_w^{(1)} & -v_1 z_w^{(1)} & -v_1 \\ \vdots & \vdots \\ x_w^{(i)} & y_w^{(i)} & z_w^{(i)} & 1 & 0 & 0 & 0 & -u_i x_w^{(i)} & -u_i y_w^{(i)} & -u_i z_w^{(i)} & -u_i \\ 0 & 0 & 0 & 0 & x_w^{(i)} & y_w^{(i)} & z_w^{(i)} & 1 & -v_i x_w^{(i)} & -v_i y_w^{(i)} & -v_i z_w^{(i)} & -v_i \\ \vdots & \vdots \\ x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & 0 & 0 & 0 & -u_n x_w^{(n)} & -u_n y_w^{(n)} & -u_n z_w^{(n)} & -u_n \\ 0 & 0 & 0 & 0 & x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & -v_n x_w^{(n)} & -v_n y_w^{(n)} & -v_n z_w^{(n)} & -v_n \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

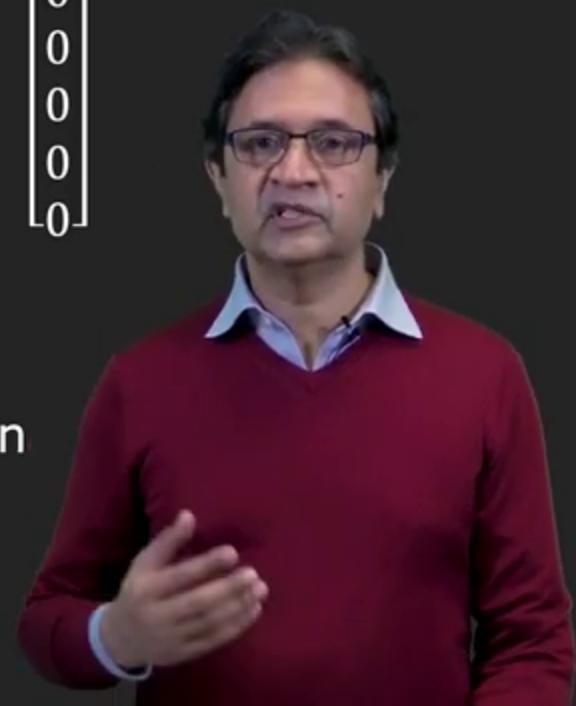


Camera Calibration Procedure

Step 4: Rearranging the terms

$$\boxed{\begin{bmatrix} x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & 0 & 0 & 0 & -u_1 x_w^{(1)} & -u_1 y_w^{(1)} & -u_1 z_w^{(1)} & -u_1 \\ 0 & 0 & 0 & 0 & x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & -v_1 x_w^{(1)} & -v_1 y_w^{(1)} & -v_1 z_w^{(1)} & -v_1 \\ \vdots & \vdots \\ x_w^{(i)} & y_w^{(i)} & z_w^{(i)} & 1 & 0 & 0 & 0 & -u_i x_w^{(i)} & -u_i y_w^{(i)} & -u_i z_w^{(i)} & -u_i \\ 0 & 0 & 0 & 0 & x_w^{(i)} & y_w^{(i)} & z_w^{(i)} & 1 & -v_i x_w^{(i)} & -v_i y_w^{(i)} & -v_i z_w^{(i)} & -v_i \\ \vdots & \vdots \\ x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & 0 & 0 & 0 & -u_n x_w^{(n)} & -u_n y_w^{(n)} & -u_n z_w^{(n)} & -u_n \\ 0 & 0 & 0 & 0 & x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & -v_n x_w^{(n)} & -v_n y_w^{(n)} & -v_n z_w^{(n)} & -v_n \end{bmatrix}} = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A
Known
p
Unknown

