Unit 2
DERIVATIVE AND ANALYTIC FUNCTION

Limit of a function of a complex variable:

A single valued function f(z) of a complex variable z is said to has limit $l = \alpha + i\beta$ if $\lim_{z \to z_0} f(z) = l$.

Continuity of a function of a complex variable:

A single valued function f(z) of a complex variable z is called continuous at a point $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

Differentiability of a complex function:

A complex valued function f(z) is called differentiable at $z = z_0$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z - z_0) - f(z_0)}{z - z_0}.$$

Analytic function:

A complex valued function f(z) is called analytic at a point $z = z_0$ in the domain D if f(z) is defined and differentiable at each point in a neighborhood of z_0.

A complex valued function f(z) is called analytic in the domain D if f(z) is defined and differentiable at each point in D.

Cauchy–Riemann's (C–R) Equation:

A function f(z) = u + iv where u = u(x, y) and v = v(x, y), is analytic in a domain D. Then we say f(z) satisfies the Cauchy–Riemann's (i.e. C–R) equation if the partial derivatives u_x, u_y, v_x, v_y exist and

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Cauchy–Riemann's (C–R) Equation in Polar Form:

A function f(z) = u + iv where u = u(x, y) and v = v(x, y) and $x = r\cos\theta$, $y = r\sin\theta$, is analytic in a domain D. Then we say f(z) satisfies the Cauchy–Riemann's (i.e. C–R) equation if the partial derivatives u_r, u_θ, v_r, v_θ exist and

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad u_\theta = -r\, v_r.$$

***Note*:** A function f(z) is analytic in D if and only if it satisfies the C–R equation.

Laplace Equation:

A function f(z) = u + iv where u = u(x, y) and v = v(x, y), is analytic in a domain D. Then we say the function u satisfies the Laplace equation if

$$\nabla^2 u = u_{xx} + u_{yy} = 0.$$

As similar, the function v satisfies the Laplace equation if

$$\nabla^2 v = v_{xx} + v_{yy} = 0.$$

Harmonic function:

Let function f(z) = u + iv where u = u(x, y) and v = v(x, y), is analytic in a domain D. Then we say the function u is harmonic function if it satisfies its Laplace equation i.e. $\nabla^2 u = u_{xx} + u_{yy} = 0$.

In this case v is called complex conjugate of u.

As similar, the function v is harmonic function if it satisfies its Laplace equation i.e. $\nabla^2 v = v_{xx} + v_{yy} = 0$.

In this case u is called complex conjugate of v.

Theorem (Cauchy–Riemann equation)

(Necessary condition for analyticity of a function)

Let* f(z) = u(x, y) + i v(x, y) *be defined and continuous in some neighborhood of a point* z = x + iy *and differentiable at z itself. Then at that point,

$$\mathbf{u_x = v_y} \quad \textit{and} \quad \mathbf{u_y = -v_x}.$$

***Hence, if* f(z) = u + iv *is analytic in a domain* D, *those partial derivatives exists and satisfy* $\frac{\partial u}{\partial x} = \frac{\partial y}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ *at all points of* D.**

2015 Spring Q. No. 1(a): Show that the necessary condition for analyticity of f(z) = u + iv, is $u_x = v_y$ and $u_y = -v_x$.

2016 Fall Q. No. 1(a): Show that if the function f(z) is analytic then show that $u_x = v_y$ and $u_y = -v_x$.

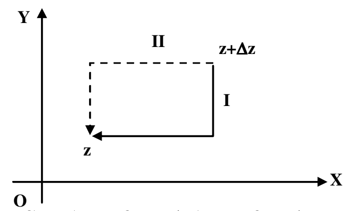
Proof: We have f(z) = u + iv is differentiable. Then f '(z) exists at z itself.

where,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x + \Delta x,\ y + \Delta y) + iv(x + \Delta x,\ y + \Delta y)] - [u(x,\ y) + iv(x,\ y)]}{\Delta x + i\Delta y} \qquad \ldots (1)$$

Here Δz approach to zero along any path in a neighborhood of z. Thus we may choose the two paths I and II in figure and then equate the results.



Choose path I. on this path we observe x is variate and y is constant. So, $\Delta x \to 0$ and $\Delta y = 0$. Then the equation (1) becomes

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x,\ y) - u(x,\ y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x,\ y) - v(x,\ y)}{\Delta x}$$

$$\Rightarrow \qquad f'(z) = u_x + iv_x \qquad \ldots (2)$$

by definition of partial derivatives.

