Vedanta

Optional Mathematics

Teacher's Guide

Grade 10

PREFACE

This is a teacher's Guide of Vedanta Excel in Optional Mathematics to help the teacher's, in teaching learning process, who are teaching Optional Mathematics in secondary level in Grade 9 and 10.

I have tried to write this book in the form to help the teachers of Optional Mathematics regarding what are objectives, how to teach, how to solve problems what are required teaching materials and how to evaluate in the classroom effectively. At the end of each chapter, there are given some questions for more practice and evaluation of the students.

I hope that the book will be one of best friend of teachers who have been using Vedanta Excel in Optional Mathematics for grade 9 and 10. It helps the teachers to make lesson plan, to use required teaching materials to evaluate the students. It also helps the teachers providing required teaching notes.

The motto of **Vedanta Publication (P) Ltd** is "read, lead and succeed". I am hopeful that the book will also help to fulfill the objectives of the publication as well as the objectives of curriculum of Optional Mathematics.

The idea how to write this book is coined by our respectable senior Mathematics text book writer, educator Hukum Pd Dahal and heartfelt gratitude to him.

I am confident that the teachers will find this book as an invaluable teaching aid. I am thankful to all the teachers who have been using Vedanta Excel in Optional Mathematics.

My hearty thanks goes to Mr. Hukum Pd Dahal, Tara Bahadur Magar and P.L Shah, the series editors, for their invaluable efforts in giving proper shape to the series. I am also thankful to my colleague Mr Gyanendra Shrestha who helped me a lot during the preparation of the book.

I would like to thank chairperson Mr Suresh Kumar Regmi, Managing Director Mr. Jiwan Shrestha, Marketing Director Manoj Kumar Regmi for their invaluable suggestion and support during the preparation of the series in Optional Mathematics.

Last but not the least I am thankful to Mr Daya Ram Dahal and Pradeep Kandel, the computer and designing senior officer for their skill in designing the book in such an attractive form.

I'm profoundly grateful to the Vedanta Publication (P) LTD to get the series published. Valuable suggestion and comments from the concerned will be highly appreciated in days ahead.

Piyush Raj Gosain

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UNIT one

Functions

Simple Algebraic Functions and Trigonometric Functions and their graphs

Estimated teaching periods : 4

1. Objectives

At the end of the topics the students will be able to do the following:

S.N.	Levels	Objectives
(i)	Knowledge (K)	To define a function. To define simple algebraic function. i) linear function. ii) constant function. iii) identity function. iv) quadratic function. To define a trigonometric function. To define periodic functions. To define trigonometric functions as periodic functions.
(ii)	Understanding (U)	To give examples of simple algebraic functions and trigonometric function.
(iii)	Application (A)	To draw graphs of simple algebraic functions mentioned above.
(iv)	Higher Ability (HA)	To draw graphs of trigonometric functions $y=\sin x$, $y=\cos x$, $y=\tan x$

2. Teaching materials

- Colour chart paper with definition of algebraic and trigonometric functions.
- Graph papers.
- Cello tape
- Scissors.
- A table with trigonometric values from $-180°$ to $180°$ (standard angles only).

3. Teaching learning strategies

- Ask the following questions to the students.
  i) What is a function?
  ii) Draw graphs of $y=x$, $y=6$, $y=4x+3$.
- The teacher defines the following functions with examples and their graphs. Linear functions, identity function, constant function, quadratic function.
- Let the students draw the graph of the following functions. $y=-x$, $y=5$, $4x+3y=12$, etc.
- Discuss to draw graph of $y=ax^2$, $y=ax^2+bx+c$ with examples $y=x^2$, $y=x^2+3x+2$
- Paste the chart paper with trigonometric value table from $-180°$ to $180°$ (standard angle only) and use it to draw graph.
- The teacher explains how to draw graphs of $y=\sin x$, $y=\cos x$, $y=\tan x$
- Defining a periodic function and links it with trigonometric function.
  $y=\sin(2\pi+x) = \sin x$, $y=\cos(2\pi+x)$, $y=\tan(2\pi+x)$.

The sine and cosine function are periodic functions with period 2π and the tangent function is a periodic function with period π.

- Discuss how $y=\tan 90^\circ$ can be shown in graph paper?
- Discuss the curve of human heart beat as at a sine curve?

Some solved problems

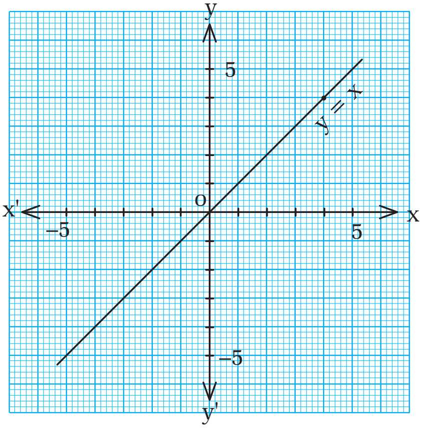
1. Define algebraic functions with examples and their graph.

Solution:

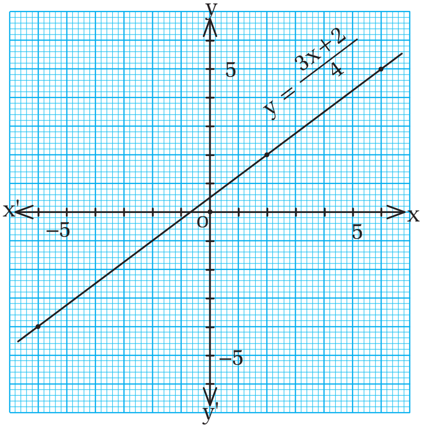
The functions which can be generated from a real variable x by a finite number of algebraic operations (addition, subtraction, multiplication, division and extraction of roots) are called algebraic functions.

Examples: i) $y=f(x)=x$ ii) $y=6$, iii) $y=f(x)=\frac{3x+2}{4}$

These algebraic functions can be plotted in the graph.



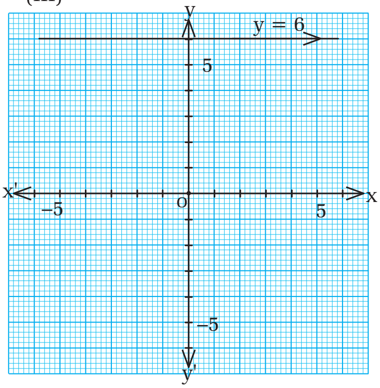
(i)



(iii)

i) $y=f(x)=x$ ii) $y=6$

x	–3	–1	0	1	4
y	–3	–1	0	1	4



(ii)

fig

iii) $y=f(x)=\frac{3x+2}{4}$

x	–6	2	6
y	–4	2	5

2. Define transcendental function with examples.

Solution:

The functions which are not algebraic are called transcendental functions. These types of functions are widely used and include will known functions like trigonometric functions, inverse trigonometric functions, exponential and logarithmic functions.

Examples: d) $y=f(x)=\sin x$, ii) $y=f(x)=\tan x$, iii) $y=e^x$ etc.

3. Define periodic functions with examples.

Solution:

A function f is said to be a periodic function in period k if $f(x+k)=f(x)$ where k is the least positive real number.

Examples a) $y=f(x)=$ sinex

We can write $f(x+2\pi)=\sin(x+2\pi)=\sin x$

Hence the sine function is a periodic function with period 2π.

b) $y=f(x)=\cos x$

We can write, $f(x+2\pi)=\cos(x+2\pi)=\cos x$

Hence, the cosine function is a periodic function with period 2π,

c) $y=f(x)=\tan x$

We can write, $f(x+\pi)=\tan(x+\pi)=\tan x$

Hence, the tangent function is a periodic function with period π.

In trigonometry, the values of trigonometric functions repeat after certain interval of angles.

Example a) $\text{Sin}(360^\circ+\theta)=\text{Sin}, \text{Sin}(360^\circ+60^\circ)=\text{Sin}60^\circ=\frac{\sqrt{3}}{2}$

i.e. $\text{Sin}420^\circ=\text{Sin}60^\circ=\frac{\sqrt{3}}{2}$

b) $\tan240^\circ=\tan(180^\circ+60^\circ)=\tan60^\circ=\sqrt{3}$

i.e. $\tan240^\circ=\tan60^\circ=\sqrt{3}$

4. Draw graph of $y=x^2-6x+8$

Solution:

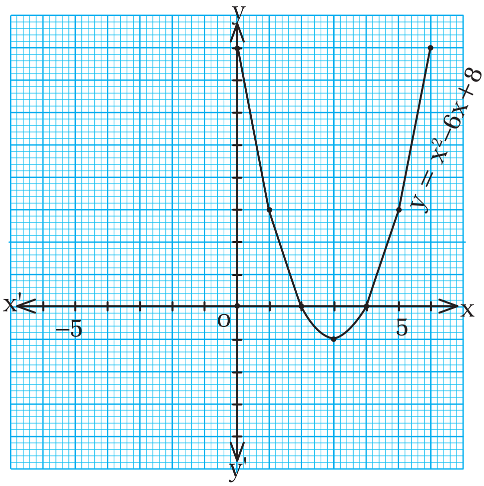
Here, $y=f(x)=x^2-6x+8$

The function is a quadratic in x. Comparing it with $y=ax^2+bx+c$, we get, $a=1$, $b=-6$, $c=8$ the equation represents a curve called parabolic.

$$\text{Vertex}=(h,k)=\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$$

$$=\left(-\frac{-6}{2.1}, \frac{4.1.8-(-6)^2}{4.1}\right)$$

$$=(3,-1)$$



Let us make a table to draw the curve

x	0	1	2	3	4	5
y	8	3	0	-1	0	3

5. Draw graph of curve from the given table.

a)

x	2	3	4	5	6
y	8	3	0	–1	0

b)

x	0	±1	±2	±3
y	0	1	4	9

c)

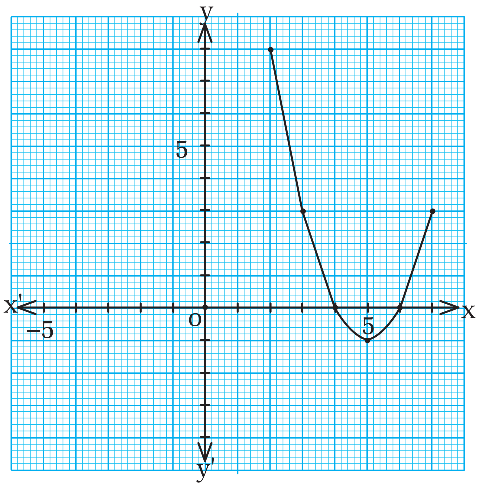
x	–3	–2	0	1
y	6	4	0	–2

Solution:

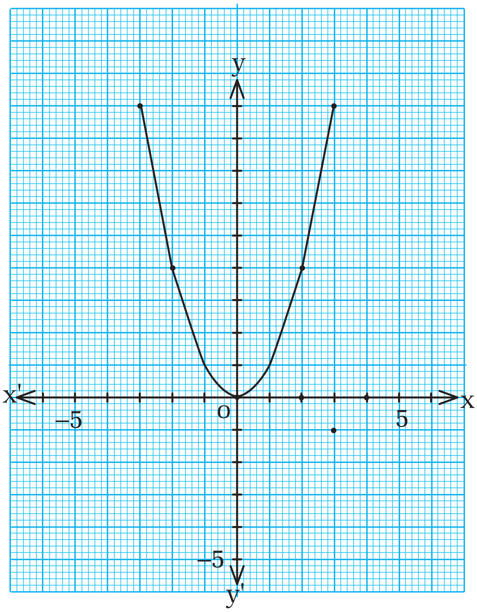
a) Here,

2	3	4	5	6
8	3	0	–1	0

Plotting the points (2,8), (4,0), (5,–1) and (6,0) joining them. We get a curve which is a parabola.



(a)



(b)

b) Here,

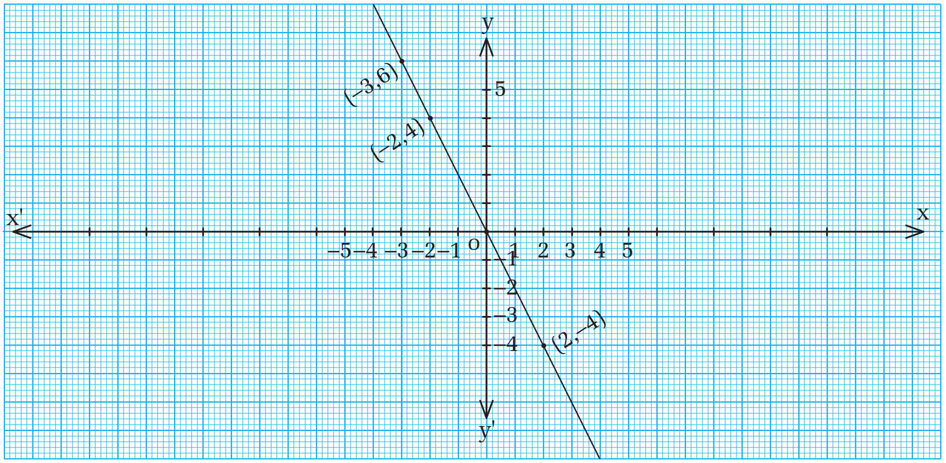
x	0	±1	±2	±3
y	0	1	4	9

Plotting the points (0,0), (1,1), (1,–1), (2,4), (–2,4), (3,9), (–3,9) joining them we get a curve which is parabola.

c) Here,

x	–3	–2	0	1	2
y	6	4	0	–2	–4

Plotting the points (–3,6), (–2,4), (0,0), (1,–2), (2,–4), we get a straight line as shown in the graph.



6. Draw the graphs of

a) $y=f(x)=\text{Sin}x$, $(-2\pi^c \le x \le 2\pi^c)$

b) $y=f(x)=\text{Cos}x$, $(-2\pi^c \le x \le 2\pi^c)$

c) $y=f(x)=\text{Tan}x$

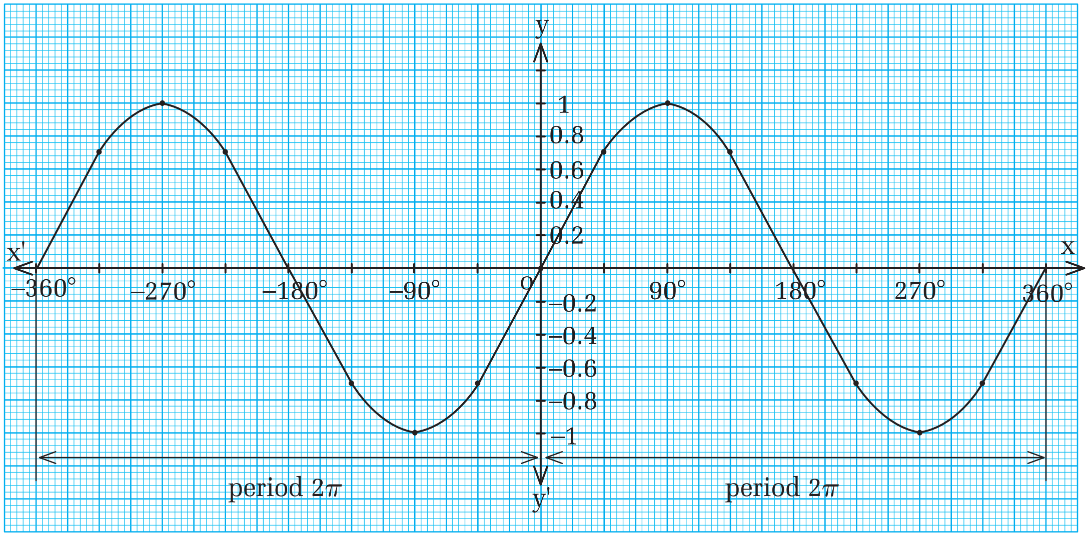
d) $y=f(x)=\tan x$, $\left(\frac{-3\pi^c}{4} \le x \le \frac{3\pi}{4}\right)$

Solution:

To draw the graphs of trigonometric functions we take angles in x-axis and their corresponding values of trigonometric functions are taken in y-axis. It will be easy to take the angles in degrees.

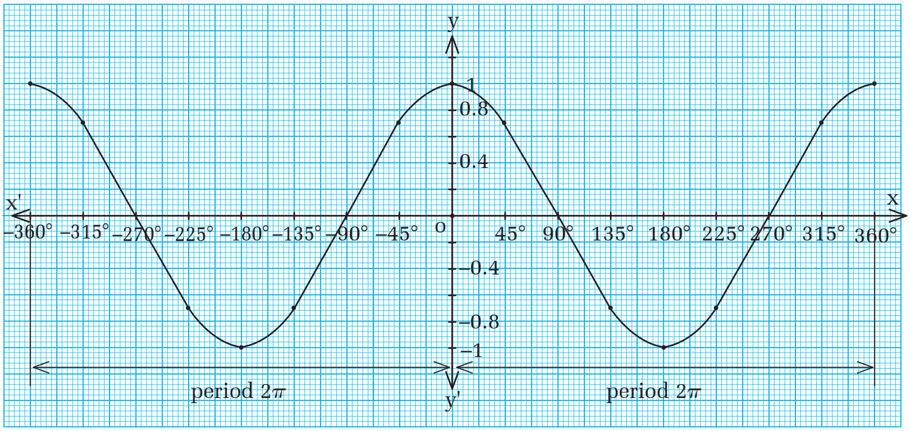
a) Let us make table to draw the graph of $y=\sin x$, $-360^\circ \le x \le 360^\circ$

x	–360	–315°	–270°	–225°	–180°	–135°	–90°	–45°	0	45°	90°	135°	180°	225°	270°	315°	360°
y	0	0.71	1	0.71	0	–0.71	–1	–0.71	0	0.71	1	0.71	0	–0.71	–1	–0.71	0



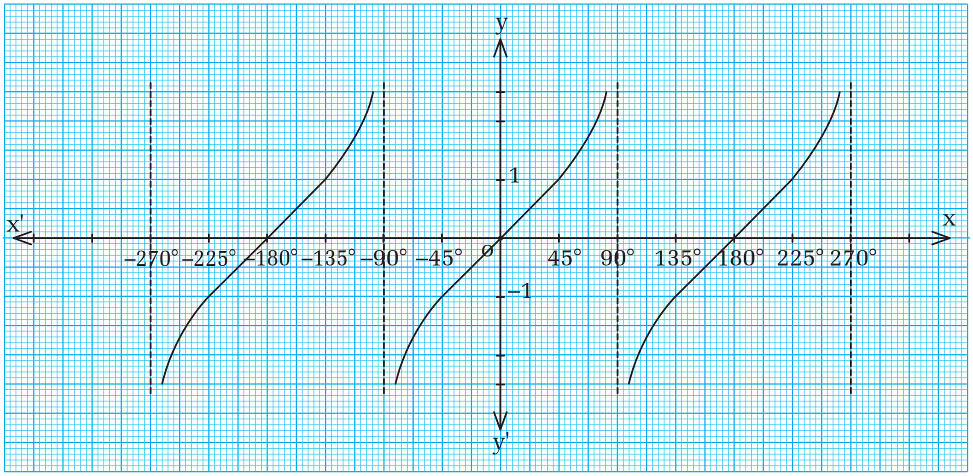
b) Let us make a table to draw graph of $y=\cos x$, $(-360^\circ \le x \le 360^\circ)$ or $(-2\pi \le x \le 2\pi)$

x	–360	–315°	–270°	–225°	–135°	–90°	–45°	0	45°	90°	135°	180°	225°	270°	315°	360°
y	1	0.71	0	–0.71	–1	0	0.71	1	0.71	0	–0.71	–1	–0.71	0	0.71	1



c) Lets us make a table to draw graph of y=tanx, $\frac{-3\pi}{2} \le x \le \frac{3\pi}{2}$

x	–180°	–135°	–90°	–45°	0	45°	90°	135°	180°
y	θ	1	∞	–1	0	1	∞	–1	0



Questions for practice

1. Draw graphs for the following functions.
   a) =2x+1 b) $f(x)=\frac{x+2}{3}$
2. Draw graphs for the following functions.
   a) $f(x) = -x^2$ b) $f(x) = x^2-3x+2$
3. Draw graphs for the following functions.
   a) $f(x) = -x^3$ b) $f(x) = -2x^3$
4. Draw graphs for the following functions.
   a) $f(x) = \sin x, -\pi \le x \le \pi$ b) $f(x) = \cos x, -\pi \le x \le \pi$
   c) $f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ d) $f(x) = \sin 2x, -180° \le x \le 180°$
5. Draw graph of $y=x^3$ and $y=-x^3$ on the same graph, take x=±1, ±2, ±3 etc.

Composite function

Estimate & teaching periods: 3 hours

1. Objectives

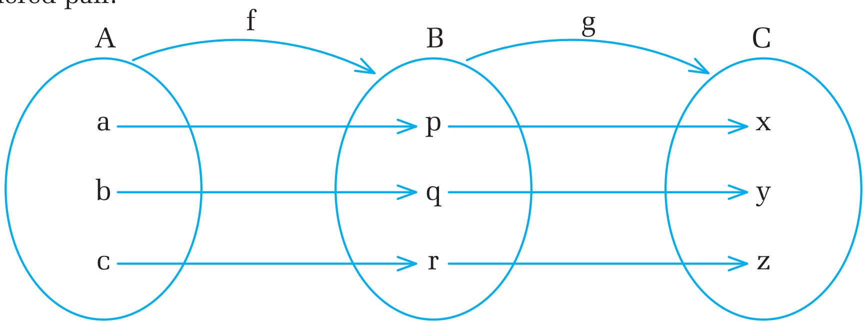
S.N.	Level	Objectives
1	Knowledge (K)	to define composite function of two functions.
2	Understanding (U)	to write diagram for composite function of two functions.
3	Application (A)	to write composite function as set of ordred pair form to show (gof) (x) $\neq$ (fog)(x)
4	Higher Ability (HA)	to write equation of composite function of given functions.

2. Teaching materials

- Chart paper with draw of composite functions.

3. Teaching learning strategies

- Discuss review concept of functions.
- Discuss water cycle (rain) as a combination of functions (evaporation and condensation).
- Teacher shows/draws arrow diagram for composite function of two functions and defines it.
- Explain how to write symbol for composition of two functions f and g, g and f.
- With the help of the arrow diagram with the elements of gof in the form of set of ordered pair.



Explain : $f:A \longrightarrow B$, $g : B \longrightarrow C$ then $gof : A \longrightarrow C$.

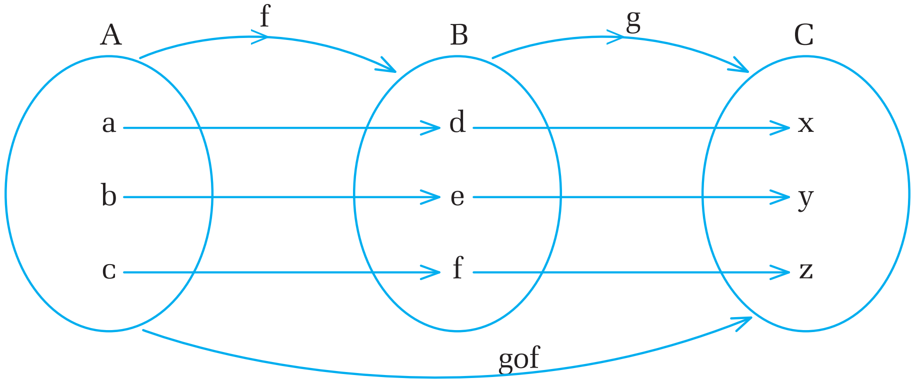
- From the above mapping diagram find f and g, then gof and fog.
- Explain in general (gof) (x) $\neq$ (fog) (x)
- Discuss the difference between fog and gof.

Note:

1. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions, then the composite function of f and g is the function of f and g and denoted by gof, gof means composite function of f and g, f is followed by g.
2. For all x€A, (gof) (x) = g(f(x))
3. Composite function is regarded as a function of a function.
4. For composite function gof the range of f is the domain of g.
5. In general (gof) (x) $\neq$ (fog) (x)

Some solved problems

1. From the adjoining figure, write the composite function gof in the ordered pair form. Also write the domain and range of gof.



Solution :

Here, the arrow diagram represents an composite function of f and g it is denoted by gof.

f = {(a,d), (b,e), (c,f)}

g = {(d,x), (e,y), (f,z)}

Here, for function f a € A $\xrightarrow{\text{corresponds}}$ f(a) = d

b $\in$ A $\longrightarrow$ f(b) = e $\in$ B

c $\in$ A $\longrightarrow$ f(c) = f $\in$ B

Again, for function g

d $\in$ B $\xrightarrow{\text{corresponds}}$ g(d) = x $\in$ C

e $\in$ B $\longrightarrow$ g(e) = y $\in$ C

f $\in$ B $\longrightarrow$ g(f) = z $\in$ C

Now, for gof (i.e. f is followed by g)

a $\in$ A $\xrightarrow{\text{corresponds}}$ x = g(d) = g(f(a))

b $\in$ A $\longrightarrow$ y = g(e) = g(f(b))

c $\in$ A $\longrightarrow$ z = g(f) = g(f(c))

Hence we write gof = {(a,x), (b,y), (c,z)}

Domain of gof = {a,b,c}

Range of gof = {x,y,z}

2. Write the composite function gof of the given functions in the set of ordered pair form if f = {(3,4), (4,5), (5,6)} and g = {(4,5), (5,6), (6,7)}

Solution :

Here, f = {(3,4), (4,5), (5,6)}, g = {(4,5), (5,6), (6,7)}

Now, gof means composite function f is followed by g.

$\therefore$ gof = {(3,5), (4,6), (5,7)}

3. Write the composite function fog of the given functions.

f = {(2,8), (4,64), (8,256)} and g = {(1,2), (2,4), (4,8)}

Solution:

Here, f = {(2,8), (4,64), (8,256)}

g = {(1,2), (2,4), (4,8)}

Now, fog means composite function g is followed by f

fog = {(1,8), (2,64), (4,256)}

4. If (fog) = {(2,1), (4,2), (6,3), (8,4)} and g = {(2,4), (4,8), (6,12), (8,16)}

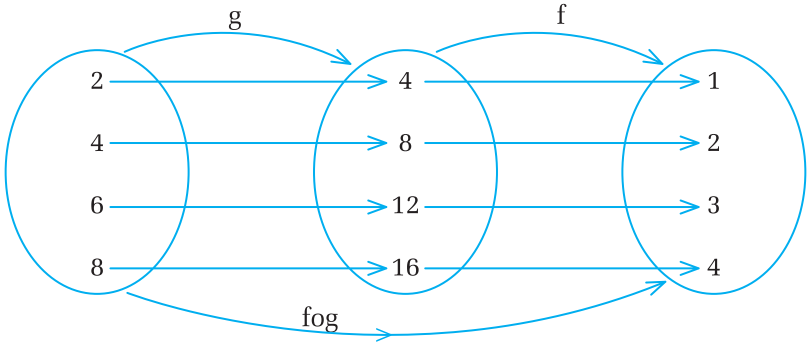
find f in ordered pair form. Also show fog in arrow diagram.

Solution:

Here, (fog) = {(2,1), (4,2), (6,3), (8,4)}

and g = {(2,4), (4,8), (6,12), (8,16)}

(fog) means composite function g is followed f.



The given information is displayed in above arrow diagram. From the diagram we can write f as set of ordered pair.

f = {(4,1), (8,2), (12,3), (16,4)}

5. (a) If f(x+2) = 4x+5, find f(x) and (fof)(x)

Solution:

Here, f(x+2) = 4x+5

x is replaced by x–2, then we get,

f(x–2+2) = 4(x–2) +5

∴f(x) = 4x–3

Now, (fof)(x) = f(f(x))

= f(4x–3)

= 4(4x–3)–3

= 16x–12–3

= 16x–15

(b)If g(x+5) = x+20, find g(x) and (gog)(x)

Solution:

Here, $g(x+5) = x+20$

x is replaced by x–5, we get,

$g(x-5+5) = (x-5)+20 = x+15$

$\therefore g(x) = x+15$

Now, $(gog)(x) = g(g(x)) = g(x+15)$

$= x+15+15 = x+30$

6. Let $f : R \longrightarrow R$ and $g : R \longrightarrow R$ be defined by $f(x) = x+1$ and $g(x) = x^3$, then i) find gof and fog ii) Is (gof)(x) = (fog)(x)?

Solution:

Here, $f(x) = x+1$ and $g(x) = x^3$

Then, $(gof)(x) = g(f(x)) = g(x+1) = (x+1)^3 = x^3+3x+3x^2+1$ and $(fog)(x) = f(g(x)) = f(x^3) = x^3+1$

This shows that $(gof)(x) \neq (fog)(x)$

It means that commutativity does not hold in composite function. i.e. gof $\neq$ fog.

7. If f : R R and g : R R are defined by $f(x) = 2x+3$ and $g(x) = 2x-1$, find (gof)(3) and (gof)(–2)

Solution:

Here, $f(x) = 2x+3$, $g(x) = 2x-1$

Let us find (gof)(x)

$(gof)(x) = g(f(x)) = g(2x+3)$

$= 2(2x+3)-1$

$= 4x+6-1$

$= 4x+5$

Now $(gof)(3) = 4.3+5 = 17$

and $(gof)(-2) = 4(-2)+5 = -8+5 = -3$

Alternate Method

we have $f(x) = 2x+3$ and $g(x) = 2x-1$

Here, $f(3) = 2.3+3 = 9$, $f(-2) = 2(-2)+3 = -1$

Now, $(gof)(3) = g(f(3)) = g(9) = 2.9-1 = 17$

$(gof)(-2) = g(f(-2)) = g(-1) = 2(-1)-1 = -3$

8. Find the value of x if $f(x) = 4x+5$ and $g(x) = 6x+3$, $(gof)(x) = 75$.

Solution:

Here, $f(x) = 4x+5$ and $g(x) = 6x+3$

Given, $(gof)(x) = 75$

or, $g(f(x)) = 75$

or, $g(4x+5) = 75$

or, $6(4x+5)+3 = 75$

or, $24x+30+3 = 75$

or, $24x = 75-33$

or, $24x = 42$

$\therefore x = \frac{7}{4}$

9. If $f(x) = \frac{7x+q}{2}$ and $g(x) = x+8$, $(fog)(4) = 24$, find the value of q.

Solution:

Here, $f(x) = \frac{7x+q}{2}$ and $g(x) = x+8$

We have, $g(4) = 4+8 = 12$

Now, $(fog)(4) = 24$

or, $f(g(4)) = 24$

or, $f(12) = 24$

or, $\frac{7\times12+q}{2} = 24$

or, $84+q = 48$

or, $q = 48-84$

$\therefore q = -36$

10. (a) If $f(x) = 4x-5$ and g(x) is a linear function, $(gof)(x) = 5x+1$, find g(x).

Solution:

Here, $f(x) = 4x-5$ and let $g(x) = ax+b$ be a linear function, where a and b are constants.

Now, $(gof)(x) = g(4x-5) = a(4x-5)+b$

But $(gof)(x) = a(4x-5)+b$

or, $4ax-5a+b = 5x+1$

or, $4ax+(b-5a) = 5x+1$

Equating the corresponding coefficients of like terms, we get

$4a = 5 \quad \therefore a = \frac{5}{4}$

and $b-5a = 1$

or, $b-5.\frac{5}{4} = 1$

or, $4b-25 = 4$

or, $b = \frac{29}{4}$

$\therefore g(x) \frac{5}{4}x+\frac{29}{4} = \frac{5x+29}{4}$

(b)If $g(x) = 5x+3$ and $(gof)(x) = 2x+5$ and f(x) if a linear function, find the value of f(x).

Solution:

Here, $g(x) = 5x+3$, $(gof)(x) = 2x+5$

Since f(x) is a linear function, let $f(x) = ax+b$

Now, $(gof)(x) = 2x+5$ (given)

or, $g(ax+b) = 2x+5$

or, $5(ax+b)+3 = 2x+5$

or, $5ax+5b+3 = 2x+5$

Equating the coefficients of like terms, we get,

$5a = 2$ or $a = \frac{2}{5}$

and $5b+3 = 5$

or, $5b = 2$

$\therefore b = \frac{2}{5}$

$\therefore f(x) = \frac{2}{5}x + \frac{2}{5} = \frac{2x+2}{5} = \frac{2}{5}(x+1)$

Alternate Method

Here, $g(x) = 5x+3$

Let $f(x) = \alpha$

Now, $(gof)(x) = 2x+5$

or, $g(f(x)) = 2x+5$

or, $g(\alpha) = 2x+5$

or, $5\alpha+3 = 2x+5$

or, $5\alpha = 2x+2$

$\therefore \alpha = \frac{2x+2}{5} = \frac{2(x+1)}{5}$

$\therefore f(x) = \frac{2}{5}(x+1)$ which is a linear function.

(c) If $f(x) = 2x+8$, $(fog)(x) = 3x+4$, find the function $g(x)$.

Solution:

Here, $f(x) = 2x+8$, Let $g(x) = \alpha$

By question, we get, $(fog)(x) = 3x+4$

or, $f(g(x)) = 3x+4$

or, $f(\alpha) = 3x+4$

or, $2\alpha+8 = 3x+4$

or, $\alpha = \frac{3x-4}{2}$

$\therefore g(x) = \frac{3x-4}{2}$

11. If $g(x) = 10-x$, show that $(gog)(x) = x$

Solution:

Here, $(gog)(x) = g(g(x))$

$= g(10-x)$

$= 10-(10-x)$

$= 10-10+x$

$= x$

$\therefore (gog)(x) = x$ proved

12. If $p(x) = \frac{3x+2}{3}$ and $q(x) = \frac{3x-2}{3}$, prove that $(poq)(x)$ is and identity function.

Solution:

Here, $p(x) = \frac{3x+2}{3}$ and $q(x) = \frac{3x-2}{3}$

Identity function means, a function in the form of f(x) = x i.e. y=x.

Now, we have to show that (poq)(x) = x

$$(poq)(x) = p\left(\frac{3x-2}{3}\right)$$

$$= \frac{3\left(\frac{3x-2}{3}\right)+2}{3}$$

$$= \frac{3x-2+2}{3}$$

$$= \frac{3x}{3}$$

$$= x$$

$\therefore$ (poq)(x) = x

Hence, (poq)(x) is an identity function proved.

13. (a) If $h(x) = \frac{1}{(x+3)^3}$, $x \neq -3$ and h(x) = (fog)(x), find the possible value of f(x) and g(x).

Solution:

Here, $h(x) = \frac{1}{(x+3)^3}$

But h(x) = (fog)(x)

or, $\frac{1}{(x+3)^3} = f(g(x))$

or, $\left(\frac{1}{x+3}\right)^3 = f(g(x))$

in which we can write, $g(x) = \frac{1}{x+3}$, $f(x) = x^3$

Also, $\frac{1}{(x+3)^3} = (fog)(x)$

or, $\frac{1}{(x+3)^3} = f(g(x))$

in which we can write, g(x) = x + 3 and $f(x) = \frac{1}{x^3}$

Note : We can check it.

f(g(x)) = f(x + 3)

$$= \frac{1}{(x+3)^3}$$

(b)If $h(x) = (2x-3)^3$ and h(x) = (fog)(x), find the possible values of f(x) and g(x).

Solution:

Here, $h(x) = (2x-3)^3$

or, h(x) = (fog)(x)

or, $(2x-3)^3 = f(g(x))$

in which we can write.

g(x) = 2x – 3 and $f(x) = x^3$

Also, $(2x - 3)^3 = (fog)(x)$

or, $\{2(x+7) - 10\}^3 = f(g(x)$

in which we can write, $g(x) = x + 7$, $f(x) = 2x - 10$

Questions for practice

1. If $f = \{(1,2), (3,5), (4,1)\}$ and $g = \{(2,5), (5,8), (1,4)\}$. Then show the function gof in an arrow diagram. Find it in an ordered pair form. Also write down the domain and range of gof.
2. Let $f = \{(1,2), (3,4), (4,5)\}$ and $g = \{(2,4), (4,2), (5,3)\}$. find the composite function (gof) in ordered pair form.
3. If $f(x +9) = 4x + 5$, find f(x) and (fof)(x).
4. Let $f : R \rightarrow R$ and $g : R \rightarrow R$ be two functions such that $f(x) = x^2 + 2$ and $g(x) = x + 2$, find (fog)(x) and (gof)(x). Is $(gof)(x) = (fog)(x)$?
5. If $f(x) = 4x - 3$, find (fof)(x) and (fof)(–2)
6. If $f(x) = 5x + 8$ and g(x) is a linear function and $(gof)(x) = 4x + 7$, find g(x).
7. If $f(x) = 3x + 5$, $(fog)(x) = 5x + 8$, find g(x).
8. Let $f : x \rightarrow 5x + p$, and $(fof)(3) = 105$, find the value of m.
9. Let $f(x) = \frac{6}{x-2}$, $(x \neq 2)$ and $g(x) = px^2 - 1$ and $(gof)(5) = t$, find the value of p.
10. If $f(x) = \frac{4x+5}{9}$ a nd $g(x) = \frac{9x-5}{4}$, then show that (fog)(x) is an identity function.
11. If $f(x) = 100 - x$, then show that $(fof)(x) = x$ comment on the result.
12. If $f(x) = (2x - 7)^3$, $f(x) = (poq)(x)$, then find the possible values of p(x) and q(x).

Inverse Function

Estimated teaching hour : 4 hours

1. Objectives

S.N.	Level	Objectives
1.	Knowledge (K)	to define inverse function. to write notation of inverse of function f. to tell conditions to have inverse of a given function.
2.	Understanding (U)	to find inverse function of given function in ordered pair form. to draw arrow diagram for given function.
3.	Application (A)	to find inverse function of given function in equation form.
4.	Higher Ability (HA)	to find composite of a function with its inverse. to find composite function of a function and inverse of other function.

2. Teaching materials:

- Chart paper with functions and their inverses.
- Chart paper with one-one function and its inverse.

3. Teaching learning strategies :

- Review concept of function with arrow diagrams and their inverse.
- Clear the concept that only one to one functions have their inverse by using chart paper with functions with their inverses.
- Discuss how to find inverse of given function with examples when

i) function is given in ordered pair form.
ii) function is given in arrow diagram.
iii) function is given in equation form.

Note :

1. If $f : A \rightarrow B$ is a one to one onto function from A to B, then there exists a function from B to A called inverse of f denoted by f^{-1} such that $f^{-1} : B \rightarrow A$.
2. If f is one-one onto, then $(f^{-1})^{-1} = f$
3. $f^{-1} \neq \dfrac{1}{f}$

Some solved problems

1. (a) Find the inverse function of the given function f = {(1,2), (2,3), (3,4), (4,5), (5,6)}. Also show f and f^{-1} in an arrow diagram.

Solution:

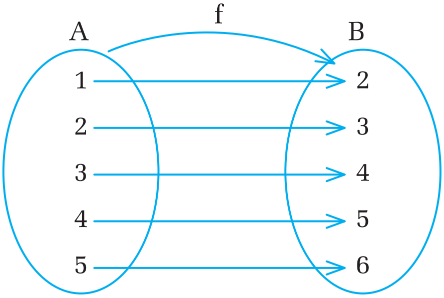
Here, f = {(1,2), (2,3), (3,4), (4,5), (5,6)}

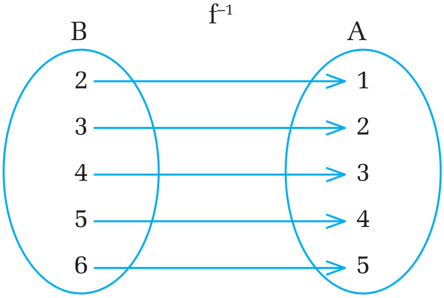
Interchanging the role of components in each ordered pair.

We get the inverse of f. It is denoted by f^{-1}.

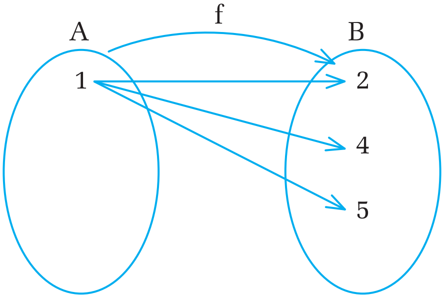
f^{-1} = {(2,1), (3,2), (4,3), (5,4), (6,5)}

We can show f and f^{-1} in arrow diagram.





2. If f = {(1,2), (1,4), (1,5)}, does f^{-1} exists.



Here, f is not a function as the first element 1 has more than 1 image. Hence f^{-1} does not exists in the sense of function. But f^{-1} exists in the sense of a relation only.

3. Let f : R → R, then find the inverse of f under given conditions. (a) f = {(x,y) : y = 4x +5}
(b) f = {(x,y) : y = $\frac{3x-1}{3-2x}$, x ≠ $\frac{3}{2}$}

Solution : (a) Here, f = {(x,y) : y = 4x +5}

We can write $y = 4x + 5$

Interchanging the role of x and y, we get

$x = 4y + 5$

Solve it for y, we get,

$y = \frac{x-5}{4}$

$\therefore f^{-1} = \frac{x-5}{4}$

(b) Here, f = {(x,y) : y = $\frac{3x-1}{3-2x}$, x ≠ $\frac{3}{2}$}}

We have,

$y = \frac{3x-1}{3-2x}, x \neq \frac{3}{2}$

Interchanging the role of x and y and solving for y,

We get,

$$x = \frac{3y-1}{3-2y}$$

or, $3x - 2xy = 3y - 1$

or, $3y + 2xy = 3x + 1$

or, $y(2x + 3) = 3x + 1$

or, $y = \frac{3x + 1}{2x + 3}$, $x \neq -\frac{3}{2}$

$\therefore f^{-1}(x) = \frac{3x + 1}{2x + 3}$

4. If $k(x) = \frac{5x + 7}{x - 2}$, $x \neq 2$, find $k^{-1}(x)$ and $k^{-1}(4)$.

Solution:

Here, $k(x) = \frac{5x + 7}{x - 2}$

or, $y = \frac{5x + 7}{x - 2}$

Interchanging the role of x and y and solving it for y, we get,

$x = \frac{5y + 7}{y - 2}$

or, $xy - 2x = 5y + 7$

or, $xy - 5y = 2x + 7$

or, $y(x - 5) = 2x + 7$

or, $y = \frac{2x + 7}{x - 5}$

$\therefore k^{-1}(x) = \frac{2x + 7}{x - 5}$, $x \neq 5$

For $x = 4$, $k^{-1}(4) = \frac{2.4 + 7}{4 - 5}$

$= -15$

5. If $f(x) = 2x + k$, $f^{-1}(4) = 20$, find the value of k.

Solution:

Here, $f(x) = 2x + k$ or $y = 2x + k$

For f^{-1}, interchanging the role of x and y, we get

$x = 2y + k$

or, $y = \frac{x - k}{2}$ $\therefore f^{-1}(x) = \frac{x - k}{2}$

But, $f^{-1}(4) = 20$

or, $\frac{4 - k}{2} = 20$

or, $4 - k = 40$

$\therefore k = -36$

6. If $f(4x + 5) = 20x + 24$, find $f^{-1}(x)$

Solution:

Here, $f(4x + 5) = 20x + 24$

or, $f(4x + 5) = 5(4x + 5) - 1$

$4x + 5$ is replaced by x

$\therefore$ fx) $= 5x - 1$

Let $y = f(x) = 5x - 1$

To find f^{-1}, interchanging the role of x and y,

we get, $x = 5y - 1$

or, $y = \frac{x+1}{5}$

$\therefore f^{-1}(x) = \frac{x+1}{5}$

7. If f(4x + 5) = 12x + 20, find $f^{-1}(x)$ and $f^{-1}(4)$.

Solution:

Here, $f(4x + 5) = 12x + 20$

$= 3(4x + 5) + 5$

Replacing (4x + 5) by x ,we get,

$f(x) = 3x + 5$

Let $y = f(x) = 3x + 5$

or $y = 3x + 5$

To find f^{-1}, interchanging the role of x and y, we get,

$x = 3y + 5$

or, $y = \frac{x-5}{3}$

$\therefore f^{-1}(x) = \frac{x-5}{3}$

and $f^{-1}(4) = \frac{4-5}{3} = -\frac{1}{3}$

8. (a) If f(x) = 2x – 3 and g(x) = 4x + 5 and (fof)(x) = $g^{-1}(x)$, find the value of x.

Solution:

Here, $f(x) = 2x - 3$, $g(x) = 4x + 5$

$(fof)(x) = f(f(x)) = f(2x - 3)$

$= 2(2x - 3) - 3 = 4x - 6 - 3$

$= 4x - 9$

To find $g^{-1}(x)$, we have, $g(x) = 4x + 5$

i.e. $y = 4x + 5$

Interchanging the role of x and y, we get,

$x = 4y + 5$

or, $y = \frac{x-5}{4}$

$g^{-1}(x) = \frac{x-5}{4}$

Now, $(fof)(x) = g^{-1}(x)$

or, $4x - 9 = \frac{x-5}{4}$

or, $16x - 36 = x - 5$

or, $15x = 31$

$\therefore x = \frac{31}{15}$

(b) If f(x) = 4x – 7 and g(x) = 3x – 5 are two one-one functions and $(fog^{-1})(x) = 15$, find the value of x.

Solution:

Here, $f(x) = 4x - 7$ and $g(x) = 3x - 5$,

For $(fog^{-1})(x)$, let us find g^{-1}

We have, $y = g(x) = 3x - 5$

i.e. $y = 3x - 5$

Interchanging the role of x and y, we get,

$x = 3y - 5$

or, $y = \frac{x+5}{3}$

$\therefore g^{-1}(x) = \frac{x+5}{3}$

Now, $(fog^{-1})(x) = 15$

or, $f(g^{-1}(x)) = 15$

or, $f\left(\frac{x+5}{3}\right) = 15$

or, $4\left(\frac{x+5}{3}\right) - 7 = 15$

or, $4x + 20 - 21 = 45$

or, $4x = 46$

$\therefore x = \frac{23}{2}$

(c) If $f(x) = 4x - 17$ and $g(x) = \frac{2x+8}{5}$ and $(fof)(x) = g^{-1}(x)$, find the value of x.

Solution

Here, $f(x) = 4x - 17$

and $g(x) = \frac{2x+8}{5}$

To find $g^{-1}(x)$, we have, $y = g(x) = \frac{2x+8}{5}$

i.e. $y = \frac{2x+8}{5}$

Interchanging the role of x and y, we get,

$x = \frac{2y+8}{5}$

or, $5x = 2y + 8$

or, $y = \frac{5x-8}{2}$ $\quad \therefore g^{-1}(x) = \frac{5x-8}{2}$

Also, $f(f(x)) = f(4x - 17)$

$= 4(4x - 17) - 17$

$= 16x - 68 - 17$

$= 16x - 85$

Now, $(fof)(x) = g^{-1}(x)$

or, $16x - 85 = \frac{5x-8}{2}$

or, $32x - 170 = 5x - 8$

or, $27x = 162$

$\therefore x = 6$

9. (a) If $f(x) = 2x - 4$, then prove that $(fof^{-1})(x)$ is an identify function.

Solution

Here, $f(x) = 2x - 4$

We have to show $(fof^{-1})(x)$ is an identity function i.e. $(fof^{-1})(x) = x$

Firstly, let us find $f^{-1}(x)$

We have, $y = f(x) = 2x - 4$

or, $y = 2x - 4$

Interchanging the role of x and y, we get

$x = 2y - 4$

or, $y = \frac{x + 4}{2}$

Now, $(fof^{-1})(x) = f(f^{-1}(x))$

$$= f\left(\frac{x + 4}{2}\right)$$

$$= 2\frac{(x + 4)}{2} - 4$$

$$= x + 4 - 4$$

$$= x$$

$\therefore (fof^{-1})(x) = x$ proved

(b)If $f(x) = \frac{2x + 5}{x + 2}$, $x \neq 2$, then prove that $(fof^{-1})(x)$ is an identity function.

Solution

Here, $f(x) = \frac{2x + 5}{x + 2}$, $x \neq -2$

We have to show that $(fof^{-1})(x)$ is an identity function i.e. $(fof^{-1})(x) = x$

Firstly let us find $f^{-1}(x)$

We have, $y = f(x) = \frac{2x + 5}{x + 2}$, $x \neq -2$

i.e. $y = \frac{2x + 5}{x + 2}$

Interchanging the role of x and y, we get,

$x = \frac{2y + 5}{y + 2}$

or, $xy + 2x = 2y + 5$

or, $y(x - 2) = 5 - 2x$

$\therefore y = \frac{5 - 2x}{x - 2} x \neq 2$

$\therefore f^{-1}(x) = \frac{5 - 2x}{x - 2}$

Now, $(fof^{-1})(x)$

$$= f(f^{-1}(x))$$

$$= f\left(\frac{5 - 2x}{x - 2}\right)$$

$$= \frac{2(5 - 2x)}{x - 2} + 5$$

$$= \frac{\frac{2(5 - 2x)}{x - 2} + 5}{\frac{5 - 2x}{x - 2}}$$

$$= 10 - 4x + 5x - 10$$

$= x$

$\therefore (fof^{-1})(x) = x$ proved

10. If $f(x) = x + 1$, $g(x) = \frac{3 - x}{x}$, $x \neq 0$, are two functions, then prove that $(f^{-1}g^{-1})(x) = 0$

Solution

Here, $f(x) = x + 1$

and $g(x) = \frac{3 - x}{x}$

To find $f^{-1}(x)$, we have, $y = x + 1$

Interchanging the role of x and y, we get,

$x = y + 1$

or, $y = x - 1$

$\therefore f^{-1}(x) = x - 1$

Again, to find $g^{-1}(x)$, we have, $y = \frac{3 - x}{x}$

Interchanging the role of x and y, we get,

$x = \frac{3 - y}{y}$

or, $xy = 3 - y$

or, $y(x + 1) = 3$

$\therefore y = \frac{3}{x + 1}$

$\therefore g^{-1}(x) = \frac{3}{x + 1}$

Now, $(f^{-1}og^{-1})(x)$

$= f^{-1}(g^{-1}(x))$

$= f^{-1}\left(\frac{3}{x + 1}\right)$

$= \frac{3}{x + 1} - 1$

$= \frac{3 - x - 1}{x + 1}$

$= \frac{3 - x}{x + 1}$

For $x = 2$,

$(f^{-1}og^{-1})(x) = \frac{2 - 2}{2 + 1}$

$(f^{-1}og^{-1})(x) = 0$ proved

Note : Formula

Conversion of degree celcius into fahrenheit or vice-versa

$\frac{C - O}{100} = \frac{F - 32}{180}$

Where C = number of degree celcius

F = number of degree fahrenheit

Example : Convert 37°C to degree fahrenheit

We write C = 37 and find F

Now, $\frac{C-O}{100}=\frac{F-32}{180}$

or, $\frac{37}{100}=\frac{F-32}{180}$

or, $\frac{37}{5}=\frac{F-32}{9}$

or, 333 = 5F – 160

or, 5F = 493

$\therefore$ F = 98.6

$\therefore$ 37°C = 98.6°F

Again, convert 98.6°F into °C, we get.

98.6°F = 37°C

This shows inverse function concept.

11. Prove that $f(x)=\frac{2x+1}{3-4x}$ and $g(x)=\frac{3x-1}{4x+2}$ are inverse to each other.

Solution

Here, $f(x)=\frac{2x+1}{3-4x}$ and $g(x)=\frac{3x-1}{4x+2}$

If f(x) and g(x) are inverse to each other, then $f^{-1}(x) = g(x)$ or $g^{-1}(x) = f(x)$

We have, $f(x)=\frac{2x+1}{3-4x}$

To find inverse function f^{-1},

We have, $y = f(x) = \frac{2x+1}{3-4x}$

i.e. $y = \frac{2x+1}{3-4x}$

Interchanging the role of x and y, we get,

$x = \frac{2y+1}{3-4y}$

or, $3x - 4xy = 2y + 1$

or, $2y + 4xy = 3x - 1$

or, $y(4x + 2) = 3x - 1$

$\therefore y = \frac{3x-1}{4x+2}$

$\therefore f^{-1}(x) = \frac{3x-1}{4x+2}$ which is also g(x)

Hence f(x) and g(x) are inverse to each other.

Alternate method

If f(x) and g(x) are inverse to each other then (fog)(x) must be an identity function.

i.e. (fog)(x) = x

Now, (fog)(x)

= f(g(x))

$$= f\left(\frac{3x-1}{4x+2}\right)$$

$$= \frac{2\left(\frac{3x-1}{4x+2}\right)+1}{3-4\left(\frac{3x-1}{4x+2}\right)}$$

$$= \frac{6x-2+4x+2}{4x+2} \times \frac{4x+2}{12x+6-12x+4}$$

$$= \frac{10x}{10}$$

$$= x$$

$\therefore$ (fog)(x) = x

Hence, f(x) and g(x) are inverse to each other. proved

Questions for practice

1. If f = {(4,8), (10, 20), (6,12), (7,14)} and g = {(8,16), (20,40), (12,24), (14,28)}
   i) find f^{-1} and g^{-1}
   ii) find gof and its inverse
2. If f = {(2,4), (3,4), (4,4)}, find f^{-1} if exists.
3. If f = {(x,y) : $y = \frac{3x+1}{2x+3}$, $x \neq \frac{-3}{2}$, find $f^{-1}(x)$.
4. If $p(x) = \frac{7x+5}{x-4}$, $x \neq 4$, find $p^{-1}(x)$ and $p^{-1}(-4)$.
5. If f(x) = 2x + p, and $f^{-1}(x) = \frac{x-7}{2}$, find the value of p.
6. If f(2x + 3) = 6x + 8, find $f^{-1}(x)$ and $f^{-1}(4)$.
7. If f(x) = 3x and g(x) = 5x – 2, $(fof^{-1})(x) = g(x)$, find the value of x.
8. If f(x) = x – 3, $g(x) = \frac{2x+1}{x+1}$, $x \neq 1$, and $g^{-1}of(x) = 4$, find the value of x.
9. If $f(x) = \frac{2x+1}{4}$, prove that $(f^{-1}f)(x)$ is an identity function.
10. If f(x) = 3x + 4 and then prove that $(f^{-1}of)(x) = (fof^{-1})(x)$.
11. If $f(x) = \frac{2x+1}{3-4x}$ and $g(x) = \frac{3x-1}{4x+2}$, then prove that $(fog^{-1})(x) = (gof^{-1})(x) = x$.

UNIT two

Polynomials

Division of polynomials using division algorithmic and synthetic division.

Estimated teaching periods : 2

1. Objectives

S.N.	Level	Objectives
1.	Knowledge (K)	To define polynomials. To tell relation among dividend, divisor, quotient and remainder.
2.	Understanding (U)	To divide a polynomial by another polynomial by division algorithm. To tell steps used in synthetic division of a polynomial by a linear polynomial.
3.	Application (A)	To apply synthetic division method to divide a polynomial by a linear polynomial.
4.	Higher Ability (HA)	To apply synthetic division method to divide a polynomial by a linear polynomial (ax+b, form).

2. Teaching materials

– Chart paper to show process of division by synthetic division method.

3. Teaching Learning Strategies

- Review definition of polynomials
- Review division in number mathematics
- Clear the concept of term - dividend, divisor, quotient and remainder.

Example : $639 \div 5$

$$\begin{array}{r} 5\ )\ 639\ (127 \\ \underline{-5} \\ 13 \\ \underline{-10} \\ 39 \\ \underline{-35} \\ 4 \end{array}$$

Here, 639 = dividend, D(x)

127 = quotient, Q(x)

5 = divisor, d(x)

4 = remainder, R(x)

- Establish the relation :

  dividend = divisor × quotient + remainder

  From above example, $639 = 5 \times 127 + 4$

- Apply the same process in division of polynomials.
- Explain about division of polynomials by synthetic division. Discuss steps performed by synthetic division (page 27 book)
- Example : $(4x^3 + 3x^2 - 7x + 5) \div (x - 2)$, Explain it ! by using synthetic division.

Coefficients of variables of dividend in descending order.

$$\begin{array}{c|cccc} 2 & 4 & 3 & -7 & 5 \\ & \downarrow & 8 & 22 & 30 \\ \hline & 4 & 11 & 15 & 35 = R \end{array}$$

leading coefficient

Here, 4 is called leading coefficient.
In coefficient degree of quotient is less than dividend by 1

Quotient $\quad Q(x) = 4x^2 + 11x + 15$

Remainder, $R(x) = 35$

Dividend, $D(x) = 4x^3 + 3x^2 - 7x + 5$

divisor, $d(x) = x - 2$

check $D(x) = Q(x) \times d(x) + R(x)$

i.e. $4x^3 + 3x^2 - 7x + 5 = (4x^2 + 11x + 15) \times (x - 2) + 35.$

2. Find the quotient and remainder when $f(x) = x^3 + 4x^2 + 3x + 5$ is divided by $d(x) = x + 3$

Solution

By using division algorithm

$$
\begin{array}{l}
x + 3\ )\ x^3 + 4x^2 + 3x + 5\ (\ x^2 + x \\
\quad\quad \underline{x^3 + 3x^2} \\
\quad\quad\quad\quad x^2 + 3x \\
\quad\quad\quad\quad x^2 + 3x \\
\quad\quad\quad\quad \underline{-\ \ -\quad\quad} \\
\quad\quad\quad\quad\quad\quad 5
\end{array}
$$

$\therefore$ Quotient, $Q(x) = x^2 + x$

Remainder $R = 5$

3. If $2x^3 - 11x^2 + 20x - 15 = (x - 6)\ Q(x) + R(x)$, find $Q(x)$ and $R(x)$.

Solution : Here, $2x^3 - 11x^2 + 20x - 15 = (x - 5)\ Q(x) + R(x)$

dividend $= 2x^3 - 11x^2 + 20x - 15$

divisor $= x - 6$

quotient $= Q(x)$

remainder $= R(x)$

By division algorithm, we get

$$
\begin{array}{l}
x - 6\ )\ 2x^3 - 11x^2 + 20x - 15\ (\ 2x^2 + x + 26 \\
\quad\quad \underline{-2x^3 \pm 12x^2} \\
\quad\quad\quad\quad x^2 + 20x \\
\quad\quad\quad\quad \underline{-x^2 \pm 6x} \\
\quad\quad\quad\quad\quad\quad 26x - 15 \\
\quad\quad\quad\quad\quad\quad \underline{-26x \pm 156} \\
\quad\quad\quad\quad\quad\quad\quad\quad 141
\end{array}
$$

$\therefore$ Quotient, $Q(x) = 2x^2 + x + 26$

Remainder, $R(x) = 141$

4. If $4x^3 - 10x^2 + 25x - 20 = (4x - 3)\ Q(x) + R(x)$

Solution

Here, dividend $= 4x^3 - 10x^2 + 25x - 20$

divisor $= 4x - 3$

By division algorithm, We get

$4x - 3\ )4x^3 - 10x^2 + 25x - 20\ (\ x^2 - 7/4x + \dfrac{79}{4}$

$\underline{-4x^3 \pm 3x^2}$

$-7x^2 + 25x$

$-7x^2 + \dfrac{-21x}{4}$

$+ \qquad -$

$\dfrac{-79}{4}x - 20$

$\dfrac{-79}{-4}x + \dfrac{277}{-16}$

$\dfrac{83}{16}$

$\therefore$ Quotient, $Q(x) = x^2 - \dfrac{7}{4}x + \dfrac{79}{16}$, Remainder $= \dfrac{-83}{16}$

5. If $p(x) = x^8 + x^4 + 1$, $q(x) = x^4 + x^2 + 1$, $r(x) = x^4 + x^2 + 6$, find the value of $\{p(x) \div q(x)\} + r(x)$

Solution

Here, $p(x) = x^8 + x^4 + 1$

$= (x^4)^2 + 2.x^4 + 1 - x^4$

$= (x^4 + 1)^2 - (x^2)^2$

$= (x^4 + 1 + x^2)(x^4 + 1 - x^2)$

$= (x^4 + x^2 + 1)(x^4 - x^2 + 1)$

Now, $\dfrac{p(x)}{q(x)} + r(x)$

$= \dfrac{(x^4 + x^2 + 1)(x^4 - x^2 + 1)}{x^4 + x^2 + 1} + x^4 + x^2 + 6$

$= x^4 - x^2 + 1 + x^4 + x^2 + 6$

$= 2x^4 + 7$

Some solved problems

1. When $f(x) = x^4 + 3x^3 + 7x^2 + 4x + 7$ is divided by $d(x) = x - 2$, write the degree of quotient.

Solution

Here, divided, $f(x) = x^4 + 3x^3 + 7x^2 + 4x + 7$

divisor, $d(x) = x - 2$

Since divisor is a linear polynomial and dividend is of degree 4,

degree of quotient = degree of dividend – degree of divisor

$= 4 - 1$

$= 3$

6. Divide f(x) by d(x) by synthetic division method.

(a) $f(x) = 3x^3 - 2x + 13x + 10$, $d(x) = x - 2$

Solution

Dividend $f(x) = 3x^3 - 2x + 13x + 10$

divisor $d(x) = x - 2$

Comparing divisor $(x - 2)$ with $(x - a)$, we get

$a = 2$

leading coefficient $= 3$

Writing the coefficients in descending order with constant

$$\begin{array}{r|rrrr} 2 & 3 & -2 & 13 & 10 \\ & & 6 & 8 & 42 \\ \hline & 3 & 4 & 21 & \big|\,52 \end{array}$$

Quotient, $Q(x) = 3x^2 + 4x + 21$,

Remainder, $R(x) = 52$

(b) $f(x) = x^4 + 3x^3 + 7x^2 + 3x + 8$, $d(x) = x + 3$

Solution

Dividend, $f(x) = x^4 + 3x^3 + 7x^2 + 3x + 8$

divisor, $d(x) = x + 3 = x - (-3)$

Synthetic division method

Writing the coefficients in dividend in order

$$\begin{array}{r|rrrrr} -3 & 1 & 3 & 7 & 3 & 8 \\ & & -3 & 0 & -21 & 54 \\ \hline & 1 & 0 & 7 & -18 & \big|\,62 = R \end{array}$$

Quotient, $Q(x) = x^3 + 7x - 18$

Remainder, $R(x) = 62$

(c) $f(x) = 2x^3 + 4x^2 + 3x + 7$, $d(x) = x + 2$

Solution

Dividend, $f(x) = 2x^3 + 4x^2 + 3x + 7$

divisor, $d(x) = x + 2 = x -(-2)$

Synthetic division method

Writing the coefficients in dividend in order

$$\begin{array}{r|rrrr} -2 & 2 & 4 & 3 & 7 \\ & & -4 & 0 & -6 \\ \hline & 2 & 0 & 3 & \big|\,1 = R \end{array}$$

$\therefore$ Quotient, $Q(x) = 2x^2 + 3$

Remainder, $R(x) = 1$

7. Find the quotient and remainder when f(x) is divided by d(x)

(a) $f(x) = 2x^3 - 9x^2 + 5x - 5$, $d(x) = 2x - 3$

Solution

dividend $f(x) = 2x^3 - 9x^2 + 5x - 5$

divisor $d(x) = 2x - 3 = 2(x - \frac{3}{2})$

Comparing $(x - \frac{3}{2})$ with $(x - a)$, we get, $a = \frac{3}{2}$

By synthetic division method

$$\begin{array}{c|cccc} \frac{3}{2} & 2 & -9 & 5 & -5 \\ & \downarrow & 3 & -9 & -6 \\ \hline & 2 & -6 & -4 & \big| \ -11 = R \end{array}$$

Quotient $= \frac{1}{2}(2x^2 - 6x - 4)$

$= x^2 - 3x - 2$

Remainder $= -11$

(b) $\mathbf{f(x) = 8x^3 + 4x^2 + 6x - 7}$

$\mathbf{d(x) = 2x - 1}$

Solution

Dividend, $f(x) = 8x^3 + 4x^2 + 6x - 7$

divisor, $d(x) = 2x - 1 = 2(x - \frac{1}{2})$

Comparing $(x - \frac{1}{2})$ with $(x - a)$, we get, $a = \frac{1}{2}$

Writing the coefficients in dividend in order

$$\begin{array}{c|cccc} \frac{1}{2} & 8 & 4 & 6 & -7 \\ & \downarrow & 4 & 4 & 5 \\ \hline & 8 & 8 & 10 & \big| \ -2 = R \end{array}$$

Since 2 is taken common in divisor, we divisor, we divide by 2 to get quotient

Quotient, $Q(x) = \frac{1}{2}(8x^2 + 8x + 10)$

$= 4x^2 + 4x + 5$

Remainder, $R = -2$

(c) $\mathbf{f(x) = 3x^3 - 3x^2 + 2x + 5}$

$\mathbf{d(x) = 2x - 1}$

Solution

Dividend, $f(x) = 3x^3 - 3x^2 + 2x + 5$

divisor, $d(x) = 2x - 1 = 2(x - \frac{1}{2})$

$$\begin{array}{c|cccc} \frac{1}{2} & 3 & -3 & 2 & 5 \\ & & \frac{3}{2} & \frac{-3}{4} & \frac{5}{8} \\ \hline & 3 & \frac{-3}{2} & \frac{5}{4} & \big| \ \frac{45}{8} = R \end{array}$$

$\therefore$ Quotient, $Q(x) = \frac{1}{2}(3 - \frac{3}{2}x + \frac{5}{4}) = \frac{3}{2}x^2 - \frac{3}{4}x + \frac{5}{8}$

Remainder, $R(x) = \frac{45}{8}$

(d) $f(x) = 4x^3 - 7x^2 + 6x - 2$

$d(x) = 3x + 2$

Solution

$f(x) = 4x^3 - 7x^2 + 6x - 2$

$d(x) = 3x + 2 \quad = 3\{x - (-\frac{2}{3})\}$

Comparing $x - (-\frac{2}{3})$ with $x - a$, we get, $a = -\frac{2}{3}$

Writing the coefficients of dividend in order.

$$\begin{array}{c|cccc} \frac{-2}{3} & 4 & -7 & 6 & -2 \\ & \downarrow & \frac{-8}{3} & \frac{58}{9} & \frac{-224}{27} \\ \hline & 4 & \frac{-29}{3} & \frac{112}{9} & \big| \ \frac{-278}{27} = R \end{array}$$

Quotient, $Q(x) = \frac{1}{3}\left(4x^2 - \frac{29x}{3} + \frac{112}{9}\right)$

$= \frac{4}{3}x^2 - \frac{29x}{9} + \frac{112}{27}$

Remainder, $R(x) = \frac{-278}{27}$

(e) $f(x) = 4x^3 - 3x^2 + 7x + 8,$

$d(x) = 2x + 3$

Solution

dividend, $f(x) = 4x^3 - 3x^2 + 7x + 8,$

divisor, $d(x) = 2x + 3 = 2\{x - (\frac{-3}{2})\}$

$$\begin{array}{c|cccc} \frac{-3}{2} & 4 & -3 & 7 & 8 \\ & \downarrow & -6 & \frac{27}{2} & \frac{-123}{4} \\ \hline & 4 & -9 & \frac{41}{2} & \big| \ \frac{-91}{4} = R \end{array}$$

$\therefore$ Quotient, $Q(x) = \frac{1}{2}\left(4x^2 - 9x + \frac{41}{2}\right)$

$= 2x^2 - \frac{9x}{2} + \frac{41}{2}$

Remainder, $R(x) = \frac{-91}{4}$

(f) $f(x) = 4x^3 + 2x^2 - 4x + 3$

$d(x) = 2x + 3$

Solution

Dividend, $f(x) = 4x^3 + 2x^2 - 4x + 3$

divisor, $d(x) = 2x + 3 = 2\{x - (\frac{-3}{2})\}$

Comparing $x - (\frac{-3}{2})$ with $x - a$, we get, $a = \frac{-3}{2}$

writing the coefficients of dividend in order.

$$\begin{array}{c|cccc} \frac{-3}{2} & 4 & 2 & -4 & 3 \\ & \downarrow & -6 & 6 & -3 \\ \hline & 4 & -4 & 2 & |\ 0 = R \end{array}$$

$\therefore$ Quotient, $Q(x) = \frac{1}{2}(4x^2 - 4x + 2)$

$= 2x^2 - 2x + 1$

Remainder, $R = 0$

8. (a) Find the value of k when the polynomial $2x^3 + 9x^2 - 7x + k$ is exactly divisible by $2x + 3$ using synthetic division method.

Solution

Here, dividend $= 2x^3 + 9x^2 - 7x + k$

divisor $= 2x + 3 = 2\{x - (\frac{-3}{2})\}$

comparing $x - (\frac{-3}{2})$ with $x - a$, we get $a = \frac{-3}{2}$

$$\begin{array}{c|cccc} \frac{-3}{2} & 2 & 9 & -7 & k \\ & \downarrow & -3 & -9 & 24 \\ \hline & 2 & 6 & -16 & |\ 24+k = R \end{array}$$

Quotient, $Q(x) = \frac{1}{2}\ (2x^2 + 6x - 16)$

$= x^2 + 3x - 8$

Since the given polynomial is exactly divisible by $2x + 3$, remainder must be zero.

$\therefore$ Remainder $= 0$

i.e. $24 + k = 0$

$\therefore k = -24$

(b) Find the value of p when the polynomial $8x^3 + 4x^2 + 6x + p$ is exactly divisible by $2x - 1$. Use synthetic division method.

Solution

Here, dividend, $f(x) = 8x^3 + 4x^2 + 6x + p$

divisor, $d(x) = 2x - 1 = 2(x - \frac{1}{2})$

By using synthetic division method

$\frac{1}{2}$	8	4	6		p
	↓	4	4		5
	8	8	10		p+5 =R

According to question, f(x) is exactly divisible by 2x – 1, remainder is zero.

i.e. $p + 5 = 0$

$\therefore p = -5$

Questions for practice

1. By using synthetic division method, find the remainder when $x^3 + 3x^2 - 2x + 16$ is divided by $x - 4$.
2. By using synthetic division method find the quotient and the remainder.

a) $(x^3 + x^2 - 20) \div (x + 2)$

b) $(2x^3 - 7x^2 - 4x + 15) \div (x - 3)$

c) $(4x^3 - 8x^2 + 12x + 25) \div (4x - 5)$

3. (a) If $x^4 - 7x^3 + 2x - 5 = (x - 7) \times Q(x) + R(x)$, find Q(x) and R(x).

(b) If $x^3 + x^2 - x + 5 = d(x)\,(x^2 + 2x + 2) + R(x)$ then find d(x) and R(x).

4. When $px^3 + 2x^2 - 3$ and $x^2 - px + 4$ leaves the equal remainder when divided by $x - 2$, find the value of p.
5. Find the value of p when $x^3 - 9x^2 + 24x + p$ is exactly divisible by $(x - 5)$.
6. Find the quotient and the remainder when $4x^3 + 2x^2 - 4x + 8$ is divided by $2x + 3$.
7. By using synthetic division method, find Q(x)

(a) $x^3 - 19x - 30 = (x + 2) \,.\, Q(x)$

(b) $x^3 - 21x - 20 = (x + 1)\, Q(x)$

8. Find the remainder R, when f(x) is divided by d(x) is the following cases

(a) $f(x) = x^3 + 4x^2 + 9x - 8$, $d(x) = x + 2$

(b) $f(x) = x^3 + 4x^2 - 22x + 22$, $d(x) = x - 3$

9. Find the value of p in each of the following when f(x), is divided by d(x).

(a) $f(x) = px^3 + 4x - 20$, $d(x) = x + 3$, $R = 4$

(b) $f(x) = x^4 + x^3 + px^2 + x + 20$, $f(2) = 20$

Remainder theorem, factor theorem, polynomial equation

Estimated teaching periods : 6

1. Objectives

S.N.	Level	Objectives
1.	Knowledge (K)	To tell statement of factor theorem. To tell statement of factor theorem. To define equation, polynomial equation. To define roots of a polynomial equation.
2.	Understanding (U)	To tell application of remainder theorem. To find remainder using remainder theorem. To tell the application of factor theorem.
3.	Application (A)	To apply remainder theorem to find remainder. To apply factor theorem to factorise given polynomials.
4.	Higher Ability (HA)	To state and prove remainder theorem. To state and prove factor theorem. To solve polynomial equations using factor theorem.

2. Teaching materials

Chart paper with statements of remainder theorem

3. Teaching learning strategies

- Review concept of remainder.
- Ask factors of $x^3 - 28$ to the students, denote it by $f(x) = x^3 - 28$
- Divide $x^3 - 28$ by $x - 3$, find the remainder.
- Again find f(3), $f(3) = -1$, discuss, is -1 remainder when f(x) is divided by $x - 3$.
- State and prove remainder theorem with examples.
- Review concept of equations and identity with illustrated examples.
- Let $p(x) = x^2 - 3x + 2$
  factorise $= x^2 - 2x - x + 2 = (x - 2)(x - 1)$
  Also $p(2) = 0$, $p(1) = 0$
  Here, $(x - 2)$ and $(x - 1)$ are the factors of p(x).
- Define roots or zeros of a polynomial.
- State and prove factor theorem with examples.
- Explain how to solve polynomial equations using factor theorem.

Note : (1) Remainder theorem

i) If a polynomial f(x) is divided by a linear polynomial $(x + a)$, then the remainder is $R = f(-a)$

ii) If a polynomial f(x) is divided by a linear polynomial $(ax + b)$, then remainder $R = f(\frac{-b}{a})$

iii) If a polynomial f(x) is divided by a linear polynomial $(ax - b)$, then remainder $R = f(\frac{b}{a})$

Note : (2) Factor theorem

i) If f(x) is a polynomial and $f(-a) = 0$, then $(x + a)$ is a factor of f(x) and vice-versa.

ii) If $(ax - b)$ is a factor of $f(x)$, then, $f(\frac{b}{a}) = 0$.

iii) If $(ax + b)$ is a factor of $f(x)$, then $f(\frac{-b}{a}) = 0$.

Some solved problems

Remainder theorem

If a polynomial $f(x)$ is divided by $x - a$, then the remainder is $R = f(a)$.

1. Find the remainder in each of following cases when f(x) is divided by d(x).

a. $f(x) = x^3 + 3x^2 + 9x - 7$, $d(x) = x - 2$

Solution

Here, $f(x) = x^3 + 3x^2 + 9x - 7$

$d(x) = x - 2$

Comparing with $(x - 2)$ we get, $a = a$

Remainder, $R = f(2) = 2^3 + 3.2^2 + 9.2 - 7$

$= 8 + 12 + 18 - 7$

$= 31$

b. $f(x) = x^3 + 4x^2 - 20x + 20$, $d(x) = x + 2$

Solution

Here, $f(x) = x^3 + 4x^2 - 20x + 20$

Comparing $x + 2$ with $x - a$, we get, $a = -2$

Remainder, $R = f(-2) = (-2)^3 + 4(-2)^2 - 20(-2) + 20$

$= -8 + 16 + 40 + 20$

$= 68$

c. $f(x) = 2x^3 + 4x^2 - 10x + 20$, $d(x) = 2x - 1$

Solution

Here, $f(x) = 2x^3 + 4x^2 - 10x + 20$

When $f(x)$ is divided by $2x - 1$, $2x - 1 = 2(x - \frac{1}{2})$

Comparing $x - \frac{1}{2}$ with $x - a$, we get $x = \frac{1}{2}$

Remainder, $R = f(\frac{1}{2}) = 2(\frac{1}{2})^3 + 4(\frac{1}{2})^2 - 10.\frac{1}{2} + 20$

$= \frac{1}{4} + 1 - 5 + 20$

$= \frac{64}{4} = 16\frac{1}{4}$

2. Find the value of p when f(x) is divided by d(x) and remainder R = 5, $f(x) = px^3 + 4x - 10$, $d(x) = x + 3$

Solution

Here, $R = 5$, $f(x) = px^3 + 4x - 10$, $d(x) = x + 3 = x - (-3)$

Comparing $x - (-3)$ with $x - a$, we get $a = -3$

Then by remainder theorem, $R = f(-3)$

$$\text{or, } 5 = p(-3)^3 + 4(-3) - 10$$

$$\text{or, } 5 = -27p - 12 - 10$$

$$\text{or, } 27 = -27p$$

$$\therefore p = -1$$

3. (a) If $4x^2 - 6x + p$ and $x^3 - 2x^2 + 7$ are polynomials, they are divided by $x + 2$ and remainders are equal. What is the value of p ?

Solution

Let $f(x) = 4x^2 - 6x + p$

$g(x) = x^3 - 2x^2 + px + 7$

When they are divided by $x + 2$, remainders are equal. So, $f(-2) = g(-2)$

or, $4.(-2)^2 - 6(-2) + p = (-2)^3 - 2(-2)^2 + p(-2) + 7$

or, $16 + 12 + p = - 8 - 8 - 2p + 7$

or, $p + 28 = - 9 - 2p$

or, $3p = 37$

$\therefore p = \dfrac{37}{3}$

(b) If $x^3 - px^2 + 10x + 11$ and $2x^2 + 7px + 23$ are divided by $x - 1$, each gives equal remainder, find the value of p.

Solution

Let $f(x) = x^3 - px^2 + 10x + 11$

$g(x) = 2x^2 + 7px + 23$

They are divided by $x - 1$, and each gives equal remainders. So, $f(1) = g(1)$

i.e. $1 - p + 10 + 11 = 2 + 7p + 23$

or, $p + 22 = 7p + 25$

or, $-6p = 3$

$\therefore p = \dfrac{-1}{2}$

(c) If $x^3p^2 + x^2p - 8$ is divided by $x - 1$, then remainder is -2, find the values of p.

Solution

Here, $f(x) = x^3p^2 + x^2p - 8$

when $f(x)$ is divided by $x - 1$, remainder is -2.

By remainder theorem, we get,

$f(1) = -2$

or, $p^2 + p - 8 = -2$

or, $p^2 + P - 6 = 0$

or, $p^2 + 3p - 2p - 6 = 0$

or, $p(p + 3) - 2(p + 3) = 0$

or, $(p + 3)(p - 2) = 0$

$\therefore p = 2, -3$

(d) Find the values of p of polynomial $2x^2p^2 - 5px + 3$ is divided by $x - 1$ and remainder is zero.

Solution

Let $f(x) = 2x^2p^2 - 5px + 3$

When f(x) is divided by $x - 1$, then remainder is zero

i.e. $f(1) = 2p^2 - 5p + 3$

or, $0 = 2p^2 - 5p + 3$

or, $2p^2 - 3p - 2p + 3 = 0$

or, $p(2p - 3) - 1(2p - 3) = 0$

or $(2p - 3)(p - 1) = 0$

$\therefore p = 1, \frac{3}{2}$

Some Solved problems

Factor Theorem :

If a polynomial f(x) is divided by $(x - a)$ and the remainder $R = f(a) = 0$, then $x - a$ is a factor of f(x).

4. (a) Is there any relation between the factor theorem and the remainder theorem when a polynomial is factorized ?

Solution

Yes, there is relationship between the factor theorem and the remainder theorem. If a polynomial f(x) is divided by $(x - a)$, then remainder is $R = f(a)$. If the remainder is zero, i.e. $R = f(a) = 0$, then $x - a$ is a factor of f(x).

(b)If $f(\frac{-b}{a}) = 0$, when f(x) is divided by a linear polynomial, find one of factor of f(x).

Solution

Here, $f(\frac{-b}{a}) = 0$, then by factor theorem, $ax + b$ will be a factor of f(x).

$\therefore ax + 3 = a(x - (\frac{-b}{a}))$

5. (a) If $(x - k)$ is a factor of $f(x) = x^2 - 9$. What will be the value of f(k) ?

Solution

As $(x - k)$ is a factor of f(x), by factor theorem, we have $f(k) = 0$

i.e. $k^2 - 9 = 0$

or, $k^2 = 9$

$\therefore k = \pm 3$

$\therefore f(\pm 3) = (\pm 3)^2 - 9$

$= 9 - 9$

$= 0$

$\therefore f(k) = 0$

(b) If $(x - a)$ is a factor of $f(x) = x^2 - 121$. What will be the value of a?

Solution

Here, $f(x) = x^2 - 121$

If $(x - a)$ is a factor of $f(x)$, then by factor theorem,

$f(x) = 0$

or, $a^2 - 121 = 0$

$\therefore a = \pm 11$

6. (a) If $(x - 2)$ is a factor of $f(x) = x^3 - 8$, find $f(2)$.

Solution

If $(x - 2)$ is a factor of $f(x)$, then by factor theorem $f(2) = 0$

i.e. $f(2) = 2^3 - 8 = 0$

(b) If $f(x) = 2x^3 + 3x^2 - 11x - 6$ has a factor $(x - 2)$, find $f(2)$.

Solution

If $(x - 2)$ is a factor of $f(x) = 2x^3 + 3x^2 - 11x - 6$ then $f(2) = 0$.

i.e. $f(2) = 2.2^3 + 3.2^2 - 11.2 - 6$

$= 16 + 12 - 22 - 6$

$= 28 - 28$

$= 0$

7. (a) By using factor theorem, check $(x - 5)$ is a factor of $2x^2 - 11x + 5$ or not.

Solution

Let $f(x) = 2x^2 - 11x + 5$

put $x = 5$,

$f(5) = 2.5^2 - 11.5 + 5$

$= 55 - 55$

$= 0$

Since $f(5) = 0$, $x - 5$ is a factor of $f(x)$.

(b)Show that $(x + 1)$ is a factor of $x^3 - 4x^2 + x + 6$.

Solution

Let $f(x) = x^3 - 4x^2 + x + 6$

$f(-1) = (-1)^3 -4(-1)^2 - 1 + 6$

$= - 6 + 6$

$= 0$

Since $f(-1) = 0$, $x + 1$ is a factor of $f(x)$. proved

8. Define zero polynomial and zero of polynomial.

Solution

Zero polynomial

The constant polynomial, $p(x) = 0$, whose coefficients are equal to zero is called zero polynomial. The corresponding polynomial function with value 0, example : $f(x) = 0$, $g(x) = 0$ are examples of zero polynomial.

Zero of polynomial

A "zero of polynomial" is a value (a number) at which the polynomial evaluates zero.

Example, Let $p(x) = x^3 - 9x^2 + 24x - 20$

put $x = 2, 5$ successively, then we get

$p(2) = 2^3 - 9.2^2 + 24.2 - 20$

$= 8 - 36 + 48 - 20$

$= 0$

Again, $p(5) = 5^3 - 9.5^2 + 24.5 - 20$

$= 125 - 225 + 120 - 20$

$= 0$

Here the values of $x = 2, 5$ are called zeros of polynomials f(x). The number 2 and 5 are also called the roots of polynomial p(x).

9. (a) What must be added in $f(x) = 2x^3 + 6x^2 + 4x + 8$ so that $(x + 3)$ is a factor of it?

Solution

Here, $f(x) = 2x^3 + 6x^2 + 4x + 8$

Let k be added in f(x) so that $(x + 3)$ is a factor of f(x)

$f(-3) = 2(-3)^3 + 6.(-3)^2 + 4.(-3) + 8 + k$

or, $0 = -54 + 54 - 12 + 8 + k$

$\therefore k = 4$

Required number to be added is 4.

(b) What must be subtracted from $f(x) = x^3 + 8x^2 + 4x + 10$ so that $(x + 2)$ is a factor of it?

Solution

Here, $f(x) = x^3 + 8x^2 + 4x + 10$,

Let k be subtracted from f(x) so that $(x + 2)$ is a factor of it.

Then by factor theorem $f(-2) = 0$

or, $(-2)^3 + 8(-2)^2 + 4(-2) + 10 - k = 0$

or, $-8 + 32 - 8 + 10 - k = 0$

or, $k = 26$

Hence the number $k = 26$ should be subtracted.

10. (a) If $4x^2 + px + 8$ has a factor $x - 2$, find the value of k.

Solution

Let $f(x) = 4x^2 + px + 8$

If $(x - 2)$ is a factor of f(x), then by factor theorem, we have $f(2) = 0$

i.e. $4.2^2 + p.2 + 8 = 0$

or, $16 + 2p + 8 = 0$

or, $2p = -24$

$\therefore p = -12$

(b)If $f(x) = 3x^3 + 3x + k$ and $f(-1) = 0$, find the value of k.

Solution

Here, $f(x) = 3.(-1)^3 + 3.(-1) + k$

or, $0 = -3 - 3 + k$

$\therefore k = 6$

11. (a) By using the factor theorem, show that $(x + 1)$, $(x - 2)$, $(x - 3)$ are the factors of $f(x) = x^3 - 4x^2 + x + 6$.

Solution

Here, $f(x) = x^3 - 4x^2 + x + 6$

From $x + 1 = x - (-1)$, $f(-1) = -1 - 4 - 1 + 6 = 0$

Hence $x + 1$ is a factor of $f(x)$

From $x - 2$, $f(2) = 2^3 - 4.2^2 + 2 + 6 = 0$

Hence $(x - 2)$ is a factor of $f(x)$.

(b) By using factor theorem, show that $(x + 2)$, $(x + 6)$ and $(2x - 3)$ are the factors of $f(x) = 2x^3 + 13x^2 - 36$.

Solution

Here, $f(x) = 2x^3 + 13x^2 - 36$

From, $x + 2 = x - (-2)$, $f(-2) = 2.(-2)^3 + 13.(-2)^2 - 36$

$= -16 + 52 - 36 = 0$

From, $x + 6 = x - (-26)$, $f(-6) = 2.(-6)^3 + 13.(-6)^2 - 36$

$= -432 + 468 - 36$

$= -468 + 468$

$= 0$

From, $2x - 3 = 2(x - \frac{3}{2})$, $f(\frac{3}{2}) = 2.(\frac{3}{2})^3 + 13.(\frac{3}{2})^2 - 36$

$= \frac{27}{4} + \frac{117}{4} - 36$

$= 36 - 36$

$= 0$

Since $f(-2) = 0$, $f(-6) = 0$ and $f(\frac{3}{2}) = 0$, $(x + 2)$, $(x + 6)$ and $(2x - 3)$ are the factors of $f(x)$.

12. Factories (by using factor theorem)

(a) $x^3 - 6x^2 + 11x - 6$

Solution

Let $f(x) = x^3 - 6x^2 + 11x - 6$

for $x = 1$, $f(1) = 1^3 - 6.(1)^2 + 11.(1) - 6$

$= 1 - 6 + 11 - 6$

$= 0$

Hence from factor theorem, $(x - 1)$ is a factor $f(x)$

$\therefore f(x) = x^3 - 6x^2 + 11x - 6$

$= x^2(x - 1) - 5x(x - 1) + 6(x - 1)$

$= (x - 1)(x^2 - 5x + 6)$

$= (x - 1)(x^2 - 3x - 2x + 6)$

$= (x - 1)\{x(x - 3) - 2(x - 3)\}$

$= (x - 1)(x - 3)(x - 2)$

$= (x - 1)(x - 2)(x - 3)$

Alternative Method

Since $f(1) = 0$

$(x-1)$ is a factor of $f(x)$.

Now using synthetic division method

1	1	–6	11	–6	
	↓	1	–5	6	
	1	–5	6		=0

$\therefore f(x) = (x-1)(x^2-5x+6)$

$= (x-1)(x^2-3x-2x-6)$

$= (x-1)(x-3)(x-2)$

(b) $x^3 - 4x^2 + x + 6$

Solution

Let $f(x) = x^3 - 4x^2 + x + 6$

for $n = -1$, $f(-1) = (-1)^3 - 4(-1)^2 - 1 + 6$

$= -1 - 4 - 1 + 6$

$= 0$

Hence by factor theorem, $(x + 1)$ is a factor of $f(x)$

$f(x) = x^3 - 4x^2 + x + 6$

$= x^2(x + 1) - 5x(x + 1) + 6(x + 1)$

$= (x + 1)(x^2 - 5x + 6)$

$= (x + 1)(x^2 - 2x - 3x + 6)$

$= (x + 1)\{x(x - 2) - 3(x - 3)\}$

$= (x + 1)(x - 2)(x - 3)$

(c) $x^3 - 11x^2 + 31x - 21$

Solution

Let $f(x) = x^3 - 11x^2 + 31x - 21$

for $x = 1$, $f(1) = (1)^3 - 11(1)^2 + 31(1) - 21$

$= 1 - 11 + 31 - 21$

$= 0$

Hence by factor theorem, $(x - 1)$ is a factor of $f(x)$

$f(x) = x^3 - 11x^2 + 31x - 21$

$= x^2(x - 1) - 10x(x - 1) + 21(x - 1)$

$= (x - 1)(x^2 - 10x + 21)$

$= (x - 1)(x^2 - 3x - 7x + 21)$

$= (x - 1)\{x(x - 3) - 7(x - 3)\}$

$= (x - 1)(x - 3)(x - 7)$

(d) $x^3 - 3x^2 + 4x - 4$

Solution

Let $f(x) = x^3 - 3x^2 + 4x - 4$

for $x = 2$, $f(2) = 2^3 - 3(2)^2 + 4(2) - 4$

$= 8 - 12 + 8 - 4$

$= 0$

Hence from factor theorem, $(x - 2)$ is a factor of $f(x)$

$f(x) = x^3 - 3x^2 + 4x - 4$

$= x^2(x - 2) - x(x - 2) + 2(x - 2)$

$= (x - 2)(x^2 - x + 2)$

(e) $x^3 + 2x^2 - 5x - 6$

Solution

Let $f(x) = x^3 + 2x^2 - 5x - 6$

for $x = -1$, $f(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6$

$= -1 + 2 + 5 - 6$

$= 0$

Hence by factor theorem, $(x + 1)$ is a factor of $f(x)$.

$f(x) = x^3 + 2x^2 - 5x - 6$

$= x^2(x + 1) + x(x + 1) - 6(x + 1)$

$= (x + 1)(x^2 + x - 6)$

$= (x + 1)(x^2 + 3x - 2x - 6)$

$= (x + 1)\{x(x + 3) - 2(x + 3)\}$

$= (x + 1)(x - 2)(x + 3)$

(f) $3x^3 - 6x^2 + 4x - 8$

Solution

Let $f(x) = 3x^3 - 6x^2 + 4x - 8$

for $x = 2$, $f(2) = 3(2)^3 - 6(2)^2 + 4(2) - 8$

$= 24 - 24 + 8 - 8$

$= 0$

Hence by factor theorem, $(x - 2)$ is a factor of $f(x)$.

$f(x) = 3x^3 - 6x^2 + 4x - 8$

$= 3x^2(x - 2) + 4(x - 2)$

$= (x - 2)(3x^2 + 4)$

(g) $6x^3 + 7x^2 - x - 2$

Solution

Let $f(x) = 6x^3 + 7x^2 - x - 2$

for $x = -1$, $f(-1) = 6(-1)^3 + 7(-1)^2 + 1 - 2$

$= -6 + 7 + 1 - 2$

$= 0$

Hence by factor theorem, $(x + 1)$ is a factor of $f(x)$.

$f(x) = 6x^3 + 7x^2 - x - 2$

$= 6x^2(x + 1) + x(x + 1) - 2(x + 1)$

$= (x + 1)(6x^2 + x - 2)$

$= (x + 1)(6x^2 + 4x - 3x - 2)$

$= (x + 1)\{2x(3x + 2) - 1(3x + 2)\}$

$= (x + 1)(2x - 1)(3x + 2)$

(h) $2x^3 + 3x^2 - 11x - 6$

Solution

Let $f(x) = 2x^3 + 3x^2 - 11x - 6$

for $x = 2$, $f(2) = 2(2)^3 + 3(2)^2 - 11(2) - 6$

$= 16 + 12 - 22 - 6$

$= 0$

Hence by factor theorem, $(x - 2)$ is a factor of $f(x)$

$f(x) = 2x^3 + 3x^2 - 11x - 6$

$= 2x^2(x - 2) + 7x(x - 2) + 3(x - 2)$

$= (x - 2)(2x^2 + 7x + 3)$

$= (x - 2)(2x^2 + 6x + x + 3)$

$= (x - 2)\{2x(x + 3) + 1(x + 3)\}$

$= (x - 2)(x + 3)(2x + 1)$

(i) $x^3 + 5x^2 - 2x - 24$

Solution

Let $f(x) = x^3 + 5x^2 - 2x - 24$

for $x = 2$, $f(2) = (2)^3 + 5(2)^2 - 2(2) - 24$

$= 8 + 20 - 4 - 24$

$= 0$

Hence by factor theorem, $(x - 2)$ is a factor of $f(x)$

$f(x) = x^3 + 5x^2 - 2x - 24$

$= x^2(x - 2) + 7x(x - 2) + 12(x - 2)$

$= (x - 2)(x^2 + 7x + 12)$

$= (x - 2)(x^2 + 4x + 3x + 12)$

$= (x - 2)\{x(x + 4) + 3(x + 4)\}$

$= (x - 2)(x + 3)(x + 4)$

(j) $x^3 - 9x^2 + 26x - 24$

Solution

Let $f(x) = x^3 - 9x^2 + 26x - 24$

for $x = 2$, $f(2) = (2)^3 - 9(2)^2 + 26(2) - 24$

$= 8 - 36 + 52 - 24$

$= 0$

Hence by factor theorem, $(x - 2)$ is a factor of $f(x)$

$f(x) = x^3 - 9x^2 + 26x - 24$

$= x^2(x - 2) - 7x(x - 2) + 12(x - 2)$

$= (x - 2)(x^2 - 7x + 12)$

$= (x - 2)(x^2 - 4x - 3x + 12)$

$= (x - 2)\{x(x - 4) - 3(x - 4)\}$

$= (x - 2)(x - 3)(x - 4)$

(k) $x^3 - 9x^2 + 23x - 15$

Solution

Let $f(x) = x^3 - 9x^2 + 23x - 15$

for $x = 1$, $f(1) = (1)^3 - 9(1)^2 + 23(1) - 15$

$= 1 - 9 + 23 - 15$

$= 0$

Hence by factor theorem, $(x - 1)$ is a factor of $f(x)$

$f(x) = x^3 - 9x^2 + 23x - 15$

$= x^2(x - 1) - 8x(x - 1) + 15(x - 1)$

$= (x - 1)(x^2 - 8x + 15)$

$= (x - 1)(x^2 - 5x - 3x + 15)$

$= (x - 1)\{x(x - 5) - 3(x - 5)\}$

$= (x - 1)(x - 3)(x - 5)$

(l) $x^3 - 19x - 30$

Solution

Let $f(x) = x^3 - 19x - 30$

for $x = -2$, $f(-2) = (-2)^3 - 19(-2) - 30$

$= -8 + 38 - 30$

$= 0$

Hence by factor theorem, $(x + 2)$ is a factor of $f(x)$

$f(x) = x^3 - 19x - 30$

$= x^2(x + 2) - 2x(x + 2) - 15(x + 2)$

$= (x + 2)(x^2 - 2x - 15)$

$= (x + 2)(x^2 - 5x + 3x - 15)$

$= (x + 2)\{x(x - 5) + 3(x - 5)\}$

$= (x + 2)(x + 3)(x - 5)$

(m) $x^3 - 5x + 4$

Solution

Let $f(x) = x^3 - 5x + 4$

for $x = 1$, $f(1) = (1)^3 - 5(1) + 4$

$= 1 - 5 + 4$

$= 0$

Hence by factor theorem, $(x - 1)$ is a factor of $f(x)$.