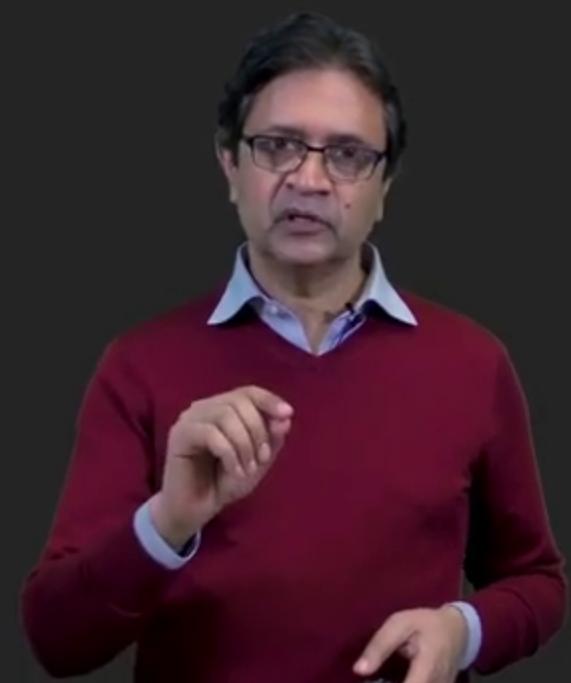


Scale of Projection Matrix

Projection matrix acts on homogenous coordinates.

We know that:

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} \equiv k \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} \quad (k \neq 0 \text{ is any constant})$$

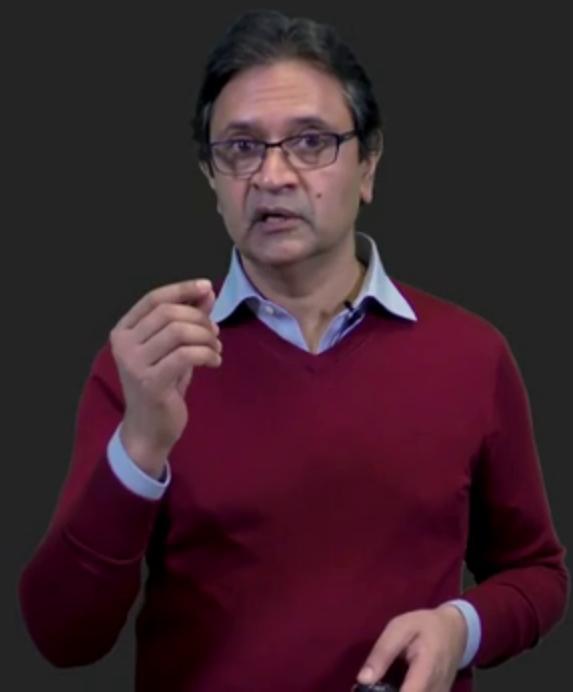


Scale of Projection Matrix

Projection matrix acts on homogenous coordinates.

We know that:

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} \equiv k \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} \quad (k \neq 0 \text{ is any constant})$$



Scale of Projection Matrix

Projection matrix acts on homogenous coordinates.

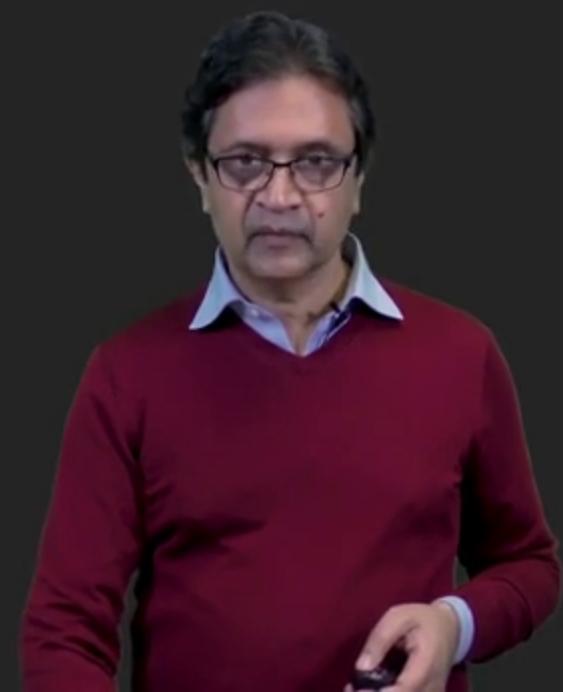
We know that:

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} \equiv k \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} \quad (k \neq 0 \text{ is any constant})$$

That is:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} \equiv k \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

Therefore, Projection Matrices P and kP produce the same homogenous pixel coordinates.