Next, choose path II. on this path we observe y is variate and x is constant. So, $\Delta y \to 0$ and $\Delta x = 0$. Then the equation (1) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x + \Delta x,\ y) - u(x,\ y)}{i\Delta y} + i \lim_{\Delta x \to 0} \frac{v(x,\ y + \Delta y) - v(x,\ y)}{i\Delta y}$$

$$\Rightarrow \qquad f'(z) = \frac{1}{i} u_y + v_y = -iu_y + v_y$$

$$\Rightarrow \qquad f'(z) = v_y - iu_y \qquad \ldots (3)$$

We have f '(z) exists, implies from (2) and (3)

$$u_x + iv_x = v_y - iu_y$$

Comparing the real and imaginary value we get

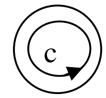
$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

Thus, we get if f(z) is analytic then $u_x = v_y$ and $u_y = -v_x$.

Unit 4

COMPLEX INTEGRATION

Contour:



A simple closed path is a contour.

Contour Integral

An integral over a contour, is a contour integral. If a curve c has a closed path then the integration over the curve c, is called line integration. It is denoted as $\int_c f(z)\,dz$ where z be any point in the curve.

Green's Theorem in Plane

Let R be a closed bounded region in xy–plane whose boundary c consists of finitely many smooth curves. Let, $F_1(x, y)$ and $F_2(x, y)$ are functions that are continuous and have continuous partial derivatives everywhere in R. Then,

$$\oint_c (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Note: (i) $\oint_c (udx - vdy) = \iint_R (-v_x - u_y)\,dxdy$ (ii) $\oint_c (udy + vdx) = \iint_R (u_x - v_y)\,dxdy$

Cauchy's Integral Theorem

If f(z) is analytic in a simply connected domain D then $\oint_c f(z)\,dz = 0$, for every simple closed path c in D.

Proof: Let $f(z) = u + iv$ and $z = x + iy$.

Then,

$$\oint_c f(z)\,dz = \oint_c (u + iv)(dx + idy)$$

$$= \oint_c (udx - vdy) + i\oint_c (udy + vdx) \qquad \ldots\ldots\ldots\ldots\ldots.. (1)$$

Since, f(z) is analytic in D. So, it satisfies the C–R equation. That means u_x, u_y, v_x and v_y exist and

$$u_x = v_y \quad \ldots (2) \quad \text{and} \quad u_y = -v_x. \quad \ldots (3)$$

By Green's theorem,

$$\oint_c (udx - vdy) = \iint_R (-v_x - u_y)\,dx\,dy = \iint_R (0)\,dx\,dy = 0 \qquad \text{[Using (3)]}$$

and,

$$\oint_c (udy + vdx) = \iint_R (u_x - v_y)\,dx\,dy = \iint_R (0)\,dx\,dy = 0 \qquad \text{[Using (2)]}$$

Now, using these results in (1) then,

$$\oint_c f(z)\,dz = 0 + i.0 = 0.$$

Cauchy's Integral Formula

Let f(z) be analytic in a simply connected domain D. Then, for any point z_0 in D and any simple closed path c in D that encloses z_0 such that

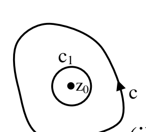
$$\oint_c \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

where, the path c is taken in counterclockwise direction.

Proof: Let c be a simple closed path in D. Consider a circle c_1 in c having radius r and centre at z_0.

Since, the function f(z) is analytic in D then $\frac{f(z)}{z - z_0}$ is analytic in the simply connected region bounded by c and c_1. Therefore by Cauchy's Integral Theorem,

$$\oint_c \frac{f(z)}{z - z_0}\,dz - \oint_{c_1} \frac{f(z)}{z - z_0}\,dz = 0.$$



$$\Rightarrow \oint_c \frac{f(z)}{z - z_0}\,dz = \oint_{c_1} \frac{f(z)}{z - z_0}\,dz \qquad \ldots\ldots\ldots\ldots (i)$$

Put $z - z_0 = re^{i\theta}$ on c_1. So, $dz = r\, ie^{i\theta}\, d\theta$. Since c_1 is a closed contour so, the, function varies from 0 to 2π. Then,

$$\oint_{c_1} \frac{dz}{z - z_0}\, dz = \oint_0^{2\pi} \frac{r\, i\, e^{i\theta}\, d\theta}{r\, e^{i\theta}} = i.2\pi \qquad \text{………… (ii)}$$

Since, f(z) is analytic, so it is continuous at $z = z_0$. Then by definition, for any $\varepsilon > 0$ there exists $\delta > 0$ such that,

$$|z - z_0| < \delta \qquad \Rightarrow |f(z) - f(z_0)| < \varepsilon \qquad \text{………… (iii)}$$

Since,

$$\oint_{c_1} \frac{f(z)}{z - z_0}\, dz = \oint_{c_1} \frac{f(z_0)}{z - z_0}\, dz + \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \qquad \text{[using (ii)]}$$

$$= 2\pi i\, f(z_0) + \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \qquad \text{……….. (iv)}$$