$f(x) = x^3 - 5x + 4$

$= x^2(x - 1) + x(x - 1) - 4(x - 1)$

$= (x - 1)(x^2 + x - 4)$

(n) $\mathbf{2x^3 + 3x^2 - 11x - 6}$

Solution

Let $f(x) = 2x^3 + 3x^2 - 11x - 6$

For $x = 2$, $f(2) = 2(2)^3 + 3(2)^2 - 11(2) - 6$

$= 16 + 12 - 22 - 6$

$= 28 - 28 \qquad = 0$

Hence by factor theorem, $(x - 2)$ is a factor of $f(x)$

Now, $f(x) = 2x^3 + 3x^2 - 11x - 6$

$= 2x^2(x - 2) + 7x(x - 2) + 3(x - 2)$

$= (x - 2)(2x^2 + 7x + 3)$

$= (x - 2)(2x^2 + x + 6x + 3)$

$= (x - 2)\{x(2x + 1) + 3(2x + 1)\}$

$= (x - 2)(2x + 1)(x + 3)$

12. Factorize:

(a) $(x + 3)(x + 5)(x + 7)(x + 9) + 1$

Solution

Let $f(x) = (x + 3)(x + 5)(x + 7)(x + 9) + 1$

$= \{(x + 5)(x + 7)\}\{(x + 3)(x + 9)\} + 1$

$= (x^2 + 12x + 35)(x^2 + 12x + 27) + 1$

$= (\alpha + 35)(\alpha + 27) + 16$

where, $\alpha = x^2 + 12x$

$f(x) = (\alpha + 35)(\alpha + 27) + 16$

$= \alpha^2 + 62\alpha + 945 + 16$

$= \alpha^2 + 2.\alpha.31 + (31)^2$

$= (\alpha + 31)^2$

$= (\alpha + 31)(\alpha + 31)$

$= (x^2 + 12x + 31)(x^2 + 12x + 31)$

(b) $\mathbf{(x + 2)(x - 3)(x - 1)(x - 6) + 56}$

Solution

Let $f(x) = (x + 2)(x - 3)(x - 1)(x - 6) + 56$

$= \{(x + 2)(x - 6)\}\{(x - 3)(x - 1)\} + 56$

$= (x^2 - 6x + 2x - 12)(x^2 - x - 3x + 3) + 56$

$= (x^2 - 4x - 12)(x^2 - 4x + 3) + 56$

Let, $x^2 - 4x = a$

$f(x) = (a - 12)(a + 3) + 56$
$= a^2 - 9a - 36 + 56$
$= a^2 - 9a + 20$
$= a^2 - 5a - 4a + 20$
$= a(a - 5) - 4(a - 5)$
$= (a - 5)(a - 4)$

replacing $a = x^2 - 4x$

$f(x) = (x^2 - 4x - 5)(x^2 - 4x - 4)$
$= (x^2 - 5x + x - 5)(x^2 - 4x - 4)$
$= \{x(x - 5) + 1(x - 5)\}(x^2 - 4x - 4)$
$= (x - 5)(x + 1)(x^2 - 4x - 4)$

(c) $(x - 1)(2x^2 + 5x + 5)$

Solution

Let $f(x) = (x - 1)(2x^2 + 5x + 5) + 4$
$= 2x^3 + 5x^2 + 5x - 2x^2 - 5x - 5 + 4$
$= 2x^3 + 3x^2 - 1$

for $x = -1$, $f(-1) = 2(-1)^3 + 3(-1)^2 - 1$
$= -2 + 3 - 1$
$= 0$

Hence by factor theorem, $(x + 1)$ is a factor of $f(x)$

$f(x) = 2x^3 + 3x^2 - 1$
$= 2x^2(x + 1) + x(x + 1) - 1(x + 1)$
$= (x + 1)(2x^2 + x - 1)$
$= (x + 1)(2x^2 + 2x - x - 1)$
$= (x + 1)\{2x(x + 1) - 1(x + 1)\}$
$= (x + 1)(x + 1)(2x - 1)$

(d) $(x - 1)(2x^2 + 15x + 15) - 21$

Solution

Let $f(x) = (x - 1)(2x^2 + 15x + 15) - 21$
$= 2x^3 + 15x^2 + 15x - 2x^2 - 15x - 15 - 21$
$= 2x^3 + 13x^2 - 36$

for $x = -2$, $f(-2) = 2(-2)^3 + 13(-2)^2 - 36$
$= -16 + 52 - 36$
$= 0$

Hence by factor theorem, $(x + 2)$ is a factor of $f(x)$.

$f(x) = 2x^3 + 13x^2 - 36$
$= 2x^2(x + 2) - 9x(x + 2) - 18(x + 2)$
$= (x + 2)(2x^2 + 9x - 18)$

$= (x + 2)(2x^2 + 12x - 3x - 18)$
$= (x + 2)\{2x(x + 6) - 3(x + 6)\}$
$= (x + 2)(x + 6)(2x - 3)$

(e) $(x - 3)(x^2 - 5x + 8) - 4x + 12$

Solution

Let $f(x) = (x - 3)(x^2 - 5x + 8) - 4x + 12$
$= x^3 - 5x^2 + 8x - 3x^2 + 15x - 24 - 4x + 12$
$= x^3 - 8x^2 + 19x - 12$

for $x = 1$, $f(1) = 1^3 - 8.1^2 + 19.1 - 12$
$= 20 - 20 \quad = 0$

Hence $(x - 1)$ is a factor of $f(x)$.

Now, $f(x) = x^3 - x^2 - 7x^2 + 7x + 12x - 12$
$= x^2(x - 1) - 7(x - 1) + 12(x - 1)$
$= (x - 1)(x^2 - 7x + 12)$
$= (x - 1)(x^2 - 4x - 3x + 12)$
$= (x - 1)(x - 4)(x - 3)$

13. Solve the following polynomial equations.

(a) $x^3 - 3x^2 - 4x + 12 = 0$

Solution

Let $f(x) = x^3 - 3x^2 - 4x + 12$

For, $x = 2$, $f(2) = 2^3 - 3.2^2 - 4.2 + 12$
$= 8 - 12 - 8 + 12$
$= 0$

Hence by factor theorem, $(x - 2)$ is a factor of $f(x)$.

Now $f(x) = x^3 - 2x^2 - x^2 + 2x - 6x + 12$
$= x^2(x - 2) - x(x - 2) - 6(x - 2)$
$= (x - 2)(x^2 - x - 6)$
$= (x - 2)(x^2 - 3x + 2x - 6)$
$= (x - 2)(x - 3)(x + 2)$

To solve

$f(x) = 0$

or, $(x - 2)(x - 3)(x + 2) = 0$

Either $x - 2 = 0$ .... (i)

or, $x - 3 = 0$ .... (ii)

$x + 2 = 0$ .... (iii)

From equation (i) $x = 2$,

From equation (ii) $x = 3$,

From equation (iii) $x = -2$

$\therefore x = 2, 3, -2$

Alternative Method

As $f(2) = 0$, $x-2$ is a factor of $f(x)$.

Now using synthetic division method

$$\begin{array}{c|cccc} 2 & 1 & -3 & -4 & 12 \\ & \downarrow & 2 & -2 & -12 \\ \hline & 1 & -1 & -6 & \vert\ 0 \end{array}$$

Now, $f(x) = (x-2)(x^2-x-6)$
$= (x-2)(x^2-3x+2x-6)$
$= (x-2)(x-3)(x+2)$

To solve, $f(x) = 0$

or, $(x-2)(x-3)(x-2) = 0$

$\therefore x = -2, 2, 3$

(b) Solve : $x^3 - 9x^2 + 24x - 20 = 0$

Solution

Let $f(x) = x^3 - 9x^2 + 24x - 20 = 0$

for $x = 2$,

$f(2) \quad = (2)^3 - 9(2)^2 + 24(2) - 20$

$= 8 - 36 + 48 - 20$

$= 0$

Hence by factor theorem, $(x - 2)$ is a factor of $f(x)$.

$f(x) \quad = x^3 - 9x^2 + 24x - 20$

$= x^2(x - 2) - 7x(x - 2) + 10(x - 2)$

$= (x - 2)(x^2 - 7x + 10)$

$= (x - 2)(x^2 - 5x - 2x + 10)$

$= (x - 2)\{x(x - 5) - 2(x - 5)\}$

$= (x - 2)(x - 5)(x - 2)$

To solve,

$f(x) = 0$

$\therefore (x - 2)(x - 5)(x - 2) = 0$

Either $x - 2 = 0$ .... (i)

or, $x - 5 = 0$ .... (ii)

or, $x - 2 = 0$ .... (iii)

From equation (i)

$x - 2 = 0$

$\therefore x = 2$

From equation (ii)

$x - 5 = 0$

$\therefore x = 5$

From equation (iii)

$x - 2 = 0$

$\therefore x = 2$

$\therefore x = 2, 5$

(c) $x^3 - 7x^2 + 7x + 15$

Solution

Let $f(x) = x^3 - 7x^2 + 7x + 15$

For $x = -1$

$f(-1) \quad = (-1)^3 - 7(-1)^2 - 7 + 15$

$= -1 - 7 - 7 + 15$

$= 0$

Hence by factor theorem, $(x + 1)$ is a factor of $f(x)$

$f(x) \quad = x^3 - 7x^2 + 7x + 15$

$= x^2(x + 1) - 8x(x + 1) + 15(x + 1)$
$= (x + 1)(x^2 - 8x + 15)$
$= (x + 1)(x^2 - 5x - 3x + 15)$
$= (x + 1)\{x(x - 5) - 3(x - 5)\}$
$= (x + 1)(x - 3)(x - 5)$

To solve,
$f(x) = 0$
$(x + 1)(x - 3)(x - 5) = 0$
Either $x + 1 = 0$ .... (i)
or, $x - 3 = 0$ .... (ii)
or, $x - 5 = 0$ .... (iii)
From equation (i)
$x + 1 = 0$
$x = -1$
From equation (ii)
$x - 3 = 0$
$x = 3$
From equation (iii)
$x - 5 = 0$
$x = 5$
$\therefore x = -1, 3, 5$

(d) $\mathbf{x^3 - 4x^2 + x + 6 = 0}$

Solution

Let $f(x) = x^3 - 4x^2 + x + 6 = 0$
for $x = -1$,
$f(-1) = (-1)^3 - 4(-1)^2 - 1 + 6$
$= -1 - 4 - 1 + 6$
$= 0$
Hence by factor theorem, $(x + 1)$ is a factor of $f(x)$
$f(x) = x^3 - 4x^2 + x + 6$
$= x^2(x + 1) - 5x(x + 1) + 6(x + 1)$
$= (x + 1)(x^2 - 5x + 6)$
$= (x + 1)(x^2 - 3x - 2x + 6)$
$= (x + 1)\{x(x - 3) - 2(x - 3)\}$
$= (x + 1)(x - 2)(x - 3)$
To solve,
$f(x) = 0$
$\therefore (x + 1)(x - 2)(x - 3) = 0$
Either $x + 1 = 0$ .... (i)

or, $x - 2 = 0$ .... (ii)
or, $x - 3 = 0$ .... (iii)
From equation (i)
$x + 1 = 0$
$\therefore x = -1$
From equation (ii)
$x - 2 = 0$
$\therefore x = 2$
From equation (iii)
$x - 3 = 0$
$\therefore x = 3$
$\therefore x = -1, 2, 3.$

(e) $\mathbf{3x^3 - 19x^2 + 32x - 16 = 0}$

Solution

Let $f(x) = 3x^3 - 19x^2 + 32x - 16$
For $x = 1$, $f(1) = 3 - 19 + 32 - 16 = 0$
Hence by factor theorem, $x - 1$ is a factor of $f(x)$.
Now, $f(x) = 3x^3 - 3x^2 - 16x^2 + 16x + 16x - 16$
$= 3x^2(x - 1) - 16x(x - 1) + 16(x - 1)$
$= (x - 1)(3x^2 - 16x + 16)$
$= (x - 1)(3x^2 - 12x - 4x + 16)$
$= (x - 1)(x - 4)(3x - 4)$
To solve, $f(x) = 0$
or, $(x - 1)(x - 4)(3x - 4) = 0$
Either $x - 1 = 0$ .... (i)
or, $x - 4 = 0$ .... (ii)
or, $3x - 4 = 0$ .... (iii)
From equation (i)
$x - 1 = 0$
or, $x = 1$
From equation (ii)
$x - 4 = 0$
or, $x = 4$
From equation (iii),
$3x - 4 = 0$
or, $x = \frac{4}{3}$
$\therefore x = 1, 4, \frac{4}{3}$

(f) $6x^3 + 7x^2 - x - 2 = 0$

Solution

Let $f(x) = 6x^3 + 7x^2 - x - 2 = 0$

for $x = -1$,

$f(-1) \quad = 6(-1)^3 + 7(-1)^2 + 1 - 2$

$= -6 + 7 + 1 - 2$

$= 0$

Hence by factor theorem, $(x + 1)$ is a factor of $f(x)$

$f(x) \quad = 6x^3 + 7x^2 - x - 2$

$= 6x^2(x + 1) + x(x + 1) - 2(x + 1)$

$= (x + 1)(6x^2 + x - 2)$

$= (x + 1)(6x^2 + 4x - 3x - 2)$

$= (x + 1)\{2x(3x + 2) - 1(3x + 2)\}$

$= (x + 1)(3x + 2)(2x - 1)$

To solve,

$f(x) = 0$

$\therefore (x + 1)(3x + 2)(2x - 1) = 0$

Either $x + 1 = 0$ .... (i)

or, $3x + 2 = 0$ .... (ii)

or, $2x - 1 = 0$ .... (iii)

$\therefore$ From (i), (ii) and (iii), we get

$x = -1, \frac{-2}{3}$ and $\frac{1}{2}$ respectively

$\therefore x = -1, -\frac{2}{3}, \frac{1}{2}$

(g) $x^3 - 6x^2 + 11x - 6 = 0$

Solution

Let $f(x) = x^3 - 6x^2 + 11x - 6 = 0$

for $x = 1$,

$f(1) \quad = 1^3 - 6(1)^2 + 11(1) - 6$

$= 1 - 6 + 11 - 6$

$= 0$

Hence by factor theorem, $(x - 1)$ is a factor of $f(x)$.

$f(x) \quad = x^3 - 6x^2 + 11x - 6$

$= x^2(x - 1) - 5x(x - 1) + 6(x - 1)$

$= (x - 1)(x^2 - 5x + 6)$

$= (x - 1)(x^2 - 3x - 2x + 6)$

$= (x - 1)\{x(x - 3) - 2(x - 3)\}$

$= (x - 1)(x - 2)(x - 3)$

To solve

$f(x) = 0$

$(x-1)(x-2)(x-3) = 0$

Either

$x - 1 = 0$ .... (i)

or,

$x - 2 = 0$ .... (ii)

or,

$x - 3 = 0$ .... (iii)

From (i), $x = 1$

From (ii), $x = 2$

From (iii), $x = 3$

$\therefore x = 1, 2, 3$

(h) $\mathbf{5x^3 + 2x^2 - 20x - 8 = 0}$

Solution

Let $f(x) = 5x^3 + 2x^2 - 20x - 8 = 0$

For $x = 2$

$$\begin{aligned} f(2) &= 5(2)^3 + 2(2)^2 - 20(2) - 8 \\ &= 40 + 8 - 40 - 8 \\ &= 0 \end{aligned}$$

Hence by factor theorem, $(x - 2)$ is a factor of $f(x)$

$$\begin{aligned} f(x) &= 5x^3 + 2x^2 - 20x - 8 \\ &= 5x^2(x-2) + 12x(x-2) + 4(x-2) \\ &= (x-2)(5x^2 + 12x + 4) \\ &= (x-2)(5x^2 + 10x + 2x + 4) \\ &= (x-2)\{5x(x+2) + 2(x+2)\} \\ &= (x-2)(x+2)(5x+2) \end{aligned}$$

To solve,

$f(x) = 0$

$(x-2)(x+2)(5x+2) = 0$

Either

$x - 2 = 0$ .... (i)

or,

$x + 2 = 0$ .... (ii)

or,

$5x + 2 = 0$........... (iii)

From (i), $x = 2$

From (ii), $x = -2$

From (iii), $x = \dfrac{-2}{5}$

$\therefore x = 2, -2, \dfrac{-2}{5}$

(i) $\mathbf{y^3 - 13y - 12 = 0}$

Solution

Let $f(y) = y^3 - 13y - 12 = 0$

for $y = -1$,

$f(-1) \quad = (-1)^3 - 13(-1) - 12$

$= -1 + 13 - 12$

$= 0$

Hence by factor theorem, $(y + 1)$ is a factor of $f(y)$

$f(y) \quad = y^3 - 13y - 12$

$= y^2(y + 1) - y(y + 1) - 12(y + 1)$

$= (y + 1)(y^2 - y - 12)$

$= (y + 1)(y^2 - 4y + 3y - 12)$

$= (y + 1)\{y(y - 4) + 3(y - 4)\}$

$= (y + 1)(y - 4)(y - 4)$

To solve,

$f(y) = 0$

$(y + 1)(y - 4)(y - 4) = 0$

Either

$y + 1 = 0$ ....(i)

$y - 4 = 0$ ....(ii)

$y + 3 = 0$ ....(iii)

From (i), $y = -1$

From (ii), $y = 4$

From (iii), $y - 3$

$\therefore y = -3, -1, 4$

(j) $y^3 - 3y - 2 = 0$

Solution

Let $f(y) = y^3 - 3y - 2 = 0$

For $y \quad = -1$

$f(-1) \quad = (-1)^3 - 3(-1) - 2$

$= -1 + 3 - 2$

$= 0$

Hence by factor theorem, $(y + 1)$ is a factor of $f(y)$.

$f(y) \quad = y^3 - 3y - 2$

$= y^2(y + 1) - y(y + 1) - 2(y + 1)$

$= (y + 1)(y^2 - y - 2)$

$= (y + 1)(y^2 - 2x + y - 2)$

$= (y + 1)\{y(y - 2) + 1(y - 2)\}$

$= (y + 1)(y + 1)(y - 2)$

To solve, $f(y) = 0$

$(y + 1)(y + 1)(y - 2) = 0$

Either

$y + 1 = 0$ .... (i)

or, $y - 2 = 0$ .... (ii)

From (i), $y = -1$

From (ii), $y = 2$

$\therefore y = -1, 2$

(k) $4y^3 - 3y - 1 = 0$

Solution

Let $f(y) = 4y^3 - 3y - 1 = 0$

for $y = 1$

$f(1) \quad = 4(1)^3 - 3 - 1$

$= 4 - 3 - 1$

$= 0$

Hence by factor theorem, $(y - 1)$ is a factor of $f(x)$

$f(y) \quad = 4y^3 - 3y - 1$

$= 4y^2(y - 1) + 4y(y - 1) + 1(y - 1)$

$= (y - 1)(4y^2 + 4y + 1)$

$= (y - 1)(2y + 1)^2$

To solve,

$f(x) = 0$

$(y - 1)(2y + 1)^2 = 0$

Either $y - 1 =$ ..... (i)

or $(2y + 1)^2 = 0$ ...... (ii)

From equation (i), $y = 1$

From equation (ii), $y = \frac{-1}{2}$

(l) $y^3 - 19y - 30 = 0$

Solution

Let $f(y) = y^3 - 19y - 30 = 0$

for $y = -2$

$f(-2) \quad = (-2)^3 - 19(-2) - 30$

$= -8 + 38 - 30$

$= 0$

Hence from factor theorem, $(y + 2)$ is a factor of $f(y)$.

$f(y) \quad = y^3 - 19y - 30$

$= y^2(y + 2) - 2y(y + 2) - 15(y + 2)$

$= (y + 2)(y^2 - 2y - 15)$

$= (y + 2)(y^2 - 5y + 3y - 15)$

$= (y + 2)\{y(y - 5) + 3(y - 5)\}$

$= (y + 2)(y + 3)(y - 5)$

To solve,

$f(y) = 0$

or, $(y + 2)(y + 3)(y - 5) = 0$

Either

$y + 2 = 0$ .... (i)

or, $y + 3 = 0$ .... (ii)
or, $y - 5 = 0$ .... (iii)
From (i), $y = -2$
From (ii), $y = -3$
From (iii), $y = 5$
$\therefore y = -2, -3, 5$

(m) $x^4 - x^3 - 19x^2 + 49x - 30 = 0$

Solution

Let $f(x) = x^4 - x^3 - 19x^2 + 49x - 30 = 0$
for $x = 1$

$$\begin{aligned} f(1) &= (1)^4 - (1)^3 - 19(1)^2 + 49(1) - 30 \\ &= 1 - 1 - 19 + 49 - 30 \\ &= 0 \end{aligned}$$

Hence by factor theorem, $(x - 1)$ is a factor of $f(x)$

$$\begin{aligned} f(x) &= x^4 - x^3 - 19x^2 + 49x - 30 \\ &= x^3(x - 1) - 19x(x - 1) + 30(x - 1) \\ &= (x - 1)(x^3 - 19x + 30) \end{aligned}$$

Let $Q(x) = (x^3 - 19x + 30)$
for $x = 2$

$$\begin{aligned} Q(2) &= 2^3 - 19(2) + 30 \\ &= 8 - 38 + 30 \\ &= 0 \end{aligned}$$

Hence from factor theorem, $(x - 2)$ is factor of $Q(x)$

$$\begin{aligned} Q(x) &= x^3 - 19x + 30 \\ &= x^2(x - 2) + 2x(x - 2) - 15(x - 2) \\ &= (x - 2)(x^2 + 2x - 15) \\ &= (x - 2)(x^2 + 5x - 3x - 15) \\ &= (x - 2)\{x(x + 5) - 3(x + 5)\} \\ &= (x - 2)(x + 5)(x - 3) \end{aligned}$$

$f(x)$ becomes,
$f(x) = (x - 1)(x - 2)(x + 5)(x - 3) = 0$
To solve, $f(x) = 0$
or, $(x - 1)(x - 2)(x + 5)(x - 3) = 0$
Either,
$x - 1 = 0$ .... (i)
or, $x - 2 = 0$ .... (ii)
or, $x + 5 = 0$ .... (iii)
or, $x - 3 = 0$ .... (iv)
From (i), (ii), (iii) and (iv),
$x = 1, 2, 3, -5$

(n) $\mathbf{2x^4 - 13x^3 + 28x^2 - 23x + 6 = 0}$

Solution

Let $f(x) = 2x^4 - 13x^3 + 28x^2 - 23x + 6 = 0$

for $x = 1$

$f(1) \quad = 2(1)^4 - 13(1)^3 + 28(1)^2 - 23(1) + 6$

$\quad = 2 - 13 + 28 - 23 + 6$

$\quad = 0$

Hence by factor theorem, $(x - 1)$ is a factor of $f(x)$

$f(x) \quad = 2x^4 - 13x^3 + 28x^2 - 23x + 6$

$\quad = 2x^3(x - 1) - 11x^2(x - 1) + 17x(x - 1) - 6(x - 1)$

$\quad = (x - 1)(2x^3 - 11x^2 + 17x - 6)$

Let $Q(x) = 2x^3 - 11x^2 + 17x - 6$

for $x = 2$

$Q(2) \quad = 2(2)^3 - 11(2)^2 + 17(2) - 6$

$\quad = 16 - 44 + 34 - 6$

$\quad = 0$

Hence by factor theorem, $(x - 2)$ is a factor of $Q(x)$

$Q(x) \quad = 2x^3 - 11x^2 + 17x - 6$

$\quad = 2x^2(x - 2) - 7x(x - 2) + 3(x - 2)$

$\quad = (x - 2)(2x^2 - 7x + 3)$

$\quad = (x - 2)(2x^2 - 6x - x + 3)$

$\quad = (x - 2)\{2x(x - 3) - 1(x - 3)\}$

$\quad = (x - 2)(x - 3)(2x - 1)$

$\therefore$ f(x) becomes

$f(x) \quad = (x - 1)(x - 2)(x - 3)(2x - 1)$

To solve, $f(x) = 0$

or, $(x - 1)(x - 2)(x - 3)(2x - 1) = 0$

Either,

$x - 1 = 0$ .... (i)

or, $x - 2 = 0$ .... (ii)

or, $x - 3 = 0$ .... (iii)

or, $2x - 1 = 0$ .... (iv)

From (i), $x = 1$

From (ii), $x = 2$

From (iii), $x = 3$

From (iv), $x = \frac{1}{2}$

$\therefore x = 1, 2, 3, \frac{1}{2}$

13. Find the values of a and b in each of the following cases

(a)(x – 1) and (x – 2) are the factors of $x^3 - ax^2 + bx - 8$.

Solution

$f(x) = x^3 - ax^2 + bx - 8$

Since (x – 1) and (x – 2) are factors of f(x)

f(1) = 0 and f(2) = 0, by factor theorem,

Now, by factor theorem, we have

$f(1) = 0$

or, $1 - a + b - 8 = 0$

or, $a - b = -7$ .... (i)

Again

$f(2) = 0$

or, $2^3 - 4a + 2b - 8 = 0$

or, $2a - b = 0$ .... (ii)

Now,

Subtracting equation (i) from (ii)

$$\begin{array}{r} 2a - b = 0 \\ \underline{\underset{-}{} a \underset{+}{-} b = \underset{+}{-}7} \\ a = 7 \end{array}$$

$\therefore a = 7$

Substituting value of a in (i)

$b = a + 7 = 7 + 7$

$ = 14$

∴ The value of a is 7 and b is 14.

(b)(x + 3) and (2x – 7) are factors of $ax^2 - bx - 21$

Solution

Let $f(x) = ax^2 - bx - 21$

Since (x + 3) and (2x – 7) are factors of f(x) $f(-3) = 0$ and $f(\frac{7}{2}) = 0$, from factor theorem.

Now

$f(-3) = 0$

or, $9a + 3b - 21 = 0$

or, $3a + b = 7$ .... (i)

Again

$f(\frac{7}{2}) = 0$

or, $(\frac{7}{2})^2 - b(\frac{7}{2}) - 21 = 0$

or, $\frac{49a}{4} - \frac{7b}{2} = 21$

or, $49a - 14b = 84$ .... (ii)

Solving equations (i) and (ii) we get,

a = 2 and b = 1

14. (a) $2x^3 + ax^2 + bx - 2$ has a factor $(x + 2)$ and leaves remainder 7 when divided by $(2x - 3)$

Solution

Let $f(x) = 2x^3 + ax^2 + bx - 2$

divider $d(x) = (2x - 3)$ with

Remainder $R = 7$

and f(x) have factor $(x + 2)$

Now

From factor theorem,

$f(-2) = 0$

or, $2(-2)^3 + a(-2)^2 + b(-2) - 2 = 0$

or, $-16 + 4a - 2b - 2 = 0$

or $2a - b = 9$ .... (i)

By remainder theorem,

$R = f(\frac{3}{2})$

or, $7 = 2(\frac{3}{2})^3 + a(\frac{3}{2})^2 + b(\frac{3}{2}) - 2$

or, $9 = \frac{27}{4} + \frac{9a}{4} + \frac{3b}{2}$

or, $36 = 27 + 9a + 6b$

or, $9a + 6b = 9$

$3a + 2b = 3$ .... (ii)

Solving equation (i) and (ii) we get

$a = 3$ and $b = -3$

(b) $2x^2 + ax^2 - 11x + b$ leaves a remainder 0 and 42 when divided by $(x - 2)$ and $(x - 3)$ respectively.

Solution

Let $f(x) = 2x^2 + ax^2 - 11x + b$

Remainder 0 when divided by $(x - 2)$

By remainder theorem,

$R = f(2)$

or, $0 = 2(2)^2 + a(2)^2 - 11(2) + b$

or, $0 = 16 + 4a - 22 + b$

or, $4a + b = 6$ .... (i)

Again,

Remainder 42 when divided by $(x - 3)$

By remainder theorem,

$R = f(3)$

or, $42 = 2(3)^2 + a(3)^2 - 11(3) + b$

or, $42 = 54 + 9a - 33 + b$

or, $9a + b = 21$ .... (ii)

Solving equation (i) and (ii)

$$\begin{array}{r} 9a + b = 21 \\ \underline{\underset{-}{}4a \underset{-}{+} b = \underset{-}{}6} \\ 5a = 15 \end{array}$$

$\therefore$ a = 3

Substituting value of a is (i)

b = 6 – 4a

= 6 – 4.3

= –6

$\therefore$ The value of a is 3 and b is –6.

Questions for practice

1. Use remainder theorem to find the remainder R when f(x) is divided by d(x) in the following cases.

(a) $f(x) = x^3 + 5x^2 - 22x + 12$, $d(x) = x - 2$

(b) $f(x) = 2x^3 + 4x^2 - 10x + 40$, $d(x) = 2x - 1$

(c) $f(x) = 16x^3 + 12x^2 + 24x + 10$, $d(x) = 2x + 3$

2. If $x^3 - px^2 + 20x + 12$ and $2x^3 - 2px^2 + 7px + 25$ are divided by $(x - 1)$, each gives equal remainder, find the value of p.

3. Find the val of p of polynomial $2x^2p - 5px + 3$ is exactly divisible by $x - 1$.

4. Use factor theorem to factories the following

(a) $x^3 - 13x - 12$

(b) $x^3 - 9x^2 + 24x - 20$

(c) $2x^3 - 4x^2 - 7x + 14$

(d) $2x^3 - 5x^2 - 6x + 9$

(e) $2x^3 - 3x^2 - 18x - 8$

5. Solve the following polynomial equations:

(a) $2x^3 - 3x^2 - 11x + 6 = 0$

(b) $6x^3 - 13x^2 + x + 2 = 0$

(c) $6x^3 - 7x^2 - 7x + 6 = 0$

(d) $3x^3 - 14x^2 - 7x + 10 = 0$

(e) $x^3 - 7x^2 + 36 = 0$

(f) $3x^3 - 7x^2 + 4 = 0$

(g) $y = x^3 - 4x^2 + x + 8$, when $y = 2$

(h) $y = 7$, $y = x^3 + 7x^2 - 21x - 20$

6. (a) When the polynomial $x^3 - ax^2 - 2a + 6$ is divided by $(x - a)$, remainder is $(a + 2)$, find the value of a.

(b) When both polynomial $ax^3 - 3x^2 + 9$ and $x^2 - ax + 4$ are divided by $(x + 1)$ leave the same remainder, find the value of a.

UNIT three

Sequence and Series

Arithmetic sequence and series

Estimated teaching periods : 5

S.N.	Level	Objectives
(i)	Knowledge (K)	to define sequence and series. to define A.S. and t_n of A.S. to define A.M.'s. to tell formula to find A.M.'s between two term. to tell formula to insert n A.M.'s between two term. to tell formula to find sum of n term of A.P.
(ii)	Understanding (U)	to find t_n term of an A.P. to find an A.M. between two terms. to find sum of n terms of an A.P.
(iii)	Application (A)	to find a, d of an A.P. when two terms are given. to find n A.M.'s between two terms.
(iv)	Higher Ability (HA)	to find number of means when two extreme term are given. to find number of terms when sum of an A.P. is given. to find a, d when sum of n terms and one of t_n of A.P. are given. to apply A.P. is practical life.

2. Teaching materials

List of formulas of i) A.P. and ii) G. P.

3. Teaching Learning strategies.

- Review definition of sequence and series.
- Ask to the students the following questions.

(i) Define natural numbers?

(ii) Is there any rule in sequence of natural numbers?

(iii) What is the difference between any two consecutive natural numbers?

- From above discussion, define an A.P., define common difference, the first term, n^{th} term of an A.P.
- Derive formula $t_n = a + (n - 1)d$ and state the meaning of the symbols used.
- define A.M. with an example.
- Derive the formula, A.M. $= \dfrac{a+b}{2}$

$$d = \frac{b-a}{n+1},$$

$$m_1 = a + d$$

$$m_2 = a + 2d$$

- Solve some problem from text book for examples.
- Discuss how to find the sum of n term of an A.P.? Derive the formula

$$s_n = \frac{n}{2}[2a + (n-1)d]$$

$$s_n = \frac{n}{2}[a + l]$$

State the meaning of the symbols used in above formulas.

- Discuss application of A.P. in our daily life.
- Solve some problem from text book.

Note:

(i) The n^{th} term/general term of an A.P., $t_n = a + (n-1)d$

(ii) The successive terms of an A.P. are a, a + d, a + 2d, a + 3d.

(iii) If t_n denotes the last term $t_n = l = a + (n-1)d$.

(iv) Common difference of A.P., $d = t_{k+1} - t_k$, $k \geq 1$

(v) The number of term should be the whole number not negative and fraction.

Some solved problems

1. Which of the following sequences are in A.P. Also write common difference of A.P.?

(a)2, 7, 12, 17, ...

Solution

Here, 2, 7, 12, 17, ...

$t_1 = 2, t_2 = 7, t_3 = 12, t_4 = 17$

$t_1 - t_2 = 7 - 2 = 5, t_3 - t_2 = 12 - 7 = 5, t_4 - t_3 = 17 - 12 = 5$

Since the difference of two successive terms in constant,

the given sequence is an arithmetic sequence.

$\therefore$ common difference (d) = 5

(b)12, 8, 6, 4, ...

Solution

Here, $t_1 = 12, t_2 = 8, t_3 = 6, t_4$

$t_2 - t_1 = 8 - 12 = -4, t_3 - t_2 = 6 - 8 = -2$

$t_4 - t_3 = 4 - 6 = -2$

Hence the difference of two successive terms is not constant throughout the whole sequence and the given sequence is not an A.P.

2. Find the general term of an A.P. 51, 45, 39, ...

Solution

Given A.P. is 51, 45, 39, ...

First term (a) = 51

Common difference = 45 – 51 = –6

Let t_n be the general term then $t_n = a + (n-1)d$

$= 51 + (n - 1).(-6)$

$= 51 - 6n + 6$

$= 57 - 6n$

$\therefore t_n = 57 - 6n$

3. Compute the first term and 10th term of the given sequence: $d = 3, t_4 = 12$

Solution

Here, is given A.P.

$d = 3, t_4 = 12$

we have, $t_4 = a + (n - 1)d$

or, $t_4 = a + (4 - 1)3$

or, $12 = a + 9$

$\therefore a = 3$

and $t_{10} = a + 9d = 3 + 9\times3 = 30$

4. Determine the number of terms in each of the following A.P.

(a) $a = 2$, common difference $= 4$, $l = 102$

Solution

Here, in given A.P. the first term (a) = 2

Common difference (d) = 4

Last term (l) = 102

We have, $l = t_n = a + (n - 1)d$

or, $102 = 2 + (n - 1).4$

or, $100 = (n - 1).4$

or, $n - 1 = 25$

$\therefore n = 26$

(b) First term (a) = 3, common difference (d) = 6, $l = 54$

Solution

Here, In given A.P., $a = 3, d = 6, l = 57$

We have, $t_n = l = a + (n - 1)d$

or, $57 = 3 + (n - 1).6$

or, $54 = (n - 1).6$

or, $n - 1 = 9$

$\therefore n = 10$

5. Which term of the series $2 + 4 + 6 + \ldots\ldots\ldots$ is 250 ?

Solution

Here, $2 + 4 + 6 + \ldots\ldots\ldots\ldots\ldots$

This is an A.P., $a = 2, d = 4 - 2 = 2$

Let $t_n = 256$, then we find the value of n.

Now, $t_n = a + (n - 1)d$

or, $256 = 2 + (n -1).2$

or, $254 = (n - 1).2$

or, $n - 1 = 127$

$\therefore n = 128$

6. (a) Is 75 a term of the series 9 + 11 + 13 + ... ?

Solution

Here, 9 + 11 + 13 + ...

This is an A.P.

$a = 9, d = 11 - 9 = 2$

Let $t_n = 75$, then we find the value of n.

Now, $t_n = a + (n - 1)d$

or, $75 = 9 + (n - 1).2$

or, $66 = (n - 1).2$

or, $n - 1 = 33$

$\therefore n = 34$

Hence 75 is the 34^{th} term of the given series.

(b) Is –100 a term of the series –10, –20, –30, ...?

Solution

Here, –10, –20, –30, ...

This is an A.P.

$a = -10, d = t_2 - t_1 = -20 - (-10) = -10$

Let $t_n = 100$, then we find the value of n.

Now, $t_n = a + (n - 1)d$

or, $-100 = -10 + (n - 1).(-10)$

or, $-90 = (n - 1) (-10)$

or, $9 = n - 1$

$\therefore n = 10$

Hence –100 is the 10^{th} term of the given series.

7. Find the common difference, the first term and the 10^{th} term of arithmetic progression.

(a) $t_4 = 13, t_6 = 7$ **(b)** $t_5 = 15, t_7 = 9$

(c) $t_4 = 75, t_{10} = 117$ **(d)** $t_3 = 40, t_{13} = 0$

(e) $t_5 = 15, t_7 = 9$

(a) Solution

$t_4 = 13$

$t_6 = 7$

We know that

$t_n = a + (n - 1)d$

or, $t_4 = a + (4 - 1)d$

or, $13 = a + 3d$ .... (i)

Again

$t_6 = a + (6 - 1)d$

or, $7 = a + 5d$ .... (ii)

Subtracting (i) from (ii)

$$\begin{array}{r} \cancel{a} + 5d = 7 \\ - \cancel{a} + 3d = 13 \\ \hline 2d = -6 \end{array}$$

$\therefore d = -3$

Substituting value of d in (i)

$a = 13 - 3\times(-3)$

$= 13 + 9$

$= 22$

$\therefore$ First term = 22 and common difference = -3

Again, We have to find t_{10}

$t_{10} = a + 9d$

$= 22 + 9 \times (-3)$

$= 22 - 27$

$= -5$

(b) Solution

$t_5 = 15$

$t_7 = 9$

We know that

$t_n = a + (n - 1)d$

$\therefore t_5 = a + ad$

or, $15 = a + 4d$ .... (i)

Again

$t_7 = a + 6d$

or, $9 = a + 6d$ .... (ii)

Now Subtracting equation (i) from (ii)

$$\begin{array}{r} \cancel{a} + 6d = 9 \\ - \cancel{a} \pm 4d = -15 \\ \hline 2d = -6 \end{array}$$

or, $d = -3$

Substituting value of 'd' in equation (i)

$15 = a + 4(-3)$

or, $a = 15 + 12$

$\therefore a = 27$

$\therefore$ The first term (a) = 27 and

The common difference (d) = –3 of arithmetic progression.

Again,

$t_{10} = a + 9d$

$= 27 + 9\times(-3)$

$= 27 - 27$

$= 0$

(c) Solution:

$t_4 = 75$

$t_{10} = 117$

We know that

$t_n = a + (n - 1)d$

$\therefore t_4 = a + 3d$

or, $75 = a + 3d$ .... (i)

Also,

$t_{10} = a + 9d$

or $117 = a + 9d$ .... (ii)

Subtracting equation (i) from (ii), we get

$$\begin{array}{r} \cancel{a} + 9d = 75 \\ \underset{-}{\cancel{a}} \underset{-}{+} 3d = \underset{-}{117} \\ \hline 6d = 42 \end{array}$$

$\therefore d = 7$

Substituting value of d in equation (i)

$75 = a + 3\times7$

or, $a = 75 - 21$

$\therefore a = 54$

$\therefore$ The first term (a) = 54 and the common difference (d) = 7 of the arithmetic progression.

$t_{10} = a + 9d$

Again, $t_{10} = 54 + 7 \times 9$

$= 54 + 63$

$= 117$

(d) Solution

$t_3 = -40$

$t_{13} = 0$

We know that

$t_n = a + (n - 1)d$

$\therefore t_3 = a + 2d$

or, $-40 = a + 2d$ .... (i)

Also

$t_{13} = a + 12d$

or, $0 = a + 12d$ .... (ii)

Subtracting equation (i) from (ii)

$$\begin{array}{r} 0 = a + 12d \\ -40 = a + 2d \\ \hline 40 = 10d \end{array}$$

$\therefore d = 4$

Substituting value of d in equation (ii), we get

$0 = a + 12\times4$

$\therefore a = -48$

$\therefore$ The first term (a) = –48 and the common difference (d) = 4 of the series.

Again,

$$\begin{aligned} t_{10} &= a + 9d \\ &= -48 + 9\times4 \\ &= -48 + 36 \\ &= -12 \end{aligned}$$

(e) Solution

$t_5 = 15$

$t_7 = 9$

We know that

$t_n = a + (n - 1)d$

$\therefore t_5 = a + 4d$

or, $15 = a + 4d$ .... (i)

Also

$t_7 = a + 6d$

or, $9 = a + 6d$ .... (ii)

Subtracting equation (i) from (ii), we get

$$\begin{array}{r} 9 = a + 6d \\ 15 = a + 4d \\ \hline -6 = 2d \end{array}$$

$\therefore d = -3$

Substituting value is equation (i)

$a = 15 + 12$

$\therefore a = 27$

$\therefore$ first term (a) = 27

$\therefore$ common difference (d) = –3

Again,

$t_{10} = a + 9d$

$= 27 + 9\times(-3)$

$= 27 - 27$

$= 0$

8. (a) In an A.P. 3rd term and 13th term are 40 and 0 respectively, which term will be 28?

Solution

$t_3 = 40$

$t_{13} = 0$

Let 28 be n^{th} term i.e. $t_n = 28$.

Now

We know that

$t_n = a + (n-1)d$

$\therefore t_3 = a + 2d$

or, $40 = a + 2d$ .... (i)

Also

$t_{13} = a + 12d$

or, $0 = a + 12d$ .... (ii)

Solving equation (i) and (ii), we get

$a = 48$ and $d = -4$

Now

$t_n = a + (n-1)d$

or, $28 = 48 + (n-1)(-4)$

or, $-20 = (n-1)(-4)$

or, $5 = n - 1$

$\therefore n = 6$

$\therefore 6^{th}$ term will be 28.

(b) 6th and 9th term of an A.P. are 23 and 35 respectively, which term is 67?

Solution

$t_6 = 23$

$t_9 = 35$

Let 67 be n^{th} term i.e. $t_n = 67$

Now,

We know that

$t_n = a + (n-1)d$

$\therefore t6 = a + 5d$

or, $23 = a + 5d$ .... (i)

Also,

$t_9 = a + 8d$

$35 = a + 8d$ .... (ii)

Subtracting equation (i) from (ii), we get

$$\begin{array}{r} 35 = \cancel{a} + 8d \\ \underline{-23 = -\cancel{a} - 5d} \\ 12 = 3d \end{array}$$

$\therefore d = 4$

Substituting value of d in equation (i)

$a = 23 - 5\times4 = 3$

Now,

$t_n = a + (n-1)d$

or, $67 = 3 + (n-1)4$

or, $64 = (n-1)4$

or, $16 = n - 1$

$\therefore n = 17$

$\therefore$ 67 is the 17^{th} term.

9. (a) If the n^{th} term of the A.S. 7, 12, 17, 22, ... is equal to the n^{th} term of another A.S. 27, 30, 33, 36, ..., find the value of n.

Solution

In the A.P., 7, 12, 17, 22, ...

$a = 7$

$d = 12 - 7 = 5$

$\therefore t_n = a + (n-1)d$

or, $t_n = a + (n-1)5$

or, $t_n = 7 + 5n - 5$

$\therefore t_n = 2 + 5n$ .... (i)

Again

In the AP 27, 30, 33, 36, ...

$a = 27$

$d = 3$

$\therefore t'_n = a + (n-1)d$

or, $t'_n = 27 + (n-1)3$

or, $t'_n = 27 + 3n - 3$

$\therefore t'_n = 24 + 3n$ .... (ii)

According to question

$t_n = t'_n$

or, $2 + 5n = 24 + 3n$

or, $5n - 3n = 24 - 2$

or, $2n = 22$

$\therefore n = 11$

$\therefore$ The value of n is 11.

(b) If the n^{th} term of an A.P. 23, 26, 29, 32, ... is equal to the n^{th} term of another A.P. 59, 58, 56, ..., find the number of terms.

Solution

In the AP, 23, 26, 29, 32, ...

$a = 23$

$d = 26 - 23 = 3$

We know that

$t_n = a + (n - 1)d$

or, $t_n = 23 + (n - 1)3$

or, $t_n = 23 + 3n - 3$

$\therefore t_n = 20 + 3n$ .... (i)

Again

In the AP, 59, 58, 56, ...

$a = 59$

$d = 58 - 59 = -1$

We know that

$t'_n = a + (n - 1)d$

or, $t'_n = 59 + (n - 1)(-1)$

or, $t'_n = 59 - n + 1$

$\therefore t'_n = 60 - n$ .... (ii)

According to question.

$t_n = t'_n$

or, $20 + 3n = 60 - n$

or, $3n + n = 60 - 20$

or, $4n = 40$

$\therefore n = 10$

$\therefore$ The value of n is 10.

10. (a) In an A.P. the 7^{th} term is four times the second term and the 10^{th} term is 29, find the progression.

Solution

: In given question of A.P.

$t_7 = 4t_2$

and $t_{10} = 29$

By using formula,

$t_n = a + (n - 1)d$

Now, $t_7 = 4\,t_2$

or, $a + 6d = 4(a + d)$

or, $a + 6d = 4a + 4d$

or, $3a - 2d = 0$ .... (i)

Again, $t_{10} = 29$

or, $a + 9d = 29$ .... (ii)

Multiplying equation (i) by 3 and subtracting (ii) from (i), we get

$$\begin{array}{r} \cancel{3a} - 2d = 0 \\ \underset{-}{}\cancel{3a} \underset{-}{+} 27d \underset{-}{=} 87 \\ \hline -29d = -87 \end{array}$$

$\therefore d = 3$

Put the value of d in (i), we get,

$3a - 2.3 = 0$

$\therefore a = 2$

Now,

$t_1 = 2,$

$t_2 = a + d = 2 + 3 = 5$

$t_3 = a + 2d = 2 + 2\times3 = 8$

$t_4 = a + 3d = 2 + 3\times3 = 11$

Hence the required progression is 2, 5, 8, 11, ...

(b) In an A.P. the 10 times of 10th term is 15 times the 15th term. If the first term is 48, find the progression.

Solution

First term (a) = 48

According to question

$10\, t_{10} = 15\, t_{15}$

We know that

$t_n = a + (n - 1)d$

$\therefore 10(a + 9d) = 15(a + 14d)$

or, $10(48 + 9d) = 15(48 + 14d)$

or, $96 + 18d = 144 + 42d$

or, $24d = -48$

$\therefore d = -2$

Now,

$\therefore t_2 = a + d = 48 + (-2) = 46$

$\therefore t_3 = a + 2d = 48 + 2(-2) = 44$

$\therefore t_4 = a + 3d = 48 + 3(-2) = 42$

$\therefore t_5 = a + 4d = 48 + 4(-2) = 40$

$\therefore$ The progression is 48, 46, 44, 42, 40,...............

11. (a) If x + 6, 3x and 2x + 9 are the first three terms in A.P., find the value of x and next three terms of the progression.

Solution

x + 6, 3x and 2x + 9 are in AP.

Now,

Common difference (d) $= 3x - (x + 6)$

Also, $d = 2x + 9 - 3x$

$\therefore 3x - x - 6 = 2x + 9 - 3x$

or, $2x - 6 = -x + 9$

or, $3x = 15$

$\therefore x = 5$

$\therefore x + 6 = 5 + 6 = 11$

$\therefore 3x = 3\times5 = 15$

$\therefore 2x + 9 = 2\times5 + 9 = 19$

Now,

$d = 15 - 11 = 4$

$a = 11$

We know,

$t_n = a + (n-1)d$

$\therefore t_4 = a + 3d = 11 + 3\times4 = 23$

$\therefore t_5 = a + 4d = 11 + 4\times4 = 27$

$\therefore t_6 = a + 5d = 11 + 5\times4 = 31$

$\therefore$ The next three terms of progression are 23, 27 and 31.

(b) If 4k + 10, 6k + 2 and 10k are the first three terms in A.P., find the value of k and terms t_7, t_8, t_9.

Solution

$4k + 10$, $6k + 2$ and $10k$ are in A.P.

$\therefore$ Common difference $(d_1) = 6k + 2 - (4k + 10)$

$= 2k - 2$

Also,

Common difference $(d_2) = 10k - (6k + 2)$

$= 4k - 2$

Since they are in AP

$d_1 = d_2$

or, $2k - 8 = 4k - 2$

or, $2 - 8 = 4k - 2k$

or, $2k = -6$

$\therefore k = -3$

$\therefore 4k + 10 = 4(-3) + 10 = -2$

$\therefore 6k + 2 = 6(-3) + 2 = -16$

$\therefore 10k = 10(-3) = -30$

Here,

$a = -2$

$d = -16 + 2 = -14$

$\therefore t_4 = a + (4-1)d = -2 + 3(-14) = -44$

$\therefore t_8 = a + 7d = -2 + 7(-14) = -100$

$\therefore t_9 = a + 8d = -2 + 8(-14) = -114$

$\therefore$ The value of k is -3 and $t_4 = -44$, $t_8 = -100$ and $t_9 = -114$.

12. (a) If the pth term of an A.S. is q and the qth term is p. Show that mth term is p + q – m.

Solution

$t_p = q$ To show: $t_m = p + q - m$

$t_q = p$

We know that

$t_n = a + (n-1)d$

$\therefore t_p = a + (p-1)d$

or, $q = a + (p-1)d$ .... (i)

Also

$t_q = a + (q-1)d$

or, $p = a + (q-1)d$ .... (ii)

Subtracting (i) from (ii),

$$\begin{array}{rcl} p &=& \cancel{a} + (q-1)d \\ \underset{-}{}q &=& \underset{-}{\cancel{a}} \underset{-}{+} (p-1)d \\ \hline \end{array}$$

$p-q = (q-1)d - (p-1)d$

or, $p - q = (q - 1 - p + 1)d$

or, $p - q = (q - p)d$

$\therefore d = -1$

Substituting value of d in (i)

$q = a + (p-1)(-d)$

or, $q = a - p + 1$

$\therefore a = p + q + 1$

Now

$t_m = a + (m-1)d$

or, $t_m = p + q - 1 + (m-1)(-1)$

or, $t_m = p + q - 1 - m + 1$

$\therefore t_m = p + q - m$

Hence proved

(b) If m times the mth term of an A.P. is equal to n times of nth term, then show that (m + n)th term of the A.P. is zero.

Solution

According to question

$m.t_m = n.t_n$

To show : $t(m + n) = 0$

Now,

$m.t_m = n.t_n$

We know that

$t_n = a + (n - 1)d$

$\therefore m\{a + (m - 1)s\} = n\{a + (n - 1)s\}$

or, $ma + m^2d - dm = na + n^2d - dn$

or, $(m - n)a + (m^2 - n^2)d - (m - n)d = 0$

or, $(m - n)a + (m + n)(m - n)d - (m - n)d = 0$

or, $a + (m + n)d - d = 0$

or, $a = d(1 - m - n)$

Now,

$t_{(m+n)} = a + (m + n - 1)d$

or, $t_{(m+n)} = d(1 - m - n) + (m + n - 1)d$

or, $t_{(m+n)} = \cancel{d} - m\cancel{d} - n\cancel{d} + m\cancel{d} + n\cancel{d} - \cancel{d}$

or, $t_{(m+n)} = 0$

Hence $(m + n)^{th}$ term of AP is zero,

Hence proved.

13. If the p^{th}, q^{th} and r^{th} terms of an A.P. are respectively a, b and c then prove that $p(b - c) + q(c - a) + r(a - b) = 0$

Solution

$t_p = a$ .... (i)

$t_q = b$ .... (ii)

$t_r = c$ .... (iii)

Now,

Subtracting (iii) from (ii),

$$\begin{aligned} b - c &= t_q - t_R \\ &= t_1 + (q - 1)d - \{t_1 + (r - 1)d\} \\ &= qd - d - rd + d \\ &= qd - rd \end{aligned}$$

Again

Subtracting (i) from (iii)

$$\begin{aligned} c - a &= t_r - t_p \\ &= t_1 + (r - 1)d - \{t_1 + (p - 1)d\} \\ &= rd - d - pd + d \\ &= rd - pd \end{aligned}$$

Again

Subtracting (ii) from (i)

$$\begin{aligned} a - b &= t_p - t_q \\ &= t_1 + (p - 1)d - \{t_1 + (q - 1)d\} \\ &= pd - d - qd + d \\ &= pd - qd \end{aligned}$$

Now,

To prove :

$p(b-c) + q(c-a) + r(a-b) = 0$

LHS

$\quad p(b-c) + q(c-a) + r(a-b)$

$= p(qd - rd) + q(rd - pd) + r(pd - qd)$

$= pqd - prd + qrd - pqd + rpd - qrd$

= - = RHS, Hence proved

Questions for practice

1. Find the 15^{th} term of the sequence 3, 6, 9, 12, ...
2. How many terms are there in the series 5 + 9 + 13 + ... + 77? Find it.
3. If the n^{th} term of series 84 + 78 + 72 + ... is 0, find the value of n.
4. If 6, p, q and 18 are in an arithmetic sequence, find the value of p and q.
5. Find the sum of the arithmetic series $\sum_{n=1}^{5}(4n+5)$.
6. If the 5^{th} and 12^{th} terms of an arithmetic progression are respectively 15 and 29, find the 30^{th} term.
7. If the 5^{th} term of an A.P. is $\frac{1}{8}$ and 8^{th} term is $\frac{1}{5}$, find the 40^{th} term of the A.P.
8. The n^{th} terms of two A.P. is –19 – 12 – 5 + 2 + 9 + ... and 1 + 6 + 11+ 16 + ... are equal, then find the value of n.
9. The n^{th} terms of two A.P.'s 61, 63, 65, ... and –4, 3, 10, 17, ... are equal, find the value of n.
10. In an arithmetic progression, the 5^{th} term is double of the third term. prove that 13^{th} term is double of the 7^{th} term.

Arithmetic Means (A.M.S)

Notes

i) If A.M. is the arithmetics mean between a and b, then A.M. $= \frac{a+b}{2}$

ii) Let $m_1, m_2, m_3, \ldots\ldots m_n$ be the n arithmetic means inserted between two numbers a and b, then common difference is $d = \frac{b-a}{n+1}$ and n arithmetic means are given by,

$m_1 = a+d,\ m_2 = a+2d, \ldots\ldots\ldots\ m_n = a+nd$

Some solved problems

1. Find an arithmetic mean between p and q

Solution

Arithmetic Mean between p and q is A.M. $= \frac{p+q}{2}$

2. If the common difference =d, first term=a, last term = b of an AP, find the three means between them in terms of a and b.

Solution

In given A.P. , first term = a

last term= b

number of means (n) =3

common difference = d = $\frac{b-a}{n+1} = \frac{b-a}{3+1} = \frac{b-a}{4}$

First mean(m_1)= a+d = a + $\frac{b-a}{4} = \frac{3a+b}{4}$

Second mean(m_2)=a + 2d = a+2.$\frac{b-a}{4}$

$= \frac{2a+b-a}{2} = \frac{a+b}{2}$

third mean (m_3)=a+3 $\frac{b-a}{4} = \frac{a+3b}{4}$

3. If A.M. between 2k+3 and 5k+8 is 6, find the value of k.

Solution

Here, a=2k+3, b=5k+8 A.M.=6

Now, using formula, A.M.= $\frac{a+b}{2}$

or, 6=$\frac{2k+3+5k+8}{2}$

or 12=7k+11

k=$\frac{1}{7}$

4. Find 5 A.M.'s between 2 and 20.

Solution

First term (a) = 2, last term (b) =20

Number of means (n)=5

Common difference (d)=$\frac{b-a}{n+1}$

$=\frac{20-2}{5+1}$

=3

Let the required 5 means be m_1, m_2, m_3, m_4 and m_5

Then, m_1=a+d=2+3=5

m_2=a+2d=2+2×3=8

m_3=a+3d =2+3×3=11

m_4 =a+4d =2+4×3=14

m_5= a+5d = 2 +5×3 =17

5(a) If 96, p, q, r, 68 are in A.P., find the values of p,q and r.

Solution

Here, 96,p,q,r,68 are in A.P.

First term (a)=96

Last term (b)=68

Number of means (n)=3

common difference (d)=$\frac{b-a}{n+1}$

$=\frac{68-96}{3+1}$

$= -7$

First mean $(m_1)=p=a+d=96-7=89$

Second mean $(m_2)=q=a+2d=96+2\times(-7)=82$

Third mean $(m_3)=r=a+3d=96+3\times(-7)=75$

(b) If x, x^2+1 and x+6 are in A.P., find the value of x.

Solution: Here, a=x, b=x+6, x^2+1 as A.M. between x and x+6.

A.M. $=\frac{a+b}{2}$

or, $x^2+1=\frac{x+x+6}{2}$

or, $2x^2+2=2x+6$

or, $x^2-x-2=0$

or, $x^2-2x+x-2=0$

or, $x(x-2)+1(x-2)=0$

or, $(x-2)(x+1)=0$

$x=-1,2$

6 (a) 6 arithmetic means between 4 and p are inserted and the fifth mean is 14, find the value of p. Also find the means.

Solution

first term(a)=4

last term (b) = p

fifth mean $(m_5)=14$

no. of means (n)=6

Now,

common difference (d)= $\frac{b-a}{n+1}=\frac{p-4}{6+1}=\frac{(p-4)}{7}$

Now We know

$m_5=a+5d$

or, $14 = 4+\frac{5(p-4)}{7}$

or, $10 = \frac{5(p-4)}{7}$

or, $14=p-4$

$p=18$

$d = \frac{p-4}{7} = \frac{18-4}{7} = 2$

Now,

$m_1=a+d = 4+2 = 6$

$m_2=a+2d = 4+2\times2 = 8$

$m_3=a+3d = 4+3\times2 = 10$

$m_4 =a+4d = 4+4\times2 = 12$

$m_5= a+5d = 4 +5\times2 = 14$

$m_6 = a+6d = 4 +6\times2 = 16$

6(b). There are three A.M's between a and b, if the first mean and the third mean are respectively 10 and 20, find the values of a and b.

Solution

First term = a

Last term = b

Number of means (n)=3

common difference $= \frac{b-a}{n+1} = \frac{b-a}{3+1} = \frac{b-a}{4}$

First mean $(m_1)=a+d$

or, $10 = a+\frac{b-a}{4}$

or, $40 = 3a+b$

$3a+b=40$...............(i)

Also, third mean $m_3 = a+3d$

or, $20 = a +3\frac{b-a}{4}$

or, $80 = a+ 3b$

$a+3b = 80$..............(ii)

From equation (i), $b= 40-3a$

Put the value of 'b' in equation (ii), we get,

$a + 3(40-3a) =80$

or, $a +120-9a=80$

or, $-8a = -40$

$a=5$

put the value of 'a' in equation (i), we get,

$b=40-3.5=25$

Alternate method :

$m_1 = a+d$ or, $10=a+d$...........(i)

and $m_3 = a+3d$

$a + 3d =20$.............(ii)

solving eqn (i) and (ii), we get , $a=5$, $d=5$, $b=25$

7(a). The 5th mean between two numbers 7 and 71 is 27 . Find the number of means.

Solution

first term (a) = 7

last term (b) =71

fifth mean (m_5) = 27

number of means (n) =?

we known,

common difference (d) $=\dfrac{b-a}{n+1} = \dfrac{71-7}{n+1} = \dfrac{64}{n+1}$

Also,

$m_5 = a + 5d$

or, $27 = 7 + 5\times\dfrac{64}{n+1}$

or, $27 = 7 + \dfrac{320}{n+1}$

or, $27n + 27 = 7n+7+320$

or, $20n = 300$

$n = 15$

The numbers of means is 15.

7. (b) There are n arithmetic means between 5 and 35. If the second mean to the last mean is 1:4, find n.

Solution

First term (a) = 5

last term (b) =35

We know, common difference (d) $=\dfrac{b-a}{n+1} = \dfrac{35-5}{n+1} = \dfrac{30}{n+1}$

According to question

$\dfrac{m_2}{m_n} = \dfrac{1}{4}$

$\dfrac{a+2d}{a+nd} = \dfrac{1}{4}$

or, $\dfrac{5+2\times\dfrac{30}{n+1}}{5+n\times\dfrac{30}{n+1}} = \dfrac{1}{4}$

or, $\dfrac{\dfrac{5n+5+60}{n+1}}{\dfrac{5n+5+30n}{n+1}} = \dfrac{1}{4}$

or, $\dfrac{5n+65}{35n+5} = \dfrac{1}{4}$

or, $20n+260=35n+5$

or, $35n-20n=260-5$

or, $15n=255$

$n=17$

The value of n is 17.

7. (c) There are n arithmetic means between 1 and 70, if the first mean : the last mean=4:67, find n.

Solution

First term (a) = 1

last term (b) =70

we known,

common difference (d) $=\frac{b-a}{n+1}=\frac{70-1}{n+1}=\frac{69}{n+1}$

According to question

$$\frac{\text{first mean}(m_1)}{\text{last mean}(m_n)}=\frac{4}{67}$$

or, $\frac{a+d}{a+nd}=\frac{4}{67}$

or, $\frac{1+\frac{69}{n+1}}{1+n\times\frac{69}{n+1}}=\frac{4}{67}$

or, $\frac{n+1+69}{n+1+69n}=\frac{4}{67}$

or, $67n+4690=280n+4$

or, $4690-4=280n-67n$

or, $4686=213n$

$n=22$

The value of n is 22.

7.(d) Find the number of arithmetic means between 2 and 37 where the second mean:last mean=3:8

Solution

Let number of arithmetic means = n

First term (a) = 2

last term (b) =37

we know,

common difference (d) $=\frac{b-a}{n+1}=\frac{37-2}{n+1}=\frac{35}{n+1}$

According to question

$$\frac{\text{second mean}(m_2)}{\text{last mean}(m_n)}=\frac{3}{8}$$

$\frac{a+2d}{a+nd}=\frac{3}{8}$

or, $\dfrac{2+2\times\dfrac{35}{n+1}}{2+n\times\dfrac{35}{n+1}}=\dfrac{3}{8}$

or, $\dfrac{2n+2+70}{2n+2+35n}=\dfrac{3}{8}$

or, $\dfrac{2n+72}{37n+2}=\dfrac{3}{8}$

or, $16n+576=111n+6$

or, $570=95n$

$n=6$

The number of arithmetic means is 6.

Questions for practice

1. If 5 is the arithmetic means of x and 8, find the value of the geometric mean.
2. If 6, x,y and 8 are in arithmetic sequence, find the values of x and y.
3. Insert 4 A.M. 's between 5 and 25.
4. Insert 6 A.M.'s between –3 and 32.
5. There are n arithmetic means between 4 and 24. If the ratio of third mean to the last mean is 4:5, find the number of means.
6. There are n A.M.'s between 3 and 39. Find the value of n so that third mean : last mean =3:7.

Sum of Arithmetic Series

Notes:

i) When the first term(a) the number of terms(n) and the common difference(d) are known, then the sum of the first n terms of an A.P. is given by

$S_n=\frac{n}{2}[2a+(n-1)d]$

ii) When the first term(a), the number of terms(n) and the last term (l) are known the sum of the first n terms is given by

$S_n=\frac{n}{2}[a+1]$

iii) Three consecutive numbers in an A.P. are taken as a–d, a, a+d, we assume that middle term is 'a' and common difference 'd' .

iv) Denote the four consecutive terms of an A.P. as a–3d, a–d, a+d, a+3d.

v) When calculating for the value of number of terms n, accept the positive integral value of n, reject the fractional and negative values of n. If the two values of n are positive integer then check the sum, accept the correct value of n whose sum of n term's is true.

Some solved problems

1. Sum to 8 terms of the A.P. –5–2+1+……….

Solution

In given A.P., $a = -5$,

common difference, $d = -2+5 = 3$

number of terms $(n) = 8$

Now, $S_n = \frac{n}{2}[2a+(n-1)d]$

$= \frac{n}{2}[2.(-5)+(8-1).3]$

$= 4[-10+21]$

$= 44$

2. Evaluate $\sum\limits_{n=2}^{8}(2n-1)$

Solution

Here, $\sum\limits_{n=2}^{8}(2n-1)$

$= (2.2-1)+(2.3-1)+(2.4-1)+(2.5-1)+(2.6-1)+(2.7-1)+(2.8-1)$

$= 3+5+7+9+11+13+15 = 63$

Alternatively, $n=7$, $a=3$, $d=2$

Now, $S_n = \frac{n}{2}[2a+(n-1)d]$

$= \frac{7}{2}[2.3+(7-1)2]$

$= \frac{7}{2}.2[3+6]$

$= 63$

3. An arithmetic series has 10 terms, if its last term is 50 and the sum of its terms is 275, find the first term.

Solution

number of terms $(n) = 10$

last term $(l) = 50$

sum of terms $(S_{10}) = 275$

first term $(a) = ?$

We know that,

$S_n = \frac{n}{2}(a+l)$

$S_{10} = \frac{10}{2}(a+50)$

or, $275 = 5(a+50)$

or, $55 = a + 50$

$\therefore a = 5$

4(a). Find the number of terms in an A.P. which has its first term 16, common difference 4 and the sum 120.

Solution

Here, the first term (a) = 16

common difference (d) = 4

sum of n terms (S_n) = 120

Now,

$S_n = \frac{n}{2}[2a+(n-1)d]$

or, $120 = \frac{n}{2}[2.16+(n-1)4]$

or, $240 = 4n[8+n-1]$

or, $60 = n^2 + 7n$

or, $n^2 + 7n - 60 = 0$

or, $n^2 + 12n - 5n - 60 = 0$

or, $(n+12)(n-5)=0$

Either $n=-12$ or $n = 5$

Since the number of terms cannot be negative, hence the required number of term $n = 5$.

(b) Find the number of terms in an A.P. with the first term 2, common difference 2 and sum 420.

Solution

Here, $a = 2$, $d= 2$, $S_n=420$

By using formula,

Now,

$S_n = \frac{n}{2}[2a+(n-1)d]$

or, $420 = \frac{n}{2}[2.2+(n-1)2]$

or, $420 = 2n + n^2 - n$

or, $n^2+n-420=0$

or, $n^2+21n-20n-420 = 0$

or, $n(n-20)-20(n+21) = 0$

Rejecting the negative value of n, we get

$n=20$

(c) Find the number of terms of the series 2+4+6+8...+... In order that the sum may be 240. Explain the double answer.

Solution

first term (a)=2.

common difference (d)=4–2=2.

Let number of terms=n.

Sum of n terms=240

Now,

We know that,

$S_n = \frac{n}{2}[2a+(n-1)d]$

$240 = \frac{n}{2}[2\times2+(n-1)2]$

$240 = n[2+n-1]$

$240 = n+n^2$

$n^2+n-240 = 0$

$n^2+16n-15n-240 = 0$

$n(n+16)-15(n+16) = 0$

$(n+16)(n-15) = 0$

Either

$n+16 = 0$ ... (i)

Or

$n-15 = 0$ ... (ii)

From (i), $n = -16$

From (ii), $n = 15$

Since number of term cannot be negative. The number of terms $n = 15$.

5. If the 9th and 29th terms of an A.P. are respectively 40 and 60, find the common difference, first term and sum of the first 30 terms.

Solution

In given A.P.

$t_9 = 40, \quad t_{29} = 60,$

By using formula,

$t_n = a+(n-1)d.$

$t_9 = a+8d$

or, $40 = a+8d$ ... (i)

$t_{29} = a+28d$

or, $60 = a+28d$ ... (ii)

Solving equations (i) and (ii), we get,

$$\begin{array}{r} a+28d = 60 \\ \underset{-}{}a\underset{-}{+}8d = \underset{-}{40} \\ \hline 20d = 20 \\ d = 1 \end{array}$$

and $a = 60-28 = 32$

Again, $S_{30} = \frac{n}{2}[2a+(n-1)d]$

or, $S_{30} = \frac{30}{2}[2.32+(30-1).1]$

$= 15\times93$

$= 1395$

6. The sum of the first ten terms of an A.P. is 520. If the 7^{th} terms in double of 3^{rd} term calculate the first term, common difference and sum of 20 terms.

Solution

Sum of the first ten terms $(S_{10}) = 520$

and $t_7 = 2t_3$

Now, $S_n = \frac{n}{2}[2a+(n-1)d]$

For, $n = 10$, $S_{10} = 520$

or, $520 = \frac{10}{2}[2a+9d]$

or, $104 = 2a+9d$ ... (i)

and $t_7 = 2t_3$

or, $a+6d = 2[a+2d]$

or, $a+6d = 2a+4d$

or, $a-2d = 0$ ... (ii)

Solving equation (i) and (ii), we get,

$a = 16$, $d = 8$.

Again, $S_{20} = \frac{20}{2}[2.16+19.8] = 1840$

7. The sum of the first six terms of an A.P. is 42 and the ratio of 10^{th} and 30^{th} term is 1:3. Find the first term and thirteenth term of the A.P.

Solution

Sum of first 6 terms $(S_6) = 42$

$\frac{t_{10}}{t_{30}} = \frac{1}{3}$ ... (i)

We know,

$t_n = a+(n-1)d$

$t_{10} = a+9d$ ... (ii)

$t_{30} = a+29d$ ... (iii)

Equation (i) becomes

$\frac{a+9d}{a+29d} = \frac{1}{3}$

$3a+27d = a+29d$

$2a-2d = 0$

$a-d = 0$ ... (iv)

Also

$S_n = \frac{n}{2}[2a+(n-1)d]$

$S_6 = \frac{6}{2}[2a+5d]$

$42 = 3(2a+5d)$

$2a+5d = 14$ ... (v)

Solving eq^n (iv) and (v) we get,

$a = 2$

$d = 2.$

Now,

$t_{13} = a+12d = 2+12\times2$

$=26$

The first term is 2 and thirteenth term is 26.

8. The sum of the first ten terms of an A.P. is 50 and its 5th term is trebble the 2nd term. Find the first term, common difference and the sum of the first 20 terms.

Solution

$S_{10} = 50$

or, $\frac{10}{2}[2a+(10-1)d]=50$

or, $2a + 9d = 10$ .... (i)

Also

$t_5 = 3t_5$

or, $a+4d=3(a+2d)$

or, $a+4d = 3a + 6d$

or, $2a + 2d = 0$

$a + d = 0$–(ii)

Solving eq^n (i) and (ii) we get

$a=\frac{-10}{7}$

$d=\frac{10}{7}$

Now,

$S_{20} = \frac{20}{2}\left(2\times\left(\frac{-10}{7}\right)+19\times\frac{10}{7}\right)$

$= \frac{20}{2}\left(\frac{-20}{7}+\frac{190}{7}\right)$

$= \frac{1700}{7}$

first term is $\frac{-10}{7}$

common difference is $\frac{10}{7}$

Sum of first 20 terms is $\frac{1700}{7}$

9(a). Find three numbers in A.P. whose sum is 9 and the product is –165.

Solution

Let the three numbers be a–d, a and a+d in A.P.

Now,

According to question,

$a-d+a+a+d=9$

or, $3a=9$

$a=3$
Also,
$(a-d)(a)(a+d)=-165$
or, $(a^2-d^2)3=-165$
or, $(3^2-d^2)3=-165$
or, $9-d^2=-55$
or, $d=\pm 8$
when $n=8$
The numbers are $a-d=3-8=-5$, $a=3$ and $a+d=3+8=11$
When $n=-8$, the required numbers are
11, 3, –5.

9(b). Find the three numbers in A.P. whose sum is 15 and their product is 80. Find them.

Solution

Let the three numbers in AP be $a-d$, a and $a+d$
Now
According to question
$a-d+a+a+d=15$
or, $3a=15$
$a=5$
Also
$(a-d)(a)(a+d)=80$
or, $(5^2-d^2)5=80$
or, $25-16=d^2$
$\therefore d=\pm 3$
When $d=3$, the required numbers are $a-d=5-3=2$, $a=5$
and $a+d=5+3=8$
When $n=-3$, the required numbers are $a-d=5+3=8$
$a=5$
$a+d=5-3=2$
$\therefore$ Required numbers are 2,5,8 or 8,5,2.

9(c). Find the three numbers in A.P. Whose sum is 12 and the sum of their squares is 50.

Solution

Let the numbers in A.P. be $a-d$, a and $a+d$.
Now,
According to question
$a-d+a+a+d=12$
$3a=12$
$a=4$

Also,

or, $(a-d)^2+a^2+(a+d)^2=50$

or, $(4-d)^2+4^2+(4+d)^2=50$

or, $16-8d+d^2+4^2+ 4^2 +8d+d^2=50$

$2d^2 =50-48=2$

$d^2 =1=(\pm1)^2\ 2$

$d^2 =\pm1$

when $d= 1$, $a-d = 4-1 =3$

$a=4$

$a+d = 4+1=5$

when $d=-1$, $a-d=4-(-1) = 5$

$a = 4$

$a+d = 4+(-1) = 3$

The numbers of AP are 3, 4, 5 or 5, 4, 3.

10. A firm produced 1000 sets of radio during the first year. The total number of radio sets produced at the end of 10 year is 14500(The sequence of production in years are in A.P.)

i) Estimate by how many units of production increased in each year.

ii) Forecast based on the estimate of the annual increment in production at the end of 15^{th} year

Solution

first term (a)=1000,(production in the first year)

S10 =14500

Now

we know

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_{10} = \frac{10}{2}[2\times1000+9d]$$

$14500 =5(2000+9d)$

$2900-2000=9d$

$900 = 9d$

$d = 100$

Each year production increased by 100 units.

Again,

Level of production at 15^{th} year is :

$t_{15} = a+(n-1)d$

$= 1000+(15-1)100$

$= 1000+1400$

$=2400$

level of production of 15^{th} year is 2400.

Questions for practice

1. Find the sum of 5+11+17+......to seventh term.
2. Find the sum of first eleven terms of the A.P. 2,6,10,.....
3. The sum of the first seven terms of an arithmetic series is 14 and the sum of the first ten terms is 125, then find the fourth term of the series.
5. The last term of an arithmetic series of 20 terms is 195 and the common difference 5. Calculate the sum of the series.
6. The sum of the first ten terms of an arithmetic series is 50 and its fifth term is treble of the second term. Calculate the first term and the sum of the first thirty terms.
7. The sum of the first 9 terms of an arithmetic series is 72 and the sum of the first 17 terms is 289. Find the sum of first 25 terms.
8. In an A.P. , the sum of the first 8 terms is 520. If its seventh term is double of its third term, calculate the first term and the common difference of the series.
9. Find the sum of the all numbers from 100 to 400 which are divisible by 6.
10. How many terms of the arithmetic sequence 50,45,40,35...........must be taken so that the sum may be 270.
11. The sum of three numbers in A.P. is 12 and the sum of whose squares is 56. Find the numbers.
12. The sum of 3 terms in A.P. is 36 and their product is 1140, find the terms.

Geometric Sequence and Series

Estimated teaching periods:6

1. Teaching Objectives

S.N.	Level	Objectives
i)	Knowledge (K)	To define G.P. To define G.M. To tell formula to (i) find G.M. between two terms. ii) find n G.M.'s between two terms. To tell formula to find the sum of n terms of a G.P.
ii)	Understanding (U)	to use the following formula $t_n = ar^{n-1}$..............(i) G.M. $=\sqrt{ab}$ ....(ii) $r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.......(ii) $S_n = \frac{a(r^n-1)}{r-1}$.......(iv) $S_n = \frac{lr-a}{r-1}$.......(v)
iii)	Application (A)	To apply above formula (i) to (v) to solve verbal problems.
iv)	Higher Ability (HA)	To find a,d,n, s_n by using given condition in verbal problems. to apply G.P. in practical life (rate of growth etc.)

2. Teaching materials

Formula charts of G.P. (t_n,S_n, G.M.)

3. Teaching Learning Strategies

- Take some sequence of numbers like 10, 20, 40, 80,......
- Take ratios of any term to its preceding term .
- Ask to the student is the ratio of any two consecutive terms constant throughout the whole sequence ?
- Give concept of common ratio, define G.P., $r=\frac{t_{k+1}}{t_k}$, $k\geq 1$
- Discuss how to get general term formula with examples.($t_n = ar^{n-1}$)
- Discuss about G.M.'s between two terms and discuss the meaning and application of G.M.
- Derive formula of sum of n terms of a G.P.

$$S_n = \frac{a(r^n-1)}{r-1}, \ r>1, \ S_n = \frac{a(1-r^n)}{1-r}, \ r < 1$$

- Discuss about the application of G.P. in our daily life.

General Terms of a G.P.

Notes :

i) The successive terms of a G.P. are a, ar, ar^2,ar^3,...............

ii) The corresponding geometric series of above G.P. is $a+ar+ar^2+ar^3+$,...............

iii) The nth or general term of a G.P.$(t_n) = ar^{n-1}$

iv) Common ratio of a G.P. $r=\frac{t_{k+1}}{t_k}, k\geq 1$

ie. $\frac{t_2}{t_1} = \frac{t_3}{t_2} = r$

1. Check the following sequences are in G.P. or not .

a) 2,4,8,32,..........................

Solution:

Here $t_1=2$, $t_2=4$, $t_3=8$, $t_4=32$

$\frac{t_2}{t_1} = \frac{4}{2} = 2$, $\frac{t_3}{t_2} = \frac{8}{4} = 2$, $\frac{t_4}{,t_3} = \frac{32}{8} = 4$

Since the ratio of two consecutive terms is not constant, the sequence is not G.P.

b) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{243}$,...................

Solution

Here $t_1=1$, $t_2=\frac{1}{3}$, $t_3=\frac{1}{9}$, $t_4=\frac{1}{27}$, $t_5=\frac{1}{243}$

Now, $t_1=1$, $t_2=\frac{1}{3}$, $t_3=\frac{1}{9}$, $t_4=\frac{1}{27}$, $t_5=\frac{1}{243}$

$\frac{t_2}{t_1} = \frac{\frac{1}{3}}{1} = \frac{1}{3}$, $\frac{t_3}{t_2} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$, $\frac{t_4}{t_3} = \frac{\frac{1}{27}}{\frac{1}{9}} = \frac{1}{3}$, $\frac{t_5}{t_4} = \frac{\frac{1}{243}}{\frac{1}{27}} = \frac{1}{9}$

Since the ratio of any two successive terms is not constant throughout the whole sequence, the given sequence is not in G.P.

c) 3, 9, 27, 81,

Solution

Here $t_1=1$, $t_2=9$, $t_3=27$, $t_4=81$

$\frac{t_2}{t_1} = \frac{9}{3} = 3$, $\frac{t_3}{t_2} = \frac{27}{9} = 3$, $\frac{t_4}{,t_3} = \frac{81}{27} = 3$

Since the ratio of any two successive terms is constant throughout the whole sequence, the given sequence is G.P.

2. If $a=4$, $r=\frac{1}{2}$, find the 5^{th} and 10^{th} terms of a G.P.

Solution

In a given G.P., $a=4$, $r=\frac{1}{2}$

Now, $t_n = ar^{n-1}$

$t5 = 4\left(\frac{1}{2}\right)^{5-1} = 4\left(\frac{1}{2}\right)^4 = 4.\frac{1}{16} = \frac{1}{4}$

or, $t10 = ar^9 = 4.\left(\frac{1}{2}\right)^9 = \frac{1}{128}$

3. If 2,x,8,y are in G.P., Find the values of x and y.

Solution

Here, 2,x,8,y are in G.P., by definition of G.P.

$\frac{x}{2}=\frac{8}{x}=\frac{y}{8}$

Taking the first two ratios

$\frac{x}{2}=\frac{8}{x}$ or $x^2=16$, $x=4$(taking positive square root)

Again taking the last two ratios,

$\frac{8}{4}=\frac{y}{8}$

or $y=16$,

$\therefore$ $x=4$, $y=16$

4. Find the number of terms in 5, –16, 45,........,–10935.

Solution

Here, 5,–15, 45,.....–10935 are in G.P.

common ratio (r) = $\frac{-15}{5}=-3$, $a=5$

Let $t_n=-10935$

or, $ar^{n-1}=-10935$

or, $5(-3)^{n-1}=-10935$

or, $(-3)^{n-1}=-2187$

or, $(-3)^{n-1}=(-3)^7$

$n-1=7$

or, $n=8$

5. If 6th term and 13th terms of a G.P. are 64 and 8192, find the first term and common ratio.

Solution

Let a and r be the first term and common ratio of a G.P. $t_6=64$, $t_{13}=8192$

Then,

Let $t_n=ar^{n-1}$

$t_6=ar^{6-1}$

$t_5=ar^5=64$........(i)

$t_{13}=ar^{13-1}$

or, $ar^{12}=8092$........(ii)

Dividing equation (ii) by (i), we get,

$\frac{ar^{12}}{ar^5}=\frac{8192}{64}$

or, $r^7=128$

or. $r^7=2^7$

$r=2$

Put the value of r in equation (i), we get,

$a.2^5 = 64$ or, $a = \frac{64}{32} = 2$

$a = 2, r = 2$

6(a). If the 3rd and 6th terms of a G.P. are 36 and $\frac{243}{2}$ respectively. Find (i) the first term (ii) common ratio (iii) 10th term.

Solution

$t_3 = 36$

$t_6 = \frac{243}{2}$

we know that

$t_n = ar^{n-1}$

$t_3 = ar^2$

or, $36 = ar^2$........(i)

Aslo

$t_6 = ar^5$

or, $= \frac{243}{2} = ar^5$ ....(ii)

Dividing (ii) by (i), we get

$$\frac{\frac{243}{9}}{36} = \frac{ar^5}{ar^2}$$

or, $\frac{243}{2\times36} = r^3$

or, $\frac{27}{8} = r^3$

$$\left(\frac{3}{2}\right)^3 = r^3$$

$r = \frac{3}{2}$

Substituting value fo r in eq (i)

$$36 = a\left(\frac{3}{2}\right)^2$$

or, $a = \frac{36\times4}{9}$

$a = 16$

Also, $t_{10} = ar^{10-1} = ar^9$

$$= 16\times\left(\frac{3}{2}\right)^9$$

$$= \frac{19683}{32}$$

i) The first term of the GP is 16

ii) The common ratio of the GP is $\frac{3}{2}$

iii) The 10^{th} term of the GP is $\frac{19683}{32}$

6(b) First and second terms of a G.P. are 32 and 8 respectively, then find (i) common on ratio (ii) tenth term.

Solution

$t_1 = 32 = a$

$t_2 = 8$

common ratio $(r) = \frac{t_2}{t_1} = \frac{8}{32} = \frac{1}{4}$

We know that

$t_n = ar^{n-1}$

$t_{10} = ar^9$

$= 32\left(\frac{1}{4}\right)^9$

$= \frac{1}{8192}$

i) The common ratio is $\frac{1}{4}$

ii) The 10^{th} term is $\frac{1}{8192}$

7(a). The 6^{th} and 9^{th} terms of a G.P. are 1 and 8 respectively which term of the sequence is 64 ?

Solution

$t_6 = 1$

$t_9 = 8$

We know that

$t_n = ar^{n-1}$

$t_6 = ar^5$

or, $1 = ar^5$......(i)

Also,

$t_9 = ar^8$

or, $8 = ar^8$......(ii)

Dividing (ii) by (i) we get

$\frac{8}{1} = \frac{ar^8}{ar^5}$

or, $8 = r^3$

or, $2^3 = r^3$

$r = 2$

Substituting value of r is eqn(i)

$1 = a(2)^5$

$a=\frac{1}{32}$

Now,

Let n^{th} term be 64

$t_n=ar^{n-1}$

or, $64 =\frac{1}{32}(2)^{n-1}$

or, $2048=(2)^{n-1}$

or, $2^{11} = 2^{n-1}$

$11=n-1$

or, $n=11+1$

$n=12$

64 is the 12^{th} term of the G.P.

7(b). Which term of the geometric sequence is $\frac{1}{9}$ if 3^{rd} and 5^{th} terms are 27 and 3 respectively.

Solution

In a given G.P.

$t_3=27$

$t_5=3$

We know that

$t_n=ar^{n-1}$

$t_3 = ar^2$

or, $27=ar^2$......(i)

Also,

$t_5 = ar^4$

or, $3=ar^4$......(ii)

Dividing (ii) by (i) we get

$\frac{3}{27}=\frac{ar^4}{ar^2}$

or, $\frac{1}{9}=r^2$

or, $\left(\frac{1}{3}\right)^2=r^2$

$r=\frac{1}{3}$

Substituting value of r is eq^n(i)

$27=a\left(\frac{1}{3}\right)^2$

or, $a = 27\times9$

$a=243$

Now,

Let $\frac{1}{9}$ be the n^{th} term of the sequence.

$t_n = ar^{n-1}$

or, $\frac{1}{9} = 243\left(\frac{1}{3}\right)^{n-1}$

or, $\frac{1}{2187} = \left(\frac{1}{3}\right)^{n-1}$

or, $\left(\frac{1}{3}\right)^7 = \left(\frac{1}{3}\right)^{n-1}$

Hence $\frac{1}{9}$ is the 8^{th} term of the sequence.

8(a). Find the geometric series whose 3^{rd} and 6^{th} terms are respectively $\frac{1}{3}$ and $\frac{1}{81}$.

Solution

In a given G.P.

$t_3 = \frac{1}{3}$

or, $ar^2 = \frac{1}{3}$ .....(i)

Also,

$t_6 = \frac{1}{81}$

or, $ar^5 = \frac{1}{81}$ .....(i)

Dividing (ii) by (i) we get

$\frac{ar^5}{ar^2} = \frac{\frac{1}{81}}{\frac{1}{3}}$

$r^3 = \frac{1}{27} = \left(\frac{1}{3}\right)^3$

$r^3 = \frac{1}{3}$

Substituting value of r is eq^n(i)

$a\left(\frac{1}{3}\right)^2 = \frac{1}{3}$

or, $a = \frac{9}{3} = 3$

Now,

Let required geometric series is

$a + ar + ar^2$

or, $3 + 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2$

$= 3 + 1 + \frac{1}{3} + \ldots\ldots\ldots$

8(b). Find the geometric series whose 4^{th} and 9^{th} terms are respectively $\frac{1}{4}$ and $\frac{1}{128}$.

Solution

Let a and r be first term and common ratio of geometric series respectively.

Now,

we know that

$t_n = ar^{n-1}$

$t_4 = ar^3$

or, $\frac{1}{4} = ar^3$......(i)

Also,

$t_9 = ar^8$

or, $\frac{1}{128} = ar^8$ .... (ii)

Dividing (ii) by (i) we get

$\frac{1}{128} \times \frac{4}{1} = \frac{ar^8}{ar^3}$

or, $\frac{1}{32} = r^5$

or, $\left(\frac{1}{2}\right)^5 = r^5$

$r = \frac{1}{2}$

Substituting value of r is eq^n(i), we get

$\frac{1}{4} = a\left(\frac{1}{2}\right)^3$

or, $a = \frac{8}{4} = 2$

Now,

Let required geometric series is

$a + ar + ar^2 + \ldots\ldots\ldots\ldots\ldots\ldots$

$= 2 + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2 + \ldots\ldots\ldots\ldots\ldots\ldots$

$= 2 + 1 + \frac{1}{2} + \ldots\ldots\ldots\ldots\ldots\ldots$

9(a). In a G.P. 5th term and 8th term are respectively 256 and 32 which term is 2 ?

Solution

In a given G.P.

$t_5 = 256$

$t_8 = 32$

We know

$t_n = ar^{n-1}$

$t_5 = ar^4$

or, $256 = ar^4$......(i)

Also,

$t_8 = ar^7$

or, $32 = ar^7$......(ii)

Dividing (ii) by (i) we get

$\frac{32}{256} = \frac{ar^7}{ar^4}$

or, $\frac{1}{8} = r^3$

or, $\left(\frac{1}{2}\right)^3 = r^3$

$r = \frac{1}{2}$

Substituting value of r is eq^n(i)

$256 = a\left(\frac{1}{2}\right)^4$

or, a = 4096

Now,

Let 2 be the n^{th} term of the G.P.

$t_n = ar^{n-1}$

or, $2 = 4096\left(\frac{1}{2}\right)^{n-1}$

or, $\frac{1}{2048} = \left(\frac{1}{2}\right)^{n-1}$

or, $\left(\frac{1}{2}\right)^{11} = \left(\frac{1}{2}\right)^{n-1}$

$11 = n-1$

or, $n = 11+1$

$n = 12$

2 is the 12^{th} term of the G.P.

9(b). In a G.P. 2^{nd} and 5^{th} terms are respectively 32 and 4 which term is 1 ?

Solution

$t_2 = 32$

$t_5 = 4$

now,

$t_2 = ar$

or, $32 = ar$......(i)

Also,

$t_5 = ar^4$

or, $4 = ar^4$......(ii)

Dividing (ii) by (i) we get

$\frac{4}{32} = \frac{ar^4}{ar^4}$

or, $\frac{1}{8} = r^3$

or, $\left(\frac{1}{2}\right)^3 = r^3$

$r = \frac{1}{2}$

Substituting value of r is eq^n(i)

$32 = a\left(\frac{1}{2}\right)$

or, a = 64

Now,

Let 1 be the nth term of the G.P.

$t_n = ar^{n-1}$

or, $1 = 64\left(\frac{1}{2}\right)^{n-1}$

or, $\frac{1}{64} = \left(\frac{1}{2}\right)^{n-1}$

or, $\left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^{n-1}$

$\therefore 6 = n-1$

or, $n = 6+1$

$n = 7$

1 is the 7th term of the G.P.

10(a). The third term of a G.P. is 27 times the 6th terms and the 4th terms is 9. Find the series.

Solution

Let a and r be the first term and common ratio of GP respectively.

We know that

$t_n = ar^{n-1}$

According to question

$t_3 = 27t_6$

or, $ar^2 = 27ar^5$

or, $\frac{1}{2} = r^3$

or, $\left(\frac{1}{3}\right)^3 = r^3$

$r=\frac{1}{3}$

Also,

$t_4=9$

or, $ar^3=9$

or, $a=\frac{9}{\left(\frac{1}{2}\right)^2}$

$a=243$

Now,

The required geometric series is given by

$a+ar+ar^2+$ ....

$=243+243\times\frac{1}{3}+243\times\left(\frac{1}{2}\right)^2+$............

$=243+81+27+$..................32

10(b). In a G.P. 7th term is 16 times the third term and fifth term is $\frac{1}{16}$. Find 3rd term.

Solution

Let a and r be the first term and common ratio of GP respectively.

We know that

$t_n=ar^{n-1}$

According to question

$t_7=16t_3$

or, $ar^6=16ar^2$

or, $r^4=16$

or, $r^4=2^4$

or, $r=2$

Also,

$t_5=\frac{1}{2}$

or, $ar^4=\frac{1}{16}$

or, $a=\frac{1}{16\times2^4}$

$=\frac{1}{256}$

Now,

Third term or, $t_3=ar^2$

$=\frac{1}{256}.2^2$

$=\frac{1}{64}$

$\therefore t_3=\frac{1}{64}$

11(a). The sum of the three numbers in G.P. is 38 and their product is 1728. Find the numbers.

Solution

Let the three numbers b in GP be $\frac{a}{r}$, a, ar

According to question

$\frac{a}{r}+a+ar=38$

or $\frac{a+ar+ar^2}{r}=38$...............(i)

Also

$\frac{a}{r}.a.ar=1728$

or, $a^3=1728$

or, $a^3=12^3$

or, $a=12$

Substituting value of r is eq^n(i)

or $\frac{12(1+r+r^2)}{r}=38$

or, $6+6r+6r^2=19r$

or, $6r^2-13r+6=0$

or, $3r(2r-3)+2(2r-3)=0$

or, $(2r-3)(3r-2)=0$

Either

$2r-3=0$............(ii)

OR,

$3r-2=0$...........(iii)

From (ii) , $r=\frac{3}{2}$

From (iii),$r=\frac{2}{3}$

when $r=\frac{3}{2}$, numbers are

$\frac{a}{r}=\frac{9}{\frac{3}{2}}$ $=8$, $a=12$ and $ar=12\times\frac{3}{2}=18$

when $r=\frac{2}{3}$

$\frac{a}{r}=\frac{12}{\frac{2}{3}}$ $=18$, $a=12$ and $ar=12\times\frac{2}{3}=18$

Hence the required numbers are 8,12 and 18 or 18, 12 and 8.

11(b). Sum of the three consecutive terms in G.P. is 28 and their product is 512. Find the numbers.

Solution

Let the three consecutive terms in GP be $\frac{a}{r}$, a, ar

According to question

$\frac{a}{r}+a+ar=28$ ....(i)

Also

$\frac{a}{r}.a.ar=512$

or, $a^3=512$

or, $a^3=8^3$

or, $a=8$

Substituting value of a is eq^n(i)

$\frac{8}{r}+8+8r=28$

or $\frac{8+8r+8r^2}{r}=28$

or, $2+2r+2r^2=7r$

or, $2r^2–5r+2=0$

or, $2r^2–4r–r+2=0$

or, $2r(r–2)–1(r–2)=0$

or, $(r–2)(2r–1)=0$

or, $(2r–3)(3r–2)=0$

Either

$r–2=0$............(ii)

OR,

$2r–1=0$...........(iii)

From (ii) , $r=2$

From (ii),$r=\frac{1}{2}$

when r=2,numbers are

$\frac{a}{r}=\frac{3}{2}=4$, $a=8$ and $ar=8\times2=16$

when $r=\frac{1}{2}$

$\frac{a}{r}=\frac{8}{\frac{1}{2}}=16$, $a=8$ and $ar=\frac{8}{2}=4$

11(c). The product of three numbers in G.P. is 729 and sum of their squares is 819 ? Find the numbers

Solution

Let the three numbers is G.P. be $\frac{a}{r}$, a and ar

According to question

$\frac{a}{r}.a.ar=729$ ....(i)

$\frac{a}{r}.a.ar=512$

or, $a^3=729$

or, $a^3=9^3$

or, $a=9$

Also.

Substituting value of a is eq^n(i)

$\left(\frac{a}{r}\right)^2+a^2+(ar)^2=819$

$\left(\frac{9}{r}\right)^2+9^2+(9r)^2=819$

$\frac{81}{r^2}=81+81r^2=819$

or, $9+9r^2+9r^4=91r^2$

or, $9r^4-81r^2-r^2+9=0$

or, $9r^2(r^2-9)-1(r^2-9)=0$

or, $(r^2-9)(9r^2-1)=0$

or, $(r+9)(r-9)(9r^2-1)=0$

or, $2r^2-5r+2=0$

Either

$r+9=0$............(ii)

or, $r-9=0$...........(iii)

or, $9r^2-1=0$.........(iii)

From (i) , $r=-9$

From (ii), $r=9$

From (iii), $r^2=\left(\frac{1}{3}\right)^2$, $r=\pm\frac{1}{3}$

where $r=-9$, numbers are

$\frac{a}{r}=\frac{9}{-9}=-1$, $a=9$ and $ar=-81$

But it doesnot satisfy the second condition. So they arenot the required numbers.

Similarly when $r=9$, numbers are 1,9 and 81.

They also don't satisfy second condition.

when $r=\frac{1}{3}$

$\frac{a}{r}=\frac{9}{\frac{1}{3}}=27$, $a=9$ and $ar=9.\frac{1}{3}=3$

It satisfies both conditions. So the numbers are 27, 9 and 3.

when $r=-\frac{1}{3}$ the required numbers are,

$\frac{a}{r}=\frac{9}{\frac{-1}{3}}=-27$, $a=9$ and $ar=9.\left(\frac{-1}{3}\right)=-3$

Hence the required numbers are 27, 9,3 or –27,9,–3

Questions for practice

1. Find the 10^{th} term of a geometric sequence 2, 4, 8, 16, 32,.....................
2. If p, p–2,p+1 are in G.P., find the value of p.
3. If $\frac{1}{3}$, p, 3 and q are in G.P., find the values of p and q.
4. If x+6, x, x–3 are three terms of a G.P.,find the value of x.
5. If the first term and third term of a G.P. are 2 and 8^{-1}, find the common ratio of the progression.
6. The 6^{th} and 10^{th} terms of a G.P. are 64 and 1024 respectiely, find the 7^{th} term of the G.P.
7. If 2^{nd} term and 4^{th} term of a G.S. are $\frac{1}{4}$ and $\frac{1}{16}$, find the 7^{th} term.
8. The product of three numbers in a G.P. is 216 and their sum is 26, find the numbers.
9. The product of three numbers in G.P. is 64 and their sum is 21 , find the numbers.
10. If the 7^{th} term of a G.P. is 1 and its 16 times of 11^{th} term is also 1, find the 20^{th} term.