Scale of Projection Matrix

Projection matrix acts on homogenous coordinates.

We know that:

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} \equiv k \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} \quad (k \neq 0 \text{ is any constant})$$

That is:

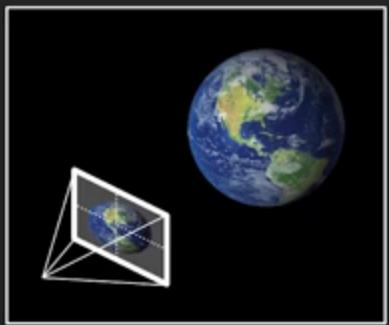
$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} \equiv k \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

Therefore, Projection Matrices P and kP produce the same homogenous pixel coordinates.

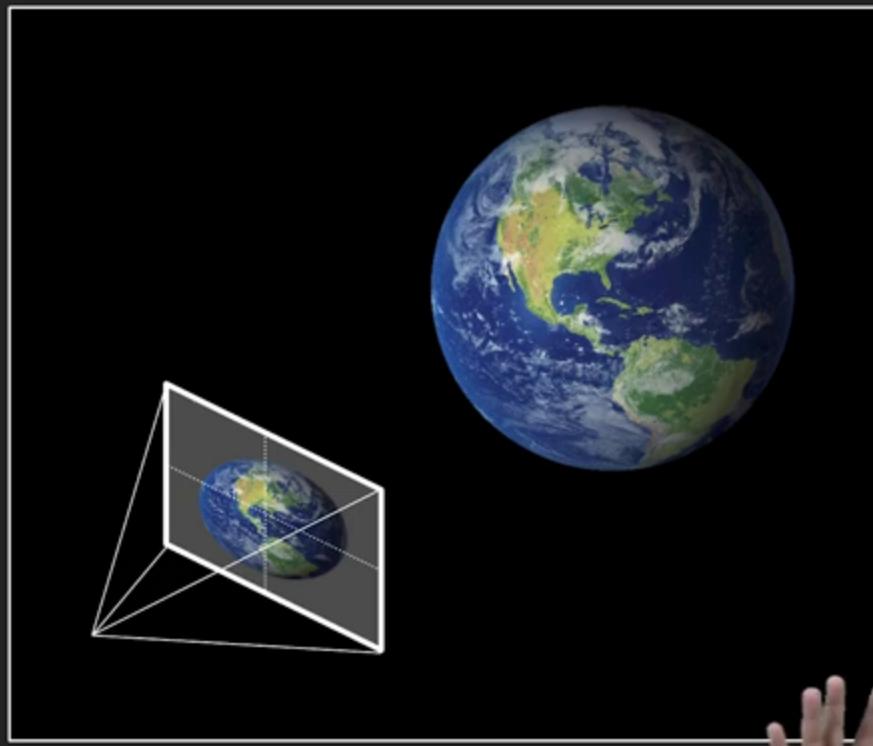
Projection Matrix P is defined only up to a scale.