Here,

$$\left| \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leq \oint_{c_1} \frac{|f(z) - f(z_0)|}{|z - z_0|}\, |dz| < \frac{\varepsilon}{\delta}.2\pi\delta = 2\pi\varepsilon$$

Since, ε be any arbitrary positive number. So, if we made ε is sufficiently small then,

$$\oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \approx 0 \qquad \text{………….. (v)}$$

Hence, from (i), (iv) and (v) then we get,

$$\oint_{c} \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0).$$

Unit 5

SEQUENCE AND SERIES

Taylor Series and Maclaurin's Series

Statement: ***If*** **f(z)** ***be analytic in*** **c** ***with centre at*** **'a'** ***and radius*** $\mathbf{r_0}$***, then at each point*** **z** ***inside*** **c*****, the series***

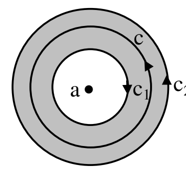
$$f(a) + f'(a)(z - a) + \frac{f''(a)}{2!}(z - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(z - a)^n + \ldots$$ ***converges to*** **f(z)*****. That is***

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n, \text{ where } a_n = \frac{f^{(n)}(a)}{n!}$$

Laurent Series

Statement: ***If*** **f(z)** ***is analytic on two concentric circles*** $\mathbf{c_1}$ ***and*** $\mathbf{c_2}$ ***with centre at*** **a*****, and also in the annular region*** **R** ***bounded by*** $\mathbf{c_1}$ ***and*** $\mathbf{c_2}$***, then at any point z in*** **R, f(z)** ***can be expressed as a convergent series of positive and negative power of*** **(z – a)** ***in the form***

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=1}^{\infty} b_n (z - a)^{-n}$$



where
$$a_n = \frac{1}{2\pi i}\int_{c'} \frac{f(w)}{(w - a)^{n+1}}\,dw, \quad n = 0, 1, 2, 3, \ldots$$

and
$$b_n = \frac{1}{2\pi i}\int_{c'} \frac{f(w)}{(w - a)^{-n+1}}\,dw, \quad n = 1, 2, 3, 4, \ldots$$

where **c** ***being any simple closed curve lying within the annular and encircling the inner boundary of the annular region*** **R.**

Unit 6

ZEROS AND SINGULARITIES OF A FUNCTION

Singularity:

A function f(z) is said to have a singularity at a point $z = z_0$ if f(z) is not analytic at that point but f(z) is analytic at all other points in the neighborhood of the point $z = z_0$.

Types of Singularity:

We have three types of singularities which are

(a) Isolated Singularity:

A function f(z) is said to have isolated singularity at $z = z_0$ if f(z) has exactly one singular point z_0 in the neighborhood of f(z).

(b) Removable Singularity:

Let, $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$ is analytic in $|z - a| < R$ but is not analytic at z = a then we call f(z) has removable singularity at z = a.

(c) Essential Singularity:

Let, $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(z - a)^n}$ then we call f(z) has essential singular point at z = a.

Pole of a Complex Function f(z):

If a complex function f(z) is defined as $f(z) = \sum_{n=0}^{m} \frac{a_n}{(z - a)^n}$ then z = a is known as pole of f(z).

Note: If m = 1 then we call the pole is simple pole.

Zeros of a Complex Function f(z):

A complex function f(z) is said to have zeros at z = a if f(a) = 0.

Note: If z = a is zeros of f(z) and f(a) = 0 but $f'(a) \neq 0$ then z = a is simple zeros of f(z).

If z = a is zeros of f(z) and $f(a) = 0 = f'(a) = f''(a) = \dots \dots \dots = f^{n-1}(a)$ but $f^{n}(a) \neq 0$ then z = a is zeros of f(z) of order n.

NOTE: If f(a) = 0 then we called f(z) has zeros at z = a and if f(a) = ∞ then we called f(z) has singularity at z = a.

Residue:

If a complex function f(z) has singular point $z = z_0$ in C then the coefficient of $\left(\frac{1}{z - z_0}\right)$ is called residue of f(z).

Cauchy's Residue Theorem

Let f(z) be analytic in a closed contour c except at finitely many points $z_1, z_2, \dots, z_k$ where each z_j for j = 1, 2, …, k lie in c. Then,

$$\oint_c f(z)\, dz = 2\pi i \sum_{j=1}^{k} \underset{z = z_j}{\text{Res}} [f(z)].$$

Proof:

We have f(z) is analytic in a closed contour except at $z_1, z_2, \dots, z_k$ where each z_j for j = 1, 2, …, k lie in c. Let us enclose each singular points z_j by a circle c_j^* with as small as possible radius so that no one circle intersects to another where c has counter clockwise direction and c_j^* has clockwise.