Geometric Means (G.M.'s)

Notes:

i) G.M. between two numbers a and b, G.M.$=\sqrt{ab}$

ii) Let $m_1, m_2, m_3, \ldots\ldots m_n$ be n G.M.'s between a and b, then common ratio $(r)=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

and geometric means are given by

$m_1=ar$

$m_2=ar^2$

.

.

.

$m_n=ar^n$

Some solved problems

1. Find the G.M. between $\frac{1}{3}$ and $\frac{1}{27}$.

Solution

Here, $a=\frac{1}{3}$,$b=\frac{1}{27}$

G.M. between a and b$=\sqrt{ab}=\sqrt{\frac{1}{3}\cdot\frac{1}{27}}=\frac{1}{9}$

2. Find the value of x,y and z if 3,x,y,z,48 are in G.P.

Solution

Here x,y,z are in G.M.'s between 3 and 48.

Numbers of G.M.'s (n)=3,a=3, b=48

Common ratio(r) $=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$=\left(\frac{48}{3}\right)^{\frac{1}{3+1}}$

$=(16)^{\frac{1}{4}}=2$

Now, first mean $(m_1)=x=ar=3.2=6$

Second mean $(m_2)=y=ar^2=3.2^2=12$

Third mean $(m_3)=z=ar^3=3.2^3=24$

3.Insert four G.M.'s between 1 and 32.

Solution: Here a=1, b=32, Common ratio(r) $=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$=\left(\frac{32}{1}\right)^{\frac{1}{4+1}}$

$=(2^5)^{\frac{1}{5}}=2$

Let required geometric means be m_1, m_2, m_3 and m_4.

$m_1=ar=1.2=2$

$m_2=ar^2=1.2^2=4$

$m_3=ar^3=1.2^3=8$

$m_4=ar^4=1.2^4=16$

4. Third geometric mean between 27 and $\frac{1}{27}$ is 1. Find the number of means.

Solution

Here, number of means(n)=?, $m^3=1$,a=27, b=$\frac{1}{27}$

common ratio (r)$=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$=\left(\frac{\frac{1}{27}}{27}\right)^{\frac{1}{n+1}}$

$=\left(\frac{1}{3^6}\right)^{\frac{1}{n+1}}$

$\left(\frac{1}{3}\right)^{\frac{6}{n+1}}$

third mean $(m_3) = ar^3$

$$\text{or, } 1 = 27\left\{\left(\frac{1}{3}\right)^{\frac{6}{n+1}}\right\}^3$$

$$\text{or, } \left(\frac{1}{27}\right) = \left(\frac{1}{3}\right)^{\frac{18}{n+1}}$$

$$\text{or,} \left(\frac{1}{3}\right)^2 = \left(\frac{1}{3}\right)^{\frac{18}{n+1}}$$

$$\text{or, } 3 = \frac{18}{n+1}$$

or, $6 = n+1$

$\therefore n=5$

5(a). There are three geometric means, between a and b. If the first mean and third mean are $\frac{1}{4}$ and $\frac{1}{64}$ respectively, find the values of a and b.

Solution

Here, first term=a, last term =b

Let r be the common ratio

$m_1 = ar$ and $m_3 = ar^3$

$$\text{or, } \frac{1}{4} = ar \text{..............(i)}$$

$$\frac{1}{64} = ar^3 \text{..............(ii)}$$

Dividing (ii) by (i), we get

$$\frac{ar^3}{ar} = \frac{\frac{1}{64}}{\frac{1}{4}}$$

$$\text{or, } r^2 = \frac{1}{16} \times 4$$

$$\text{or, } r^2 = \frac{1}{16}$$

$$\text{or, } r = \frac{1}{4}$$

put the value of r in equation (i), we get.

$$\frac{1}{4} = a.\frac{1}{4}$$

$a=1$

Again, put the value of a and r in

$r=\left(\frac{b}{1}\right)^{\frac{1}{n+1}}$

or, $\frac{1}{4}=\left(\frac{b}{1}\right)^{\frac{1}{3+1}}$

or, $\frac{1}{4}=\left(b\right)^{\frac{1}{4}}$

Taking fourth power on both sides, we get.

$\frac{1}{256}=b$

$b=\frac{1}{256}$

$a=1,\ b=\frac{1}{256}$

5(b). There are three G.M.'s between p and q . If the first mean and the third mean are respectively 1 and 125, find the values of p and q.

Solution

Here, first mean$(m_1)=1$

Third mean $(m_3)=125$

Let r be the common ratio.

Then

$r=\left(\frac{q}{p}\right)^{\frac{1}{n+1}}=\left(\frac{q}{p}\right)^{\frac{1}{3+1}}=\left(\frac{q}{p}\right)^{\frac{1}{4}}$

$m_1=ar$ or, $1=pr$..............(i)

and $m_3=ar^3$ or $125=pr^3$...............(ii)

Dividing (ii) by (i), we get

$\frac{pr^3}{pr}=\frac{125}{1}$

or, $r^2=125$

$r=5\sqrt{5}$

Put the value of r in equation(i)

$1=p.5\sqrt{5}$

$p=\frac{1}{5\sqrt{5}}$

Again, $r=\left(\frac{q}{\frac{1}{5\sqrt{5}}}\right)^{\frac{1}{4}}$

or, $5\sqrt{5}=\left(\frac{q}{\frac{1}{5\sqrt{5}}}\right)^{\frac{1}{4}}$

Taking fourth power on both sides,

$15625=5\sqrt{5}\ q$

or, $q=\dfrac{15625}{5\sqrt{5}}=625\sqrt{5}$

$p=\dfrac{1}{5\sqrt{5}}$ and $q=625\sqrt{5}$

6(a). Some G.M.'s are inserted between 4 and 128. Find the number of means where the ratio of first mean and last mean is 1:8.

Solution

Let r be common ratio. a=4, b=128.

$r=\left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

$=\left(\dfrac{128}{4}\right)^{\frac{1}{n+1}}$

$= 2^{\frac{5}{n+1}}$

But,

$\dfrac{m_1}{m_n}=\dfrac{1}{8}$

or, $\dfrac{ar}{ar^n}=\dfrac{1}{8}$

or, $r^{n-1} = 2^3$

But $r= 2^{\frac{5}{n+1}}$

$\left(2^{\frac{5}{n+1}}\right)^{n-1}=2^3$

or $\dfrac{5(n-1)}{n+1}=3$

or, $5n-5=3n+3$

o, $2n=8$

$\therefore n=4$

(b). There are n G.M.'s between $\dfrac{1}{25}$ and 25. If the ratio of the first mean to the third mean is 1:25, find the value of n.

Solution

Here, $a=\dfrac{1}{25}$, $b=25$, common ratio$(r)=\left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

$=\left(\dfrac{25}{\frac{1}{25}}\right)^{\frac{1}{n+1}}$

$\left(\dfrac{625}{4}\right)^{\frac{1}{n+1}}$

$= 5^{\frac{4}{n+1}}$

$\therefore\ r=5^{\frac{4}{n+1}}$

By question, $\dfrac{\text{first mean}}{\text{last mean}} = \dfrac{1}{25}$

or, $\dfrac{ar}{ar^n} = \dfrac{1}{25}$

or, $r^{n-1} = 5^2$

But $r = 5^{\frac{4}{n+1}}$

$\left(5^{\frac{4}{n+1}}\right)^{n-1} = 5^2$

$\therefore\ \dfrac{4(n-1)}{n+1} = 2$

or, $4n-4=2n+2$

or, $2n = 6$

$\therefore\ n=3$

7(a). Find the two numbers whose A.M. and G.M. are respectively 10 and 8.

Solution

Let required two numbers be a and b.

Then, A.M. $= \dfrac{a+b}{2} = 10$

$a+b = 20$...........(i)

G.M. $= \sqrt{ab} = 8$

or, $ab=64$.............(ii)

we have, $a-b=\sqrt{(a+b)^2-4ab}$

$=\sqrt{400-256}$

$=\sqrt{144}$

$a-b=12$............(iii)

adding(i) and (iii),we get,

$2a = 32$

$\therefore\ a=16$

put the value of a in equation(i), we get,

$b=20-16=4$

$\therefore\ a=16, b=4$

(b). Find two numbers whose A.M. and G.M. are respectively 25 and 20.

Solution

Let a and b two required numbers

Then, A.M. $=\frac{a+b}{2}=25$

$a+b=50$...........(i)

G.M. $=\sqrt{ab}=20$

or, $ab=400$............(ii)

we have, $a-b=\sqrt{(a+b)^2\text{-}4ab}$

$=\sqrt{2500\text{-}1600}$

$=\sqrt{900}$

$a-b=30$............(i)

adding (i) and (ii),we get,

$2a=80$

$\therefore a=40$

put the value of a in equation(i), we get,

$b=20-16=410$

$\therefore a=40,\ b=10$

(c). Find the two numbers whose A.M. and G.M. are respectively 13 and 8.

Solution

Let a and b two required numbers

Then, A.M. $=\frac{a+b}{2}=13$

$a+b=26$...........(i)

G.M. $=\sqrt{ab}=12$

or, $ab=144$............(ii)

we have, $a-b=\sqrt{(a+b)^2\text{-}4ab}$

$=\sqrt{676\text{-}576}$

$=\sqrt{100}=10$

$a-b=10$............(iii)

solving equations (ii) and (iii),we get,

$a=18,\ b=8$

put the value of a in equation(i), we get,

$b=20-16=4$

$\therefore a=16,b=4$

(d). Find two numbers whose A.M. is 34 and G.M. is 16.

Solution

Let the required two numbers be a and b.

Then, A.M. $=\frac{a+b}{2}=34$

$a+b=68$...........(i)

Also, G.M. $=\sqrt{ab}=16$

or, $ab=256$............(ii)

we have, $a-b=\sqrt{(a+b)^2-4ab}$

$=\sqrt{4624-1024}$

$=\sqrt{3600}$

$a-b=60$............(iii)

adding(i) and (iii),we get,

$a=64$

put the value of a in (i),b=4

$b=20-16=4$

$\therefore a=64,\ b=4$

8(a). The ratio of two numbers is 1:16, their geometric mean is $\frac{1}{4}$. Find the numbers.

Solution

Let the required numbers be a and b.

Also, Let a=k, b=16k

Then, G.M. $=\sqrt{ab}=\frac{1}{4}$

or, $\sqrt{k.16k}=\frac{1}{4}$

or, $4k=\frac{1}{4}$

$\therefore k=\frac{1}{16}$

Hence $a=k=\frac{1}{16}$

$b=16k=16.\frac{1}{16}=1$

$\therefore a=\frac{1}{16}, b=1$

(b). The ratio of two numbers is 1:81, their geometric mean is1 . Find the numbers.

Solution

Let two numbers be ak and b=81k.

Then, G.M. $=\sqrt{ab}=1$

or, $\sqrt{k.81k}=1$

or, 9k=1

$\therefore k=\frac{1}{9}$

Hence the required numbers are $a=\frac{1}{9}$ and $b=81.\frac{1}{9}=9$

Questions for practice

1. Find the G.M. between 100 and 400.
2. If p+2, P+8 and 17+p are the geometric sequence, find the value of m.
3. Find the values of x,y,z from the given geometric sequence $\frac{1}{8}$,x,y,z,2.
4, Insert 3 G.M.'s between 3 and 48.
5. Insert 5 G.M.'s between 35 and 2240.
6. There are some geometric means between 5 and 80. If the second mean is 20, find the number of means. Also find the last mean.
7. Find the number of geometric means inserted between 1 and 64 in which the ratio of first mean to the last means is 1:16.
8. There are 5 geometric means between a and b. If the second mean and last means are 63 and 1701 respectively, find the values of a and b.

Sum of Geometric Series

Notes

i) In a G.P. if the first term =a, common ratio =r, number of terms =n are given, the sum of the first n terms is given by

$$S_n = \frac{a(r^n-1)}{r-1}, \text{ when } r>1$$

$$S_n = \frac{a(1-r^n)}{1-r}, \text{ when } r<1$$

ii) When the last term = l, common ratio =r and the first term =a, then the sum of the first n terms is given by

$$S_n = \frac{lr-a}{r-1}, \text{ when } r>1$$

$$S_n = \frac{a-lr}{1-r}, \text{ when } r<1$$

iii) Three consecutive terms of a G.P. are given by

$\frac{a}{r}$, a, ar

Some solved problems

1. Find the sum of series

(a). $1+\frac{1}{2}+\frac{1}{4}$............ upto 8 terms.

Solution

Given series is a G.P.

First term (a)=1,common ratio (r)= $\frac{1}{2}$ <1

number of terms (n)=8

Sum of the 8 terms is given by,

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_8 = \frac{1\left\{1-\left(\frac{1}{2}\right)^8\right\}}{1-\frac{1}{2}} = \frac{1-\frac{1}{256}}{\frac{1}{2}} = 0\frac{255}{128} = 1\frac{127}{128}$$

(b). $81+27+\ldots\ldots\ldots+\frac{1}{243}$

Solution

Here. First term (a)=81,

common ratio (r)= $\frac{27}{81} = \frac{1}{3}$ <1

last term (l)= $\frac{1}{243}$

Now, $S_n = \frac{a-lr}{1-r}$

$$= \frac{81+\frac{1}{243}\cdot\frac{1}{3}}{1-\frac{1}{3}}$$

$$= \frac{59049}{729}\times\frac{1}{3}$$

$$= \frac{59048}{243\times2}$$

$$= \frac{29524}{243}$$

3. Find the first n term of a geometric series whose last term is 1792, common ratio 2, sum 3577

Solution

Here, in a given series, a =?, r=2,

$l=1792$, $s_n=3577$

By using formula,

Now, $S_n = \frac{lr-a}{r-1}$

or, $3577=\frac{1792\times2-a}{2-1}$

or, 3577=3584–a

or, –7=–a

∴ a=7

4(a). How many terms of G.P. must be taken in the series 128+64+32+................ so that the sum may be 254.

Solution

In the given G.S.

First term (a)=128,common ratio (r)= $\frac{64}{128} = \frac{1}{2}$

$S_n = 254$

By using formula,

$S_n = \frac{a(1-r^n)}{1-r}$

or, $254 = \frac{128\left\{1-\left(\frac{1}{2}\right)^n\right\}}{1-\frac{1}{2}}$

or, $\frac{127}{64} = 2\left\{1-\left(\frac{1}{2}\right)^n\right\}$

or, $\frac{127}{128} = \left\{1-\left(\frac{1}{2}\right)^n\right\}$

or, $\left(\frac{1}{2}\right)^n = 1-\frac{127}{128}$

or, $\left(\frac{1}{2}\right)^n = 1-\frac{1}{128}$

or, $\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^7$

$\therefore n=7$

(b) How many terms of a G.P. must be taken in the series 64+96+144+216+......... so that the sum may be 2059 ?

Solution

In the given G.S., a=64, r = $\frac{96}{64} = \frac{3}{2}$

$S_n = 254$, n= ?

By using formula,

$S_n = \frac{a(r^n-1)}{r-1}$

or,$2059 = \frac{64\left\{\left(\frac{3}{2}\right)^n-1\right\}}{\frac{3}{2}-1}$

or, $\frac{2059}{64} = 2\left\{\left(\frac{3}{2}\right)^n-1\right\}$

or, $\frac{2059}{128}=\left(\frac{3}{2}\right)^{n}-1$

or, $\frac{2059}{128}+1=\left(\frac{3}{2}\right)^{n}$

or, $\frac{2187}{128}=\left(\frac{3}{2}\right)^{n}$

or, $\left(\frac{3}{2}\right)^{7}=\left(\frac{3}{2}\right)^{n}$

$\therefore$ n=7

(c). How many terms of G.P. must be taken in the series 3–6+12..... so that the sum may be –63 ?

Solution

Let $S_n = -63$

$r=-\frac{6}{3}=-2$

a=3

By using formula,

$S_n = \frac{a(1-r^n)}{1-r}$

or, $-63 = \frac{3(1-(-2)^n)}{1+2}$

or, $-63 = 1-(-2)^n$

or, $64 = (-2)^n$

or, $(-2)^6 = (-2)^n$

$\therefore$ n=6

5(a). The first term of a G.P. is 5 and the sum of the first four terms is 780, find the common ratio.

Solution

Here, a=5, $S_4 = 780$

Now, $S_n = \frac{a(r^n-1)}{r-1}$

or, $780 = \frac{5(r^4-1)}{r-1}$

or, $156 = \frac{(r^2+1)(r-1)(r+1)}{r-1}$

or, $156=r^3+r^2+r+1$

or, $r^3+r^2+r-155=0$

or, $r^3-5r^2+6r^2-30r+31r-155=0$

or, $r^2(r-5)+6r(r-5)+31(r-5)=0$

or, $(r^2+6r+31)(r-5)=0$

Either $r=5$ or $r^2+6r+31=0$

Here, $r^2+6r+31$ doesnot give real solution

$\therefore r=5$

5(b). The first term of a G.P. is 3 and the sum of the first four terms is 120, find the common ratio.

Solution

Here, in given G.P. , $a=3$, $S_4=120$, $n=4$

Now, $S_n = \dfrac{a(r^n-1)}{r-1}$

or, $120 = \dfrac{3(r^4-1)}{r-1}$

or, $40 = \dfrac{(r^2+1)(r-1)(r+1)}{r-1}$

or, $40=r^3+r+r^2+1$

or, $r^3+r^2+r-39=0$

for $r=3$, $r^3+r^2-39=0$

By factor theorem, $r-3$ is a factor of

$r^3+r^2+-39=0$

Now, $r^3-3r^2+4r^2-12r+13r-39=0$

or, $r^2(r-3)+4r(r-3)+13(r-3)=0$

or, $(r-3)(r^2+4r+13)=0$

Either $r-3=0=3$ or $r^2+6r+31=0$

Here, $r^2+4r+13=0$ does not give real value.

$\therefore r=3$

6(a). The sum of the first four terms of a G.P. with common ratio 3 is $\dfrac{118}{3}$. Find the sum of the first ten terms of the progression.

Solution

Here, $S_4 = \dfrac{118}{3}$, $r=3$

Now,

$S_n = \dfrac{a(r^n-1)}{r-1}$, or, $S_n = \dfrac{a(3^4-1)}{3-1}$

or, $\dfrac{118}{3} = \dfrac{a.80}{2}$

or, $\dfrac{118}{3}=40a$

or, $a=\dfrac{118}{120}$

$=\dfrac{59}{60}$

Again,

$S_{10} = \dfrac{a(r^{10}-1)}{r-1}$, or, $S_n = \dfrac{a(3^4-1)}{3-1}$

$$= \frac{\frac{59}{120}\{3^{10}-1\}}{\frac{59}{120}\ 3-1}$$

$$= \frac{\frac{59}{120}\{59049-1\}}{2}$$

$$= \frac{59.59048}{240}$$

$$= \frac{435479}{30}$$

$$\therefore S_{10} = \frac{435479}{30}$$

6(b). The sum of the first three terms of a G.P. with common ratio 2 is 112. Find the sum of the first 8 terms of the progression.

Solution

Here, $S_3 = 112$, $r = 2$

Now, $S_n = \frac{a(r^n - 1)}{r-1}$

or, $S_3 = \frac{a(2^3 - 1)}{2-1}$

or, $112 = \frac{7a}{1}$

$\therefore a = \frac{112}{7} = 16$

Again, $S_8 = \frac{16(2^8 - 1)}{2-1}$

$= 4080$

7(a). In a G.P. the sum of three consecutive terms is 21 and their product is 64. Find them.

Solution

Let three terms in be $\frac{a}{r}$, a, ar

then,

$\frac{a}{r} + a + ar = 64$

or, $a^3 = 64$

$a = 4$

Again,

$\frac{a}{r}.a.ar = 21$

or, $\frac{4}{r} + 4 + 4r = 21$

or, $4 + 4r + 4r^2 = 21r$

or, $4r^2 - 17r + 4 = 0$

or, $4r^2 - 16r - r + 4 = 0$

or, $4r(r-4) - 1(r-4) = 0$

or, $(r-4)(4r-1)=0$

$\therefore r=4, \frac{1}{4}$

Find $r=4$, the required numbers are,

$\frac{a}{r}=\frac{4}{4}=1$, $a=4$, $ar=4.4=16$

when, $r=\frac{1}{4}$, the required numbers are,

$\frac{a}{r}=\frac{4}{\frac{1}{4}}=16$, $a=4$, $ar=4.\frac{1}{4}=16$

Hence the required numbers are 1, 4, 16 or 16,4,1

7(b). The sum of three consecutive terms in G.P. is 62 and their product is 1000, find them.

Solution

Let three terms in G.P. be $\frac{a}{r}$, a, ar

then,

$\frac{a}{r}+a+ar=1000$

$\therefore a=10$

Again,

$\frac{a}{r}.a.ar=62$

or, $\frac{10}{r}+10+10r=62$

or, $10+10r+10r^2=62r$

or, $10r^2-52r+10=0$

or, $5r^2-26r+5=0$

or, $5r^2-25r-r+5=0$

or, $5r(r-5)-1(r-5)=0$

or, $(r-5)(5r-1)=0$

$\therefore r=5, \frac{1}{5}$

Find $r=5$, the required numbers are,

$\frac{a}{r}=\frac{10}{5}=2$, $a=10$, $ar=10.5=50$

when, $r=\frac{1}{5}$, the required numbers are,

$\frac{a}{r}=\frac{10}{\frac{1}{5}}=50$, $a=10$, $ar=10.\frac{1}{5}=2$

Hence the required numbers are 2, 10 and 50 or 50,10 and 2.

8(a). The sum of the first 2 terms of G.P. is 6 and that of the first four terms is $\frac{15}{2}$. Find the sum of the first six terms.

Solution

Here, $S_2=6$, $S_4=\frac{15}{2}$

Now, $\dfrac{S_2}{S_4} = \dfrac{9}{\frac{15}{2}}$

or, $\dfrac{\frac{a(r^n-1)}{r-1}}{\frac{a(r^n-1)}{r-1}} = \dfrac{12}{15}$

or, $\dfrac{1}{r^2+1} = \dfrac{12}{15}$

or, $15 = 12r^2 + 12$

or, $3 = 12r^2$

$\therefore r^2 = \dfrac{1}{4}$ or, $r = \pm\dfrac{1}{2}$

Since sum is positive, we take positive value of $r = \dfrac{1}{2}$

Put the value r in $s_2 = 6$

By using formula,

or, $\dfrac{a(r^2-1)}{r-1} = 6$

or, $\dfrac{a(r^2-1)}{r-1} = 6$

or, $\dfrac{a\left(\frac{1}{4} - 1\right)}{\frac{1}{2} - 1} = 6$

or, $a.\dfrac{3}{4}.2 = 6$

$\therefore$ $a = 4$

Now, $S_n = \dfrac{a(1-r^n)}{1-r}$

or, $S_6 = \dfrac{4\left\{1 - \left(\frac{1}{4}\right)^6\right\}}{1-\frac{1}{2}} = \dfrac{4\left\{1 - \frac{1}{64}\right\}}{1 - \frac{1}{2}} = \dfrac{63}{8}$

(b). The sum of the first 2 terms of G.P. is 8 and that of the first four terms is 80 . Find the sum of the first 6 terms.

Solution

Here, $S_2 = 8$ and $s_4 = 80$

Now,

$S_n = \dfrac{a(r^n-1)}{r-1}$,

or, $S_2 = \dfrac{a(r^2-1)}{r-1}$

and

$S_4 = \dfrac{a(r^4-1)}{r-1}$

or, $\dfrac{S_2}{S_4} = \dfrac{\dfrac{a(r^4-1)}{r-1}}{\dfrac{a(r^2-1)}{r-1}}$

or, $\dfrac{12}{15} = r^2+1$

or, $10 = r^2+1$

or, $r^2 = 9$

$r = \pm 3$

Since sum are positive, taking positive square roots only.

$r = 3$

put the value of r in

$S_2 = \dfrac{a(r^2-1)}{r-1}$

or, $8 = \dfrac{a(9-1)}{3-1}$

$\therefore a = 2$

Again,

$S_6 = \dfrac{a(r^6-1)}{r-1}$,

$= \dfrac{2(3^6-1)}{3-1}$

$= 728$

9(a). The sum of the first 8 terms of G.P. is 5 times the sum of the first 4 terms. Find the common ratio.

Solution

Here, $S_6 = 5\ s_4$

or. $\dfrac{a(r^8-1)}{r-1} = 5\ \dfrac{a(r^4-1)}{r-1}$

or, $(r^4+1)(r^4-1) = 5(r^4-1)$

or, $r^4+1 = 5$

or, $r^4 = 4$

$\therefore r = \sqrt{2}$

9(b). The sum of the first 6 terms of G.P. is 9 times the sum of the first 3 terms. Find the common ratio.

Solution

Here, $S_6 = 9\ S_3$

or. $\dfrac{a(r^6-1)}{r-1} = 9.\dfrac{a(r^3-1)}{r-1}$

or, $r^3+1=9$

or, $r^3=8$

$\therefore r=2$

10(a). The sum of the first 2 terms of G.P. is $\frac{9}{8}$ and the sum of the last 2 terms is 1152. There are altogether 12 terms. Find the common ratio and the first term.

Solution

In given G.P. $S_2=\dfrac{9}{8}$

or. $\dfrac{a(r^2-1)}{r-1}=\dfrac{9}{8}$ or, $a(r+1)=\dfrac{9}{8}$ ....(i)

Total number of terms $=12$

and the sum of last two term $=1152$

$t_{11}+t_{12}=1152$

or, $ar^{10}+ar^{11}=1152$

or, $ar^{10}(1+r)=1152$............(ii)

Dividing (ii) by (i), we get,

$\dfrac{ar^{10}(1+r)}{a(1+a)} = \dfrac{1152}{918}$

or, $r^{10}=1024$

or, $r^{10}=2^{10}$

$\therefore r=2$

put the value of r in equation (i), we get

$a(2+1)=\dfrac{9}{8}$

or,$a=\dfrac{9}{8\times 3} =\dfrac{3}{8}$

$\therefore r=2,\ a=\dfrac{3}{8}$

10(b). The sum of the first four terms of G.P. is 30 and that of the last four terms is 960.If its first terms is 2 and the last terms is 512,find the common ratio and the number of terms.

Solution

In a given G.P.

First term (a)=3, $S_4=30$

Let us find common ratio r,

$S_n = \dfrac{a(r^n-1)}{r-1}$

or,$S_4 = \dfrac{2(r^4-1)}{r-1}$

or,$30 = \dfrac{2(r^2+1)(r+1)(r-1)}{r-1}$

or, $15=r^3+r^2+r+1$

or, $r^3+r^2+r-14=0 \rightarrow$ For $r = 2$, $2^3 + 2^2 + 2 - 14 = 0$

Hence,by factor theorem,

$(r-2)$ is a factor of r^3+r^2+r-14

Now,

$r^3-2r^2+3r^2-6r+7r-14=0$

or, $r^2(r-2)+3r(r-2)+7(r-2)=0$

or, $(r-2)(r^2+3r+7)=0$

Either $r-2=0 \Rightarrow r=2$

or, $r^2+3r+7=0$ It doesnot give real valueof r.

Again, last term, $l=ar^{n-1}$

or, $512=2.2^{n-1}$

or, $256=2^{n-1}$

or, $2^8=2^{n-1}$

$\therefore 8=n-1$

or, $n=9$

Then sum of the last four terms is given by

$ar^5+ar^6+ar^7+ar^8=960$(True).

ie.

$2(2^5+2^6+2^7+2^8)$

$=2(32+64+128+256)$

$=960$

Questions for practice

1. Find the sum of the series $27+9+3+.......+\frac{1}{81}$.
2. How many terms are there in the series $1+2+4+.....+256$. Calculate the sum.
3. A geometric series has 8 terms whose sum of the first three terms 7 and the sum of the sum of the last three terms is 224. Find the first term, common ratio and sum of the series.
4. In a geometric series if the fifth term is 8 times the second term and the sum of the first four terms is 60, then find the positive common ratio and the first term of the series.
5. In a G.P. of the sixth term is 16 times the second term and the sum of the first seven terms is $\frac{127}{4}$ then find the positive common ratio and the first term of the series.
6. Find two numbers whose A.M. is 5 and G.M. 4.
7. Find two numbers whose A.M. is 85 and G.M. 40.
8. The sum of three numbers in A.P. 45 and if 17, 11 and 3 are subtracted from them respectively, the resulting numbers will be in G.P. Find the numbers.
9. Find the sum of the first 8 terms of a G.P. whose 5^{th} and 8^{th} terms are respectively 81 and 2187.
10. The ratio of the fifth term and eighth term of a G.P. is 1:8. If its seventh term is 64, find the sum of the first 10 ten terms.

UNIT four

Linear Programming

Estimated teaching periods : 4

1. Objectives

S.N.	Level	Objectives
(i)	Knowledge(K)	To define inequality To draw graph of linear equation To define linear programming problem. To define feasible region.
(ii)	Understanding(U)	To draw graph of linear inequality To draw graph of set of inequalities.
(iii)	Application(A)	To draw graph of system of linear inequalities to find feasible/convex region. To find the vertices of convex polygon.
(iv)	Higher Ability(HA)	To find maximum and minimum values of an objective function under given conditions graphically.

2. Teaching materials

Graph papers, pencils

3. Teaching Learning Strategies:

- Review on real number lines.
- Draw graphs of the following inequalities (as given in text book).
  $x\geq0$, $x\leq0$, $x\geq4$, $x\leq4$, $y\geq4$, $y\leq4$
- Discuss on the graph of $x>4$ and $x\geq4$.
- Define linear inequalities with examples
- Discuss how to draw graph of $3x+4y \leq 12$, with origin test and shade solution region.
- Define system of linear inequalities and their solution, on the basis of graphs of system of linear inequalities–define feasible region and convex polygon, find the vertices of convex polygon.
- Define linear programming graphically.
- Discuss about mathematical formulation of linear programming problems. (given in text book).

Note :

(i) The equation of boundary line corresponding to inequalities $ax+by \leq c$ and $ax+by \geq c$.

(ii) The inequalities $x \geq 0$ and $y \geq 0$ represent a feasible region on the first quadrant.

(iii) $x=0$ represents y–axis and $y=0$ represent x–axis.

(iv) Each of the boundary line is drawn on the graph paper.

(v) To find the feasible plane region formed by given equatities $ax + by + \leq c$ and $ax + by$ 7 c, we use origin o(0,0) as a testing point, if the boundary line does not pass through the origin.

(vi) Find the feasible region by shading the area which is satisfied by all the constraints.

(vii) Find the coordinates of all the vertices of feasible region / convex polygon.

(viii) Find the value of the objective function at each vertex of the feasible region / convex polygon.

(ix) Identify the value of x and y of the point for which the objective function has maximum or minimum values.

(x) Write the conclusion at which point the objective function has maximum or minimum value.

Some solved problems

1. Draw graph of $4y-5x \geq 20$,.

Solution

Given inequality is $4y-5x \geq 20$

The corresponding boundary line of given inequality is

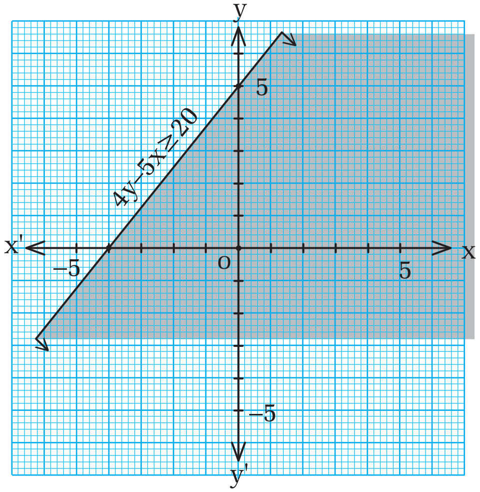
$4y-5x \geq 20$

when x=0, then, y=5

when y=0, then x = –4

Hence the boundary line passes through the points (–4,0) and (5,0).

Take 0(0,0) as a testing point for $4y-5x \geq 20$, we get, $0 \geq 20$ which is false. Hence the solution region is the half plane not containing the origin.



Alternatively,

The points on the boundary line

$-5x + 4y = 20$ can be obtained as

$$\frac{x}{-4}+\frac{y}{5}=1$$

which is in the form of $\frac{x}{a}+\frac{y}{b}=1$

Hence the boundary line passes through the points (–4,0) and (0,5).

2. Graph the following system of inequalities and find the vertices of convex polygon of exists.

(a) $x+y+2 \geq 0$, $y \leq 2x+4$, $y \leq 4-4x$.

Solution

The boundary line equations of given system of inequalities are given by

$x+y+2=0$..........(i)

$y = 2x+4$..........(ii)

$y = 4-4x$ ....(iii)

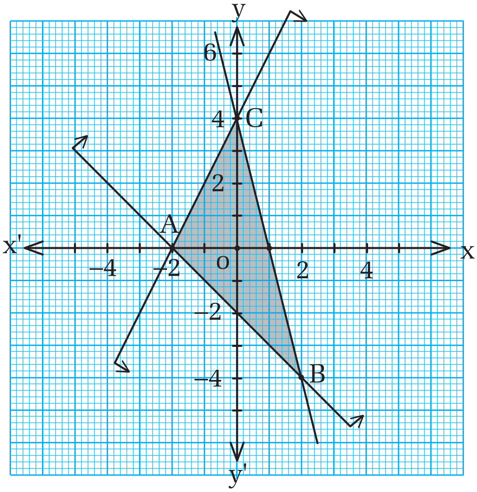
From boundary line equation (i)

$x+y+2=0$

when x=0, they y=–2

when $y=0$, they $x=-2$

The boundary line (i) passes through the points $(-2,0)$ and $(0,-2)$. Take $0(0,0)$ as a testing point for $x+y+2 \geq 0$, we get, $2 \geq 0$, which is true . Hence the solution of $x+y+2 \geq 0$, is the half plane containing the origin.



From the boundary line equation (ii), we get,

$y=2x+4$,

For $x=0$, $y=4$ and for $y=0$, $x=-2$ the boundary line (ii) passes through the points $(0,4)$ and $(-2,0)$. Take $0(0,0)$ as a testing point for $y \leq 2x+4$ we get, $0 \leq 4$ which is true.

Hence the solution of $y \leq 2x+4$ is the half plane containing the origin.

Again from the boundary line (iii), we get,

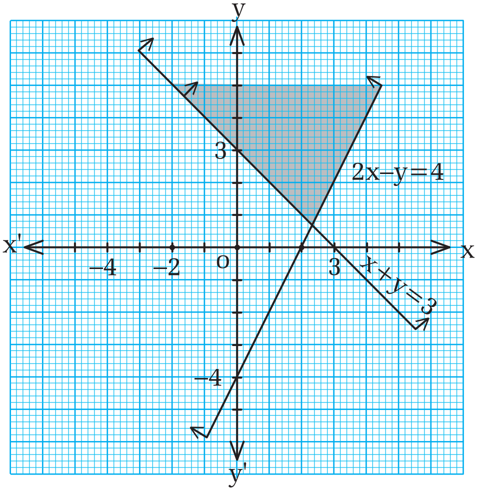
$y = 4-4x$

For $x=0$, $y=4$ and for $y=0$, $x=1$. The boundary line (iii) passes through the points $(0,4)$ and $(1,0)$. Take $0(0,0)$ as a testing point for $y \leq 4-4x$. we get, $0 \leq 4$ which is true.

Hence the solution of $y \leq 4-4x$ is the half plane containing the origin. All the boundary lines are drawn on the graph the feasible region is ΔABC whose vertices $A(-2,0)$, $B(2,-2)$ and $C(0,4)$ which is shaded in the graph.

(b) $x+y \geq 3$ and $2x-y \leq 4$

Solution



The boundary line equations of the corresponding inequalities are

$x+y=3$..........(i)

$2x-y=4$.........(ii)

From boundary line equation (i), we get,

$\frac{x}{3}+\frac{y}{3}=1$.

The boundary line equation (i) passes through the points $(3,0)$ and $(0,3)$. Take $0(0,0)$ as a testing point for $x+y \geq 3$, we get $0 \geq 3$ which is false. Hence solution of $x+y \geq 3$ is the half plane not containing the origin.

Again from boundary equation (ii), we get,

$\frac{x}{2}+\frac{y}{-4}=1$

The boundary line (ii), passes through the points $(2,0)$ and $(0,-4)$. Take $0(0,0)$ as a testing point for $2x–y≤4$, we get , $0≤4$ which is true.

Hence the solution of $2x–y≤4$ is the half plane containing the origin. The boundary lines are plotted on the graph and common solution region is shaded in the graph which is not closed. The convex polygon does not exist.

3. Maximize $z=2x+3y$ subject to the constraints $2x+y \leq 14$, $x+2y \leq 10$, $x \geq 0$, $y \geq 0$.

Solution

The given linear constraints are

$2x+y \leq 14$ ....(i)

$x+2y \leq 10$ ....(ii)

$x \geq 0, y \geq 0$ ....(iii)

The corresponding linear equations are

$2x+y = 14$ ....(i)

$x+2y =10$ ....(ii)

$x = 0$.......(iii)

From equation(i),

if $x=0$, then $y=14$

if $y=0$, then $x=7$

Hence the line (i) passes through the points (0,14) and (7,0).

From equation (ii), we get, if $x=0$ then $y=5$ and

if $y=0$, then $x=10$

Hence the boundary line (ii) passes through the points (0,5) and (10,0)

For determination of the plane region if both of the linear constraints, we get, by using origin as testing point (0,0), put $x=0$, $y=0$, $2.0 +0 \leq 14$

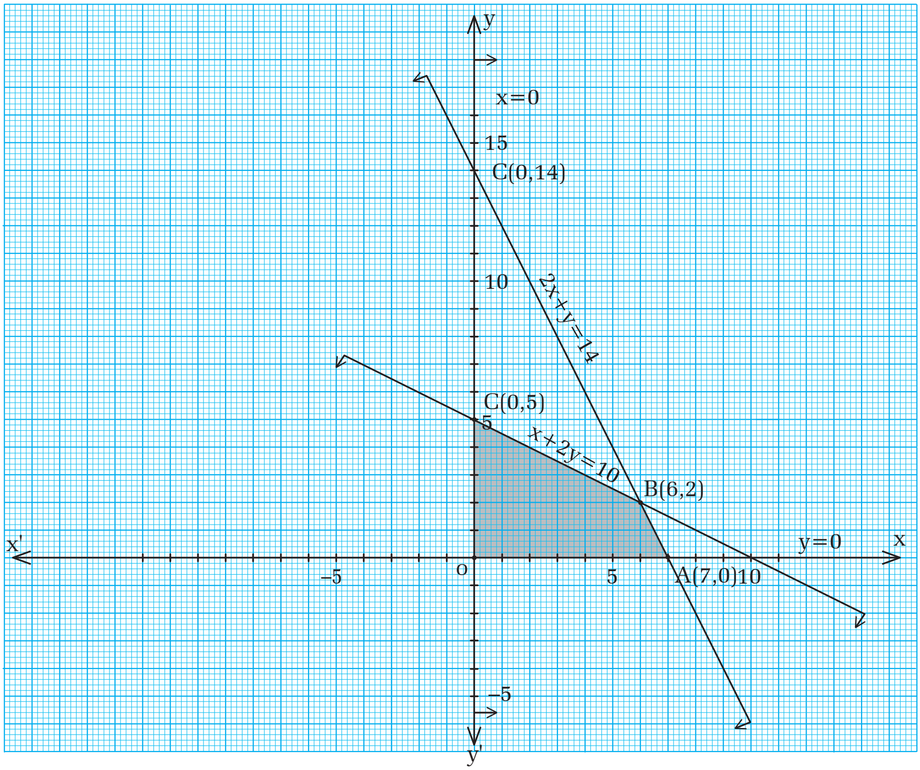
ie. $0≤14$ which is true also put the point in $x+2y≤10$, we get, $0≤10$ which is true.

Hence the solutions of both $2x+y \leq 14$ and $x+2y \leq 10$ contains the origin.

Here, $x=0$ represents equation of y–axis $x \geq 0$ has solution of the right half plane of y–axis. All the boundary lines are plotted in the graph. the feasible region is quadrilateral OABCD with vertices O(0,0), A(7,0), B(6,2) and C(0,5).

S.N.	vertices	value of $z=2x+3y$	Remarks
1.	O(0,0)	0	Minimum
2.	A(7,0)	14	
3.	B(6,2)18	18	maximum
4.	C(0,5)	15	

Hence z has the maximum value 18 at B(6,2).



4. Minimize F=2x+y subject to the constraints 2x+y ≤20, 2x+3y≤24. x ≥ 0, y ≥ 0.

Solution

The given linear constraints are 2x+y ≤20, 2x+3y≤24. x ≥ 0, y ≥ 0.

The corresponding boundary line equations of above constraints are

2x+y =20..................(i)

2x+3y=24.................(ii)

x= 0...................(iii)

y = 0...................(iv)

From equation (i), if x=0, then y=20

if y=0, then x=10

The boundary line equation(i) passes through the points (0,20) and (10,0). Taking O(0,0) for 2x+y ≤20, we get, 0≤ 20 which is true. Hence the solution of 2x+y ≤ 0 is the half plane containing the origin.

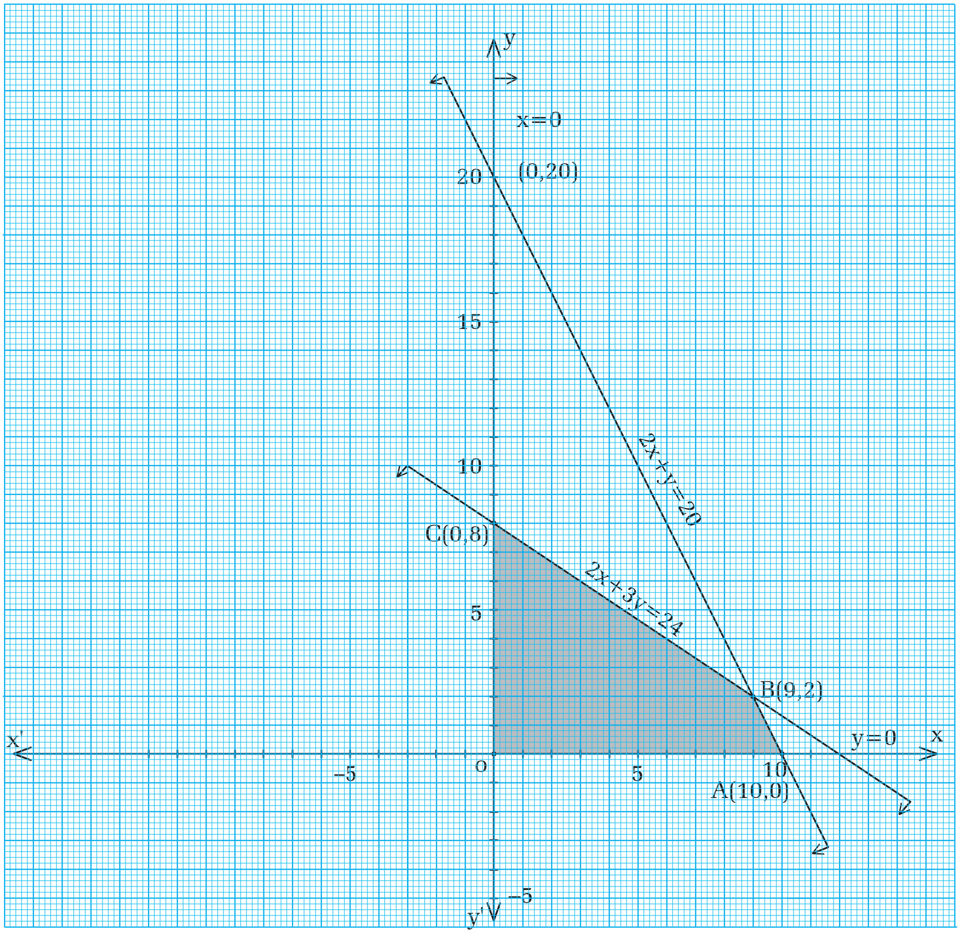
From equation (ii), if x=0, they y=8, if y=0, then x=10.

Hence the boundary line (ii) passes through the paints.(0,8) and (10,0)

Taking O(0,0) as a testing paint for 2x+3y≤24, we get 0≤ 24, which is true. x=0 and y=0 represents y–axis and x–axis respectively. As x ≥ 0 and y ≥ 0 a feasible region lies in the first quadrant. Hence the solution of 2x+3y ≤ 24 is the half plane containing the origin. All the boundary lines are plotted in the graph. The feasible region of given system of constraints is quadrilateral OABC with vertices O(0,0), A(10,0), B(9,2) and C(0,8).

S.N.	vertices	value of Objectives function F=2x+y	Remarks
1.	O(0,0)	0	Minimum
2.	A(10,0)	20	Maximum
3.	B(9,2)	20	maximum
4.	C(0,8)	8	

Hence the given objective function F has minimum value o at origin O(0,0).



5. Find the maximum and minimum values (extreme values) of F=16x–2y+40 subject to the constraints 3x+5y ≤ 24, 0≤ x ≤ 7, 0 ≤ y ≤ 4.

Solution

Given objective function is F=16x–2y+40

Given linear constraints are

$3x+5y \leq 24$

$0\leq x \leq 7$ ie, $x \geq 0$, $x\leq7$

$0 \leq y \leq 4$ ie. $y \geq 0$, $y\leq4$

The corresponding boundary line equations of above consttraints are

$3x+5y = 24$,...................(i)

$x=7$..................(ii)

$y=4$....................(iii)

$x=0$, $y = 0$....................(iv)

From equation (i), if $y=0$, then $x=8$

if $x=-2$, then $y=6$

Hence the boundary line equation(i) passes through the points (8,0) and (–2,6). Taking O(0,0) for $3x+5y \leq 24$, we get, $0 \leq 24$, which is true. So, solution of $3x+5y \leq 24$ is the half plane containing the origin.

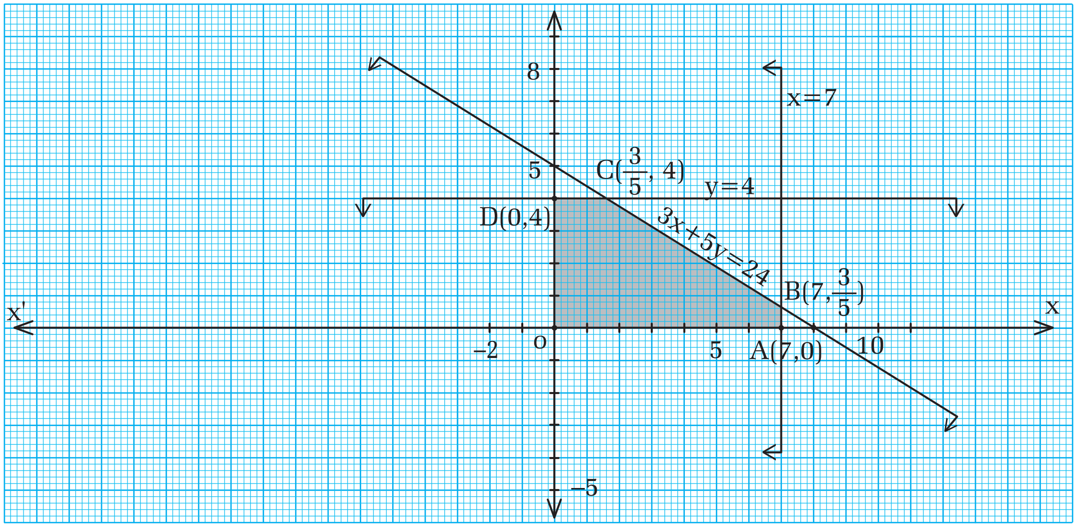
$x=7$, is a line parallel to y–axis and $x \leq 7$ has the solution left if $x=7$.

$y=4$ is the boundary line parallel to x–axis and $y \leq 4$ has the solution lower half plane of $y=4$.

As $x \geq 0$ and $y \geq 0$, a feasible region lies in the first quadrant. All the boundary lines are plotted in the graph and the feasible region is pentagon OABC with vertices O(0,0), A(7,0), B$(7, \frac{3}{5})$ and C$(\frac{4}{3}, 4)$ and D(0,4)

S.N.	vertices	value of Objectives function F=2x+3y	Remarks
1.	O(0,0)	40	
2.	A(7,0)	152	Maximum
3.	B$(7, \frac{3}{5})$	150.8	
4.	C$(\frac{4}{3}, 4)$	53.3	
5.	D(0,4)	32	Minimum

From above table, we conclude that the maximum value is 152 at A(7,0) and minimum value 32 at (0,4).



6(a). In the given graph the coordinates of the points O, A, B, C, D and E are respectively

(0,0), (2,0), (0,3). (0,5), ($\frac{20}{19}$, $\frac{45}{19}$), (5,0) respectively. OADB is feasible region. Find all the inequalities and minimize p=6x+10y+20

Solution

In the graph feasible region is OADB. We have to find the equations of linear constraints. For line AC, it cuts x–axis at A(2,0) and y–axis at c(0,5). So equation of boundary line AC is

$$\frac{x}{2}+\frac{y}{5}=1.$$

or, $5x+2y \leq 10$

Let $5x+2y \leq 10$, taking O(0,0) as the testing point we get, $0 \leq 10$ which is true. So $5x+2y \leq 10$ is the inequality associated with line AC. For line BE, it cuts X–axis at E (5,0) and Y–axis at B (0,3).So equation of boundary line is given by

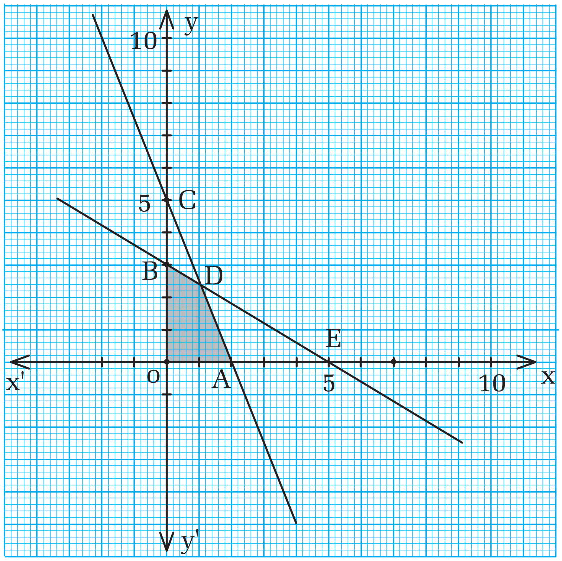
$$\frac{x}{5}+\frac{y}{3}=1$$

or, $5x+3y \leq 15$

Let $5x+3y \leq 15$, taking O(0,0) as the testing point we get, $0 \leq 15$ which is true. So $5x+3y \leq 15$ is the inequality associated with boundary line BE. Feasible region is quadrilateral OADB which is the first quatrant. Hence $x \geq 0$ and $y \geq 0$. Now, we find the minimum value of p $=6x+10y+20$.

S.N.	vertices	value of Objectives function F=2x+3y	Remarks
1.	O(0,0)	20	
2.	A(2,0)	32	Maximum
3.	B($\frac{20}{19}$, $\frac{45}{19}$)	50	
4.	B(0,4)	50	

From above table, we conclude that P has minimum value is 20 at (0,0) .



6(b) text book 8b (Page 95)

Solution

In the graph ΔABC is the feasible region is AB is parallel to the y–axis at distance of 5 units right of the y–axis. Hence its equation is x=5. . As feasible region is left of AB,we write $x \leq 5$. F. For line AC, we have A(5,0) and C(0,5), equation of AC is given by

$$\frac{x}{5}+\frac{y}{5}=1.$$

or, $x+y = 5$

Let $x+y = 5$, taking testing point O(0,0), we get, $0 \leq 5$ which is true. Hence $x+y = 5$ is the linear constraint associated with line AC. Again for line OB it passes through the origin, we have O(0,0) and B (5,10). The equation of OB is given by

$y=mx$

It passes through the point (5,10), So we get,

$10=m.5$

$\therefore m=2$

Hence the equation of OB is $y=2x$.

let $2x-y \leq 0$, take the testing point (1,0),

we get, $2 \leq 0$, which is false . Hence required linear constraints is

$2x-y \geq 0$

Questions for practice

1. Find the maximum value of the objective function $Z=5x+8y$ subject to constraints $2x +5y \leq 15$, $x \geq 2$, $y \geq 1$. (Ans 33 at (5,1))

2. Maximum the objective function $F=x+2y$ subject to constraints $2x +y \leq 14$, $x +2y \leq 10$ and $x \geq 0$, $y \geq 0$. (Ans 10 at (6,2)and (0,5))

3. Minimize $F=100x+600y$ subject to constraints $x +y \leq 10$, $x +3y \leq 16$, $x \leq 3$, $y \geq 2$. (Ans 1500 at (3,2))

4. Minimize the function $F(x,y)=5x+2y$ subject to constraints $2x +y \leq 4$, $x -2y \leq 2$, $x \geq 0$, $y \geq 0$ (Ans 0 at (3,2))

5. Find the extreme values of the function $Z=5x+7y$ subject to constraints $x + 2y \leq 20$, $x +y \leq 16$, $x \geq 0$, $y \geq 0$.

UNIT five

Quadratic Functions and Graphs

Estimated teaching periods : 3

1. Objectives

S.N.	Level	Objectives
(i)	Knowledge(K)	To define a quadratic function To define a Cubic function To define a line of symmetry of a parabola. To write vartex of parabola with equation $y=ax^2+bx+c$, $a \neq 0$. To define the point of intersection of a curve and a straight line.
(ii)	Understanding(U)	To draw graph of a straight line
(iii)	Application(A)	To draw graph of given equation of parabola. To draw graph of given Cubic function. To find solution of a parabola and a straight line graphically. To find the equation of parabola passing through given points.
(iv)	Higher Ability(H.A)	To find equation of given curve of parabola.

2. Teaching materials

i) Graph papers, functions

ii) Chart papers with graph of parabola and cubic function.

3. Teaching learning strategies.

- Review definitions of linear function, quadratic function and cubic function with graphs.
- Discuss graph of a straight line like $4x+3y=12$.
- Discuss graph of a parabola ln the form of
  $y=ax^2$, put $a=\pm1, \pm2, \pm3$.
- Discuss and conclude the conclusions about nature of curves. eg. Draw graph of $y=x^2+4$, $y=4x^2-8x+3$, $y=(x+1)^3$
- Review the solution of a straight line and a quadratic equation by substitution method.
- Discuss how to solve a linear equations and quadratic equation by using graph, with an example ($y=x^2+2x-8$, $y=-5$)
- Let the student give to do some problems in each class as class work and the teacher supervise them and give necessary feedback.

Notes :

1. General equation of parabola is $y=ax^2+bx+c$, $a\neq0$

2. Equations of parabola	vertex
i) $y=ax^2$, $a\neq0$	O(0,0)
ii) $y=a(x-h)^2+k$, $a\neq0$,	(h,k)

iii) $y=ax^2+c \quad a\neq0,$ $(0,c)$

iv) $y=ax^2+bx+c, a\neq0,$ $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

v) $y=ax^2+bx, \ a\neq0,$ $\left(-\frac{b}{2a}, \frac{-b^2}{4a}\right)$

3. General Equation of cubic function is $y=ax^3+bx^2+cx+d, a\neq0$

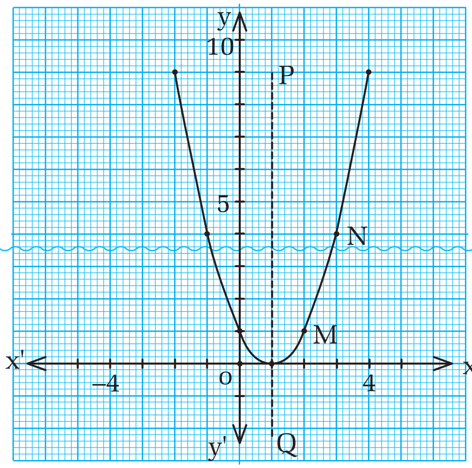
Some solved problems

1. Observe the given curve and answer the following questions.

(a) Name the curve (b) Find the vertex of the curve

(c) Write the coordinates of points M and N.

(d) Name the line PQ which is parallel to y-axis.



Solutions

(a) The name of curve is parabola.

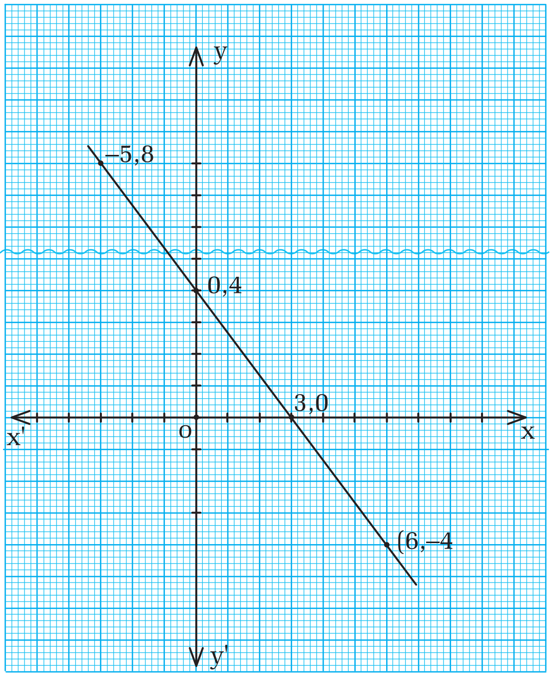
(b) The vertex or turning point of the parabola is A(1,0)

(c) The coordinates of M and N are (2,1) and (3,4) respectively.

(d) The line PQ which divides the curve into two equal parts is called the line of symmetry. It is parallel to y–axis. Its equation in this figure is $x=1$.

2(a). Draw graph of the given linear equations

$4x+3y=12$



Solution

Here, $4x+3y=12$

or, $3y=12-4x$

or, $y=\frac{1}{3}(12-4x)$

take

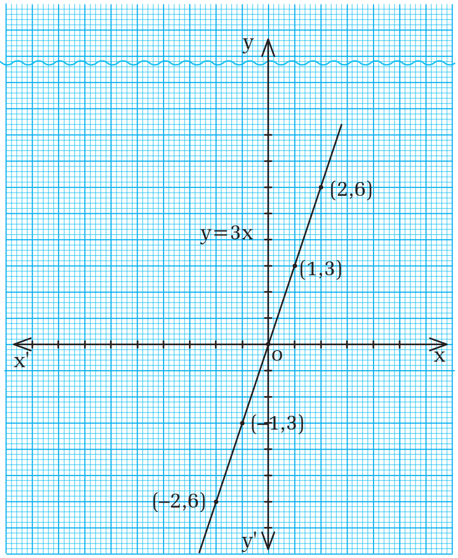
x	0	3	6	–3
y	4	0	–4	8

The points (0,4), (3,0),(6,–4) and (–3,8) are plotted in a graph paper and joined them, we get a straight line.

(b) $y = 3x$

Solution,

x	0	1	2	–1	–2
y	0	3	6	–3	–6



3. Find the graphs for the quadratic equations

(a) $y=x^2$

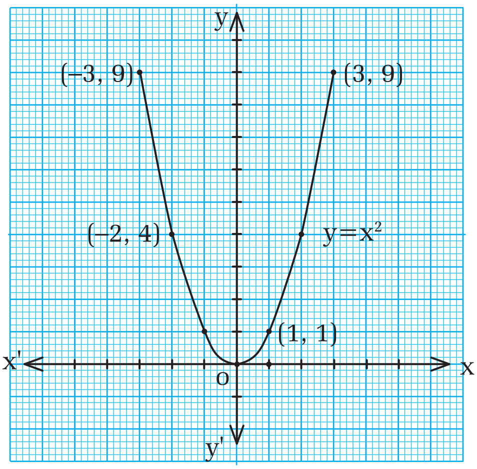
Solution

Here, $y=x^2$ represents a parabola

take

x	0	±1	±2	±3
y	0	1	4	9

The points from above table are plotted in a graph paper and joined them, we get a curve called parabola.



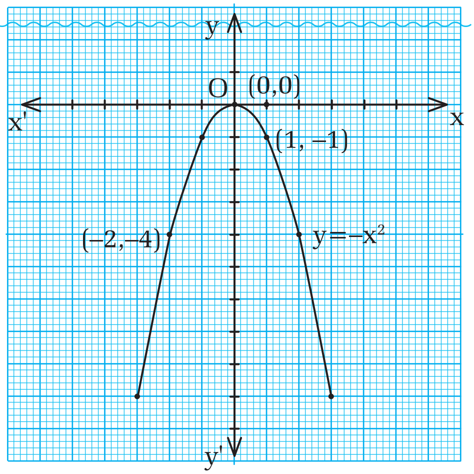
(b) $y=-x^2$

Solution

Here, $y=-x^2$ it represents a parabola with vertex at the origin.

x	0	±1	±2	±3
y	0	–1	–4	–9

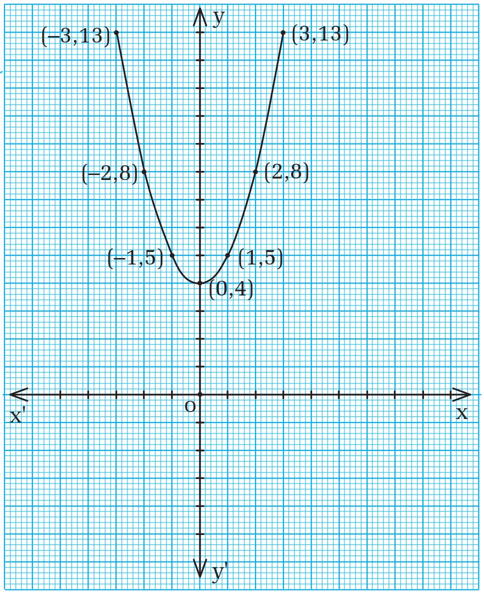
The points from above table are plotted in a graph paper and joined them, we get a curved as shown in the graph along side.



(c) $y=x^2+4$

Solution

Here, $y=x^2+4$, comparing it with $y=ax^2+bx=c$,we get, a=1, b=0, c=4



$$\text{Vertex}=\left(-\frac{b}{2a},\frac{4ac-b^2}{4a}\right)$$

$$=\left(-\frac{0}{2.1},\frac{4.1.4-0^2}{4.1}\right)$$

$=(0,4)$

x	0	±1	±2	±3
y	4	5	8	13

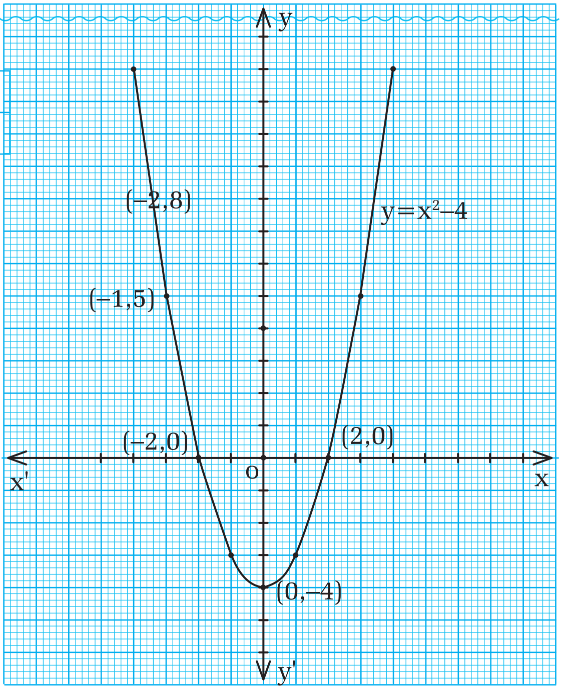
The points from above table are plotted in a graph paper and joined them, we get a curved graph along side.

(d) y = x2–4

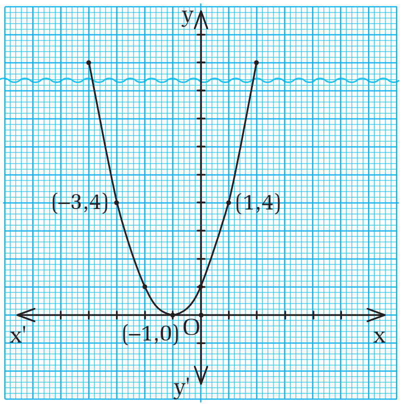
Solution

Here, $y=x^2-4$

x	±1	0	±2	±3	±4
y	–3	–4	0	5	12



(e) $y=(x+1)^2$



Solution

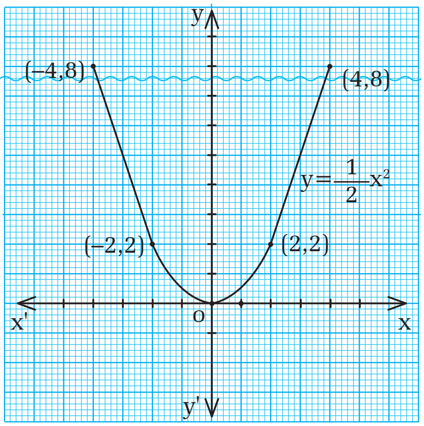
Here, $y=(x+1)^2$. It is in the form of $y=a(x-h)^2+k$
vertex $=(h,k)=(-1,0)$, where , $a=1$, $h=-1$, $k=0$

Table

x	0	1	2	−1	−2	−3	−4
y	1	4	9	0	1	4	9

The points from above table are plotted in a graph paper and joined them, we get a curved as shown along side.

(f) $y=\frac{1}{2}x^2$



Solution

Here, $y=\frac{1}{2}x^2$. It is in the form of $y=ax^2$, where ,

$a=$ $y=\frac{1}{2}$, vertex $=(0,0)$

$y=\frac{1}{2}x^2$

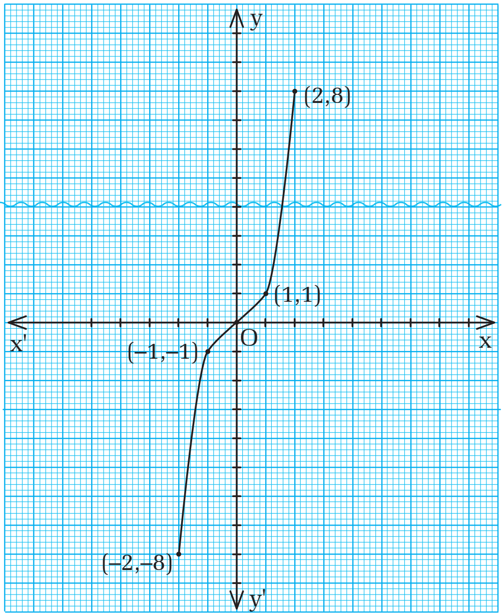
Table

x	0	±1	±2	±4
y	0	$\frac{1}{2}$	2	8

The points from above table are plotted in a graph paper and joined them, we get a curved as shown along side.

4. Draw the graphs for the following equations

(a) $y=x^3$



Solution

Here, $y=x^3$. This is a equation of cubic function

Table

x	−2	−1	0	1	2
y	−8	−1	0	1	−8

The points from the above table are plotted on a graph and joined them, we get a curve as shown alongside.

(b) $y=-x^3$

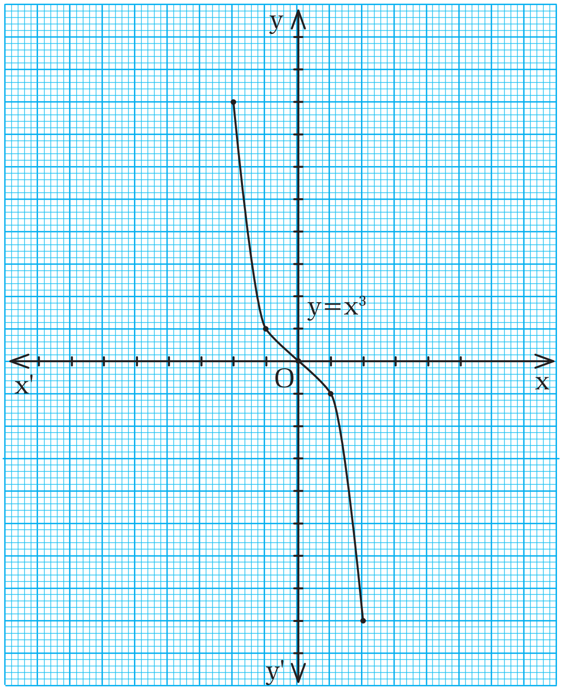
Solution

Here, $y=-x^3$............(i). It is an equation of cubic function

Table

x	–2	–1	0	1	2
y	8	1	0	–1	–8

The points from above table are plotted in a graph and joined them, we get a curve as shown the graph alongside.



(c) $y=(x+1)^3$

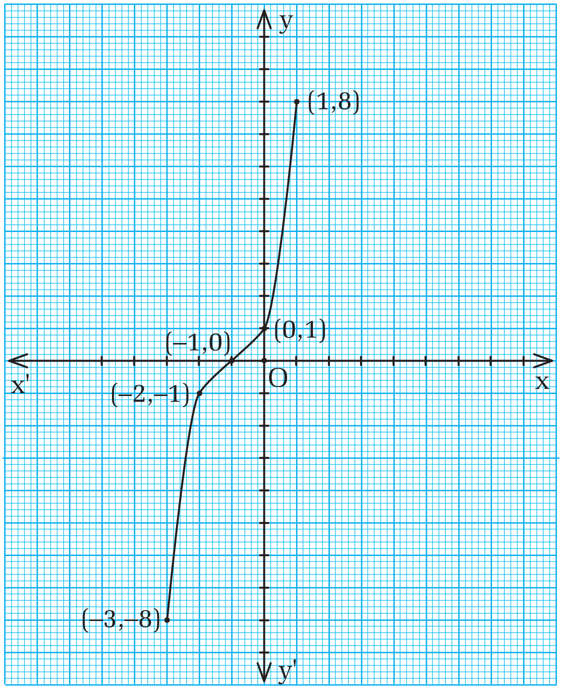
Solution

Here, $y=(x+1)^3$..............(i)

Table

x	–3	–2	–1	0	2
y	–8	–1	0	1	8

The points from the above table are plotted ia graph and joined them, we get a curve as shown in the graph alongside.



(d) $y=-(x+1)^3$

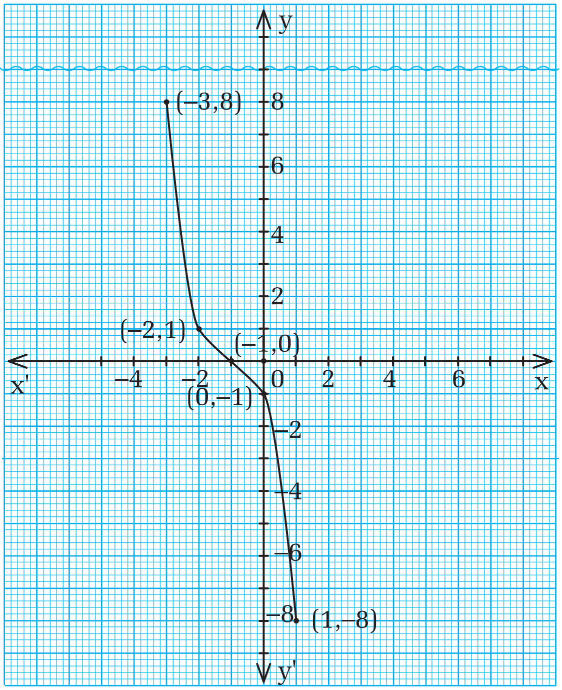
Solution

Here, $y=-(x+1)^3$.=.............(i)

Table

x	–3	–2	–1	0	1
y	8	1	0	–1	–8

The points from the above table are plotted ia graph and joined them, we get a curve as shown in the graph alongside.



5.Draw the graphs for the following quadratic functions

(a) $y=4x^2-8x+3$

Solution

This equation is in the form of $y=bx^2$ $-8b+c$, where, a=4, b=8 and c=3.

$$\text{Vertex}=\left(-\frac{b}{2a},\frac{4ac-b^2}{4a}\right)$$

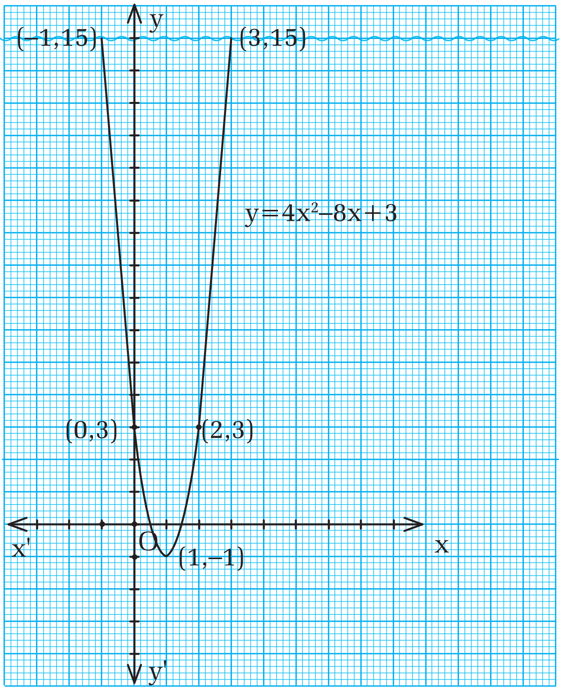
$$=\left(-\left(\frac{-8}{2.4}\right),\frac{4.1.4-(-8)^2}{4.1}\right)$$

$=(1,-1)$

Table

x	–3	–2	–1	0	1	2	3
y	69	35	15	3	–1	3	15

Plotting the above points in the graph paper and joining them, we get a curved curve as shown alongside.

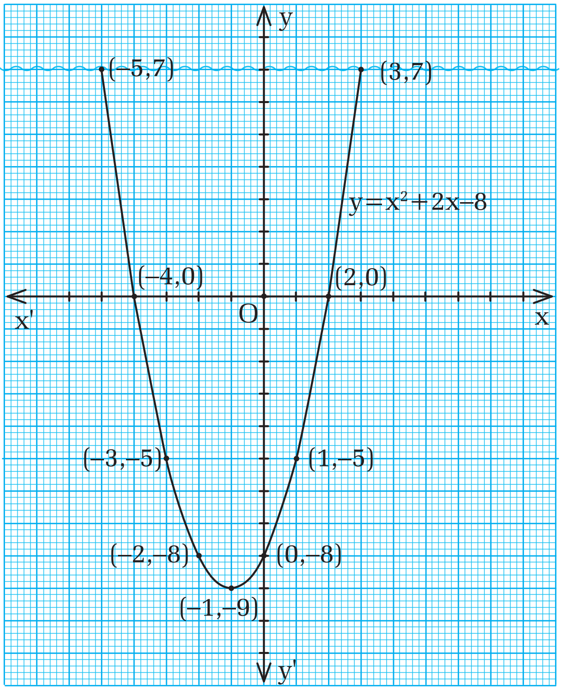


(b) $y=x^2+2x-8$

Solution

Here,

$y=x^2+2x-8$.............(i). It is in the form of $y=bx^2+8b+c$, where, a=1, b=2 and c=–8.



$$\text{Vertex}=\left(-\frac{b}{2a},\frac{4ac\text{-}b^2}{4a}\right)$$

$$=\left(-\frac{2}{2.1},\frac{4.1.(-8)-2^2}{4.1}\right)$$

$=(-1,-9)$

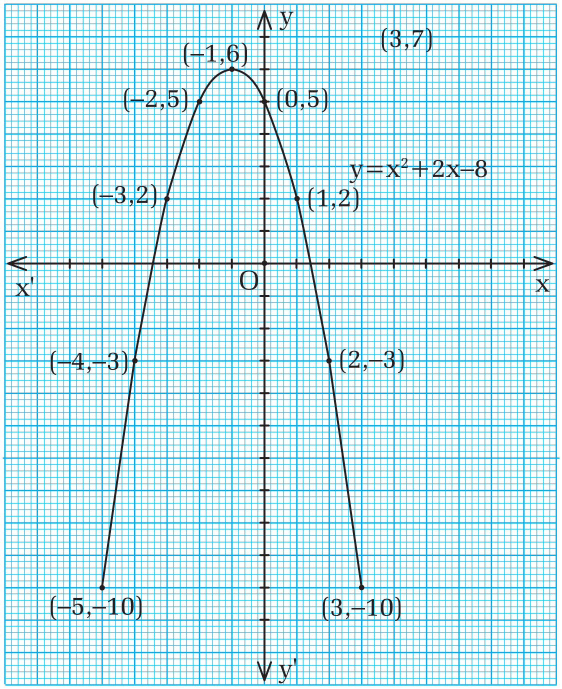
x	1	2	3	0	–1	4	–2	–3	–4	–5
y	–5	0	7	–8	–9	16	–8	–5	0	7

Plotting the points from above table and joining them, we get a curved curve as shown alongside.

(c) $y=-x^2-2x+5$

Solution

The equation is, $y=-x^2-2x+5$.............(i). It is in the form of $y=bx^2+bx+c$, where, a=–1, b=–2 and c=5.



$$\text{Vertex}=\left(\text{-}\frac{b}{2a},\frac{4ac\text{-}b^2}{4a}\right)$$

$$=\left(\frac{-2}{2.1},\frac{4(-1)(5)-(-2)^2}{4\times(-1)}\right)$$

$=(-1,6)$

Table

x	–5	–4	–3	–2	–1	0	1	2	3
y	–10	–3	2	5	6	5	2	–3	–10

Plotting the points from above table and joining them, we get a curved curve as shown alongside.

(d) $y=4x^2+8x+5$

Solution

The eq^n. is $y=4x^2+8x+5$..............(i). It is in the form of $y=ax^2-bx+c$, where, a=4, b=8 and c=5.

$$\text{Vertex}=\left(-\frac{b}{2a},\frac{4ac-b^2}{4a}\right)$$

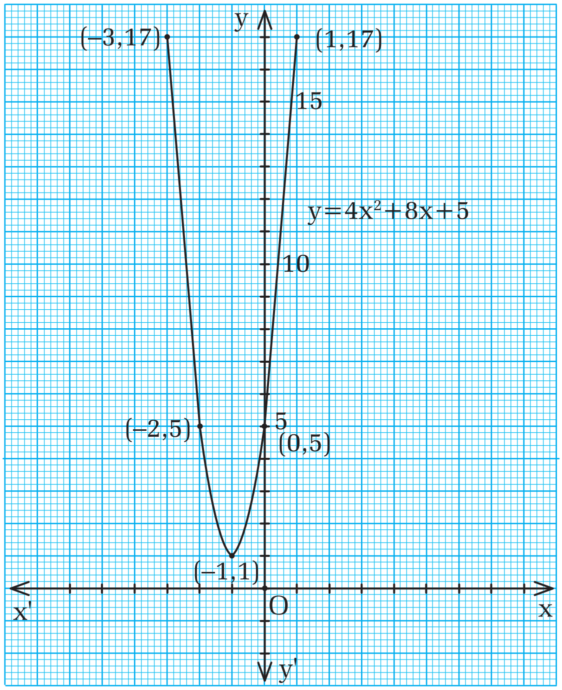
$$=\left(-\frac{-8}{2.4},\frac{4\times4\times5-(8)^2}{4\times(-1)}\right)$$

$=(-1,1)$

Table

x	–3	–2	–1	0	1
y	17	5	1	5	17

Plotting the points from above table and joining them, we get a curve as shown alongside.



(e) $y=-(x+3)^2$

Solution

The eqn is $y=-(x+3)^2$..............(i).

or, $y=-[x^2+6x+9]$

or, $y=-x^2-6x-9$..............(ii)

To find vertex of parabola,

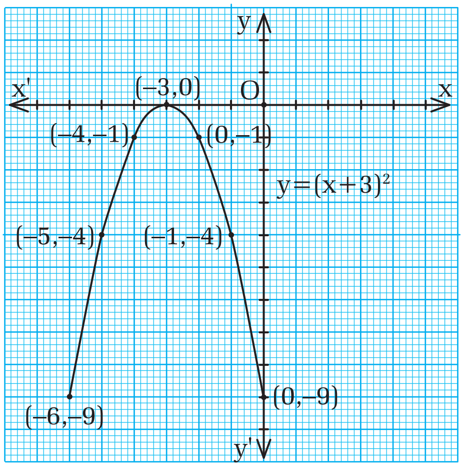
$$\text{Vertex}=\left(-\frac{b}{2a},\frac{4ac-b^2}{4a}\right)$$

$$=\left(\frac{6}{-2},\frac{4\times9-(-6)^2}{-4}\right)$$

$=(-3,0)$

eq^n. (i) in table;

x	–6	–5	–4	–3	–2	–1	0
y	–9	–4	–1	0	–1	–4	–9



6. Find the equation of parabola under the given conditions

(a). Vertex at O(0,0) and passing through (2,3).

Solution

The eq^n. of the parabola whose vertex is at origin is given by:–

(i) $y=ax^2$

The line/parabola passes through (2,3)

$\therefore\ 3=a.(2)^2$

or, $\frac{3}{4}=a$

$\therefore a=\frac{3}{4}$

$\therefore$ The eq^n. is, $y=ax^2$

put the value of a in the eq^n.

$y=\frac{3}{4}.x^2$

$\therefore y=\frac{3x^2}{4}$ is the required equation.

6.(b) Vertex =(2,–1), passing through (1,0).

Solution

Let, the eq^n. of the parabola be,

$y=a(x-h)^2+k$............(i)

vertex=(h,k)=(2,–1)

$\therefore$ the eq^n. is

$y=a(x-2)^2-1$............(ii)

The parabola passes through (1,0)

$\therefore$ Put value of x and y such that x=1.y=0,

$0=a(1-2)^2-1$

or. $0=a(-1)^2-1$

or, $1=a$

Put the value of a in (ii)

$y=1(x-2)^2-1$

or, $y=x^2-4x+4-1$

or, $y=x^2-4x+3$

$\therefore y=x^2-4x+3$, is the required eq^n.

6.(c) Vertex =(–1,–4), passing through (2,5).

Solution

let, vertex =(–1,–4)=(h,k).

Let, the eq^n. of the parabola be,

$y=a(x–h)^2 +k$............(i)

Put value of hand k in equation (i), we get

$y=a(x+1)^2 +(–4)$

or, $y=a(x+1)^2 –4$............(ii)

(2,5) lies in the parabola.

when $x=2$, $y=5$ in (ii)

$\therefore$ Put value of x and y such that $x=1.y=0$,

$5=a(x+1)^2 –4$

or. $9=a(2+1)^2$

or, $9=a\times9$

$\therefore a=1$

Put the value of a in (ii)

$y=1(x+1)^2 –4$

or, $y=x^2\ 2x+1–4$

or, $y=x^2 +2x33$

$\therefore y=x^2 +2x–3$, is the required eq^n.

6.(d) passes through the points (2,4),(–1,–2) and (3,10).

Solution

let, the eq^n. of parabola be :

$y=ax^2 +bx+c$............(i)

(2,4) lies in eq^n. (i)

when $x=2$, $y=4$ in (i);

$4=a.(2)^2 +b(2)+c$

or, $4=a\times4 +2b+c$

$\therefore$ $4=4a +2b+c$............(ii)

(–1,–2) lies in eq^n. (i)

when $x=–1$, $y=–2$ in (i);

$–2=(2)^2 a +b(–1)+c$

$–2=a – b+c$......(iii)

(3,10) lies in eq^n. (i)

ie, when $x=3$, $y=10$ in (i);

$10=a(3)^2 +b(3)+c$

$\therefore$ $10=9a +3b+c$......(iv)

Subtract (ii) from (iii)

$$
\begin{array}{rcl}
-2 &=& a - b + c \\
4 &=& 4a + 2b + c \\
- && - \quad - \quad - \\
\hline
-6 &=& -4a+a-2b-b
\end{array}
$$

or, $-6=-3a-3b$

or, $a+b=2$............(v)

Subtract (iv) from (iii)

$$\begin{array}{rcl} -2 &=& a-b+c \\ 10 &=& 9a+3b+c \\ - && - \quad - \quad - \\ \hline -12 &=& -9a+a-3b-b \end{array}$$

or, $-12=-8a-4b$

or, $3=2a+b$

$2a+b=3$ .... (vi)

Subtract (vi) from (v);

$$\begin{array}{rcl} a+b &=& 2 \\ 2a+b &=& 3 \\ - \quad - && - \\ \hline -a &=& -1 \end{array}$$

$\therefore a=1$

Put the value of a in (iv);

$3=2(1)+b$

$\therefore b=1$

Put the value of a and b in (iii);

$-2 = 1-1+c$

$\therefore c=-2$

Put the value of a, b and c in (i)

$y=1\times x^2+1\times x-2$

or, $y=x^2+x-2$

which is the required equation.

6(e). passes through the points (1,4),(2,1) and (4,1).

Solution

let, the parabola represented by (a) (i)

$y=ax^2+bx+c$............(a)

(1,4) lies in eq^n. (i)

when $x=1$, $y=4$ in (i);

$4=a.(1)^2+b(1)+c$

or, $4=1.a+1.b+c$

$\therefore 4=a+b+c$............(i)

(2,1) lies in eq^n. (a)

when $x=2$, $y=1$

or, $1=a.(2)^2+2.b+c$

or, $1=4a+2b+c$......(ii)

Subtract (ii) from (i)

$$\begin{array}{r} 4 = a + b + c \\ \underset{-}{1} = \underset{-}{4a} + \underset{-}{2b} + \underset{-}{c} \\ \hline 3 = -3a - b \end{array}$$

$\therefore 3a+b=-3$....................(iv)

(4,1)0) lies in eq^n. (i)

ie, when x=4, y=1;

$1=a.(4)^2 +4.b+c$

$\therefore$ 1=16a +4b+c......(iv)

Subtract (iv) from (iii)

$$\begin{array}{r} 1 = 4a + 2b + c \\ \underset{-}{1} = \underset{-}{16a} + \underset{-}{4b} + \underset{-}{c} \\ \hline 0 = -12a - 2b \end{array}$$

or, 6a + b =0

or, 3=2a+b.............(v)

Subtract (iii) from (v);

$$\begin{array}{r} 6a + b = 0 \\ \underset{-}{3a} + \underset{-}{b} = \underset{+}{-3} \\ \hline 3a = 3 \end{array}$$

$\therefore$ a=1

Put the value of a in (v);

$6 \times 1 + b = 0$

$\therefore$ b=–6

Put the value of a , b and c in (i);

a + b+ c =4

or, 1–6+c=4

or, –5+c =4

$\therefore$ c= 9

Put the value of a, b and c in (a)

$y=1\times x^2 - 6\times x +9$

or, $y=x^2- 6x + 2$

$\therefore y=x^2- 6x + 2$ which is the required equation.

7. (a) From the given graphs of parabola, find their equations.

Solution

From the graph, it can be cleared that

(0,0) is the vertex of the parabola (1,1),(2,4) and (3,9) lies in the parabola.

Since the vertex is origin,

the eq^n. is given by,

$y=ax^2$............(i)
(1,1) lies in the parabola,
when y=1, x=1,
$1=a\times1$
$\therefore a=1$
The value of a also satisfies the condition
For (2,4) and (3,9) according to the equation.
$\therefore$ The eq^n. of the parabola
$y=1\times x^2$
i.e., $y=x^2$

(b)

Solution
here, vertex =(–1,)
let, (–1,–4) = (h,k)
Also, (–2,–3),(0,–3), (–3,0), (1,0) and (–4,5) lies in the parabola.
let, the eq^n. of the parabola be
$y=a(x-h)^2 + k$...............(i)
where,
(h,k)=vertex
Then,
When h=–1, k=–4 in (i),
$y=a(x+1)^2 + k$...............(ii)
(–2,–3) lies in the parabola,
when x=–2, y=–3 in (ii);
$-3=a(-2+1)^2 -4$
or, $1 = a\times1$
$\therefore a=1$
put the value of a , b and k in (i);
$y=1(x+1)^2 -4$
or, $y=x^2 +2x+1 - 4$
or, $y=x^2 +2x - 3$ which is the required eq^n.

Solution of linear and Quadratic Equations

Some solved problems

1. Solve the following equations by substitution method.

(a) $y=x^2-3x$ and $y=x$

Solution

Given equation are

$y=x^2-3x$ ....(i)

and $y=x$..............(ii)

put the value of y in equation (i), from (ii), we get,

$x=x^2-3x$

or, $x^2-4x=0$

or, $x(x-4)=0$

Either $x=0$

or, $x=4$

$\therefore x=0,4$

when $x=0$, then $y=0$

when $x=4$, then $y=4$

The required solutions are (0,0) and (4,4).

(b) $y=x^2+7x+12$. $x-y+7=0$

Solution

Given equation are

$y=x^2+7x+12$ ....(i)

or, . $x-y+7=0$..............(ii)

From equation (ii) $y=x+7$

put the value of y in equation (i), from (ii), we get,

$x+7=x^2+7x+12$

or, $x^2+6x+5=0$

or, $x^2+5x+x+5=0$

or, $x(x+5)+1(x+5)=0$

or, $(x+5)(x+1)=0$

Either $x+5=0$, $\therefore x=-5$

or, $x+1=0$ $\therefore x=-1$

$\therefore x=0,4$

when $x=-5$, then $y=-5+7=2$

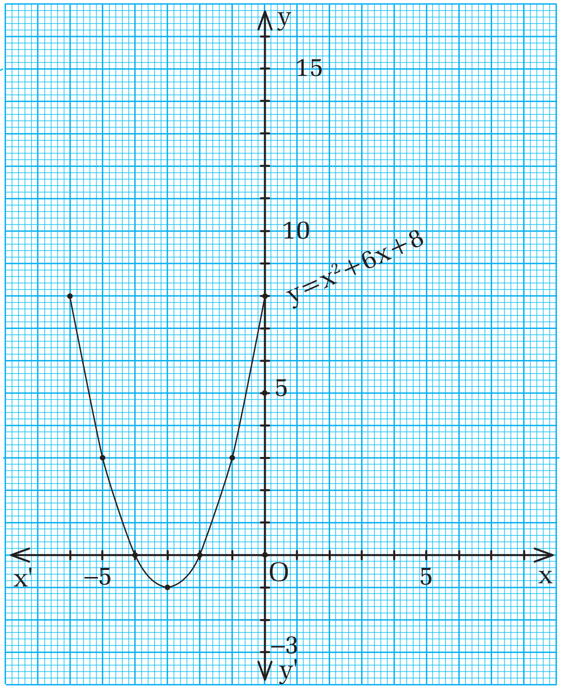
when $x=-1$, then $y=-1+7=6$

Hence the required solution (–5,2) and (–1,6).

2. Solve the folowing equations graphically

(a) $x^2 + 6x + 8 = 0$

Solution



Let $y=x^2+6x+8=0$, i.e. $y=0$

$y=x^2+6x+8=0$ represents an equation of parabola, comparing at with

$y=ax^2+bx+c$, we get, a=1, b=6 and c=8

First let us find the vertex of the parabola.

x–coordinate of vertex (h)$=-\dfrac{b}{2a}$

$=-\dfrac{6}{2.1}$

$=-3$

y–coordinate of vertex (k) $=\dfrac{4ac-b^2}{4a}$

$=\dfrac{4\times1\times8-6^2}{4.1}$

$=\dfrac{32-36}{4}$

$=-1$

$\therefore$ Vertex (h,k)=(–3,–1)

Table to draw curve of parabola

x	–5	–4	–3	–2	–1	0	1
y		0	–1	0	3	8	15

From the above curve we observed that the curve cuts the x–axis at points (–2,0) and (–4,0).

$\therefore$ x = –2,–4

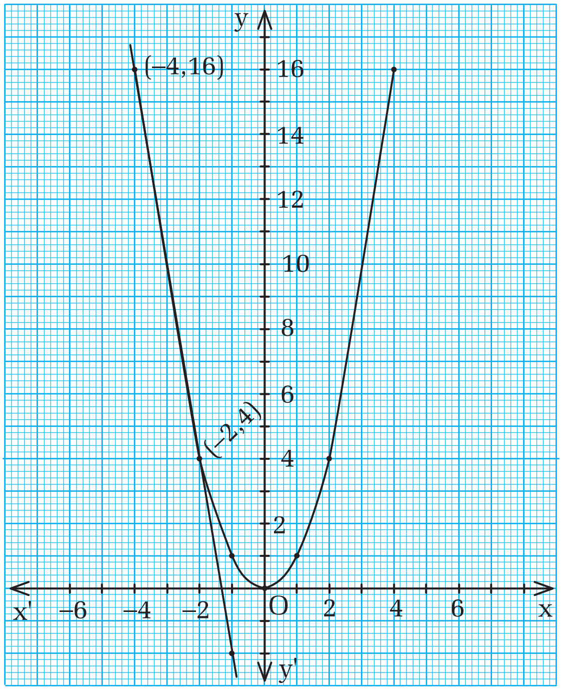
Alternate Method

Here, $x^2 + 6x + 8 = 0$ or, $x^2 = -6x = 8$

Let $y = x^2 = -6x-8$

Then, $y=x^2$................(i) and $y=-6x-8$.............(ii)

The equation(i) represents a parabola and (ii) represents a straight line. From equation (i)



x	±4	±3	±2	±1	0
y	16	9	4	1	0

plotting the above points on graph paper and joining them, we get a curve of parabola.

From equation (ii) $y=-6x-8$

x	–1	–2	–3
y	–2	4	10

plotting there points on the same graph, we get a straight line. The straight line cuts the curve of parabola at points (–2,4) and (–4,16). Hence the required solution of given equation area =–2,–4

(b) $x^2 - x - 2 = 0$

Solution

Let $x^2 - x - 2 = 0$,

i.e. $y = x^2 - x - 2$, $y=0$

It represents an equation of parabola, in the term of $y=ax^2+bx+c$, where, $a=1$, $b=-1$, $c=-2$

Let us find the vertex of the parabola.