Scale of Projection Matrix

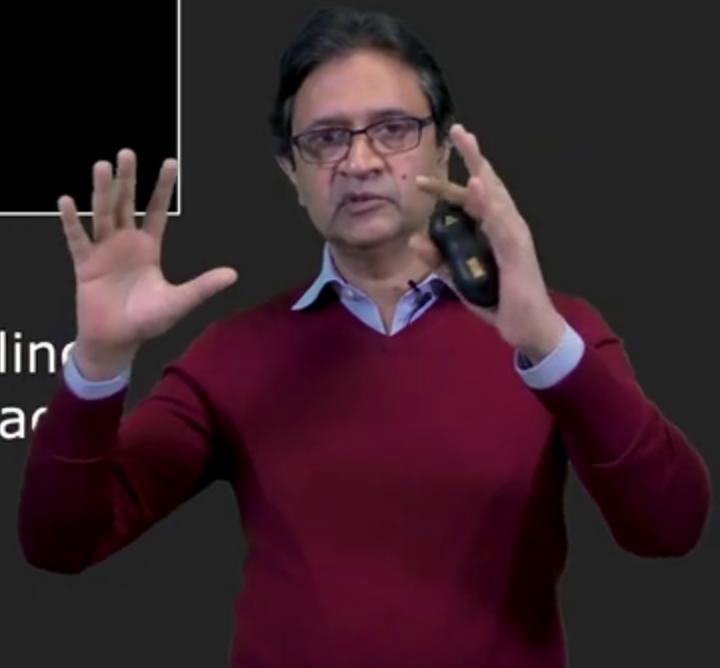


Scale = k_1

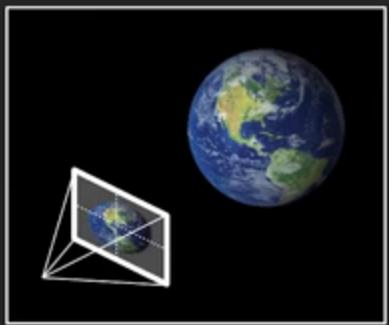


Scale = k_2

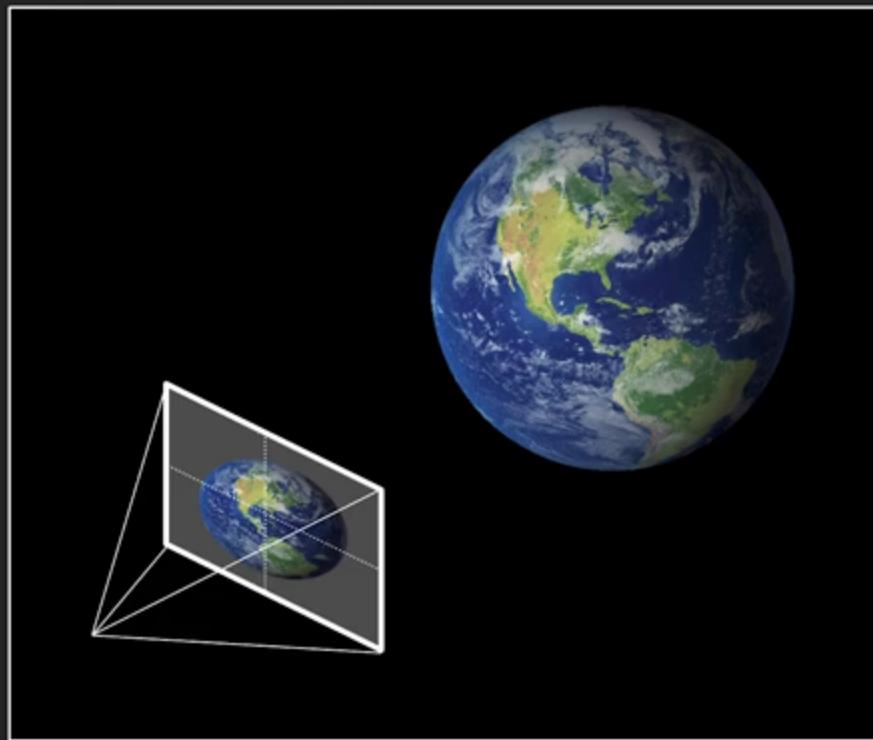
Scaling projection matrix, implies simultaneously scaling the world and camera, which does not change the image