Then f(z) is analytic in the multiply connected domain D bounded by c and $c_1^*, c_2^*, \dots, c_k^*$ and on the entire boundary of D.

Then by Cauchy's integral theorem,

$$\oint_c f(z)\, dz + \oint_{c_1^*} f(z)\, dz + \oint_{c_2^*} f(z)\, dz + \dots + \oint_{c_k^*} f(z)\, dz = 0.$$

$$\Rightarrow \oint_{c} f(z)\,dz + \sum_{j=1}^{k} \oint_{c_j^*} f(z)\,dz = 0.$$

where the integral c being taken in counter clockwise and the other integrals c_j^* has clockwise direction.
Now, reversing the path of integration in each integral c_j^*. Then

$$\oint_{c} f(z)\,dz = \sum_{j=1}^{k} \oint_{c_j} f(z)\,dz. \qquad \dots (1)$$

where all the integrals are now taken in counterclockwise direction.
Since the function has singular point $z = z_j$ in c_j for $j = 1, 2, \dots, k$. So, by definition of residue,

$$\oint_{c_j} f(z)\,dz = \operatorname*{Res}_{z = z_j} f(z) \quad \text{for } j = 1, 2, \dots, k.$$

Then (1) becomes,

$$\oint_{c} f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z = z_j} f(z).$$

This completes the proof.

Unit 9

Z TRANSFORM AND ITS APPLICATION

Definition (Z–Transform)

Let f(t) be the function of time t defined in the discrete time invariant system 0, T, 2T, …, then the Z–Transform of f(t) is defined by

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n},$$

where z be the complex number in the region where $\left|\frac{1}{z}\right| < R$ and t = nT.

Linearity property of Z–Transform

Statement: ***If f(t) and g(t) are any two function of t in the discrete time period* 0, T, 2T, … *then,***

$$\mathbf{Z\,[af(t) + bg(t) = a\,Z[f(t)] + b\,Z[g(t)],}$$

where* a *and* b *are two constants.

Proof: Let f(t) and g(t) are any two function of t in the discrete time period 0, T, 2T,……then,

$$Z\,[af(t) + bg(t) = az\,[f(t)] + bg(t)]\,z^{-n} = a\sum_{n=0}^{\infty} f(t)\,z^{-n} + b\sum_{n=0}^{\infty} g\,(t)\,z^{-n} = a\,z[f(t)] + b\,z[g(t)]$$

This proves that Z–Transform is linear.

First Shifting Theorem

Statement: If Z[f(t)] = F(z) then Z[e^{-at} f(t)] = F(e^{aT}z) = [f(z)]$_{z \to z\,e^{aT}}$.

Proof: Let,

$$Z\,[f(t)] = F(z) = \sum_{n=0}^{\infty} f(nT)\,z^{-n}$$

Now,

$$Z\,[e^{-at} f(t)] = \sum_{n=0}^{\infty} e^{-at} f(t)\,z^{-n} = \sum_{n=0}^{\infty} e^{-anT} f(nT)\,z^{-n} = \sum_{n=0}^{\infty} f(nT)\,(e^{aT}z)^{-n} = F[ze^{aT}] = [f(z)]_{z\to ze^{aT}}$$

Thus, $Z\,[e^{-at} f(t)] = [f(z)]_{z\to ze^{aT}}$.

Second Shifting Theorem:

Statement: If Z[f(t)] = F(z) then Z[f(t + T)] = z [F(z) – f(0)]

Proof: Let

$$Z[f(t)] = F(z) = \sum_{n=0}^{\infty} f(t)\,z^{-n} = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

Now,

$$Z\,[f(t + T)] = \sum_{n=0}^{\infty} f(t + T)\,z^{-n} = \sum_{n=0}^{\infty} f(nT + T)\,z^{-n} = \sum_{n=0}^{\infty} f[(n + 1)T]\,z^{-n}$$

Put, n + 1 = k then,

$$= \sum_{k=1}^{\infty} f(kT)\,z^{-(k-1)}$$

$$= z\sum_{k=1}^{\infty} f(kT)\,z^{-k}$$

$$= z\left[\sum_{n=0}^{\infty} f(kT)z^{-k} - f(0)\right] = z\,[F(z) - f(0)].$$

Thus, Z [f(t + T)] = z [F(z) – f(0)].

Initial value theorem:

If Z{f(t)} = F(z) then, f(0) = $\lim_{z\to\infty}$ F(z) and $\lim_{z\to\infty}$ z F(z) = f(1) as f(0) = 0.