Vertex $(h,k) = \left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

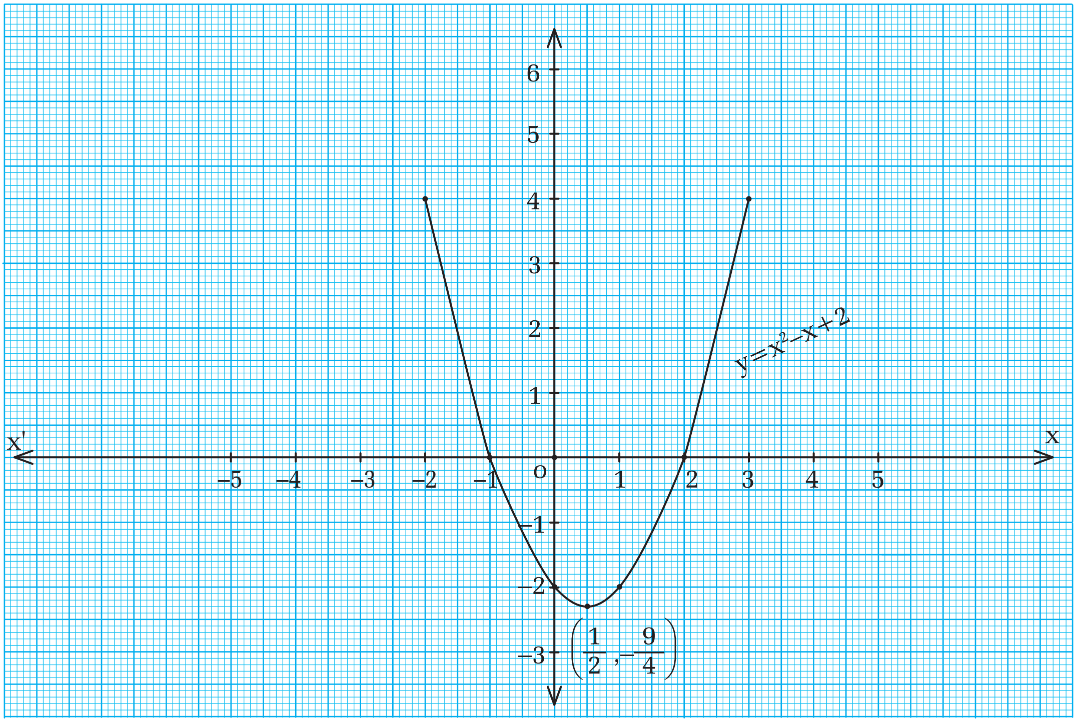
$= \left(-\frac{(-1)}{2.1}, \frac{4.1.(-2)-(-1)^2}{4.1.(-2)}\right)$

$= \left(\frac{1}{2}, -\frac{9}{4}\right)$

To draw a curve of parabola

x	–2	–1	0	1	2	3
y	4	0	–2	–2	0	4

The points from the above table are plotted in a graph and joined them, we get a curve of parabola.



The curve cuts x–axis at points (–1,0) and (2,0). Hence the required solutions are x=–1,2.

3. Solve the following equations graphically.

(a) $y= x^2 - 5$ and $y=2x+3$

Solution

Given equations are

$y= x^2 - 5$ ....(i)

and $y=2x+3$ ....(ii)

Equation (i) represents a parabola and (ii) represents a straight line. The point of the intersection of the curve of parabola and straight line is the solution of given equations.

From equation (i),

Vertex $(h,k)=\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

$=\left(-\frac{0}{2.1}, \frac{4.1.(-5)-0}{4.1}\right)$

$=(0,-5)$

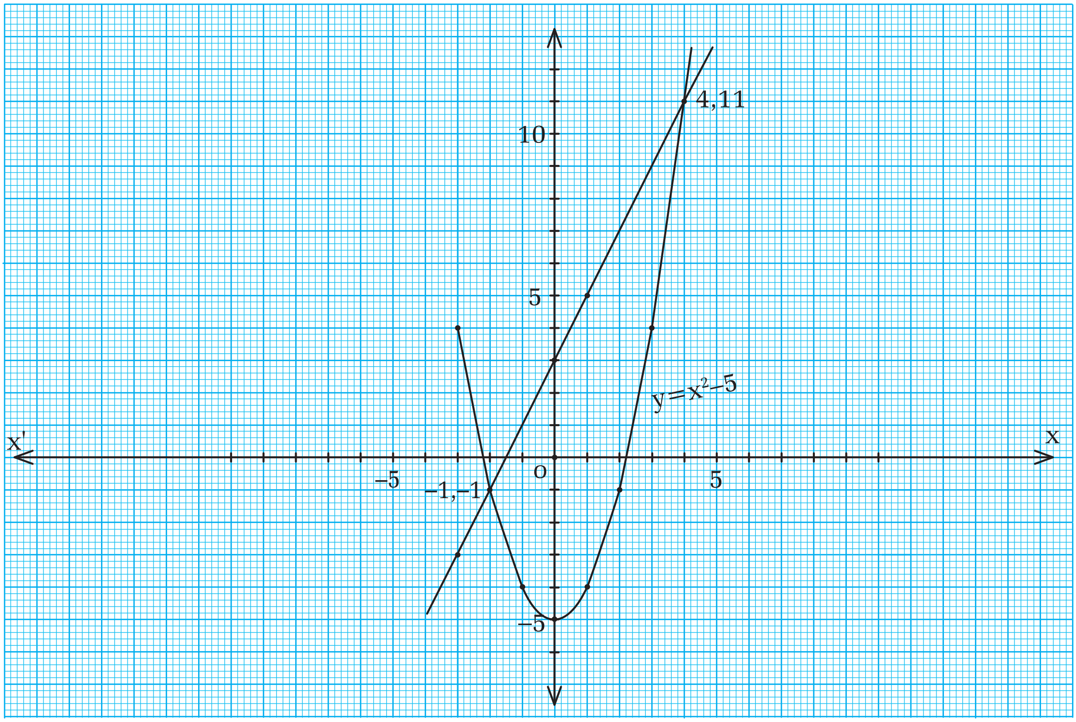
To draw the curve of parabola

x	–3	–2	–1	0	1	2	3
y	4	–1	–4	–5	–4	–1	4

Plotting the points on graph and joining them we get a curve of parabola.

From equation (i) $y=2x + 3$

x	–3	0	1
y	3	3	5



From the graph, we observed that the point of intersection of the parabola and the straight lines are (–2,–1) and (4,11).

Hence the required solutions are (–2,–1) and (4,11).

(g) $y=4x^2+8x+5$, $x+y=3$

Solution

Given equations are

$4x^2+8x+5$ ....(i)

and $x + y=3$...............(ii)

In above equation, equation (i) represents a parabola and equation (ii), represents a straight line.

From equation (i), $y=4x^2+8x+5$

x	–3	–2	–1	0	1
y	17	5	1	5	17

Vertex of parabola (h,k)

$$=\left(-\frac{b}{2a},\frac{4ac-b^2}{4a}\right)$$

$$=\left(-\frac{8}{2.4},\frac{4.4.5-8^2}{4.4}\right)$$

$=(-1,1)$

Then plotting above points from table and joining them, we get a curve of parabola.

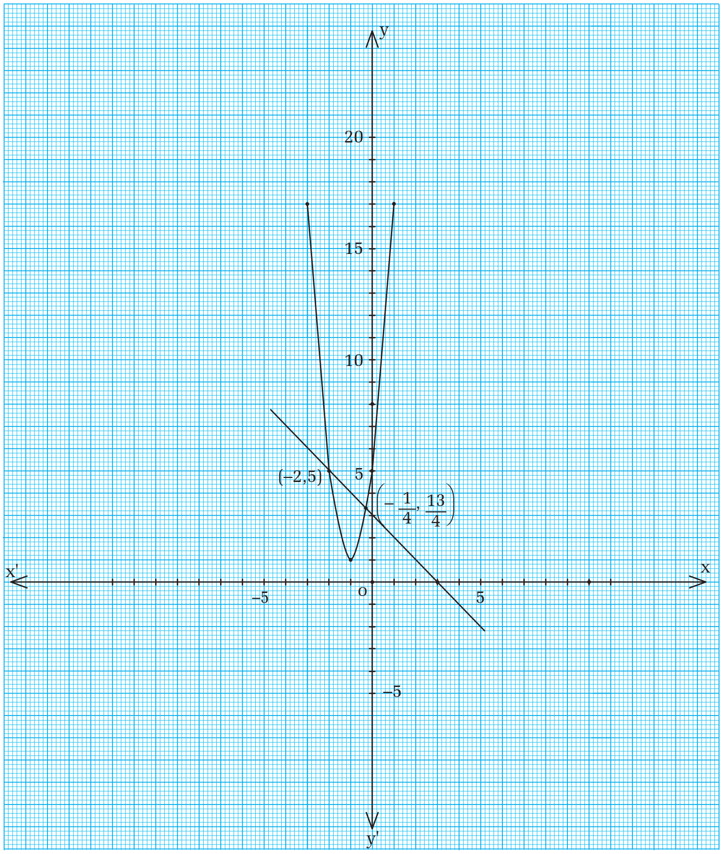
From equation(ii), $y=3-x$

x	0	–1	3
y	3	4	0

plotting these points in same graph, we get a straight line.

From the graph, the curve and the straight lines intersects at points (–2,5) and $\left(-\frac{1}{4},\frac{13}{4}\right)$.

Note. If the point of intersection of a curve and a straight line is not clear to read their coordinates, we can solve them by substitution method in rough.



Questions for practice

1. Solve graphically (a) $x^2 - 8x + 12 = 0$

(b) $x^2 - 9x - 10 = 0$

(c) $x^2 + 4x + 3 = 0$

2. Solve the following equations graphically.

(a) $y = x^2$, $y = 2$

(b) $y = x^2 - 6x + 9$, $3x + 4y = 12$

(c) $y = x^2 - 2x$, $y = x + 2$

(d) $y = x^2 - 2x - 15$, $25x - 8y + 20 = 0$

UNIT six

Continuity

1. Objectives

S.N.	Level	Objectives
i)	Knowledge (K)	– To define natural numbers, rational numbers, whole numbers – To define continuity by using graphs. – To define discontinuity by using graphs.
ii)	Understanding(U)	To say meaning of continuity. To define and check continuity or discontinuity of numbers in number line. To identify continuity or discontinuity of given graphical figures. To define continuity of a function.
iii)	Application (A)	To discuss the continuity or discontinuity of given functions given in graphs or equations.
iv)	Higher Ability (HA)	To examine the continuity or discontinuity of given functions at given points calculating functional values and limits.

2. Teaching materials

- Number lines with natural numbers, whole numbers, integers.
- diagrams in graphs to discuss continuity or discontinuity with intervals.

3. Teaching Learning Strategies

- Review the concept of real number system.
- Draw the number lines to show the following.

i) natural numbers ii) integers iii) whole numbers.

- Discuss the continuity or discontinuity of the number drawn in above number lines.
- Give concept intervals with diagrams (open interval, left open interval, right open intervals, closed intervals)
- Discuss the continuity or discontinuity of given graphical diagrams with intervals.
- Review the meaning of , $\lim\limits_{x \to a^+} f(x)$, $\lim\limits_{x \to a^-} f(x)$, f(a) with an example.
- Define continuity of a function at a point.
- Review the concept of existence of limit of a function at a point.
- Explain continuity or discontinuity of a function at a given point calculating
- functional value f(a), at x=a
- right hand limit, $\lim\limits_{x \to a^+} f(x)$.
- left hand limit, $\lim\limits_{x \to a^-} f(x)$.

Note :

1) In mathematics, the word "continuous " applies to functions not in sets.
2) The continuity of a simple function can be checked by drawing a curve. If there is no breakage at any point on the curve, then the function is continuous.
3. If there is a breakage or a hole on the given curve, then it is discontinuous at that point.
4. A function f(x) is said to be continuous at x =0, if the following conditions are satisfied.

i) f(a) exists or f(a) is finite

ii) $\lim_{x \to a} f(x)$ exists ie. $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

iii) $\lim_{x \to a} f(x) = f(a)$

If any one of the above conditions fails, then the function is said to be discontinuous at that point.

Some solved problems

1. In the following given curves.

(a) (i) Find the initial and the terminating points of the curve.

(ii) State the continuity or discontinuity of the curve.

Solution

(Graph 1(a) page 117)

i) The initial point is x =1 and terminating point is x=14.

ii) The given curve is discontinuous at x =10

graph 1(b)

(b) i) The initial paint is x=0 and the terminating point is x=6.

ii) The the points of discontinuity are x=2 and x=4.

graph 1(c)

(c) i) The initial point is x=0 and terminating point is x=7

ii) The straight line is continuous.

2. Discuss the continuity and discontinuity of the following curves from point x=–6 and x=6. (stating the intervals for continuity and points of discontinuity for discontinuity).

(a) Page 118 graph of 2 (a)

interval	point of discontinuous
Continuous in interval [–6,–1]	x=1
Continuous in interval [–1,6]	

(b) Graph of page 118 2(c)

interval	point of discontinuous
Continuous in interval (–6,3)	x=–3

Continuous in interval (–3,2)	x=2
Continuous in interval (2,6)	

3. Write a sentance for each of the following notation.

(a) $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^{-0}} f(x)$

Solution

Left hand limit of function f(x) at x=a is denoted by $\lim_{x \to a^-} f(x)$

or

It denotes the left hand limit of f(x) at x=a

(b) $\lim_{x \to a} f(x)$

Solution

The limit of function f(x) at x=a is denoted by $\lim_{x \to a} f(x)$

It denotes the limit of f(x) at x =a.

(c) $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

Solution

The limit of f(x) at x=a exists.

The left hand limit and the right hand limit of f(x) at x = a are equal.

(d) write conditions for continuity of a function f(x) at x=a, using notations.

Solution

The following are the required conditions for continuity of a function f(x) at x=a.

i) f(a) is finite

ii) limit of f(x) at x=a exists

ie. $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, $\lim_{x \to a} f(x)$ is finite.

iii) $f(a) = \lim_{x \to a} f(x)$

page 121(2) long question

4. Let f:R⟶R be a real valued function defined by f(x)=x+4

(a) For x=3.9, 3.99, 3.999, 3.9999, find the value of f(x).

(b) For x=4.1, 4.01, 4.001,4.0001, find the value of f(x)

(c) Find the value of f(x) at x=4.

(d) Find the values of $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$

(e) Does limit of the function f(x) exists at x=4 ?

(f) Write the notation to show above function is continuous at x=4.

Solution

Here, f(x)= x+4

(a) f(3.9)=3.9+4=7.9

f(3.99)=3.99+4=7.99

f(3.999)=3.999+4=7.999

f(3.9999)=3.9999+4=7.9999

(b) f(4.1) =4.1 +4 =8.1

f(4.01) =4.01 +4 =8.01

f(4.001) =4.001 +4 =8.001

f(4.0001) =4.0001 +4 =8.0001

(c) f(4) =4+4 =8

(d) $\lim\limits_{x \to 4^-} f(x) = 4 + 4 = 8$

$\lim\limits_{x \to 4^+} f(x) = 4+4=8$

(e) (i) f(4)=8

(ii) Now, $\lim\limits_{x \to 4^+} f(x) = 8$

$\lim\limits_{x \to 4^-} f(x) = 8$

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^+} f(x)$

limit of the function exists at x =4

ie. $\lim\limits_{x \to 4} f(x) = 8$

(iii) $f(4)= \lim\limits_{x \to 4} f(x)$

Hence f(x) is continuous at x =4.

5. Let $f : R \to R$ be a real valued function defined by $f(x) = \begin{cases} x+3, & 1 \le x < 2 \\ 4x-3 & x \ge 2. \end{cases}$ at x=2

(a) Find $\lim\limits_{x \to 2^-} f(x)$

(b) $\lim\limits_{x \to 2^+} f(x)$

(c)Is $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$

(d) Find f(2).

(e) Draw your conclusions

Solution

Here, $f(x) = \begin{cases} x+3, & 1\le x<2 \\ 4x-3, & x\ge 2. \end{cases}$ at $x=2$

(a) For left hand limit, we take

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+3) \qquad (\therefore 1\le x<2)$$

$=2+3=5$

(b) For right hand limit, we take

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x-3) \qquad (\therefore x\ge 2)$$

$=4\times 2-2= 8-3=5$

(c) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

(d) For functional value, we take

$f(x) = 4x-3, \qquad (\therefore x\ge 2)$

$f(2) = 4\times 2-3 =5$

(e) From (a), (b), (c) and (d), we get.

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) =5$$

i.e $\lim_{x\to 2} f(x) = 5$

$f(2) =5$

and $\lim_{x\to 2} f(x) =f(2)$

Hence f(x) is continuous at x=2.

6. Discuss the continuity of the function f(x) at x=2.

$f(x) = \begin{cases} 2x - 1, & \text{when } x<2 \\ 3 & \text{, when } x=2 \\ x+1 & \text{, when } x > 2. \end{cases}$ at $x = 2$.

Solution

Here, $f(x) = \begin{cases} 2x - 1, & \text{when } x<2 \\ 3 & \text{, when } x=2 \\ x+1 & \text{, when } x > 2. \end{cases}$ at $x = 2$.

For $x < 2$, we take $f(x)=2x-1$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2x-1) = 2.2 - 1 = 3$

For x=3, we have, $f(2) = 3$

For $x>2$, we take, $f(x) = x + 1$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} x + 1 = 2 + 1 = 3$

we have,

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x) = 3$

i.e. $\lim\limits_{x \to 2} f(x) = 3$

From above, we get

$f(2) = \lim\limits_{x \to 2} f(x)$

Hence the given function f(x) is continuous at $x = 2$.

Some solved problems

1. Examine the continuity or discontinuity of the following functions at the points mentioned.

(a) $f(x) = 4x + 1$, at $x=3$.

Solution

Functional value at $x = 3$, $f(3)=4\times3 + 1=13$

Also, $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} (4x+1)$

$= 4\times3 + 1$

$= 12+1$

$= 13$

$\therefore f(3) = \lim\limits_{x \to 3} f(x)$

Hence f(x) is continuous at $x = 3$.

(b) $f(x)=\dfrac{x^2-64}{x-8}$

Solution

Here, $f(x)=\dfrac{x^2-64}{x-8}$

For x=8, $f(8)=\dfrac{x^2-64}{x-8}=\dfrac{0}{0}$ which is not finite.

ie. the functional value of f(x) at $x = 8$ does not exists. Hence f(x) is discontinuous at $x = 8$.

2. Examine the continuity or discontinuity of the following functions at the points mentioned.

(a) $f(x) = \begin{cases} \dfrac{x^2 - 7x}{x - 7}, & \text{when } x \neq 7 \\ 3, & \text{When } x = 7 \end{cases}$ at $x = 7$

For $x \neq 7$, we take limit of the function when $x \to 7$.

$\lim\limits_{x \to 7} f(x) = \lim\limits_{x \to 7} \dfrac{x^2 - 7x}{x - 7}$

$= \lim\limits_{x \to 7} \dfrac{x(x-7)}{(x-7)}$

$= \lim\limits_{x \to 7} x$

$= 7$

Functional value at $x = 7$ is given as 3.

ie. $f(7) = 3$

$\therefore f(7) \neq \lim\limits_{x \to 7} f(x)$

Hence the function f(x) is discontinuous at x=7.

Note:

To calculate limit of a function at x=a, if the function take the form of $\frac{0}{0}$, we factorize the numerator and denominator if possible. In this case we do not put the value of x directly.

Example Evaluate $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x - 5}$

Here, if we put x=5, we get $\frac{0}{0}$ forms which is not finite. In sense of limit, $x \to 5$ means, the value of x is slightly equal to 5 but not exactly equal to 5.

Now, $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x - 5}$ ($\frac{0}{0}$ forms)

$= \lim\limits_{x \to 5} \dfrac{(x+5)(x-5)}{(x-5)}$

$= \lim\limits_{x \to 5} (x+5)$

$= 5 + 5$

$= 10$

$\therefore$ 10 is the limit of f(x) at x=5.

(b) $f(x) = \begin{cases} \dfrac{x^2 - 2x}{x - 2}, & \text{when } x \neq 2 \\ 2, & \text{When } x = 2 \end{cases}$ **at x = 2**

Solution

Here, $f(x) = \begin{cases} \dfrac{x^2 - 2x}{x - 2}, & \text{when } x \neq 2 \\ 2, & \text{When } x = 2 \end{cases}$ at x = 2

Functional value at x = 2

$f(2) = 2$

Limit of f(x) at x = 2

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 2x}{x - 2} \quad \left(\frac{0}{0} \text{ forms}\right)$$

$$= \lim_{x \to 2} \frac{x(x-2)}{x-2}$$

$$= \lim_{x \to 2} x$$

$= 2$

$\therefore f(2) = \lim_{x \to 2} f(x)$

Hence f(x) is continuous at x=2.

(C) $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{when } x \neq 3 \\ 5, & \text{When } x = 3 \end{cases}$ **at x = 3**

Solution

Here, $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{when } x \neq 3 \\ 5, & \text{When } x = 3 \end{cases}$ at x = 3

Functional value at x = 3

$f(3) = 5$

For limit of f(x) at x = 5

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} \quad \left(\frac{0}{0} \text{ forms}\right)$$

$$= \lim_{x \to 3} \frac{x^2-3x+2x-6}{x-3}$$

$$= \lim_{x \to 3} \frac{x(x-3)+2(x-3)}{x-3}$$

$$= \lim_{x \to 3} \frac{(x-3)(x+2)}{(x-3)}$$

$$= \lim_{x \to 3} (x+2)$$

$$= 3+2$$

$$= 5$$

$$\therefore f(3) = \lim_{x \to 3} f(x)$$

Hence f(x) is continuous at x=3.

(d) $\mathbf{f(x)} = \begin{cases} \dfrac{x^2-3x+2}{x^2+x-6}, & \text{when } x \neq 2 \\ \dfrac{1}{5}, & \text{When } x = 2 \end{cases}$ **at x = 2**

Solution

Here, $f(x) = \begin{cases} \dfrac{x^2-3x+2}{x^2+x-6}, & \text{when } x \neq 2 \\ \dfrac{1}{5}, & \text{When } x = 2 \end{cases}$ at x = 2

when $x = 2$, $f(x) = f(2) = \dfrac{2^2-3.2+2}{2^2+2-6}$

$= \dfrac{6-6}{6-6} = \dfrac{0}{0}$ form

For limit of f(x) at x = 2 we factorize the denominator and numerator of the function.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-3x+2}{x^2+x-6}$$

$$= \lim_{x \to 2} \frac{x^2-2x-x+2}{x^2+3x-2x-6}$$

$$= \lim_{x \to 2} \frac{x(x-2)-1(x-2)}{x(x+3)-2(x+3)}$$

$$= \lim_{x \to 2} \frac{(x-3)(x-1)}{(x+3)(x-2)}$$

$$= \lim_{x \to 2} \frac{(x-1)}{(x+3)}$$

$$= \frac{2-1}{2+3}$$

$$= \frac{1}{5}$$

Functional value is $\frac{1}{5}$ at x=2(given)

$$\therefore \lim_{x \to 2} f(x) = f(2)$$

Hence f(x) is continuous at x=2.

3. Examine the continuity or discontinuity the following functions at the points muntioned by calculating left hand limit (LHL), right hand limit (RHL) and functioned value.

(a) $f(x) = \begin{cases} 3 - x, & \text{for } x \le 0 \\ 3, & \text{for } x > 0 \end{cases}$ at $x = 0$

Solution

Here, $f(x) = \begin{cases} 3 - x, & \text{for } x \le 0 \\ 3, & \text{for } x > 0 \end{cases}$ at $x = 0$

For functional value at x = 0, we take

$f(x) = 3 - x$ ($\because$ $x \le 0$)

$f(0) = 3 - 0 = 3$

For right hand limit, we take

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3 = 3$$

$$=2+3=5$$

For left hand limit, we take

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3-x$$

$$= 3 - 0$$

$$= 3$$

$$\therefore \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

Hence the limit of f(x) at x = 0 exists.

Now, we take $f(0) = \lim_{x \to 0} f(x)$

Hence the function f(x) is continuous at x = 0.

(b) $f(x) = \begin{cases} 2x^2 + 1, & \text{for } x < 2 \\ 9 \quad , & \text{for } \ x=2 \\ 4x + 1, & \text{for } x > 2. \end{cases}$ at x = 2.

Solution

Here, $f(x) = \begin{cases} 2x^2 + 1, & \text{for } x < 2 \\ 9, & \text{for } \ x=2 \\ 4x + 1, & \text{for } x > 2. \end{cases}$ at x = 2.

For functional value at x = 2, we take

$f(2) = 9$

For right hand limit at x = 2, we have

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} 4x + 1$

$= 4\times2 + 1$

$= 9$

For left hand limit at x = 2, we have,

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 2x^2+1$

$= 2.2^2 + 1$

$= 9$

$\therefore \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x)$

The limit of the function f(x) exists at x = 2.

$\lim\limits_{x \to 2} f(2) = 9$

Now, $f(2) = \lim\limits_{x \to 2} f(x)$

$\therefore$ The function f(x) is continuous at x = 2.

4. Show that the following functions are continuous at the points mentioned.

$f(x) = \begin{cases} \dfrac{x^3 - 8}{x-2}, & \text{for } x \neq 2 \\ 12 \quad , & \text{for } x = 2 \end{cases}$ at x = 2

Solution

$$\text{Here}, f(x) = \begin{cases} \dfrac{x^3-8}{x-2}, & \text{for } x \neq 2 \\ 12, & \text{for } x = 2 \end{cases} \quad \text{at } x = 2$$

For $x \neq 2$, we take limit of the function f(x) at $x = 2$.

For limit of f(x) at $x = 2$ we factorize the denominator and numerator of the function.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3-8}{x-2}$$

$$= \lim_{x \to 2} \frac{x^3-2^3}{x-2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{x-2}$$

$$= \lim_{x \to 2} (x^2+2x+4)$$

$=2^2+2.2+4$

$=4+4+4$

$=12$

Functional value at $x = 2$ is given as 12

i.e. $f(2) = 12$

$$\therefore f(2) = \lim_{x \to 2} f(x)$$

$\therefore$ The function is continuous at $x = 2$.

5. A function is defined as follow

$$f(x) = \begin{cases} 3+2x, & \text{for } -\frac{3}{2} \leq x < 0 \\ 3-2x, & \text{for } 0 \leq x < \frac{3}{2} \\ -3-2x, & \text{for } x \geq \frac{3}{2} \end{cases}$$

Show that f(x) is continuous at $x = 0$ and discontinuous at $x = \frac{3}{2}$.

Solution

$$\text{Here}, f(x) = \begin{cases} 3+2x, & \text{for } -\frac{3}{2} \leq x < 2 \\ 3-2x, & \text{for } 0 \leq x < \frac{3}{2} \\ -3-2x, & \text{for } x \geq \frac{3}{2} \end{cases}$$

First let us discuss the continuity of f(x) at $x = 0$

Functional value for $x = 0$ is given by

$f(0) = 3 - 2\times0 = 3$ (as $x \geq 0$)

Left hand limit of f(x) at $x = 0$ is given by

$\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} (3+2x)$ (as $x < 2$)

$= 3 + 2\times0$

$= 3$

Right hand limit of f(x) at $x = 0$ is given by

$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} (3-2x)$ (as $x \geq 0$)

$= 3 - 2\times0$

$= 3$

$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^-} 4x-3$

i.e. the limit of the function f(x) exists at x = 0. ie. $\lim\limits_{x\to0} f(x) = 3$

Finally, we have,

$f(0) = \lim\limits_{x\to0} f(x)$

$\therefore$ f(x) is continuous at x=0.proved

Again let us discuss the continuity of the function f(x) at $x = \frac{3}{2}$.

Functional value of f(x) at $x = \frac{3}{2}$ is given by

$f(x) = -3-2x$ (as $x \geq \frac{3}{2}$)

$f(\frac{3}{2}) = -3-2\times\frac{3}{2}$

$= -6$

Left hand limit of f(x) at $x = \frac{3}{2}$ is given by

$\lim\limits_{x\to\frac{3}{2}^-} f(x) = \lim\limits_{x\to\frac{3}{2}^-} (3-2x)$ (as $x \leq \frac{3}{2}$)

$= 3 - 2.\frac{3}{2}$

$= 0$

Right hand limit of f(x) at $x = \frac{3}{2}$ is given by

$\lim\limits_{x\to\frac{3}{2}^+} f(x) = \lim\limits_{x\to\frac{3}{2}^+} f(-3-2x)$ (as $x \geq \frac{3}{2}$)

$= -3 - 2\times\frac{3}{2}$

$= -6$

$$\therefore \lim_{x \to \frac{3}{2}^-} f(x) \neq \lim_{x \to \frac{3}{2}^+} f(4x-3)$$

i.e. the limit of the function does not exists at $x = \frac{3}{2}$.

$\therefore$ The function is discontinuous at $x = \frac{3}{2}$. proved.

6. Find the value of m of f(x) is continuous at x =5

$$f(x) = \begin{cases} \dfrac{x^2-2x}{x-2}, & \text{for } x \neq 5 \\ m, & \text{for } x = 5 \end{cases} \quad \text{at } x = 5$$

Solution

Here, $$f(x) = \begin{cases} \dfrac{x^2-2x}{x-2}, & \text{for } x \neq 5 \\ m, & \text{for } x = 5 \end{cases} \quad \text{at } x = 5$$

Functional value of f(x) at x =5 is m.

i.e. f(5) =m

For limit of the function at x = 5, we take

$$\lim_{x \to 5} f(x) = \lim_{x \to 5} \frac{x^2-2x}{x-2}$$

$$= \lim_{x \to 5} \frac{x(x-2)}{(x-2)}$$

$$= \lim_{x \to 2} x$$

$=5$

Since the given function is continous at x =5, we take

$$\lim_{x \to 2} f(x) = f(5)$$

ie. 5=m

$\therefore$ m=5

7. Discuss the continuity of given function at x =2

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{for } x \neq 2 \\ 5, & \text{for } x = 2 \end{cases}$$

If f(x) is not continuous, how can your make it continuous at the point x =2.

Solution:

Here, $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{for } x \neq 2 \\ 5, & \text{for } x = 2 \end{cases}$

Functional value of f(x) at x =2 is 5.

i.e. f(2) =5

For limit of the function, we have

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} \quad \left(\frac{0}{0} \text{ forms}\right)$$

$$= \lim_{x \to 2} \frac{(x+3)(x-2)}{(x-2)}$$

$$= \lim_{x \to 2} (x+2)$$

$=2+2$

$=4$

$\therefore f(2) \neq \lim_{x \to 2} f(x)$

i.e. the function is discontinuous at x =2. To make the above function continuous at x =2, we can redefine at as follows

Here, $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{for } x \neq 2 \\ 4, & \text{for } x = 2 \end{cases}$

Questions for practice

1. Discuss the continuity of the following

(a) Growth of plants for certain period of time.

(b) A snake crawling on a ground.

(c) Motion of wheels of a motorcycle on the road when it is in motion.

(d) A frog jumping on a ground.

(e) traces of feet of a man when he walking on the road.

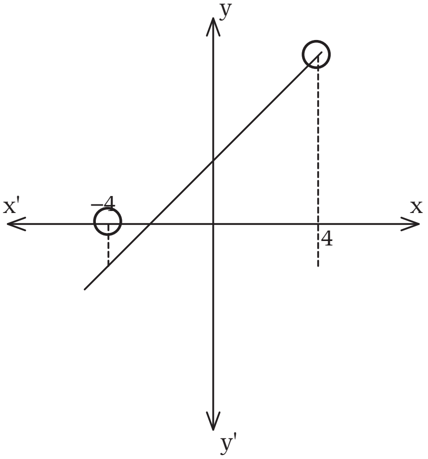
(AM:(a) Continuous (b) Continuous (c) Continuous (d) discontinuous (e) discontinuous

(f) Write the continuity of number line of set of real numbers. (continuity)

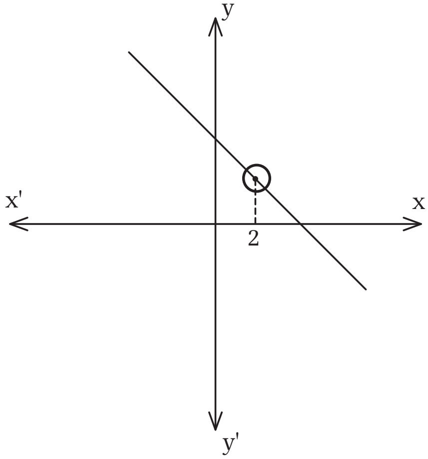
(g) Write the continuity of number in a number line.

2. State the continuity or discontinuity of the function from the given graph.

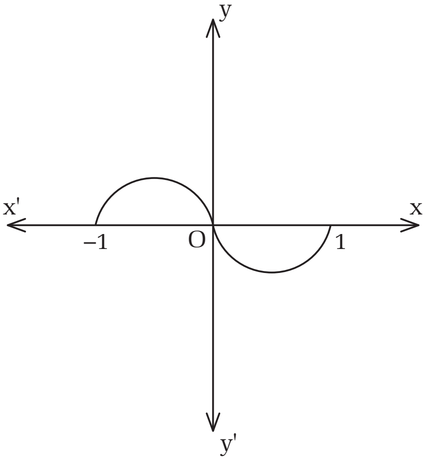
(a)



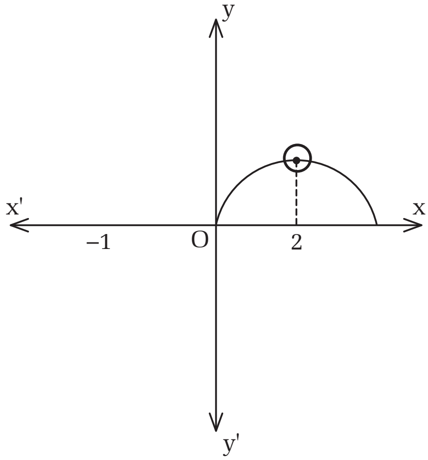
(b)



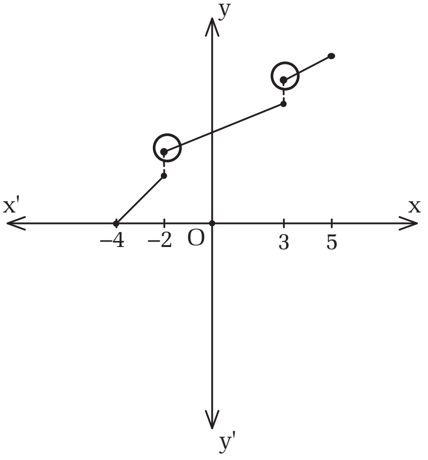
(c)



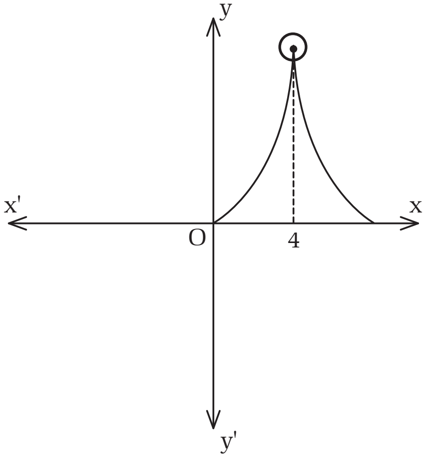
(d)



(e)



(f)



3. Let f be a real valued function defined by f(x)=x +8

(a) What are the values of f(x), for x=1.9, 1.99,1.999,1.9999.

(b) What are the values of f(x), for x=2.1,2.01,2.001,2.0001

(c) What are the left hand and right hand limit of the above function ? Write the limit of the function.

(d) Does the limit of the function exists at x =2 ?

(e) Can you say the function f(x) is continuous at x =2 ?
Also state the reason.

4. For a real valued function f(x)=2x+3

(a) Find the values f(2.95), f(2.99), f(2.999), f(3.01), f(3.001), f(3.0001), f(3).

(b) Is the function continuous for x=3 ?

(c) Write the conditions of continuity of above functions ?

5. For a real valued function f(x)=2x +3,

(a) Find f(4,9), f(4.99), f(4.999), f(4.9999).

(b) Write the left hand limit of the f(x) at x=5 with symbol.

(c) Find the values of f(5.1), f(5.01), f(5.001), f(5.0001)

(d) Write the right hand limit of the f(x) at x=5 with symbol.

(e) Find f(5).

(f) what conclusion can you draw from above ?

6. Given that $f(x) = \begin{cases} 3x + 1, & \text{for } x < 1 \\ 4, & \text{for } x = 1 \\ 5x - 1, & \text{for } x \geq 1. \end{cases}$

(a) Find the left hand and right hand limit of the function at x=1.

(b) Find the value of f(x) at x=1.

(c) What is the meaning of existence of limit of a function at x=1?

(d) Is the above function continuous at x=1 ?

Give your reasons ?

UNIT
seven

Matrices

1. Objectives

S,N.	Level	Objectives
(i)	Knowledge(k)	To define determinant To define determinant of order 2×2 . To define singular and non – singular matrices. To define inverse matrix of a given matrix To define Cramer's rule.
(ii)	Understanding(U)	To find determinant of order 2×2. To check given matrices singular or non – singular. To write formula of inverse of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Write Cramer's rule to solve simultaneous equations with two variables.
(ii)	Application(A)	To solve problems of determinant of order 2×2. To solve simultaneous equation with two variables by using inverse matrix method . To solve simultaneous equation of two variables by Cramer's rule..
(iv)	Higher Ability (HA)	To solve verbal problems in two variables by – inverse matrix method – Cramer's rule.

2. Required Teaching Materials:

Chart papers with definitions

– determinants

– inverse matrix

– Cramer's rule.

2. Teaching learning strategies:

– Review definitions of a matrix.

– Discuss on definition of determinants.

– To calculate determinants of order 1×1 and 2×2 with suitable examples.

– To differentiate det. $|-7|$ and absolute value $|-7|$.

– Discuss singular and non – singular matrices with examples.

– If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$, discuss and derive the formula

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where, } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

– Solve at least two examples of solution of simultaneous equations with two variables by matrix inverse method.

– Discuss about Cramer's rule to solve simultaneous equation with two variables x and y.

– Demonstrate solution of simultaneous equation in two variables by Cramer's rule.

Notes :

1. A square matrix A is called a singular matrix if its determinant is zero, ie. $|A|=0$, otherwise it is non – singular.
2. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non – singular matrix, then inverse of A is given by

$$A^{-1}=\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ where, } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \neq 0$$

3. $|A|=0$, then A^{-1} does not exist.
4. If AB = BA =I, then A and B are said to be inverse of each other.
5. Let two simultaneous equations be

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$ and $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$

then solution will be, $x = \frac{D_1}{D}$ where, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$

$x = \frac{D_2}{D}$ where, $D_2 = \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}$

Some solved problems

Determinants

1. If $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix}$, find the determinant of P + Q + I.

Solution

: Here, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, P + Q + I

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 3 + 1 & 0 + 1 + 0 \\ 3 + 5 + 0 & 1 + 3 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ 8 & 5 \end{bmatrix}$$

Now, $|P + Q + I|$

$$= \begin{vmatrix} 5 & 1 \\ 8 & 5 \end{vmatrix} = 25 - 8$$

$= 17$

2. Evaluate $= \begin{bmatrix} \mathbf{a^2 + ab + b^2} & \mathbf{b^2 + bc + c^2} \\ \mathbf{b - c} & a - b \end{bmatrix}$

Solution

We have, $a^3 - b^3 = (a - b)(a^2 - ab + b^2)$

Here, $= \begin{bmatrix} a^2 + ab + b^2 & b^2 + bc + c^2 \\ b - c & a - b \end{bmatrix}$

$= (a^3 - b^3) - (b^3 - c^3)$

$= a^3 - b^3 - b^3 + c^3$

$= a^3 - 2b^3 + c^3$

3. Show that $|AB| = |A||B|$

if A = $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$

Solution

Here, $AB = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 0 + 15 & -2 + 6 \\ 0 - 5 & -4 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 4 \\ -5 & -6 \end{bmatrix}$$

$= -90 + 20$

$= -70$

Again, $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = -2 - 12 = -14$

$|B| = \begin{vmatrix} 0 & -1 \\ 5 & 2 \end{vmatrix} = 0 + 5 = 5$

$\therefore |A||B| = -14 \times 5 = -70$

$\therefore$ |AB| = |A||B| proved.

4. If M = $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ and N = $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, find the determinants of

(a) $M^T + N^T$ (b) $(MN)^T$

Solution

Here, $M^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ and $N^T = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

Now, $M^t+N^t = \begin{bmatrix} 1+2 & 4+3 \\ 2+3 & 5+5 \end{bmatrix}$

$= \begin{bmatrix} 3 & 7 \\ 5 & 10 \end{bmatrix}$

$|M^t + N^t| = \begin{vmatrix} 3 & 7 \\ 5 & 10 \end{vmatrix} = 30 - 35 = -5$

(b) Here, $MN = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

$= \begin{bmatrix} 2+6 & 3+10 \\ 8+15 & 12+25 \end{bmatrix}$

$= \begin{bmatrix} 8 & 13 \\ 23 & 37 \end{bmatrix}$

Now, $|MN| = \begin{vmatrix} 8 & 13 \\ 23 & 37 \end{vmatrix} = 296 - 299 = -3$

5. Find the value of x if $\begin{vmatrix} x & x \\ 3x & 4x \end{vmatrix} = 9$

Solution

Here, $\begin{vmatrix} x & x \\ 3x & 4x \end{vmatrix} = (4x^2 - 3x^2) = 9$

or, $x^2 = 9$

$\therefore x = \pm 3$

6. If P = $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ and Q = $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

is $|(P + Q)^2| = |P^2 + 2PQ + Q^2|$.

Solution

Here, $P = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

Now, $P^2 = P.P = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 1+8 & 2+10 \\ 4+20 & 8+25 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 12 \\ 24 & 33 \end{bmatrix}$$

$$Q^2 = Q.\ Q = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4+12 & 6+15 \\ 8+20 & 12+25 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}$$

$$PQ = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+8 & 3+10 \\ 8+20 & 12+25 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 12 \\ 28 & 37 \end{bmatrix}$$

$$2PQ = 2 \begin{bmatrix} 10 & 12 \\ 28 & 37 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 24 \\ 56 & 74 \end{bmatrix}$$

$$P + Q = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & 10 \end{bmatrix}$$

$$(P + Q)^2 = (P + Q)(P + Q)$$

$$\text{Now, } (P + Q)\ (P + Q) \begin{bmatrix} 3 & 5 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 8 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 9+40 & 15+50 \\ 24+80 & 40+100 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & 65 \\ 104 & 140 \end{bmatrix}$$

Now, LHS $= |\ (P + Q\ )^2|$

$$= \begin{vmatrix} 45 & 57 \\ 108 & 144 \end{vmatrix} \begin{bmatrix} 49 & 65 \\ 104 & 140 \end{bmatrix}$$

$= 6860 - 6760$

$= 100$

Also. $P^2 + 2PQ + Q^2$

$$= \begin{bmatrix} 9 & 12 \\ 24 & 33 \end{bmatrix} + \begin{bmatrix} 20 & 24 \\ 56 & 74 \end{bmatrix} + \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}$$

$$= \begin{bmatrix} 45 & 57 \\ 108 & 144 \end{bmatrix}$$

RHS $=| P^2 + 2PQ + Q^2 |$

$$= \begin{bmatrix} 45 & 57 \\ 108 & 144 \end{bmatrix}$$

$= 6480 - 6156$

$= 324$

$\therefore (P + Q)^2 \neq | P^2 + 2PQ + Q^2 |$

7. If $P = \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix}$, find the determinant of $2P^2 - 5P + 4I$, where, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution

Here, $\begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix}$

Now, $P^2 = P.P = \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 0 - 6 & 0 - 8 \\ 0 + 12 & -6 + 16 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -8 \\ 12 & 10 \end{bmatrix}$$

$$2P^2 = 2 \begin{bmatrix} -6 & -8 \\ 12 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -16 \\ 24 & 20 \end{bmatrix}$$

$$5P = 5 \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ 15 & 20 \end{bmatrix}$$

$$4I = 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Now, $2P^2 - 5P + 4I$

$$= \begin{bmatrix} -12 & -16 \\ 24 & 20 \end{bmatrix} - \begin{bmatrix} 0 & -10 \\ 15 & 20 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -12 - 0 + 4 & -16 + 10 + 0 \\ 24 - 15 + 0 & 20 - 20 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -6 \\ 9 & 4 \end{bmatrix}$$

$$|2P^2 - 5P + 4I| = \begin{vmatrix} -8 & -6 \\ 9 & 4 \end{vmatrix} = -32 + 54$$

$= 22$

Inverse Matrix

1. Find the adjoint matrices of A = $\begin{bmatrix} \mathbf{10} & \mathbf{5} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$**.**

Solution

Here, $A = \begin{bmatrix} 10 & 5 \\ 2 & 3 \end{bmatrix}$

Adjoint of $A = \begin{bmatrix} 3 & -5 \\ -2 & 10 \end{bmatrix}$

2. Find the inverse of given matrices.

(a) $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ **(b)** $C = \begin{bmatrix} 10 & 5 \\ 12 & 3 \end{bmatrix}$

Solution

(a) Here, $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$, $|A| = \begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} = 12 - 12 = 0$

$|A| = 0$, the inverse of matrix A ie. A^{-1} does not exists.

(b) Here, $C = \begin{bmatrix} 10 & 5 \\ 12 & 3 \end{bmatrix}$

$|C| = = \begin{vmatrix} 10 & 5 \\ 12 & 3 \end{vmatrix} = 30 - 60 = -30.$

Since $|C| \neq 0$,

$\therefore C^{-1}$ exists

adjoint of $C = \begin{bmatrix} 3 & -5 \\ -12 & 10 \end{bmatrix}$

$$= \frac{1}{-30} \begin{bmatrix} 3 & -5 \\ -12 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{10} & \frac{1}{6} \\ \frac{2}{5} & -\frac{1}{3} \end{bmatrix}$$

3. Show that $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ are inverse to each other.

Solution

Let $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} . \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

$= \begin{bmatrix} 6-5 & -3+3 \\ 10-10 & -5+6 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Also, $BA = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

$= \begin{bmatrix} 6-5 & 2-2 \\ -15+15 & -5+6 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ AB = BA = I

By definition A and B are inverse of each other

Alternative Method

Let $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

$|A| = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = 6 - 5 = 1$

$\therefore |A| \neq 0.$

Hence inverse of martix A exists.

Adjoint of $A = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}$ adjoint of A

$= \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = B$

$\therefore$ B is the inverse of A. Similarly , we can show that B is the inverse of A. **proved**

4. If $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, then show that

i) $(A^{-1})^{-1} = A$

ii) $A^{-1}A = AA^{-1} = I$

Solution

i) $(A^{-1})^{-1} = A$

Let us find A^{-1}

Here, $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $|A| = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = 10 - 9$

$= 1$

$\therefore A^{-1} = \frac{1}{|A|}$ adjoint of $A = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

Again, let $A^{-1} = B$

Let us find B^{-1} = ie. $(A^{-1})^{-1}$

$|B| = \begin{vmatrix} 2 & -3 \\ -3 & 5 \end{vmatrix} = 10 - 9 = 1$

$B^{-1} = \frac{1}{|B|}$ adj. B

$= \frac{1}{1}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = A.$

$\therefore B^{-1} = (A^{-1})^{-1} = A$ proved

ii) $A^{-1}A = AA^{-1} = I$

LHS $= A^{-1}A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 10 - 9 & 6 - 6 \\ -15 + 15 & -9 + 10 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Again, $AA^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

$= \begin{bmatrix} 10-9 & -15-15 \\ 6-6 & -9+10 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{-1}A = AA^{-1} = I$ proved.

5. For what value of x, the product of matrix $\begin{bmatrix} 3 & 2 \\ x & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ does not have its inverse matrix.

Solution

: Here, $\begin{bmatrix} 3 & 2 \\ x & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 6+6 & -3+4 \\ 2x+12 & -x+8 \end{bmatrix}$

$= \begin{bmatrix} 12 & 1 \\ 2x+12 & 8-x \end{bmatrix}$

Now, $\begin{vmatrix} 12 & 1 \\ 2x+12 & 8-x \end{vmatrix}$

$= 96 - 12x - 2x - 12 = 84 - 14x$

If $= \begin{vmatrix} 12 & 1 \\ 2x+12 & 8-x \end{vmatrix} = 0$, the inverse matrix does not exists.

ie. $84 - 14x = 0$

$\therefore x = \dfrac{84}{14} = 6$

$\therefore$ For $x = 6$, the inverse of given matrix does not exist.

Solution of system of Linear Equations by Inverse matrix method

1. Factorize : $\begin{bmatrix} 2x + 4y \\ 5x + y \end{bmatrix}$

Solution

Here, $\begin{bmatrix} 2x + 4y \\ 5x + y \end{bmatrix}$

$= \begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

2. If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then answer the following question :

(a) Find the determinant of matrix A

(b) Under which condition A does not have its inverse ?

Solution

(a) Here, A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(b) If $|A| \neq 0$, then inverse of matrix A exists. It means that if $|A| = 0$, then the given matrix A does not have its inverse.

3. If $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, find the values of x and y.

Solution

Here, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

or, $= \begin{bmatrix} x+0 \\ 0+y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Equating the corresponding elements of equal matrices

$x = 4$ and $y = 5$.

4. Check whether the system of linear equations have unique solution or not

(a) $4x + 2y = 8$

$x - y = 1$

Solution

$4x + 2y = 8$ ie. $2x + y = 4$ ....(i)

and $x - y = 1$ ....(ii)

writing teh above equations in the matrix form, we get

$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

or, AX = C,

where,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -2 - 1 = -3$

Since $|A| = -3 \neq 0$, A^{-1} exists and the given system of linear equations have unique solution

5. Solve the following system of linear equations by matrix method.

(a) $x = 2y - 1$ and $y = 2x$

Solution

Here, $x = 2y - 1$

or, $x - 2y = -1$ ....(i)

and $y = 2x$

or, $2x - y = 0$..........(ii)

Writing the above equation in matrix form, we get,

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

or, $AX = C$ ....(iii)

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ C = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

where,

$|A| = \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = -1 + 4 = 3$

Since $|A| = 3 \neq 0$, $\overline{A}^{\,1}$ exists. There is a unique solution of given system of linear equations.

From (ii), we get,

$X = \overline{A}^{\,1}\ C$

To find $\overline{A}^{\,1}$, we have $|A| = 3$,

adj. $A = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$

$$\overline{A}^{\,1} = A^{-1} = \frac{1}{|A|} \text{adj} . A = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$$

Now, $X = A^{-1} C$

$$= \frac{1}{3}\begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 + 0 \\ 2 + 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Equating the corresponding elements of equal matrices, we get,

$x = = \frac{1}{3}$ and $y = = \frac{2}{3}$,

(b) $\mathbf{\frac{3x}{2} - \frac{y}{3} = 1}$

$\frac{x}{3} - \frac{y}{3}$ **=1**

Solution :

Here, $\frac{3x}{2} + 2y = 1$ or $3x + 4y = 2$ ....(i)

$\frac{x}{3} - \frac{y}{3} = 1$ or $x - y = 3$......(ii)

Writing above equation in matrix form, we get,

$$\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or, $AX = c$ or $X = A^{-1} C$ ....(iii),

where,

$$A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{and } C = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

where,

$$|A| = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} = -3 - 4 = -7$$

$\therefore |A| = -7 \neq 0.$

Hence $\overline{A}^{1}$ exists and there is a unique solution of given system of linear equaiton.

From (ii), we get,

$$A^{-1} = \frac{1}{|A|} \text{adj} . A = \frac{1}{-7} \begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix}$$

Now, $X = A^{-1} C$

$$= \frac{1}{-7} \begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{7} \begin{bmatrix} -2-12 \\ -2+9 \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} -14 \\ 7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\therefore$ x = 2 and y = −1

(c) $\mathbf{\frac{4}{x} - \frac{3}{y} = 1}$ **and** $\mathbf{\frac{3}{x} - \frac{2}{y} = \frac{1}{24}}$

Solution

Writing the given equation in matrix form, we get

$$\begin{bmatrix} 4 & 3 \\ 3 & - \end{bmatrix}\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{24} \end{bmatrix}$$

or, AX = C

or, $X = A^{-1}C$

where,

$$A = \begin{bmatrix} 4 & 3 \\ 3 & - \end{bmatrix}, X = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 \\ \frac{1}{24} \end{bmatrix}$$

where,

$$|A| = \begin{vmatrix} 4 & -3 \\ 3 & -2 \end{vmatrix} = -8 - 9 = -17$$

Now, from $X = A^{-1}C$

$$A^{-1} = \frac{1}{|A|}\text{ adj . A } = \frac{1}{-17}\begin{bmatrix} -2 & -3 \\ -3 & 4 \end{bmatrix}$$

Now, $X = A^{-1}C$

$$= \frac{1}{-17}\begin{bmatrix} -2 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{24} \end{bmatrix}$$

$$= \frac{1}{-17}\begin{bmatrix} -2 - \frac{1}{6} \\ -3 + \frac{1}{8} \end{bmatrix}$$

$$= \frac{1}{-17}\begin{bmatrix} -\frac{17}{8} \\ -\frac{17}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{8} \\ -\frac{1}{6} \end{bmatrix}$$

$$\therefore \ = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \\ -\frac{1}{6} \end{bmatrix}$$

ie. $\frac{1}{x} = \frac{1}{8}$ or $x = 8$

and $\frac{1}{y} = \frac{1}{6}$ or $x = 6$

$\therefore \ x = 8, y = 6$

(d) $\frac{x}{3} + \frac{2}{y} = 2$ and $x + \frac{4}{y} = 5$

Solution

Given equations can be written in the matrix form,

$$\begin{bmatrix} \frac{1}{3} & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

or, $AX = C$

or, $X = A^{-1} C$

where,

$$A = \begin{bmatrix} \frac{1}{3} & 2 \\ 1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

where,

$$|A| = \begin{vmatrix} \frac{1}{3} & 2 \\ 1 & 4 \end{vmatrix} = \frac{4}{3} - 2 = \frac{-2}{3}$$

Since $|A| \neq 0$, $\overline{A}^{1}$ exists and there is a unique solution of given system of equations.

$$A^{-1} = \frac{1}{|A|} \text{adj} . A$$

$$= \frac{-2}{3}\begin{bmatrix} 4 & 2 \\ -1 & \frac{1}{3} \end{bmatrix}$$

$$= -\frac{3}{2}\begin{bmatrix} 4 & 2 \\ -1 & \frac{1}{3} \end{bmatrix}$$

Now, $X = \overline{A}^{1} C$

$$= -\frac{3}{2}\begin{bmatrix} 4 & 2 \\ -1 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= -\begin{bmatrix} 8-10 \\ -2+\frac{5}{3} \end{bmatrix}$$

$$= -\frac{3}{2}\begin{bmatrix} -2 \\ \frac{-1}{3} \end{bmatrix}$$

$$\therefore = \begin{bmatrix} x \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}$$

ie. $x = 3$ and $\frac{1}{y} = \frac{1}{2}$ or $y = 2$.

(e) $3y + 4x = 2xy$ $\quad$ $7y + 5x = 29\,xy$

Solution

Here, $4x + 3y = 2xy$

dividing both sides by xy we get,

$\frac{4}{y} + \frac{3}{x} = 2$

or, $\frac{3}{x} + \frac{4}{y} = 2$ ....(i)

and $5x + 7y = 29xy$

or, $\frac{5}{y} + \frac{7}{x} = 29$

or, $\frac{7}{x} + \frac{5}{y} = 29$ ....(ii)

Writing above equations in matrix form,

$$\begin{bmatrix} 3 & 4 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} 2 \\ 29 \end{bmatrix}$$

or, AX = C or, $X = A^{-1} C$

where $X = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix}$, $A = \begin{bmatrix} 3 & 4 \\ 7 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ 29 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 \\ 7 & 5 \end{vmatrix} = 15 - 28 = -13$$

$\therefore \overline{A}^{1}$ exists and there is a unique solution of given system of equations.

$$A^{-1} = \frac{1}{|A|} \text{adj} . A$$

$$= \frac{1}{-13} \begin{bmatrix} 5 & -4 \\ -7 & 3 \end{bmatrix}$$

Now, $X = \frac{1}{-13} \begin{bmatrix} 5 & -4 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 29 \end{bmatrix}$

$$= \frac{1}{-13} \begin{bmatrix} 10 - 116 \\ -14 + 87 \end{bmatrix}$$

$$= \frac{1}{-13} \begin{bmatrix} -106 \\ 73 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} \frac{106}{13} \\ -\frac{73}{13} \end{bmatrix}$$

ie. $x = \frac{13}{106}$, $y = -\frac{13}{73}$

(f) $\mathbf{\frac{3x + 5y}{4} = \frac{7x + 3y}{5} = 4}$

Solution

Here, $\frac{3x + 5y}{4} = \frac{7x + 3y}{5} = 4$

Taking $\frac{3x + 5y}{4} = 4$

or, $3x + 5y = 16$............(i)

and taking $\dfrac{7x + 3y}{5} = 4$

and $7x + 3y = 20$............(ii)

Writing above equations in matrix form,we get,

$$\begin{bmatrix} 3 & 5 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$$

or, $AX = C$ or, $X = A^{-1}C$ ....(iii)

where $X = \begin{bmatrix} 3 & 5 \\ 7 & 3 \end{bmatrix}$, $A = \begin{bmatrix} x \\ y \end{bmatrix}$ and $C = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 \\ 7 & 5 \end{vmatrix} = 9 - 35 = -26$$

$\therefore |A| \neq 0$, $\overline{A}^{\,1}$ exists and there is a unique solution of given system of equations.

Now, $= \dfrac{1}{|A|}$ adj . A

$$= \frac{1}{-26}\begin{bmatrix} 3 & -5 \\ -7 & 3 \end{bmatrix}$$

From (iii), we get

$$\text{Now, } X = \frac{1}{-26}\begin{bmatrix} 3 & -5 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 16 \\ 20 \end{bmatrix}$$

$$= \frac{1}{-26}\begin{bmatrix} 48 - 100 \\ -112 + 60 \end{bmatrix}$$

$$= \frac{1}{-26}\begin{bmatrix} -52 \\ -52 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\therefore x = 2$, $y = 2$.

(g) $\mathbf{\dfrac{2x + 4}{5} = y = \dfrac{40 - 3x}{4} = 4}$

Solution

Here, $\dfrac{2x + 4}{5} = y = \dfrac{40 - 3x}{4} = 4$

Taking $\dfrac{2x + 4}{5} = y$

or, $2x + 4 = 5y$

$\therefore 2x - 5y = -4$.........(i)

and taking $y = \dfrac{40 - 3x}{4}$

or, $4y = 40 - 3x$

$\therefore 3x + 4y = 40$..........(ii)

and $7x + 3y = 20$............(ii)

Writing above equations in matrix form,we get,

$$\begin{bmatrix} 2 & -5 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 40 \end{bmatrix}$$

or, $AX = C$

$\therefore X = A^{-1}C$ ....(iii)

where $X = \begin{bmatrix} 2 & -5 \\ 3 & 4 \end{bmatrix}$, $A = \begin{bmatrix} x \\ y \end{bmatrix}$ and $C = \begin{bmatrix} -4 \\ 40 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -5 \\ 3 & 4 \end{vmatrix} = 8 + 15 = 23$$

$\therefore$ Since $|A| \neq 0$, A^{-1} exists

Now, $A^{-1} = \dfrac{1}{|A|}$ adj . A

$$= \frac{1}{23}\begin{bmatrix} 4 & 5 \\ -3 & 2 \end{bmatrix}$$

From (iii), we get,

Now, $X = \dfrac{1}{23}\begin{bmatrix} 4 & 5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ 40 \end{bmatrix}$

$$= \frac{1}{23}\begin{bmatrix} -16 + 200 \\ 12 + 80 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{184}{23} \\ \frac{92}{23} \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

ie. $x = 8$ and $y = 4$.

6. Equations of pair of lines are $2x - y = 5$ and $x - 2y = 1$

(a) Write the equation in matrix form.

(b) Is there unique solution of above given equations.

(c) Solve the equations.

(d) Check the solutions to show that the values of x and y so obtained are true.

Solution

(a) Given equations are

$2x - y = 5$

$x - 2y = 1$

writing above equations in matrix form,

$$\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

or, $AX = C$

$\therefore X = A^{-1} C$ ....(iii)

where $X = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$, $A = \begin{bmatrix} x \\ y \end{bmatrix}$ and $C = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(b) $|A| = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -4 + 1 = -3$

$\therefore$ Since $|A| \neq 0$, $\overline{A}^{\,1}$ exists and there is unique solution of given equations.

Now, $\overline{A}^{\,1} = \frac{1}{|A|}$ adj . A

$= \frac{1}{-3}\begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}$

Now, $X = A^{-1} C = \frac{1}{-3}\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$= \frac{1}{-3}\begin{bmatrix} -10 + 1 \\ -5 + 2 \end{bmatrix}$

$= \frac{1}{-3}\begin{bmatrix} -9 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

ie. $x = 3$ and $y = 1$.

(d) put $x = 3$ and $y = 1$ in above equation from equation $2x - y = 5$

or, $2.3 - 1 = 5$

$\therefore 5 = 5$ (True).

Again from equation, $x - 2y = 1$

$3 - 2.1 = 1$

$\therefore 1 = 1$ (True)

$\therefore x = 3$ and $y = 1$ are true.

Cramer's Rule

1. Equations are $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$
What are the determinants represented by D, D_x and D_y.

Solution

here, $a_1x + b_1y = c_1$

and $a_2x + b_2y = c_2$

coefficient of x	coefficient of y	constants
a_1	b_1	c_1
a_2	b_2	c_2

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = b_2c_1 - b_1c_2$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$

2. If $D = 4$, $D_x = \frac{1}{2}$, $D_y = \frac{1}{4}$, find the values of x and y.

ting D, D_x and D_y.

Solution

Here, $D = 4$, $D_x = \frac{1}{2}$, $D_y = \frac{1}{4}$

$$x = \frac{D_x}{D} = \frac{\frac{1}{2}}{4} = \frac{1}{8}$$

$$y = \frac{D_y}{D} = \frac{\frac{1}{4}}{4} = \frac{1}{16}$$

$$\therefore x = \frac{1}{8}, y = \frac{1}{16}$$

3. Solve the following system of equations by using cramer's rule.

(a) $3x - 2y = 1$ and $-x + 4y = 3$

Solution

coefficient of x   coefficient of y   constants

3	– 2	1
– 1	4	3

$$D = \begin{vmatrix} 3 & -2 \\ -1 & -4 \end{vmatrix} = 12 - 2 = 10$$

$$D_1 = \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} = 4 + 6 = 10$$

$$D_2 = \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} = 9 + 1 = 10$$

$$x = \frac{D_1}{D} = \frac{10}{10} = 1$$

$$y = \frac{D_2}{D} = \frac{10}{10} = 1$$

(b) $\frac{3}{x} + \frac{5}{y} = 1, \ \frac{4}{x} + \frac{3}{y} = \frac{29}{30}$

Solution

coefficient of $\frac{1}{x}$	coefficient of $\frac{1}{y}$	constants
3	5	1
4	3	3

$$D = \begin{vmatrix} 3 & 5 \\ 4 & 3 \end{vmatrix} = 9 - 20 = -11$$

$$D_1 = \begin{vmatrix} 1 & 5 \\ \frac{29}{30} & 3 \end{vmatrix} = 3 - \frac{29}{30} \times 5 = -\frac{11}{6}$$

$$D_2 = \begin{vmatrix} 3 & 1 \\ 4 & \frac{29}{30} \end{vmatrix} = 3 \times \frac{29}{30} - 4 = -\frac{11}{10}$$

$$\frac{1}{x} = \frac{D_1}{D} = \frac{\frac{-11}{6}}{-11} = \frac{1}{6}$$

or, $x = 6$ and

$$\frac{1}{y} = \frac{D_2}{D} = \frac{\frac{-11}{10}}{-11} = \frac{1}{10} \quad \therefore y = 10$$

(c) $\frac{10}{x} - 2y = -1$ **and** $\frac{4}{x} + 3y = 11$

Solution

coefficient of x	coefficient of y	constants
10	– 2	– 1
4	3	11

$$D = \begin{vmatrix} 10 & -2 \\ 4 & 3 \end{vmatrix} = 30 + 8 = 38$$

$$D_1 = \begin{bmatrix} -1 & -2 \\ 11 & 3 \end{bmatrix} = -3 + 22 = 19$$

$$D_2 = \begin{bmatrix} 10 & -1 \\ 4 & 11 \end{bmatrix} = 110 + 4 = 114$$

$$\frac{1}{x} = \frac{D_1}{D} = \frac{19}{38} = \frac{1}{2} \quad \therefore\ x = 2$$

$$y = \frac{D_y}{D} = \frac{114}{38} = 3$$

$\therefore$ x = 2 and y = 3.

(d) 7 (x –y) = x + y and 5(x + y) = 35 (x – y)

Solution

Here, 7 (x –y) = x + y

or, 7x – 7y = x + y

or, 6x – 8y = 0 ....(i)

and 5(x + y) = 35 (x – y)

or, x + y = 7x – 7y

or, 6x – 8y = o ....(ii)

coefficient of x	coefficient of y	constants
6	– 8	0
6	– 8	0

$$D = \begin{vmatrix} 6 & -8 \\ 6 & -8 \end{vmatrix} = -48 + 48 = 0$$

Since D = 0, there is no solution of given equations.

(e) 2(3x – y) =5(x – 2) and 3(x +4y) = 2(y – 3)

Solution

Here, 2(3x – y) =5(x – 2)

or, 6x – 2y – 5x = –10

or, x – 2y = – 10 ....(i)

and 3(x +4y) = 2(y – 3)

or, $3x + 12y = 2y - 6$

or, $3x + 10y = -6$ ......(ii)

coefficient of x	coefficient of y	constants
1	– 2	– 10
3	10	– 6

Now, $D = \begin{vmatrix} 1 & -2 \\ 3 & 10 \end{vmatrix} = 10 + 6 = 16$

$D_1 = \begin{vmatrix} -10 & -2 \\ -6 & 10 \end{vmatrix} = -100 - 12 = -112$

$D_2 = D_2 = \begin{vmatrix} 1 & -10 \\ 3 & -6 \end{vmatrix} = -6 + 30 = 24$

$x = \dfrac{D_1}{D} = \dfrac{-112}{16} = -7$

$y = \dfrac{D_2}{D} = \dfrac{24}{16} = \dfrac{3}{2}$

$\therefore x = -7$ and $y = \dfrac{3}{2}$

4(a) A helicopter has 4 seats for passengers. Those willing to pay first class fares can take 60 kg of baggage each but tourist class passengers are restricted to 20 kg each. The helicopter can carry only 120 kg baggage all together. To find the number of passengers of each kind, use Cramer's rule.

Solution

Let x and y be the number of passengers of first class and tourist class respectively. Then by question, we get,

$x + y = 4$ ....i)

$60x + 20y = 120$

or, $3x + y = 6$ ii

To solve equations (i) and (ii) by Cramer's rule

coefficient of x	coefficient of y	constants
1	1	4
3	1	6

Now, $D = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = 1 - 3 = -2$

$D_1 = \begin{vmatrix} 4 & 1 \\ 6 & 1 \end{vmatrix} = 4 - 6 = -2$

$D_2 = \begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} = 6 - 12 = -6$

$$x = \frac{D_1}{D} = \frac{-2}{-2} = 1$$

$$y = \frac{D_2}{D} = \frac{-6}{-2} = 3$$

$\therefore x = 1$ and $y = 3$

Questions for practice

Determinants

1. Find the determinants of given matrices

(a) $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$, (b) $P = \begin{bmatrix} -4 & -5 \\ 6 & 7 \end{bmatrix}$

2. If $P = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$,find the determinants of

a) $P + Q$ b) $2P + 3Q$ (c) $P + Q + I$.

3. If $M = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & 2 \\ 6 & 3 \end{bmatrix}$, then find the determinants of MN and NM.

4. If $A = \begin{bmatrix} 2 & 3 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 3 & 6 \end{bmatrix}$, then verify that

$|AB| = |A||B|$.

5. If $P = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$, then find the determinants of (a) $(P + Q)^T$ (b) $P^T + Q^T$.

6. If $P = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, then find the determinant of $P^2 + 4P - 5I$.

7. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, is $|(A + B)|^2 = |A^2 + 2AB + B^2|$.

8. If $A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$, show that $|A|^2 = |A|^2$.

Inverse Matrix

1. Find the adjoint matrix of $\begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$.

2. Find the inverse of the following matrices.

(a) $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ (b) $P = \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$ (c) $R = \begin{bmatrix} 5 & 7 \\ 5 & 7 \end{bmatrix}$

3. Show that $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ are inverse to each other.

4. If A = $\begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix}$ and AB = I, then find the matrix B.

5. If P = $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, then show the following

a) $PP^{-1} = P^{-1}P = I$

b) $(P^{-1})^{-1} = P$

6. If A = $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$, then show that $(AB)^{-1} = B^{-1}A^{-1}$.

7. Find the matrix P when

(a) $(3P^{-1}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $(3P)^{-1} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $(I + 3P) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

8.Solve for matrix A under the following conditions.

(a) $\begin{bmatrix} -3 & 2 \\ 6 & 5 \end{bmatrix} A = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 2 \\ -6 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution of system of Linear Equations by Inverse matrix method.

1. Factorize $\begin{bmatrix} 4x + 3y \\ 6x + 5y \end{bmatrix}$

2. Solve the following matrices by inverse matrix method.

(a) $3x + 2y = 20$ and $2x - y = 4$.

(b) $4x - 5y = 2$ and $\frac{x}{4} + \frac{y}{3} = 4$

(c) $\frac{3x + 5y}{8} = \frac{5x - 2y}{5} = 4$

(d) $\frac{3x + 5y}{4} = \frac{7x + 3y}{5} = 4$

(e) $\frac{5}{x} + 3y = 7$ and $7y - \frac{10}{x} = 12$

(f) $\frac{x}{4} + \frac{y}{3} = 2$, $x + y = 7$

(g) $\frac{1}{x} + \frac{2}{y} = 2$, $\frac{3}{x} + \frac{4}{y} = 5$

Cramer's rule :

Solve the following equations using Cramer's rule.

1. $x + y = 5$ and $x - y = 3$
2. $2x + 3y = 5$ and $3x - 2y = 1$
3. $5x - 4y = 1$ and $4x + 5y = 9$
4. $\frac{x}{4} + \frac{y}{10} = 1$ and $\frac{x}{5} + \frac{3y}{25} = 1$
5. $\frac{1}{x} + \frac{1}{y} = 2$ and $\frac{2}{x} - \frac{3}{y} = 5$
6. $\frac{3}{x} - \frac{7}{y} = 1$ and $\frac{5}{x} + \frac{4}{y} = 17$
7. $\frac{4}{x} + \frac{6}{y} = 0$ and $\frac{3}{x} - \frac{4}{y} = -\frac{17}{6}$

6\. $\frac{4}{x} + \frac{5}{y} = 58$ and $\frac{7}{x} + \frac{3}{y} = 67$

UNIT eight

Coordinate Geometry

Angle between two lines

Estimated Teaching periods : 7

1. Objectives :

S.N.	Level	Objectives
(i)	Knowledge (K)	To tell formula of angle between two lines. To tell conditions for parallelism and perpendicularity of two lines.
(ii)	Understanding(U)	To find angle between two lines when their slopes are given. To identify given lines parallel or perpendicular when their slopes are given.
(iii)	Application(A)	To use formula to find angle between two lines when their equations are given
(iv)	Higher Ability (HA)	To derive formula to find angle between the lines $y=m_1x+c_1$ and $y=m_2x+c_2$. Find conditions for parallelism and perpendicularity of two lines.

2. Teaching Materials

Chart papers with angle between two lines, condition of parallelism and perpendicularity.

3. Teaching Learning Activities

- First review the formula of equation straight lines that the students have studied in class 9.
- Draw figure to derive the formula to find angle between two lines $y=m_1x+c_1$ and $y=m_2x+c_2$ and derive formula to find the angle between them.
- Discuss the conditions for parallelism and perpendicularity of two lines.
- Discuss the meaning of $m_1 = m_2$ and $m_1.m_2 = -1$
- Let the students do some problems after the teacher solved some problems as examples.

Notes :

1. Three standard forms of equations of straight lines are

(i) Slope–intercept form $y=mx+c$.

(ii) Double intercept form $\frac{x}{a}+\frac{y}{b}=1$

(iii) Normal / perpendicular form

$x\cos\alpha + y\sin\alpha = p$

2. Slope of general equation of straight line ax + by +c =0 is given by

$m = -\dfrac{\text{Coefficient of x}}{\text{Coefficient of y}} = -\dfrac{a}{b}$

3. Slope of a line joining two points (m)= $\dfrac{y_2 - y_1}{x_2 - x_1}$

4. Special forms of equation of straight line.

i) slope–point form $y - y_1 = m(x - x_1)$

ii) Two –points form $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

5(a) Angle between two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is given by

$$\theta = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$$

(i) Condition for parallelism, $m_1 = m_2$

(ii) Condition for parallelism, $m_1 - m_2 = -1$

(b) Equation of straight line parallel to the line ax +by +c =0 is ax +by +k =0, where k is a constant

(b) Equation of a straight line perpendicular to the line ax +by +c =0 is bx – ay +k =0

Some solved problems

1. Show that the lines mx +my +p=0 and 2mx + 2ny +r =0 are parallel to each other.

Solution

Given equations of lines are

$mx + my + p = 0$ ....(i)

$2mx + 2ny + r = 0$..........(ii)

From equation (i), its slope$(m_1) = -\dfrac{\text{Coefficient of x}}{\text{Coefficient of y}} = -\dfrac{m}{n}$

From equation (ii), its slope$(m_2) = -\dfrac{2m}{2n} = -\dfrac{m}{n}$

since $m_1 = m_2$, the given two lines are parallel to each other.

2.Find the slope of line parallel to 4x +3y +12=0 .

Solution

Given equations of lines $4x + 3y + 12 = 0$..............(i)

its slope$(m_1) = -\dfrac{\text{Coefficient of x}}{\text{Coefficient of y}} = -\dfrac{-4}{3}$

Equation of any line parallel to (i) has the same slope. Hence the required slope is $(m_1)=-\frac{-4}{3}$

From equation (ii), its slope$(m_2)=-\frac{2m}{2n}=-\frac{m}{n}$

since $m_1=m_2$, the given two lines are parallel to each other.

3.Find the slope of line perpendicular to $3x+2y+20=0$.

Solution

Given equations of lines $3x+2y+20=0$.............(i)

slope of given line is $(m_1)=-\frac{\text{Coefficient of } x}{\text{Coefficient of } y}=-\frac{3}{2}$

Let m_2 be the slope of the line perpendicular to line (i) then

$m_1 \cdot m_2=-1$

or, $\left(\frac{-3}{2}\right).m_2=-1$

$\therefore m_2=\frac{2}{3}$

4. Find the acute angle between the two given lines.

(a) $y-(2-\sqrt{3})x=5$ and $y=(2+\sqrt{3})x+8$

Solution

Equation of the given lines are

$(2-\sqrt{3})x-y+5=0$..........(i)

and $(2+\sqrt{3})x-y+8=0$.......(ii)

From equation (i), its slope is $(m_1)=-\frac{\text{Coefficient of } x}{\text{Coefficient of } y}=-\frac{(2-\sqrt{3})}{-1}=2-\sqrt{3}$

From equation (ii), its slope is $(m_2)=-\frac{(2-\sqrt{3})}{-1}=2+\sqrt{3}$

Let be the angle between the lines (i) and (ii), we get

$\tan\theta=\pm\frac{m_1-m_2}{1+m_1m_2}$

$=\pm\frac{(2-\sqrt{3}-2-\sqrt{3})}{1+(2-\sqrt{3})(2+\sqrt{3})}$

$=\pm\frac{-2\sqrt{3}}{1+4-3}=\pm\sqrt{3}$

we require acute angle , taking positive sign,

$\tan\theta = \sqrt{3}$

$= \tan 60^o$

$\therefore \theta = 60^o$

(b) $x\cos\alpha + y\sin\alpha = p$ and $x\sin\alpha - y\cos\alpha = q$

Solution

Given equation of line are

$x\cos\alpha + y\sin\alpha - p = 0$..........(i)

$x\sin\alpha - y\cos\alpha - q = 0$.........(ii)

From equation(i),slope $(m_1) = -\frac{\text{Coefficient of x}}{\text{Coefficient of y}} = \frac{-\cos\alpha}{\sin\alpha}$

From equation (ii), slope $(m_2) = -\frac{\sin\alpha}{-\cos\alpha} = \frac{\sin\alpha}{\cos\alpha}$

Let θ be the angle between the given lines.

Then, $\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$

$$= \pm \frac{-\frac{\cos\alpha}{\sin\alpha} - \frac{\sin\alpha}{\cos\alpha}}{1 + \left(\frac{-\cancel{\cos\alpha}}{\cancel{\sin\alpha}}\right)\left(\frac{\cancel{\sin\alpha}}{\cancel{\cos\alpha}}\right)}$$

$= \pm\infty$

Taking positive sign, we get $\tan\theta = \infty$

$\tan\theta = \tan 90^\circ$

$\therefore \theta = 90^\circ$

Hence the angle between lines (i) and (ii) as 90°.

5. Find the obtuse angle between the lines .

(a) $y + \sqrt{3}\ x + 8 = 0$ and $y + 20 = 0$

Solution

The given equations of lines are

$y + \sqrt{3}\ x + 8 = 0$ ....(i)

$y + 20 = 0$..........(ii)

Slope of line (i), $m_1 = -\frac{\text{Coefficient of x}}{\text{Coefficient of y}} = -\frac{\sqrt{3}}{1} = -\sqrt{3}$

Slope of line (ii), $m_2 = -\frac{\text{Coefficient of x}}{\text{Coefficient of y}} = -\frac{0}{1} = 0$

Let θ be the angle between the lines. Then,

$\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$

$$=\pm\left(\frac{-\sqrt{3}\ -0}{1+0}\right)$$

$$=\pm\sqrt{3}$$

Taking negative sign, we get

$\tan\theta=-\sqrt{3}$

$\tan\theta=\tan 120^\circ$

$\therefore \theta=120^\circ$

Hence the angle between lines (i) and (ii) as 120°.

(b) 2x+y = 3 and x +2y =1

Solution

Given equations are

$2x+y+3=0$ ....(i)

$x+2y-1=0$..........(ii)

From equation (i),slope of line (i), $m_1=-\dfrac{\text{Coefficient of x}}{\text{Coefficient of y}}=-\dfrac{2}{1}=-2$

From equation (ii), slope of line (ii), $m_2=-\dfrac{\text{Coefficient of x}}{\text{Coefficient of y}}=-\dfrac{3}{2}$

Let θ be the angle between the lines, (i) and (ii)

Then, $\tan\theta=\pm\dfrac{m_1-m_2}{1+m_1m_2}$

$$=\pm\left(\frac{-2+\frac{3}{2}}{1+2\ .\ \frac{3}{2}}\right)$$

$$=\pm\frac{1}{2\times 4}$$

$$=\pm\frac{1}{8}$$

Taking positive sign, we get

$\tan\theta=-\dfrac{1}{8}$

$\theta=\tan^{-1}\left(\dfrac{-1}{8}\right)=172.87^\circ$

6. Show that the lines x – y +2 =0 and the line joining the points (4,6) and (10,12) are parallel to each other.

Solution

Given equations of lines is $x-y+2=0$

its slope is, $m_1 = -\frac{\text{Coefficient of x}}{\text{Coefficient of y}}$

$= -\frac{1}{-1} = 1$

slope of line joining points (4,6) and (10,12), $m_2 = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{12-6}{10-4} = \frac{6}{6} = 1$

Here, $m_1 = m_2$, hence the line (i) and the line joining points (4,6) and (10,12) are parallel.

7, Show that lines joining the points (7, –5) and (3,4) is perpendicular to the line 4x–2y +7=0.

Solution

Slope of line joining points (7,–5) and (3,4) is given by, $m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{4+5}{3-7} = \frac{9}{-4} = -\frac{9}{4}$

Slope of the line 4x –9y+7=7 is $m_2 = -\frac{4}{-9} = \frac{4}{9}$

now, $m_1 . m_2 = \frac{9}{-4} . \frac{4}{9} = -1$

Since the product of slopes is –1, the lines are perpendicular to each other.

8. Find the value of k is that the lines represented by kx +3y+5 =0 and 4x=3y+10=0 are perpendicular to each other.

Solution

Here given equations of lines are

kx +3y+5 =0 ....(i)

and 4x=3y+10=0.............(ii)

slope of line (i), $m_1 = -\frac{\text{Coefficient of x}}{\text{Coefficient of y}}$

$= -\frac{k}{3}$

Slope of line, $m_2 = -\frac{4}{-3} = \frac{4}{3}$

As the two lines are perpendicular to each other

$m_1 . m_2 = -1$

or, $-\frac{k}{3} . \frac{4}{3} = -1$

or, $k = \frac{9}{4}$

9. Find the equation of the straight line that passes through the point (4,5) and perpendicular to the line having slope $\frac{6}{5}$.

Solution

Let the required line be

$y - y_1 = m(x - x_1)$

It passes through the point (4,5)

$y - 5 = m(x - 4)$.........(i)

slope ot given line is $(m_1) = \frac{6}{5}$

As the lines are perpendicular to each other

$m.m_1 = -1$

or, $m.\left(\frac{6}{5}\right) = -1$

$\therefore m = -\frac{5}{6}$

put the value of m is equation (i), we get

$y - 5 = -\frac{5}{6}(x - 4)$

or, $6y - 30 = -5x + 20$

$\therefore 5x + 6y = 50$ which is the required equation.

10. (a) Find the equation of straight line which passes through the point (2,3) and parallel to the line $x - 2y - 2 = 0$.

Solution

Given equations of straight line is $x - 2y - 2 = 0$

its slope is $(m_1) = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$

$= -\frac{1}{-2} = \frac{1}{2}$

we find the equation of straight line which is parallel to given line. So its slope $(m) = m_1 = \frac{1}{2}$

Now, equation of required line passing through (2,3) and with slope m is given by

$y - y_1 = m(x - x_1)$

or, $y - 3 = \frac{1}{2}(x - 2)$

or, $2y - 6 = x - 2$

or, $x - 2y + 4 = 0$

$\therefore x - 2y + 4 = 0$ is the required equation.

(b) Find the equation if the straight line which passes through the point of intersection of the straight line $3x+4y=7$ and $5x-2y=3$ and perpendicular to the line $2x+3y=5$.

Solution

For the point of intersection of the straight lines

$3x+4y=7$ and $5x-2y=3$,

from 1st equation, and from 2nd equation

$3x+4y=7$ 　　　 $5x-2y=3$

or, $x = \frac{7-4y}{3}$ 　　　 or, $x = \frac{3+2y}{5}$

from both

$\frac{7-4y}{3} = \frac{3+2y}{5}$

or, $35 - 20y = 9 + 6y$

$\therefore y = 1$

now, $x=\frac{3+2\times 1}{5} = 1$

Let $(x,y)=(1,1)$ be (x_1,y_1) and the slope of required line be m,

m $\times$slope of equation $(2x+3y=5)=-1$ (pep^r to each other)

or, $m\times\frac{-2}{3}=-1$

$\therefore m=\frac{3}{2}$

now, eqn. of st. lines is,

$y-y_1=m(x-x_1)$

or, $y-1=\frac{3}{2}(x-1)$

or, $2y-2=3x-3$

or, $3x-2y-1=0$

$\therefore$ $3x-2y-1=0$ is the required equation.

(c) Find the equation if the straight line which passes through the point of intersection of $2x-3y+1=0$ and $x+2y=3$ and parallel to the line $4x+3y=12$.

Solution

For the point of intersection of the straight lines

$2x-3y+1=0$ and $x+2y=3$

from 1st equation, and from 2nd equation

$2x-3y=-1$ $\qquad$ $x=3-2y$

or, $x=\frac{3y-1}{2}$

from both

$\frac{3y-1}{2}= 3-2y$

or, $3y-1=6-4y$

$\therefore y = 1$

Also, $x=3-2\times 1=1$

Let $(x,y)=(1,1)$ be (x_1,y_1)

for slope (m), two lines passing through (1,1) and $4x+3y=12$ are parallel.

Now, eqn. of straight lines is,

$y-y_1=m(x-x_1)$

or, $y-1=\frac{-4}{3}(x-1)$

or, $3y-3=-4x+4$

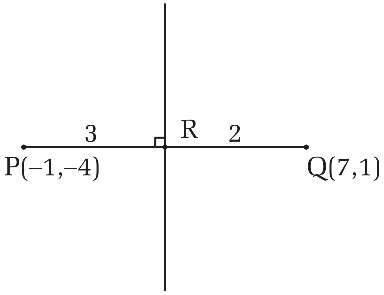
$\therefore$ $4x+3y=7$ is the required equation.

(d) Find the equation of the straight line that divides the line joining the points p(–1,–4) and

Q(7,1) in the ratio of 3:2 and perpendicular to it.

Solution

The given two points are P(–1,–4) and Q(7,1)



Slope of PQ $(m_1)=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{1+4}{7+1}=\dfrac{5}{8}$

Let R(x,y) be the point of PQ which divides at in 3:2 ratio.

$$(x,y)=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$

$$=\left(\frac{3.7+2.(-1)}{3+2},\frac{3.1+2.(-4)}{3+2}\right)$$

$$=\left(\frac{19}{5},-1\right)$$

Since the required line is perpendicular to PQ and passes through $\left(\frac{19}{5},-1\right)$, its slope (m_2) is given by $m_1.m_2=-1$

or, $\frac{5}{8}.m_2=-1$ or, $\therefore m_2=-\frac{8}{5}$

Now, required equation is

$y-y_1=m(x-x_1)$

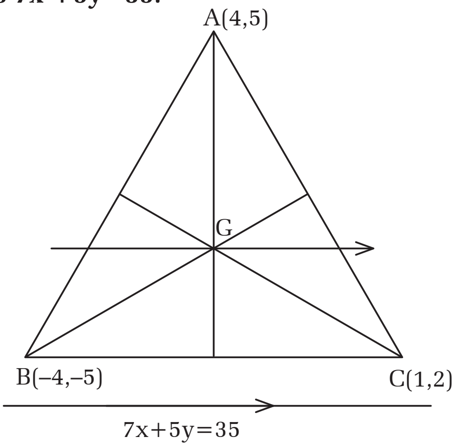
$$y+1=-\frac{8}{5}\left(x-\frac{19}{5}\right)$$

or, $40x+25y=127$

$\therefore$ $40x+25y=127$ is the required equation.

(e) Find the equation of the straight line which passes through the centroid of ΔABC with vertices A(4,5), B(–4,–5) and C(1,2) and parallel to 7x +5y=35.

Solution



For centroid of ΔABC,

$$G(x,y)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_2}{3}\right)$$

$$=\left(\frac{4-4+1}{3},\frac{5-5+2}{3}\right)$$

$$=\left(\frac{1}{3},\frac{2}{3}\right)$$

Let m be the slope of required line which is perpendicular to 7x +5y=35 and passes through G

$\left(\frac{1}{3}, \frac{2}{3}\right)$

m =slope of eqn. (7x +5y=35) [∴ parallel to each other]

$\therefore m_2 = -\frac{7}{5}$

Now, required equation is

$y - y_1 = m(x - x_1)$

$y - \frac{2}{3} = -\frac{7}{5}\left(x - \frac{1}{3}\right)$

or, $5y - \frac{10}{3} = -7x + \frac{7}{3}$

or, $7x + 5y = \frac{17}{3}$

∴ 21x+15y=17 is the required equation.

11. If the angle between the lines $(a^2-b^2)x-(p+q)y=0$ and $(p^2-q^2)x+(a+b)y=0$ is 90°, prove that $(a-b)(p-q)=1$.

Solution

Given equations of the lines are

$(a^2-b^2)x-(p+q)y=0$ and ....(i)

and $(p^2-q^2)x+(a+b)y=0$.............(ii)

From equation (i), slope$(m_1) = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{(a^2-b^2)}{-(p+q)} = \frac{a^2-b^2}{p+q}$

From equation (ii), slope $(m_2) = -\frac{(p^2-q^2)}{a+b}$

Since the lines (i) and (ii) are perpendicular to each other

$m_1.m_2 = -1$

or, $\frac{a^2-b^2}{p+q}.\left(-\frac{p^2-q^2}{a+b}\right) = -1$

or, $\frac{(a+b)(a-b)(p-q)(p+q)}{(p+q)(a+b)} = 1$

∴ $(a-b)(p-q) = 1$ proved

12.(a) From the point p(–2,4), perpendicular PQ is drawn to the line AB:7x–4y+15=0. Find the equation of PQ.

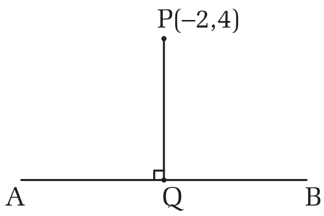
Solution

Equation of AB is

$7x - 4y + 15 = 0$

Its Slope is $(m_1) = \frac{7}{(-4)} = \frac{7}{4}$

PQ is perpendicular to AB,



So slope of PQ $(m_2)=-\frac{4}{7}$ $(\therefore m_1.m_2=-1)$

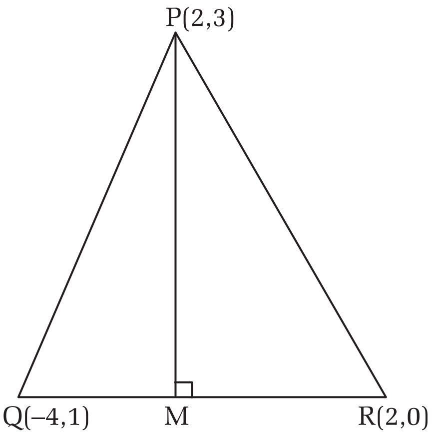
Now the equaton of line PQ is given by

$y-y_1=m(x-x_1)$

$y-4=\frac{-4}{7}(x+2)$

$\therefore$ 4x+7y=20 is the required equation.

12(b). Find the equation of the altitude PM drawn from the vertex P to QR in ΔPQR p(2,3),Q(–4,1) and R(2,0).



Solution

Here PM is perpendicular to QR.

Slope of line QR $(m_2)=\frac{y_2-y_1}{x_2-x_1}=\frac{0-1}{2+4}$

$=\frac{1}{-6}=\frac{-1}{6}$

Since they PM is perpendicular to QR perpendicular to QR, $m_1\times m_2=-1$, where m_2 is the slope of PM

$+\frac{1}{6}\times m_2=+1$

$m_2=6$

Equation of PM is

$y-y_1=m(x-x_1)$

$y-3=6(x-2)$

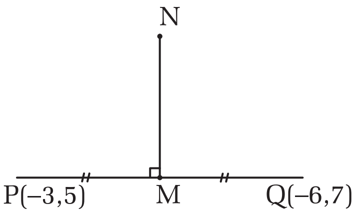
$y-3=6x-12$

or, $6x-y-9=0$

$\therefore$ 6x–y=9 is the required equation.

13. Find the equation of the line perpendicular bisector to the join of the following two points.

(a) P(–3,5) and Q(–6,7)



Solution

Given points are P(–3,5) and Q(–6,7)

Mid point PQ$=M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$=M\left(\frac{-3-6}{2},\frac{5+7}{2}\right)$

$=M\left(\frac{-9}{2},6\right)$

Slope of PQ $(m_1)=\dfrac{y_2\text{-}y_1}{x_2\text{-}x_1}$

$= \dfrac{7\text{-}5}{\text{-}6+3} = -\dfrac{2}{3}$

Let MN be the perpendicular bisector of PQ with slope(m_2).

then, $m_1 - m_2 = -1$

or, $-\dfrac{2}{3}.m_2 = -1$

$\therefore m_2 = \dfrac{3}{2}$

Now, equation of MN is

$y-y_1=m(x-x_1)$, $\left(\because m=m_2=\dfrac{3}{2}\right)$

or, $y - 6= \dfrac{3}{2}\left(x + \dfrac{9}{2}\right)$

or, $4y - 24=6x +27$

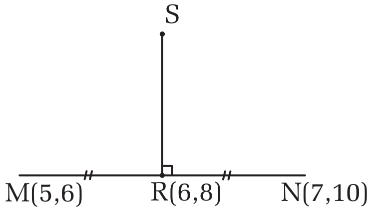
or, $6x-4y+51=0$

$\therefore$ $6x-4y +51=0$ is the required equation.

(b) P(5,6) and Q(7,10)

Solution

We have to find the equation of perpendicular bisector of MN



Mid point PQ$=M\left(\dfrac{x_1 +x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

$=M\left(\dfrac{5+7}{2}, \dfrac{6+10}{2}\right)$

$=(6,8)$

Slope of MN is:

$m_1 = \dfrac{y_2-y_1}{x_2-x_1}$

$= \dfrac{10-6}{7-5}$

$= \dfrac{4}{2}$

$=2$

Let RS be the perpendicular of MN

Since the lines are perpendicular, slope of MN$=m_2$ is given by

$m_1 \times m_2 = -1$

or, $-\dfrac{2}{3}.m_2 = -1$

or, $2\times m_2=-1$

$\therefore m_2=\frac{-1}{2}$

Now, equation of st. line RS is:

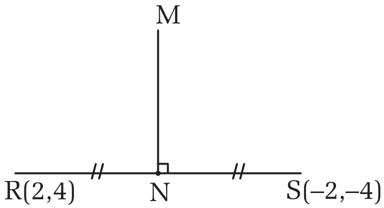
$y-y_1=m(x-x_1)$,

or, $y-8=-\frac{1}{2}(x-6)$

$\therefore x+2y=22$ is the required equation.

(c) P(2,4) and Q(–2,–4)

Solution



Let N be the mid point of RS

Then the coordinates of S

are $\left(\frac{2-2}{2},\frac{4-4}{2}\right)=(0,0)$

Slope of PQ $(m_1)=\frac{y_2-y_1}{x_2-x_1}$

$=\frac{-4-4}{-2-2}$

$=2$

Let MN be the perpendicular bisector of RS, then we get slope of MN is $(m_2)=-\frac{1}{2}$, ($\therefore m_1.m_2=-1$)

Now, equation of RS is

$y-y_1=m(x-x_1)$, where $m=m_2=-\frac{1}{2}$

or, $y-0=-\frac{1}{2}(x-0)$

or, $2y=-x$

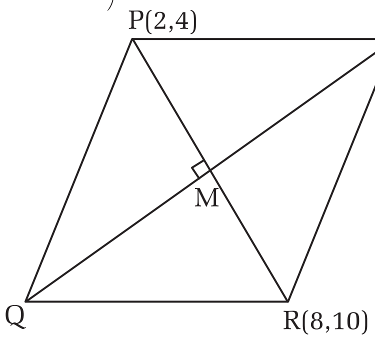
$\therefore x+2y=0$ is the required equation.

14 (a) In rhombus PQRS P(2,4) and R(8,10) are the opposite vertices. Find the equation of diagonal QS.

Solution

In rhombus PQRS the diagonals bisect each other at right angles. Let M be the mid point of the diagonals. Then the coordinates of M are $\left(\frac{2+8}{2},\frac{4+10}{2}\right)=(5,7)$

Slope of PR $(m_1)=\frac{y_2-y_1}{x_2-x_1}=\frac{10-4}{8-2}=1$



Since PR $\perp$ QS, slope of QS $(m_2)=-1$ ($\therefore m_1.m_2=-1$)

Now, equation of QS is

$y-y_1=m(x-x_1)$,

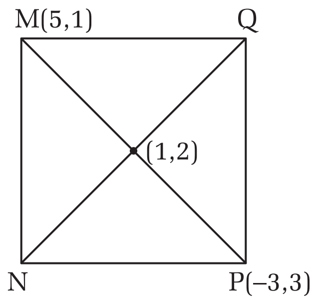
or, $y-7=-1(x-5)$

or, $y-7=-x+5$

$\therefore x+y=12$ is the required equation.

14(b) M(5,1) and P(–3,3) are two opposite vertices of square MNPQ. Find the equation of diagonal QN.