Scale of Projection Matrix



Scale = k_1

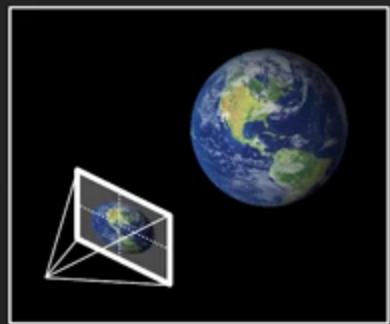


Scale = k_2

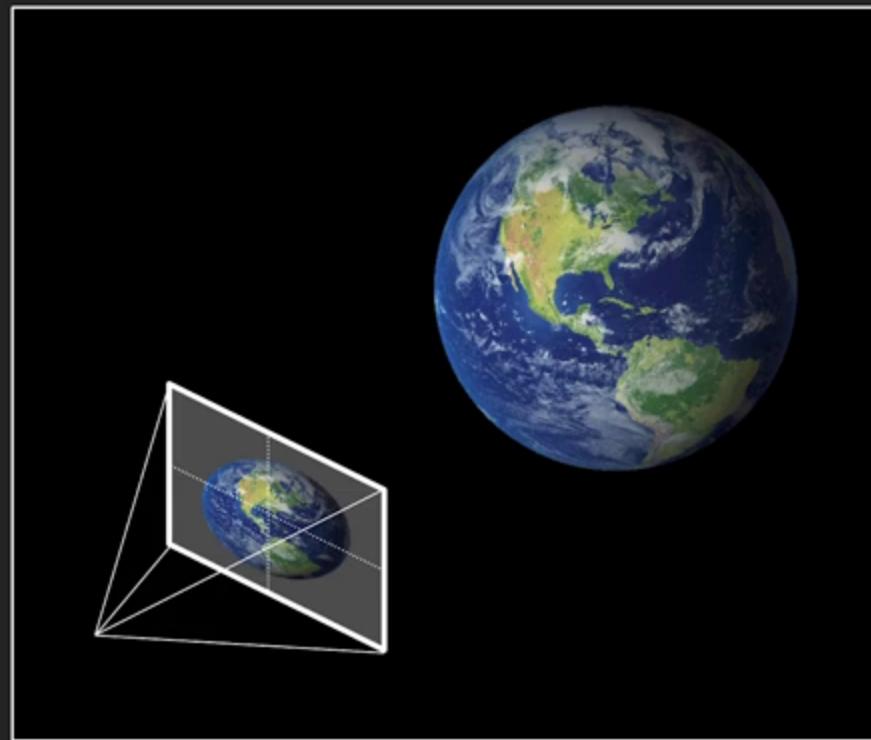
Scaling projection matrix, implies simultaneously scaling the world and camera, which does not change the image.



Scale of Projection Matrix



Scale = k_1



Scale = k_2

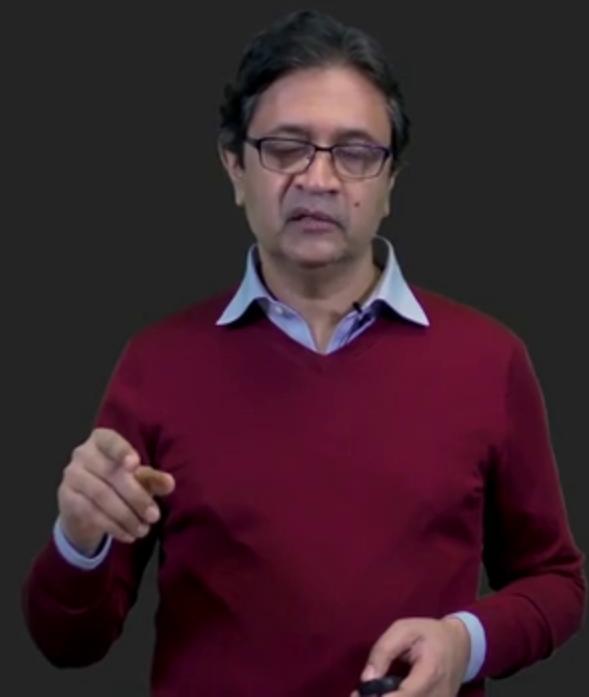
Scaling projection matrix, implies simultaneously scaling the world and camera, which does not change the image.

Set projection matrix to some arbitrary scale!



Least Squares Solution for P

Option 1: Set scale so that: $p_{34} = 1$



Least Squares Solution for P

Option 1: Set scale so that: $p_{34} = 1$

Option 2: Set scale so that: $\|\mathbf{p}\|^2 = 1$



Least Squares Solution for P

Option 1: Set scale so that: $p_{34} = 1$

Option 2: Set scale so that: $\|\mathbf{p}\|^2 = 1$

We want $A\mathbf{p}$ as close to 0 as possible and $\|\mathbf{p}\|^2 = 1$:

$$\min_{\mathbf{p}} \|A\mathbf{p}\|^2 \text{ such that } \|\mathbf{p}\|^2 = 1$$



Least Squares Solution for P

Option 1: Set scale so that: $p_{34} = 1$

Option 2: Set scale so that: $\|\mathbf{p}\|^2 = 1$

We want $A\mathbf{p}$ as close to 0 as possible and $\|\mathbf{p}\|^2 = 1$:

$$\min_{\mathbf{p}} \|A\mathbf{p}\|^2 \text{ such that } \|\mathbf{p}\|^2 = 1$$

$$\min_{\mathbf{p}} (\mathbf{p}^T A^T A \mathbf{p}) \text{ such that } \mathbf{p}^T \mathbf{p} = 1$$