Proof: Let, $Z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(t)\, z^{-n} = f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + \ldots\ldots$

So,

$$\lim_{z\to\infty} F(z) = f(0) \qquad [\because 1/z \to 0 \text{ as } z \to\infty]$$

Also, set f(0) = 0 then,

$$F(z) = f(1)\, z^{-1} + f(2)\, z^{-2} + \ldots\ldots$$

So, $\qquad z\, F(z) = f(1) + f(2)\, z^{-1} + \ldots..$

Then, $\qquad \lim_{z\to\infty} z\, F(z) = f(1).$

Final value theorem:

If $Z\{f(t)\} = F(z)$ then, $\lim_{t\to\infty} f(t) = \lim_{z\to 1} \{(z-1)\, F(z)\}$

Proof: Let, $Z\{f(t)\} = F(z)$ and we have $Z\{f(t+T)\} = z\,(F(z) - f(0))$.

Now,

$$\begin{aligned}
\lim_{z\to 1} [z\, F(z) - z\, f(0) - F(z)] &= \lim_{z\to 1} [Z\{f(t+T)\} - F(z)] \\
&= \lim_{z\to 1} \left(\sum_{n=0}^{\infty} f(t+T)\, z^{-n} - F(z) \right) \\
&= \lim_{z\to 1} \sum_{n=0}^{\infty} [f(nT+T) - f(nT)]\, z^{-n} \\
&= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] \qquad [\because z^{-n} \to 1 \text{ as } z \to 1] \\
&= \lim_{n\to\infty} [f(T) + f(2T) + f(3T) + \ldots. - f(0) - f(T) - f(2T) - \ldots.] \\
&= \lim_{n\to\infty} f[(n+1)T] - f(0) \\
&= f(\infty) - f(0)
\end{aligned}$$

$$\Rightarrow \quad \lim_{z\to 1} (z-1)\, F(z) - f(0) = \lim_{t\to\infty} f(t) - f(0)$$

$$\Rightarrow \quad \lim_{z\to 1} (z-1)\, F(z) = \lim_{t\to\infty} f(t).$$

Convolution of functions:

Let f(t) and g(t) are any two functions. Then, the convolution of the functions is denoted by $f*g$ and defined as

$$(f*g)(n) = \sum_{k=0}^{\infty} f(kT)\, g[(n-k)\, T]$$

Convolution Theorem

If F(z) and G(z) are Z–Transform of f(t) and g(t) respectively then

$Z[f(t)*g(t)] = F(z)\, G(z).$

Proof: Let, F(z) and G(z) are Z–Transform of f(t) and g(t) respectively. Then,

$$\begin{aligned}
F(z)\, G(z) &= \sum_{n=0}^{\infty} f(kt)\, z^{-n} \sum_{n=0}^{\infty} g\,(mT)\, z^{-n} \\
&= [f(0) + f(T)z^{-1} + f(2T)z^{-2} + \ldots]\; [g(0) + g(T)z^{-1} + g(2T)z^{-2} + \ldots..] \\
&= [f(0)\, g(0)] + [f(0)\, g(T) + f(T)\, g(0)]z^{-1} + [f(0)g(2T) + f(T)g(T) + f(2T)\, g(0)]z^{-2} + \ldots.. \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} f(kT)g[(n-k)T] \right] z^{-n} \\
&= \sum_{n=0}^{\infty} (f*g)(t)z^{-n} \\
&= Z\{f*g\}
\end{aligned}$$

Thus, $Z[f(t)*g(t)] = F(z)\, G(z).$

Unit 3

CONFORMAL MAPPING

Problems

Q. Transform the rectangular region ABCD in z plane bounded by x = 1, x = 3, y = 0, and y = 3 under the transformation f(z) = w = z + 2 + i.

Solution: Given, function is,

$$f(z) = w = z + 2 + i = x + iy + 2 + i = (x + 2) + i\,(y + 1)$$

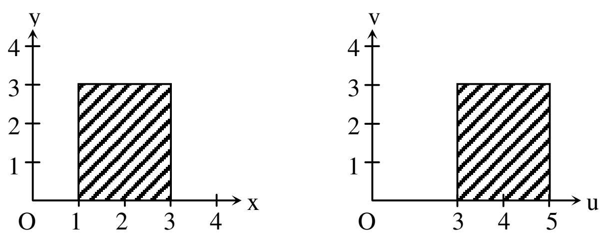
Since $f(z) = u + iv$ and $z = x + iy$. So, we get,

$$u = x + 2,\ v = y + 1$$

The transformation is as

in z–plane	$x = 1$	$x = 3$	$y = 0$	$y = 3$
in w–plane	$u = 3$	$u = 5$	$v = 1$	$v = 4$

Thus, the region in z–plane is mapped into w–plane as in figure.



z-plane W-plane

Q. If $u = 2x^2 + y^2$ and $v = \dfrac{y^2}{x}$, show that the curves u = constant and v = constant cuts orthogonally at all intersections but that the transformation w = u + iv is not conformal.