Solution



MNPQ is a square with opposite vertices M(5,1) and P(–3,3).
In a square the diagonals bisect each other. Mid point of MP is

$$\left(\frac{5-3}{2}, \frac{1+3}{2}\right)=(1,2)$$

Slope of MP $(m_2)=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{-3-5}=-\frac{1}{4}$

MP is perpendicular to NQ. Slope of QN $(m_2)=4$

Hence equation of QN passing through (1,2) and wiht slope 4 is given by,

$y-y_1=m(x-x_1)$,

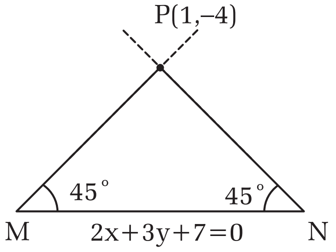
or, $y-2=4(x-2)$

or, $y-2=4x-8$

$\therefore$ $4x-y=6$ is the required equation.

15(a) Determine the equation of straight lines through (1,–4) that make an angle of 45° with the straight line 2x +3y+7=0.

Solution



Let MN be the given line with equation $2x+3y+7=0$

Its slope is $(m_1)=-\frac{\text{Coefficient of x}}{\text{Coefficient of y}}$

$=-\frac{2}{3}$

Let PM and PN be two lines passing through P(1,–4) which make 45° with MN.

Let m_2 be the slope of PM or PN

Then required equation are given by

$y-y_1=m(x-x_1)$

or, $y+4=m(x-1)$............(i)

Angle between the lines $\theta=45°$

Now, $\tan\theta=\pm\frac{(m_1-m_2)}{1+m_1m_2}$

$$\tan45^o=\pm\frac{\left(-\frac{2}{3}-m_2\right)}{1+\left(\frac{-2}{3}\right)m_2}$$

or, $1=\pm\frac{(2+3m_2)}{3-2m_2}$

or, $3-2m_2 = \pm(2+3m_2)$

Taking positive sign, we get

$3-2m_2 = 2+3m_2$

or, $-5m_2 = -1$

$\therefore m_2 = \frac{1}{5}$

Taking negative sign, we get

$3-2m_2 = 2+3m_2$

$\therefore m_2 = -5$

When $m_2 = \frac{1}{5}$, then from equation(i), we get

$y + 4 = \frac{1}{5}(x-1)$

or, $5y + 20 = (x-1)$

$\therefore x - 5y = 21$

When $m_2 = -5$, then, from equation(ii), we get

$y + 4 = -5(x-1)$

or, $5x + y = 1$

Hence the required equations are $x - 5y = 21$ and $5x + y = 1$.

(b) Find the equation of the straight lines passing through the point (3,2) and making angle of 45° with the line x–2y–3=0.

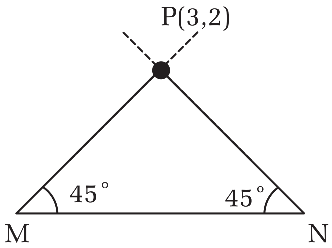
Solution

Let MN be the line with equation $x-2y-3=0$..........(i)

slope is $(m_1) = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$

$= -\frac{1}{-2}$

$= \frac{1}{2}$



Let MP and NP be two lines passing through P(3,2) which

make 45° with MN. and the slope m.

Let the slope of MP or NP be m.

Then required equation are given by

$y - y_1 = m(x - x_1)$

or, $y-2 = m(x-3)$............(i)

Angle between the lines $\theta = 45°$

Now, $\tan\theta = \pm\frac{(m_1 - m_2)}{1 + m_1 m_2}$, where $m_2 = m$

$$\tan 45^o = \pm\frac{\left(m - \frac{1}{2}\right)}{1 + \frac{1}{2}m}$$

or, $1= \pm \dfrac{(1-2m)}{2+m}$

or, $2+m=\pm(1-2m)$

Taking positive sign,

$2+m=1-2m$

or, $3m=-1$

$\therefore m= -\dfrac{1}{3}$

Taking negative sign, we get,

$2+m=-1+2m$

$\therefore m= 3$

When $m= -\dfrac{1}{3}$, then from equation(i),we get

$y-2 = -\dfrac{1}{3}(x-3)$

or, $3y-6 =-x+3$

$\therefore x +3y=9$

When m=3, then from equation(i),we get

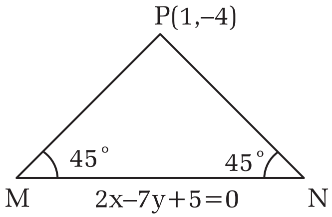
$y-2 =3(x-3)$

or, $3x-y =7$

Hence the required equations are $x + 3y =9$ and $3x-y =7$.

(c) Find the equation of two lines passing through the point (1,–4) and making an angle of 45° with the lines $2x-7y+5=0$.

Solution



Let MN be the line with equation $2x-7y+5=0$

Its slope is $(m_1) = -\dfrac{\text{Coefficient of x}}{\text{Coefficient of y}}$

$=\dfrac{2}{7}$

Let required lines be MP and NP passing through (1,–4) and the slope m.

Let the slope of MP or NP be m.

Then required equation is given by

$y-y_1=m(x-x_1)$

or, $y+4=m(x-1)$............(i)

Angle between the lines $\theta=45^\circ$

Now, $\tan\theta = \pm \dfrac{(m_1-m_2)}{1+m_1m_2}$

$\tan45^o = \pm\dfrac{\left(\dfrac{2}{7}-m\right)}{1+\dfrac{3}{7}m}$

or, $1 = \pm \dfrac{(2-7m)}{7+2m}$

Taking positive sign, we get

$7+2m=2-7m$

$\therefore m = -\dfrac{5}{9}$

Taking negative sign, we get,

$7+2m=-2+7m$

$\therefore m = \dfrac{9}{5}$

When $m = -\dfrac{5}{9}$, from equation(i),we get

$y+4 = -\dfrac{5}{9}(x-1)$

or, $9y+36 = -5x+5$

$\therefore 5x + 9y + 31 = 0$

When $m = \dfrac{9}{5}$, from equation(i),we get

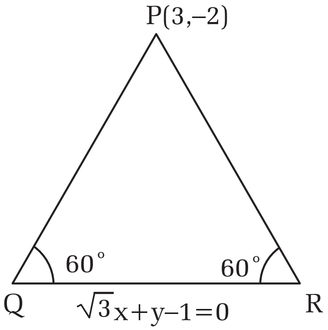
$y+4 = \dfrac{9}{5}(x-1)$

or, $9x-5y = 29$

Hence the required equations are $5x + 9y + 31$ and $9x-5y = 29$.

(d)Find the eq^n. of straight line passing through the point (3,–2) and making an angle of 60° with the line $\sqrt{3}x+y-1=0$.

Solution



Let the given line be QR with equation $\sqrt{3}x+y-1=0$.

Its slope is $(m_1) = -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } y}$

$= -\sqrt{3}$

Let QP and RP be two lines passing through P(3,–2) making angle 60° with QR.

Let the slope of QP or RP be m.

Then the required equation is

$y - y_1 = m(x-x_1)$

or, $y+2=m(x-3)$............(i)

Now, $\tan\theta = \pm \dfrac{m_1-m_2}{1+m_1m_2}$

$\tan 60^o = \pm \dfrac{(-\sqrt{5}-m)}{1+(-\sqrt{3})m}$

or, $\sqrt{3} = \pm \dfrac{(\sqrt{3}+m)}{1-\sqrt{3}m}$

Taking negative sign, we get

$\sqrt{3}-3m=-\sqrt{3}-m$

or, $2m=-2\sqrt{3}$

$\therefore m=\sqrt{3}$

Taking positive sign, we get,

$\sqrt{3}-3m=\sqrt{3}+m$

$\therefore$ m=0

When $m=\sqrt{3}$, then from equation(i),

$y+2=\sqrt{3}(x-3)$

or, $y+2=\sqrt{3}x-3\sqrt{3}$

or, $\sqrt{3}x-y=3\sqrt{3}+2$

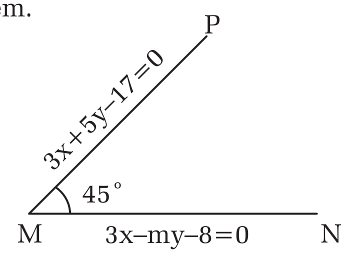
When m = 0, then from equation (i), y + 2 = 0

Hence the required equations are y+2=0 and $\sqrt{3}x-y=3\sqrt{3}+2$.

(e) Determine the value of m so that 3x–my–8=0 will make an angle of 45° with the line 3x+5y–17=0.

Solution

:Let MN and MP be the given lines with angle 45° between them.



Equation MN : 3x–5y–8=0.

slope of MN $(m_1)=-\frac{3}{-m}$

$=\frac{3}{m}$

Equation MN : 3x+5y–17=0

slope of MP(m2)= $-\frac{3}{5}$

Now, $\tan\theta=\pm\dfrac{\left(\frac{3}{m}+\frac{3}{5}\right)}{1+\frac{3}{m}\left(-\frac{3}{5}\right)}$

or, $\tan45°=\pm\dfrac{\frac{3}{m}+\frac{3}{5}}{1+\frac{3}{m}.\left(-\frac{3}{5}\right)}$

or, $1=\pm\dfrac{15+3m}{5m-9}$

or, $5m-9=\pm(15+3m)$

Taking positive sign, we get

$5m-9=15+3m$

or, $2m=24$

$\therefore m=12$

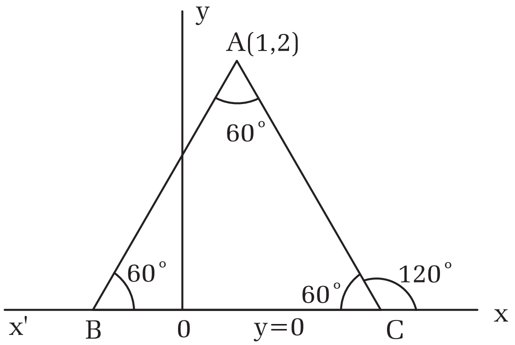
Taking negative sign, we get,

$5m-9=-15-3m$

or, $8m=-6$

$\therefore m=12$ or $-\frac{3}{4}$.

16(a). Find the equation of the sides of an equilateral triangle whose vertex is (1,2) and base is y =0.



Solution

Let ABC be an equilateral triangle with vertex A(1,2) with base y=0

there . $\angle ABC=\angle BCA=\angle CAB=60°$

$\therefore$ slope of $BA=\tan 60°=\sqrt{3}$

$\therefore$ slope at $CA=\tan 120° =-\sqrt{3}$

Now, for equation of BA

let, $(x_1,y_1)=(1,2)$

$(y-2)=\sqrt{3}\ (x-1)$

or, $\sqrt{3}\ x-y+2-\sqrt{3}\ =0$

$\therefore \sqrt{3}\ x-y+2-\sqrt{3}\ =0$

Again, for equation of CA

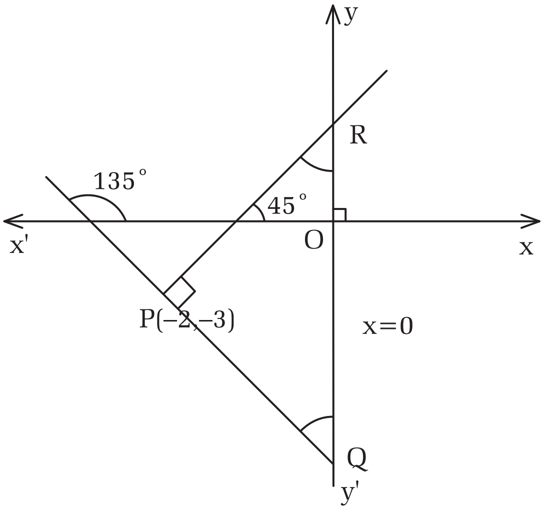
$(y-2)=-\sqrt{3}\ (x-1)$

or, $y-2=-\sqrt{3}\ x-\sqrt{3}$

$\therefore \sqrt{3}\ x+y+\sqrt{3}\ -2=0$

Hence the required equations are $\sqrt{3}\ x-y+2-\sqrt{3}\ =0$ and $\sqrt{3}\ x+y+\sqrt{3}\ -2=0$.

(b). Find the equation of the sides of right angled isoceles triangle whose vertex is at (–2,–3) and base is x =0.



Solution: Let P(–2,–3) be the vertex of right angled isoceles triangle PQR with $\angle P=90°$

For side PR, its slope $(m_1)=\tan 45°$

Equation of PR is

$y-y_1=m(x-x_1)$

or, $y+3=1(x+2)$

$\therefore x-y=1$

Again, for equation of side QP, slope $(m)=\tan 135°=-1$

Equation of QP is $y-y_1=m(x-x_1)$

or, $y+3=-1(x+2)$

or, $x+y+5=0$

$\therefore$ Hence the required equation are $x-y=1$ and $x+y+5=0$.

(c). Find the equation of the sides of right angled isoceles triangle whose vertex is at (2,4) and equation of base x =0.

Solution

Let ΔMNP be an isoceles right angled triangle with $\measuredangle p=90°$

For side NP, slope $(m_1)=\tan 45°=1$

Now, equation of NP is

$y-y_1=m(x-x_1)$,

or, $y-4=1(x-2)$

$\therefore x-y+2=0$

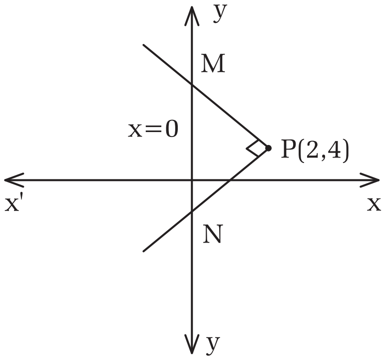
For side MP, slope $(m_2)=\tan 135° = -1$

Equation of MP is $y-y_1=m(x-x_1)$

or, $y-4=-1(x-2)$

$\therefore x+y=6$

Hence the required equation are $x+y+2=0$ and $x+y=6$.



Questions for practice

1. The line $px+3y+5=0$ is perpendicular to the line joining the points (4,3) and (6,–3), find the values of P.
2. Find the equation of a straight line passing through the point of intersection of the straight lines $x-y=7$ and $x+y=15$ and parallel to the line $4x+3y=10$.
3. Find the equation of the line segment PQ which passes through the point (3,4) and the mid point of line segment joining M(–4,–5) and N(7,8).
4. If the line $\frac{x}{a}+\frac{y}{b}=1$ passes through the point of intersection of the lines $x+y=3$ and $2x-3y=1$ and is parallel to the line $x-y=6$, then find the values of a and b.
5. Find the equation of perpendicular bisector of line segment joining M(–4,–5) and N(8,9).

Pair of straight lines

Estimated Teaching periods : 8

1. Objectives

S. N.	Level	Objectives
1.	Knowledge(K)	To define equation of pair of lines. To define homogeneous equation. To tell general equation of second degree. To tell formula to find angle between two lines represented by $ax^2+2hxy+by^2=0$ To state conditions for coincidence and perpendicularity.

2.	Understanding(U)	To find a single equation of a pair of lines. To find pair of lines of given homogeneous equation in x and y. To identify given homogeneous equation represents a pair of perpendicular or coincident lines.
3.	Application(A)	To find angle between two lines represented by general equation of second degree. To find separate equation of two lines represented by general eq^n. of second degree.
4.	Higher Ability (HA)	To derive formula to find angle between two lines represented by $ax^2+2hxy+by^2=0$ and find condition for coincidence and perpendicularity. To show $ax^2+2hxy+by^2=0$ represents a pair of lines through origin.

2. Required teaching materials :

Chart paper with figure of angle between two lines represented by $ax^2+2hxy+by^2=0$.

3. Teaching Learning Activities:

- Take two equations like $x+2y=0$ and $x-2y=0$ and multiply them to get $x^2-4y^2=0$, discuss the conclusion.
- Find two separate equations represented by $x^2+8xy+12y^2=0$, discuss the conclusion.
- Define general equation of second degree in x and y .
- Define a homogeneous equation with examples.
- Show that $ax^2+2hxy+by^2=0$. represents a pair of lines through the origin.
- Discuss how to find the angle between the lines represented by $ax^2+2hxy+by^2=0$. ?
- Discuss the conditions for perpendicularity and coincidence of the lines represented by $ax^2+2hxy+by^2=0$.
- Do some problems from exercise to guide the students.

Note:

i) The equation $ax^2+2hxy+by^2=0$ always represents a pair of lines through the origin in the form of $y=m_1x$ and $y=m_2x$.

2. The angle between the lines represented by $ax^2+2hxy+by^2=0$ is

$$\theta=\tan^{-1}\left(\pm\frac{2\sqrt{h^2-ab}}{a+b}\right)$$

3. The $ax^2+2hxy+by^2=0$ represents a pair of lines, then

   i) Condition for coincidence of the lines $h^2=ab$

   ii) Condition for perpendicularity/orthogonality $a+b=0$

4. The roots of quadratic equation $ax^2+bx+c=0$ are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Some solved problems

1. Find the single equation represented by the pair of straight lines.

$3x+4y=0$ and $2x-3y=0$

Solution

Given equations of lines are

$3x+4y=0$ and $2x-3y=0$

the single equation represented by above equations is

$(3x+4y)(2x-3y)=0$

or, $6x^2-9xy+8xy-12y^2=0$

or, $6x^2-xy-12y^2=0$

$\therefore$ $6x^2-xy-12y^2=0$ is the required equation.

2. Find the separate equations of of straight lines represented by the following equations.

$x^2+y^2-2xy+2x-2y=0$

Solution

Here, $x^2+y^2-2xy+2x-2y=0$

or, $(x^2-2xy+y^2)+2(x-y)=0$

or, $(x-y)^2+2(x-y)=0$

or, $(x-y)(x-y+2)=0$

Hence $x-y=0$ and $x-y+2=0$ are the required equations.

3. Show that $6x^2-5xy-6y^2=0$ represents a pair of lines.

Solution

Here, $6x^2-5xy-6y^2=0$

or, $6x^2-9xy+4xy-6y^2=0$

or, $3x(2x-3y)+2y(2x-3y)=0$

or, $(2x-3y)(3x-2y)=0$

Either $2x-3y=0$ or $3x-2y=0$,

each of which are equation of straight lines.

4. Determine the lines represented by each of the given equations.

(a) $x^2+2xy+y^2-2x-2y-15=0$

Solution

Here, $x^2+2xy+y^2-2x-2y-15=0$

or, $(x+y)^2-2(x+y)-15=0$

or, $(x^2+y)^2-5(x+y)+3(x+y)-15=0$

or, $(x+y)$ $(x+y-5)+3(x+y-5)=0$

or, $(x+y-5)$ $(x+y+3)=0$

Either, or, $(x+y-5)=0$ or $(x+y+3)=0$

Each of which represents a straight line

Hence the given equation represents a pair of lines.

(b) $2x^2+xy-3y^2+2y-8=0$

Solution

Here, $2x^2+xy-3y^2+2y-8=0$

or, $2x^2+xy+(-3y^2+10y-8)=0$

which is in the form of $ax^2+bx+c=0$,

where, $a=2$, $b=y$ and $c=-3y^2+10y-8=0$

By using formula,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-y\pm\sqrt{y^2-4.2.(-3y^2+10y-8)}}{2.2}$$

$$=\frac{-y\pm\sqrt{y^2+24y^2-80y+64}}{2.2}$$

$$=\frac{-y\pm\sqrt{(5y-8)^2}}{4}$$

Taking positive sign, we get

$4x=-y+5y-8$

or, $4x-4y+8=0$

or, $x-y+2=0$

Taking negative sign, we get

$4x=-y-5y+8$

or, $4x+6y-8=0$

or, $2x+3y-4=0$

Hence the required equation are $x-y+2=0$ and $2x+3y=4$

(c) $x^2+9y^2+6xy+4x+12y-5=0$

Solution

Here, $x^2+6xy+9y^2+4x+12y-5=0$

or, $x^2+2.x.3y+(3y)^2+4(x+3y)-5=0$

or, $(x+3y)^2+4(x+3y)-5=0$

or, $(x+3y)^2+5(x+3y)-(x+3y)-5=0$

or, $(x+3y)(x+3y+5)-1(x+3y+5)=0$

or, $(x+3y+5)(x+3y-1)=0$

Hence the required equation are $x+3y+5=0$ and $(x+3y-1)=0$

5. Prove that two lines represented by the following equations are perpendicular to each other.

$9x^2-13xy-9y^2+2x-3y+7=0$

Solution

Here, $9x^2-13xy-9y^2+2x-3y+7=0$

Comparing at with $ax^2-2hxy+by^2+2gx+2fy+c=0$

we get, $a=9$, $h=-\frac{13}{2}$, $b=-9$, $g=1$, $f=-\frac{3}{2}$, $c=7$

condition for perpendicularity

$a+b=0$

Here, $a+b=9-9=0$

$\therefore a+b=0$

Hence the lines represented by given equation are perpendicular to each other.

6. Show that the given equations represents a pair of coincident lines.

$9x^2-6xy+y^2 = 0$

Solution

Here, $9x^2-6xy+y^2 = 0$

Comparing it with $ax^2-2hxy+by^2=0$ we get,

$a=9$, $b=1$, $g=1$, $h=-3$

condition for coincident,

$h^2=ab$

ie. $(-3)^2 = 9.1$

$\therefore 9=9$(true) proved.

7. Find the value of λ when the lines represented by each of the following equations are coincident.

$(10\lambda-1)x^2+(5\lambda+3)xy+(\lambda-1)y^2=0$

Solution: Comparing given equation

$(10\lambda-1)x^2+(5\lambda+3)xy+(\lambda-1)y^2=0$ with

$ax^2+2hxy+by^2+2gx+2fy+c=0$, we get,

$a= 10\lambda-1$, $h=\frac{5\lambda+3}{2}$, $b= \lambda-1$

Now, condition for coincident

$h^2=ab$

or, $\left(\frac{5\lambda+3}{2}\right)^2 = (10\lambda-1).(\lambda-1)$

or, $\frac{25\lambda^2+30\lambda+9}{4} = 10\lambda^2-10\lambda-\lambda+1$

or, $25\lambda^2+30\lambda+9= 40\lambda^2-44\lambda+4$

or, $15\lambda^2-74\lambda-5=0$

or, $15\lambda^2-75\lambda+\lambda-5=0$

or, $15\lambda(\lambda-5)+1(\lambda-5)=0$

or, $(\lambda-5)(15\lambda+1)=0$

Either $\lambda-5=0$ or $\lambda=5$

or, $15\lambda+1=0$ or, $\lambda=-\dfrac{1}{15}$

$\therefore \lambda=5,-\dfrac{1}{15}$

8. Find the value of λ when the give equation represents a pair of lines perpendicular to each other.

$(3\lambda+4)x^2-48xy-\lambda^2y^2=0$

Solution

Comparing the given equation with

$ax^2+2hxy+by^2=0$, we get

$a= 3\lambda+4,\ h=-24,\ b=-\lambda^2$

Condition for perpendicularity, $a+b=0$

or, $3\lambda+4-\lambda^2= 0$

or, $\lambda^2-3\lambda-4= 0$

or, $\lambda^2- 4\lambda+\lambda - 4=0$

or, $\lambda(\lambda- 4)+1(\lambda - 4)=0$

or, $(\lambda- 4)(\lambda +1)=0$

Either, $\lambda- 4=0$ or $\lambda=4$

or, $\lambda+1=0$ or $\lambda=-1$

$\therefore \lambda=4,-1$

9. Find the angle between the following pair of lines.

(a) $x^2+9xy+14y^2=0$

Solution

Here, $x^2+9xy+14y^2=0$

comparing it with $ax^2+2hxy+by^2=0$, we get

$a=1,\ h=\dfrac{9}{2},\ b=14$

Let θ be the angle between them.

$$\tan\theta=\frac{\pm 2\sqrt{h^2-ab}}{a+b}$$

$$=\frac{\pm 2\sqrt{\frac{81}{4}-14}}{1+14}$$

$$=\pm 2\,\frac{\sqrt{81-56}}{2\times 15}$$

$$=\pm\frac{5}{15}$$

$=\pm\frac{1}{3}$

taking positive sign, $\tan\theta=\frac{1}{3}$

$\theta=71.57°$

taking negative sign, $\tan\theta=-\frac{1}{3}$

$\theta=108.43°$

(b) $2x^2+7xy+3y^2=0$

Solution

Comparing it with $ax^2+2hxy+by^2=0$, we get

$a=2,\ h=\frac{7}{2},\ b=3$

Let θ be the angle between the pair of lines, we get,

$$\tan\theta=\frac{\pm 2\sqrt{h^2-ab}}{a+b}$$

$$=\frac{\pm 2\sqrt{\frac{49}{4}-2.3}}{2+3}$$

$$=\pm 2\frac{5}{2.5}$$

$=\pm 1$

taking positive sign, we get,

$\tan\theta=1$

or, $\tan\theta=\tan 45°$

$\therefore\ \theta=45°$

taking negative sign, we get

$\tan\theta=-1$

or, $\tan\theta=\tan 135°$

$\therefore\ \theta=135°$

Hence, $\theta=45°, 135°$.

(c) $x^2-7xy+y^2=0$

Solution

Comparing it with $ax^2+2hxy+by^2=0$, we get

$a=1,\ h=-\frac{7}{2},\ b=1$

Let θ be the angle between the pair of lines, we get,

$$\tan\theta=\frac{\pm 2\sqrt{h^2-ab}}{a+b}$$

$$=\frac{\pm 2\sqrt{\frac{49}{4}-1}}{1+1}$$

$= \pm\frac{3\sqrt{5}}{2}$

taking positive sign, we get,

$\tan\theta=\frac{3\sqrt{5}}{2}$

$\therefore \theta=73.4°$

taking negative sign, we get

$\tan\theta=-\frac{3\sqrt{5}}{2}$

or, $\tan\theta=\tan135°$

$\therefore \theta=106.6°$

Hence, $\theta=73.4°, 106.6°$.

(d) $9x^2-13xy-9y^2+2x-3y+7=0$

Solution

comparing it with $ax^2+2hxy+by^2+2gx+2fy+c=0$, we get

$a=9,\ h=-\frac{13}{2},\ b=-9,\ g=1,\ f=-\frac{3}{2},\ c=7$

Let θ be the angle between the pair of lines, we get,

$\tan\theta=\frac{\pm2\sqrt{h^2-ab}}{a+b}$

$=\frac{\pm2\sqrt{\frac{169}{4}+81}}{9-9}$

$=\infty$

$=\tan90°$

$\therefore \theta=90°$

(e) $x^2+6xy+9y^2+4x+12y-5=0$

Solution

comparing it with $ax^2+2hxy+by^2+2gx+2fy+c=0$, we get

$a=1,\ h=3,\ b=9,\ g=2,\ f=6,\ c=-5$

Let θ be the angle between the pair of lines,

$\tan\theta=\frac{\pm2\sqrt{h^2-ab}}{a+b}$

$=\frac{\pm2\sqrt{9-9}}{1+9}$

$=0$

or, $\tan\theta=\tan0°$

$\therefore \theta=0°$

10. Find the angle between the following pair of lines.

(a) $x^2-2xy\cot\alpha-y^2=0$

Solution

comparing given equation with $ax^2+2hxy+by^2=0$, we get

$a=1$, $h=-\cot\alpha$, $b=-1$

Let θ be the angle between the pair of lines,

$$\tan\theta=\frac{\pm 2\sqrt{h^2-ab}}{a+b}$$

$$=\frac{\pm 2\sqrt{\cot^2\alpha+1}}{1-1}$$

$=\infty$

or, $\tan\theta=\tan 90°$

$\therefore\ \theta=90°$

(b) $x^2+2xy\operatorname{cosec}\alpha+y^2=0$

Solution

Comparing given equation with $ax^2+2hxy+by^2=0$, we get

$a=1$, $h=\operatorname{cosec}\alpha$, $b=1$

Let θ be the angle between the pair of lines,

$$\tan\theta=\frac{\pm 2\sqrt{h^2-ab}}{a+b}$$

$$=\frac{\pm 2\sqrt{\operatorname{cosec}^2\alpha-1}}{1+1}$$

$=\pm\cot\alpha$

or, $\tan\theta=\tan(90°\pm\alpha)$

$\therefore\ \theta=90°\pm\alpha$

(c) $y^2+2xy\cot\alpha-x^2=0$

Solution

The given equation can be written as

$y^2+2xy\cot\alpha-x^2=0$

Comparing given equation with $ax^2+2hxy+by^2=0$, we get

$a=1$, $h=-\cot\alpha$, $b=-1$

Let θ be the angle between the pair of lines,

$$\tan\theta=\frac{\pm 2\sqrt{h^2-ab}}{a+b}$$

$$=\frac{\pm 2\sqrt{\cot^2\alpha+1}}{1-1}$$

$=\infty$

$\therefore\ \theta=90°$

11. Find the separate equation of two lines represented by the following equations.

(a) $x^2+2xy\cot\alpha-y^2=0$

Solution

Here, $x^2+2xy\cot\alpha-y^2=0$

or, $x^2-2xy\cot\alpha-y^2(\operatorname{cosec}^2\alpha-\cot^2\alpha)=0$

or, $x^2+2xy\cot\alpha+y^2\cot^2\alpha-y^2\operatorname{cosec}^2\alpha=0$

or, $(x+y\cot\alpha)^2-(y\operatorname{cosec}\alpha)^2=0$

or, $(x+y\cot\alpha+y\operatorname{cosec}\alpha)(x+y\cot\alpha-y\operatorname{cosec}\alpha)=0$

Hence the required separate equations are

$x+y(\cot\alpha+\operatorname{cosec}\alpha)=0$

and $x-y(\cot\alpha-\operatorname{cosec}\alpha)=0$

(b) $x^2+2xy\sec\alpha+y^2=0$

Solution

Here, $x^2+2xy\sec\alpha+y^2=0$

or, $x^2+2xy\sec\alpha+y^2(\sec^2\alpha-\tan^2\alpha)=0$

or, $x^2+2xy\sec\alpha+y^2\sec^2\alpha-y^2\tan^2\alpha=0$

or, $(x+y\sec\alpha)^2-(y\tan\alpha)^2=0$

or, $(x+y\sec\alpha+y\tan\alpha)(x+y\sec\alpha-y\tan\alpha)=0$

Hence the required separate equations are

$x+y(\sec\alpha+\tan\alpha)=0$

and $x+y(\sec\alpha-\tan\alpha)=0$

(c) $x^2+2xy\operatorname{cosec}\alpha+y^2=0$

Solution

Here, $x^2+2xy\operatorname{cosec}\alpha+y^2=0$

or, $x^2+2xy\operatorname{cosec}\alpha+y^2(\operatorname{cosec}^2\alpha-\cot^2\alpha)=0$

or, $(x^2+2xy\operatorname{cosec}\alpha+y^2\operatorname{cosec}\alpha)-y^2\cot^2\alpha=0$

or, $(x+y\operatorname{cosec}\alpha)^2-(y\cot\alpha)^2=0$

or, $(x+y\operatorname{cosec}\alpha+y\cot\alpha)(x+y\operatorname{cosec}\alpha-y\cot\alpha)=0$

Hence the required separate equations are

$x+y(\operatorname{cosec}\alpha+\cot\alpha)=0$

and $x+y(\operatorname{cosec}\alpha-\cot\alpha)=0$

(d) $x^2-2xy\cot2\alpha-y^2=0$

Solution

Here, $x^2-2xy\cot2\alpha-y^2=0$

or, $x^2-2xy\cot2\alpha-y^2(\operatorname{cosec}^22\alpha-\cot^22\alpha)=0$

or, $x^2-2xy\cot2\alpha+y^2\cot^22\alpha-y^2\operatorname{cosec}^22\alpha=0$

or, $(x-y\cot2\alpha)^2-(y\operatorname{cosec}2\alpha)^2=0$

or, $(x - y\cot 2\alpha + y\text{cosec}2\alpha)(x - y\cot 2\alpha - y\text{cosec}2\alpha)=0$

or, $\{x - y(\cot 2\alpha - \text{cosec}2\alpha)\}\{x - y(\cot 2\alpha + y\text{cosec}2\alpha)\}=0$

Hence the required separate equations are

$x - y(\cot 2\alpha - \text{cosec}2\alpha)=0$

and $x - y(\cot 2\alpha + \text{cosec}2\alpha)=0$

14(a). Find the equation of two lines represented $2x^2 + 7xy + 3y^2=0$. Find the point of intersection . Also find the angle between them.

Solution

Here, $2x^2 + 7xy + 3y^2=0$

or, $2x^2 + 6xy + xy + 3y^2=0$

or, $2x(x + 3y) + y(x+3y) =0$

or, $(x + 3y)(2x + y)=0$

Hence the equation of two lines are

$x + 3y = 0$.................(i)

$2x + y = 0$..............(ii)

solving equations (i) and (ii) , we get $x=0$ and $y = 0$

$\therefore$ (0,0) is the point of intersection of lines (i) and (ii)

To find the angle between the lines.

comparing the given equation $2x^2 + 7xy + 3y^2=0$

we get, $a= 2$, $h =\frac{7}{2}$ and $b =3$

Let θ be the angle between the lines

$$\tan\theta = \frac{\pm 2\sqrt{h^2-ab}}{a+b}$$

$$= \frac{\pm 2\sqrt{\frac{49}{4} - 2.3}}{1+1}$$

$$= \pm\frac{2 . \frac{5}{2}}{5}$$

$= \pm 1$

taking positive sign, we get,

$\tan\theta= 1$

$= \tan 45°$

$\therefore \theta=45°$

taking negative sign, we get

$t\tan\theta= -1$

$= \tan 135°$

$\theta=45°$

$\therefore \theta= 45°$ or $135°$

14(b). Find the separate equation of two lines represented $x^2 - 5xy + 4y^2 = 0$. Also find the angle between them and their point of intersection.

Solution

Here, $x^2 - 5xy + 4y^2 = 0$

or, $x^2 - 5xy + 4y^2 = 0$

or, $x(x - 4y) - y(x - 4y) = 0$

or, $(x - 4y)(x - y) = 0$

Hence the equation of two lines are

$x - 4y = 0$.................(i)

$x - y = 0$..............(ii)

solving equations (i) and (ii) , we get

$(x, y) = (0,0)$

comparing the given equation $x^2 - 5xy + 4y^2 = 0$, we get,

$a = 1,\ h = -\frac{5}{2}$ and $b = 4$

Let θ be the angle between the lines

$$\tan\theta = \frac{\pm 2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{\pm 2\sqrt{\frac{25}{4} - 1.4}}{1+4}$$

$$= \pm\frac{5}{2}$$

$$\theta = \tan^{-1}\left(\pm\frac{3}{5}\right),$$

$\therefore \theta = 31°, 149°$

(c) Find the separate equations of two lines represented by the equation x2– 2xycosecα + y2=0 . Also find the angle between them.

Solution

Here, $x^2 - 2xy\,\text{cosec}\alpha + y^2 = 0$

or, $x^2 - 2xy\,\text{cosec}\alpha + y^2(\text{cosec}^2\alpha - \cot^2\alpha) = 0$

or, $(x^2 - 2xy\,\text{cosec}\alpha + y^2\text{cosec}\alpha) - y^2\cot^2\alpha = 0$

or, $(x - y\,\text{cosec}\alpha)^2 - (y\cot\alpha)^2 = 0$

or, $(x - y\,\text{cosec}\alpha + y\cot\alpha)(x - y\,\text{cosec}\alpha - y\cot\alpha) = 0$

Either

$x - y(\text{cosec}\alpha - \cot\alpha) = 0$...............(i)

$x - y(\text{cosec}\alpha + y\cot\alpha) = 0$...............(ii)

Which are the required equation of straight lines

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 1$, $h = -\text{cosec}\alpha$, $b = 1$

Let θ be the angle between the lines

$$\tan\theta = \frac{\pm 2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{\pm 2\sqrt{\text{cosec}^2\alpha - 1}}{1 + 1}$$

$= \pm\cot\alpha$

$= \pm\tan(90° \pm \alpha)$

$= \tan(90° \pm \alpha)$

$\therefore \theta = 90° \pm \alpha$

15(a). Find the pair of lines parallel to the lines $x^2 - 3xy + 2y^2 = 0$ and passing through the origin.

Solution

Here, $x^2 - 3xy + 2y^2 = 0$

or, $x^2 - 2xy - xy + 2y^2 = 0$

or, $x(x - 2y) - y(x - 2y) = 0$

or, $(x - 2y)(x - y) = 0$

$x - 2y = 0$..........(i)

and $x - y = 0$ ....(ii) are the required equations.

Equations of the lines parallel to (i) and (ii) and passing through the origin are,

$x - 2y = 0$ and $x - y = 0$

(b). Find the pair of lines parallel to the $2x^2 - 7xy + 5y^2 = 0$ and passing through the point (1,2).

Solution

Here, $2x^2 - 7xy + 5y^2$

or, $x^2 - 5xy - 2xy + 5y^2 = 0$

or, $x(2x - 5y) - y(2x - 5y) = 0$

or, $(2x - 5y)(x - y) = 0$

$2x - 5y = 0$..........(i)

and $x - y = 0$ ....(ii)

Equations of the lines parallel to (i) and (ii)

are, $2x - 5y + k_1 = 0$ ....(iii)

and $x - y + k_2 = 0$ ....(iv)

The lines (i) and (ii) passes through the point (1,2), we get,

$2 \times 1 - 5 \times 2 + k_1 = 0 \Rightarrow k_1 = 8$

and $1 - 2 + k_2 = 0 \Rightarrow k_2 = 1$

put the values of k_1 and k_2 in eq^n. (iii) and (iv), we get,

$2x - y + 8 = 0$ and $x - y + 1 = 0$.

16(a). Find the pair of lines perpendicular to the lines $x^2 - 5xy + 4y^2 = 0$ and passing through the origin.

Solution

Here, $x^2 - 5xy + 4y^2 = 0$

or, $x^2 - 4xy - xy + 4y^2 = 0$

or, $x(x - 4y) - y(x - 4y) = 0$

or, $(x - 4y)(x - y) = 0$

Either, $x - 4y = 0$..........(i)

or, $x - y = 0$ ....(ii)

From equations (i), equation of the lines perpendicular to it and passing through the origin are,

$4x + y = 0$

Also, from equation (ii), equation of line perpendicular to it is

$x + y = 0$

$\therefore$ Required pair of lines are

$4x + y = 0$ and $x + y = 0$.

(b). Find the equation of two straight lines which pass through the point (2,3) and perpendicular to the lines $x^2 - 6xy + 8y^2 = 0$.

Solution

Here, $x^2 - 6xy + 8y^2 = 0$

or, $x^2 - 4xy - 2xy + 8y^2 = 0$

or, $x(x - 4y) - 2y(x - 4y) = 0$

or, $(x - 4y)(x - 2y) = 0$

Either $x - 4y = 0$..........(i)

or, $x - 2y = 0$ ....(ii)

Equations of the lines perpendicular to above lines are,

$4x + y + k_1 = 0$ ....(iii)

and $2x - y + k_2 = 0$ ....(iv)

Both of above lines passes through the point (2,3), we get,

$\therefore 4 \times 2 + 3 + k_1 = 0 \rightarrow k_1 = -11$

and $2 + 2\times3 + k_2 = 0 \rightarrow k_2 = -7$

put the values of k_1 and k_2 in eq^n. (iii) and (iv), we get,

$4x + y - 11 = 0$ and $2x+y - 7 = 0$ which are the required equation of the lines.

17 (a). Find the two separate equations when the lines represented by $kx^2 + 8xy - 3y^2 = 0$ are perpendicular to each other .

Solution

Here, $kx^2 + 8xy - 3y^2 = 0$ ....(i)

Comparing it with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = k, b = -3, h = 4$

Conditions for perpendicularity is

$a + b = 0$

or, $k - 3 = 0$ or, $k \Rightarrow 3$

put the value of k in eq^n.(i), we get

$3x^2 + 8xy - 3y^2 = 0$

or, $3x^2 + 9xy - xy - 3y^2 = 0$

or, $3x(x + 3y) - y(x + 3y) = 0$

or, $(x + 3y)(3x - y) = 0$

$\therefore$ $x + 3y = 0$ and $3x - y = 0$ are the required equation of the lines.

(b). Find the two separate equations when the lines represented by $6x^2 + 5xy - ky^2 = 0$ are perpendicular.

Solution

Here, $6x^2 + 5xy - ky^2 = 0$ ....(i)

Comparing it with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 6, b = -k, h = \frac{5}{2}$

conditions for perpendicularity is

$a + b = 0$

or, $6 - k = 0$

$\therefore k = 6$

put the value of k in eq^n.(i), we get

$6x^2 + 5xy - 6y^2 = 0$

or, $6x^2 + 9xy - 4xy - 6y^2 = 0$

or, $3x(2x + 3y) - 2y(2x + 3y) = 0$

or, $(2x + 3y)(2x - 2y) = 0$

$\therefore$ $2x + 3y = 0$ and $2x - 2y = 0$ are the required equation of the lines.

18(a). Show that the pair of lines $3x^2 - 2xy - y^2 = 0$ are parallel to the lines $3x^2 - 2xy - y^2 - 5x + y + 2 = 0$

Solution

Here, $3x^2 - 2xy - y^2 = 0$

or, $3x^2 - 3xy + xy - y^2 = 0$

or, $3x(x - y) + y(x - y) = 0$

or, $(x - y)(3x + y) = 0$

Either, $x - y = 0$ ....(i)

$3x + y = 0$ ....(ii)

Also, $3x^2 - 2xy - y^2 - 5x + y + 2 = 0$

$3x^2 - (2y + 5)x + (y - y^2 + 2) = 0$

which is in the form of

$ax^2 + bx + c = 0$

where a = 3, b = – (2y + 5), c = y – y^2 + 2

Comparing it with $ax^2 + 2hxy + by^2 = 0$, we get,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{(2y+5)\pm\sqrt{(2y+5)^2-4\,.\,3.(y-y^2+2)}}{2\,.\,3}$$

$$=\frac{(2y+5)\pm\sqrt{4y^2+20y+25-12y+12y^2-12)}}{6}$$

$$=\frac{2y\pm\sqrt{16y^2+8y+1}}{6}$$

$$=\frac{2y\pm\sqrt{(4y+1)^2}}{6}$$

= 2y ± (4y + 1)

Taking positive sign, we get

6x = 2y+4y + 1

or, 6x – 6y – 1=0 .... (iii)

or, x–y+2=0

Taking negative sign, we get

6x = 2y – 4y – 1

or, 6x + 2y + 1=0 .... (iv)

From eq^n.(i), slope (m_1) = 1

From equation (iii), slope (m_3) = 1 $\quad\therefore m_1 = m_3$

The lines (i) and (iii) are parallel to each other.

Again, from equation (ii), slope(m_2) = –3

from equation (iv), slope (m_4) = $-\frac{6}{2}=-3$

$\therefore m_2 = m_4$

$\therefore$ The lines (ii) and (iv) are parallel to each other .

proved

18(b). Show that the pair of lines $4x^2 - 9y^2 = 0$ and $9x^2 - 4y^2 = 0$ are perpendicular to each other.

Solution

Here, $4x^2 - 9y^2 = 0$

or, (2x – 3y) (2x + 3y) = 0

Either, 2x – 3y = 0 ....(i)

or, 2x + 3y = 0 ....(ii)

Again ,$9x^2 - 4y^2 = 0$

$(3x + 2y)(3x - 2y) = 0$

Either, $3x + 2y = 0$ ....(iii)

or, $3x - 2y = 0$ ....(iv)

From equation (i), slope $(m_1) = \frac{2}{3}$

From equation (iii), slope $(m_3) = -\frac{3}{2}$

$\therefore m_1 . m_3 = \left(\frac{2}{3}\right)\left(-\frac{3}{2}\right) = -1$

The lines (i) and (iii) are perpendicular to each other.

Again, from equation (ii), slope$(m_2) = -\frac{2}{3}$

from equation (iv), slope $(m_4) = \frac{3}{2}$

Now, $m_2 . m_4 = \left(-\frac{2}{3}\right)\left(\frac{3}{2}\right) = -1$

$\therefore$ The lines (ii) and (iv) are perpendicular to each other . **proved**

Questions for practice

1. Find the single equation representing the pair of lines $y = x$ and $y = -x$.
2. Find the two separate equations represented by

i) $2x^2 + 7xy + 3y^2 = 0$ ii) $x^2 + 2xy\sec\theta + y^2 = 0$

3. Determine the two straight lines represented by $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$
4. Find the equation to the straight lines through the origin and at right angles to the lines $x^2 - 5xy + 4y^2 = 0$
5. Find the value of k when the pair of lines represented by $(k + 2)x^2 + 8xy + 4y^2 = 0$ are coincident.
6. Show that the angle between a pair of lines represented by $2x^2 - 7xy + 3y^2 + 2x - 6y=0$ is $60°$.
7. Find the two separate equations of straight lines passing through the point (1,0) and parallel to the lines represented by the equation

   $x^2 + 3xy + 2y^2 = 0$
8. Find the equation of two lines which pass through the point (3, –1) and perpendicular to the pair $x^2 - xy - 2y^2 = 0$.
9. Find the value of p so that the two lines represented by the equation $(p + 1)x^2 - 12xy + 9y^2 = 0$ are coincident.
10. Prove that the lines represented by $x^2 - 7xy + 12y^2 = 0$ are perpendicular to the lines represented by $12x^2 + 7xy + y^2 = 0$

Conic Section and Circle

Estimated Teaching periods : 10 Hours

1. Teaching Objectives

S,N.	Level	Objectives
1.	Knowledge(K)	To define terms vertex axis, generator of a cone. To define ellipse, parabola, hyperbola.
2.	Understanding (U)	To identify types of conic sections from given figures. – To derive equation of circle $x^2 + y^2 = a^2$ and $(x - h)^2 + (y - k)^2 = r^2$ by using distance formula.
3.	Application (A)	To use equation of circles $(x - h)^2 + (y - k)^2 = r^2$ $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ to find equation of circles.
4.	Higher Ability (HA)	To solve verbal problems related to circle – eq^n of circle passing through three or more given points. To show given four points concyclic.

2. Required teaching materials

- diagrams of conic sections.
- graph papers.

3. Teaching strategies

- Discuss different types of conic sections
- Circle, parabola, ellipse, hyperbola by using plane figures of intersection of a plane and a cone.
- Discuss to derive the following equations of circles

  $x^2 + y^2 = a^2$
- $(x - h)^2 + (y - k)^2 = r^2$, $(x - h)^2 + (y - h)^2 = h^2$, $(x - k)^2 + (y - k)^2 = k^2$
- $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
- $x^2 + y^2 + 2gx + 2fy + c = 0$

To each of formula, illustrated examples are to be given.

Notes :

Equations of circles in different forms.

i) Equation of circle with centre at the origin 0(0,0) and radius r : $x^2 + y^2 = a^2$

ii) Equation of circle with centre at (h, k) and radius r; $(x - h)^2 + (y - k)^2 = r^2$

iii) Equation of circles touching x – axis ie. r = k,

$(x - k)^2 + (y - k)^2 = r^2$

iv) Equation of a circle touching y–axis, ie. r = h:

$(x - h)^2 + (y - k)^2 = r^2$

v) Equation of a circle touching both axis : h = k = r

$(x - h)^2 + (y - h)^2 = h^2$,

vi) Equation of a circle in a diameter form

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

vii) General Equation of circle : $x^2 + y^2 + 2gx + 2fy + c = 0$

where center $(h,k) = (-g, -f)$

radius $(r) = \sqrt{g^2 + f^2 - c}$

If circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through origin, then $c = 0$, equation of circle is

$x^2 + y^2 + 2gx + 2fy = 0$.

viii) The point of intersection of two diameters is the center of the circle.

Some solved problems

1. Write radius and centre of circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution

Equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Radius $(r) = \sqrt{g^2 + f^2 - c}$

centre $(h,k) = (-g, -f)$

2. Find the equation of circle with centre (–2, –3) and radius 6 units.

Solution

Here, centre $(h,k) = (-2, -3)$

radius $(r) = 6$ units

Equations of circle is $(x - h)^2 + (y - k)^2 = r^2$

ie. $(x + 2)^2 + (y + 3)^2 = 6^2$

or, $x^2 + 2.x.2 + 2^2 + y^2 + 2.y.3 + 3^2 = 36$

or, $x^2 + y^2 + 4x + 6y = 36$

3. Find the equation of circle whose end of diameters are (4,5) and (–2, –3).

Solution

Here, centre $(x_1, x_2) = (4,5)$, $(y_1, y_2) = (-2, -3)$

Equations of the required circle is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

ie. $(x - 4)(x + 2) + (y - 5)(y + 3) = 0$

or, $x^2 - 2x - 8 + y^2 - 2y - 15 = 0$

ie. $x^2 + y^2 - 2x - 2y = 23$

$\therefore x^2 + y^2 - 2x - 2y = 23$ is the required equation.

4. Find the centre and radius of the circles.

(a) $x^2 + y^2 + 6x + 4y - 12 = 0$

Solution

Here, $x^2 + y^2 + 6x + 4y - 12 = 0$

comparing it with $x^2 + y^2 + 2gx + 2fy + c = 0$

$g = 3, f = 2, c = -12$

centre $(-g, -f) = (-3, -2)$

Radius (r) $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{9 + 4 - 12}$

$= \sqrt{5}$

= 5 units.

(b) $9x^2 + 9y^2 - 36x + 6y = 107$

Solution

Here, $9x^2 + 9y^2 - 36x + 6y = 107$

dividing both sides by 9, we get,

$x^2 + y^2 - 4x + \frac{2y}{3} = \frac{107}{9}$

comparing it with $x^2 + y^2 + 2gx + 2fy + c = 0$

we get, $g = -2$, $f = \frac{1}{3}$, $c = -\frac{107}{9}$

centre $(-g, -f) = \left(2, -\frac{1}{3}\right)$

Radius (r) $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{2^2 + \left(-\frac{1}{3}\right)^2 + \frac{107}{9}}$

$= \sqrt{4 + \frac{1}{9} + \frac{107}{9}}$

$= \sqrt{\frac{36 + 1 + 107}{9}}$

$= \sqrt{\frac{144}{9}}$

$= \sqrt{16}$

$= 4$

(c) $x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta = 0$

Solution

Here, $x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta = 0$

comparing it with $x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -a\cos\theta$, $h = -a\sin\theta$, we get centre $(-g,-f) = (a\sin\theta, a\sin\theta)$

Radius $(r) = \sqrt{g^2 + f^2 - c}$

$= \sqrt{(-a\cos\theta)^2+(-a\sin\theta)^2}$

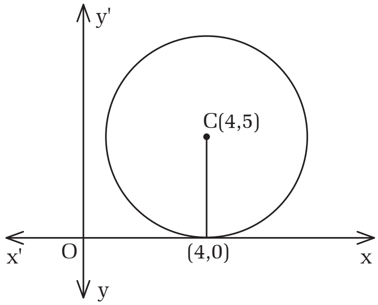
$= \sqrt{a^2\cos^2\theta + a^2\sin^2\theta}$

$= \sqrt{a^2(\sin^2\theta + \cos^2\theta)}$

= a units.

5. Find the equation of circle whose centre is (4,5) and touches x – axis.

Solution



Here centre $(h,k) = (4,5)$

The required circle touches x – axis at (4,0) and radius (r) = k=5

Now, the equation of the circle is, $(x-h)^2 + (y-k)^2 = k^2$

ie. $(x-4)^2 + (y-5)^2 = 5^2$

$x^2 - 8x + 16 + y^2 - 10y + 25 = 25$

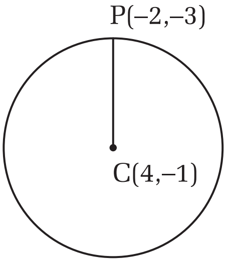
$x^2 + y^2 - 8x - 10y + 16 = 0$ is the required equation.

6. Find the equation of a circle whose centre is (4,–1) and passing through (–2,–3) .

Solution

Distance between the centre (4,–1) and point (–2,–3) is the radius of the circle.

radius (r) = distance between C(4,–1) and P(–2,–3)



$= \sqrt{(-2-4)^2+(-3+1)^2}$

$= \sqrt{36 + 4}$

$= \sqrt{40}$

$= 2\sqrt{10}$ units.

Now, the equation of the circle is, $(x-h)^2 + (y-k)^2 = r^2$

ie. $(x-4)^2 + (y+1)^2 = 40$

$x^2 - 8x + 16 + y^2 + 2y + 1 = 40$

$x^2 + y^2 - 8x + 2y = 23$ is the required equation of the circle.

7. Find the equation of a circle whose centre is the p oint of intersection of $x + 2y - 1 = 0$ and $2x - y - 7 = 0$ and passing through the point (6,4) .

Solution

Given equation of lines are,

$x + 2y - 1 = 0$ ....(i)

$2x - y - 7 = 0$ ....(ii)

Solving equations (i) and (ii) we get, $(x, y) = (3, -1)$.

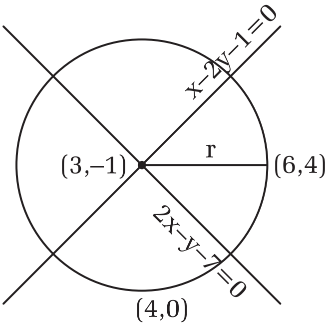
The point of intersection of the lines (i) and (ii) is the centre of the circle.

$\therefore$ centre $(h, k) = (3, -1)$

The point (6, 4) is on the circumference of the circle.

radius (r) = distance between (3, –1) and a point on circumference.

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



$= \sqrt{(6 - 3)^2 + (4 + 1)^2}$

$= \sqrt{3^2 + 5^2}$

$= \sqrt{9 + 25}$

$= \sqrt{34}$ units.

Hence the equation of circle is,

$(x - h)^2 + (y - k)^2 = r^2$

ie. $(x - 3)^2 + (y + 1)^2 = 34$

$\therefore$ $x^2 + y^2 - 6x + 2y = 24$ is the required equation of the circle.

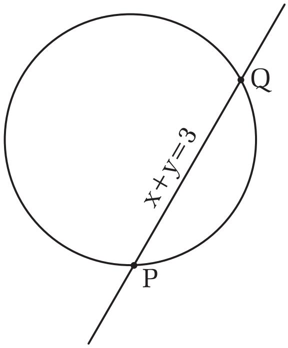
7. Determine the points of intersections of a straight line and the circle $x + y = 3$, $x^2 + y^2 - 2x - 3 = 0$. Also find the length of the intercepts(chord).

Solution

Given equation of lines are,

$x + y = 3$........(i)(a line)

$x^2 + y^2 - 2x - 3 = 0$ ....(ii)(a circle)



Solving equations (i) and (ii) we get, the point of intersection of the line and the circle.

From equation (i), $y = 3 - x$ ....(iii)

put the value of y in equation (ii), we get

$x^2 + (3 - x)^2 - 2x - 3 = 0$

or, $x^2 + 9 - 6x + x^2 - 2x - 3 = 0$

or, $2x^2 - 8x + 6 = 0$

or, $x^2 - 4x + 3 = 0$

or, $x^2 - 3x - x + 3 = 0$

or, $(x-3)(x-1)=0$
$\therefore x = 1, 3$
From equation(i), when $x = 1$, then $y = 2$
when $x = 3$, then $y = 0$
Hence the required points of intersection are (1,2) and (3,0)
Also, the length of chord, $PQ = \sqrt{(3-1)^2+(0-2)^2}$
$= \sqrt{4+4}$
$= \sqrt{8}$
$= 2\sqrt{2}$ units.

9. Find the centre and radius of circle passing through the points P(2, –1), Q(2,3) and R(4,1). Also find the equation of the circle.

Solution
Let the equation of required circle be
$(x-h)^2+(y-k)^2 = r^2$ ....(i)
where centre = c(h,k), radius = r
The circle (i) passes through the points P(2, –1), Q(2,3) and R(4,1).
Now, CP = CQ = CR
ie. $CP^2 = CQ^2 = CR^2$
Taking $CP^2 = CQ^2$
ie. $(2-h)^2+(-1-k)^2 = (2-h)^2+(3-k)^2$
ie. $h^2+k^2-4h+2k+5 = 4+h^2-4h+9-6k+k^2$
or, $8k = 8$
$\therefore k = 1$

again,
Taking $CP^2 = CR^2$
ie. $(2-h)^2+(1+k)^2 = (4-h)^2+(1-k)^2$
ie. $h^2+k^2-4h+2k+5 = h^2+k^2-8h-2k+17$
put $k = 1$,
$4h + 4.1 = 12$
or, $4h = 8$
$\therefore h = 2$
$\therefore$ centre $(h,k) = (2, 1)$
Radius $(r) = \sqrt{(x-h)^2+(y-k)^2}$ $\quad (\therefore (x,y)=(2,-1),(h,k)=(2,1)$

$= \sqrt{(2-2)^2+(-1-1)^2}$
$= 2$ units.
Equation of the required circle is, (from (i))
$(x-2)^2+(y-1)^2 = 4$

or, $x^2 + y^2 - 4x - 2y + 5 = 4$

$\therefore x^2 + y^2 - 4x - 2y + 1 = 0$ is the required equation of the circle.

Note:

Alternatively above question can be solved by taking equation $x^2 + y^2 + 2gx + 2fy + c = 0$ putting the points and solving three obtained equations for g,f and c.

10. Find the equation of the circle passing through the points (5,7), (6,6) and (2,–2).

Solution

Let the required equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ....(i)

The circle passes through the points (5,7), (6,6) and (2,–2), we get

$5^2 + 7^2 + 2.\ g.\ 5 + 2\ .f\ .7 + c = 0$

or, $10g + 14f + 74 + c = 0$ .... (ii)

Similarly,

$12g + 12f + 72 + c = 0$ ....(iii)

and $4g - 4f + 8 + c = 0$ ....(iv)

Subtracting (iii) from (ii), we get,

$-2g + 2f + 2 = 0$

or, $g - f = 1$ ....(v)

Again subtracting (iv) from (iii)

$8g + 16f + 64 = 0$

or, $g + 2f + = -8$ ....(vi)

Solving equation (v) and (vi), we get,

$f = -3$ and $g = -2$

$\therefore$ centre $= (-g, -f) = (2,3)$

Put the values of g and f in equation (ii),

we get, $10\times(-2) + 14 \times (-3) + 74 + c = 0$

or, $-20 - 42 + 74 + c = 0$

$\therefore c = -12$

put the values of g, f and c in eq^n(i), we get

$x^2 + y^2 - 4x - 6y - 12 = 0$

which is the required equation.

11. Find the equation of the circle passing through the points (3,2) and (5,4) and centre lies on the line 3x – 2y = 1.

Solution

Let c (h,k) be the centre of the circle.

It lies on the line $3x - 2y = 1$.

ie. $3h - 2k = 1$ ....(1)

Let A(3,2) and B (5,4) be the points on the circumference of the circle. Then

CA = CB

or, $CA^2 = CB^2$

$(3-h)^2 + (2-k)^2 = (5-h)^2 + (4-k)^2$

or, $9 - 6h + h^2 + 4 - 4k + k^2 = 25 - 10h + h^2 + 16 - 8k + k^2$

or, $13 + 4h + 4k = 41$

or, $4h + 4k = 28$

$\therefore$ $h + k = 7$ ....(ii)

solving equation (i) and (ii), we get

$h = 3$ and $k = 4$

$\therefore$ centre = (h,k) =(3,4)

Now, equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$

$(x-3)^2 + (y-4)^2 = r^2$

where, $r = \sqrt{(3-3)^2 + (2-4)^2} = 2$

or, $x^2 - 6x + 9 + y^2 - 8y + 16 = 4$

or, $x^2 - 6x + 9 + y^2 - 8y + 16 = 4$

$\therefore$ $x^2 + y^2 - 6x - 8y + 21 = 0$ is the required equation of the circle.

12. Find the equation of the circle which touches the x –axis at (4, 0) and cuts off an intercepts of 6 units from the y – axis positively.

Solution

Let c(h,k) be the centre of the required circle. Let A(4,0) be the point on x –axis and DE = 6 units from y – axis.

The point of (4,0) on the line $3x - 2y = 1$.

ie. $3h - 2k = 1$ ....(1)

Let A(3,2) and B (5,4) be the points on the circumference of the circle. Then

Draw CB perpendicular on y – axis and CA $\perp$ OX .

CA = k = radius = r

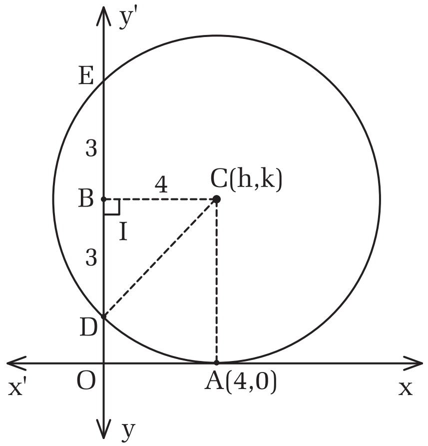
h = BC = 4 untis = OA, BD = 3

Now, $CD = K = \sqrt{BC^2 + BD^2}$

$= \sqrt{4^2 + 3^2}$

$= \sqrt{25} = 5$

$\therefore$ centre = (h,k) =(4, 5)



Now, equation of the required circle is given by

$(x-h)^2 + (y-k)^2 = k^2$, r = k = 5

$(x-4)^2 + (y-5)^2 = 5^2$

or, $x^2 - 8x + 9 + y^2 - 105 + 16 = 4$

$\therefore x^2 + y^2 - 8x - 105 + 16 = 0$

13. Prove that the points A(2, –4), B(3, –1), C(3, –3) and D(0, 0) are concyclic.

Solution

Let P(h,k) be the centre of the circle and A(2, –4), B(3, –1), C(3, –3) and D(0, 0) be three points on the circumference of the circle.

Then

$PA = PB = PC$

or, $PA^2 = PB^2 = PC^2$

Taking $PA^2 = PB^2$

ie. $(h-2)^2 + (k+4)^2 = (h-3)^2 + (k+1)^2$

or, $h^2 - 4h + 4 + k^2 + 8k + 16 = h^2 - 6h + 9 + k^2 + 2k + 1$

or, $2h + 6k = -10$

or, $h + 3k = -5$ ....(i)

Taking $PB^2 = PC^2$

ie. $(h-3)^2 + (k+1)^2 = (h-3)^2 + (k+3)^2$

or, $k^2 + 2k + 1 = k^2 + 6k + 9$

or, $-4k = 8$

$\therefore k = -2$

Put the value of k in equation (i), we get

$h + 3(-2) = -5$

or, $h = 1$

$\therefore$ $(h,k) = (1, -2)$

radius(r) $= PA = \sqrt{(1-2)^2 + (-2+4)^2}$

$= \sqrt{1+4}$

$= \sqrt{5}$

Now, equation of required circle is

$(x-h)^2 + (y-k)^2 = r^2$

$(x-1)^2 + (y+2)^2 = 5$

$\therefore$ $x^2 + y^2 - 2x + 4y = 0$ ....(ii)

put the point D(0,0) in equation (ii), we get,

$0 = 0$ (true)

$\therefore$ D(0,0) also lies on the circle.

Hence the points A, B, C and D are concyclic.

14. Show that the two circles $x^2 + y^2 = 36$ and $x^2 + y^2 - 12x - 16y + 84 = 0$ touch externally.

Solution

Let the given circles be p_1 and p_2 as shown in the given figure.

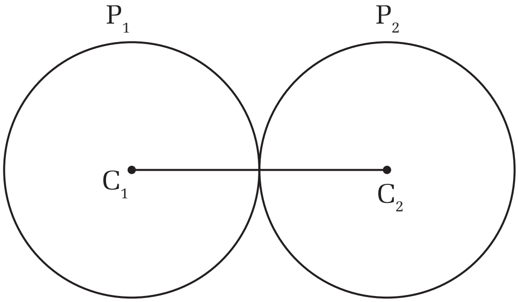
$x^2 + y^2 = 36$ ....(i)

$x^2 + y^2 - 12x - 16y + 84 = 0$ ....(ii)

From equation (i), radius $= r_1 = 6$, centre $c_1 = (0,0)$

Again, from circle (ii),

radius $(r_2) = \sqrt{g^2 + f^2 - c}$, centre $= c_2 = (6,8)$



$= \sqrt{(-6)^2 + (-8)^2 - 84}$

$= \sqrt{36 + 64 - 84}$

$= \sqrt{16} = 4$ units

Distance between c_1 and $c_2 = \sqrt{6^2 + 8^2}$

$= \sqrt{100} = 10$ units

sum of the radii $= r_1 + r_2 = 6 + 4 = 10$

Since the distance between the centres of two circles is equal to the sum of radii of the circles. So given two circles touch externally. **proved**

15. Show that two circles touch internally. $x^2 + y^2 = 81$ and $x^2 + y^2 - 6x - 8y + 9 = 0$.

Solution

Given equations of circles are

$x^2 + y^2 = 81$ ....(i)

$x^2 + y^2 - 6x - 8y + 9 = 0$ ....(ii)

From equation (i), radius $= r_1 = 9$ imoyd, centre $(h,k) = (0,0)$

Again, from equation (ii),

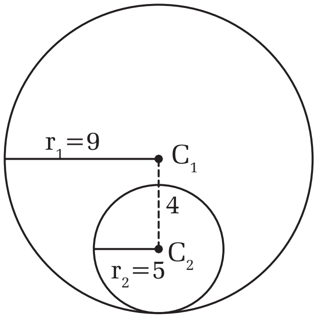
radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{(-3)^2 + (-4)^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25}$

$= 5$ units



radius $(r_2) = 5$

centre $= c_2 = (3, 4)$

Difference of raddi $= r_1 - r_2 = 9 - 5 = 4$

Distance between the centres of the circles $= \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Since the distance between the centres of two circles is equal to the difference of the of radii of the circles, the two circles touch internally. **proved**

Conic Section

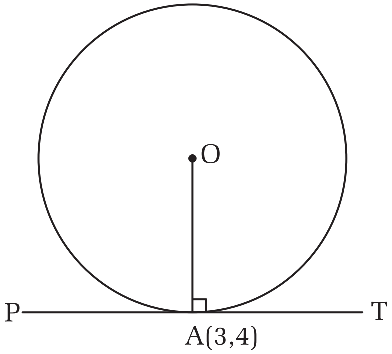
16. Text book Q. N. 3(paper 184)

Solution

(a) (b) (c)Ellipse (d) Hyperbola

17. Find the equation of tangent to the circles $x^2 + y^2 = 25$ at (3, 4).

Solution



Let PT be the tangent to the circle

$x^2 + y^2 = 25$............(i)

at the point A(3, 4).

Now, radius of circle = 5

centre = O(0,0)

Slope OA $(m_2) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{3-0} = -\frac{4}{3}$

Let slope of tangent PT be m_2

By plane geometry, we know that OA is perpendicular to PT

$\therefore m_1 . m_2 = -1$

ie. $m_c = -\frac{3}{4}$

Now, equation of tangent PT is given by

$y - y_1 = m(x - x_1)$

where $m = m_2 = -\frac{3}{4}$

ie. $y - 4 = -\frac{3}{4}(x - 3)$

or, $4y - 16 = -3x + 9$

$\therefore 3x + 4y = 25$

Note: Equation of tangent to the circles $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$

Questions for practice

1. Find the equation of the circle with centre (2,3) and radius 5.
2. Find the centre and radius of the circle whose equation is $x^2 + y^2 + 4x - 6y + 4 = 0$
3. Find the equation of the circle whose centre is the point of intersection of $x + 2y - 1 = 0$ and $2x - y - 7 = 0$ and it passes through (3,1).
4. Find the equation of the circle which touches positive axes of x and y and whose radius is 6 units.
5. Find the equation of the circle of the ends of a diameter are (3,2) and (7, –2).
6. (a). Find the equation of the circle passing through the points (1,0),(2,–2) and (3,1).

   (b). Find the equation of the circle through the points (1,2),(3,1) and (–3,–1).
7. Show that the points (3,3),(3,–3) and (–3,3) and (–3,–3) are concyclic.
8. Find the equation of the circle passing through the points (1, –5) and concentric with the circles, $x^2 + y^2 - 4x - 8y - 81 = 0$ (2,–2).
9. If $y = x + 2$ is the equation of a chord of the circle $x^2 + y^2 + 2x = 0$. Find the equation of the circle of which this chord as an a diameter. Also find the length of the chord.
10. Find the equation of the tangent to the circle $x^2 + y^2 = 100$ at (6,8).
11. Find the point of intersection of the line $x + y = 3$ and the circle $x^2 + y^2 - 2x - 3 = 0$. Also find the length hord.(Ans: (1,2), (3,0), $2\sqrt{2}$)
12. Show that the two circle touch externally. $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 - 22x - 4y + 109 = 0$

UNIT
nine

Trigonometry

Multiple Angles

Estimated Periods : 7

1. Objectives

S,N.	Level	Objectives
(i)	Knowledge(K)	To define multiple angles of angle A as 2A, 3A ect. To tell the formulae of trigonometric ratios of multiple angles.
(ii)	Understanding(U)	To explain to derive the formulae of multiple angles by using compound angle formulae.
(ii)	Application(A)	To solve problems of trigonometric identities of multiple angles.
(iv)	Higher Ability (HA)	To interpreter sin2A, cos2A, tan2A geometrically. To solve trigonometric identities of difficult questions of multiple angles.

2. Teaching Materials

Formula chart of compound angles and multiple angles.

3. Learning stategies

- Review the formula of trigonometric ratios of compound angles studied in class 9.
- Define multiple angles of A as 2A, 3A, ...
- Show how to derive formulae of multiple angles of trigonometric ratios.
  eg. $\sin 2A = 2 \sin A \cdot \cos A$, $\sin 3A = 3 \sin A - 4\sin^3 A$
- Discuss how to solve trigonometric identition wiht examples.

4 List of formulae

1. $\sin 2A = 2 \sin A \cdot \cos A = \dfrac{2 \tan A}{1 + \tan^2 A} = \dfrac{2 \cot A}{1 + \cot^2 A}$

2. $\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A} = = \dfrac{\cot^2 A - 1}{\cot^2 A + 1}$

3. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

4. $\cot 2A = \dfrac{\cot^2 A - 1}{2 \cot A}$

5. $\sin 3A = 3 \sin A - 4 \sin^3 A$

   $\Rightarrow 4 \sin^3 A = 3 \sin A - \sin 3A$

6. $\cos 3A = 4 \cos^3 A - 3 \cos A$

   $\Rightarrow 4 \cos^3 A = 3 \cos A + \cos 3A$

7. $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

8. $\cot 3A = \dfrac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}$

Some solved problems

1. If $\cos\theta = \frac{1}{2}\left(p + \frac{1}{p}\right)$, then show that:

i) $\cos 2\theta = \frac{1}{2}\left(p^2 + \frac{1}{p^2}\right)$

ii) $\cos 3\theta = \frac{1}{2}\left(p^3 + \frac{1}{p^3}\right)$

Solution

i) Here, $\cos 2\theta = 2\cos^2\theta - 1$

$$= 2\left\{\frac{1}{4}\left(p + \frac{1}{p}\right)^2\right\} - 1$$

$$= \frac{1}{2}\left(p^2 + 2 \,.\, p \,.\, \frac{1}{p} + \frac{1}{p^2}\right) - 1$$

$$= \frac{1}{2}\left(p^2 + 2 + \frac{1}{p^2}\right) - 1$$

$$= \frac{1}{2}\left(p^2 + 2 + \frac{1}{p^2} - 2\right)$$

$$= \frac{1}{2}\left(p^2 + \frac{1}{p^2}\right)$$

ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$= \cos\theta(4\cos^2\theta - 3)$$

$$= \frac{1}{2}\left(p + \frac{1}{p}\right)\left\{4 \,.\, \frac{1}{4}\left(p + \frac{1}{p}\right)^2 - 3\right\}$$

$$= \frac{1}{2}\left(p + \frac{1}{p}\right)\left(p^2 + 2 \,.\, p. \frac{1}{p} + \frac{1}{p^2}\right) - 3$$

$$= \frac{1}{2}\left(p + \frac{1}{p}\right)\left(p^2 + 2 + \frac{1}{p^2} - 3\right)$$

$$= \frac{1}{2}\left(p + \frac{1}{p}\right)\left(p^2 - 1 + \frac{1}{p^2}\right)$$

$$= \frac{1}{2}\left(p + \frac{1}{p}\right)\left(p^2 - p \,.\, \frac{1}{p} + \frac{1}{p^2}\right)$$

$$= \frac{1}{2}\left(p^3 + \frac{1}{p^3}\right)$$

2. a) **If $\cos\theta = \frac{6}{5\sqrt{2}}$, Show that $\cos 2\theta = \frac{11}{25}$.**

b. **If $\tan\theta = \frac{5}{12}$, show that $\tan 2\theta = \frac{120}{119}$**

Solution

i) **Here, $\cos\theta = \frac{6}{5\sqrt{2}}$**

LHS = $\cos\theta = 2\cos^2\theta - 1$

$$= 2\left(\frac{6}{5\sqrt{2}}\right)^2 - 1$$

$$= 2\,\frac{36}{50} - 1$$

$$= \frac{36}{25} - 1$$

$$= \frac{36-25}{25}$$

$$= \frac{11}{25} = \text{RHS proved}$$

ii) **Here, $\tan\theta = \frac{5}{12}$**

$$\text{LHS} = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\cdot\frac{5}{12}}{1-\frac{25}{144}}$$

$$= \frac{5}{6} \times \frac{144}{144-25}$$

$$= \frac{120}{119} = \text{RHS proved}$$

3. Prove that following.

a. $$\frac{\cos 2\theta}{1+\sin 2\theta} = \frac{1-\tan\theta}{1+\tan\theta}$$

Solution

$$\text{LHS} = \frac{\cos 2\theta}{1+\sin 2\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{(\cos\theta + \sin\theta)^2}$$

$$= \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$$

(dividing numerator and denominator by $\cos\theta$)

$$= \frac{1 - \tan\theta}{1 + \tan\theta} = \text{RHS proved}$$

Alternate Method

$$\text{LHS} = \frac{\cos 2\theta}{1 + \sin 2\theta}$$

$$= \frac{\frac{1 - \tan^2\theta}{1 + \tan^2\theta}}{1 + \frac{2\tan\theta}{1 + \tan^2\theta}}$$

$$= \frac{1 - \tan\theta}{1 + \tan\theta} = \text{RHS proved}$$

b. $\mathbf{\dfrac{\cos\theta}{\cos\theta - \sin\theta} - \dfrac{\cos\theta}{\cos\theta + \sin\theta} = \tan 2\theta}$

Solution

$$\text{LHS} = \frac{\cos\theta}{\cos\theta - \sin\theta} - \frac{\cos\theta}{\cos\theta + \sin\theta}$$

$$= \frac{\cos\theta(\cos\theta + \sin\theta) - \cos\theta(\cos\theta - \sin\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$$

$$= \frac{\cos^2\theta + \sin\theta . \cos\theta - \cos^2\theta + \sin\theta . \cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{2\sin\theta . \cos\theta}{\cos 2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = \quad \textbf{RHS proved}$$

c. $\mathbf{\dfrac{1 + \sin 2A}{\cos 2A} = \dfrac{\sin A + \cos A}{\cos A - \sin A}}$

Solution

$$\text{LHS} = \frac{1 + \sin 2A}{\cos 2A}$$

$$= \frac{\sin^2 A + \cos^2 A + 2\sin A . \cos A}{\cos^2 A - \sin^2 A}$$

$$= \frac{(\cos A + \sin A)^2}{(\cos A + \sin A)(\cos A - \sin A)}$$

$$= \frac{\sin A + \cos A}{\cos A - \sin A} = \text{RHS proved}$$

d. $\dfrac{\sin 5\theta}{\sin\theta} - \dfrac{\cos 5\theta}{\cos\theta} = 4\cos 2\theta$

Solution

$$\text{LHS} = \frac{\sin 5\theta}{\sin\theta} - \frac{\cos 5\theta}{\cos\theta}$$

$$= \frac{\cos\theta \,.\, \sin 5\theta - \sin\theta \,.\cos 5\theta}{\sin\theta \,.\, \cos\theta}$$

$$= \frac{\sin(5\theta - \theta)}{\sin\theta \,.\, \sin\theta}$$

$$= \frac{\sin 4\theta}{\sin\theta \,.\, \cos\theta} \quad = \frac{\sin(2\theta)}{\sin\theta \,.\, \cos\theta}$$

$$= \frac{2\sin 2\theta \,.\, \cos 2\theta}{\sin\theta \,.\, \cos\theta}$$

$$= \frac{4\sin\theta \,.\, \cos\theta \,.\, \cos 2\theta}{\sin\theta \,.\, \cos\theta}$$

$= 4\cos 2\theta =$ **RHS proved**

e. $\dfrac{\cos 2\theta}{1 + \sin 2\theta} = \tan\left(\dfrac{\pi}{4} - \theta\right)$

Solution

$$\text{LHS} = \frac{\cos 2\theta}{1 + \sin 2\theta}$$

$$= \frac{\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}}{1 + \dfrac{2\tan\theta}{1 + \tan^2\theta}}$$

$$= \frac{1 - \tan^2\theta}{1 + \tan^2\theta} \times \frac{1 + \tan^2\theta}{1 + \tan^2\theta + 2\tan\theta}$$

$$= \frac{(1 - \tan\theta)(1 + \tan\theta)}{(1 + \tan\theta)^2}$$

$$= \frac{1 - \tan\theta}{1 + \tan\theta}$$

$$= \frac{\tan\dfrac{\pi}{4} - \tan\theta}{1 + \tan\dfrac{\pi}{4} \,.\, \tan\theta}$$

$= \tan\left(\dfrac{\pi}{4} - \theta\right) =$ **RHS proved**

4. Prove that following.

a. $\dfrac{\sin2\theta - \cos\theta}{1 - \sin\theta - \cos2\theta} = \cot\theta$

Solution

$$\text{LHS} = \frac{\sin2\theta - \cos\theta}{1 - \sin\theta - \cos2\theta}$$

$$= \frac{2\sin\theta \,.\, \cos\theta - \cos\theta}{1 - \sin\theta - 1 + 2\sin^2\theta}$$

$$= \frac{\cos\theta(2\sin\theta - 1)}{\sin\theta(2\sin\theta - 1)}$$

$$= \cot\theta = \text{RHS proved}$$

b. $\dfrac{1 + \sin2\theta - \cos2\theta}{1 + \sin2\theta + \cos2\theta} = \tan\theta$

Solution

$$\text{LHS} = \frac{1 + \sin2\theta - \cos2\theta}{1 + \sin2\theta + \cos2\theta}$$

$$= \frac{(1 - \cos2\theta) + \sin2\theta}{(1 + \cos2\theta) + \sin2\theta}$$

$$= \frac{2\sin^2\theta + 2\sin\theta \,.\, \cos\theta}{2\cos^2\theta + 2\sin\theta \,.\, \cos\theta}$$

$$= \frac{2\sin\theta(\sin\theta + \cos\theta)}{2\cos\theta(\sin\theta + \cos\theta)}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta = \textbf{RHS proved}$$

c. $\dfrac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} + \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\sec2\theta$

Solution

$$\text{LHS} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} + \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$$

$$= \frac{(\sin\theta + \cos\theta)^2 + (\cos\theta - \sin\theta)^2}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$$

$$= \frac{\sin^2\theta + 2\sin\theta \,.\, \cos\theta + \cos^2\theta + \cos^2\theta - 2\sin\theta \,.\, \cos\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{2(\sin^2\theta + \cos^2\theta)}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{2}{\cos2\theta}$$

$$= 2\sec2\theta \quad = \textbf{RHS proved}$$

d. $(1 + \sin2\theta + \cos2\theta)^2 = 4\cos^2\theta(1 + \sin2\theta)$

Solution

LHS $= (1 + \sin2\theta + \cos2\theta)^2$

$= (1 + 2\sin\theta \,.\, \cos\theta + 2\cos^2\theta - 1)^2$

$= [2\cos\theta(\sin\theta + \cos\theta)]^2$

$= 4\cos^2\theta(\sin\theta + \cos\theta)^2$

$= 4\cos^2\theta(\sin^2\theta + 2\sin\theta \,.\, \cos\theta + \cos^2\theta)$

$= 4\cos^2\theta(1 + \sin2\theta) =$ RHS proved

e. $\dfrac{1}{\tan2\theta - \tan\theta} - \dfrac{1}{\cot2\theta - \cot\theta} = \cot\theta$

Solution

$$\text{LHS} = \frac{1}{\tan2\theta - \tan\theta} - \frac{1}{\cot2\theta - \cot\theta}$$

$$= \frac{1}{\tan2\theta - \tan\theta} - \frac{1}{\dfrac{1}{\tan2\theta} - \dfrac{1}{\tan\theta}}$$

$$= \frac{1}{\tan2\theta - \tan\theta} - \frac{\tan2\theta \,.\, \tan\theta}{\tan\theta - \tan2\theta}$$

$$= \frac{1 + \tan2\theta \,.\, \tan\theta}{\tan2\theta - \tan\theta}$$

$$= \frac{1 + \dfrac{2\tan\theta}{1 - \tan^2\theta}\tan\theta}{\dfrac{2\tan\theta}{1 - \tan^2\theta} - \tan\theta}$$

$$= \frac{1 - \tan^2\theta + 2\tan^2\theta}{2\tan\theta - \tan\theta + 1 - \tan^3\theta}$$

$$= \frac{1 + \tan^2\theta}{\tan\theta + \tan^3\theta}$$

$$= \frac{1 + \tan^2\theta}{\tan\theta(1 + \tan^2\theta)}$$

$$= \frac{1}{\tan\theta}$$

$= \cot\theta =$ **RHS proved**

5. Prove that:

a. $\dfrac{1 - \sin2\theta}{1 + \sin2\theta} = \left(\dfrac{\cot\theta - 1}{\cot\theta + 1}\right)^2$

Solution

$$\text{LHS} = \frac{1 - \sin 2\theta}{1 + \sin 2\theta}$$

$$= \frac{1 + \dfrac{2\tan\theta}{1 + \tan^2\theta}}{1 + \dfrac{2\tan\theta}{1 + \tan^2\theta}}$$

$$= \frac{1 + \tan^2\theta - 2\tan\theta}{1 + \tan^2\theta + 2\tan\theta}$$

$$= \left(\frac{1 + \tan\theta}{1 - \tan\theta}\right)^2$$

$$= \left(\frac{1 - \dfrac{1}{\cot\theta}}{1 + \dfrac{1}{\cot\theta}}\right)^2$$

$$= \left(\frac{\cot\theta - 1}{\cot\theta + 1}\right)^2 \quad = \textbf{RHS proved}$$

b. $$\frac{\mathbf{1 + \tan^2\left(\dfrac{\pi}{4} - \theta\right)}}{\mathbf{1 - \tan^2\left(\dfrac{\pi}{4} - \theta\right)}} = \mathbf{cosec 2\theta}$$

Solution

$$\text{LHS} = \frac{1 + \tan^2\left(\dfrac{\pi}{4} - \theta\right)}{1 - \tan^2\left(\dfrac{\pi}{4} - \theta\right)}$$

$$= \frac{1}{\dfrac{1 - \tan^2\left(\dfrac{\pi}{4} - \theta\right)}{1 + \tan^2\left(\dfrac{\pi}{4} - \theta\right)}}$$

$$= \frac{1}{\cos\left(\dfrac{\pi}{4} - \theta\right)}$$

$$= \frac{1}{\cos\left(\dfrac{\pi}{2} - 2\theta\right)}$$

$$= \frac{1}{\sin 2\theta}$$

$$= \text{cosec}\, 2\theta$$

= RHS proved

6. Prove that:

a. $\cos^4\theta + \sin^4\theta = \frac{1}{4}(3 + \cos4\theta)$

Solution

$$
\begin{aligned}
\text{LHS} &= \cos^4\theta + \sin^4\theta \\
&= (\cos^2\theta - \sin^2\theta)^2 + 2\sin^2\theta \,.\, \cos^2\theta \\
&= (\cos2\theta)^2 + \frac{1}{2}(2\sin\theta \,.\, \cos\theta)^2 \\
&= \cos^2 2\theta + \frac{1}{2}\sin^2 2\theta \\
&= \frac{1}{2}(2\cos^2 2\theta + \sin^2 2\theta) \\
&= \frac{1}{2}(\cos^2 2\theta + \sin^2 2\theta + \cos^2 2\theta) \\
&= \frac{1}{2}(1 + \cos^2 2\theta) \\
&= \frac{1}{4}(2 + 2\cos^2 2\theta) \\
&= \frac{1}{4}(2 + 1 + \cos4\theta) \\
&= \frac{1}{4}(3 + \cos4\theta) \qquad (\because \ 2\cos^2 x = 1 + \cos2x)
\end{aligned}
$$

= RHS proved

b. $\cos^6\theta - \sin^6\theta = \cos2\theta\left(1 - \frac{1}{4}\sin^2 2\theta\right)$

Solution

$$
\begin{aligned}
\text{LHS} &= \cos^6\theta - \sin^6\theta \\
&= (\cos^2\theta - \sin^2\theta)(\cos^4\theta + \cos^2\theta \,.\, \sin^2\theta + \sin^4\theta) \\
&= \cos2\theta\,\{(\cos^2\theta - \sin^2\theta)^2 + 2\sin^2\theta \,.\, \cos^2\theta + \sin^2\theta \,.\, \cos^2\theta\} \\
&= \cos2\theta\,(\cos^2 2\theta + 3\sin^2\theta \,.\, \cos^2\theta) \\
&= \cos2\theta\left\{\cos^2 2\theta + \frac{3}{4}(2\sin\theta \,.\, \cos\theta)^2\right\} \\
&= \cos2\theta\left\{\frac{4\,\text{sins}^2 2\theta + 3\sin^2 2\theta}{4}\right\} \\
&= \cos2\theta\left\{\frac{4 - 4\cos^2 2\theta + 3\sin^2 2\theta}{4}\right\} \\
&= \cos2\theta\left(1 - \frac{1}{4}\sin^2 2\theta\right)
\end{aligned}
$$

= RHS proved

c. $\sin^4\theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$

Solution

$$\text{RHS} = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$$
$$= \frac{1}{8}\{3 - 4(1 - 2\sin^2\theta) + (1 - 2\sin^2 2\theta)\}$$
$$= \frac{1}{8}\{3 - 4 + 8\sin^2\theta + 1 - 2\sin^2 2\theta\}$$
$$= \frac{1}{8}\{8\sin^2\theta - 2(2\sin\theta \,.\, \cos\theta)^2\}$$
$$= \frac{1}{8}\{8\sin^2\theta - 8\sin^2\theta \,.\, \cos^2\theta\}$$
$$= \frac{1}{8} \,.\, 8\sin^2\theta(1 - \cos^2\theta)$$
$$= \sin^2\theta \,.\, \sin^2\theta$$
$$= \sin^4\theta \;= \textbf{LHS proved}$$

d. $\cos^8\theta + \sin^8\theta = 1 - \sin^2 2\theta + \frac{1}{8}\cos^4 2\theta$

Solution

$$\text{LHS} = \cos^8\theta + \sin^8\theta$$
$$= (\cos^4\theta + \sin^4\theta)^2 - 2\sin^4\theta \,.\, \cos^4\theta$$
$$= \{(\cos^2\theta - \sin^2\theta)^2 + 2\sin^2\theta \,.\, \cos^2\theta\}^2 - 2\sin^4\theta \,.\, \cos^4\theta$$
$$= \left(\cos^2 2\theta + \frac{1}{2}\sin^2 2\theta\right)^2 - \frac{1}{8}\sin^4 2\theta$$
$$= \cos^4 2\theta + \cos^2 2\theta \,.\, \sin^2 2\theta + \frac{1}{4}\sin^4 2\theta - \frac{1}{8}\sin^4 2\theta$$
$$= (1 - \sin^2 2\theta)^2 + (1 - \sin^2 2\theta) \,.\, \sin^2 2\theta + \frac{1}{8}\sin^4 2\theta$$
$$= 1 - 2\sin^2 2\theta + \sin^4 2\theta + \sin^2 2\theta - \sin^4 2\theta + \frac{1}{8}\sin^4 2\theta$$
$$= 1 - \sin^2 2\theta + \frac{1}{8}\sin^4 2\theta$$
$$= \textbf{RHS proved}$$

6. Prove that:

a. $\dfrac{\sqrt{3}}{\sin 40^\circ} + \dfrac{1}{\cos 40^\circ} = 4$

Solution

$$\text{LHS} = \frac{\sqrt{3}}{\sin 40^\circ} + \frac{1}{\cos 40^\circ}$$

$$= \frac{\sqrt{3}\cos40° - \sin40°}{\sin40° . \cos40°}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos40° + \frac{1}{2}\sin40°\right)}{\sin40° . \cos40°}$$

$$= \frac{2(\cos30° . \cos40° + \sin30° . \sin40°)}{\sin40° . \cos40°}$$

$$= \frac{4\cos(40° - 30°)}{2\sin40° . \cos40°}$$

$$= 4\frac{\cos10°}{\sin80°}$$

$$= 4\frac{\cos10°}{\sin(90° - 10°)}$$

$$= 4\frac{\cos10°}{\cos10°}$$

$= 4 =$ **RHS proved**

b. $\mathbf{cosec10° - \sqrt{3}\,sec10° = 4}$

Solution

LHS $= \text{cosec}10° - \sqrt{3}\sec10°$

$$= \frac{1}{\sin10°} - \frac{\sqrt{3}}{\cos10°}$$

$$= \frac{\cos10° - \sqrt{3}\sin10°}{\sin10° . \cos10°}$$

$$= \frac{4\left(\frac{1}{2}\cos10° - \frac{\sqrt{3}}{2}\sin10°\right)}{2\sin10° . \cos10°}$$

$$= 4\frac{\sin30° . \cos10° - \cos30° . \sin10°}{\sin20°}$$

$$= 4\frac{\sin(30° - 10°)}{\sin20°}$$

$$= 4\frac{\sin20°}{\sin20°}$$

$= 4 \;=$ **RHS proved**

7. Prove that:

a. $\mathbf{(2\cos\theta + 1)(2\cos\theta - 1) = 2\cos2\theta + 1}$

Solution

LHS $= (2\cos\theta + 1)(2\cos\theta - 1)$

$= 4\cos^2\theta - 1$

$= 2(2\cos^2\theta - 1) + 1$

$= 2\cos 2\theta + 1$

RHS proved

b. $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

Solution

$$\text{LHS} = \frac{\sec 8\theta - 1}{\sec 4\theta - 1}$$

$$= \frac{\dfrac{1}{\cos 8\theta} - 1}{\dfrac{1}{\cos 4\theta} - 1}$$

$$= \frac{1 - \cos 8\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{1 - \cos 4\theta}$$

$$= \frac{2\sin^2 4\theta \,.\, \cos 4\theta}{\cos 8\theta \,.\, 2\sin^2 2\theta}$$

$$= \frac{(2\sin 4\theta \,.\, \cos 4\theta)\sin 4\theta}{\cos 8\theta \,.\, 2\sin^2 2\theta}$$

$$= \frac{\sin 8\theta}{\cos 8\theta} \,.\, \frac{2\sin 2\theta \,.\, \cos 2\theta}{2\sin^2 2\theta}$$

$$= \frac{\tan 8\theta}{\dfrac{\sin 2\theta}{\cos 2\theta}}$$

$$= \frac{\tan 8\theta}{\tan 2\theta} = \textbf{RHS proved}$$

c. $\tan\theta + 2\tan\theta + 4\tan 4\theta + 8\cot 8\theta = \cot\theta$

Solution

Example (14) text book

d. $\sin^2\alpha - \cos^2\alpha \,.\, \cos 2\beta = \sin^2\beta - \cos^2\beta \,.\, \cos 2\alpha$

Solution

LHS $= \sin^2\alpha - \cos^2\alpha \,.\, \cos 2\beta$

$= \sin^2\alpha - \cos^2\alpha(2\cos^2\beta - 1)$

$= \sin^2\alpha - 2\cos^2\alpha \,.\, \cos^2\beta + \cos^2\alpha$

$= \sin^2\alpha + \cos^2\alpha - 2\cos^2\beta \,.\, \cos^2\alpha$

$= 1 - 2\cos^2\alpha \cdot \cos^2\beta$

$= \sin^2\beta + \cos^2\beta - 2\cos^2\alpha \cdot \cos^2\beta$

$= \sin^2\beta - \cos^2\beta(2\cos^2\alpha - 1)$

$= \sin^2\beta - \cos^2\beta \cdot \cos2\alpha$ = **RHS proved**

e. $\sqrt{2 + \sqrt{\mathbf{2} + \sqrt{2 + 2\cos8\theta}}}$ **= 2 cosθ**

Solution

$$\text{LHS} = \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos8\theta}}}$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2(1 + \cos8\theta)}}}$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2 \cdot 2\cos^2 4\theta}}}$$

$$= \sqrt{2 + \sqrt{2 + 2\cos4\theta}}$$

$$= \sqrt{2 + \sqrt{2(1 + \cos4\theta)}}$$

$$= \sqrt{2 + \sqrt{4\cos^2 2\theta}}$$

$$= \sqrt{2 + 2\cos2\theta}$$

$$= \sqrt{2(1 + \cos2\theta)}$$

$$= \sqrt{2 \cdot 2\cos^2\theta}$$

$= 2\cos\theta$ = **RHS proved**

f. $\dfrac{\sin^2\alpha - \sin^2\beta}{\sin\alpha \cdot \cos\alpha - \sin\beta \cdot \cos\beta} = \tan(\alpha + \beta)$

Solution

$\text{RHS} = \tan(\alpha + \beta)$

$$= \frac{\tan\alpha - \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

$$= \frac{\tan\alpha - \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha} \cdot \dfrac{\sin\beta}{\cos\beta}}$$

$$= \frac{\sin\alpha \cdot \cos\beta + \sin\beta \cdot \cos\alpha}{\cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta} \times \frac{\sin\alpha \cdot \cos\beta - \sin\beta \cdot \cos\alpha}{\sin\alpha \cdot \cos\beta - \sin\beta \cdot \cos\alpha}$$

$$= \frac{\sin^2\alpha \,.\, \cos^2\beta - \sin^2\beta \,.\, \cos^2\alpha}{\sin\alpha \,.\, \cos\alpha \,.\, \cos^2\beta - \cos^2\alpha \,.\, \sin\beta \,.\, \cos\beta - \sin^2\alpha \,.\, \sin\beta \,.\, \cos\beta + \sin^2\beta \,.\, \sin\alpha \,.\, \cos\alpha}$$

$$= \frac{\sin^2\alpha(1 - \sin^2\beta) - \sin^2\beta(1 - \sin^2\alpha)}{\sin\alpha \,.\, \cos\alpha(\cos^2\beta + \sin^2\beta) - \sin\beta \,.\, \cos\beta(\cos^2\alpha + \sin^2\alpha)}$$

$$= \frac{\sin^2\alpha - \sin^2\beta}{\sin\alpha \,.\, \cos\alpha - \sin\beta \,.\, \cos\beta}$$

= LHS proved

8. a) $4(\cos^3 10° + \sin^2 20°) = 3(\cos 10° + \sin 20°)$

Solution

LHS $= 4(\cos^3 10° + \sin^2 20°)$

$= 4 \cos^3 10° + 4 \sin^2 20°$

$= 3 \cos 10° + \cos(3.10°) + 3 \sin 20° - \sin(3.20°)$

$= 3(\cos 10° + \sin 20°) + 3(\cos 30° - \sin 60°)$

$= 3(\cos 10° + \sin 20°) + 3\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$