(Similar to Solving Homography in Image Stitching)



Least Squares Solution for P

Option 1: Set scale so that: $p_{34} = 1$

Option 2: Set scale so that: $\|\mathbf{p}\|^2 = 1$

We want $A\mathbf{p}$ as close to 0 as possible and $\|\mathbf{p}\|^2 = 1$:

$$\min_{\mathbf{p}} \|A\mathbf{p}\|^2 \text{ such that } \|\mathbf{p}\|^2 = 1$$

$$\min_{\mathbf{p}} (\mathbf{p}^T A^T A \mathbf{p}) \text{ such that } \mathbf{p}^T \mathbf{p} = 1$$



(Similar to Solving Homography in Image Stitching)

Least Squares Solution for P

Option 1: Set scale so that: $p_{34} = 1$

Option 2: Set scale so that: $\|\mathbf{p}\|^2 = 1$

We want $A\mathbf{p}$ as close to 0 as possible and $\|\mathbf{p}\|^2 = 1$:

$$\min_{\mathbf{p}} \|A\mathbf{p}\|^2 \text{ such that } \|\mathbf{p}\|^2 = 1$$

$$\min_{\mathbf{p}} (\mathbf{p}^T A^T A \mathbf{p}) \text{ such that } \mathbf{p}^T \mathbf{p} = 1$$

Define Loss function $L(\mathbf{p}, \lambda)$:

$$L(\mathbf{p}, \lambda) = \mathbf{p}^T A^T A \mathbf{p} - \lambda(\mathbf{p}^T \mathbf{p} - 1)$$

(Similar to Solving Homography in Image Stitching)



Least Squares Solution for P

Option 1: Set scale so that: $p_{34} = 1$

Option 2: Set scale so that: $\|\mathbf{p}\|^2 = 1$

We want $A\mathbf{p}$ as close to 0 as possible and $\|\mathbf{p}\|^2 = 1$:

$$\min_{\mathbf{p}} \|A\mathbf{p}\|^2 \text{ such that } \|\mathbf{p}\|^2 = 1$$

$$\min_{\mathbf{p}} (\mathbf{p}^T A^T A \mathbf{p}) \text{ such that } \mathbf{p}^T \mathbf{p} = 1$$

Define Loss function $L(\mathbf{p}, \lambda)$:

$$L(\mathbf{p}, \lambda) = \mathbf{p}^T A^T A \mathbf{p} - \lambda(\mathbf{p}^T \mathbf{p} - 1)$$

(Similar to Solving Homography in Image Stitching)



Constrained Least Squares Solution

Taking derivatives of $L(\mathbf{p}, \lambda)$ w.r.t \mathbf{p} : $2A^T A \mathbf{p} - 2\lambda \mathbf{p} = \mathbf{0}$

$$A^T A \mathbf{p} = \lambda \mathbf{p}$$

Eigenvalue Problem



Constrained Least Squares Solution

Taking derivatives of $L(\mathbf{p}, \lambda)$ w.r.t \mathbf{p} : $2A^T A \mathbf{p} - 2\lambda \mathbf{p} = \mathbf{0}$

$$A^T A \mathbf{p} = \lambda \mathbf{p}$$

Eigenvalue Problem

Eigenvector \mathbf{p} with **smallest eigenvalue** λ of matrix $A^T A$ minimizes the loss function $L(\mathbf{p})$.



Constrained Least Squares Solution

Taking derivatives of $L(\mathbf{p}, \lambda)$ w.r.t \mathbf{p} : $2A^T A \mathbf{p} - 2\lambda \mathbf{p} = \mathbf{0}$

$$A^T A \mathbf{p} = \lambda \mathbf{p}$$

Eigenvalue Problem

Eigenvector \mathbf{p} with **smallest eigenvalue λ** of matrix $A^T A$ minimizes the loss function $L(\mathbf{p})$.