Solution: Given, function is,

$$u = 2x^2 + y^2 \text{ and } v = \frac{y^2}{x}$$

Then, $\quad u_x = 4x,\ u_y = 2y,\ v_x = -\dfrac{y^2}{x^2}, \qquad v_y = \dfrac{2y}{x}$

Here, $u_x = 4y \neq v_y$. So, by Cauchy–Riemann equation for analytic in a complex plane, the given function $f(z) = u + iv$ with $u = 2x^2 + y^2$, $v = \dfrac{y^2}{x}$, is not analytic. This implies $w = f(z)$ is not conformal mapping.

Q. Obtain the image of the infinite strip $0 < y < \dfrac{1}{2}$ under the transformation $w = \dfrac{1}{z}$.

Solution: Given that

$$f(z) = w = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

Comparing it with $f(z) = u + iv$ then,

$$u = \frac{x}{x^2 + y^2},\ v = \frac{-y}{x^2 + y^2}$$

imples

$$x = \frac{u}{u^2 + v^2},\ y = \frac{-v}{u^2 + v^2}$$

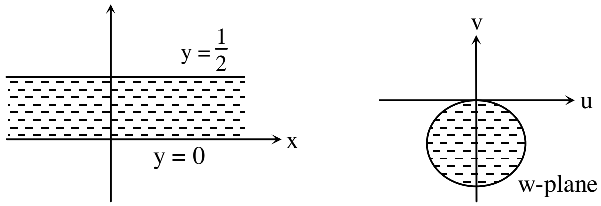
Here we have to observe the transformed image of the infinite strip $0 < y < 1/2$ under $w = 1/z$.

The transformation will as:

in z–plane	$y = 0$	$y = 1/2$
in w–plane	$v = 0$	$u^2 + (v + 1)^2 =$

Since $u^2 + (v + 1)^2 = 1$ is a circle with radius 1 and having center at $(0, -1)$.

Thus, the region in z–plane and image image in w–plane is as in figure.



Q. Determine the region of transformation $w = 2ze^{i(\pi/3)}$, where the region in the z–plane be bounded by x = 0, x = 1, y = 0, y = 2.

Solution: Given that

$$f(x) = w = 2ze^{i\pi/3} = 2z\,(\cos\pi/3 + i\sin\pi/3)$$
$$= 2z\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$= (x + iy)\,(1 + i\sqrt{3}\,)$$
$$= x - y\sqrt{3} + i\,(x\sqrt{3} + y)$$

Comparing it with f(z) = u + iv then,

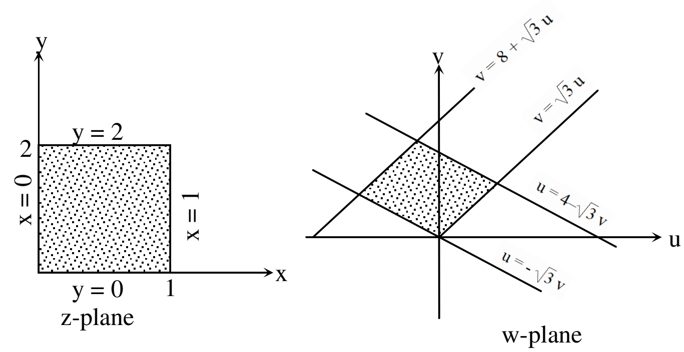
$$u = x - y\sqrt{3}\,,\quad v = x\sqrt{3} + y.$$

So that $x = \frac{u + \sqrt{3}\,v}{4}$,

Then the transformation form z (= x + iy)–plane to w (= u + iv) plane is as tabled.

in z–plane	x = 0	x = 1	y = 0	y = 2
in w–plane	$u = -\sqrt{3}\,v$	$u = 4 - \sqrt{3}\,v$	$v = \sqrt{3}\,u$	$v = 8 + \sqrt{3}\,u$

Now, the region in z–plane is mapped into w–plane as in figure.



Since (i) $u = -\sqrt{3}\,v$ passes through (0, 0) and $(-\sqrt{3}, 1)$. (ii) $u = u - \sqrt{3}\,v$ passes through (4, 0) and $(0, 4/\sqrt{3})$ (ii) $v = \sqrt{3}\,u$ passes through (0, 0) and $(1, \sqrt{3})$ (iv) $v = 8 + \sqrt{3}\,u$ passes through (0, 8) and $(1, 8 + \sqrt{3})$.

Q. Find the bilinear transformation which maps the points of z = 1, z = i, z = – i into the points w = i, w = 0, w = –i and find the image of |z| < 1.