$= 3(\cos 10° + \sin 20°) + 3.0$

$= 3(\cos 10° + \sin 20°)$ **= RHS proved**

b) $\sin^3 10° + \cos^3 20° = \frac{3}{4}(\cos 20° + \sin 10°)$

Solution

LHS $= \sin^3 10° + \cos^3 20°$

By using formula,

$\cos^3\theta = \frac{1}{4}(3 \cos\theta + \cos 3\theta)$

$\sin^3\theta = \frac{1}{4}(3 \sin\theta - \sin 3\theta)$

LHS $= \frac{1}{4}[3 \sin 10° - \sin(3.10°) + 3 \cos 20° + \cos(3.20°)]$

$= \frac{1}{4}[3 \sin 10° - \sin 30° + 3 \cos 20° + \cos 60°]$

$= \frac{3}{4}(\sin 10° + \cos 20°) + \frac{1}{4}\left(-\frac{1}{2} + \frac{1}{2}\right)$

$= \frac{3}{4}(\sin 10° + \cos 20°)$

= RHS proved

9. Prove that:

a) $\cot(A + 45°) - \tan(A - 45°) = \frac{2 \cos 2A}{1 + \sin 2A}$

Solution

$$\text{LHS} = \cot(A + 45°) - \tan(A - 45°)$$

$$= \frac{1}{\tan(A + 45°)} - \tan(A - 45°)$$

$$= \frac{1}{\dfrac{\tan A + \tan 45°}{1 - \tan A \,.\, \tan 45°}} - \frac{\tan A - \tan 45°}{1 + \tan A \,.\, \tan 45°}$$

$$= \frac{1 - \tan A}{1 + \tan A} - \frac{\tan A - 1}{1 + \tan A}$$

$$= \frac{1 - \tan A \;- \tan A + 1}{1 + \tan A}$$

$$= \frac{2(1 - \tan A)}{1 + \tan A}$$

$$= 2\frac{1 - \dfrac{\sin A}{\cos A}}{1 + \dfrac{\sin A}{\cos A}}$$

$$= 2\frac{\cos A - \sin A}{\cos A + \sin A} \times \frac{\cos A + \sin A}{\cos A + \sin A}$$

$$= 2\frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A + 2\sin A \,.\, \cos A}$$

$$= \frac{2\cos 2A}{1 + \sin 2A} \quad = \textbf{RHS proved}$$

b) tan(A + 45°) + tan(A – 45°) = 2 tan2A

Solution

$$\text{LHS} = \tan(A + 45°) + \tan(A - 45°)$$

$$= \frac{\tan A + \tan 45°}{1 - \tan A \,.\, \tan 45°} + \frac{\tan A - \tan 45°}{1 + \tan A \,.\, \tan 45°}$$

$$= \frac{\tan A + 1}{1 - \tan A} + \frac{\tan A - 1}{1 + \tan A}$$

$$= \frac{(1 + \tan A)^2 + (\tan A - 1)(1 - \text{tna}A)}{1 - \tan^2 A}$$

$$= \frac{1 + 2\tan A + \tan^2 A + \tan A - \tan^2 A - 1 + \tan A}{1 - \tan^2 A}$$

$$= \frac{4\tan A}{1 - \dfrac{\sin^2 A}{\cos^2 A}}$$

$$= 4\frac{\sin A}{\cos A} \times \frac{\cos^2 A}{\cos^2 A - \sin^2 A}$$

$$= 2\frac{2\sin A\cos A}{\cos 2A}$$

$$= 2\frac{\sin 2A}{\cos 2A}$$

$= 2\tan 2A$ **= RHS proved**

c) $\tan(A + 45°) - \tan(A - 45°) = 2\sec 2A$

Solution

$$\text{LHS} = \tan(A + 45°) - \tan(A - 45°)$$

$$= \frac{\tan A + \tan 45°}{1 - \tan A\,.\,\tan 45°} - \frac{\tan A - \tan 45°}{1 + \tan A\,.\,\tan 45°}$$

$$= \frac{\tan A + 1}{1 - \tan A} - \frac{\tan A - 1}{1 + \tan A}$$

$$= \frac{\tan A + 1}{1 - \tan A} + \frac{1 - \tan A}{1 + \tan A}$$

$$= \frac{(1 + \tan A)^2 + (1 - \tan A)^2}{1 - \tan^2 A}$$

$$= \frac{1 + 2\tan A + \tan^2 A + 1 - 2\tan A - \tan^2 A}{1 - \tan^2 A}$$

$$= \frac{2(1 + \tan^2 A)}{1 - \tan^2 A}$$

$$= 2\frac{1}{\frac{1 - \tan^2 A}{1 + \tan^2 A}}$$

$$= \frac{2}{\cos 2A}$$

$= 2\sec 2A$ **= RHS proved**

d) $\tan A + \tan\left(\frac{\pi}{3} + A\right) - \tan\left(\frac{\pi}{3} - A\right) = 3\tan 3A$

Solution

$$\text{LHS} = \tan A + \tan\left(\frac{\pi}{3} + A\right) - \tan\left(\frac{\pi}{3} - A\right)$$

$$= \tan A + \frac{\tan^2\frac{\pi}{3} + \tan A}{1 - \tan\frac{\pi}{3}\,.\,\tan A} - \frac{\tan\frac{\pi}{3} - \tan A}{1 + \tan\frac{\pi}{3}\,.\,\tan A}$$

$$= \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

$$= \tan A + \frac{(\sqrt{3} + \tan A)(1 + \sqrt{3}\tan A) - (\sqrt{3} - 3\tan A)(1 - \sqrt{3}\tan A)}{1 - 3\tan^2 A}$$

$$= \tan A + \frac{\sqrt{3} + 3\tan A + \tan A + \sqrt{3}\tan^2 A - \sqrt{3} + \tan A + \tan A - \sqrt{3}\tan^2 A}{1 - 3\tan^2 A}$$

$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A}$$

$$= \frac{\tan A - 3\tan^3 A + 8\tan A}{1 - 3\tan^2 A}$$

$$= \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A}$$

$$= \frac{3(3\tan A - \tan^3 A)}{1 - 3\tan^2 A}$$

$= 3\tan 3A$ **= RHS proved**

10.a) If $2\tan\alpha = 3\tan\beta$, prove that : $\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$

Solution

Given, $2\tan\alpha = 3\tan\beta$

or, $\tan\alpha = \frac{3}{2}\tan\beta$

LHS $= \tan(\alpha - \beta)$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$$

$$= \frac{\frac{3}{2}\tan\beta - \tan\beta}{1 + \frac{3}{2}\tan\beta \cdot \tan\beta}$$

$$= \frac{3\tan\beta - 2\tan\beta}{2 + 3\tan^2\beta}$$

$$= \frac{\frac{\sin\beta}{\cos\beta}}{2 + \frac{3\sin^2\beta}{\cos^2\beta}}$$

$$= \frac{\sin\beta}{\cos\beta} \times \frac{\cos^2\beta}{2\cos^2\beta + 3\sin^2\beta}$$

$$= \frac{2\sin\beta \cdot \cos\beta}{4\cos^2\beta + 6\sin^2\beta}$$

$$= \frac{\sin 2\beta}{4 - 4\sin^2\beta + 6\sin^2\beta}$$

$$= \frac{\sin 2\beta}{4 + 2\sin^2\beta}$$

$$= \frac{\sin 2\beta}{5 - 1 + 2\sin^2\beta}$$

$$= \frac{\sin 2\beta}{5 - (1 - 2\sin^2\beta)}$$

$$= \frac{\sin 2\beta}{5 - \cos 2\beta} \quad = \textbf{RHS proved}$$

b) $\tan\theta = \frac{1}{7}$ and $\tan\beta = \frac{1}{3}$, prove that : $\cos 2\theta = \sin 4\beta$

Solution

Here, $\tan\theta = \frac{1}{7}$, $\tan\beta = \frac{1}{3}$

LHS $= \cos 2\theta \quad = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$

$$= \frac{1 - \left(\frac{1}{7}\right)^2}{1 + \left(\frac{1}{7}\right)^2}$$

$$= \frac{1 - \frac{1}{49}}{1 + \frac{1}{49}}$$

$$= \frac{48}{49} \times \frac{49}{50}$$

$$= \frac{24}{25}$$

RHS $= \sin 4\beta \quad = \sin 2(2\beta)$

$$= 2\sin 2\beta \, . \, \cos 2\beta$$

$$= 2\,\frac{2\tan\beta}{1 + \tan^2\beta} \times \frac{1 - \tan^2\beta}{1 + \tan^2\beta}$$

$$= 4 \times \frac{\frac{1}{3}}{1 + \frac{1}{9}} \times \frac{1 - \frac{1}{9}}{1 + \frac{1}{9}}$$

$$= \frac{4}{3} \times \frac{9}{10} \times \frac{8}{9} \times \frac{9}{10}$$

$$= \frac{24}{25}$$

$\therefore$ LHS = **RHS proved**

11. Prove that :

a) $$\frac{\cos A - \sqrt{1 + \sin 2A}}{\sin A - \sqrt{1 + \sin 2A}} = \tan A$$

Solution

$$\text{LHS} = \frac{\cos A - \sqrt{1 + \sin 2A}}{\sin A - \sqrt{1 + \sin 2A}}$$

$$= \frac{\cos A - \sqrt{\sin^2 A + \cos^2 A + 2 \sin A \cos A}}{\sin A - \sqrt{\sin^2 A + \cos^2 A + 2 \sin A \cos A}}$$

$$= \frac{\cos A - \sqrt{(\sin A + \cos A)^2}}{\sin A - \sqrt{(\sin A + \cos A)^2}}$$

$$= \frac{\cos A - \sin A - \cos A}{\sin A - \sin A - \cos A}$$

$$= \frac{-\sin A}{-\cos A}$$

$= \tan A \quad = \text{RHS proved}$

b. $\dfrac{1}{\tan 3\theta + \tan\theta} - \dfrac{1}{\cot 3\theta + \cot\theta} = \cot 4\theta$

Solution

$$\text{LHS} = \frac{1}{\tan 3\theta + \tan\theta} - \frac{1}{\cot 3\theta + \cot\theta}$$

$$= \frac{1}{\tan 3\theta + \tan\theta} - \frac{1}{\dfrac{1}{\tan 3\theta} + \dfrac{1}{\tan\theta}}$$

$$= \frac{1}{\tan 3\theta + \tan\theta} - \frac{\tan\theta \, . \, \tan 3\theta}{\tan\theta + \tan 3\theta}$$

$$= \frac{1 - \tan\theta \, . \, \tan 3\theta}{\tan 3\theta + \tan\theta}$$

$$= \frac{1}{\dfrac{\tan 3\theta + \tan\theta}{1 - \tan\theta \, . \, \tan 3\theta}}$$

$$= \frac{1}{\tan(3\theta + \theta)}$$

$= \cot 4\theta \quad =$ **RHS proved**

c. $\dfrac{\cot A}{\cot A - \cot 3A} - \dfrac{\tan A}{\tan 3A - \tan A} = 1$

Solution

$$\text{LHS} = \frac{\cot A}{\cot A - \cot 3A} - \frac{\tan A}{\tan 3A - \tan A}$$

$$= \frac{\dfrac{1}{\tan A}}{\dfrac{1}{\tan A} - \dfrac{1}{\tan 3A}} - \frac{\tan A}{\tan 3A - \tan A}$$

$$= \frac{1}{\tan A}\ \frac{\tan A \,.\, \tan 3A}{\tan 3A - \tan A} - \frac{\tan A}{\tan 3A - \tan A}$$

$$= \frac{\tan A}{\tan 3A - \tan A} - \frac{\tan A}{\tan 3A - \tan A}$$

$$= \frac{\tan 3A - \tan A}{\tan 3A - \tan A}$$

$= 1 \ \ =$ **RHS proved**

12. Prove that:

a) $\mathbf{8\left(1 + \sin\frac{\pi}{8}\right)\left(1 + \sin\frac{3\pi}{8}\right)\left(1 - \sin\frac{5\pi}{8}\right)\left(1 - \sin\frac{7\pi}{8}\right) = 1}$

Solution

$$\text{LHS} = 8\left(1 + \sin\frac{\pi}{8}\right)\left(1 + \sin\frac{3\pi}{8}\right)\left(1 - \sin\frac{5\pi}{8}\right)\left(1 - \sin\frac{7\pi}{8}\right)$$

$$= 8\left(1 + \sin\frac{\pi}{8}\right)\left(1 + \sin\frac{3\pi}{8}\right)\left\{1 - \sin\left(\pi - \frac{3\pi}{8}\right)\right\}\left\{1 - \sin\left(\pi - \frac{\pi}{8}\right)\right\}$$

$$= 8\left(1 + \sin\frac{\pi}{8}\right)\left(1 + \sin\frac{3\pi}{8}\right)\left(1 - \sin\frac{3\pi}{8}\right).\left(1 - \sin\frac{\pi}{8}\right)$$

$$= 8\left(1 - \sin^2\frac{\pi}{8}\right).\left(1 - \sin^2\frac{3\pi}{8}\right)$$

$$= 8\cos^2\frac{\pi}{8}\,.\,\cos^2\frac{3\pi}{8} \qquad \left(\therefore \cos^2\frac{3\pi}{8} = \cos^2\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \sin^2\frac{\pi}{8}\right)$$

$$= 8\cos^2\frac{\pi}{8}\,.\,\sin^2\frac{\pi}{8}$$

$$= 2\left(4\sin^2\frac{\pi}{8}\,.\,\cos^2\frac{\pi}{8}\right)$$

$$= 2\left(2\sin\frac{\pi}{8}\,.\,\cos\frac{\pi}{8}\right)^2$$

$$= 2\sin^2\frac{\pi}{4}$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)^2$$

$= 1 \ \ =$ **RHS proved**

b) $\mathbf{\sin^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{3\pi}{8}\right) + \sin^4\left(\frac{5\pi}{8}\right) + \sin^2\left(\frac{7\pi}{8}\right) = \frac{3}{2}}$

Solution

$$\text{LHS} = \sin^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{3\pi}{8}\right) + \sin^4\left(\frac{5\pi}{8}\right) + \sin^2\left(\frac{7\pi}{8}\right)$$

$$= \sin^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{3\pi}{8}\right) + \sin^4\left(\frac{\pi}{2} + \frac{\pi}{8}\right) + \sin^4\left(\frac{\pi}{2} + \frac{3\pi}{8}\right)$$

$$= \sin^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{3\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{3\pi}{8}\right)$$

$$= \sin^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{\pi}{2} - \frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{2} - \frac{\pi}{8}\right)$$

$$= \sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{\pi}{8}\right)$$

$$= 2\left[\sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right)\right]$$

$$= 2\left[\left\{\sin^2\left(\frac{\pi}{8}\right) + \cos^2\left(\frac{\pi}{8}\right)\right\}^2 - 2\sin^2\left(\frac{\pi}{8}\right).\cos^2\left(\frac{\pi}{8}\right)\right]$$

$$= 2\left[1 - \frac{1}{2}\left\{4\sin^2\left(\frac{\pi}{8}\right).\cos^2\left(\frac{\pi}{8}\right)\right\}\right]$$

$$= 2\left[1 - \frac{1}{2}\sin^2\frac{\pi}{4}\right]$$

$$= 2\left(1 - \frac{1}{2}.\frac{1}{2}\right)$$

$$= 2\left(1 - \frac{1}{4}\right)$$

$$= 2.\frac{3}{4}$$

$$= \frac{3}{2}$$ = **RHS proved**

Questions for practice

1. If $\cos\theta = \frac{1}{2}$, then find the values of $\sin2\theta$, $\cos2\theta$ and $\tan2\theta$.
2. Prove that $\cot\theta = \pm\sqrt{\frac{1+\cos2\theta}{1-\cos2\theta}}$
3. Prove the following :

   (a) $2\sin^2\left(\frac{\pi}{4} - A\right) = (1 - \sin A)$

   (b) $\tan\alpha + \cot\alpha = 2\operatorname{cosec}2\alpha$

   (c) $\tan\left(\frac{\pi}{4} + \theta\right) = \frac{\cos2\theta}{1-\sin2\theta}$

   (d) $\dfrac{1-\tan^2\left(\frac{\pi}{4} - \theta\right)}{1+\tan^2\left(\frac{\pi}{4} - \theta\right)} = \sin2\theta$

   (e) $\frac{1+\sin2A}{1-\sin2A} = \left(\frac{\cot A+1}{\cot A-1}\right)^2$
4. Prove the following:

   (a) $\cos^2\theta + \sin^2\theta.\cos2ß = \cos^2ß + \sin^2ß.\cos2\theta$

   (b) $\operatorname{cosec}20° + \cot40° = \cot10° - \operatorname{cosec}40°$

   (c) $4\operatorname{cosec}2\theta.\cot2\theta = \operatorname{cosec}^2\theta - \sin^2\theta$
5. Prove that : $\tan\theta + 2\tan2\theta + 4\cot4\theta = \cot\theta$
6. Prove that : $\tan\theta + \tan\left(\frac{\pi}{3} + \theta\right) + \tan\left(\frac{2\pi}{3} - \theta\right) = 3\tan3\theta$.

Sub multiple angles

Estimated Periods: 7

1. Objectives

S,N.	Level	Objectives
(i)	Knowledge(k)	To define sub-multiple angles of an angle. To tell the formulae of trigonometric ratios of sub-multiple angles.
(ii)	Understanding(U)	To explain to derive the formulae of sub-multiple angles by using compound angle formulae.
(ii)	Application(A)	To solve problems of trigonometric identities of sub-multiple angles.
(iv)	Higher Ability (HA)	To solve very long question of trigonometric ratios multiple angles.

2. Teaching Materials

Formula chart of trigonometric ratios sub-multiple angles.

Teaching Strategies

- Review the formulae of trigonometric ratios of compound angles and multiple angles.
- Define sub-multiple angles of θ as $\frac{\theta}{2}, \frac{\theta}{3}, \frac{\theta}{4}$ etc.
- Show how to derive the formulae of trigonometric ratios of sub-multiple angles by using compound angle formulae.
- Compare formula of multiple and sub-multiple angle formulae of rigonometry like

$$\sin2\theta = 2\sin\theta \cdot \cos\theta$$

$$\sin\theta = \sin2\left(\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}.$$

$$\cos2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\cos\theta = \cos2\left(\frac{\theta}{2}\right) - \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

5. Discuss how to evaluate values of sinθ, cosθ, tanθ if $\sin\theta = \frac{1}{\sqrt{2}}$.

6. **Discuss to solve problems related to trigonometric ratios of sub-multiple questions given in exercises.**

List of formulae:

1. $\sin\theta = 2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

2. $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2} = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

3. $\sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3}$

$\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$

4. $\tan\theta = \dfrac{3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3}}{1 - 3\tan^2\frac{\theta}{3}}$

5. $\cot\theta = \dfrac{\cot^2\frac{\theta}{2} - 1}{2\cot\frac{\theta}{2}}$

6. $\cot\theta = \dfrac{\cot^3\frac{\theta}{3} - 3\cot\frac{\theta}{3}}{3\cot^2\frac{\theta}{3} - 1}$

Some solved problems

Prove the following :

1. $\cot x = \dfrac{\cot^3\left(\frac{x}{3}\right) - 3\cot\left(\frac{x}{3}\right)}{3\cot^2\left(\frac{x}{3}\right) - 1}$

Solution

$$\text{LHS} = \cot x = \cot\left(\frac{x}{3} + \frac{2x}{3}\right)$$

$$= \frac{\cot\frac{x}{3}\,.\,\cot\frac{2x}{3} - 1}{\cot\frac{2x}{3} + \cot\frac{x}{3}}$$

$$= \cot\frac{x}{3}\,.\,\frac{\cot^2\frac{x}{3} - 1}{2\cot\frac{x}{3}} - 1$$

$$= \frac{\cot^2\frac{x}{3} - 1}{2\cot\frac{x}{3}} + \cot\frac{x}{3}$$

$$= \frac{\cot^3\left(\frac{x}{3}\right) - 3\cot\left(\frac{x}{3}\right)}{3\cot^2\left(\frac{x}{3}\right) - 1}$$

= **RHS proved**

2. a) If $\cos\frac{\theta}{3} = \frac{1}{2}\left(p + \frac{1}{p}\right)$, then prove that : $\cos\theta = \frac{1}{2}\left(p^3 + \frac{1}{p^3}\right)$

Solution

Here, $\cos\frac{\theta}{3} = \frac{1}{2}\left(p + \frac{1}{p}\right)$

$$\begin{aligned}
\text{LHS} = \cos\theta &= 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3} \\
&= 4 \cdot \frac{1}{8}\left(p + \frac{1}{p}\right)^3 - \frac{3}{2}\left(p + \frac{1}{p}\right) \\
&= \frac{1}{2}\left(p + \frac{1}{p}\right)\left\{\left(p + \frac{1}{p}\right)^2 - 3\right\} \\
&= \frac{1}{2}\left(p + \frac{1}{p}\right)\left\{\left(p^2 + 2 \cdot p \cdot \frac{1}{p} + \frac{1}{p^2}\right) - 3\right\} \\
&= \frac{1}{2}\left(p + \frac{1}{p}\right)\left(p^2 + 2 + \frac{1}{p^2} - 3\right) \\
&= \frac{1}{2}\left(p + \frac{1}{p}\right)\left(p^2 - 1 + \frac{1}{p^2}\right) \\
&= \frac{1}{2}\left(p^3 + \frac{1}{p^3}\right)
\end{aligned}$$

= **RHS proved**

b) If $\sin\frac{\theta}{2} = \frac{1}{2}\left(p + \frac{1}{p}\right)$, then prove that : $\cos\theta = -\frac{1}{2}\left(p^2 + \frac{1}{p^2}\right)$

Solution

$$\begin{aligned}
\text{LHS} = \cos\theta &= 1 - 2\sin^2\frac{\theta}{2} \\
&= 1 - 2 \cdot \frac{1}{4}\left(p + \frac{1}{p}\right)^2 \\
&= 1 - \frac{1}{2}\left(p^2 + 2 \cdot p \cdot \frac{1}{p^2} + \frac{1}{p^2}\right) \\
&= 1 - \frac{1}{2}\left(p^2 + \frac{1}{p^2} + 2\right) \\
&= \frac{1}{2}\left(2 - p^2 - \frac{1}{p^2} - 2\right) \\
&= -\frac{1}{2}\left(p^2 + \frac{1}{p^2}\right)
\end{aligned}$$

$= \textbf{RHS proved}$

3. Prove the following

a) $$\frac{1-\sin A}{\cos A} = \frac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}}$$

Solution

$$\text{LHS} = \frac{1-\sin A}{\cos A}$$

$$= \frac{1-\dfrac{2\tan A/2}{1+\tan^2 A/2}}{\dfrac{1-\tan^2 A/2}{1+\tan^2 A/2}}$$

$$= \frac{1+\tan^2\frac{A}{2}-2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

$$= \frac{\left(1-\tan\frac{A}{2}\right)^2}{\left(1+\tan\frac{A}{2}\right)\left(1-\tan\frac{A}{2}\right)}$$

$$= \frac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}}$$

$= \textbf{RHS proved}$

b) $1-2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \sin\theta$

Solution

$$\text{LHS} = 1-2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \cos 2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \cos\left(\frac{\pi}{2}-\theta\right)$$

$= \sin\theta$ $= \textbf{RHS proved}$

c) $$\frac{1-\tan^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)}{1+\tan^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)} = \sin\frac{\theta}{2}$$

Solution

$$\text{LHS} = \frac{1 - \tan^2\left(\frac{\pi}{4} - \frac{\theta}{4}\right)}{1 + \tan^2\left(\frac{\pi}{4} - \frac{\theta}{4}\right)}$$

$$= \cos 2\left(\frac{\pi}{4} - \frac{\theta}{4}\right)$$

$$= \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$= \sin\frac{\theta}{2} \quad \textbf{= RHS proved}$$

4. Prove that

a) $\cos^4\left(\frac{\theta}{2}\right) - \sin^4\left(\frac{\theta}{2}\right) = \cos\theta$

Solution

$$\text{LHS} = \cos^4\left(\frac{\theta}{2}\right) - \sin^4\left(\frac{\theta}{2}\right)$$

$$= \left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)$$

$$= 1 \,.\, \cos\theta$$

$$= \cos\theta \;= \textbf{RHS proved.}$$

b) $\dfrac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta} = \tan$

Solution

$$\text{LHS} = \frac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta}$$

$$= \frac{2\sin\theta - 2\sin\theta \,.\, \cos\theta}{2\sin\theta + 2\sin\theta \,.\, \cos\theta}$$

$$= \frac{2\sin\theta\,(1 - \cos\theta)}{2\sin\theta\,(1 + \cos\theta)}$$

$$= \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= \tan^2\frac{\theta}{2} \;= \textbf{RHS proved.}$$

c) $\dfrac{\sin 2\theta}{1+\cos 2\theta} \cdot \dfrac{\cos\theta}{1+\cos\theta} = \tan\dfrac{\theta}{2}$

Solution

$$\text{LHS} = \frac{\sin 2\theta}{1+\cos 2\theta} \cdot \frac{\cos\theta}{1+\cos\theta}$$

$$= \frac{\sin 2\theta \cos\theta}{2\cos^2\theta} \cdot \frac{\cos\theta}{1+\cos\theta}$$

$$= \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2} \quad = \textbf{RHS proved}$$

d) $\dfrac{1+\sin\theta}{1-\sin\theta} = \tan^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$

Solution

$$\text{RHS} = \tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$= \left\{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}^2$$

$$= \left(\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4} \cdot \tan\frac{\theta}{2}}\right)^2$$

$$= \frac{\left(1+\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)^2}{\left(1-\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)^2}$$

$$= \frac{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}$$

$$= \frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+2\cos\frac{\theta}{2} \cdot \sin\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}-2\cos\frac{\theta}{2} \cdot \sin\frac{\theta}{2}}$$

$$= \frac{1 + \sin\theta}{1 - \sin\theta} = \textbf{LHS proved}$$

e) $\dfrac{\cos\frac{\theta}{2} - \sqrt{1 + \sin\theta}}{\sin\frac{\theta}{2} - \sqrt{1 + \sin\theta}} = \tan\frac{\theta}{2}$

Solution

$$\text{LHS} = \frac{\cos\frac{\theta}{2} - \sqrt{1 + \sin\theta}}{\sin\frac{\theta}{2} - \sqrt{1 + \sin\theta}}$$

$$= \frac{\cos\frac{\theta}{2} - \sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2} . \cos\frac{\theta}{2}}}{\sin\frac{\theta}{2} - \sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2} . \cos\frac{\theta}{2}}}$$

$$= \frac{\cos\frac{\theta}{2} - \sqrt{\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2}}{\sin\frac{\theta}{2} - \sqrt{\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2}}$$

$$= \frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2} - \cos\frac{\theta}{2}}{\sin\frac{\theta}{2} - \sin\frac{\theta}{2} - \cos\frac{\theta}{2}}$$

$$= \frac{-\sin\frac{\theta}{2}}{-\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2} = \textbf{RHS proved.}$$

5. Prove the following

a) $\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \sec\theta + \tan\theta$

Solution

$$\text{LHS} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4} . \tan\frac{\theta}{2}}$$

$$= \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}$$

$$= \frac{1 + \frac{\sin\theta/2}{\cos\theta/2}}{1 - \frac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}} \times \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}$$

$$= \frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$= \frac{1 + \sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$= \sec\theta + \tan\theta$ **= RHS proved.**

b) $\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \sqrt{\frac{1 - \sin\theta}{1 + \sin\theta}}$

Solution

$$\text{RHS} = \sqrt{\frac{1 - \sin\theta}{1 + \sin\theta}}$$

$$= \frac{\sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}}{\sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}}$$

$$= \frac{\sqrt{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}}{\sqrt{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}}$$

$$= \frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}$$

Dividing numerator and denominator by $\cos\frac{\theta}{2}$ **, we get**

$$= \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\theta}{2}}{1 + \tan\frac{\pi}{4}\,.\,\tan\frac{\theta}{2}}$$

$$= \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \quad = \textbf{LHS proved}$$

c) $\mathbf{sec\left(\frac{\pi}{4} + \frac{\theta}{2}\right).\ sec\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = 2\ sec\theta}$

Solution

$$\text{LHS} = \sec\left(\frac{\pi}{4} + \frac{\theta}{2}\right).\ \sec\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$= \frac{1}{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} - \frac{1}{\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}$$

$$= \frac{1}{\cos\frac{\pi}{4}\,.\,\cos\frac{\theta}{2} + \sin\frac{\pi}{4}\,.\,\sin\frac{\theta}{2}}\,.\,\frac{1}{\cos\frac{\pi}{4}\,.\,\cos\frac{\theta}{2} + \sin\frac{\pi}{4}\,.\,\sin\frac{\theta}{2}}$$

$$= \frac{1}{\frac{1}{\sqrt{2}}\,.\,\cos\frac{\theta}{2} + \frac{1}{\sqrt{2}}\,.\,\sin\frac{\theta}{2}}\,.\,\frac{1}{\frac{1}{\sqrt{2}}\,.\,\cos\frac{\theta}{2} + \frac{1}{\sqrt{2}}\,.\,\sin\frac{\theta}{2}}$$

$$= \frac{2}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}$$

$$= \frac{2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$= \frac{2}{\cos\theta}$$

$$= 2\sec\theta \quad = \textbf{RHS proved.}$$

d) $\mathbf{tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{cos\theta}{1 + sin\theta}}$

Solution

$$\text{LHS} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\theta}{2}}{1 + \tan\frac{\pi}{4}\,.\,\tan\frac{\theta}{2}}$$

$$= \frac{1 - \frac{\sin \theta/2}{\cos \theta/2}}{1 + \frac{\sin \theta/2}{\cos \theta/2}}$$

$$= \frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}$$

$$= \frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}} \times \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}$$

$$= \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\left(\sin \frac{\theta}{2} + \cos \frac{\theta}{2}\right)^2}$$

$$= \frac{\cos\theta}{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} + 2 \sin \frac{\theta}{2} . \cos \frac{\theta}{2}}$$

$$= \frac{\cos\theta}{1 + \sin\theta} \quad = \text{RHS proved.}$$

e) $\boldsymbol{\cot\left(\frac{\theta}{2} + \frac{\pi}{4}\right) - \tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right) = \frac{2 \cos\theta}{1 + \sin\theta}}$

Solution

$$\text{LHS} = \cot\left(\frac{\theta}{2} + \frac{\pi}{4}\right) - \tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$$

$$= \frac{\cot \frac{\theta}{2} . \cot \frac{\pi}{4} - 1}{\cot \frac{\theta}{2} + \cot \frac{\pi}{4}} - \frac{\tan \frac{\theta}{2} - \tan \frac{\pi}{4}}{1 + \tan \frac{\theta}{2} . \tan \frac{\pi}{4}}$$

$$= \frac{\frac{\cos \theta/2}{\sin \theta/2} - 1}{\frac{\cos \theta/2}{\sin \theta/2} + 1} - \frac{\frac{\sin \theta/2}{\cos \theta/2} - 1}{1 + \frac{\sin \theta/2}{\cos \theta/2}}$$

$$= \frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}} - \frac{\sin \frac{\theta}{2} - \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}$$

$$= \frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}} + \frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}$$

$$= \frac{2\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}$$

$$= 2\,\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}} \times \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}$$

$$= 2\,\frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}$$

$$= \frac{2\cos\theta}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\,.\cos\frac{\theta}{2}}$$

$$= \frac{2\cos\theta}{1 + \sin\theta} \quad = \textbf{RHS proved.}$$

f) $\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = 2\sec\theta$

Solution

$$\text{LHS} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4}\,.\tan\frac{\theta}{2}} + \frac{\tan\frac{\pi}{4} - \tan\frac{\theta}{2}}{1 + \tan\frac{\pi}{4}\,.\tan\frac{\theta}{2}}$$

$$= \frac{1 + \frac{\sin\theta/2}{\cos\theta/2}}{1 - \frac{\sin\theta/2}{\cos\theta/2}} + \frac{1 - \frac{\sin\theta/2}{\cos\theta/2}}{1 + \frac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}} + \frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}$$

$$= \frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2 + \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}$$

$$= \frac{1 + \sin\theta + 1 - \sin\theta}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$= \frac{2}{\cos\theta}$$

$$= 2\sec\theta \quad = \textbf{RHS proved.}$$

6. Prove that

a) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha - \beta}{2}\right)$

Solution

$$\begin{aligned}
\text{LHS} &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\
&= \cos^2\alpha - 2\cos\alpha \, . \cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha \, . \sin\beta + \cos^2\beta \\
&= 2(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha \, . \cos\beta + \sin\alpha \, . \sin\beta) \\
&= 1 + 1 - 2\cos(\alpha - \beta) \\
&= 2 - 2\cos(\alpha - \beta) \\
&= 2[1 - \cos(\alpha - \beta)] \\
&= 2 \, . \, 2\sin^2\left(\frac{\alpha - \beta}{2}\right) \\
&= 4\sin^2\left(\frac{\alpha - \beta}{2}\right) \quad = \textbf{RHS proved}
\end{aligned}$$

b) $(\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2 = 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$

Solution

$$\begin{aligned}
\text{LHS} &= (\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2 \\
&= \sin^2\alpha + \sin^2\beta + 2\sin\alpha \, . \sin\beta + \cos^2\alpha + \cos^2\beta - 2\cos\alpha \, . \cos\beta \\
&= (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \sin^2\alpha) + 2(\sin\alpha \, . \sin\beta + \cos\alpha \, . \cos\beta) \\
&= 1 + 1 + 2\cos(\alpha - \beta) \\
&= 2 + 2\cos(\alpha - \beta) \\
&= 2[1 + \cos(\alpha - \beta)] \\
&= 2 \, . \, 2\cos^2\left(\frac{\alpha - \beta}{2}\right) \\
&= 4\cos^2\left(\frac{\alpha - \beta}{2}\right) \quad = \textbf{RHS proved}
\end{aligned}$$

7. Prove the following

a) $\cos\left(\frac{2\pi}{15}\right) . \cos\left(\frac{4\pi}{15}\right) . \cos\left(\frac{8\pi}{15}\right) . \cos\left(\frac{16\pi}{15}\right) = \frac{1}{16}$

Solution

$$\begin{aligned}
\text{LHS} &= \cos\left(\frac{2\pi}{15}\right) . \cos\left(\frac{4\pi}{15}\right) . \cos\left(\frac{8\pi}{15}\right) . \cos\left(\frac{16\pi}{15}\right) \\
&= \frac{1}{2\sin\frac{2\pi}{15}}\left[\left(2\sin\frac{2\pi}{15} . \cos\frac{2\pi}{15}\right) . \cos\left(\frac{4\pi}{15}\right) . \cos\left(\frac{8\pi}{15}\right) . \cos\left(\frac{16\pi}{15}\right)\right]
\end{aligned}$$

$$= \frac{1}{2\sin\frac{2\pi}{15}}\left[\sin\left(\frac{4\pi}{15}\right).\cos\left(\frac{4\pi}{15}\right).\cos\left(\frac{8\pi}{15}\right).\cos\left(\frac{16\pi}{15}\right)\right]$$

$$= \frac{1}{4\sin\frac{2\pi}{15}}\left[\sin\left(\frac{8\pi}{15}\right).\cos\left(\frac{8\pi}{15}\right).\cos\left(\frac{16\pi}{15}\right)\right]$$

$$= \frac{1}{8\sin\frac{2\pi}{15}}\left[\sin\left(\frac{16\pi}{15}\right).\cos\left(\frac{16\pi}{15}\right)\right]$$

$$= \frac{1}{16\sin\frac{2\pi}{15}}\sin\left(\frac{32\pi}{15}\right)$$

$$= \frac{1}{16\sin\frac{2\pi}{15}}.\sin\left(2\pi + \frac{2\pi}{15}\right)$$

$$= \frac{1}{16\sin\frac{2\pi}{15}}.\sin\frac{2\pi}{15}$$

$$= \frac{1}{16} \quad = \textbf{RHS proved.}$$

8. Proved that:

$$\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right) = \frac{1}{8}$$

Solution

$$\text{LHS} = \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right)$$

$$= \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left\{1 + \cos\left(\pi - \frac{3\pi}{8}\right)\right\}\left\{1 + \cos\left(\pi - \frac{\pi}{8}\right)\right\}$$

$$= \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)$$

$$= \left(1 - \cos^2\frac{\pi}{8}\right)\left(1 - \cos^2\frac{3\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}.\sin^2\frac{3\pi}{8}$$

$$= \sin^2\frac{\pi}{8}.\sin^2\left(\frac{\pi}{2} - \frac{\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}.\cos^2\frac{\pi}{8}$$

$$= \frac{1}{4}\left(2 \sin \frac{\pi}{8} . \cos \frac{\pi}{8}\right)^2$$

$$= \frac{1}{4}\left[\sin 2\left(\frac{\pi}{8}\right)\right]^2$$

$$= \frac{1}{4}\left(\sin \frac{\pi}{4}\right)^2$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{8} \quad = \textbf{RHS proved.}$$

8. Prove that:

$$\mathbf{\tan 7\frac{1^\circ}{2} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2}$$

Solution

$$\text{LHS} = \tan 7\frac{1^\circ}{2}$$

$$= \tan \frac{15^\circ}{2}$$

$$= \frac{\sin \frac{15^\circ}{2}}{\cos 15^\circ} \times \frac{2 \sin \frac{15^\circ}{2}}{2 \sin \frac{15^\circ}{2}}$$

$$= \frac{2 \sin^2 \frac{15^\circ}{2}}{\sin 15^\circ}$$

$$= \frac{1 - \cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1 - \cos(45^\circ - 30^\circ)}{\sin(45^\circ - 30^\circ)}$$

$$= \frac{1 - \cos 45^\circ . \cos 30^\circ - \sin 45^\circ . \sin 30^\circ}{\sin 45^\circ . \cos 30^\circ - \cos 45^\circ . \sin 30^\circ}$$

$$= \frac{1 - \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} . \frac{1}{2}}{\frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} . \frac{1}{2}}$$

$$= \frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{2\sqrt{6} - 3 - \sqrt{3} + 2\sqrt{2} - \sqrt{3} - 1}{3 - 1}$$

$= \dfrac{2\sqrt{6} - 2\sqrt{3} + 2\sqrt{2} - 4}{2}$

$= \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$ **= RHS proved**

Questions for practice

1. **Prove that:** $\tan\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$

2. **If** $\cos 330° = \frac{\sqrt{3}}{2}$, **prove that :** $\sin 165° = \frac{1}{2}\sqrt{2 - \sqrt{3}}$

3. **Prove that:** $\frac{1 + \cos\theta + \sin\theta}{1 - \cos\theta + \sin\theta} = \cot\frac{\theta}{2}$

4. **Prove that:** $\sec\theta + \tan\theta = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

5. **If** $\sin\frac{\theta}{3} = \frac{1}{2}\left(a + \frac{1}{a}\right)$, **prove that:** $\sin\theta = \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$

6. **Prove that:** $\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}} = \sec\theta + \tan\theta$

7. **Prove that:** $\frac{\sin 2A}{1 + \cos 2A} \times \frac{\cos A}{1 + \cos A} = \tan\frac{A}{2}$

8. **Prove that:** $\cot 7\frac{1°}{2} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$

Transformation of Trigonometric Formula of sine and cosine

Estimated Periods: 5

1. Objectives

S,N.	Level	Objectives
(i)	Knowledge(k)	To tell the formulae of transformation of trigonometric ratios of sine and cosine.
(ii)	Understanding(U)	To explain to derive the formulae of trannsformation of trigonometric ratios.
(ii)	Application(A)	To solve the problems of transformation of trigonometric formulae.
(iv)	Higher Ability (HA)	To solve harder problems of transformation of trigonometric formulae.

2. Required teaching materials

Formula chart of trigonometric ratio of compound angles and transformation formula of trigonometric ratios.

Teaching strategies

- Review the formulae of trigonometric ratios of compound angles.
- List the trigonometric ratios of compound angles as given below:

$$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B \ \ldots\ldots\ldots \text{(i)}$$
$$\sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B \ \ldots\ldots\ldots \text{(ii)}$$
$$\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B \ \ldots\ldots\ldots \text{(iii)}$$
$$\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B \ \ldots\ldots\ldots \text{(iv)}$$

- Adding and subtracting above identities we get the required formulae.

Example adding (i) and (ii), we get

$$2 \sin A \cdot \cos B = \sin(A + B) + \sin(A - B)$$

Subtracting (ii) from (i), we get

$$2 \cos A \cdot \sin B = \sin(A + B) - \sin(A - B)$$

Similarly, explain to get the following results

$$2 \cos A \cdot \cos B = \cos(A - B) + \cos(A - B)$$

$$2 \sin A \cdot \sin B = \sin(A - B) - \cos(A + B)$$

- Again, discuss how to derive the following formulae

$$\sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right) \cdot \cos\left(\frac{C - D}{2}\right)$$
$$\sin C - \sin D = 2 \cos\left(\frac{C + D}{2}\right) \cdot \sin\left(\frac{C - D}{2}\right)$$
$$\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right) \cdot \cos\left(\frac{C - D}{2}\right)$$
$$\cos C - \cos D = 2 \sin\left(\frac{C + D}{2}\right) \cdot \sin\left(\frac{C - D}{2}\right)$$

List of formula:

1. $2\sin A \,.\, \cos B = \sin(A + B) + \sin(A - B)$
2. $2\cos A \,.\, \sin B = \sin(A + B) - \sin(A - B)$
3. $2\cos A \,.\, \cos B = \cos(A + B) + \cos(A - B)$
4. $2\sin A \,.\, \sin B = \cos(A - B) - \cos(A + B)$
5. $\sin C + \sin D = 2\sin\left(\frac{C + D}{2}\right) . \cos\left(\frac{C - D}{2}\right)$
6. $\sin C - \sin D = 2\cos\left(\frac{C + D}{2}\right) . \sin\left(\frac{C - D}{2}\right)$
7. $\cos D - \cos C = 2\sin\left(\frac{C + D}{2}\right) . \sin\left(\frac{C - D}{2}\right)$
8. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ is called componendo and dividendo.

Some solved problems

1. Prove the following:

a) $\cos 75° + \cos 15° = \sqrt{\frac{3}{2}}$

Solution

$$LHS = \cos 75° + \cos 15°$$

$$= 2\cos\left(\frac{75° + 15°}{2}\right)\cos\left(\frac{75° - 15°}{2}\right)$$

$$= 2\cos 45° . \cos 30°$$

$$= 2 . \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2}$$

$$= \sqrt{\frac{3}{2}} \quad = \textbf{RHS proved.}$$

b) $\sin 75° - \sin 15° = \frac{1}{\sqrt{2}}$

Solution

$$LHS = \sin 75° - \sin 15°$$

$$= 2\cos\left(\frac{75° + 15°}{2}\right) . \sin\left(\frac{75° - 15°}{2}\right)$$

$$= 2\cos 45° . \sin 30°$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{1}{\sqrt{2}} \quad = \textbf{RHS proved.}$$

c) $\cos 52° + \cos 68° + \cos 172° = 0$

Solution

LHS $= \cos 52° + \cos 68° + \cos 172°$

$$= 2\cos\left(\frac{52° + 68°}{2}\right) . \cos\left(\frac{52° - 68°}{2}\right) + \cos(180° - 8°)$$

$$= 2\cos 60° . \cos(-8°) - \cos 8°$$

$$= 2 . \frac{1}{2} . \cos 8° - \cos 8°$$

$$= \cos 8° - \cos 8°$$

$$= 0 \quad = \textbf{RHS proved.}$$

2. Prove that:

a) $\dfrac{\cos 5A + \sin 3A}{\sin 5A - \sin 3A} = \cot A$

Solution

$$\text{LHS} = \frac{\cos 5A + \sin 3A}{\sin 5A - \sin 3A}$$

$$= \frac{2\cos\left(\frac{5A + 3A}{2}\right) . \cos\left(\frac{5A - 3A}{2}\right)}{2\cos\left(\frac{5A + 3A}{2}\right) . \sin\left(\frac{5A - 3A}{2}\right)}$$

$$= \frac{\cos 4A . \cos A}{\cos 4A . \sin A} \qquad \cos(-\theta) = \cos\theta$$

$$= \cot A \quad = \textbf{RHS proved.}$$

b) $\dfrac{\cos 40° - \cos 60°}{\sin 60° - \sin 40°} = \tan 50°$

Solution

$$\text{LHS} = \frac{\cos 40° - \cos 60°}{\sin 60° - \sin 40°}$$

$$= \frac{2\sin\left(\frac{40° + 60°}{2}\right) . \sin\left(\frac{40° - 60°}{2}\right)}{2\cos\left(\frac{40° + 60°}{2}\right) . \sin\left(\frac{40° - 60°}{2}\right)}$$

$$= \frac{\sin 50° . \sin 10°}{\cos 50° . \sin 10°}$$

$$= \tan 50° \quad = \textbf{RHS proved.}$$

c) $\dfrac{\cos80° + \cos20°}{\sin80° - \sin20°} = \sqrt{3}$

Solution

$$\text{LHS} = \frac{\cos80° + \cos20°}{\sin80° - \sin20°}$$

$$= \frac{2\cos\left(\frac{80° + 20°}{2}\right).\cos\left(\frac{80° - 20°}{2}\right)}{2\cos\left(\frac{80° + 20°}{2}\right).\sin\left(\frac{80° - 20°}{2}\right)}$$

$$= \frac{\cos30°}{\sin30°}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{2} \times 2$$

$$= \sqrt{3} \quad = \textbf{RHS proved.}$$

d) $\dfrac{\cos8° + \sin8°}{\cos8° - \sin8°} = \tan53°$

Solution

$$\text{LHS} = \frac{\cos8° + \sin8°}{\cos8° - \sin8°}$$

$$= \frac{\cos8° + \sin(90° - 82°)}{\cos8° - \sin(90° - 82°)}$$

$$= \frac{\cos8° + \cos82°}{\cos8° - \cos82°}$$

$$= \frac{2\cos\left(\frac{8° + 82°}{2}\right).\cos\left(\frac{8° - 82°}{2}\right)}{2\sin\left(\frac{8° + 82°}{2}\right).\sin\left(\frac{82° - 8°}{2}\right)}$$

$$= \frac{\cos37°}{\sin37°}$$

$$= \cot37°$$

$$= \cot(90° - 53°)$$

$$= \tan53° \quad \textbf{= RHS proved.}$$

e) $\dfrac{\cos10° - \sin10°}{\cos10° + \sin10°} = \cot55°$

Solution

$$\text{LHS} = \frac{\cos10° - \sin10°}{\cos10° + \sin10°}$$

$$= \frac{\cos10° - \sin(90° - 80°)}{\cos10° + \sin(90° - 80°)}$$

$$= \frac{\cos10° - \cos80°}{\cos10° + \cos80°}$$

$$= \frac{2\sin\left(\frac{10° + 80°}{2}\right) . \sin\left(\frac{80° - 10°}{2}\right)}{2\cos\left(\frac{10° + 80°}{2}\right) . \cos\left(\frac{10° - 80°}{2}\right)}$$

$$= \frac{\sin45° . \sin35°}{\cos45° . \cos35°}$$

$$= \tan35°$$

$$= \tan(90° - 55°)$$

$= \cot55°$ **= RHS proved.**

f) $\dfrac{\cos(40° + A) + \cos(40° - A)}{\sin(40° + A) - \sin(40° - A)} = \cot A$

Solution

$$\text{LHS} = \frac{\cos(40° + A) + \cos(40° - A)}{\sin(40° + A) - \sin(40° - A)}$$

$$= \frac{2\cos\left(\frac{40° + A + 40° - A}{2}\right) . \cos\left(\frac{40° + A - 40° + A}{2}\right)}{2\cos\left(\frac{40° + A + 40° - A}{2}\right) . \sin\left(\frac{40° + A - 40° + A}{2}\right)}$$

$$= \frac{\cos A}{\sin A}$$

$= \cot A$ = **RHS proved.**

3. Prove the following.

a) $\dfrac{\sin A . \sin2A + \sin3A . \sin6A}{\sin A . \cos2A + \sin3A . \cos6A} = \tan5A$

Solution

$$\text{LHS} = \frac{\sin A . \sin2A + \sin3A . \sin6A}{\sin A . \cos2A + \sin3A . \cos6A}$$

$$= \frac{2\sin A . \sin2A + 2\sin3A . \sin6A}{2\sin A . \cos2A + 2\sin3A . \cos6A}$$

$$= \frac{\cos(A - 2A) - \cos(A + 2A) + \cos(3A - 6A) - \cos(3A + 6A)}{\sin(A + 2A) + \sin(A - 2A) + \sin(3A + 6A) + \sin(3A - 6A)}$$

$$= \frac{\cos A - \cos 3A + \cos 3A - \cos 9A}{\sin 3A - \sin A + \sin 9A - \sin 3A}$$

$$= \frac{\cos A - \cos 9A}{\sin 9A - \sin A}$$

$$= \frac{2 \sin\left(\frac{A + 9A}{2}\right) . \sin\left(\frac{A + 9A}{2}\right)}{2 \cos\left(\frac{A + 9A}{2}\right) . \sin\left(\frac{A + 9A}{2}\right)}$$

$= \tan 5A \quad =$ **RHS proved.**

b) $\dfrac{\mathbf{\cos 2A . \cos 3A - \cos 2A . \cos 7A}}{\mathbf{\sin 4A . \sin 3A - \sin 2A . \sin 5A}} = \dfrac{\mathbf{\sin 7A + \sin 3A}}{\mathbf{\sin A}}$

Solution

$$\text{LHS} = \frac{\cos 2A . \cos 3A - \cos 2A . \cos 7A}{\sin 4A . \sin 3A - \sin 2A . \sin 5A}$$

$$= \frac{2 \cos 2A . \cos 3A - 2 \cos 2A . \cos 7A}{2 \sin 2A . \sin 3A - 2 \sin 2A . \sin 5A}$$

$$= \frac{\cos(2A + 3A) + \cos(2A - 3A) - \cos(2A + 7A) - \cos(2A - 7A)}{\cos(4A - 3A) - \cos(4A + 3A) - \cos(2A - A) + \cos(2A + 5A)}$$

$$= \frac{\cos 5A + \cos A - \cos 9A - \cos 5A}{\cos A - \cos 7A - \cos 3A + \cos 7A}$$

$$= \frac{\cos A - \cos 9A}{\cos A - \cos 3A}$$

$$= \frac{2 \sin\left(\frac{A + 9A}{2}\right) . \sin\left(\frac{9A - A}{2}\right)}{2 \sin\left(\frac{A + 3A}{2}\right) . \sin\left(\frac{3A - A}{2}\right)}$$

$$= \frac{\sin 5A . \sin 4A}{\sin 2A . \sin A}$$

$$= \frac{\sin 5A . 2 \sin 2A . \cos 2A}{\sin 2A . \sin A}$$

$$= \frac{2 \sin 5A . \cos 2A}{\sin A}$$

$$\text{RHS} = \frac{\sin 7A + \sin 3A}{\sin A}$$

$$= \frac{2 \sin\left(\frac{7A + 3A}{2}\right) . \cos\left(\frac{7A - 3A}{2}\right)}{\sin A}$$

$$= \frac{2 \sin 5A . \cos 2A}{\sin A}$$

$\therefore$ **LHS = RHS proved**

c) $\dfrac{\cos7A + \cos3A - \cos5A - \cos A}{\sin7A - \sin3A - \sin5A + \sin A} = \cot2A$

Solution

$$\text{LHS} = \frac{\cos7A + \cos3A - \cos5A - \cos A}{\sin7A - \sin3A - \sin5A + \sin A}$$

$$= \frac{(\cos7A - \cos A) + (\cos3A - \cos5A)}{(\sin7A + \sin A) - (\sin3A + \sin5A)}$$

$$= \frac{2\sin\left(\frac{7A + A}{2}\right).\sin\left(\frac{7A - A}{2}\right) + 2\sin\left(\frac{3A + 5A}{2}\right).\sin\left(\frac{5A - 3A}{2}\right)}{2\sin\left(\frac{7A + A}{2}\right).\cos\left(\frac{7A - A}{2}\right) - 2\sin\left(\frac{3A + 5A}{2}\right).\cos\left(\frac{3A - 5A}{2}\right)}$$

$$= \frac{2\sin4A\,[\sin3A + \sin A]}{2\sin4A\,[\cos3A - \cos A]}$$

$$= \frac{\sin A - \sin3A}{\cos3A - \cos A}$$

$$= \frac{2\cos\left(\frac{A + 3A}{2}\right).\sin\left(\frac{A - 3A}{2}\right)}{2\sin\left(\frac{3A + A}{2}\right).\sin\left(\frac{A - 3A}{2}\right)}$$

$$= \frac{\cos2A}{\sin2A}$$

$= \cot2A$ **= RHS proved.**

d) $\dfrac{\sin A + \sin3A + \sin5A + \sin7A}{\cos A + \cos3A + \cos5A + \cos7A} = \tan4A$

Solution

$$\text{LHS} = \frac{\sin A + \sin3A + \sin5A + \sin7A}{\cos A + \cos3A + \cos5A + \cos7A}$$

$$= \frac{(\sin A + \sin7A) + (\sin3A + \sin5A)}{(\cos A + \cos7A) + (\cos3A + \cos5A)}$$

$$= \frac{2\sin\left(\frac{A + 7A}{2}\right).\cos\left(\frac{A - 7A}{2}\right) + 2\sin\left(\frac{3A + 5A}{2}\right).\cos\left(\frac{3A - 5A}{2}\right)}{2\cos\left(\frac{A + 7A}{2}\right).\cos\left(\frac{A - 7A}{2}\right) + 2\cos\left(\frac{3A + 5A}{2}\right).\cos\left(\frac{3A - 5A}{2}\right)}$$

$$= \frac{2\sin4A\,[\cos3A + \cos A]}{2\cos4A\,[\cos3A + \cos A]}$$

$= \tan4A$ **= RHS proved.**

e) $\dfrac{\sin5A - \sin7A - \sin4A + \sin8A}{\cos4A - \cos5A - \cos8A + \cos7A} = \cot6A$

Solution

$$\text{LHS} = \frac{\sin5A - \sin7A - \sin4A + \sin8A}{\cos4A - \cos5A - \cos8A + \cos7A}$$

$$= \frac{(\sin 5A - \sin 7A) - (\sin 4A - \sin 8A)}{(\cos 4A - \cos 8A) - (\cos 5A - \cos 7A)}$$

$$= \frac{2\cos\left(\frac{5A + 7A}{2}\right) . \sin\left(\frac{5A - 7A}{2}\right) - 2\cos\left(\frac{4A + 8A}{2}\right) . \sin\left(\frac{4A - 8A}{2}\right)}{2\sin\left(\frac{4A + 8A}{2}\right) . \sin\left(\frac{8A - 4A}{2}\right) - 2\sin\left(\frac{5A + 7A}{2}\right) . \sin\left(\frac{7A - 5A}{2}\right)}$$

$$= \frac{\cos 6A\,[-\sin A + \sin 2A]}{\sin 6A\,[\sin 2A - \sin A]}$$

$= \cot 6A$ = **RHS proved.**

f) $\mathbf{\dfrac{\sin(p + 2)\theta - \sin p\theta}{\cos p\theta - \cos(p + 2)\theta} = \cot(p + 1)\theta}$

Solution

$$\text{LHS} = \frac{\sin(p + 2)\theta - \sin p\theta}{\cos p\theta - \cos(p + 2)\theta}$$

$$= \frac{2\cos\left(\frac{p + 2 + p}{2}\right)\theta . \sin\left(\frac{p + 2 - p}{2}\right)\theta}{2\sin\left(\frac{p + p + 2}{2}\right)\theta . \sin\left(\frac{p + 2 - p}{2}\right)\theta}$$

$$= \frac{\cos(p + 1)\theta . \sin\theta}{\sin(p + 1)\theta . \sin\theta}$$

$= \cot(p + 1)\theta$ = RHS proved.

g) $\mathbf{\dfrac{(\sin 4A + \sin 2A) . (\cos 4A - \cos 8A)}{(\sin 7A + \sin 5A) . (\cos A - \cos 5A)} = 1}$

Solution

$$\text{LHS} = \frac{(\sin 4A + \sin 2A) . (\cos 4A - \cos 8A)}{(\sin 7A + \sin 5A) . (\cos A - \cos 5A)}$$

$$= \frac{2\sin\left(\frac{4A + 2A}{2}\right) . \cos\left(\frac{4A - 2A}{2}\right) . 2\sin\left(\frac{4A + 8A}{2}\right) . \sin\left(\frac{8A - 4A}{2}\right)}{2\sin\left(\frac{7A + 5A}{2}\right) . \cos\left(\frac{7A - 5A}{2}\right) . 2\sin\left(\frac{A + 5A}{2}\right) . \sin\left(\frac{5A - A}{2}\right)}$$

$$= \frac{\sin 3A . \cos A . \sin 6A . \sin 2A}{\sin 6A . \cos A . \sin 3A . \sin 2A}$$

$= 1$ = **RHS proved.**

4. Prove that:

a) $\mathbf{\sin 20° . \sin 40° . \sin 60° . \sin 80° = \dfrac{3}{16}}$

Solution

$$\text{LHS} = \sin20° \,.\, \sin40° \,.\, \sin60° \,.\, \sin80°$$

$$= \frac{\sqrt{3}}{2} \,.\, \frac{1}{2}\,(2\sin20° \,.\, \sin40°) \,.\, \sin80°$$

$$= \frac{\sqrt{3}}{4}\left\{\cos(20° - 40°) - \cos(20° + 40°)\right\} \,.\, \sin80°$$

$$= \frac{\sqrt{3}}{4}\,\{\cos20° - \cos60°\} \,.\, \sin80°$$

$$= \frac{\sqrt{3}}{4}\cos20° \,.\, \sin80° - \frac{\sqrt{3}}{4} \,.\, \frac{1}{2} \,.\, \sin80°$$

$$= \frac{\sqrt{3}}{8}\left\{\sin(20° + 80°) - \sin(20° - 80°)\right\} - \frac{\sqrt{3}}{8}\sin80°$$

$$= \frac{\sqrt{3}}{8}\sin100° + \frac{\sqrt{3}}{8}\sin60° - \frac{\sqrt{3}}{8}\sin80°$$

$$= \frac{\sqrt{3}}{8}\sin(180° - 80°) - \frac{\sqrt{3}}{8}\sin80° + \frac{\sqrt{3}}{8} \,.\, \frac{\sqrt{3}}{2}$$

$$= \frac{3}{16} + \frac{\sqrt{3}}{8}\sin80° - \frac{\sqrt{3}}{8}\sin80°$$

$$= \frac{3}{16} \quad \textbf{= RHS proved.}$$

b) $\mathbf{\sin10° \,.\, \sin30° \,.\, \sin50° \,.\, \sin70° = \frac{1}{16}}$

Solution

$$\text{LHS} = \sin10° \,.\, \sin30° \,.\, \sin50° \,.\, \sin70°$$

$$= \sin10° \,.\, \frac{1}{2} \,.\, \sin50° \,.\, \sin70°$$

$$= \frac{1}{4}\,(2\sin10° \,.\, \sin50°)\sin70°$$

$$= \frac{1}{4}\,\{\cos(10° - 50°) - \cos(10° + 50°)\} \,.\, \sin70°$$

$$= \frac{1}{4}\{\cos40° - \cos60°\} \,.\, \sin70°$$

$$= \frac{1}{4}\cos40° \,.\, \sin70° - \frac{1}{4} \,.\, \frac{1}{2} \,.\, \sin70°$$

$$= \frac{1}{8}(2\cos40° \,.\, \sin70°) - \frac{1}{8}\sin70°$$

$$= \frac{1}{8}\{\sin110° - \sin(-30°)\} - \frac{1}{8}\sin70°$$

$$= \frac{1}{8}\sin110° + \frac{1}{8} \,.\, \frac{1}{2} - \frac{1}{8}\sin70°$$

$$= \frac{1}{8}\sin(180° - 70°) + \frac{1}{16} - \frac{1}{8}\ \sin 70°$$

$$= \frac{1}{8}\sin 70° - \frac{1}{8}\ \sin 70° + \frac{1}{16}$$

$$= \frac{1}{16}$$ = **RHS proved.**

d) $\cos 40° \,.\, \cos 100° \,.\, \cos 160° = \frac{1}{8}$

Solution

$$\text{LHS} = \cos 40° \,.\, \cos 100° \,.\, \cos 160°$$

$$= \frac{1}{2}[2\cos 40° \,.\, \cos 100°] \,.\, \cos 160°$$

$$= \frac{1}{2}\ [\cos 40° + 100°) + \cos(40° - 100°)] \,.\, \cos 160°$$

$$= \frac{1}{2}\ [\cos 140° + \cos 60°] \,.\, \cos 160°$$

$$= \frac{1}{2}\cos 140° \,.\, \cos 160° + \frac{1}{2} \,.\, \frac{1}{2} \,.\, \cos 160°$$

$$= \frac{1}{4}\ (\cos 300° + \cos 20°) + \frac{1}{4} \,.\, \cos 160°$$

$$= \frac{1}{4} \,.\, \frac{1}{2} + \frac{1}{4}\cos 20° - \frac{1}{4}\cos 20°$$

$$= \frac{1}{8}$$ **= RHS proved.**

e) $\tan 20° \,.\, \tan 40° \,.\, \tan 80° = \sqrt{3}$

Solution

$$\text{LHS} = \tan 20° \,.\, \tan 40° \,.\, \tan 80°$$

$$= \frac{\sin 20°}{\cos 20°} \,.\, \frac{\sin 40°}{\cos 40°} \,.\, \frac{\sin 80°}{\cos 80°}$$

Numerator = $\sin 20° \,.\, \sin 40° \,.\, \sin 80°$

Simplify it to get, $\frac{\sqrt{3}}{8}$

Again, denominator = $\cos 20°, \cos 40°, \cos 80°$

Simplify it to get, $\frac{1}{8}$

Then, LHT = $\dfrac{\frac{\sqrt{3}}{8}}{\frac{1}{8}} = \sqrt{3}$ **= RHS proved.**

5. Prove that:

a) $\mathbf{\sin(45° + \theta) . \sin(45° - \theta) = \frac{1}{2}\cos2\theta}$

Solution

$$\text{LHS} = \sin(45° + \theta) . \sin(45° - \theta)$$
$$= \frac{1}{2}\{2 \sin(45° + \theta) . \sin(45° - \theta)\}$$
$$= \frac{1}{2}\{\cos(45° + \theta - 45° + \theta) - \cos(45° + \theta + 45° - \theta)\}$$
$$= \frac{1}{2}\{\cos2\theta - \cos90°\}$$
$$= \frac{1}{2}\cos2\theta \quad \textbf{= RHS proved.}$$

b) $\mathbf{\cos(45° + \theta) . \cos(45° - \theta) = \frac{1}{2}\cos2\theta}$

Solution

$$\text{LHS} = \cos(45° + \theta) . \cos(45° - \theta)$$
$$= \frac{1}{2}\{\cos(45° + \theta + 45° - \theta) + \cos(45° + \theta - 45° + \theta)\}$$
$$= \frac{1}{2}[\cos90° + \cos2\theta]$$
$$= \frac{1}{2}\cos2\theta \quad \textbf{= RHS proved.}$$

c) $\mathbf{\cos\theta . \cos(60° - \theta) . \cos(60° + \theta) = \frac{1}{4}\cos3\theta}$

Solution

$$\text{LHS} = \cos\theta . \cos(60° - \theta) . \cos(60° + \theta)$$
$$= \frac{1}{2}\cos\theta\{2 \cos(60° - \theta) . \cos(60° + \theta)\}$$
$$= \frac{1}{2}\cos\theta\{\cos(60° - \theta + 60° + \theta) + \cos(60° - \theta - 60° - \theta)\}$$
$$= \frac{1}{2}\cos\theta\{\cos120° + \cos2\theta\}$$
$$= \frac{1}{2}\cos\theta\left(\frac{-1}{2}\right) + \frac{1}{2}\cos\theta . \cos2\theta$$
$$= -\frac{1}{4}\cos\theta + \frac{1}{4}\{\cos3\theta . \cos\theta\}$$
$$= -\frac{1}{4}\cos\theta + \frac{1}{4}\cos3\theta + \frac{1}{4}\cos\theta$$
$$= \frac{1}{4}\cos3\theta \quad \textbf{= RHS proved.}$$

6. **Prove that: $\cos(36° - \theta) . \cos(36° + \theta) + \cos(54° + \theta) . \cos(54° - \theta) = \cos2\theta$**

Solution

$$\text{LHS} = \cos(36° - \theta) . \cos(36° + \theta) + \cos(54° + \theta) . \cos(54° - \theta)$$

$$= \frac{1}{2}\{\cos(36° - \theta + 36° + \theta) + \cos(36° - \theta - 36° - \theta)\}$$

$$+ \frac{1}{2}\{\cos(54° + \theta + 54° - \theta) + \cos(54° + \theta - 54° + \theta)\}$$

$$= \frac{1}{2}\{\cos72° + \cos2\theta + \cos108° + \cos2\theta\}$$

$$= \frac{1}{2} . 2\cos2\theta + \frac{1}{2}\{\cos72° + \cos(180° - 72°)\}$$

$$= \cos2\theta + \frac{1}{2}\{\cos72° - \cos72°\}$$

$$= \cos2\theta + \frac{1}{2} . 0$$

$= \cos2\theta$ $=$ **RHS proved.**

7. **Prove that:**

a) $\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right) = 2\sec2\theta$

Solution

$$\text{LHS} = \tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)$$

$$= \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4} . \tan\theta} - \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4} . \tan\theta}$$

$$= \frac{1 + \frac{\sin\theta}{\cos\theta}}{1 - \frac{\sin\theta}{\cos\theta}} + \frac{1 - \frac{\sin\theta}{\cos\theta}}{1 + \frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} + \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$$

$$= \frac{(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta + 2\sin\theta.\cos\theta + \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}{\cos2\theta}$$

$$= \frac{1 + 1}{\cos2\theta}$$

$$= 2\frac{1}{\cos2\theta}$$

$= 2\sec2\theta$ **= RHS proved.**

b) $\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)=2\tan2\theta$

Solution

$$\text{LHS} = \tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\,.\,\tan\theta}-\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\,.\,\tan\theta}$$

$$= \frac{1+\frac{\sin\theta}{\cos\theta}}{1-\frac{\sin\theta}{\cos\theta}}-\frac{1-\frac{\sin\theta}{\cos\theta}}{1+\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}-\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$$

$$= \frac{(\cos\theta+\sin\theta)^2-(\cos\theta-\sin\theta)^2}{\cos^2\theta-\sin^2\theta}$$

$$= \frac{\cos^2\theta+\sin^2\theta+2\sin\theta.\cos\theta-\cos^2\theta-\sin^2\theta+2\sin\theta\cos\theta}{\cos2\theta}$$

$$= \frac{\sin2\theta+\cos2\theta}{\cos2\theta}$$

$$= 2\,\frac{\sin2\theta}{\cos2\theta}$$

$= 2\tan2\theta$ **= RHS proved.**

8. Prove the following.

a) $(\cos A+\cos B)^2+(\sin A+\sin B)^2=4\cos^2\left(\frac{A-B}{2}\right)$

Solution

$$\text{LHS} = (\cos A+\cos B)^2+(\sin A+\sin B)^2$$

$$= \left\{2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)\right\}^2+\left\{2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)\right\}^2$$

$$= 4\cos^2\left(\frac{A-B}{2}\right)\left\{\cos^2\left(\frac{A+B}{2}\right)+\sin^2\left(\frac{A+B}{2}\right)\right\}$$

$= 4\cos^2\left(\frac{A-B}{2}\right)$ **= RHS proved.**

b) $(\text{Cos}B-\cos A)2+(\sin A-\sin B)2=4\sin2\left(\frac{A-B}{2}\right)$

Solution

$$\text{LHS} = (\text{Cos}B-\cos A)^2+(\sin A-\sin B)^2$$

$$= \left\{2\sin\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)\right\}^2+\left\{2\cos\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)\right\}^2$$

$$= 4\sin^2\left(\frac{A+B}{2}\right)\sin^2\left(\frac{A-B}{2}\right) + 4\cos^2\left(\frac{A+B}{2}\right).\sin^2\left(\frac{A-B}{2}\right)$$

$$= 4\sin^2\left(\frac{A-B}{2}\right)\left\{\sin^2\left(\frac{A+B}{2}\right) + \cos^2\left(\frac{A+B}{2}\right)\right\}$$

$$= 4\sin^2\left(\frac{A-B}{2}\right).1$$

$$= 4\sin^2\left(\frac{A-B}{2}\right) \quad \textbf{= RHS proved.}$$

c) $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$

Solution

$$\text{LHS} = \sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)$$

$$= \frac{1}{2}\left\{1-\cos2\left(\frac{\pi}{8}+\frac{A}{2}\right)\right\} - \frac{1}{2}\left\{1-\cos2\left(\frac{\pi}{8}-\frac{A}{2}\right)\right\}$$

$$= \frac{1}{2} - \frac{1}{2}\left\{\cos\left(\frac{\pi}{4}+A\right)\right\} - \frac{1}{2} + \frac{1}{2}\left\{\cos\left(\frac{\pi}{4}-A\right)\right\}$$

$$= \frac{1}{2}\left\{\cos\left(\frac{\pi}{4}-A\right) - \cos\left(\frac{\pi}{4}-A\right)\right\}$$

$$= \frac{1}{2}\left\{2\sin\frac{\left(\frac{\pi}{4}-A+\frac{\pi}{4}+A\right)}{2}.\sin\frac{\left(\frac{\pi}{4}+A-\frac{\pi}{4}+A\right)}{2}\right\}$$

$$= \sin\frac{\pi}{4}.\sin A$$

$$= \frac{1}{\sqrt{2}}\sin A \quad \textbf{= RHS proved.}$$

d) $\sin^2A + \sin^2(A+120°) + \sin^2(A-120°) = \frac{3}{2}$

Solution

$$\text{LHS} = \sin^2A + \sin^2(A+120°) + \sin^2(A-120°)$$

$$= \sin^2A + \frac{1}{2}\{1-\cos2(A+120°) + 1 - \cos2(A-120°)\}$$

$$= \frac{1}{2} + \frac{1}{2} + \sin^2A - \frac{1}{2}\{\cos(2A+240°) + \cos(2A-240°)\}$$

$$= 1 + \sin^2A - \frac{1}{2}.2\cos\left(\frac{2A+240°+2A-240°}{2}\right)\cos\left(\frac{2A+240°-2A-240°}{2}\right)$$

$$= 1 + \sin^2A - \cos2A.\cos240°$$

$$= 1 + \sin^2A - \cos2A\cos240°$$

$$= 1 + \sin^2A - (1-2\sin^2A).\left(\frac{-1}{2}\right)$$

$$= 1 + \sin^2A + \frac{1}{2} - \sin^2A$$

$$= 1 + \frac{1}{2} = \frac{3}{2} \quad \textbf{= RHS proved.}$$

9. a) If $\sin\alpha + \sin\beta = \frac{1}{4}$ and $\cos\alpha + \cos\beta = \frac{1}{2}$, prove that: $\tan\left(\frac{\alpha + \beta}{2}\right) = \frac{1}{2}$

Solution

Here, $\sin\alpha + \sin\beta = \frac{1}{4}$

or, $2\sin\left(\frac{\alpha+\beta}{2}\right) . \cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{4}$ ... ... (i)

and $\cos\alpha + \cos\beta = \frac{1}{2}$

or, $2\cos\left(\frac{\alpha+\beta}{2}\right) . \cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}$ ... ... (ii)

Dividing (i) by (ii), we get,

$\therefore \tan(\alpha + \beta) = \frac{1}{2}$ **proved**.

b) If $\sin x = k \sin y$, then prove that: $\tan\left(\frac{x-y}{2}\right) = \frac{k-1}{k+1}\tan\left(\frac{x+y}{2}\right)$

Solution

Here, $k = \frac{\sin x}{\sin y}$

$$\text{RHS} = \frac{k-1}{k+1}\tan\left(\frac{x+y}{2}\right)$$

$$= \frac{\frac{\sin x}{\sin y} - 1}{\frac{\sin x}{\sin y} + 1}\tan\left(\frac{x+y}{2}\right)$$

$$= \frac{\sin x - \sin y}{\sin y} \times \frac{\sin y}{\sin x + \sin y}\tan\left(\frac{x+y}{2}\right)$$

$$= \frac{2\cos\left(\frac{x+y}{2}\right) . \sin\left(\frac{x-y}{2}\right)}{2\sin\left(\frac{x+y}{2}\right) . \cos\left(\frac{x-y}{2}\right)}\tan\left(\frac{x+y}{2}\right)$$

$$= \cot\left(\frac{x+y}{2}\right) . \tan\left(\frac{x+y}{2}\right) . \tan\left(\frac{x-y}{2}\right)$$

$$= \tan\left(\frac{x-y}{2}\right) = \text{RHS proved.}$$

c) If $\sin(A + B) = k \sin(A - B)$, then prove that: $(k - 1) \tan A = (k + 1) \tan B$.

Solution

Here, $= \frac{\sin(A+B)}{\sin(A-B)} = \frac{k}{1}$

By using componendo and dividendo

$$\frac{\sin(A+B) + \sin(A-B)}{\sin(A+B) - \sin(A-B)} = \frac{k+1}{k-1}$$

or, $\dfrac{2\sin A \,.\, \cos B}{2\cos A \,.\, \sin B} = \dfrac{k+1}{k-1}$

or, $\dfrac{\tan A}{\tan B} = \dfrac{k+1}{k-1}$

$\therefore$ $(k-1)\tan A = (k+1)\tan B$ **proved.**

10. Prove that: $\dfrac{\sin^2 x - \sin^2 y}{\sin x \,.\, \cos x - \sin y \,.\, \cos y} = \tan(x\ \ y)$

Solution

$$\text{LHS} = \frac{\sin^2 x - \sin^2 y}{\sin x \,.\, \cos x - \sin y \,.\, \cos y}$$

$$= \frac{2\sin^2 x - \sin^2 y}{2\sin x \,.\, \cos x - 2\sin y \,.\, \cos y}$$

$$= \frac{1 - \cos 2x - 1 + \cos 2y}{\sin 2x - \sin 2y}$$

$$= \frac{\cos 2y - \cos 2x}{\sin 2x - \sin 2y}$$

$$= \frac{2\sin\left(\frac{2y+2x}{2}\right) . \sin\left(\frac{2x-2y}{2}\right)}{2\cos\left(\frac{2x+2y}{2}\right) . \sin\left(\frac{2x-2y}{2}\right)}$$

$= \tan(x\ \ y)$ **= RHS proved.**

11. Prove that $\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7} = -\dfrac{1}{2}$

Solution

$$\text{LHS} = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[2\cos\frac{2\pi}{7} . \sin\frac{\pi}{7} + 2\cos\frac{4\pi}{7} . \sin\frac{\pi}{7} + 2\cos\frac{6\pi}{7} . \sin\frac{\pi}{7}\right]$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[\left\{\sin\left(\frac{2\pi}{7}+\frac{\pi}{7}\right) - \sin\left(\frac{2\pi}{7}-\frac{\pi}{7}\right)\right\} + \left\{\sin\left(\frac{4\pi}{7}+\frac{\pi}{7}\right) - \sin\left(\frac{4\pi}{7}-\frac{\pi}{7}\right)\right\} + \left\{\sin\left(\frac{6\pi}{7}+\frac{\pi}{7}\right) - \sin\left(\frac{6\pi}{7}-\frac{\pi}{7}\right)\right\}\right]$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[\sin\frac{3\pi}{7} - \sin\frac{\pi}{7} + \sin\frac{5\pi}{7} - \sin\frac{3\pi}{7} + \sin\pi - \sin\frac{5\pi}{7}\right]$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[-\sin\frac{\pi}{7} + 0\right]$$

$= -\dfrac{1}{2}$ = RHS proved.

11. Prove that: $x = y \cot\left(\frac{\alpha + \beta}{2}\right)$, if $x \cos\alpha + y \sin\alpha = x \cos\beta + y \sin\beta$.

Solution

Here, $x \cos\alpha + y \sin\alpha = x \cos\beta + y \sin\beta$

or, $x(\cos\alpha - \cos\beta) = y(\sin\beta - \sin\alpha)$

or, $x \,.\, 2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\beta - \alpha}{2}\right) = y\, 2 \cos\left(\frac{\beta + \alpha}{2}\right) \sin\left(\frac{\beta - \alpha}{2}\right)$

$\therefore$ $x = y \cot\left(\frac{\alpha + \beta}{2}\right)$ proved.

Questions for practice

Prove the following

1. $\cos 15° \,.\, \sin 75° = \frac{2 + \sqrt{3}}{4}$
2. $\frac{\sin 5x - \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$
3. $\frac{\cos 20° - \sin 20°}{\cos 20° + \sin 25°} = \tan 25°$
4. $\frac{\cos\theta - \cos 2\theta + \cos 3\theta}{\sin\theta - \sin 2\theta + \sin 3\theta} = \cot 2\theta$
5. $\frac{\sin 8\theta \,.\, \cos\theta - \sin 6\theta \,.\, \cos 3\theta}{\cos 2\theta \,.\, \cos\theta - \sin 3\theta \,.\, \sin 4\theta} = \tan 2\theta$
6. $\operatorname{cosec}\left(\frac{\pi}{4} - \theta\right) . \operatorname{cosec}\left(\frac{\pi}{4} + \theta\right) = \frac{2}{\cos\theta}$
7. $\sin 10° \,.\, \sin 50° \,.\, \sin 60° \,.\, \sin 70° = \frac{\sqrt{3}}{16}$
8. $\cos 80° \,.\, \cos 140° \,.\, \cos 160° = \frac{1}{8}$
9. $\cos\theta \,.\, \cos(60° - \theta) \,.\, \cos(60° + \theta) = \frac{1}{4} \cos 3\theta$
10. $\cos 20° \,.\, \cos 40° \,.\, \cos 80° \,.\, \cos 240° = -\frac{1}{16}$
11. $\cos \frac{\pi}{7} \,.\, \cos \frac{2\pi}{7} \,.\, \cos \frac{4\pi}{7} = -\frac{1}{8}$
12. If $\sin 2x + \sin 2y = \frac{1}{3}$ and $\cos 2x + \cos 2y = \frac{1}{2}$, then show that: $\tan(x + y) = \frac{2}{3}$.

Conditional Trigonometric Identities

1. Objectives

S,N.	Level	Objectives
(i)	Knowledge(k)	To state the conditional identities related to angles of a triangle.
(ii)	Understanding(U)	To explain to derive the conditional trigonometric identities.
(ii)	Application(A)	To solve the problems related to conditional identities in trigonometry.
(iv)	Higher Ability (HA)	To solve the order problems related to conditional identities.

2. Teaching Materials

List of formula of conditional trigonometric identity in a chart paper.

3. Teaching Learning Stategies:

- Review the formulae of transformation of trigonometric ratios.
- Discuss components of a triangle 3 sides and 3 angles.
- If A, B and C are angles of a triangle. Then $A + B + C = \pi$ or $A + B = \pi - C$

Taking sin, cosine and tangent ratios on both side, find different identities.

Example $\sin(A + B) = \sin(\pi - C) = \sin C$

$\cos(A + B) = \cos(\pi - C) = -\cos C$

$\tan(A + B) = \tan(\pi - C) = -\tan C$

- Similarly, from $\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

  find different conditional identities.
- Again from $2A + 2B = 2\pi - 2C$

  find different conditional identities.
- Solve some question from exercise of the text book and give guidance to the students to solve the problems in the same exercise.

List of formula:

1. $2\sin A \,.\, \cos B = \sin(A + B) + \sin(A - B)$
2. $2\cos A \,.\, \sin B = \sin(A + B) - \sin(A - B)$
3. $2\sin A \,.\, \sin B = \cos(A - B) - \cos(A + B)$
4. $2\cos A \,.\, \cos B = \cos(A + B) + \cos(A - B)$
5. $\sin C + \sin D = 2\sin\left(\frac{C + D}{2}\right) . \cos\left(\frac{C - D}{2}\right)$
6. $\sin C - \sin D = 2\cos\left(\frac{C + D}{2}\right) . \sin\left(\frac{C + D}{2}\right)$

7. **$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$**

8. **$\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right).\sin\left(\frac{D-C}{2}\right)$**

$$= -2\sin\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)$$

9. **$\sin(A+B) = \sin(\pi - C) = \sin C$**

$\sin\left(\frac{A}{2}+\frac{B}{2}\right) = \sin\left(\frac{\pi}{2}-\frac{C}{2}\right) = \cos\frac{C}{2}$

$\cos(A+B) = \cos(\pi - C) = -\cos C$

$\cos\left(\frac{A}{2}+\frac{B}{2}\right) = \cos\left(\frac{\pi}{2}-\frac{C}{2}\right) = \sin\frac{C}{2}$

$\tan(A+B) = \tan(\pi - C) = -\tan C$

$\tan\left(\frac{A}{2}+\frac{B}{2}\right) = \tan\left(\frac{\pi}{2}-\frac{C}{2}\right) = \tan\frac{C}{2}$

$\sin(2A+2B) = \sin(2\pi - 2C) = -\sin 2C$

$\cos(2A+2B) = \cos(2\pi - 2C) = \cos 2C$

$\tan(2A+2B) = \tan(2\pi - 2C) = -\tan 2C$

Some solved problems

1. If A + B + C = 180°, show that: $\sin(B+2C) + \sin(C+2A) + \sin(A+2B) = \sin(A-C) + \sin(B-A) + \sin(C-B)$

Solution

Here, $A + B + C = 180°$

or, $A + B = 180° - C$

$\therefore$ $\sin(A+B) = \sin(180° - C) = \sin C$

LHS $= \sin(B+2C) + \sin(C+2A) + \sin(A+2B)$ $(\therefore 180° = \pi)$

$= \sin(B+C+C) + \sin(C+A+A) + \sin(A+B+B)$

$= \sin(\pi - A + C) + \sin(\pi - B + A) + \sin(\pi - C + B)$

$= \sin\{\pi - (A-C)\} + \sin\{\pi - (B-A)\} + \sin\{\pi - (C-B)\}$

$= \sin(A-C) + \sin(B-A) + \sin(C-B)$

= RHS proved.

2. If $A + B + C = \pi$, prove that:

a) $\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

b) $\tan\frac{A}{2}.\tan\frac{B}{2}+\tan\frac{B}{2}.\tan\frac{C}{2}+\tan\frac{C}{2}.\tan\frac{A}{2}=1.$

Solution

a) $\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$

Here, $A+B+C=\pi^c$

or, $\frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$

$\therefore\quad \cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)$

or, $\dfrac{\cot\frac{A}{2}.\cot\frac{B}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}=\tan\frac{C}{2}$

or, $\dfrac{\cot\frac{A}{2}.\cot\frac{B}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}=\dfrac{1}{\cot\frac{C}{2}}$

$\therefore\quad \cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$ **proved**.

b) $\tan\frac{A}{2}.\tan\frac{B}{2}+\tan\frac{B}{2}.\tan\frac{C}{2}+\tan\frac{C}{2}.\tan\frac{A}{2}=1.$

Solution

Here, $A+B+C=\pi^c$

or, $\frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$

$\therefore\quad \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)$

or, $\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}.\tan\frac{B}{2}}=\cot\frac{C}{2}$

or, $\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}.\tan\frac{B}{2}}=\dfrac{1}{\tan\frac{C}{2}}$

$\therefore\quad \tan\frac{A}{2}.\tan\frac{B}{2}+\tan\frac{B}{2}.\tan\frac{C}{2}+\tan\frac{C}{2}.\tan\frac{A}{2}=1$ **proved**.

3. If $A+B+C=\pi$, prove that:

a) $\cos2A+\cos2B-\cos2C=1-4\sin A.\sin B.\cos C$

Solution

Here, $A + B + C = \pi$

or, $A + B = \pi - C$

$\therefore \quad \cos(A + B) = \cos(\pi - C) = -\cos C$

LHS $= \cos 2A + \cos 2B - \cos 2C$

$= 2\cos\left(\frac{2A + 2B}{2}\right) . \cos\left(\frac{2A - 2B}{2}\right) - (2\cos^2 C - 1)$

$= 2\cos(A + B) . \cos(A - B) - 2\cos^2 C + 1$

$= 2(-\cos C) . \cos(A - B) - 2\cos^2 C + 1$

$= 1 - 2\cos C\,[\cos(A - B) + \cos C]$

$= 1 - 2\cos C\,[\cos(A - B - \cos(A + B)]$

$= 1 - 2\cos C . 2\sin A . \sin B$

$= 1 - 4\sin A . \sin B . \cos C$

= **RHS proved.**

4. If $A + B + C = \pi^c$, prove the following:

a. $\text{Sin}A + \sin B - \sin C = 4\sin\frac{A}{2} . \sin\frac{B}{2} . \cos\frac{C}{2}$

Solution

Here, $A + B + C = \pi^c$

$\frac{A}{2} + \frac{B}{2} = \frac{\pi^c}{2} - \frac{C}{2}$

$\therefore \quad \sin\left(\frac{A}{2} - \frac{B}{2}\right) = \sin\left(\frac{\pi^c}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}$

and $\cos\left(\frac{A}{2} - \frac{B}{2}\right) = \cos\left(\frac{\pi^c}{2} - \frac{C}{2}\right) = \sin\frac{C}{2}$

Here, LHS $= \sin A + \sin B - \sin C$

$= 2\sin\left(\frac{A + B}{2}\right) . \cos\left(\frac{A - B}{2}\right) - 2\sin\frac{C}{2} . \cos\frac{C}{2}$

$= 2\cos\frac{C}{2} . \cos\left(\frac{A - B}{2}\right) . 2\sin\frac{C}{2} . \cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A - B}{2}\right) - \sin\frac{C}{2}\right]$

$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A - B}{2}\right) - \cos\left(\frac{A + B}{2}\right)\right]$

$= 2\cos\frac{C}{2} . 2\sin\frac{A}{2} . \sin\frac{B}{2}$

$= 4\sin\frac{A}{2} . \sin\frac{B}{2} . \cos\frac{C}{2}$

= RHS proved.

b. $-\sin A + \sin B + \sin C = 4\cos\frac{A}{2} . \sin\frac{B}{2} . \sin C$

Solution

Here, $A + B + C = \pi^c$

$\frac{A}{2} + \frac{B}{2} = \frac{\pi^c}{2} - \frac{C}{2}$

$\therefore \quad \sin\left(\frac{A}{2} + \frac{B}{2}\right) = \sin\left(\frac{\pi^c}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}$

and $\quad \cos\left(\frac{A}{2} + \frac{B}{2}\right) = \sin\frac{C}{2}$

LHS $= \sin B + \sin C - \sin A$

$= 2\sin\left(\frac{B + C}{2}\right) . \cos\left(\frac{B - C}{2}\right) - 2\sin\frac{A}{2} . \cos\frac{A}{2}$

$= 2\cos\frac{A}{2}\left[\cos\left(\frac{B - C}{2}\right) - \sin\frac{A}{2}\right]$

$= 2\cos\frac{A}{2}\left[\cos\left(\frac{B - C}{2}\right) - \cos\left(\frac{B + C}{2}\right)\right]$

$= 2\cos\frac{A}{2}\left[2\sin\frac{B}{2} . \sin\frac{C}{2}\right]$

$= 4\cos\frac{A}{2} . \sin\frac{B}{2} . \sin\frac{C}{2}$

= RHS proved.

5. If $A + B + C = \pi^c$, prove that

a. $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A . \sin B . \cos C$

Solution

Here, $A + B + C = \pi$

$A + B = \pi - C$

$\therefore \quad \cos(A + B) = \cos(\pi - C) = -\cos C$

LHS $= \cos 2A + \cos 2B - \cos 2C$

$= 2\cos\left(\frac{2A + 2B}{2}\right) . \cos\left(\frac{2A - 2B}{2}\right) - (2\cos^2 C - 1)$

$= 2\cos(A + B) . \cos(A - B) - 2\cos^2 C + 1$

$= -2\cos C . \cos(A - B) - 2\cos^2 C + 1$

$= 1 - 2\cos C[\cos(A - B) + \cos C]$

$= 1 - 2\cos C[\cos(A - B) - \cos(A + B)]$

$= 1 - 2\cos C . 2\sin A . \sin B$

$= 1 - 4\sin A . \sin B . \cos C$

= RHS proved.

b. sin2A – sin2B + sin2C = 4cosA . sinB . cosC

Solution

LHS $= \sin2A - \sin2B + \sin2C$

$= 2\cos\left(\frac{2A + 2B}{2}\right) . \sin\left(\frac{2A - 2B}{2}\right) + 2\sin C . \cos C$

$= 2\cos(A + B) . \sin(A - B) + 2\sin C . \cos C$

$= -2\cos C . \sin(A - B) + 2\sin C \cos C$

$= 2\cos C[-\sin(A - B) + \sin C]$

$= 2\cos C[\sin(A + B) - \sin(A - B)]$

$= 2\cos C . 2\sin B . \cos A$

$= 4\cos A . \sin B . \cos C$

= RHS proved.

6. If A + B + C = 180°, then prove that:

a. $\mathbf{\cos A - \cos B + \cos C = 4\cos\frac{A}{2} . \sin\frac{B}{2} . \cos\frac{C}{2} - 1}$

Solution

LHS $= \cos A - \cos B + 2\cos^2\frac{C}{2} - 1$

$= 2\sin\left(\frac{A + B}{2}\right) . \sin\left(\frac{B - A}{2}\right) + 2\cos^2\frac{C}{2} - 1$

$= 2\cos\frac{C}{2} . \sin\left(\frac{B - A}{2}\right) + 2\cos^2\frac{C}{2} - 1$ $\left(\because \sin\frac{(A + B)}{2} = \cos\frac{C}{2}\right)$

$= 2\cos\frac{C}{2}\left[\sin\left(\frac{B - A}{2}\right) + \cos\frac{C}{2}\right] - 1$

$= 2\cos\frac{C}{2}\left[\sin\left(\frac{B - A}{2}\right) + \sin\left(\frac{A + B}{2}\right)\right] - 1$

$= 2\cos\frac{C}{2}\left[2\sin\frac{B}{2} . \cos\frac{A}{2}\right]$

$= 4\cos\frac{A}{2} . \sin\frac{B}{2} . \cos\frac{C}{2} - 1$

= RHS proved.

b. $\mathbf{\cos A - \cos B - \cos C = 1 - 4\sin\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}}$

Solution

LHS $= \cos A - \cos B - \cos C$

$= 1 - 2\sin^2\frac{A}{2} - [\cos B + \cos C]$

$= 1 - 2\sin^2\frac{A}{2} - 2\cos\left(\frac{B + C}{2}\right) . \cos\left(\frac{B - C}{2}\right)$

$$= 1 - 2\sin^2\frac{A}{2} - 2\sin\frac{A}{2} \cdot \cos\left(\frac{B-C}{2}\right)$$

$$= 1 - 2\sin\frac{A}{2}\left[\sin\frac{A}{2} - \cos\left(\frac{B-C}{2}\right)\right]$$

$$= 1 - 2\sin\frac{A}{2}\left[\cos\left(\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right)\right]$$

$$= 1 - 2\sin\frac{A}{2} \cdot 2\cos\frac{B}{2} \cdot \cos\frac{C}{2}$$

$$= 1 - 4\sin\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2}$$

= RHS proved.

7. If $A + B + C = \pi^c$, then prove that:

a. $\sin^2A - \sin^2B - \sin^2C = -2\cos A \cdot \sin B \cdot \sin C$

Solution

Here, $A + B + C = \pi^c$

$A + B = \pi^c - C$

$\therefore$ $\sin(A + B) = \sin(\pi^c - C) = \sin C$

and $\cos(A + B) = \cos(\pi^c - C) = -\cos C$

LHS $= \sin^2A - \sin^2B - \sin^2C$

$$= \frac{1}{2}[2\sin^2A - 2\sin^2B] - \sin^2C$$

$$= \frac{1}{2}[1 - \cos2A - 1 + \cos2B] - \sin^2C$$

$$= \frac{-1}{2}[\cos2A - \cos2B] - \sin^2C$$

$$= \frac{-1}{2}\left[2\sin\left(\frac{2A-2B}{2}\right) \cdot \sin\left(\frac{2B-2A}{2}\right)\right] - \sin^2C$$

$$= -\sin(A + B) \cdot \sin(B - A) - \sin^2C$$

$$= -\sin C[\sin(B - A) + \sin C]$$

$$= -\sin C[\sin(B - A) + \sin(A + B)]$$

$$= -\sin C \cdot 2\sin B \cdot \cos A$$

$$= -2\cos A \cdot \sin B \cdot \sin C$$

= RHS proved.

b. $\sin^2\frac{A}{2} - \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 2\sin\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2} - 1$

Solution

LHS $= \sin^2\frac{A}{2} - \sin^2\frac{B}{2} - \sin^2\frac{C}{2}$

$$= \frac{1}{2}\left[2\sin^2\frac{A}{2} - 2\sin^2\frac{B}{2}\right] - 1 + \cos^2\frac{C}{2}$$

$$= \frac{1}{2}[1 - \cos A - 1 + \cos B] - 1 + \cos^2\frac{C}{2}$$

$$= \frac{1}{2}[\cos B - \cos A] + \cos^2 C - 1$$

$$= \frac{1}{2} . 2\sin\left(\frac{B + A}{2}\right) . \sin\left(\frac{A - B}{2}\right) + \cos^2\frac{C}{2} - 1$$

$$= \cos\frac{C}{2} . \sin\left(\frac{A - B}{2}\right) + \cos^2\frac{C}{2} - 1$$

$$= \cos\frac{C}{2}\left[\sin\left(\frac{A - B}{2}\right) + \sin\left(\frac{A + B}{2}\right)\right] - 1$$

$$= \cos\frac{C}{2} . 2\sin\frac{A}{2} . \cos\frac{B}{2} - 1$$

$$= 2\sin\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2} - 1$$

= RHS proved.

c. $\boldsymbol{\cos^2\frac{A}{2} - \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2\cos\frac{A}{2} . \sin\frac{B}{2} . \cos\frac{C}{2}}$

Solution

LHS $= \cos^2\frac{A}{2} - \cos^2\frac{B}{2} + \cos^2\frac{C}{2}$

$$= \frac{1}{2}\left[2\cos^2\frac{A}{2} - 2\cos^2\frac{B}{2}\right] + \cos^2\frac{C}{2}$$

$$= \frac{1}{2}[1 + \cos A - 1 - \cos B] + \cos^2\frac{C}{2}$$

$$= \frac{1}{2}[\cos A - \cos B] + \cos^2\frac{C}{2}$$

$$= \frac{1}{2} . 2\sin\left(\frac{A + B}{2}\right) . \sin\left(\frac{B - A}{2}\right) + \cos^2\frac{C}{2}$$

$$= \cos\frac{C}{2}\left[\sin\left(\frac{B - A}{2}\right) + \cos\frac{C}{2}\right]$$

$$= \cos\frac{C}{2}\left[\sin\left(\frac{B - A}{2}\right) + \sin\left(\frac{A + B}{2}\right)\right]$$

$$= \cos\frac{C}{2}\left[-\sin\left(\frac{A - B}{2}\right) + \sin\left(\frac{A + B}{2}\right)\right]$$

$$= \cos\frac{C}{2} . 2\sin\frac{B}{2} . \cos\frac{A}{2}$$

$$= 2\cos\frac{A}{2} . \sin\frac{B}{2} . \cos\frac{C}{2}$$

= RHS proved.

d. $\boldsymbol{\cos^2\frac{A}{2} - \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = -2\sin\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}}$

Solution

Here, $A + B + C = \pi^c$

$$A + B = \pi^c - C$$
$\therefore$ $\sin(A + B) = \sin(\pi^c - C) = \sin C$
and $\cos(A + B) = \cos(\pi^c - C) = -\cos C$

LHS $= \cos^2\frac{A}{2} - \cos^2\frac{B}{2} - \cos^2\frac{C}{2}$

$= \frac{1}{2}\left[2\cos^2\frac{A}{2} - 2\cos^2\frac{B}{2}\right] - \cos^2\frac{C}{2}$

$= \frac{1}{2}[1 + \cos A - 1 - \cos B] - \cos^2\frac{C}{2}$

$= \frac{1}{2}[\cos A - \cos B] - \cos^2\frac{C}{2}$

$= \frac{1}{2} . 2\sin\left(\frac{A + B}{2}\right) . \sin\left(\frac{B - A}{2}\right) - \cos^2\frac{C}{2}$

$= \cos\frac{C}{2} . \sin\left(\frac{B - A}{2}\right) - \cos^2\frac{C}{2}$

$= \cos\frac{C}{2}\left[\sin\left(\frac{B - C}{2}\right) - \cos\frac{C}{2}\right]$

$= \cos\frac{C}{2}\left[\sin\left(\frac{B - A}{2}\right) - \sin\left(\frac{A + B}{2}\right)\right]$

$= \cos\frac{C}{2}\left[2\cos\left(\frac{B - A + A + B}{4}\right) . \sin\left(\frac{B - A - A - B}{4}\right)\right]$

$= \cos\frac{C}{2}\left[2\cos\frac{B}{2} . \sin\left(\frac{-A}{2}\right)\right]$

$= -2\sin\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}$

= RHS proved.