Solution: Given that

$$z = 1,\ z = i,\ z = -\,i \text{ maps into } w = i,\ w = 0,\ w = -\,i.$$

We know the bilinear transformation of z, z_1, z_2, z_3 in z–plane and w, w_1, w_2, w_3 in w–plane is

$$\frac{(w - w_1)\,(w_2 - w_3)}{w_2 - w_1} = \frac{(z - z_1)\,(z_2 - z_3)}{(z - z_3)\,(z_2 - z_1)} \quad \ldots \text{(i)}$$

Substituting the given values in (i) then

$$\frac{(w - i)\,(0 + i)}{(w + i)\,(0 - i)} = \frac{(z - 1)\,(i + i)}{(z + i)\,(i - 1)}$$

$$\Rightarrow \frac{-\,(w - i)}{(w + i)} = \frac{(z - 1)\,(2i)}{(z + i)\,(i - 1)}$$

$$\Rightarrow \; -(w-i)(z+i)(i-1) = (w+i)(z-1)(2i)$$

$$\Rightarrow \; -w[(z+i)(i-1) + (z-1)2i] = i(z-1)2i - i(z+i)(i-1)$$

$$\Rightarrow \; -w(zi - z - 1 - i + 2iz - 2i) = -2z + 2 - i(iz - z - 1 - i)$$

$$\Rightarrow \; w[z + 1 + i(3 - 3z)] = -2z + 2 + z + iz + i - 1$$

$$\Rightarrow \; w = \frac{(1-z) + i(z+1)}{(z+1) + i(3-3z)} \qquad \ldots (i)$$

This is required bilinear transformation.

Problems

Q. Give the Cartesian representation of $(t, t^2 + 2, 0)$.

Solution: Given that,

$$(x, y, z) = (t, t^2 + 2, 0)$$

$$\Rightarrow \quad x = t,\ y = t^2 + 2,\ z = 0$$

This gives that $y = x^2 + 2$, $z = 0$

Q. Give the parametric representation of $x^2 + y^2 = 1$, $y = z$

OR 2005 Spring Q. No. 7(v)

Find the position vector of the curve $x^2 + y^2 = 1$, $y = z$ in parametrical representation.

Solution: Given that,

$$x^2 + y^2 = 1,\ y = z$$

Clearly, the given curve is ellipse in xyz-plane.

Put $x = r\cos t$, $y = r\sin t$, and $y = z$ with $r = 1$.

So, $x = \cos t$, $y = \sin t$, $z = \sin t$

Thus, $(x, y, z) = (\cos t, \sin t, \sin t)$

Q. Find a tangent vector and the corresponding unit tangent vector

$\vec{r}(t) = \cos t\ \vec{i} + 2\sin\ \vec{j}$ **at P** $\left(\frac{1}{2}, \sqrt{3}, 0\right)$. **[2008 Spring Q. No. 7(b)]**

Solution: Given that,

$$\vec{r}(t) = \cos t\ \vec{i} + 2\sin\ \vec{j} + 0.\vec{k} = (x, y, z)$$

So, $x = \cos t$, $y = 2\sin t$, $z = 0$.

And given point be, $P\left(\frac{1}{2}, \sqrt{3}, 0\right)$.

So, $\frac{1}{2} = \cos t$, $\sqrt{3} = 2\sin t$, $0 = 0$

$$\Rightarrow \quad \cos 60° = \cos t,\ \frac{\sqrt{3}}{2} = \sin t \text{ i.e. } \sin 60° = \sin t.$$

Then, $t = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sin 60°}{\cos 60°}\right) = 60°$

Since, $\vec{r}(t) = \cos t\ \vec{i} + 2\sin t\ \vec{j}$

So, $\vec{r}\,'(t) = -\sin t\ \vec{i} + 2\cos t\ \vec{j}$

$$= -\sin 60°\ \vec{i} + 2\cos 60°\ \vec{j} \quad \text{at P}$$

$$= -\frac{\sqrt{3}}{2}\ \vec{i} + 2\cdot\frac{1}{2}\ \vec{j} \quad \text{at P}$$

$$= \left(-\frac{\sqrt{3}}{2}, 1, 0\right)$$

Thus, tangent vector at P is, $\left(-\frac{\sqrt{3}}{2}, 1, 0\right)$

And, unit tangent vector at P is

$$\hat{r}'(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\left(-\frac{\sqrt{3}}{2}, 1, 0\right)}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + 1^2 + 0^2}}$$

$$= \frac{1}{\sqrt{\frac{3}{4} + 1}}\left(-\frac{\sqrt{3}}{2}, 1, 0\right)$$

$$= \frac{2}{\sqrt{7}}\left(-\frac{\sqrt{3}}{2}, 1, 0\right)$$

$$= \left(\frac{-\sqrt{3}}{\sqrt{7}}, \frac{2}{\sqrt{7}}, 0\right)$$

2003 Fall Q. No. 7(a) : What do you mean by tangent to a curve?

Solution: Tangent to a Curve:

Let C be a curve in space. The tangent on C at a point P of C is the limiting position of a straight line that through P and a point Q of C as Q tends to P along C.

2005 Fall Q. No. 7(a)

Define tangent and tangent plane of a curve at a point.

Solution: Tangent to a Curve:

Let C be a curve in space. The tangent on C at a point P of C is the limiting position of a straight line that through P and a point Q of C as Q tends to P along C.

Tangent Plane to a Curve:

Let C be a curve in space. The tangents on C at a point P of C, is called tangent plane.

Equation of and paraboloid, ellipsoid, hyperboloid.

Equation of Paraboloid:

The equation of paraboloid is $x^2 + y^2 = z$.

Equation of Ellipsoid:

The equation of hyperboloid of two sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Equation of Hyperboloid:

The equation of hyperboloid of is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Note:

The equation of hyperboloid of two sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$.

2005 Fall Q. No. 7(b); 2016 Fall Q. No. 7(d): Sketch the paraboloid $z = x^2 + y^2$.

Solution: The equation of paraboloid is $z = x^2 + y^2$ … (i)

For sketch:

(i) Symmetry: Since the paraboloid has x and y with degree 2, so it is symmetrical about x-axis and y-axis.

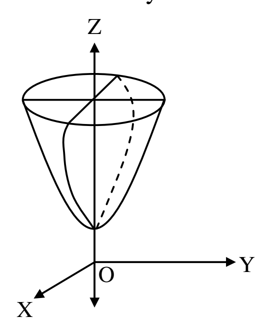
(ii) Intercept: Clearly the paraboloid has vertex at origin. So, it does not intersect the x-axis and y-axis and z-axis except at origin.

(iii) Plane section:

(a) In yz-plane: When x = 0 then (i) gives, $z = y^2$. This is a parabola having openward toward the positive z-axis.

(b) In zx-plane: When y = 0 then (i) gives, $z = x^2$. This is a parabola having openward toward the positive z-axis.

With the information the sketch of the hyperboloid is as in figure.



2006 Spring Q. No. 7(d); 2006 Fall Q. No. 7(e); 2007 Spring Q. No. 7(c)

Write equation of an ellipsoid. Sketch it with centre and axis of symmetry.

Solution: The equation of ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ … (i)

For sketch:

(i) Centre: Clearly (i) has centre at (0, 0, 0).

(ii) Symmetry: Since all three variables have same degree, so it is symmetrical about all three axes.

(iii) Intercept:

When y = 0 = z then $x^2 = a^2$. So, $x = \pm a$. Therefore, the figure cuts x-axis at $x = \pm a$.

When x = 0 = z then $y^2 = b^2$. So, $y = \pm b$. Therefore, the figure cuts y-axis at $y = \pm b$.

When x = 0 = y then $z^2 = c^2$. So, $z = \pm c$. Therefore, the figure cuts z-axis at $z = \pm c$.

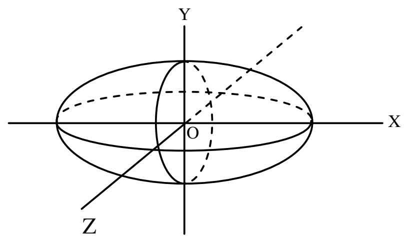
(iv) Plane section:

(a) In xy-plane: When z = 0 then (i) gives, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is an ellipse.

(b) In yz-plane: When x = 0 then (i) gives, $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. This is an ellipse.

(c) In zx-plane: When y = 0 then (i) gives, $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$. This is an ellipse.

With the information, the sketch of the ellipsoid is as in figure.



<u>2003 Fall Q. No. 7(b):</u> Write the equation of hyperboloid of two sheet and then sketch.

Solution: The equation of hyperboloid of two sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ … (i)

For sketch:

(i) Centre: Clearly (i) has centre at (0, 0, 0).

(ii) Symmetry: Since all three variables have same degree, so it is symmetrical about all three axes.

(iii) Intercept: When y = 0 = z then $x^2 = -a^2$. So, x has only imaginary value which is non-acceptable for sketch.

When x = 0 = z then $y^2 = -b^2$. So, y has only imaginary value which is also non-acceptable for sketch.

When x = 0 = y then $z^2 = c^2$. So, $z = \pm c$. Therefore, the figure cuts z-axis at $z = \pm c$.

(iv) Plane section:

(a) In xy-plane: When z = 0 then (i) gives, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$.

This is an imaginary ellipse.

(b) In yz-plane: When x = 0 then (i) gives,

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \quad \Rightarrow \quad \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1.$$

This is a hyperbola.

(c) In zx-plane: When y = 0 then (i) gives,

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = -1 \quad \Rightarrow \quad \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1.$$

This is a hyperbola.

With the information the sketch of the hyperboloid is as in figure.