8. If $A + B + C = \pi^c$, prove that

a. $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\left(\frac{\pi - A}{4}\right) . \sin\left(\frac{\pi - B}{4}\right) . \sin\left(\frac{\pi - C}{4}\right)$

$= 1 + 4\sin\left(\frac{B + C}{4}\right) . \sin\left(\frac{C + A}{4}\right) . \sin\left(\frac{A + B}{4}\right)$

Solution

LHS $= \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} + 1 - 1$

$= 1 + \left(\sin\frac{A}{2} + \sin\frac{B}{2}\right) + \sin C - \sin\frac{\pi}{2}$

$= 1 + 2\sin\left(\frac{A + B}{4}\right) . \cos\left(\frac{A - B}{4}\right) + 2\cos\left(\frac{C + \pi}{4}\right) . \sin\left(\frac{C - \pi}{4}\right)$

$= 1 + 2\sin\left(\frac{\pi - C}{4}\right) \cos\left(\frac{A - B}{4}\right) - 2\cos\left(\frac{\pi + C}{4}\right) . \sin\left(\frac{\pi - C}{4}\right)$

$= 1 + 2\sin\left(\frac{\pi - C}{4}\right)\left[\cos\left(\frac{A - B}{4}\right) - \cos\left(\frac{\pi + C}{4}\right)\right]$

$= 1 + 2\sin\left(\frac{\pi - C}{4}\right) . 2\sin\left(\frac{A - B + \pi + C}{8}\right) . \sin\left(\frac{\pi + C - A + B}{8}\right)$

$$= 1 + 4\sin\left(\frac{\pi - C}{4}\right) . \sin\left(\frac{A + C - B + \pi}{8}\right) . \sin\left(\frac{B + C - A + \pi}{8}\right)$$

$$= 1 + 4\sin\left(\frac{\pi - C}{4}\right) . \sin\left(\frac{\pi - B - B + \pi}{8}\right) . \sin\left(\frac{\pi - A - A + \pi}{8}\right)$$

$$= 1 + 4\sin\left(\frac{\pi - C}{4}\right) . \sin\left(\frac{\pi - B}{4}\right) . \sin\left(\frac{\pi - A}{4}\right)$$

$$= 1 + 4\sin\left(\frac{\pi - A}{4}\right) . \sin\left(\frac{\pi - B}{4}\right) . \sin\left(\frac{\pi - C}{4}\right) = MS$$

$$= 1 + 4\sin\left(\frac{B + C}{4}\right) . \sin\left(\frac{C + A}{4}\right) . \sin\left(\frac{A + B}{4}\right)$$

= RHS proved.

b. $\mathbf{\cos A + \cos B + \cos C = 1 + 4\cos\left(\frac{\pi - A}{2}\right) . \cos\left(\frac{\pi - B}{2}\right) . \cos\left(\frac{\pi - C}{2}\right)}$

$$\mathbf{= 1 + 4\cos\left(\frac{B + C}{2}\right) . \sin\left(\frac{C + A}{2}\right) . \sin\left(\frac{A + B}{2}\right)}$$

Solution

LHS $= \cos A + \cos B + \cos C$

$= \cos A + \cos B + \cos C + \cos\pi + 1 \qquad (\because \cos\pi = -1)$

$= 1 + (\cos A + \cos B) + (\cos C + \cos\pi)$

$$= 1 + 2\cos\left(\frac{A + B}{2}\right) . \cos\left(\frac{A - B}{2}\right) + 2\cos\left(\frac{\pi + C}{2}\right) \cos\left(\frac{\pi - C}{2}\right)$$

$$= 1 + 2\cos\left(\frac{\pi - C}{2}\right) . \cos\left(\frac{A - B}{2}\right) + 2\cos\left(\frac{\pi + C}{2}\right) . \cos\left(\frac{\pi - C}{2}\right)$$

$$= 1 + 2\cos\left(\frac{\pi - C}{2}\right)\left[\cos\left(\frac{A - B}{2}\right) + \cos\left(\frac{\pi + C}{2}\right)\right]$$

$$= 1 + 2\cos\left(\frac{\pi - C}{2}\right)\left[2\cos\left(\frac{A - B + \pi + C}{4}\right) . \cos\left(\frac{A - B - \pi - C}{4}\right)\right]$$

$$= 1 + 4\cos\left(\frac{\pi - C}{2}\right) . \cos\left(\frac{A + C - B + \pi}{4}\right) . \cos\left(\frac{A - \pi + (B + C)}{4}\right)$$

$$= 1 + 4\cos\left(\frac{\pi - C}{2}\right) . \cos\left(\frac{\pi - B - B + \pi}{4}\right) . \cos\left(\frac{A - \pi - \pi + A}{4}\right)$$

$$= 1 + 4\cos\left(\frac{\pi - C}{2}\right) . \cos\left(\frac{\pi - B}{2}\right) . \cos\left(\frac{A - \pi}{2}\right)$$

$$= 1 + 4\cos\left(\frac{\pi - A}{2}\right) . \cos\left(\frac{\pi - B}{2}\right) . \cos\left(\frac{\pi - C}{2}\right)$$

$$= 1 + 4\cos\left(\frac{B + C}{2}\right) . \sin\left(\frac{C + A}{2}\right) . \sin\left(\frac{A + B}{2}\right)$$

= RHS proved.

9. If $A + B + C = \pi^c$, prove that
$\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4 \sin A . \sin B . \sin C$

Solution

Here, $A + B + C = \pi$

$A + B = \pi - C$

$\therefore$ $\sin(A + B) = \sin(\pi - C) = \sin C$

and $\cos(A + B) = \cos(\pi - C) = -\cos C$

LHS $= \sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C)$

$= \sin(\pi - 2A) + \sin(\pi - 2B) + \sin(\pi - 2C)$

$= \sin 2A + \sin 2B + \sin 2C$

$= 2\sin\left(\dfrac{2A + 2B}{2}\right) . \cos\left(\dfrac{2A - 2B}{2}\right) + \sin C \cos C$

$= 2\sin(A + B) . \cos(A - B) + 2\sin C . \cos C$

$= 2\sin C . \cos(A - B) + 2\sin C . \cos C$

$= 2\sin C[\cos(A - B) + \cos C]$

$= 2\sin C[\cos(A - B) - \cos(A + B)]$

$= 2\sin C . 2\sin A . \sin B$

$= 4\sin A . \sin B . \sin C$

= RHS proved.

10. If $A + B + C = \pi^c$, prove that

$\cos(B + 2C) + \cos(C + 2A) + \cos(A + 2B) = 1 - 4\cos\left(\dfrac{A - B}{2}\right) . \cos\left(\dfrac{B - C}{2}\right) . \cos\left(\dfrac{C - A}{2}\right)$

Solution

Here, $A + B + C = \pi$

$A + B = \pi - C$

$\therefore$ $\sin(A + B) = \sin(\pi - C) = \sin C$

and $\cos(A + B) = \cos(\pi - C) = -\cos C$

LHS $= \cos(B + 2C) + \cos(C + 2A) + \cos(A + 2B)$

$= \cos(A + C + C) + \cos(C + A + A) + \cos(A + B + B)$

$= \cos(\pi - A + C) + \cos(\pi - B + A) + \cos(\pi - C + B)$

$= \cos\{\pi - (A - C)\} + \text{ocs}\{\pi - (B - A)\} + \cos\{\pi - (C - B)\}$

$= \cos(A - C) - \cos(B - A) - \cos(C - B)$

$= -[\cos(A - C) + \cos(B - A)] - \cos(C - B)$

$= -2\cos\left(\dfrac{A - C + B - A}{2}\right) . \cos\left(\dfrac{A - C - B + A}{2}\right) - 2\cos^2\left(\dfrac{B - C}{2}\right) + 1$

$= -2\cos\left(\dfrac{B - C}{2}\right) . \cos\left(\dfrac{2A - C - B}{2}\right) - 2\cos^2\left(\dfrac{B - C}{2}\right) + 1$

$= 1 - 2\cos\left(\dfrac{B - C}{2}\right)\left[\cos\left(\dfrac{2A - C - B}{2}\right) + \cos\left(\dfrac{B - C}{2}\right)\right]$

$= 1 - 2\cos\left(\dfrac{B - C}{2}\right)\left[2\cos\left(\dfrac{2A - C - B + B - C}{4}\right) + \cos\left(\dfrac{2A - C - B - B + C}{4}\right)\right]$

$$= 1 - 4\cos\left(\frac{B-C}{2}\right) . \cos\left(\frac{A-C}{2}\right) . \cos\left(\frac{A-B}{2}\right)$$

$$= 1 - 4\cos\left(\frac{A-B}{2}\right) . \cos\left(\frac{B-C}{2}\right) . \cos\left(\frac{C-A}{2}\right)$$

= RHS proved.

11. If $\alpha + \beta + \gamma = \frac{\pi^c}{2}$, prove that

$\cos(\alpha - \beta - \gamma) + \cos(\beta - \gamma - \alpha) + \cos(\gamma - \alpha - \beta) = 4\cos\alpha . \cos\beta . \cos\gamma$

Solution

Here, $\alpha + \beta + \gamma = \frac{\pi^c}{2}$

$\alpha + \beta = \frac{\pi^c}{2} - \gamma$

$\therefore$ $\sin(\alpha + \beta) = \sin\left(\frac{\pi^c}{2} - \gamma\right) = \cos\gamma$

and $\cos(\alpha + \beta) = \sin\gamma$

LHS $= \cos(\alpha - \beta - \gamma) + \cos(\beta - \gamma - \alpha) + \cos(\gamma - \alpha - \beta)$

$$= 2\cos\left(\frac{\alpha - \beta - \gamma + \beta - \gamma - \alpha}{2}\right) . \cos\left(\frac{\alpha - \beta - \gamma - \beta + \gamma + \alpha}{2}\right) . \cos\left(\gamma - \frac{\pi}{2} + \gamma\right)$$

$= 2\cos\gamma . \cos(\alpha - \beta) + \sin2\gamma$

$= 2\cos\gamma . \cos(\alpha - \beta) + 2\sin\gamma . \cos\gamma$

$= 2\cos\gamma\ [\cos(\alpha - \beta) + \sin\gamma]$

$= 2\cos\gamma\ [\cos(\alpha - \beta) + \cos\ (\alpha + \beta)]$

$= 2\cos\gamma . 2\cos\alpha . \cos\beta$

$= 2\cos\alpha . \cos\beta . \cos\gamma$

= RHS proved.

12. If $A + B + C = \pi^c$, prove that

a. $$\frac{\sin2A + \sin2B + \sin2C}{4\cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}} = 8\sin\frac{A}{2} . \sin\frac{B}{2} . \sin C$$

Hints :

Numerator = $\sin2A + 2\sin B + 2\sin C$

It gives $4\sin A . \sin B . \sin C$

and $4\sin A . \sin B . \sin C$

$$= 16\sin\frac{A}{2} . \sin\frac{B}{2} . \sin\frac{C}{2} . \cos\frac{A}{2} . \cos B . \cos\frac{C}{2}$$

b. $\cos A . \cos B . \sin C + \cos B . \sin C . \sin A + \cos C . \sin A . \sin B = 1 + \cos A . \cos B . \cos C$

Solution

Here, $A + B + C = \pi^c$

$A + B = \pi^c - C$

$\therefore$ $\sin(A + B) = \sin(\pi^c - C) = \sin C$

and $\cos(A + B) = \cos(\pi^c - C) = -\cos C$

LHS $= \cos A \cdot \cos B \cdot \sin C + \cos B \cdot \sin C \cdot \sin A + \cos C \cdot \sin A \cdot \sin B$

$= \sin C(\cos A \cdot \sin B + \cos B \cdot \sin A) + \cos C \cdot \sin A \cdot \sin B$

$= \sin C \cdot \sin(A + B) + \cos C \cdot \sin A \cdot \sin B$

$= \sin C \cdot \sin C + \cos C \cdot \sin A \cdot \sin B$

$= 1 - \cos^2 C + \cos C \cdot \sin A \cdot \sin B$

$= 1 - \cos C\,[\cos C - \sin A \cdot \sin B]$

$= 1 - \cos C\,[-\cos(A + B) - \sin A \cdot \sin B]$

$= 1 + \cos C\,[\cos(A + B) + \sin A \cdot \sin B]$

$= 1 + \cos C\,[\cos A \cdot \cos B - \sin A \cdot \sin B + \sin A \cdot \sin B]$

$= 1 + \cos A \cdot \cos B \cdot \cos C$

= RHS proved.

Questions for practice

If $A + B + C = \pi$, prove the following:

1. $\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$
2. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$
3. $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \sin C$
4. $\cos 2A - \cos 2B - \cos 2C = 4\cos A \sin B \sin C - 1$
5. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2}$
6. $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cdot \cos B \cdot \cos C$
7. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \sin\frac{C}{2}$
8. $\cos(B + C - A) + \cos(C + A - B) + \cos(A + B - C) = 1 + 4\cos A \cdot \cos B \cdot \cos C$
9. $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\left(\frac{\pi - A}{4}\right) \cdot \sin\left(\frac{\pi - B}{2}\right) \cdot \sin\left(\frac{\pi - C}{2}\right)$
10. $\tan\frac{A}{2} + \tan\left(\frac{B + C}{2}\right) = \sec\frac{A}{2} \cdot \sec\left(\frac{B + C}{2}\right)$
11. $\cot\frac{A}{2} + \cot\left(\frac{B + C}{2}\right) = \operatorname{cosec}\frac{A}{2} \cdot \operatorname{cosec}\left(\frac{B + C}{2}\right)$
12. $\frac{\sin B + \sin C - \sin A}{\sin A + \sin B + \sin C} = \tan\frac{B}{2} \cdot \tan\frac{C}{2}$

Trigonometric Equations

Estimated Periods : 4

1. Objectives

Knowledge (K)	To define trigonometric equation. To define principal solutions of trigonometric equations. To say limitations of variables in trigonometric equations.
Understanding (U)	To explain the rule of 'CAST' To find the values of unknown angles from trigonometric equations.
Skill/Application (S/A)	To solve simple trigonometric equations.
Higher Ability (HA)	To solve harder trigonometric equations. To check solutions trigonometric equations. To check solutions are true or false when equations are solved squaring on both sides of equations.

2. Teaching Materials

Chart paper with rule of 'CAST' and table of trigonometric values of standard angles.

3. Teaching Learning Strategies

→ Review the concept of equations and identities with some appropriate examples.

→ Explain the rule of 'CAST' with examples

Give some basic ideas for solution of equations, define principal solutions.

→ Ask the solution of simple equations like $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan^2\theta = 1$, $0° \leq \theta \leq 180°$

→ Discuss how equation in the form of a $\sin^2\theta + b \sin\theta + c = 0$ can be solved, when it can be factorized. Also state limitations of values of trigonometric equations.

Example: $2 \sin^2\theta - 3 \sin\theta + 1 = 0$, $0° \leq \theta \leq 360°$.

Give idea how to check roots of equations are true or false as in algebra solving simultanesous equations in two variables.

→ Solve an equation, for example $\sin x + \cos x = 1$, $0° \leq \theta \leq 360°$. Solve it in two ways:

i) dividing by $\sqrt{1^2 + 1^2} = \sqrt{2}$

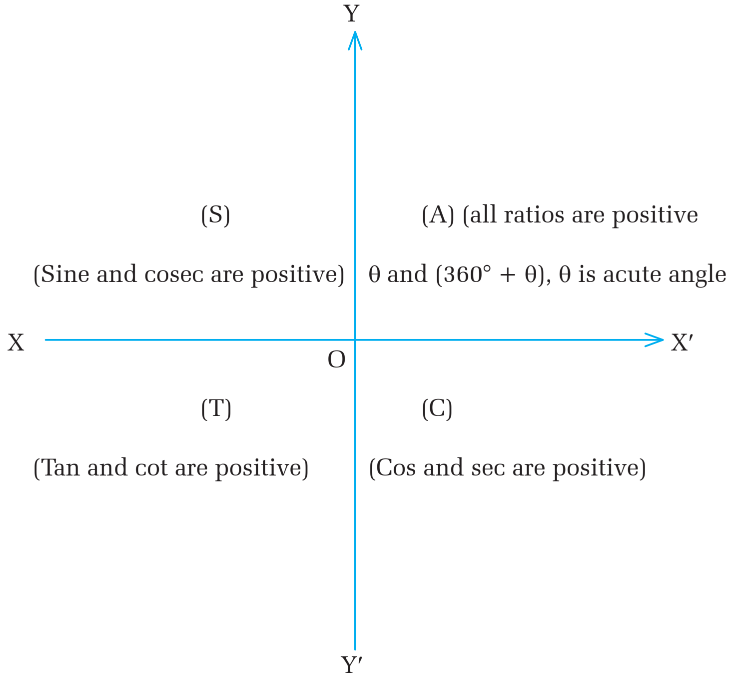
ii) squaring on both sides.

Suggest how to check roots so obtained are true or false.

→ Give ideas to solve some harder questions. For example:

$2 \tan 3x \cos 2x + 1 = \tan 3x + 2\cos x$, $0° \leq \theta \leq 360°$.

Rule of CAST (From Book)



Some Basic Ideas (From Book page 245)

Working rules for solutions of trigonometric equations (From Book page 246)

Some special cases (From Book page 297)

Some solved problems

1. Solve $0° \leq \theta \leq 180°$

a. $2\cos\theta + 1 = 0$

Solution

Here, $2\cos\theta + 1 = 0$

or, $\cos\theta = \dfrac{-1}{2}$

Since $\cos\theta$ is negatyive, θ lies on the second and third quadrant. But $0° \leq \theta \leq 180°$.

$\cos\theta = \cos 120°$

$\therefore \theta = 120°$.

b. $\sqrt{3}\tan\theta - 1 = 0$

Solution

Here, $\sqrt{3}\tan\theta - 1 = 0$

or, $\sqrt{3}\tan\theta = 1$

or, $\tan\theta = \frac{1}{\sqrt{3}}$

Since $\tan\theta$ is positive, it lies on the first quadrant.

$\tan\theta = \tan 30°$

$\therefore$ $\theta = 30°$

2. Find the value of θ, $0° \le \theta \le 90°$

a. $\tan\theta = \cot\theta$

Solution

Here, $\tan\theta = \cot\theta$

or, $\tan\theta = \tan(90° - \theta)$

$\therefore$ $\theta = 90° - \theta$

or, $2\theta = 90°$

$\therefore$ $\theta = 45°$

b. $\sin 4\theta = \cos 2\theta$

Solution

Here, $\sin 4\theta = \cos 2\theta$

or, $\sin 4\theta = \sin(90° - 2\theta)$

$\therefore$ $4\theta = 90° - 2\theta$

or, $6\theta = 90°$

$\therefore$ $\theta = 15°$

3. Solve: ($0° \le \theta \le 360°$)

a. $3\tan^2\theta = 1$

Solution

Here, $3\tan^2\theta = 1$

or, $\tan^2\theta = \frac{1}{3}$

$\therefore$ $\tan\theta = \pm\frac{1}{\sqrt{3}}$

Taking positive sign, we get.

$\tan\theta = \frac{1}{\sqrt{3}}$

or, $\tan\theta = \tan 30°, \tan(180° + 30°)$

$\therefore$ $\theta = 30°, 210°$

Taking negative sign, we get.

$$\tan\theta = -\frac{1}{\sqrt{3}}$$

or, $\tan\theta = \tan 150°, \tan(360° - 30°)$

$\therefore$ $\theta = 150°, 330°$

b. $\sec^2\theta = 2\tan^2\theta$

Solution

Here, $\sec^2\theta = 2\tan^2\theta$

or, $1 + \tan^2\theta = 2\tan^2\theta$

or, $\tan^2\theta = 1$

$\therefore$ $\tan\theta = \pm 1$

Taking positive sign, we get.

$\tan\theta = 1$

or, $\tan\theta = \tan 45°, \tan(180° + 45°)$

$\therefore$ $\theta = 45°, 225°$

Taking negative sign, we get.

$\tan\theta = -1$

or, $\tan\theta = \tan 135°, \tan(360° - 45°)$

$\therefore$ $\theta = 135°, 315°$

Hence the required values of θ are 45°, 135°, 225°, 315°.

4. Solve : ($0° \leq \theta \leq 180°$)

a. $\sin 4\theta = \cos\theta - \cos 7\theta$

Solution

Here, $\sin 4\theta = \cos\theta - \cos 7\theta$

or, $\sin 4\theta = 2\sin\left(\frac{\theta + 7\theta}{2}\right) . \sin\left(\frac{7\theta - \theta}{2}\right)$

or, $\sin 4\theta = 2\sin 4\theta . \sin 3\theta = 0$

or, $\sin 4\theta (1 - 2\sin 3\theta) = 0$

Either, $\sin 4\theta = 0$ ... ... (i)

$\therefore$ $1 - 2\sin 3\theta = 0$ ... ... (ii)

From equation (i) $\sin 4\theta = 0$

$\sin 4\theta = \sin 0°, \sin 180°, \sin 360°, \sin 540°$

$\therefore$ $4\theta = 0°, 180°, 360°, 540°$

$\therefore$ $\theta = 0°, 45°, 90°, 135°$

From equation (ii), we get

$2\sin3\theta = 1$

or, $\sin3\theta = \frac{1}{2}$

or, $\sin3\theta = \sin30^\circ, \sin150^\circ, \sin(360^\circ + 30^\circ)$

$\therefore 3\theta = 30^\circ, 150^\circ, 390^\circ$

$\Rightarrow \theta = 10^\circ, 50^\circ, 130^\circ$

b. $\sin\theta + \sin2\theta + \sin3\theta = 0$

Solution

Here, $\sin\theta + \sin2\theta + \sin3\theta = 0$

or, $(\sin\theta + \sin3\theta) + \sin2\theta = 0$

or, $2\sin\left(\frac{\theta + 3\theta}{2}\right) . \cos\left(\frac{\theta - 3\theta}{2}\right) + \sin2\theta = 0$

or, $2\sin2\theta . \cos\theta + \sin2\theta = 0$

or, $2\sin2\theta (\cos\theta + 1) = 0$

or, $\sin2\theta (\cos\theta + 1) = 0$

Either, $\sin2\theta = 0$ ... ... (i)

$\therefore$ $\cos\theta + 1 = 0$ ... ... (ii)

From equation (i) $\sin2\theta = 0$

$\sin2\theta = \sin0^\circ, \sin180^\circ, \sin360^\circ$

$\therefore$ $\theta = 0^\circ, 90^\circ, 180^\circ$

From equation (ii) $\cos\theta = -1$

$\cos\theta = \cos180^\circ$

$\therefore$ $\theta = 180^\circ$

Hence the required values of θ are $0^\circ, 180^\circ$.

c. $3\cot\theta - \tan\theta = 2$

Solution

Here, $3\cot\theta - \tan\theta = 2$

or, $\frac{3}{\tan\theta} - \tan\theta = 2$

or, $3 - \tan^2\theta = 2\tan\theta$

or, $\tan^2\theta + 2\tan\theta - 3 = 0$

or, $\tan^2\theta + 3\tan\theta - \tan\theta - 3 = 0$

or, $\tan\theta(\tan\theta + 3) - 1(\tan\theta + 3) = 0$

or, $(\tan\theta + 3)(\tan\theta - 1) = 0$

Either, $\tan\theta - 1 = 0$ ... ... (i)

$\therefore$ $\tan\theta + 3 = 0$ ... ... (ii)

From equation (i) $\tan\theta = 1$

$\tan\theta = \tan 45°$

$\therefore$ $\theta = 45°$

From equation (ii) $\tan\theta = -3$

$\tan\theta = \tan 108.43°$

$\therefore$ $\theta = 108.43°$

Hence the required values of θ are 45°, 108.43°.

5. Solve : ($0° \leq \theta \leq 360°$)

a. $2\cos^2\theta - 3\sin\theta = 0$

Solution

Here, $2\cos^2\theta - 3\sin\theta = 0$

or, $2 - 2\sin^2\theta - 3\sin\theta = 0$

or, $2\sin^2\theta + 3\sin\theta - 2 = 0$

or, $2\sin^2\theta + 4\sin\theta - \sin\theta - 2 = 0$

or, $2\sin\theta(\sin\theta + 2) - 1(\sin\theta + 2) = 0$

or, $(\sin\theta + 2)(2\sin\theta - 1) = 0$

Either, $\sin\theta + 2 = 0$ ... ... (i)

$\therefore$ $2\sin\theta - 1 = 0$ ... ... (ii)

From equation (i) $\sin\theta = -2$

It has no solution as $-1 \leq \sin\theta \leq 1$

From equation (ii) $\sin\theta = \frac{1}{2}$

$\sin\theta = \sin 30°, \sin 150°$

$\therefore$ $\theta = 30°, 150°$

Hence the required values of θ are 30°, 150°.

b. $4\cos^2\theta + 4\sin\theta = 5$

Solution

Here, $4\cos^2\theta + 4\sin\theta = 5$

or, $4 - 4\sin^2\theta + 4\sin\theta - 5 = 0$

or, $4\sin^2\theta - 4\sin\theta + 1 = 0$

or, $(2\sin\theta - 1)^2 = 0$

or, $2\sin\theta - 1 = 0$

or, $\sin\theta = \frac{1}{2}$

or, $\sin\theta = \sin30°, \sin150°$

$\therefore$ $\theta = 30°, 150°$

c. $3 - 2\sin^2\theta = 3\cos\theta$

Solution

Here, $3 - 2\sin^2\theta = 3\cos\theta$

or, $3 - 2 + 2\cos^2\theta = 3\cos\theta$

or, $2\cos^2\theta - 3\cos\theta + 1 = 0$

or, $2\cos^2\theta - 2\cos\theta - \cos\theta + 1 = 0$

or, $2\cos\theta(\cos\theta - 1) - 1(\cos\theta - 1) = 0$

or, $(\cos\theta - 1)(2\cos\theta - 1) = 0$

Either, $\cos\theta - 1 = 0$ ... ... (i)

$\therefore$ $2\cos\theta - 1 = 0$ ... ... (ii)

From equation (i) $\cos\theta = 1$

$\cos\theta = \cos0°, \cos360°$

$\therefore$ $\theta = 0°, 360°$

From equation (ii) $\cos\theta = \frac{1}{2}$

$\cos\theta = \cos60°, \cos300°$

$\therefore$ $\theta = 60°, 300°$

Hence the required values of θ are $0°, 60°, 300°, 360°$

d. $\tan^2\theta + (1 - \sqrt{3})\tan\theta = \sqrt{3}$

Solution

Here, $\tan^2\theta + (1 - \sqrt{3})\tan\theta = \sqrt{3}$

or, $\tan^2\theta + \tan\theta - \sqrt{3}\tan\theta - \sqrt{3} = 0$

or, $\tan\theta(\tan\theta + 1) - \sqrt{3}(\tan\theta + 1) = 0$

or, $(\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$

Either, $\tan\theta + 1 = 0$ ... ... (i)

$\therefore$ $\tan\theta - \sqrt{3} = 0$ ... ... (ii)

From equation (i) $\tan\theta = -1$

$\tan\theta = \tan135°, \tan315°$

$\therefore$ $\theta = 135°, 315°$

From equation (ii) $\tan\theta = \sqrt{3}$

$\tan\theta = \tan 60°, \tan 240°$

$\therefore \quad \theta = 60°, 240°$

Hence the required values of θ are 60°, 135°, 240°, 315°

e. $\cot^2\theta + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\cot\theta + 1 = 0$

Solution

Here, $\cot^2\theta + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\cot\theta + 1 = 0$

or, $\sqrt{3}\cot^2\theta + 3\cot\theta + \cot\theta + \sqrt{3} = 0$

or, $\sqrt{3}\cot\theta(\cot\theta + \sqrt{3}) + 1(\cot\theta + \sqrt{3}) = 0$

or, $(\cot\theta + \sqrt{3})(\sqrt{3}\cot\theta + 1) = 0$

Either, $\cot\theta + \sqrt{3} = 0$ ... ... (i)

$\therefore \quad \sqrt{3}\cot\theta + 1 = 0$ ... ... (ii)

From equation (i) $\cot\theta = -\sqrt{3}$

$\cot\theta = \cot 150°, \cot 330°$

$\therefore \quad \theta = 150°, 330°$

From equation (ii) $\cot\theta = \frac{1}{\sqrt{3}}$

$\cot\theta = \cot 120°, \cot 300°$

$\therefore \quad \theta = 120°, 300°$

Hence the required values of θ are 120°, 150°, 300°, 330°

6. Solve (0° ≤ x ≤ 360°)

a. $\sqrt{3}\cos x + \sin x = \sqrt{3}$

It can be solved in two ways:

First Method

Here, $\sqrt{3}\cos x + \sin x = \sqrt{3}$

It is in the form of $a\sin\theta + b\cos\theta = c$

Where, $a = 1, b = \sqrt{3}$

$\sqrt{a^2 + b^2} = \sqrt{1 + 3} = \sqrt{4} = 2$

Dividing the given equation on both sides by '2', we get,

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{3}}{2}$$

or, $\cos x \cos 30° + \sin x \sin 30° = \frac{\sqrt{3}}{2}$

or, $\cos(x - 30°) = \cos 30°. \cos(360° - 30°)$

or, $\cos(x - 30°) = \cos 30°, \cos 330°$

or, $x - 30° = 30°, 330°$

$\therefore$ $x = 60°, 360°$

Second Method

Here, $\sqrt{3}\cos x + \sin x = \sqrt{3}$

$\sqrt{3^2 + 1^2} = 2$

Dividing the given equation by '2', on both sides, we get,

$$\frac{\sqrt{3}}{2}\cos x + \frac{\sin x}{2} = \frac{\sqrt{3}}{2}$$

or, $\sin x \cos 60° + \cos x \sin 60° = \frac{\sqrt{3}}{2}$

or, $\sin(x + 60°) = \sin 60°, \sin 120°, \sin 420°$

or, $x + 60° = 60°, 120°, 420°$

$\therefore$ $x = 0°, 60°, 360°$

Hence, the required values of x are 0°, 60°, 360°.

Alternative Method

In this method, we square on both sides, the roots so obtained must be checked whether it is true or false, only true values are accepted.

Here, $\sqrt{3}\cos x + \sin x = \sqrt{3}$

$\sqrt{3}\cos x = \sqrt{3} - \sin x$

Squaring on both sides, we get

$(\sqrt{3}\cos x)^2 = (\sqrt{3} - \sin x)^2$

or, $3\cos^2 x = 3 - 2\sqrt{3}\sin x + \sin^2 x$

or, $3 - 3\sin^2 x = 3 - 2\sqrt{3}\sin x + \sin^2 x$

or, $-4\sin^2 x + 2\sqrt{3}\sin x = 0$

or, $2\sin^2 x - \sqrt{3}\sin x = 0$

$\therefore$ $\sin x(2\sin x - \sqrt{3}) = 0$

Either, $\sin x = 0$ ... ... (i)

$\therefore \quad 2 \sin x - \sqrt{3} = 0 \ldots \ldots \ldots$ (ii)

From equation (i) $\sin x = 0$

$\sin x = \sin 0°, \sin 180°, \sin 360°$

$\therefore \quad x = 0°, 180°, 360°$

From equation (ii) $\sin x = \dfrac{\sqrt{3}}{2}$

$\sin x = \sin 60°, \sin 120°$

$\therefore \quad x = 60°, 120°$

In checking for $x = 0°, 120°, 180°, 360°$

For	$x = 0°$	$\sqrt{3} \cdot 1 + 0 = \sqrt{3}$		(true)
For	$x = 120°$	$\sqrt{3} \cdot \left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2} = \sqrt{3}$	$0 = \sqrt{3}$	(false)
For	$x = 60°$	$\sqrt{3} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$	$\sqrt{3} = \sqrt{3}$	(true)
For	$x = 180°$	$\sqrt{3} \cdot (-1) + 0 = \sqrt{3}$	$-\sqrt{3} = \sqrt{3}$	(false)
For	$x = 360°$	$\sqrt{3} \cdot 1 + 0 = \sqrt{3}$		(true)

Hence, the required values of x are 0°, 60°, 360°.

b. $\sqrt{3}\sin x + \cos x = 2, 0° \leq x \leq 360°$

Solution

Here $\quad \sqrt{3} \sin x + \cos x = 2$

$\sqrt{(\sqrt{3})^2 + 1} = 2$

Dividing on both sides by 2, we get,

$\dfrac{\sqrt{3}}{2} \sin x + \dfrac{1}{2} \cos x = 1$

or, $\quad \cos x \cdot \cos 60° + \sin x \cdot \sin 60° = 1$

or, $\quad \cos(x - 60°) = \cos 0°$

or, $\quad x - 60° = 0°$

$\therefore \quad x = 60°$

Alternatively,

we can solve it by squaring on both sides,

$\sqrt{3} \sin x = 2 - \cos x$

Squaring on both sides, we get,

$3 \sin^2 x = 4 - 4 \cos x + \cos^2 x$

or, $\quad 3 - 3 \cos^2 x = 4 - 4 \cos x + \cos^2 x$

or, $-4\cos^2x + 4\cos x - 1 = 0$

or, $4\cos^2x - 4\cos x + 1 = 0$

or, $(2\cos x - 1)^2 = 0$

or, $2\cos x - 1 = 0$

or, $\cos x = \frac{1}{2}$

or, $\cos x = \cos 60°, \cos 300°$

$\therefore$ $x = 60°, 300°$

On checking,

When $x = 60$	$\frac{\sqrt{3}}{2} . \frac{\sqrt{3}}{2} + \frac{1}{2} = 2$	$\frac{3}{4} + \frac{1}{2} = 2$	$2 = 2$	(true)
When $x = 300$	$\sqrt{3} . \left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{2} = 2$	$\frac{-3 + 1}{2} = 2$	$-1 = 2$	(false)

Hence, the required value of x is 60°.

7. Solve for x, (0° ≤ x ≤ 360°)

a. $\frac{\sqrt{3}}{\sin 2x} + \frac{1}{\cos 2x} = 0$

Solution

Here, $\frac{\sqrt{3}}{\sin 2x} + \frac{1}{\cos 2x} = 0$

or, $\sqrt{3}\cos 2x + \sin 2x = 0$

or, $\frac{\sqrt{3}}{2}\cos 2x + \frac{1}{2}\sin 2x = 0$

or, $\cos 2x . \cos 30° + \sin 2x . \sin 30° = 0$

or, $\cos(2x - 30°) = \cos 90°, \cos 270°$

or, $2x - 30° = 90°, 270°$

or, $2x = 120°, 300°$

$\therefore$ $x = 60°, 150°$

b. $\frac{\sqrt{3}}{\sin 2x} + \frac{1}{\cos 2x} = 4$

Solution

Here, $\frac{\sqrt{3}}{\sin 2x} + \frac{1}{\cos 2x} = 4$

or, $\sqrt{3}\cos 2x + \sin 2x = 4\sin 2x . \cos 2x$

or, $\frac{\sqrt{3}}{2}\cos 2x + \frac{1}{2}\sin 2x = 2\sin 2x . \cos 2x$

or, $\cos 2x . \sin 60° + \sin 2x . \cos 60° = \sin 4x$

or, $\cos(2x + 60°) = \sin 4x, \sin(180° – 4x)$

$\therefore$ $2x + 60° = 4x, 180° – 4x$

$2x = 60°$ or, $x = 30°$

and $2x + 60° = 180° – 4x$

or, $x = 20°$

$\therefore$ $x = 20°, 30°$

c. $\sqrt{2}$ secx + tanx = 1

Solution

Here, $\sqrt{2} \sec x + \tan x = 1$

or, $\dfrac{\sqrt{2}}{\cos x} + \dfrac{\sin x}{\cos x} = 1$

or, $\sqrt{2} + \sin x = \cos x$

or, $\sqrt{2} = \cos x – \sin x$

Squaring on both sides, we get,

$2 = \cos^2 x – 2 \cos x \sin x + \sin^2 x$

or, $2 = (\sin^2 x + \cos^2 x) – \sin 2x$

or, $2 = 1 – \sin 2x$

or, $\sin 2x = –1$

or, $\sin 2x = \sin 270°, \sin(360° + 270°)$

or, $2x = 270°, 630°$

$\therefore$ $x = 135°, 315°$

8. Solve : (0° ≤ x ≤ 360°)

a. 2 tan3x . cos2x + 1 = tan3x + 2 cos2x

Solution

Here, $2 \tan 3x . \cos 2x + 1 = \tan 3x + 2 \cos 2x$

or, $2 \tan 3x . \cos 2x – \tan 3x + 1 – 2 \cos 2x = 0$

or, $\tan 3x(2 \cos 2x – 1) – 1(2 \cos 2x – 1) = 0$

$\therefore$ $(\tan 3x – 1) (2 \cos 2x – 1) = 0$

Either, $\tan 3x – 1 = 0$ … … … (i)

$\therefore$ $2 \cos 2x – 1 = 0$ … … … (ii)

From equation (i) $\tan 3x = 1$

$\tan 3x = \tan 45°, \tan 225°, \tan 405°, \tan 765°, \tan 945°$

or, $3x = 45°, 225°, 405°, 765°, 945°$

$\therefore$ $x = 15°, 75°, 135°, 225°, 315°$

From equation (ii) $2 \cos 2x = 1$

$$\cos 2x = \frac{1}{2}$$

or, $\cos 2x = \cos 60°, \cos 300°, \cos 420°, \cos 660°$

or, $2x = 60°, 300°, 420°, 660°$

$\therefore$ $x = 30°, 150°, 210°, 330°$

Hence, the required values of x are 15°, 30°, 75°, 135°, 150°, 210°, 225°, 315°, 330°

b. sin2x . tanx + 1 = sin2x + tanx

Solution

Here, $\sin 2x \cdot \tan x + 1 = \sin 2x + \tan x$

or, $\sin 2x \cdot \tan x - \sin 2x + 1 - \tan x = 0$

or, $\sin 2x(\tan x - 1) - 1(\tan x - 1) = 0$

$\therefore$ $(\tan x - 1)(\sin 2x - 1) = 0$

Either, $\tan x - 1 = 0$ ... ... (i)

$\therefore$ $\sin 2x - 1 = 0$ ... ... (ii)

From equation (i) $\tan x = 1$

$\tan x = \tan 45°, \tan 225°$

$\therefore$ $x = 45°, 225°$

From equation (ii) $\sin 2x = 1$

$\sin 2x = \sin 90°, \sin 450°$

or, $2x = 90°, 450°$

$\therefore$ $x = 45°, 225°$

Hence, the required values of x are 45°, 225°

9. Solve for θ, (0° ≤ θ ≤ 360°)

a. tanθ – 3 cotθ = 2 tan3θ

Solution

Here, $\tan\theta - 3\cot\theta = 2\tan 3\theta$

or, $\tan\theta - \dfrac{3}{\tan\theta} = 2 \cdot \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

or, $\dfrac{\tan^2\theta - 3}{\tan\theta} = \dfrac{6\tan\theta - 2\tan^3\theta}{1 - 3\tan^2\theta}$

or, $\tan^2\theta - 3 - 3\tan^4\theta + 9\tan^2\theta = 6\tan^2\theta - 2\tan^4\theta$

or, $\quad -\tan^4\theta + 4\tan^2\theta - 3 = 0$

or, $\quad \tan^4\theta - 4\tan^2\theta + 3 = 0$

or, $\quad \tan^4\theta - 3\tan^2\theta - \tan^2\theta + 3 = 0$

or, $\quad \tan^2\theta(\tan^2\theta - 3) - 1(\tan^2\theta - 3) = 0$

or, $\quad (\tan^2\theta - 3)(\tan^2\theta - 1) = 0$

Either, $\quad \tan^2\theta - 3 = 0$ ... ... (i)

$\therefore \quad \tan^2\theta - 1 = 0$ ... ... (ii)

From equation (i) $\tan^2\theta = 3$

$$\tan^2\theta = (\sqrt{3})^2$$

$$\therefore \quad \tan\theta = \pm\sqrt{3}$$

Taking positive sign,

$\quad \tan\theta = \sqrt{3}$

or, $\quad \tan\theta = \tan60°, \tan240°$

$\therefore \quad \theta = 60°, 240°$

Taking negative sign,

$\quad \tan\theta = -\sqrt{3}$

or, $\quad \tan\theta = \tan120°, \tan300°$

$\therefore \quad \theta = 120°, 300°$

From equation (ii)

$\quad \tan\theta = \pm1$

Taking positive sign,

$\quad \tan\theta = 1$

or, $\quad \tan\theta = \tan45°, \tan225°$

$\therefore \quad \theta = 45°, 225°$

Taking negative sign,

$\quad \tan\theta = -1$

or, $\quad \tan\theta = \tan135°, \tan315°$

$\therefore \quad \theta = 135°, 315°$

Hence, the required values of θ are 45°, 60°, 120°, 135°, 225°, 240°, 300°, 315°

Height and Distance

1. Objectives

Knowledge (K)	To define angle of depression and angle of eleration.
Understanding (U)	To draw figures o show angle of eleration and angle of depression.
Skill/Application (S/A)	To solve simple problems on height and distance with drawing diagrams.
Higher Ability (HA)	To solve harder problems on height and distance with verbal expressions.

2. Teaching Materials Required

Chart paper with illustration of angle of elevation and angle of depression, theodolite, sextant, clinometor, hypsometre etc.

3. Teaching Activities:

→ Discuss about the components of a triangle as review and fundamental trigonometric ratios.

→ Review on solution of a right angled triangle.

→ Define angle of eleration and angle of depression with draw for illustration.

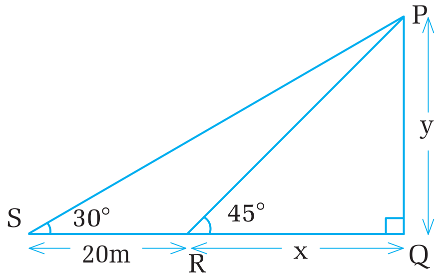
→ Give problems from exercise of the text book Q.N. 2 and 3.

→ The teacher solves some verbal problems as examples.

Some solved problems

1. Find the values of x and y from the given figure.

Solution



Here, from right angled triangle PQR,

$$\tan 45° = \frac{y}{x}$$

or, $1 = \frac{y}{x}$

$\therefore$ $x = y$ ... ... (i)

Again from right angled triangled PQR

$$\tan 30° = \frac{PQ}{SQ}$$

or, $\frac{1}{\sqrt{3}} = \frac{y}{20 + x}$

$\therefore$ $20 + x = \sqrt{3}y$ ... ... (ii)

Put the value of x from (i) in (ii), we get

$$20 + y = \sqrt{3}y$$

or, $y(1 - \sqrt{3}) = -20$

or, $y = \dfrac{20}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

or, $y = \dfrac{20(\sqrt{3}+1)}{2}$

or, $y = 10 \times 2.732$

$\therefore$ $y = 27.32\text{m}$

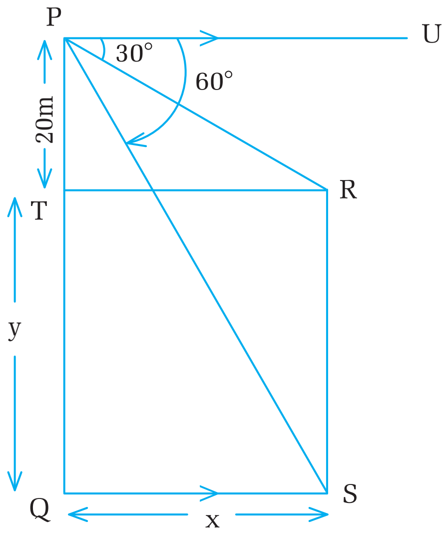
b.

Solution

Here, $\measuredangle UPS = \measuredangle PSQ = 60°$, PU//QS, corresponding angles.

$\measuredangle UPR = \measuredangle PRT = 30°$, PU//TR, corresponding angles.

From right angled ΔPRT,



$$\tan 30° = \frac{PT}{TR}$$

or, $\dfrac{1}{\sqrt{3}} = \dfrac{20}{x}$ $\quad (\because \ TR = QS = x)$

$\therefore$ $x = 20\sqrt{3}\text{m}$

Again from right angled ΔPQR,

$$\tan 60° = \frac{PQ}{RS}$$

or, $\sqrt{3} = \dfrac{20 + y}{x}$

$\therefore$ $\sqrt{3}x = 20 + y$ ... ... (ii)

Put the value of x in eqn (ii), from (i)

$$\sqrt{3} \,.\, 20\sqrt{3} = 20 + y$$

or, $60 = 20 + y$

$\therefore$ $y = 40\text{m}$

Hence, $x = 20\sqrt{3}\text{m}$ and $y = 40\text{m}$.

2. **The angle of elevation of the top of a tower from a point on ground was observed to be 45° on walking 30m away from that point it was found to be 30°. Find the height of the house.**

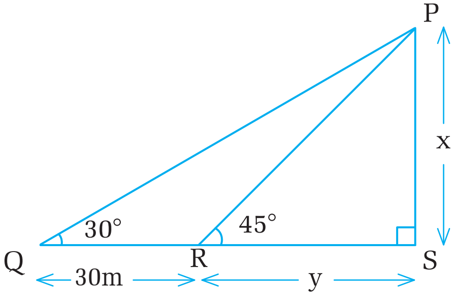
Solution

Let, PS be the height of the house and R the point of observer.

$\measuredangle PRS = 45°$, $\measuredangle PQS = 30°$, $QR = 30m$

Let, PS = xm, RS = ym

From right angled ΔPQS,



$$\tan 30° = \frac{PS}{QS}$$

or, $$\frac{1}{\sqrt{3}} = \frac{x}{30 + y}$$

$\therefore$ $$30 + y = \sqrt{3}x \text{ ... ... (i)}$$

Again from right angled triangle PRS,

$$\tan 45° = \frac{PS}{RS}$$

or, $$1 = \frac{x}{y}$$

$\therefore$ $$x = y \text{ ... ... (ii)}$$

Put the value of x from eqn (ii) in (i), we get

$$30 + y = \sqrt{3}y$$

or, $$y(\sqrt{3} - 1) = 30$$

or, $$y = \frac{30}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

or, $$y = \frac{30(\sqrt{3} + 1)}{3 - 1}$$

or, $$y = 15(\sqrt{3} + 1)$$

or, $$y = 15 \times 2.732$$

$\therefore$ $$y = 40.98m$$

Put the value of y in eqn (i)

$$x = 40.98m$$

Hence the height of the house was 40.98m.

3. From the top of a tower of 200m the angle of depression of two boats which are in a straight line on the same side of the tower are to be 30° and 45°. Find the distance between the boats.

Solution

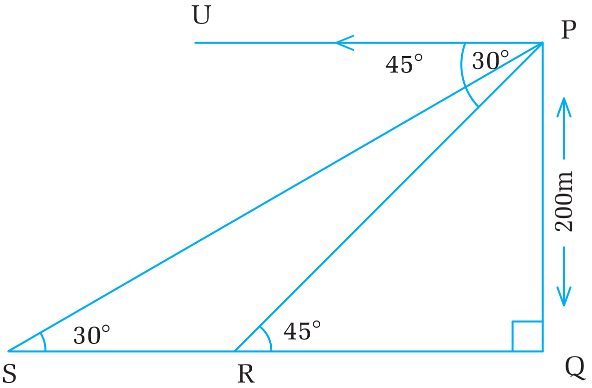
Let, PQ = 200m, the height of the tower,

R and S are the positions of the boats such that

$\measuredangle UPS = \measuredangle PSQ = 30°$, UP//SQ, the corresponding angles

$\measuredangle UPR = \measuredangle PRQ = 45°$

From right angled ΔPRQ,



$$\tan 45° = \frac{PQ}{RQ}$$

or, $$1 = \frac{200}{RQ}$$

$\therefore$ $$RQ = 200m$$

Again from right angled triangle PQR,

$$\tan 30° = \frac{PQ}{SQ}$$

or, $$\frac{1}{\sqrt{3}} = \frac{200}{SR + RQ}$$

or, $$SR + RQ = 200\sqrt{3}$$

or, $$SR = 200 = 200\sqrt{3}$$

or, $$SR = 200(\sqrt{3} - 1)$$

or, $$SR = 200(1.732 - 1)$$

or, $$SR = 200 \times 0.732$$

$\therefore$ $$SR = 146.4m$$

Hence the distance between the two boats is 146.4m.

4. From a helicopter flying vertically above a straight road, the angles of depression of two consecutive kilometer stone on the same side are found to be 45° and 60°. Find the height of the helicopter.

Solution

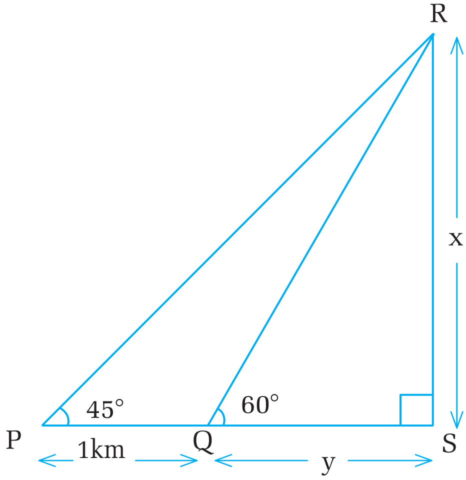
Let, R be the position of the helicopter and RS the height of it from the ground.

Let, P and Q be the positions of two stones on the ground such that PQ = 1km = 1000m

$\measuredangle RPS = 45°$, $\measuredangle RQS = 60°$

Let, QS = ym, RS = xm

From right angled ΔRQS,



$$\tan 60° = \frac{RS}{QS}$$

or, $$\sqrt{3} = \frac{RS}{QS}$$

or, $$\sqrt{3} = \frac{x}{y}$$

$\therefore$ $$x = \sqrt{3}y \text{ ... ... (i)}$$

Again, from right angled triangle RPS,

$$\tan 45° = \frac{RS}{PS}$$

or, $1 = \frac{x}{y + 1000}$

$\therefore$ $x = y + 1000$ ... ... (ii)

From equation (i) and (ii), we get

$$y + 1000 = \sqrt{3}y$$

or, $1000 = (\sqrt{3} - 1)y$

or, $y = \frac{1000}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$

or, $y = \frac{1000(\sqrt{3} + 1)}{3 - 1}$

or, $y = \frac{1000(2.732)}{2}$

or, $y = 500 \times 2.732$

$\therefore$ $y = 1366\text{m}$

Put the value of y in eqn (ii)

$x = 1366 + 1000$
$= 2366\text{m}.$

Hence the height of the helicopter was 2366m.

5. From the top of 21m high cliff; the angles of depression of top and bottom of a towers are observed to be 45° and 60° respectively. Find the height of the tower.

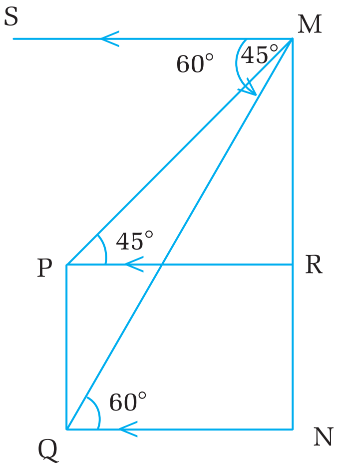
Solution

Let, MN and PQ be the heights of cliff and the tower respectively and MN = 21m

Then, draw SM//PR and PR//QN

$\measuredangle SMP = \measuredangle MPR = 45°$, $\measuredangle SMQ = \measuredangle MQN = 60°$, MN = 21m

From right angled ΔMQN,



$\tan 60° = \frac{MN}{QN}$

or, $\sqrt{3} = \frac{21}{QN}$

$\therefore$ $QN = 7\sqrt{3}\text{m}$

Again from right angled triangle MPR,

$\tan 45° = \frac{MR}{PR}$

But $PR = QN = 7\sqrt{3}\text{m}$

or, $1 = \frac{MR}{7\sqrt{3}}$

$\therefore$ $MR = 7\sqrt{3}\text{m}$

Hence the height of the tower is PQ $= RN$

$= MN - MR$

$= 21 - 7\sqrt{3}$

$= 21 - 7 \times 1.732 = 8.87\text{m}$

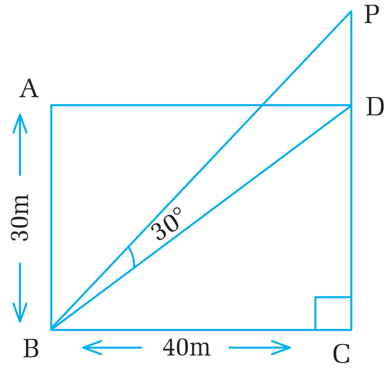
6. A flagstaff is placed at one corner of a rectangular 40m long and 30m wide. If the angle of elevation of the top of the flagstaff from the opposite corner is 30°. Find the height of the flagstaff.

Solution

Let, ABCD be rectangular garden of length 40m and breadth 30m and PD be the height of the flagstaff.

The angle of elevation of the flagstaff PD is 30°

Diagonel BD is drawn. By using pythagoras theorem,



$$BD = \sqrt{BC^2 + CD^2} = \sqrt{1600 + 900} = \sqrt{2500} = 50\text{m}$$

From right angled ΔPBD,

$$\tan 30° = \frac{PD}{BD}$$

or, $$\frac{1}{\sqrt{3}} = \frac{PD}{50}$$

$\therefore$ $PD = 28.86\text{m}$

9. From the top and bottom of a tower, the angle of depression of the top of the house and angle of elevation of the house are found to be 60° and 30° respectively. If the height of the building is 20m, find the height of the tower.

Solution

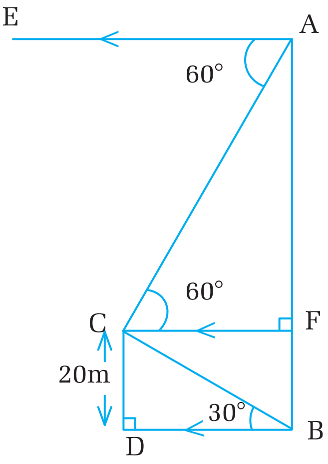
In the figure, EA//CF//DB

AB = the height of the tower

CD = 20m, the height of the house = FB

$\measuredangle EAC = \measuredangle ACF = 60°$, $\measuredangle DBC = 30°$

EA//CF the corresponding angles.



From right angled ΔCDB,

$$\tan 30° = \frac{CD}{DB}$$

or, $$\frac{1}{\sqrt{3}} = \frac{20}{DB}$$

$\therefore$ $DB = 20\sqrt{3}$

Again, from right angled triangle ACF,

$$\tan 60° = \frac{AF}{CF}$$

or, $\sqrt{3} = \frac{AF}{20\sqrt{3}}$ $(\because \quad CF = DB)$

$\therefore$ $AF = 60m$

Hence, the height of the tower $= AB$

$= AF + FB$

$= 60m + 20m$

$= 80m.$

10. A ladder of 18m reaches a point 18m below the top of a vertical flagstaff. From the foot of the ladder the angle of elevation of the flagstaff is 60°. Find the height of the flagstaff.

Solution

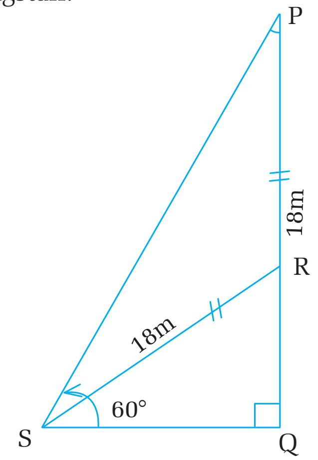
Let, SR be the ladder of length 18 and PQ the height of the flagstaff.

$SR = 18m$, $PR = 18m$

$\measuredangle PSR = 60°$. ΔPRS is an isoceles triangles.

$\therefore$ $\measuredangle PSR = \Delta RPS = 30°$

$\measuredangle RSQ = 60° - 30° = 30°$



Now, from the right angled triangle RSQ,

$$\sin 30° = \frac{RQ}{SR}$$

or, $\frac{1}{2} = \frac{RQ}{18}$

$\therefore$ $RQ = 9m$

Hence, the height of the flagstaff is $18m + 9m = 27m$.

11. AB is a vertical pole with its foot B on a level of ground. A point C on AB divides such that AC:CD = 3:2. If the parts AC and CB subtend equal angles at a point on the ground which is at a distance of 20m from the foot of the pole. Find the height of the pole.

Solution

In the figure, AC:CB = 3:2.

Let, $AC = 3x$, $BC = 2x$

D is a point 20m away from the foot of the pole B. $DB = 20m$

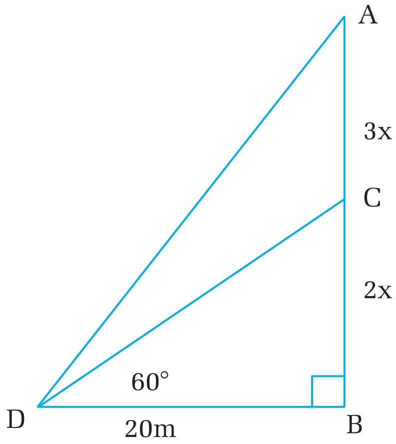
$\measuredangle BDC = \theta$, $\measuredangle ADC = \theta$, $\measuredangle ADB = 20\theta$

From the right angled triangle CDB,

$$\tan\theta = \frac{CB}{DB}$$

$$\text{or,} \quad \tan\theta = \frac{2x}{20} = \frac{x}{10}$$

Again, from right angled triangle ADB,



$$\tan 2\theta = \frac{3x + 2x}{20}$$

$$\text{or,} \quad \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{5x}{20}$$

$$\text{or,} \quad \frac{2 \cdot \frac{x}{10}}{1 - \frac{x^2}{100}} = \frac{x}{4}$$

$$\text{or,} \quad \frac{1}{5} \times \frac{100}{100 - x^2} = \frac{1}{4}$$

$$\text{or,} \quad 80 = 100 - x^2$$

$$\text{or,} \quad x^2 = 20$$

$$\therefore \quad x = 2\sqrt{5}\,\text{m}$$

$$\therefore \quad AB = 5x = 5.2\sqrt{5} = 10\sqrt{5} = 22.36\text{m}$$

12. A man 1.75m stands at a distance of 8.5m from a lamp post and it is observed that his shadow 3.5m long. Find the height of the lamp post.

Solution

Let, PQ the height of the lamp post and MN the height of the man.

$$MN = 1.75\text{m}$$

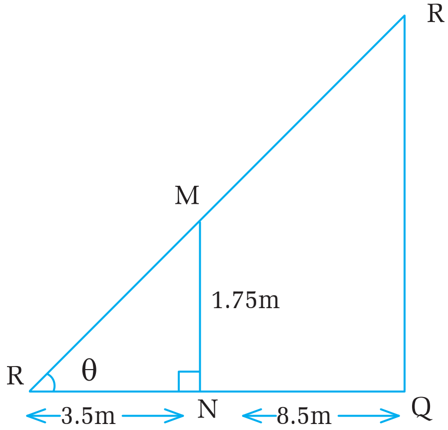
RS = 3.5m, the shadow of the man.

From the right angled triangle MRN,

Let, $\theta = \measuredangle MRN$

$$\tan\theta = \frac{MN}{RN} = \frac{1.75}{3.5} = \frac{1}{2}$$

Again, from right angled triangle PRQ,



$$\tan\theta = \frac{PR}{RQ}$$

$$\text{or,} \quad \frac{1}{2} = \frac{PQ}{3.5 + 8.5}$$

$$\text{or,} \quad \frac{1}{2} = \frac{PQ}{12}$$

$$\therefore \quad PQ = 6\text{m}$$

13. The angle of elevation of the top of a tower is 45° from a point 10m above the water level of a lake. The angle of depression of its image in the lake from the same point is 60°. Find the height of the tower above the water level.

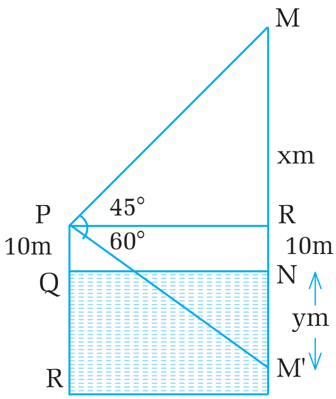
Solution

Let, MN be the height of the tower from the water level of the lake.

P is the position of the observer which is 10m above the water level.

Let, M' be the image of M in water of the lake

PQNR is a rectangle.



$PQ = RN = 10m$

Let, $MR = xm$

$NM' = ym$

From the right angled triangle MPR,

$$\tan 45° = \frac{MR}{PR}$$

or, $MR = PR$

or, $PR = x$

Again, from right angled triangle PRM',

$$\tan 60° = \frac{RM'}{PR}$$

or, $\sqrt{3} = \dfrac{10 + y}{x}$

or, $\sqrt{3}x = 10 + y$ ... ... (i)

But by definition of reflection, MN = NM'

i.e. image distance = object distance

So, we can write $x + 10 = y$ ... ... (ii)

From the equation (i) and (ii), we get

$$x + 10 = \sqrt{3}x - 10$$

or, $20 = (\sqrt{3} - 1)x$

or, $x = \dfrac{20}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1} = \dfrac{20(\sqrt{3}+1)}{2} = 10 \times 2.732 = 27.32m$

Now, the height of the tower above the level of water

$= MN$

$= x + 10$

$= 27.32 + 10$

$= 37.32m$

14. The angle of elevation of an aeroplane from a point in the ground is 45°. After 15 seconds of flight the angle of elevation changes to 30°. If the aeroplane is flying horizontally at a height of 4000m, in the same direction, find the speed of aeroplane.

Solution

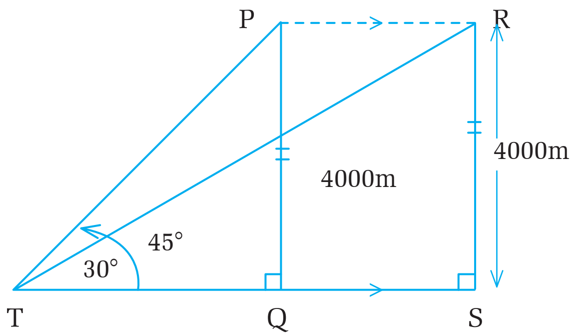
Let, $PQ = RS = 4000m$, the constant height of the aeroplane.

P the starting position of the aeroplane when the observer saw it.

$\measuredangle PTQ = 45°$

Let, R be the position of the aeroplane after 15 second $\measuredangle RTS = 30°$

Now, from the right angled triangle PTQ,



$$\tan 45° = \frac{PQ}{TQ}$$

or, $1 = \frac{PQ}{TQ}$

or, $PQ = TQ = 4000m$

Again, from right angled triangle RTS,

$$\tan 30° = \frac{RS}{TS}$$

or, $\frac{1}{\sqrt{3}} = \frac{4000}{TQ + QS}$

or, $4000\sqrt{3} = 4000 + QS$

or, $4000(\sqrt{3} - 1) = QS$

or, $QS = 4000(\sqrt{3} - 1)$

$= 4000(1.732 - 1)$

$= 4000 \times 0.732$

$= 2928m$

But $PR = QS = 2928m$

The distance covered by the aeroplane in 15 second is 2928m.

Hence, speed of the aeroplane (v) $= \frac{\text{distance}}{\text{time}}$

$= \frac{2928m}{15s}$

$= 195.2ms^{-1}$

15. From the foot of mountain, the elevation of its summit is 45°. After going up at a distance of 1km towards the top of the mountain at an angle of 30°, the elevation changes to 60°. Find the height of the mountain.

Hints:

Solution

$PQ = 1km$, initially climbed part.

Again, from right angled triangle PQR,

$$\tan 30° = \frac{QR}{PQ}$$

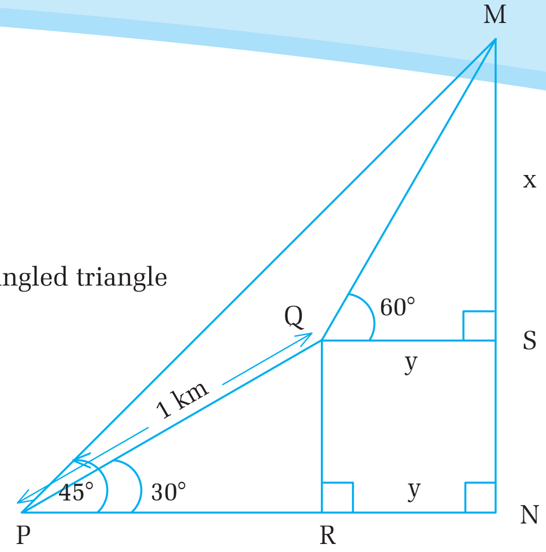
or, $\frac{1}{2} = \frac{QR}{1}$

$\therefore$ QR = 0.5km

QRNS is a rectangle $\therefore$ QR = SN = 0.5

But $\measuredangle$MPN = 45°, ΔMPN is an isoceles right angled triangle

MN = PN



Also, $\cos30° = \frac{PR}{PQ}$

or, $PR = \frac{\sqrt{3}}{2} = 0.8660$ km.

$\tan60° = \frac{x}{y}$

or, $x = \sqrt{3}y = 1.7321y$

Again from right angled trinalge PMN

$\tan45° = \frac{MN}{PN}$

or, PN = MN

or, PR + RN = MS + SN

or, 0.8660 + y = x + 0.5

or, 0.8660 + y = 1.7321y + 0.5

or, $y = \frac{0.3660}{0.7321} = 0.5$

and x = 1.7321 × 0.5 = 0.866

Total height of the mountain = 0.5 + 0.866 = 1.366km = 1366m.

Questions for practice

1. A chimney is $10\sqrt{3}$m. high. Find the angle of elevation of its top from a point 100m away from its foot.
2. From the top of a building 45m high, a man observes the angle of depression of a stationary bus is 30°. Find the distance of the bus from the building.
3. The angle of elevation of a tower was observed to be 60° from a point. on walking 200m away from the point, it was found to be 30°. Find the height of the tower.
4. A boy standing between two pillars of equal height observes the angle of elevation of the top of a pillar to be 30°. Approaching 15m, towards anyone of the pillars the angle of elevation is 45°. Find the height of the pillars and distance between them.
5. A cloud is observed above a river from opposite banks at angles of elevation are found to be 45° and 60°. The cloud is vertically above the line joining the points of observation and the river is 80m wide. Find the height of the cloud.
7. The angles of elevation of the top of a tower from two points 'a' and 'b' m from the base in the name straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ m.
8. The angles of depression and elevation of the top and the bottom of a tele-communication tower are observed respectively 45° and 30° from the root of the house. The height of the house is 40m. Find the height of the tower and house and tower are on the same plans.

UNIT ten

Vectors

Scalar Product of two vectors

Estimated Teaching Periods : 5

1. Objectives

Level	Objectives
Knowledge (K)	To define dot product of two vectors. Tell meaning of $\vec{a}.\vec{b} = \lvert\vec{a}\rvert\,\lvert\vec{b}\rvert\cos\theta$
Understanding (U)	To establish relation $\cos\theta = \dfrac{\vec{a}.\vec{b}}{\lvert\vec{a}\rvert\,\lvert\vec{b}\rvert}$ To state conditions of perpendicularity and parallelism of two vectors.
Skill/Application (S/A)	To solve problems involving dot product of two vectors.
Higher Ability (HA)	To solve harder problems of dot product of vectors.

2. Teaching Materials

Graph papers, list of formula of scaler product of two vectors.

3. Teaching Learning Strategies

→ Review the concept of vectors and scalars.

→ Take two position vectors $\overrightarrow{OA} = (4, 5)$ and $\overrightarrow{OB} = (-5, 4)$ plot them in a graph papers. Find angle between them. Multiply (4, 5) and (–5, 4) to get $4 \times -5 + 5 \times 4 = 0$. Draw the conclusion from it.

→ Define dot product of two vectors $\vec{a}$ and $\vec{b}$ as $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$ with figure.

→ Discuss to find angle between two vectors by using formula

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

→ If $\vec{a} = x_1\vec{i} + y_1\vec{j}$ and $\vec{b} = x_2\vec{i} + y_2\vec{j}$ and show that $\vec{a}.\vec{b} = x_1x_2 + y_1y_2$.

→ Discuss the conditions of perpendicularity and parallelism of two vectors by using above formula and also discuss the meaning of $\vec{a}.\vec{b} = 0$ and $\vec{a} = k\vec{b}$.

→ Also show $\vec{i}.\vec{j} = 0$ and $\vec{i}.\vec{i} = 1$ as review.

→ Discuss the some properties of dot product of two vectors.

List of Formula

1. $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

   or, $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$

2. If $\vec{a} = x_1\vec{i} + y_1\vec{j}$ and $\vec{b} = x_2\vec{i} + y_2\vec{j}$ then $\vec{a}.\vec{b} = x_1x_2 + y_1y_2$.

3. If $\vec{a}.\vec{b} = 0$, then the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

i.e. $\cos 90° = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|} \Rightarrow \vec{a}.\vec{b} = 0$

4. If $\vec{a} = m\vec{b}$ or $\vec{b} = k\vec{a}$, where m and k are scalars, then two vectors $\vec{a}$ and $\vec{b}$ are parallel to each other.

5. $(\vec{a} + \vec{b})^2 = a^2 + 2\vec{a}.\vec{b} + b^2 = a^2 + b^2 + ab\cos\theta$.

Some solved problems

1. Find the dot product of :

a. $\vec{a} = \begin{pmatrix}3\\4\end{pmatrix}$ **and** $\vec{b} = \begin{pmatrix}2\\1\end{pmatrix}$

Solution

Here, $\vec{a} = \begin{pmatrix}3\\4\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}2\\1\end{pmatrix}$

$$\vec{a}\,.\,\vec{b} = \begin{pmatrix}3\\4\end{pmatrix}.\begin{pmatrix}2\\1\end{pmatrix}$$
$$= (3, 4)\,.\,(2, 1)$$
$$= 3 \times 2 + 4 \times 1$$
$$= 6 + 4 \quad = 10$$

b. $\vec{a} = \vec{i} + 2\vec{j}$ **and** $\vec{b} = 3\vec{i} - \vec{j}$

Solution

Here, $\vec{a} = \vec{i} + 2\vec{j}$ and $\vec{b} = 3\vec{i} - \vec{j}$

$$\vec{a}\,.\,\vec{b} = (\vec{i} + 2\vec{j})\,.\,(3\vec{i} - \vec{j})$$
$$= 1 \times 3 + 2 \times (-1)$$
$$= 3 - 2 \quad = 1$$

2. If $|\vec{a}| = 2$, $|\vec{b}| = 3$, angle between them is 45°, find $\vec{a}\,.\,\vec{b}$.

Solution

Here, $|\vec{a}| = 2$, $|\vec{b}| = 3$, $\theta = 45°$

$$\vec{a}\,.\,\vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta$$
$$= 2 \times 3 \times \cos 45°$$
$$= 6 \times \frac{1}{\sqrt{2}}$$
$$= \frac{6}{\sqrt{2}} \quad = \frac{3 \times \sqrt{2} \times \sqrt{2}}{\sqrt{2}} \quad = 3\sqrt{2}$$

3. Show that $\vec{p} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ are perpendicular to each other.

Solution

Here, $\vec{p} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

$$\vec{p} \cdot \vec{q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$= 3 \times (-4) + 4 \times 3$$

$$= -12 + 12 \quad = 0$$

Since $\vec{p} \cdot \vec{q} = 0$, hence $\vec{p}$ and $\vec{q}$ are perpendicular to each other. **Proved**

Alternatively

Let, θ be the angle between $\vec{p}$ and $\vec{q}$, then

$$\cos\theta = \frac{\vec{p} \cdot \vec{q}}{|\vec{p}|\ |\vec{q}|}$$

$$\vec{p} \cdot \vec{q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$= -12 + 12 \quad = 0$$

$$|\vec{p}| = \sqrt{3^2 + 4^2} = 5$$

$$|\vec{q}| = \sqrt{(-4)^2 + 3^2} = 5$$

Now, $\cos\theta = \dfrac{0}{5 \times 5} = 0 = \cos 90°$

$\therefore \quad \theta = 90°$

This shows that $\vec{p}$ and $\vec{q}$ are perpendicular to each other.

4. Find the value of k, if the $\vec{a} = 4\vec{i} + k\vec{j}$ and $\vec{b} = 3\vec{i} - 6\vec{j}$ are perpendicular to each other.

Solution

Here, $\vec{a} = 4\vec{i} + k\vec{j}$, $\vec{b} = 3\vec{i} - 6\vec{j}$

Since $\vec{a}$ and $\vec{b}$ are perpendicular to each other, their dot product is zero.

i.e. $\quad \vec{a} \cdot \vec{b} = 0$

or, $\quad (4\vec{i} + k\vec{j}) \cdot (3\vec{i} - 6\vec{j}) = 0$

or, $\quad 4 \times 3 + k(-6) = 0$

or, $\quad 12 - 6k = 0$

$\therefore \quad k = 2$

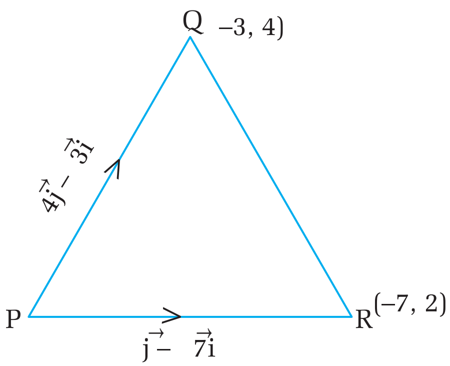
5. In ΔPQR, if $\overrightarrow{PQ} = 4\vec{j} - 3\vec{i}$, $\overrightarrow{PR} = \vec{j} - 7\vec{i}$, prove that ΔPQR is an isoceles right angled triangle.

Solution

Let, P be taken as origin,

then $\overrightarrow{OQ} = \overrightarrow{PQ} = 4\vec{j} - 3\vec{i} = (-3, 4)$

$\overrightarrow{OR} = \overrightarrow{PR} = \vec{j} - 7\vec{i} = (-7, 1)$



$$\overrightarrow{QR} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \begin{pmatrix} -7 + 3 \\ 1 - 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} = (-4, -3)$$

Now, $\overrightarrow{QR} \cdot \overrightarrow{PQ} = (-4, -3) \cdot (-3, 4) = 12 - 12 = 0$

$\therefore \quad \measuredangle PQR = 90°$

Also, $|\overrightarrow{PQ}| = \sqrt{(-3)^2 + 4^2} = 5$

$|\overrightarrow{QR}| = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = 5$

$\therefore \quad |\overrightarrow{PQ}| = |\overrightarrow{QR}|$

and $\measuredangle PQR = 90°$

Hence ΔPQR is an isoceles right angled triangle.

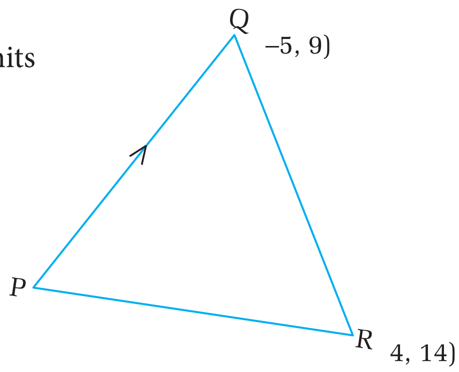
6. In ΔPQR, if $\overrightarrow{PQ} = 5\vec{i} - 9\vec{j}$ and $\overrightarrow{QR} = 4\vec{i} + 14\vec{j}$, prove that ΔPQR is an isoceles right angled triangle.

Solution

Here, $\overrightarrow{PQ} = 5\vec{i} - 9\vec{j}$, $|\overrightarrow{PQ}| = \sqrt{5^2 + (-9)^2} = \sqrt{25 + 81} = \sqrt{106}$ units

or, $\overrightarrow{QP} = -5\vec{i} + 9\vec{j}$

and $\overrightarrow{QR} = 4\vec{i} + 14\vec{j}$, $|\overrightarrow{QR}| = \sqrt{4^2 + 14^2} = \sqrt{212}$ units



Let, Q be origin O(0, 0)

$\overrightarrow{QP} = \overrightarrow{OP} = (-5, 9)$

$\overrightarrow{QR} = \overrightarrow{OR} = (4, 14)$

Now, $\overrightarrow{PR} = (4 + 5, 14 - 9) = (9, 5)$

$|\overrightarrow{PR}| = \sqrt{9^2 + 5^2} = \sqrt{106}$ units

$\overrightarrow{QP} \cdot \overrightarrow{QR} = (-5, 9) \cdot (9, 5) = -45 + 45 = 0$

$\therefore \quad \measuredangle PQR = 90°$ and $|\overrightarrow{PQ}| = |\overrightarrow{PR}|$

Hence ΔPQR is an right angled triangle. **proved**

7. a. Find the angle between $\vec{a} = 4\vec{i} - 3\vec{j}$ and x-axis.

Solution

We know that a unit vector along x-axis is $\vec{i}$.

Let, $\overrightarrow{OB} = \vec{i}$

and $\vec{a} = \overrightarrow{OA} = 4\vec{i} - 3\vec{j}$

Let, θ be the angle between $\overrightarrow{OB}$ and $\overrightarrow{OA}$.

$$\cos\theta = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}|\ |\overrightarrow{OB}|} = \frac{(4\vec{i} - 3\vec{j}) \cdot \vec{i}}{\sqrt{4^2 + (-3)^2} \cdot \sqrt{1^2}} = \frac{4}{5}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{4}{5}\right).$$

b. Find the angle between $2\vec{i} + \vec{j}$ and y-axis.

Solution

Let, unit vector along y-axis be $\overrightarrow{OB} = \vec{j}$

and given vector $\overrightarrow{OA} = 2\vec{i} + \vec{j}$

Let, θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then

$$\cos\theta = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}|\ |\overrightarrow{OB}|} = \frac{(2\vec{i} + \vec{j}) \cdot \vec{j}}{\sqrt{2^2 + 1} \cdot \sqrt{1}} = \frac{1}{\sqrt{5}}$$

$$\therefore \quad \theta = 63.43°.$$

8. Show that the angle between two vector $\vec{a}$ and $\vec{c}$ is 90°, if $\vec{a} + \vec{b} + \vec{c} = (0, 0)$.

Solution

Let, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 4$

Here, $\vec{a} + \vec{b} + \vec{c} = (0, 0)$

or, $\vec{a} + \vec{c} = -\vec{b}$

Squaring on both sides, we get,

$$(\vec{a} + \vec{c})^2 = b^2$$

or, $a^2 + 2\vec{a} \cdot \vec{c} + c^2 = b^2$

or, $9 + 2|\vec{a}|\ |\vec{c}| \cos\theta + 16 = 25$

(Where θ is the angle between $\vec{a}$ and $\vec{c}$.)

$2 \times 3 \times 4 \cos\theta = 0$

or, $\cos\theta = 0$

$\therefore$ $\cos\theta = \cos 90°$

$\theta = 90°$. **proved**

9. If $\vec{a} + \vec{b} + \vec{c} = O\ (0, 0)$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, show that the angle between $\vec{a}$ and $\vec{b}$ is 60°.

Solution

Here, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$

$\vec{a} + \vec{b} + \vec{c} = O(0, 0)$

or, $\vec{a} + \vec{b} = -\vec{c}$

Squaring on both sides, we get,

$(\vec{a} + \vec{b})^2 = (-\vec{c})^2$

or, $a^2 + 2\vec{a} \cdot \vec{b} + b^2 = c^2$

or, $9 + 2\ |\vec{a}|\ |\vec{b}| \cos\theta + 25 = 49$

Where θ is the angle between $\vec{a}$ and $\vec{b}$.

$2 \times 3 \times 5 \cos\theta = 49 - 34$

or, $\cos\theta = \frac{15}{30} = \frac{1}{2} = \cos 60°$

$\therefore$ $\theta = 60°$. proved

10. Find the angle between $\vec{a}$ and $\vec{b}$, if $\vec{a} + \vec{b} + \vec{c} = O\ (0, 0)$, $|\vec{a}| = 6$, $|\vec{b}| = 7$ and $\vec{c} = \sqrt{127}$.

Solution

Here, $|\vec{a}| = 6$, $|\vec{b}| = 7$ and $\vec{c} = \sqrt{106}$

Now, $\vec{a} + \vec{b} + \vec{c} = O\ (0, 0)$

$(\vec{a} + \vec{b}) = -\vec{c}$

Squaring on both sides, we get,

$a^2 + b^2 + 2\vec{a} \cdot \vec{b} = c^2$

or, $36 + 49 + 2|\vec{a}|\ |\vec{b}| \cos\theta = 127$

or, $2 \times 6 \times 7 \cos\theta = 42$

or, $\cos\theta = \frac{1}{2} = \cos 60°$

$\therefore$ $\theta = 60°$.

11. If $\vec{a} + \vec{b}$ and $2\vec{a} - \vec{b}$ are orthogonal vectors, $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a}$ and $\vec{b}$.

Solution

Here, $\vec{a} + \vec{b}$ and $2\vec{a} - \vec{b}$ are orthogonal vectors, $(\vec{a} + \vec{b}) \cdot (2\vec{a} - \vec{b}) = O(0, 0)$ and

$|\vec{a}| = 1, \; |\vec{b}| = 1$

Now, $(\vec{a} + \vec{b}) \cdot (2\vec{a} - \vec{b}) = 0$

or, $2a^2 + 2\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} - b^2 = 0$

or, $2 + |\vec{a}|\,|\vec{b}| \cos\theta - 1 = 0$

or, $\cos\theta = -1$

or, $\cos\theta = \cos 180°$

$\therefore$ $\theta = 180°$.

12. If $|\vec{a} - 3\vec{b}| = |\vec{a} + 3\vec{b}|$, prove that $\vec{a}$ and $\vec{b}$ are orthogonal vectors.

Solution

Here, $|\vec{a} - 3\vec{b}| = |\vec{a} + 3\vec{b}|$

Squaring on both sides, we get,

$(\vec{a} - 3\vec{b})^2 = (\vec{a} + 3\vec{b})^2$

or, $a^2 + 9b^2 - 2\vec{a} \cdot 3\vec{b} = a^2 + 9b^2 + 2\vec{a} \cdot 3\vec{b}$

or, $-6\vec{a} \cdot \vec{b} = 6\vec{a} \cdot \vec{b}$

or, $12\vec{a} \cdot \vec{b} = 0$

or, $\vec{a} \cdot \vec{b} = 0$

Hence, $\vec{a}$ and $\vec{b}$ are orthogonal vectors.

Questions for practice

1. **If $|\vec{a}| = 4$, $|\vec{b}| = 5$ and $\vec{a} \cdot \vec{b} = 10$, find the angle between $\vec{a}$ and $\vec{b}$.**
2. **If $\vec{p} + \vec{q} + \vec{r} = 0$, $|\vec{p}| = 6$, $|\vec{q}| = 7$ and $|\vec{r}| = \sqrt{127}$,** find the angle between $\vec{p}$ **and** $\vec{q}$**.**
3. **If $\vec{x} = 3\vec{i} + m\vec{j}$ and $\vec{x} = 4\vec{i} - 2\vec{j}$ are perpendicular to each other, find the value of m.**
4. **In ΔABC if $\vec{AB} = 3\vec{i} + 4\vec{j}$ and $\vec{BC} = \vec{i} - 7\vec{j}$, show that the triangle ABC is a right angled triangle.**
5. **Prove that $\vec{p} = 3\vec{i} + 5\vec{j}$, $\vec{q} = 5\vec{i} + 3\vec{j}$ and $\vec{r} = 8\vec{i} - 2\vec{j}$ form an isosceles right angled triangle.**

Vector Geometry

Estimated Periods : 13

1. Objectives

Knowledge (K)	To state triangle law of vector addition. To state mid point theorem, section formula, centroid formula.
Understanding (U)	To state vector geometry theorems : mid point theorem, section formula, centroid formula, theorems related to triangles, theorems on quadrilateral, semi-circle.
Skill/Application (S/A)	To prove vector geometry theorems and problems based on the theorems.
Higher Ability (HA)	To prove the following vector geometry problems. – the diagonals of a rectangle are equal. – the diagonals of a parm bisect each other. – the diagonals of a rhombus bisec each other at right angle. – the mid point of the hypotenuse of a right angled triangle is at equidistance from its vertices.

2. Required Teaching Materials

Chart paper with statements of vector geometry to prescribed course by CDC.

3. Teaching Learning Strategies

→ Review the triangle law, parallelogram law of vector geometry.

→ State and prove each of theorems stated as in above objectives.

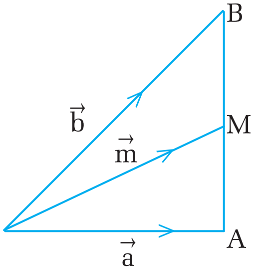
→ Show relations of each of above stated theorems with same theorems on plane geometry.

→ After proving each of above theorems let the students do the some theorems with figure labelled differnty for further practice for them.

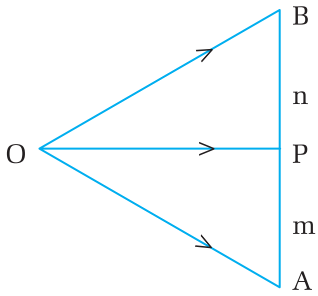
List of Formula

1. Mid point formula

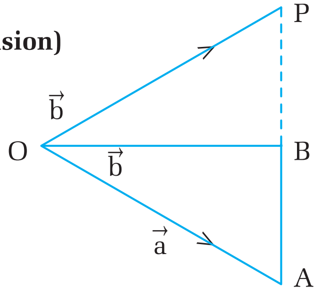
$$\vec{m} = \overrightarrow{OM} = \frac{1}{2}(\vec{a} + \vec{b})$$



2. Section formula



$$\overrightarrow{OP} = \frac{m\vec{b} + n\vec{a}}{m + n}, \text{ (internal division)}$$



3. Section formula,

$$\overrightarrow{OP} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

4. **Centroid of triangle**

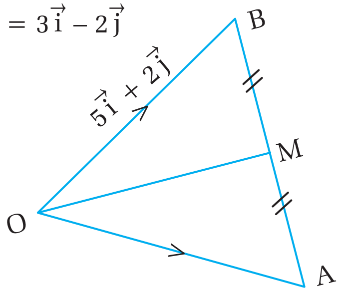
$$\vec{g} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

(Study all theorem of vector geometry from text book Vedanta Excel In opt. Mathematics-10)

Some solved problems

1. **If the position vector of the mid point of the line segment AB is $(3\vec{i} - 2\vec{j})$, where the position vector of B is $(5\vec{i} + 2\vec{j})$. Find the position vector of A.**

Solution



Let, $\overrightarrow{OM}$ be the position vector of the mid point of AB. $\overrightarrow{OM} = 3\vec{i} - 2\vec{j}$

then $\overrightarrow{OB} = 5\vec{i} + 2\vec{j}$

Now, $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

or, $2OM = \overrightarrow{OA} + \overrightarrow{OB}$

or, $\overrightarrow{OA} = 2\overrightarrow{OM} - \overrightarrow{OB}$

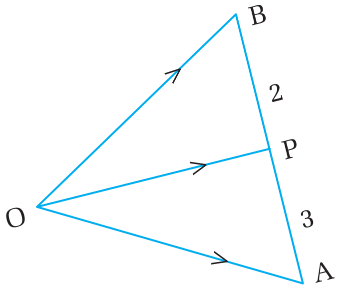
$= 2(3\vec{i} - 2\vec{j}) - (5\vec{i} + 2\vec{j})$

$= 6\vec{i} - 4\vec{j} - 5\vec{i} - 2\vec{j}$

$\therefore$ $\overrightarrow{OA} = \vec{i} - 6\vec{j}$

2. **The position vectors of the points A and B of the line segment AB are respectively $\vec{a} = 3\vec{i} + 4\vec{j}$ and $\vec{b} = \vec{i} - 2\vec{j}$. If C divides AB in the ratio of 3:2 internally, find the position vector of C.**

Solution



Here, $\overrightarrow{OA} = \vec{a} = 3\vec{i} + 4\vec{j}$

$\overrightarrow{OB} = \vec{i} - 2\vec{j}$

m:n = 3:2

i.e. m = 3, n = 2

C divides AB in ratio of 3:2 internally.

Now, $\overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m + n}$

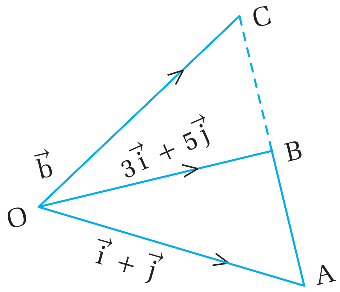
$= \frac{3(\vec{i} - 2\vec{j}) + 2(3\vec{i} + 4\vec{j})}{3 + 2}$

$= \frac{9\vec{i} + 2\vec{j}}{5}$

$= \frac{9}{5}\vec{i} + \frac{2}{5}\vec{j}$

3. The position vectors of A and B are respectively $(\vec{i} + \vec{j})$ and $(3\vec{i} + 5\vec{j})$. Find the position vector of P which divides AB in ratio of 2:1 externally.

Solution



Here, $\overrightarrow{OA} = \vec{i} + \vec{j}$

$\overrightarrow{OB} = 3\vec{i} + 5\vec{j}$

C divides AB in ratio of 2:1 externally.

i.e. $m = 2, n = 1$

then, $\overrightarrow{OC} = \dfrac{m\vec{b} - n\vec{a}}{m - n}$

$= \dfrac{2(3\vec{i} + 5\vec{j}) - 1(\vec{i} + \vec{j})}{2 - 1}$

$= \dfrac{6\vec{i} + 10\vec{j} - \vec{i} - \vec{j}}{1}$

$= 5\vec{i} + 9\vec{j}$

4. Find the position vector of centroid of ΔPQR whose position vector of vertices are respectively $(3\vec{i} + 4\vec{j})$, $(4\vec{i} + 5\vec{j})$ and $(5\vec{i} + 6\vec{j})$.

Solution

Let, $\overrightarrow{OP} = \vec{p} = 3\vec{i} + 4\vec{j}$

$\overrightarrow{OQ} = \vec{q} = 4\vec{i} + 5\vec{j}$

$\overrightarrow{OR} = \vec{r} = 5\vec{i} + 6\vec{j}$

Let, $\overrightarrow{OG} = \vec{g}$ be the position vector of the centroid of ΔPQR, then

$\overrightarrow{OG} = \vec{g} = \dfrac{\vec{p} + \vec{q} + \vec{r}}{3}$

$= \dfrac{3\vec{i} + 4\vec{j} + 4\vec{i} + 5\vec{j} + 5\vec{i} + 6\vec{j}}{3}$

$= \dfrac{12\vec{i} + 15\vec{j}}{3}$

$= 4\vec{i} + 5\vec{j}$

5. In ΔLMN, OL $= 4\vec{i} - 5\vec{j}$, $\overrightarrow{OM} = 6\vec{i} + 4\vec{j}$ and the position vector of centroid G, $\overrightarrow{OG} = 2\vec{i} + \vec{j}$. Find the $\overrightarrow{ON}$.

Solution

Let, $OL = 4\vec{i} - 5\vec{j}$

$$\overrightarrow{OM} = 6\vec{i} + 4\vec{j}$$

$$\overrightarrow{OG} = 2\vec{i} + \vec{j}$$

Using formula, $\overrightarrow{OG} = \dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$

$$\overrightarrow{OG} = \frac{\overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OL}}{3}$$

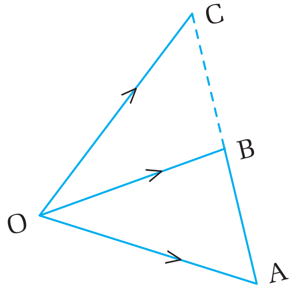
or, $3\overrightarrow{OG} = 6\vec{i} + 4\vec{j} + \overrightarrow{ON} + 4\vec{i} - 5\vec{j}$

or, $3(2\vec{i} + \vec{j}) = 10\vec{i} - \vec{j} + \overrightarrow{ON}$

$\therefore$ $\overrightarrow{ON} = -4\vec{i} + 4\vec{j}$

5. If the position vector of A and B are respectively $\vec{a}$ and $\vec{b}$. Find the position vector of C in AB produced such that $\overrightarrow{AC} = 3\overrightarrow{BC}$.

Solution



Here, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$

C is a point on AB produced and $\overrightarrow{AC} = 3\overrightarrow{BC}$

i.e. $\dfrac{\overrightarrow{AC}}{\overrightarrow{BC}} = \dfrac{1}{3}$

It means that C divides AB in ratio of 3:1 externally.

Hence, $\overrightarrow{OC} = \dfrac{m\vec{b} - n\vec{a}}{m - n}$

Where, m = 3, n = 1

$$\overrightarrow{OC} = \frac{3\vec{b} - \vec{a}}{3 - 1} = \frac{3\vec{b} - \vec{a}}{2}$$

Alernatively,

Here, $\overrightarrow{AC} = 3\overrightarrow{BC}$

or, $\overrightarrow{OC} - \overrightarrow{OA} = 3\,(\overrightarrow{OC} - \overrightarrow{OB})$

or, $\overrightarrow{OC} - \vec{a} = 3\overrightarrow{OC} - 3\vec{b}$

or, $3\vec{b} - \vec{a} = 2\overrightarrow{OC}$

$\therefore$ $\overrightarrow{OC} = \dfrac{3\vec{b} - \vec{a}}{2}$.

$= \vec{c}$, find the vector $\overrightarrow{PQ}$ and show that $\overrightarrow{PQ} // \overrightarrow{OB}$.

Solution

Here, OABC is a parallelogram, CP:PO = CQ:QB = 1:3

$$\overrightarrow{OA} = \vec{a},\ \overrightarrow{OC} = \vec{c}$$

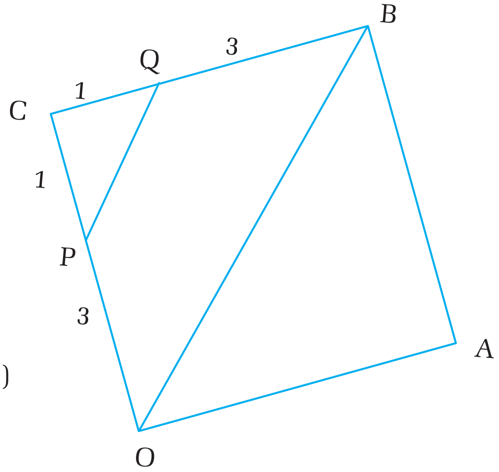
or, $\overrightarrow{CQ} = \frac{1}{4}\overrightarrow{CB},\ \overrightarrow{PC} = \frac{1}{4}\overrightarrow{OC}$

Now, $\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{CQ}$

$= \frac{1}{4}\overrightarrow{OC} + \frac{1}{4}\overrightarrow{CB}$

$= \frac{1}{4}(\overrightarrow{OC} + \overrightarrow{CB})$

$= \frac{1}{4}\overrightarrow{OB} \quad (\because \overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB})$



$\therefore \overrightarrow{PQ} = \frac{1}{4}\overrightarrow{OB}$

and $\overrightarrow{PQ} // \overrightarrow{OB}$. **proved**

7. In the given figure, $\frac{|\overrightarrow{CB}|}{|\overrightarrow{CD}|} = \frac{3}{2}$, then show that : $2\overrightarrow{AB} + \overrightarrow{AC} = 3\overrightarrow{AD}$.

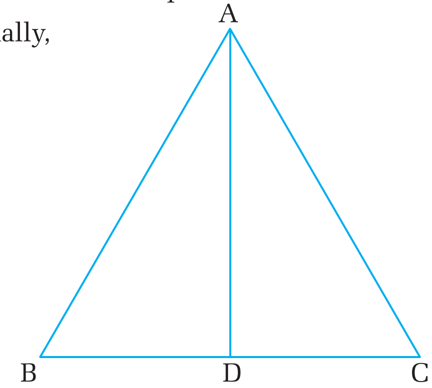
Solution

Here, CB:CD = 3:2

It shows that BC = 3 parts, CD = 2 parts and BD = 3 – 2 = 1 part

It means that D divides BC in ratio of 1:2 internally,

$m = 1, n = 2$



Now, $AD = \frac{m\vec{b} + n\vec{a}}{2 + 1}$

where, $\vec{b} = \overrightarrow{AC},\ \vec{a} = \overrightarrow{AB}$

or, $\overrightarrow{AD} = \frac{1 . \overrightarrow{AC} + 2 . \overrightarrow{AB}}{3}$

$\therefore \overrightarrow{AD} = 2\overrightarrow{AB} + \overrightarrow{AC}$. **Proved**

8. In ΔABC, the medians AD, BE and CF are drawn from the vertices A, B and C respectively. G is centroid, then prove that:

i) $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = O$

ii) $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = O$

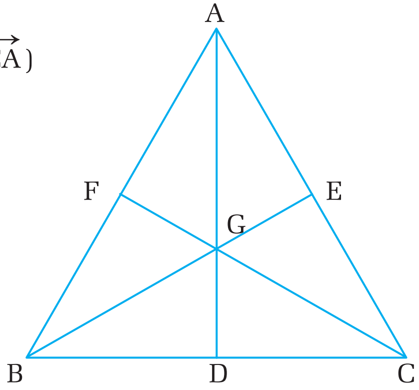
Solution

i) To prove: $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = O$

By using mid point theorem, we get

$\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}),\ \overrightarrow{BE} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{BC}),\ \overrightarrow{CF} = \frac{1}{2}(\overrightarrow{CB} + \overrightarrow{CA})$

LHS $= \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$



$= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{BC}) + \frac{1}{2}(\overrightarrow{CB} + \overrightarrow{CA})$

$= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{CB} + \overrightarrow{CA})$

$= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CA} + \overrightarrow{BC} + \overrightarrow{CB})$

$(\because \overrightarrow{AB} = -\overrightarrow{BA})$

$= \frac{1}{2} . 0$

$= 0$

ii) To prove: $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = O(0, 0)$

Since, AD, BE and CF are the medians of triangle ABC and G is the point of intersection of medians. i.e. centroid. G divides the medians in ratio of 2:1.

LHS $= \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}$

$= \left(\frac{2}{3}\overrightarrow{AD} + \frac{2}{3}\overrightarrow{BE} + \frac{2}{3}\overrightarrow{CF}\right)$

$= \frac{2}{3}(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{CB} + \overrightarrow{CA})$

$= \frac{2}{3} . 0$ (as in (i))

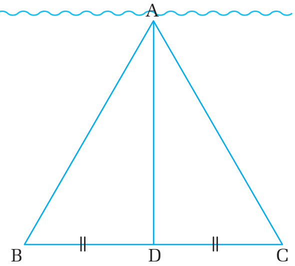
$= 0$ **Proved**

9. In the figure, ΔABC is an isoceles triangle AD is median, then show that : $\overrightarrow{AD} . \overrightarrow{BC} = 0$.

or

In an isoceles triangle, the median drawn from the vertex to the base is perpendicular to the base.

Solution



Here, AD is the median from vertex A to the base BC of ΔABC.

ΔABC is an isoceles triangle i.e. AB = AC

Now, by mid point theorem,

$\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$

Also, $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} + \overrightarrow{AC}$

Taking the dot product of $\overrightarrow{AD}$ and $\overrightarrow{BC}$, we get

$$\overrightarrow{AD} \cdot \overrightarrow{BC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) \cdot (-\overrightarrow{AB} + \overrightarrow{AC})$$

$$= \frac{1}{2}(AC^2 - AB^2)$$

$$= \frac{1}{2} \cdot 0 \qquad \because \ |\overrightarrow{AC}| = |\overrightarrow{AB}|$$

$\therefore \quad \overrightarrow{AD} \cdot \overrightarrow{BC} = 0$

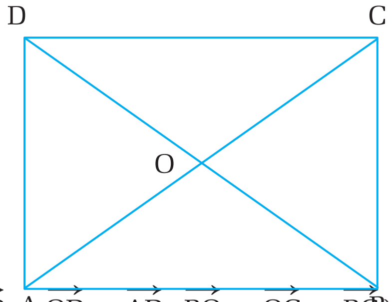
As the dot product of $\overrightarrow{AD}$ and $\overrightarrow{BC}$ is zero, AD is perpendicular to BC. **Proved**

10. If the diagonals of a quadrilateral bisect each other, prove by vector method that it is a parallelogram.

Solution

Let, ABCD is a quadrilateral in which the diagonals AC and BD bisect at O.

Then, we can write,



$\overrightarrow{AO} = \overrightarrow{OC}$ ... ... (i)

$\overrightarrow{OD} = \overrightarrow{BO}$ ... ... (ii)

Adding (i) and (ii), we get

$\overrightarrow{AO} + \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{BO}$

or, $\overrightarrow{AD} = \overrightarrow{BC}$, By using triangle law of vector addition $(\overrightarrow{AO} + \overrightarrow{OD} = \overrightarrow{AD},\ \overrightarrow{BO} + \overrightarrow{OC} = \overrightarrow{BC})$

$\therefore \quad \overrightarrow{AD} /\!/ \overrightarrow{BC}$.

Also, we can show that $\overrightarrow{DC} = \overrightarrow{AB}$ and $\overrightarrow{DC} /\!/ \overrightarrow{AB}$ as above.

Hence, ABCD is a parallelogarm. **Proved**

11. If a line is drawn from the centre of a circle to the mid point of a chord, prove by vector method that the line is perpendicular to the chord.

Solution

Let, O be the centre of the circle and PQ be a chord.

M is the mid point of PQ.

Join OP and OQ.

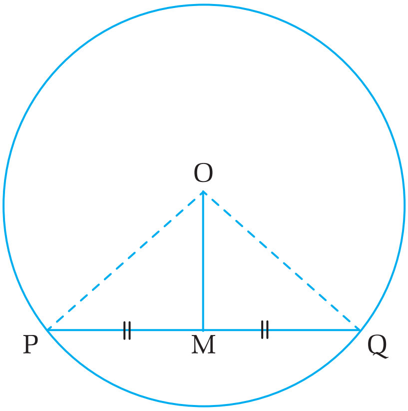
1. Since M is the mid point of PQ. We have by mid point theorem.

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ})$$

2. By using triangle law of vector addition

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$

$$= (-\overrightarrow{OP}) + \overrightarrow{OQ}$$



3. Taking the dot product of $\overrightarrow{OM}$ and $\overrightarrow{PQ}$, we get,

$$\overrightarrow{OM} \cdot \overrightarrow{PQ} = \frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ}) \cdot (-\overrightarrow{OP} + \overrightarrow{OQ})$$

$$= \frac{1}{2}(OQ^2 - OP^2)$$

$\therefore$ $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$, radii of same circle

$$= \frac{1}{2} \cdot 0$$

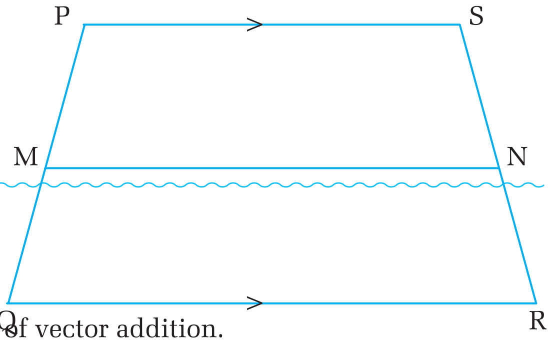
$\therefore$ $\overrightarrow{OM} \cdot \overrightarrow{PQ} = 0$

Hence, OM is perpendicular to PQ. **Proved**

12. In the given figure, PQRS is a trapezium where PS//QR. M and N are the mid points of $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ respectively. Prove vectorically that:

i) $\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{PS} + \overrightarrow{QR})$

ii) $\overrightarrow{MN} // \overrightarrow{QR}$



Solution

Here, In the figure PQRS is a trapezium.

M and N are the mid points of PQ and SR.

1. $\overrightarrow{MN} = \overrightarrow{MP} + \overrightarrow{PS} + \overrightarrow{SN}$, by polygon law of vector addition.

2. $\overrightarrow{MN} = \overrightarrow{MQ} + \overrightarrow{QR} + \overrightarrow{RN}$

3. $2\overrightarrow{MN} = \overrightarrow{MP} + \overrightarrow{PS} + \overrightarrow{SN} + \overrightarrow{MQ} + \overrightarrow{QR} + \overrightarrow{RN}$ (adding (1) and (2)

$$= (\overrightarrow{MP} + \overrightarrow{MQ}) + (\overrightarrow{SN} + \overrightarrow{RN}) + (\overrightarrow{PS} + \overrightarrow{QR})$$

$$= 0 + 0 + \overrightarrow{PS} + \overrightarrow{QR} \qquad (\because \overrightarrow{MP} = -\overrightarrow{MQ}, \overrightarrow{SN} = -\overrightarrow{RN})$$

$$= \overrightarrow{PS} + \overrightarrow{QR}$$

$\therefore$ $\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{PS} + \overrightarrow{QR})$

4. Let, $\overrightarrow{PS} = k\overrightarrow{QR}$, where k is a scalar, $\overrightarrow{PS} // \overrightarrow{QR}$

$\therefore$ $\overrightarrow{MN} = \frac{1}{2}(k\overrightarrow{QR} + \overrightarrow{QR})$

$$= \frac{1}{2}(k + 1) \cdot \overrightarrow{QR}$$

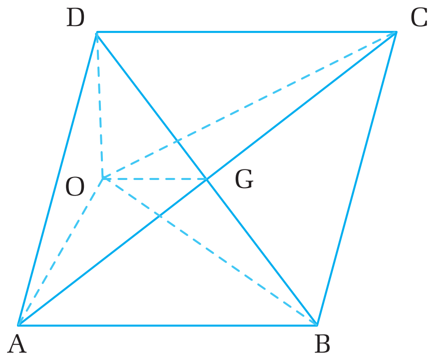
$\therefore \quad \overrightarrow{MN} // \overrightarrow{QR}$

Also, $\overrightarrow{MN} // \overrightarrow{QR}$. **Proved**

13. In the adjoining figure, PQRS is a parallelogram. G is the point of intersection of the diagonals. If O is any point, prove that : $\overrightarrow{OG} = \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD})$.

Solution

Here, In the given figure, PQRS is a parallelogram and G is the point of intersection of diagonals AC and BD.



1. by using mid point theorem, we get

i) $\overrightarrow{OG} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OB})$, G being mid point of BD.

or, $2\overrightarrow{OG} = \overrightarrow{OD} + \overrightarrow{OD}$

ii) $\overrightarrow{OG} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$, G being mid point AC.

or, $2\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{OC}$

2. Adding (i) and (ii) of (1), we get

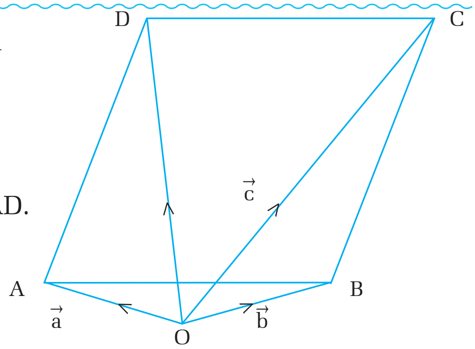
$$4\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OB}$$

$\therefore \quad \overrightarrow{OG} = \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD})$. Proved

14. ABCD is a parallelogram and O is the origin. If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$, find $\overrightarrow{OD}$ is terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Solution

Here, In the given figure, ABCD is a parallelogram



$$\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b} \text{ and } \overrightarrow{OC} = \vec{c}$$

1. $\overrightarrow{AD} = \overrightarrow{BC}$, opposite sides of a parm.
2. By using triangle law of vector addition in ΔOAD.

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$

$$= \overrightarrow{OA} + \overrightarrow{BC} \quad (\because \text{ using (1)})$$

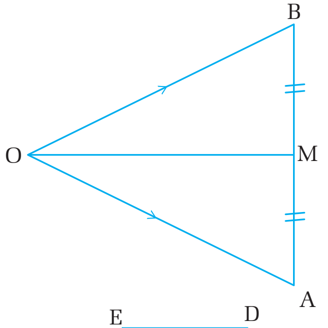
$$= \vec{a} + (\overrightarrow{OC} - \overrightarrow{OB})$$

$$= \vec{a} + \vec{c} - \vec{b}$$

$$= \vec{a} - \vec{b} + \vec{c}.$$

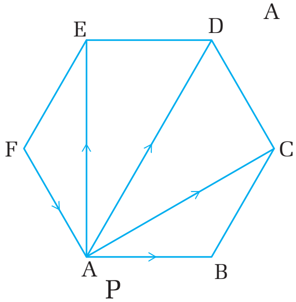
Questions for practice

1. If $(3\vec{i} + 6\vec{j})$ and $(5\vec{i} + 2\vec{j})$ are the position of vectors of the points P and Q respectively, find the position vector of M which divides PQ internally in the ratio of 2:3.
2. In the given figure, PQR = 90°, prove vectorically that $PR^2 = PQ^2 + QR^2$
3. If M is the mid point of AB with $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Find the position vector of M.

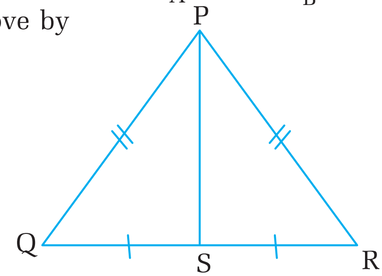


4. In the given figure ABCDEF is a regular hexagon prove that

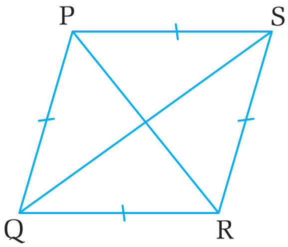
$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{EA} + \overrightarrow{FA} = 4\overrightarrow{AB}$



5. In the given figure PQ = PR and QS = RS, then prove by vector method that PS is perpendicular to QR.



6. In the given figure PQ = QR = RS = SP, prove by vector method PR is perpendicular to QS.



UNIT eleven Transformation

Combined Translation

Estimated Teaching Periods : 3

1. Objectives

Level	Objectives
Knowledge (K)	To define combined geometric transformations. To define combined translations. To tell meaning of T_1oT_2 and T_2oT_1.
Understanding (U)	To explain combined transformation, as a composite function. To link combined translations with combined transformations.
Skill/Application (S/A)	To solve problems on combined translations.
Higher Ability (HA)	To find images of given geometric objects by drawing method and using coordinates due combined translations.

2. Required Teaching Materials

List of formula of fundamental geometrical transformation as review on a chart paper, graph papers.

3. Teaching Learning Strategies

→ Ask meaning of transformation -as a review concept.

→ Define transformation as a function.

→ Review meaning of (gof) (x) an (fog) (x) and link them with combined transformation with an example.

→ Explain meaning of T_1oT_2 and T_2oT_1 with examples.

→ If $T_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ and $T_2 = \begin{pmatrix} c \\ d \end{pmatrix}$, then $T_1oT_2 = \begin{pmatrix} c + a \\ d + b \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix} = T_2oT_1$

→ Discuss how to draw image of a triangle ABC, due to translations $T_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ followed by $T_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Where A(–0, 0), B(–2, 4) and C $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ by using graph.

Review List of Formula

(From text book page 299 and 302)

Combined translation:

a. Translation vector $T = \begin{pmatrix} a \\ b \end{pmatrix}$ = image coordinator – object coordinates

b. If $T_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ and $T_2 = \begin{pmatrix} c \\ d \end{pmatrix}$, then

$T_1oT_2 = T_1 + T_2 = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$

c. If $T_1 = \begin{pmatrix} a \\ b \end{pmatrix}$, then $T_1^2 = T_1 + T_1 = \begin{pmatrix} a + a \\ b + b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix}$

d. In combined translation, $T_1 + T_2 = T_2 + T_1$

Some solved problems

1. Let $T_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ be two transformations, find the image of points A(4, 5) and B(–6, 7) under combined transformations T_1oT_2. Show them in graph.

Solution

Here, $T_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

T_1oT_2 means the translation T_2 followed by T_1.

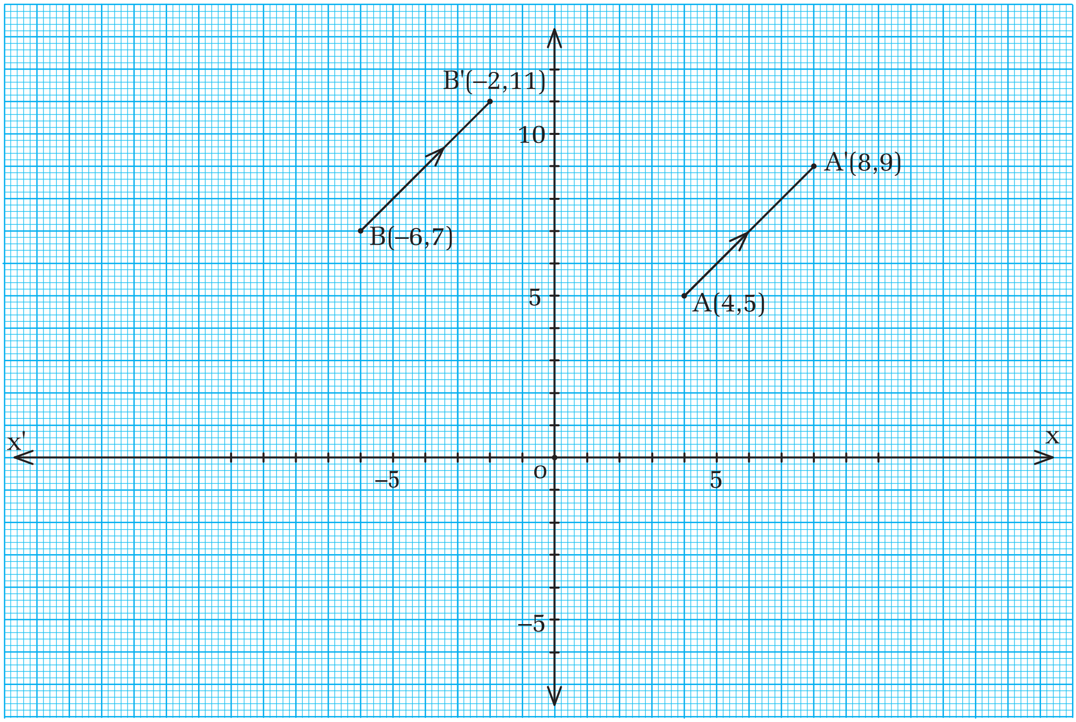
Using combined translation formula

Let, $T = T_1oT_2 = T_1 + T_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

Now,

$A(4, 5) \xrightarrow{T = \begin{pmatrix} 4 \\ 4 \end{pmatrix}} A'(4 + 4, 4 + 5) = A'(8, 9)$

$B(-6, 7) \xrightarrow{T = \begin{pmatrix} 4 \\ 4 \end{pmatrix}} B'(-6 + 4, 7 + 4) = B'(-2, 11)$



Alternate Solution

Here, $T_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

T_1oT_2 means T_2 is followed by T_1. It means the given points are firstly translated by T_2 and the image so obtained is translated by T_1.

Now,

$$A(4, 5) \xrightarrow{T_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}} A'(4 + 1, 5 + 2) = A'(5, 7)$$

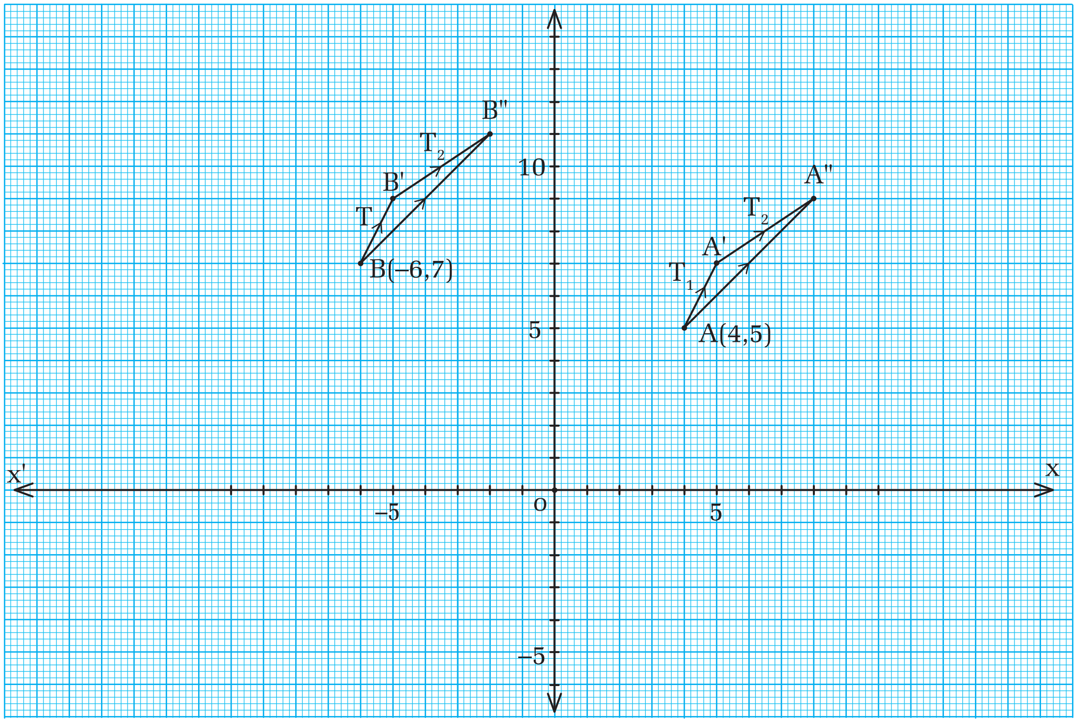
$$B(-6, 7) \xrightarrow{T_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}} B'(-6 + 1, 7 + 2) = B'(-5, 9)$$

Again,

$$A'(5, 7) \xrightarrow{T_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}} A''(5 + 3, 7 + 2) = A''(8, 9)$$

$$B'(-5, 9) \xrightarrow{T_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}} B''(-5 + 3, 9 + 2) = B''(-2, 11)$$

The object points and their images are shown in the graph.



2. If $T_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $T_2 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and (T_1oT_2) (x, y) = (8, 8), find the values of x and y.

Solution

Here, $T_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $T_2 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

then, $T_1oT_2 = T_1 + T_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

Let, given point be P(x, y)

Now,

$$P(x, y) \xrightarrow{\begin{pmatrix} -1 \\ 6 \end{pmatrix}} P'(x - 1, y + 6)$$

But P'(8, 8)

$\therefore$ (8, 8) = (x – 1, y + 6)

Equating the corresponding elements, we get,

8 = x – 1 or, x = 2

and 8 = y + 6 or. y = 2

$\therefore$ x = 9, y = 2.

3. If T_1 and T_2 are two translations defined by $T_1(x, y) = (x + 6, y - 3)$ and $T_2 = (x, y) = (x - 3, y + 4)$. Find the image of P(1, 3) under T_2oT_1.

Solution

Here, $T_1(x, y) = (x + 6, y - 3)$

i.e. $(x, y) \longrightarrow (x + 6, y - 3)$

$\therefore \quad T_1 = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$

and $T_2 = (x, y) = (x - 3, y + 4)$

i.e. $(x, y) \longrightarrow (x - 3, y + 4)$

$\therefore \quad T_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$\therefore \quad T_1oT_2 = T_1 + T_2 = \begin{pmatrix} 6 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Let, $T = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Hence, P(1, 3) is translated by $T = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$P(1, 3) \xrightarrow{T = \begin{pmatrix} 3 \\ 1 \end{pmatrix}} P'(3 + 1, 3 + 1) \quad = P'(4, 4).$

4. If T_1 and T_2 are two translations defined by $T_1(x, y) = (x + 3, y + 2)$ and $T_2\,(x, y) = (x - 4, y - 7)$. Find $T_2oT_1(x, y)$ and $T_2oT_1(x, y)$. Also find the images of A(1, 2), B(7, 2) and C(4, 7) of ΔABC under the composite transformation T_1oT_2 and T_2oT_1. Are images due to T_1oT_2 and T_2oT_1 same ? Show ΔABC and its image on the same graph paper.

Solution

Here, $T_1(x, y) = (x + 3, y + 2) \Rightarrow T_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

and $T_2 = (x, y) = (x - 4, y - 7) \Rightarrow T_2 = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$

Now,

$T_2oT_1 = T_2 + T_1 = \begin{pmatrix} -4 \\ -7 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

Again, $T_1oT_2 = T_1 + T_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -4 \\ -7 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

This shows that translation vector due to T_1oT_2 and T_2oT_1 is same $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$

Let, $T_1 = T_1oT_2 = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

Hence, we translate triangle ABC by translation $T = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$.

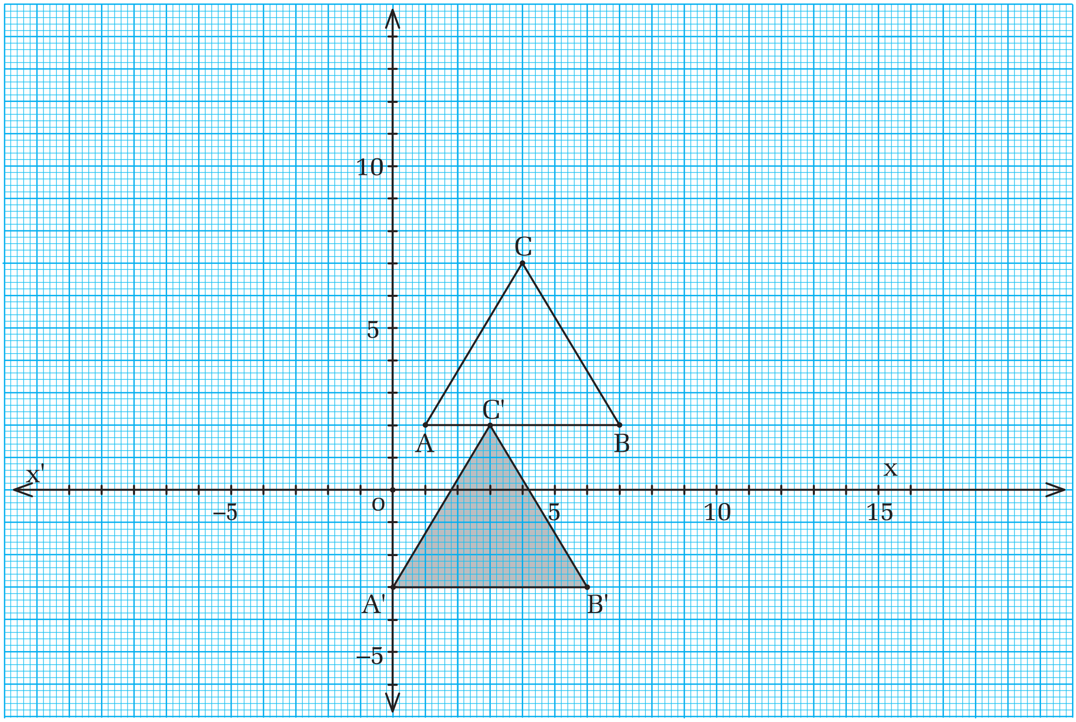
i.e. image formed due to T_1oT_2 and T_2oT_1 is same.

$A(1, 2) \xrightarrow{T = \begin{pmatrix} -1 \\ -5 \end{pmatrix}} A'(1 - 1, 2 - 5) \quad = A'(0, -3).$

$$B(7, 2) \xrightarrow{T = \begin{pmatrix}-1\\-5\end{pmatrix}} B'(7-1, 2-5) = B'(6, -3).$$

$$C(4, 7) \xrightarrow{T = \begin{pmatrix}-1\\-5\end{pmatrix}} C'(4-1, 7-5) = C'(3, 2).$$

The object triangle ABC and image triangle A'B'C' are plotted in the graph. The image triangle is shaded.



Questions for practice

1. Let $T_1 = \begin{pmatrix}2\\3\end{pmatrix}$ and $T_2 = \begin{pmatrix}-5\\-6\end{pmatrix}$, find the image of P(5, 8) due to combined translation T_2oT_1.

2. If $T_1 = \begin{pmatrix}4\\5\end{pmatrix}$ and $T_2 = \begin{pmatrix}-6\\-8\end{pmatrix}$ and T_1oT_2 (x, y) = (15, 16), find the values of x and y.

3. If T_1 and T_2 are two translations defined by $T_1(x, y) = (x + 5, y + 2)$ and $T_2(x, y) = (x - 7, y - 8)$, find the image of M(6, 7) due to T_2oT_1.

4. If T_1 and T_2 are two translations defined by $T_1(x, y) = (x + 6, y - 3)$ and $T_2(x, y) = (x - 3, y + 3)$, find the image of ΔABC with vertices A(1, 3), B(4, 1) and C(8, 4) under the translation T_2 followed by T_1. Present the object triangle and its image on the same graph. The images formed due to T_2oT_1 and T_1oT_2 same ?

Combinations of Rotation and Rotation

Notes:

1) **A rotation $\alpha°$ followed by another rotation $\beta°$ about the same centre is equivalent to the rotation of $(\alpha° + \beta°)$ about the same centre.**

Some solved problems

1. **Let P(2, 3) be a point and R_1 = rotation through +90° about origin. R_2 = rotation through 180° about origin. Then find the image point under the following transformation.**

   (i) $R_2 o R_1$ **(ii) $R_1 o R_2$**

Solution

Here, R_1 and R_2 are the rotations about the same centre origin.

Combined angle of rotation = 90° + 180° = 270° about origin.

$P(x, y) \xrightarrow{R[0, +270°]} P'(y, -x).$

$P(2, 3) \xrightarrow{R[0, +270°]} P'(3, -2).$

Alternate Method

Here, $P(2, 3) \xrightarrow{R_1[0, +90°]} P'(-3, 2)$

Again, $P'(-3, 2) \xrightarrow{R_2[0, 180°]} P''(3, -2)$

$\therefore$ **Required image due to $R_2 o R_1$ is P''(3, –2)**

(ii) $R_1 o R_2$

Solution

Here, R_1 and R_2 are the rotations about the same centre origin.

Combined angle of rotation = 180° + 90° = 270° about origin.

$P(x, y) \xrightarrow{R[0, 270°]} P'(y, -x).$

Now, $P(2, 3) \xrightarrow{R[0, 270°]} P'(3, -2).$

2. **Let T = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and R = rotation of +90° about 0. Then find the images of given points.**

   (i) ToR(2, 7) **(ii) RoT(–2, –3)**

Solution

i) To find ToR(2, 7)

Here, Let given point be P(2, 7)

Now, P' = ToR(2, 7) = T(–7, 2) = (–7 + 2, 2 + 3) = (–5, 5)

Alternate Method

Let the given point be P(2, 7)

ToR means the first transformation is rotation and then it is again translated by T.

Now, $\quad$ **P(2, 7)** $\xrightarrow{R[0,\ +90°]}$ **P'(–7, 2).**

Again, $\quad$ P'(–7, 2) $\xrightarrow{T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$ P''(–7 + 2, 2 + 3) $\quad$ = P''(–5, 5).

ii) RoT

It means the first transformation is translation T and then it is rotated.

P'(2, 7) $\xrightarrow{T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$ P'(2 + 2, 7 + 3) $\quad$ = P'(4, 10).

Again, $\quad$ P'(4, 10) $\xrightarrow{R[0,\ +90°]}$ P''(–10, 4).

Note: RoT ≠ ToR

3. Points A(5, 2), B(4, 5) and C(8, 4) are the vertices of ΔABC. Find the image of the vertices of ΔA'B'C' under the combination of –180° followed by +90° about the centre origin. Draw ΔABC and its image in the same graph paper. Write a single transformation of combined transformation.

Solution

Here, A(5, 2), B(4, 5) and C(8, 4) are the vertices of ΔABC.

First the ΔABC is rotated through –180° about origin.

We have, P(x, y) $\xrightarrow{R[0,\ -180°]}$ P'(–x, –y).

Now, $\quad$ A(5, 2) $\longrightarrow$ A'(–5, –2)

B(4, 5) $\longrightarrow$ B'(–4, –5)

C(8, 4) $\longrightarrow$ C'(–8, –4)

Again, ΔA'B'C' is rotated through +90° about origin.

We have, P(x, y) $\xrightarrow{R[0,\ +90°]}$ P'(–y, x).

Now, $\quad$ A'(–5, –2) $\longrightarrow$ A''(2, –5)

$$B'(-4, -5) \longrightarrow B''(5, -4)$$

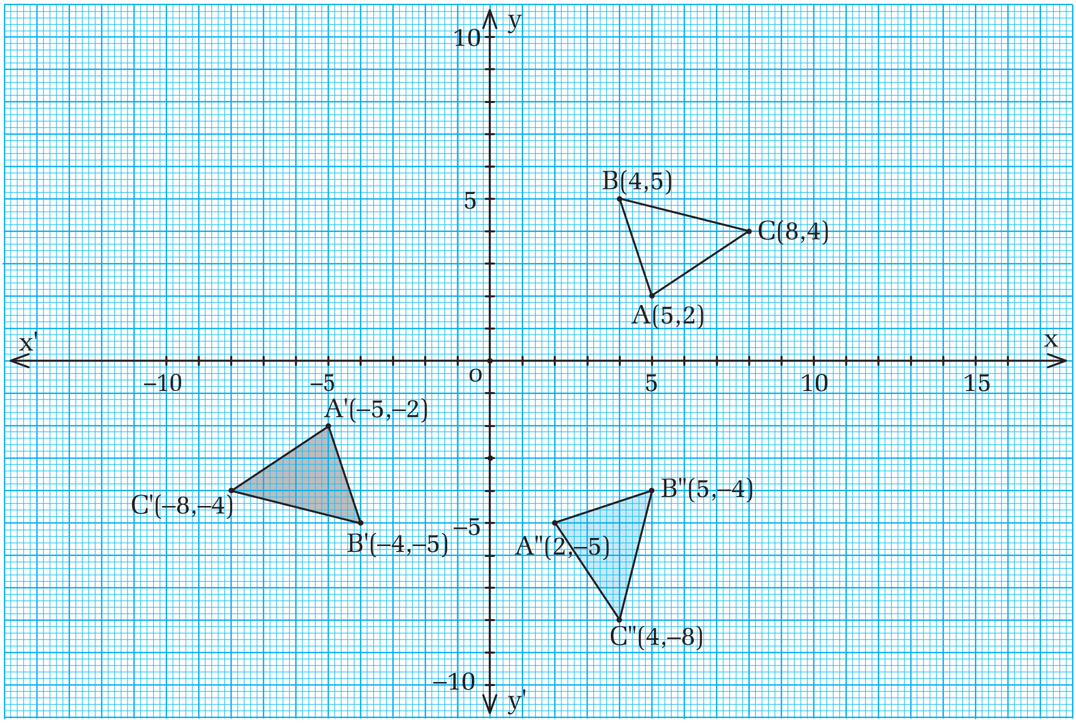
$$C'(-8, -4) \longrightarrow C''(4, -8)$$

Since, both of the rotations are about the same centre origin,

the combined rotation is $(-180° + 90°) = -90°$

about the origin i.e. $R[(0, 0), -90°]$

ΔABC, $\Delta A'B'C'$ are plotted on the graph given below.



4. Draw a triangle with vertices A(1, 2), B(–2, 3) and C(2, 4) and its images on a graph paper.

a) Find the image ΔA'B'C' under rotation of –90° about origin.

b) Find the image ΔA"B"C" of ΔA'B'C' under rotation through 180° about (2, 2)

c) Find the single transformation of above transformation.

Solution

a) The vertices of ΔABC are A(1, 2), B(–2, 3) and C(2, 4)

When ΔABC is rotated through –90° about origin, its image vertices are given below.

$$P(x, y) \xrightarrow{R[(0,\ 0),\ -90°]} P'(y, -x).$$

Now, A(1, 2) $\longrightarrow$ A'(2, –1)
B(–2, 3) $\longrightarrow$ B'(3, 2)
C(2, 4) $\longrightarrow$ C'(4, –2)

b) Again, ΔA'B'C' is rotated through 180° about (2, 2) its image is ΔA"B"C".

$P(x, y) \xrightarrow{R[(a, b), +180^\circ]} P'(2a - x, 2b - y)$.

$A'(2, -1) \xrightarrow{R[(2, 2), 180^\circ]} A''(2\times2 - 2, 2\times2 + 1) = A''(2, 5)$

$B'(3, 2) \xrightarrow{R[(2, 2), 180^\circ]} B''(2\times2 - 3, 2\times2 - 2) = B''(1, 2)$

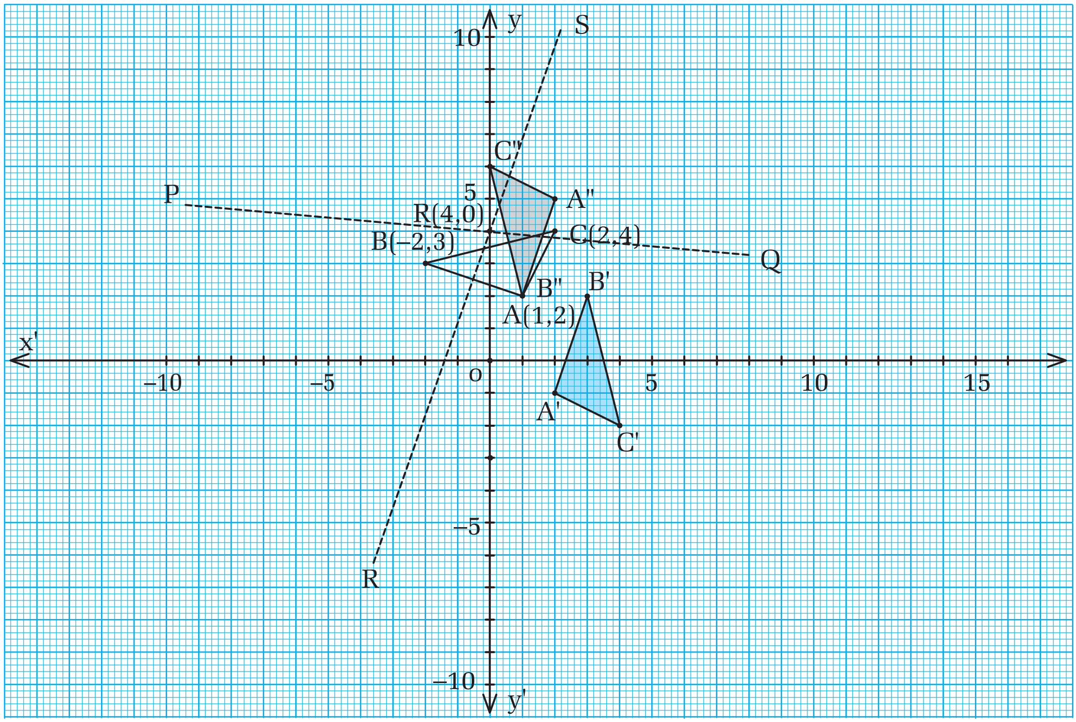
$C'(4, -2) \xrightarrow{R[(2, 2), 180^\circ]} C''(2\times2 - 4, 2\times2 + 2) = C''(0, 6)$

All three triangles ΔABC, ΔA'B'C' and ΔA"B"C" are plotted on the same graph.

c) To find the single transformation of above transformation join AA", BB" and CC" and draw their perpendicular bisectors.

We get the new centre R(0, 4) as shown in the graph.

It is observed that the single transformation of above transformation is a rotation of 90° about centre R(0, 4).



5. ΔPQR having vertices P(–2, 2), Q(2, 2) and R(6, 6) is rotated through +90° about origin. The image ΔP'Q'R' so formed is translated by T = $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the vertices of ΔP'Q'R' and

ΔP"Q"R" wiht coordinates. Draw graph of ΔPQR, ΔP'Q'R' and ΔP"Q"R" on the same graph.

Solution

We have, $P(x, y) \xrightarrow{R[0,\ +90°]} P'(-y, x)$.

$P(-2, 2) \xrightarrow{R[0,\ +90°]} P'(-2, -2)$.

$Q(2, 2) \longrightarrow Q'(-2, 2)$

$R(6, 6) \longrightarrow R'(-6, 6)$

Again, ΔP'Q'R' is translated by $T = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, we get,

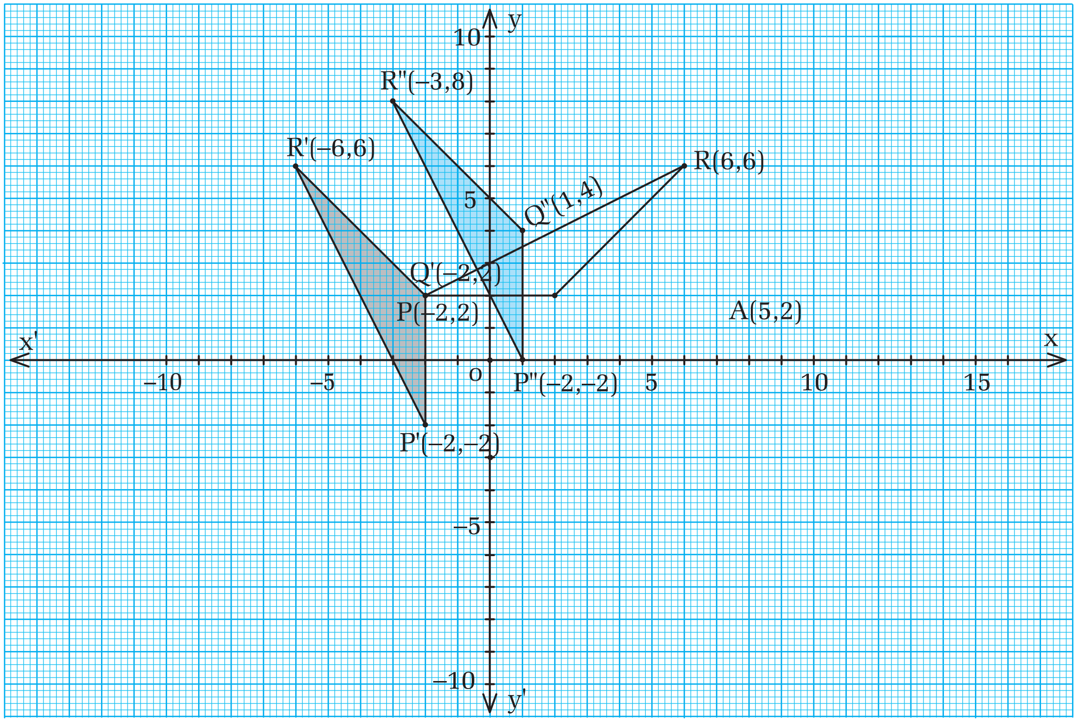
$P(x, y) \xrightarrow{T = \begin{pmatrix} a \\ b \end{pmatrix}} P'(x + a, y + b)$

$P'(-2, -2) \xrightarrow{T = \begin{pmatrix} 3 \\ 2 \end{pmatrix}} P''(-2 + 3, -2 + 2) \quad = P''(1, 0)$

$Q'(-2, 2) \longrightarrow Q''(-2 + 3, 2 + 2) \quad = Q''(1, 4)$

$R'(-6, 6) \longrightarrow R''(-6 + 3, 6 + 2) \quad = R''(-3, 8)$

ΔPQR, ΔP'Q'R' and ΔP"Q"R" are plotted on the same graph given below.



Questions for practice

1. P(4,5) is rotated through +90° about to origin O and the image so obtained is rotated through +180° about the same centre. Find the final image of P.
2. P(2,4), Q(–2,4) and R(0,–4) are the vertices of ΔPQR. ΔPQR is rotated through +90° about origin. The image triangle is again rotated through +180° about the same centre. Find the coordinates of the final image obtained. Plot the triangle and its images on the same graph.
3. P(2,1), Q(5,3) and R(7,–1) are vertices of ΔPQR. ΔPQR is rotated through +180° about origin and again it is translated by $T = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Stating the coordinates of the images, plot them in the same graph.

Combinations of Reflection and Reflection

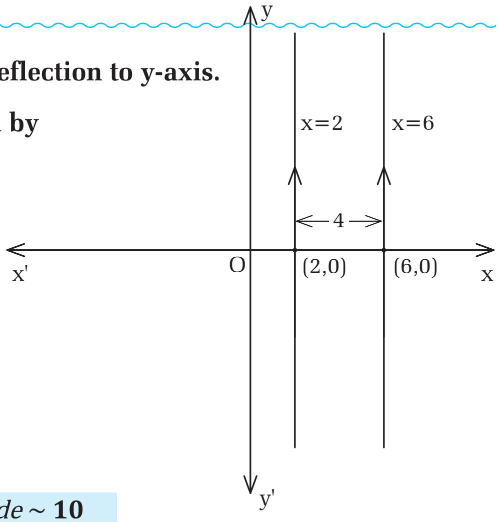
Notes:

1. **When the axes of reflections are parallel, a reflection followed by another reflection is equivalent to the translation. Translation is two times the displacement between the axes of reflectin and the direction is perpendicular to the axes of reflection.**
2. **When the axes of reflections intersect at a point, then the combined reflection is equivalent to a rotation about the centre, the point of intersection of axes; the angle of rotation is the angle between the axes.**

Some solved problems

1. Write the combined reflection in x = 2 followed by x = 6.

Solution



Here, x = 2 and x = 6 are the parallel axes of reflection to y-axis.

Hence, combined reflection is a translation given by

$$T = 2\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

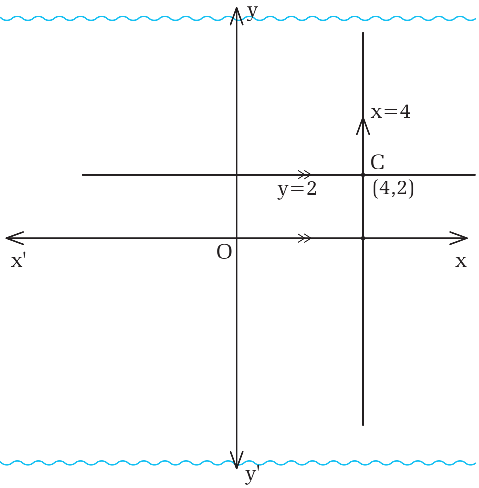
$$= 2\begin{pmatrix} 6-2 \\ 0-0 \end{pmatrix}$$

$$= 2\begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

2. Write the combined reflection in x = 4 followed by reflection in y = 2.

Solution



Here, the lines x = 4 and y = 2.

cut each other at (4, 2). The angle between them is 90°.

Hence the combined reflection is a rotation of 2 × 90° = 180° about centre (4, 2).

3. Let, R_x = reflection in x-axis, R_y = reflection in y-axis, R_1 = reflection in y = x, R_2 = reflection in y=–x and T = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, translation. Find the image of points under the following combined transformation.

Solution

a) $R_x o R_y(2, 3)$

Here, $R_x o R_y$, means the point (2, 3) is firstly reflected on y-axis and then it is reflected on x-axis.

$R_x o R_y(2, 3) = R_x(R_y(2, 3)) = R_x(-2, 3) = (-2, -3)$

∴ (–2, –3) is the required image.

b) $R_y o R_x(2, 3)$

Here, $R_y o R_x$, means the point (2, 3) is firstly reflected on x-axis and then it is reflected on y-axis.

Now, $R_y o R_x(2, 3) = R_y(R_x(2, 3)) = R_y(2, -3) = (-2, -3)$

∴ (–2, –3) is the required image.

Note:

Since x-axis and y-axis cut at origin and angle between them is 90°, combined rotation is of 180° about origin.

3. A(–4, 0), B(–6, 2) and C(–4, 3) are the vertices of ΔABC. The triangle ABC is reflected successively on the line x = –3 and x = 1.Find the final image and describe the single transformation equivalent to the combination of these transformations.

Solution

Here, A(–4, 0), B(–6, 2) and C(–4, 3) are the vertices of ΔABC.

ΔABC is reflected on line x = –3

$$P(x, y) \xrightarrow{x = h} P'(2h - x, y).$$

$A(-4, 0) \xrightarrow{x = -3} A'(2\times-3 + 4, 0) = A'(-2, 0)$

$B(-6, 2) \xrightarrow{x = -3} B'(2\times-3 + 6, 2) = B'(0, 2)$

$C(-4, 3) \xrightarrow{x = -3} C'(2\times-3 + 4, 3) = C'(-2, 3)$

Again, $\Delta A'B'C'$ is reflected on line $x = 1$

$A'(-2, 0) \xrightarrow{x = 1} A''(2\times1 + 2, 0) = A''(4, 0)$

$B'(0, 2) \xrightarrow{x = 1} B''(2\times1 - 0, 2) = B''(2, 2)$

$C'(-2, 3) \xrightarrow{x = 1} C''(2\times1 + 2, 3) = C''(4, 3)$

Here, $x = -3$ and $x = 1$ are the parallel lines.

Hence the combined reflection is a translation.

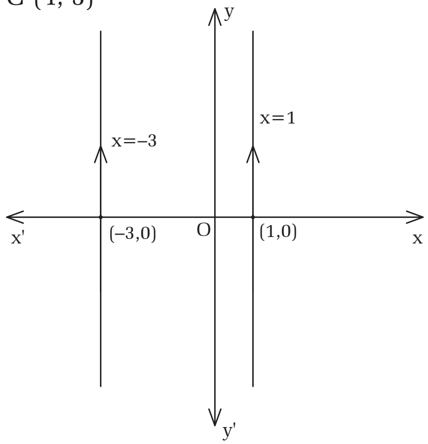
The translation is given by T.

$$T = 2\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 + 3 \\ 0 - 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$



4. On a graph paper draw ΔABC having the vertices A(5, 4), B(2, 2) and C(5, 2). Find the image of ΔABC by stating coordinates and graphing them after successive reflections on x-axis followed by the reflection on the line y = x.

Solution

Here, A(5, 4), B(2, 2) and C(5, 2) are the vertices of ΔABC.

ΔABC is reflected on x-axis

$P(x, y) \xrightarrow{\text{x-axis}} P'(x, -y)$.

$A(5, 4) \longrightarrow A'(5, -4)$

$B(2, 2) \longrightarrow B'(2, -2)$

$C(5, 2) \longrightarrow C'(5, -2)$

Again, $\Delta A'B'C'$ is reflected on the line $y = x$

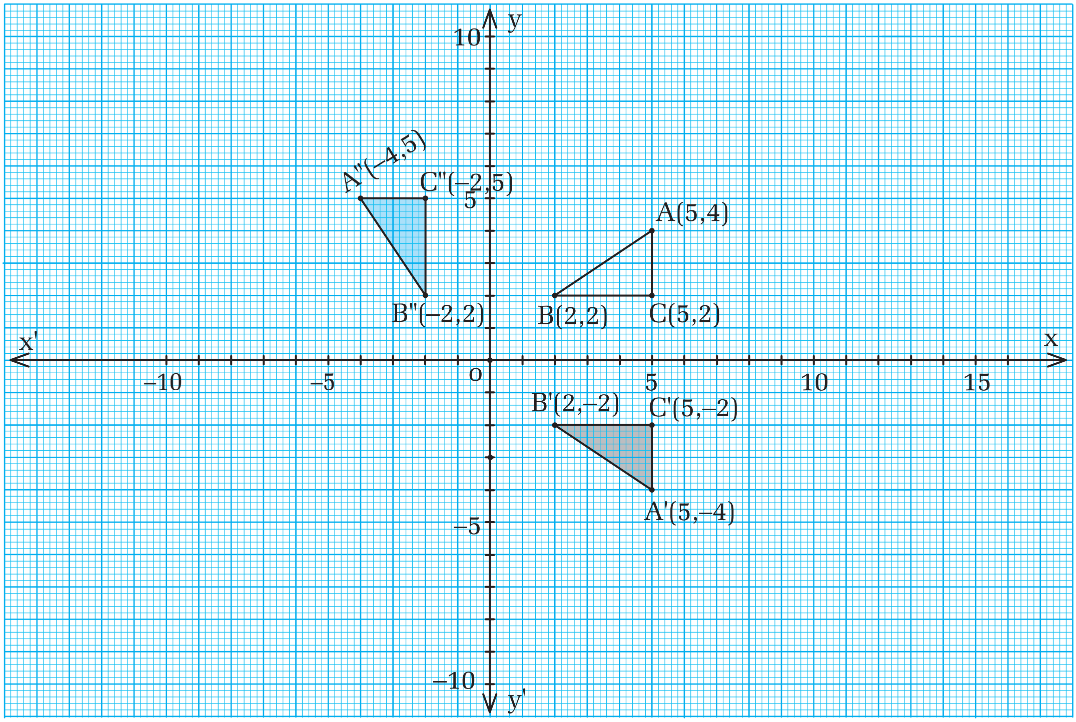
$P(x, y) \xrightarrow{y = x} P'(y, x)$.

$A'(5, -4) \longrightarrow A''(-4, 5)$

$B'(2, -2) \longrightarrow B''(-2, 2)$

$C'(5, -2) \longrightarrow C''(-2, 5)$

ΔABC, ΔA'B'C' and ΔA"B"C" are plotted on graph paper.



5. O(0, 0), A(2, 0), B(3, 2) and C(1, 2) are the vertices of quadrilateral OABC. Translate the quadrilateral by translation vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Reflection the image so formed on the line x = 3. Represent the images and object on the same graph.

Solution

Here, O(0, 0), A(2, 0), B(3, 2) and C(1, 2) are the vertices of quadrilateral OABC.

Firstly it is translated by T = $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

We have,

$$P(x, y) \xrightarrow{T = \begin{bmatrix} a \\ b \end{bmatrix}} P'(x + a, y + b)$$

$$O(0, 0) \xrightarrow{T = \begin{bmatrix} 0 \\ 2 \end{bmatrix}} O'(0 + 0, 0 + 2) = O'(0, 2)$$

$$A(2, 0) \xrightarrow{T = \begin{bmatrix} 0 \\ 2 \end{bmatrix}} A'(2 + 0, 0 + 2) = A'(2, 2)$$

$$B(3, 2) \xrightarrow{T = \begin{bmatrix} 0 \\ 2 \end{bmatrix}} B'(3 + 0, 2 + 2) = B'(3, 4)$$

$$D(1, 2) \xrightarrow{T = \begin{bmatrix} 0 \\ 2 \end{bmatrix}} D'(1 + 0, 2 + 2) = C'(1, 4)$$

Again, the quadrilateral O'A'B'C' is reflected on line x = 3.

We have,

$$P(x, y) \xrightarrow{x = h} P'(2h - x, y).$$

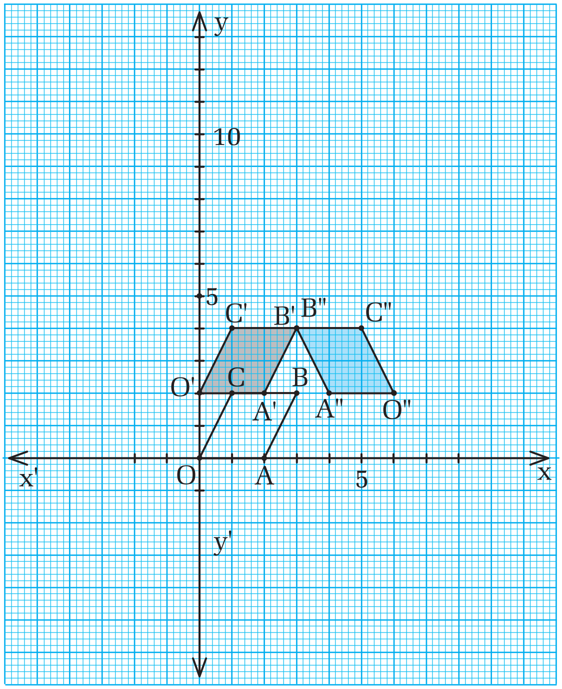
$$O'(0, 2) \xrightarrow{x = 3} O''(2\times3 - 0, 2) = O''(6, 2)$$

$$A'(2, 2) \xrightarrow{x = 3} A''(2\times3 - 2, 2) = A''(4, 2)$$

$$B'(3, 4) \xrightarrow{x = 3} B''(2\times3 - 3, 4) = B''(3, 4)$$

$$C'(1, 4) \xrightarrow{x = 3} C''(2\times3 - 1, 4) = C''(5, 4)$$

The quadrilaterals OABC, O'A'B'C' and O"A"B"C" are plotted on a same graph given below.



Combinations of Enlargement and Enlargement

Notes:

1. **If $E_1[(0, 0), k_1]$ and $E_2[(0, 0), k_2]$ are two enlargements, then the combined enlargement is $E[(0, 0), k_1 k_2]$**
2. **If $E_1[(a, b), k_1]$ and $E_2[(0, 0), k_1]$ are two enlargements, then the combined enlargement is $E[(a, b), k_1 k_2]$**

Some solved problems

1. Let $E_1[(2, 1), 3]$ and $E_2[(2, 1), 2]$, what is the single enlargement due to E_1oE_2.

Solution

Here, $E_1[(2, 1), 3]$ and $E_2[(2, 1), 2]$ are enlargements with the same centre .

So, the single combined enlargement is $E[(1, 2), 3 \times 2]$

i.e. $E[(1, 2), 6]$

2. Let $E_1[(0, 0), 2]$ and $E_2[(0, 0), 3]$, be two enlargements. Find the image of P(2, 5) under E_2oE_1.

Solution

Here, E_1 and E_2 both enlargements have the same centre (0, 0).

So, their combined enlargement is equivalent to a single enlargement $E[(0, 0), 2 \times 3]$

i.e. $E[(0, 0), 6]$

We have,

$P(x, y) \xrightarrow{E[(0,\,0),\,k]} P'(kx, ky)$

Now, $P(2, 5) \xrightarrow{E[(0,\,0),\,6]} P'(6\times2, 6\times5) = P'(12, 30)$

3. P(3, 0), Q(4, 2), R(2, 4) and S(1, 2) are the vertices of a parallelogram PQRS. Draw PQRS on a graph paper. Enlarge the parallelogram under $E_1[(0, 0),$ 2] followed by $E_2\left[(0, 0), 1\frac{1}{2}\right]$. Find the images P'Q'R'S' and P"Q"R"S" on the same graph.

Solution

Here, P(3, 0), Q(4, 2), R(2, 4) and S(1, 2) are the vertices of a parallelogram PQRS.

Parm PQRS is enlarged under $E_1[(0, 0), 2]$.

We have,

$P(x, y) \xrightarrow{E[(0,\,0),\,k]} P'(kx, ky)$

Now, $P(3, 0) \xrightarrow{E[(0,\,0),\,2]} P'(6, 0)$

$Q(4, 2) \longrightarrow Q'(8, 4)$

$R(2, 4) \longrightarrow R'(4, 8)$

$S(1, 2) \longrightarrow S'(2, 4)$

Again, the parm PQRS is enlarged under $E_2\left[(0, 0), 1\frac{1}{2}\right]$

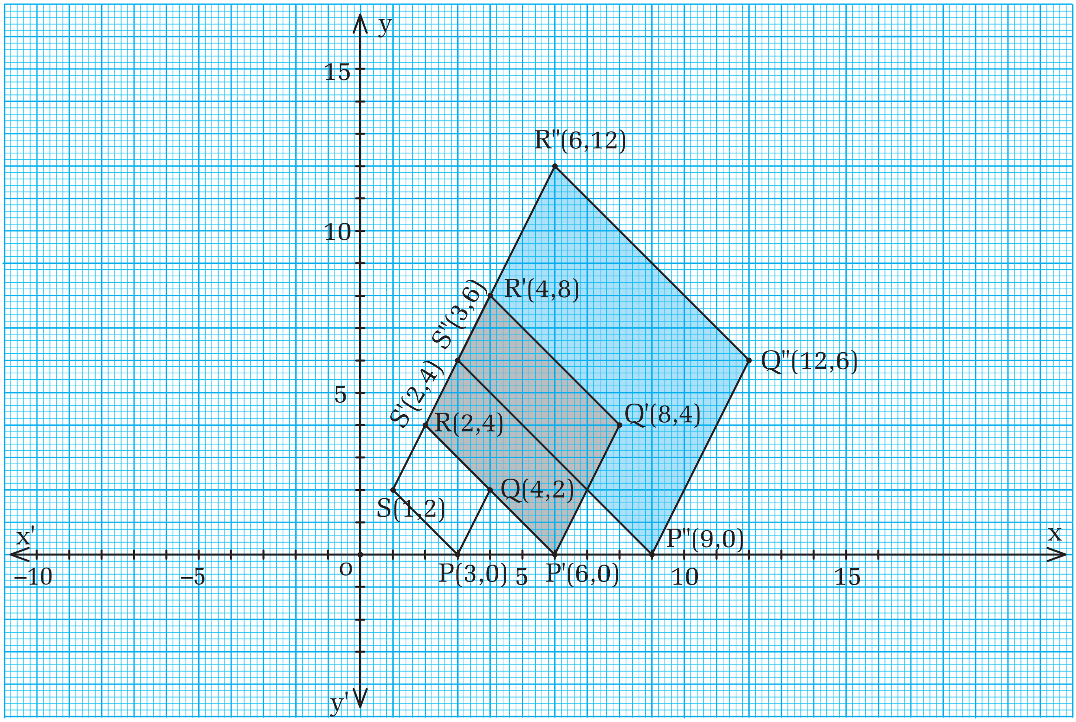
$$P'(6, 0) \xrightarrow{E_2\left[(0,0),\ 1\frac{1}{2}\right]} P''(9, 0)$$

$$Q'(8, 4) \longrightarrow Q''(12, 6)$$

$$R'(4, 8) \longrightarrow R''(6, 12)$$

$$S'(2, 4) \longrightarrow S''(3, 6)$$

The parm PQRS and its images P'Q'R'S' and P"Q"R"S" are plotted on same graph as shown below.



4. A(2, 5), B(–1, 3) and C(4, 1) are the vertices of ΔABC. Find the coordinates of the image of ΔABC under rotation of positive 90° about the origin followed by the enlargement E[(0, 0), 2]

Solution

Here, A(2, 5), B(–1, 3) and C(4, 1) are the vertices of ΔABC.

ΔABC is rotated through +90° about the origin.

We have,

$$P(x, y) \xrightarrow{R[(0,0),\ +90^\circ]} P'(-y, x)$$

$$A(2, 5) \xrightarrow{R[(0,0),\ +90^\circ]} A'(-5, 2)$$

$B(-1, 3) \longrightarrow B'(-3, -1)$

$C(4, 1) \longrightarrow C'(-1, 4)$

Again, $\Delta A'B'C'$ is enlarged under E[(0, 0), 2]

We have,

$P(x, y) \xrightarrow{E[(0,0),k]} P'(kx, ky)$

$A'(-5, 2) \xrightarrow{E[(0,0),2]} A''(-10, 4)$

$B'(-3, -1) \longrightarrow B''(-6, -2)$

$C'(-1, 4) \longrightarrow C''(-2, 8)$

5. Let E denote enlargement about centre (–3, –4) with scale factor 2 and R denote a reflection on the line y = x. ΔABC with vertices A(2, 0), B(3, 1) and C(1, 1) is mapped under the combined transformation RoE. Find the image of ΔABC and draw both figures on the same graph.

Solution

Here, A(2, 0), B(3, 1) and C(1, 1) are the vertices of ΔABC.

Combined transformation RoE means first enlargement E[(–3, –4), 2], then reflection R on y = x.

ΔABC is enlarged under E[(–3, –4), 2]

We have,

$P(x, y) \xrightarrow{E[(a,b),k]} P'(k(x-a)+a, k(y-b)+b)$

$A(2, 0) \xrightarrow{E[(-3,-4),2]} A'(2(2+3)-3, 2(0+4)-4) = A'(7, 4)$

$B(3, 1) \longrightarrow B'(2(3+3)-3, 2(1+4)-4) = B'(9, 6)$

$C(1, 1) \longrightarrow C'(2(1+3)-3, 2(1+4)-4) = C'(5, 6)$

Again, ΔA'B'C' is reflected about line y = x.

We have,

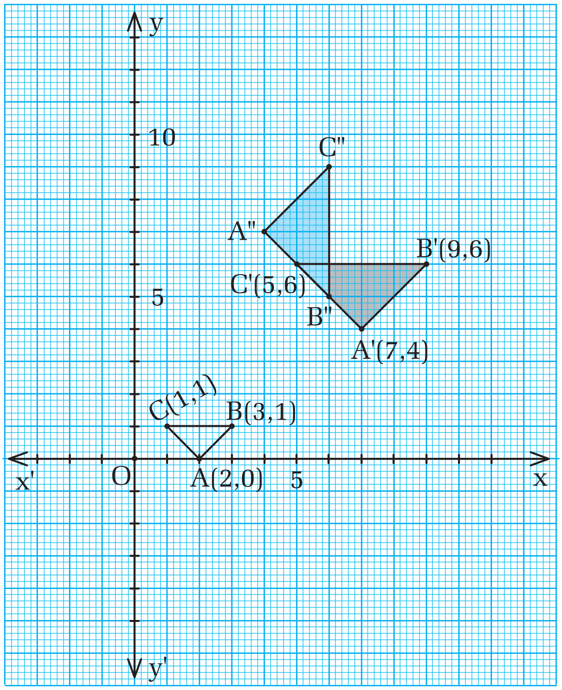
$P(x, y) \xrightarrow{x=y} P'(y, x)$

$A'(7, 4) \xrightarrow{y=x} A''(4, 7)$

$B'(9, 6) \xrightarrow{y=x} B''(6, 9)$

$C'(5, 6) \xrightarrow{y=x} C''(6, 5)$

ΔABC, ΔA'B'C' and ΔA"B"C" are plotted on the same graph as shown in the figure below.



6. M (3,4), N (1,1) and P (4,1) are the vertices of ΔMNP. Find the image of ΔMNP under the enlargement with centre (1–1) and scale factor –2 fallowed by the rotation about the origin through negative quarter turn. Also show the images on the same graph paper.

Solution

Here, M(3,4), N (1,1) and P (4,1) are the vertices of ΔMNP.

ΔMNP is enlarged about centre (1,1) and scale factor –2.

i.e. E[(1,1), –2]

We have,

$$P(x,y) \xrightarrow{E[(a,b),\ k]} P'(k\,(x-a) + a,\ k(y-b) + b)$$

$$M(3,4) \xrightarrow{E[(1,-1),\ -2]} M'(-2(3-1) + 1,\ -2\,(4+1) - 1) = M'(-3,-11)$$

$$N(1,1) \xrightarrow{E[(1,1),\ -2]} N'(-2(1-1) + 1,\ -2(1+1) - 1) = N'(1,-5)$$

$$P(4,1) \xrightarrow{E[(1,1),\ -2]} P'(-2(4-1) + 1,\ -2(1+1) - 1) = P'(-5,-5)$$

Again

ΔM′N′P′ is rotated through negative quarter turn or –90° about the origin.

We have,

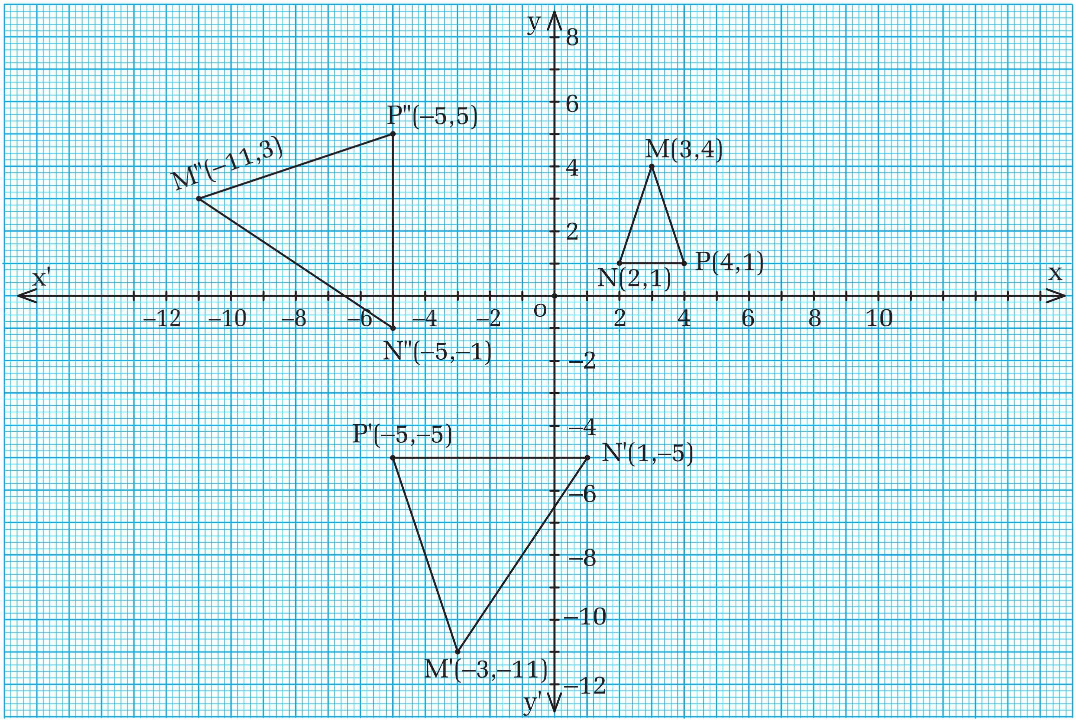
$$P(x,y) \xrightarrow{R[(0,0),\ -90^\circ]} P'(y,-x)$$

Now, $M'(-3,-11) \xrightarrow{R[(0,0),\ -90^\circ]} M''(-11,3)$

$N'(1,-5) \longrightarrow N''(-5,-1)$

$P'(-5,-5) \longrightarrow P''(-5,5)$

The given object ΔMNP and its images ΔM′N′P′ and ΔM′′P′′N′′ are plotted on the same graph paper.



Questions for practice

1. a. What is the single line transformation which represents the reflection in X–axis followed by y =x ?

b. Write the single transformation which represents $x=h_1$ f0llowed by reflection in $x=h_2$?

c. What is the single transformation which represents reflection in x=2 is followed by reflection in x=–4?

d. What is the single transformation which represents the reflection in $y=k_1$ followed by reflection in $y= k_2$?

e. What is the single transformation of the rotation of $\alpha°$ about the origin followed by the rotation of ß° about same centre ?

f. Find the image of point P(4,5) when it is reflected in x=2 followed by reflection in x=6.

g. If r_1 is the reflection about y– axis and r_2 is the rotation of –90° about the origin, find the image of the point P(3,–4) under the combined transformation $r_1 o r_2$.

h. Find the coordinates of the point P whose image after reflection about the line y=x followed by reflection about the line x=0 is (6,–5).

i. Point P(6,7) is firstly reflected on x–axis. The image P′ thus obtained is rotated through +90° about the origin. Find the coordinates of fined image p″.

2. A triangle with vertices A(2,3), B(5,8) and C(5,0) is reflected successfully in the line x=–2 and y=2. Find by stating coordinates and represent the images graphically under these transformations. State also the single transformation given by the combination of these transformations.

3. P(1,4), Q(6,1), and R(6,6) are the vertices of ΔPQR. Find the coordinates of the vertices of the triangle P′Q′R′ under the reflection on the line x=y followed by the enlargement E[(1,2),2].

4. State the single transformation equivalent to the combination of reflections on the x–axis and y–axis respectively. Using this single transformation find the coordinates of the vertices of the image of ΔPQR having vertices P(4,0), Q(4,4) and R(7,5). also draw the object and image on the same graph.

5. ΔPQR with the vertices P(4,5), Q(–2,4) and R(–2,0) is translated by $T=\begin{pmatrix}4\\2\end{pmatrix}$. The image so obtained is enlarged by E[(0,0),2]. stating the coordinates of the vertices of the images thus formed, represent the ΔPQR and its images in the same graph paper.

Inversion Transformation and Inversion Circle

Note

S.N	circle	centre	radius	point	inversion point
1.	$x^2+y^2=r^2$	(0,0)	r	P(x,y)	$x' = \frac{r^2x}{x^2+y^2}$, $y' = \frac{r^2x}{x^2+y^2}$
2.	$(x-h)^2 + (y-k)^2 = r^2$	(h,k)	r	P(x,y)	$x' = \frac{r^2(x-h)}{(x-h)^2+(y-k)^2} + h$ $y' = \frac{r^2(y-k)}{(x-h)^2+(y-k)^2} + k$

6. Main features of inversion transformation:

a. If P′=P, then P is on the circle of inversion.

b. If OP < r, then P lies inside the circle.

c. If OP>r, then P lies outside the circle.

d. (P')' = P. It is called symmetric features pf inversion.

Some solved problems

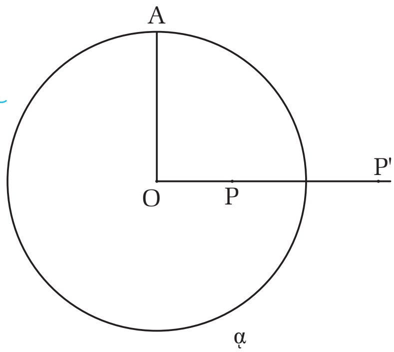
1. Find the following from the given figure:

a. Inversion circle

b. Inversion radius

c. Centre of inversion

d. Relation among OP, OP' and r.



Solution

From the adjoint figure,

we have,

a. given circle α is inversion circle.

b. OA= r is the inversion radius.

c. O is the center of inversion

d. OP.OP'= r^2, this is the relation among OP, OP' and OA

2. Find the inverse points of the following points with respect to the inversion circle with center at the origin

a. P(2,3),r=5

b. S(3,0),r=3

Solution

a. P(2,3), i.e. x=2, y=3, radius=5 centre of inversion circle= O(0,0),

Let P' (x',y') be the required inverse point.

using formula, we get,

$$x' = \frac{r^2x}{x^2 + y^2} = \frac{5^2.2}{2^2 + 3^2} = \frac{50}{4+9} = \frac{50}{13}$$

$$y' = \frac{r^2y}{x^2 + y^2} = \frac{5^2.3}{2^2 + 3^2} = \frac{75}{13}$$

$\therefore$ P' $\left(\frac{50}{13}, \frac{75}{65}\right)$ is the required inverse point.

b. S(3,0), radius(r)=3, centre =O(0,0)

here, x=3, y=0,

Let S'(x'y') be the required inverse point.

$$x' = \frac{r^2x}{x^2 + y^2} = \frac{3^2.3}{3^2 + 0^2} = 9$$

$$y' = \frac{r^2y}{x^2 + y^2} = \frac{3^2.0}{3^2 + 0^2} = 0$$

$\therefore$ S'(9,0) is the required inverse point.

3. Find the distance of inverse points M,N and P which are at distance 2,4 and 8 units respectively from the centre O of the circle with radius 8 units.

Solution

Here,

OM=2, OM'=?, r=8

where M' is the inverse point of M.

By definition of inverse point of inverse circle, we get.

we have,

$OM.OM' = r^2$

or $2 \times OM' = 8^2$

$OM' = 32$ units. Here $r < OM'$

M is outside the inverse circle.

similarly

For point N, ON $=4$, ON' $=?$, $r=8$

Then, $ON.ON' = r^2$

or, $4 \times ON' = 8^2$

$ON' = 16$ units.

Here, $r < ON'$; N' is outside the circle.

For point P, OP$=8$,OP'$=?$, $r=8$

Then, $OP.OP' = r^2$

or,$8 \times OP' = 8^2$

OP'$=8$units

Here, OP=OP'=r, the point Pis on the Circle.

4. Find the inverse of the point(4,5) with respect to the circle, $x^2+y^2=25$

Solution

Given, equation of inverse circle $x^2+y^2=25$

radius(r)=5units.

Let P(x,y)=P(4,5)

Let P'(x',y') be the required inverse point.

Then,

$$x' = \frac{r^2x}{x^2 + y^2} = \frac{5^2.4}{4^2 + 5^2} = \frac{100}{16+25} = \frac{100}{41}$$

$$y' = \frac{r^2x}{x^2 + y^2} = \frac{5^2.5}{4^2 + 5^2} = \frac{125}{41}$$

$\therefore P\left(\frac{100}{41}, \frac{125}{41}\right)$ is the required inverse point.

5. Find the inverse point of the point(5,10) with respect to a circle with centre the at the point(3,4) of radius 6 units.

Solution

Let given point be P(x,y)=P(5,10).

Radius of inverse Circle (r)=6 units.

Centre (h,k)=(3,4)

Here, $x-h=5-3=2$, $(x-h)^2=2^2=4$

$y-k=10-4=6$,$(y-k)^2=6^2=36$

Let p'(x',y') be the inverse points.

$$x' = \frac{r^2(x-h)}{(x-h)^2 + (y-K)^2} + h$$

$$= \frac{6^2.2}{4 + 36} + 3$$

$$= \frac{72}{40} + 3$$

$$= \frac{9}{5} + 3$$

$$= \frac{24}{5}$$

$$y' = \frac{r^2(y-k)}{(x-h)^2 + (y-K)^2} + k$$

$$= \frac{6^2.6}{4 + 36} + 4$$

$$= \frac{216}{40} + 4$$

$$= \frac{27}{5} + 4$$

$$= \frac{47}{5}$$

$$\therefore p(x',y') = \left(\frac{24}{5}, \frac{47}{5}\right)$$

6. Find the inverse line segment of AB with A(1,2) and B(3,3) with respect to inversion circle $x^2+y^2+8xy+8y+24=0$

Solution

Equation of given circle is $x^2+y^2+8x+8y+24=0$

comparing it with $x^2+y^2+ 2gx + 2fy + c = 0$

$g=4, f=4, c=24$

centre $(-g,-f)=-(-4,-4)$, radius$(r)=\sqrt{g^2+f^2-c}$

$=\sqrt{16+16-24}$

$=\sqrt{8}$

$=2\sqrt{2}$

Let us find the inversion points of A and B with respect to the inversion circle.

Let A'(x',y') be inverse point of A(1,2)

$(x,y)=(1,2)$, $(h,k)=(-4,-4)$

$x-h=1+4=5$, $(x-h)^2=5^2=25$

$y-k=2+4=6, (y-k)^2=6^2=36$

Now,

$$=x' = \frac{r^2(x-h)}{(x-h)^2 + (y-k)^2} + h$$

$$= \frac{8.5}{25+36} - 4$$

$$= \frac{40}{61} - 4$$

$$= -\frac{204}{61}$$

$$y' = \frac{r^2(y-k)}{(x-h)^2 + (y-K)^2} + k$$

$$= \frac{8.6}{25+36} - 4$$

$$= \frac{48}{61} - 4$$

$$= -\frac{196}{61}$$

Let B'(x',y') be the inverse point of B(3,3).

B(x,y)=B(3,3), (h,k)=(–4,–4)

x–h=3+4=7, $(x-h)^2=7^2=49$

y–k=3+4=7, $(y-k)^2=49$

$r=2\sqrt{2}$

Now,

$$x' = \frac{r^2(x-h)}{(x-h)^2 + (y-K)^2} + h$$

$$= \frac{8.7}{49+49} - 4$$

$$= \frac{56}{98} - 4$$

$$= -\frac{168}{49}$$

$$= -\frac{24}{7}$$

$$y' = \frac{r^2(y-k)}{(x-h)^2 + (y-K)^2} + k$$

$$= \frac{8.7}{49+49} - 4$$

$$= \frac{56}{98} - 4$$

$$= -\frac{196}{49}$$

$$= -\frac{196}{49}$$

$$B(x',y') = \left(-\frac{24}{7}, -\frac{24}{7}\right)$$

Hence, the required inverse line segment is A'B'.

7. **Find the inverse line segment of PQ where PC(4,5) and Q(4,3) with respect to the circle $x^2+y^2-4x-6y=3$. Show both of the line segments on the line segments on the same graph with the inverse circle.**

Solution

Equation of inverse circle is

$x^2+y^2-4x-6y=3$

or, $(x-2)^2+(y-3)^2=3+2^2+3^2$

or, $(x-2)^2+(y-3)^2=16$

or, $(x-2)^2+(y-3)=4^2$

comparing it with $(x-h)^2+(y-k)^2=r^2$, we get

centre $(h,k)=(2,3)$, radius$(r)=4$

Let P'(x',y') be the inverse point of P(4,5)

Here, $(h,k)=(2,3)$, $(x,y)=(4,5)$

$x-h=4-2=2$, $(x-h)^2=2^2=4$

$y-k=5-3=2$, $(y-k)^2=2^2=4$

Now,

$$x'=\frac{r^2(x-h)}{(x-h)^2+(y-k)^2}+h$$

$$=\frac{16.2}{4+4}+2$$

$$=6$$

$$y'=\frac{r^2(y-k)}{(x-h)^2+(y-k)^2}+k$$

$$=\frac{16.2}{4+4}+3$$

$$=7$$

P'(x',y')=P'(6,7)

Again, Q'(x',y') be the inverse point of Q(4,3).

$(x,y)=(4,3)$, $(h,k)=(2,3)$

$x-h=3-3=0$, $(x-h)^2=2^2=4$

$y-k=3-3=0$, $(y-k)^2=0^2=0, r=4$

Now,

$$x'=\frac{r^2(x-h)}{(x-h)^2+(y-k)^2}+h$$

$$=\frac{16.2}{4+0}+2$$

$$=10$$

$$y'=\frac{r^2(y-k)}{(x-h)^2+(y-k)^2}+k$$

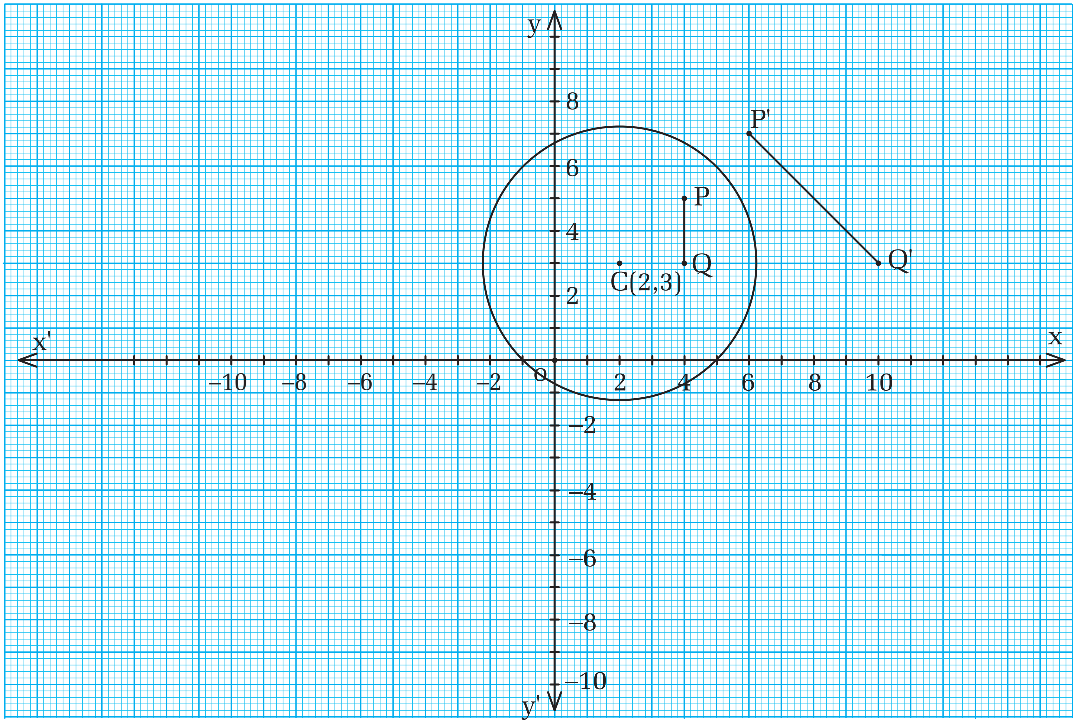
$$=\frac{16.0}{4+0}+3$$

$$=3$$

$Q'(x',y') = Q'(10,3)$

Hence, the image of the line segment PQ is P'Q'

The inversion circle, line segment PQ, inverse line segment P'Q' are plotted on the graph given below.



Questions for practice

1. Find the inverse point of P(4,5) with respect to an inversion circle with centre origin and radius 4 units.
2. O is the centre of an inversion circle $x^2+y^2=64$ and OP=6 units. find OP'.
3. Find the inversion point of (5,6) with respect to an inversion circle with radius 8 units and centre (2,3).
4. Find the inverse of point P(–3,2) with respect to an inversion circle $x^2+y^2=10$.
5. Find the inverse of point of S(2,3) with respect to an inversion circle $x^2+y^2+6x-8y=0$.
6. Find the inverse of line segment joining point P(2,3) and Q(6,8) with respect to an inversion circle $x^2+y^2-6x-8y-24=0$.

Use of Matrix in Transformation

Note:
(Class 10 book Page no 338)

Some solved problems

1. Let P(x,y) be any point on the plane and P'(x',y') be the image of P. Write the transformation matrices under the following cases:

a. (x',y')= (–y,–x)

Solution

Here $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix} = \begin{pmatrix} 0.x+(-1)y \\ (-1)x + 0.y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Hence, the required transformation matrix is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

b. (x',y')=(–x,–y)

Solution

Here, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} (-1).+0.y \\ 0.x + (-1)y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$\therefore$ Required transformation matrix is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

c. (x',y')= (Kx,Ky)

Solution

Here, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore$ Required transformation matrix is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

d (x',y')= (x+a,y+b) $= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore$ Required transformation matrix is $\begin{pmatrix} a \\ b \end{pmatrix}$.

2. Write down 2× 2 transformation matrices in each of the following transformation and using the matrices. Find the image of given point.

a. Reflection on y– axis, A(3,4)

Solution

Transformation matrices for reflection on y–axis = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Now, $\overset{A'}{\begin{pmatrix} x' \\ y' \end{pmatrix}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$\therefore$ A′ (x′, y′) = A′ (–3, 4)

b. Rotation about origin through +90° about origin is

$$\text{Now, } \overset{A'}{\begin{pmatrix} x' \\ y' \end{pmatrix}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \overset{E}{\begin{pmatrix} -4 \\ -4 \end{pmatrix}} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$\therefore E'(x'\ y') = E'(4,-4)$

c. Enlargement with centre at the origin by scale factor 3, G(4,7)

Solution

Transformation matrix for enlargement with centre 0(0,0) and scale factor 3 is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Now, for given point G(4,7).

$$\text{Now, } \overset{G'}{\begin{pmatrix} x' \\ y' \end{pmatrix}} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \overset{G}{\begin{pmatrix} 4 \\ 7 \end{pmatrix}} = \begin{pmatrix} 12 \\ 21 \end{pmatrix}$$

$\therefore G'(x',y') = G'(12,21)$.

3.a. If a matrix $\begin{bmatrix} 1 & 6 \\ 5 & 8 \end{bmatrix}$ maps a point (3,4) onto the point (15,17), Find the value of a.

Solution:

Here,

$$\begin{bmatrix} a & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} 3a + & 0 \\ 9 + & 8 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} 3a \\ 17 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \end{bmatrix}$$

$\therefore 3a = 15$

$a = 5$

b. Find a 2×2 matrix which transform a point (8,–4) to (4,8).

Solution:

$$\text{Here, } \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represent a transformation matrix of rotation of +90° about origin.

By using above transformation matrix, We get,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 8 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$\therefore$ The required transformation matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

4. a. Find the image of A(6,7) under the translation matrix $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ followed by translation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Solution

Here, A(6,7) is translated by matrix $T=\begin{pmatrix}3\\2\end{pmatrix}$

$$\overset{A}{\begin{pmatrix}6\\7\end{pmatrix}} \xrightarrow{T=\begin{pmatrix}3\\2\end{pmatrix}} \begin{pmatrix}6+3\\7+2\end{pmatrix}=\begin{pmatrix}9\\9\end{pmatrix}$$

Again, A'(9,9) is translate by $\begin{bmatrix}3\\2\end{bmatrix}$

$$\overset{A'}{\begin{pmatrix}9\\9\end{pmatrix}}+\begin{pmatrix}3\\2\end{pmatrix}=\overset{A''}{\begin{pmatrix}12\\11\end{pmatrix}}$$

b. What does the matrix $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ represent. Find the image of P(4,5) using the matrix

Solution

Here,

$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ **represent reflection on X–axis.**

Now, $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}=\begin{bmatrix}4\\-5\end{bmatrix}$

5. P(2,1), Q(5,1), R(5,4) and S(2,4) are the vectors of a square PQRS. The square is transformed by matrix $\begin{bmatrix}1 & 2\\1 & -2\end{bmatrix}$ into a parallelogram. Find the vertice of the parallelogram.

Solution

Here, the square PQRS can be written in the matrix form.

$$\begin{array}{cccc}P & Q & R & S\end{array}$$
$$\begin{bmatrix}2 & 5 & 5 & 2\\1 & 1 & 4 & 4\end{bmatrix}$$

Now,

$$\begin{bmatrix}1 & 2\\1 & -2\end{bmatrix}\begin{bmatrix}2 & 5 & 5 & 2\\1 & 1 & 4 & 4\end{bmatrix}=\begin{array}{c}\begin{array}{cccc}P' & Q' & R' & S'\end{array}\\\begin{bmatrix}2+2 & 5+2 & 5+8 & 2+8\\2-2 & 5-2 & 5-8 & 2-8\end{bmatrix}\end{array}$$

$$=\begin{bmatrix}4 & 7 & 13 & 10\\0 & 3 & -3 & -6\end{bmatrix}$$

6. ΔPQR with vertices P(5,1), Q(12,4), and R(4,5) maps onto the ΔP'Q'R' with vertices P'(–5,–1) Q'(–12,–4) and R'(–4,–5). Which single transformation for this mapping ? Also find the 2×2 matrix that represents the transformation.

Solution

Let $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ be the required transformation matrix.

then,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 5 & 12 & 4 \\ 1 & 4 & 5 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} P' & Q' & R' \end{array} \\ \begin{bmatrix} 5a+b & 12a+4b & 4a+5b \\ 5c+d & 12c+d & 4c+5d \end{bmatrix} \end{array}$$

$$\text{or, } \begin{array}{c} \begin{array}{ccc} P' & Q' & R' \end{array} \\ \begin{bmatrix} -5 & -12 & -4 \\ -1 & -4 & -5 \end{bmatrix} \end{array} = \begin{bmatrix} 5a+b & 12a+4b & 4a+5b \\ 5c+d & 12c+d & 4c+5d \end{bmatrix}$$

Equating the corresponding elements of equal matrices.

$5a+b=-5$..........(i)

$12a+4b=-12$....(ii)

$4a+5b=-4$.........(iii)

$5c+d=-1$..........(iv)

$12c+d=-4$.........(v)

$4c+5d=-5$.........(vi)

Solving equation (i) and (iii), we get

$a=-1, \ b=0$

again solving equation (iv) and (v) we get,

$c=0, \ d=1$

Hence required transformation matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ which is rotation of 180° about the origin.

7. A unit square having vertices A(0,0), B(1,0), C(1,1) and D(0,1) is mapped to the parallelogram A'B'C'D' by a 2×2 matrix is that the vertices of parallelogram are A'(0,0),B'(3,0),C'(4,)and D'(1,1).Find the 2×2 matrix.

Solution

Let required transformation matrix be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A' & B' & C' & D' \end{array} \\ \begin{bmatrix} 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\text{or, } \begin{bmatrix} 0 & a+0 & a+b & 0+b \\ 0 & c+0 & c+d & 0+d \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} A' & B' & C' & D' \end{array} \\ \begin{bmatrix} 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\text{or, } \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix} = \begin{bmatrix} 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Equating the corresponding elements of equal matrices we get,

$a=3, \quad b=1, \quad a+b=4$

$c=0, \quad d=1, \quad c+d=1$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

8. Verify by the matrix method that the reflection on the line y=x followed by the reflection on the y–axis is equivalent to the rotation about origin through +90°

Solution

Transformation matrix on the line y=x is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
Let us take a point P(x,y)

$$\therefore \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{array}{c} P' \\ \begin{bmatrix} y \\ x \end{bmatrix} \end{array}$$

Again, P' is reflection on y–axis. Its transformation matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{array}{c} P' \\ \begin{bmatrix} y \\ x \end{bmatrix} \end{array} = \begin{array}{c} P'' \\ \begin{bmatrix} -y \\ x \end{bmatrix} \end{array} ....(i)$$

Transformation matrix for rotation through +90°about the origin is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{array}{c} P \\ \begin{bmatrix} y \\ x \end{bmatrix} \end{array} = \begin{array}{c} P^{I} \\ \begin{bmatrix} -y \\ x \end{bmatrix} \end{array} ....(ii)$$

From (i) and (ii), we get,

$P'' = P^{I}$

It means that the reflection on the line y=x followed by the reflection on the y–axis is equivalent to the rotation about the origin through +90'. **proved**.

Questions for practice

1. If a point (x,y) is transformed into point (–y,–x) by a transformation matrix of order 2×2. Write the matrix.
2. Find the transformation matrix from given relations

   a. $\begin{array}{c} P \\ \begin{pmatrix} x \\ y \end{pmatrix} \end{array} = \begin{array}{c} P' \\ \begin{pmatrix} 2x \\ -3 \end{pmatrix} \end{array}$ b. $\begin{array}{c} P \\ \begin{pmatrix} a \\ b \end{pmatrix} \end{array} = \begin{array}{c} P' \\ \begin{pmatrix} -b \\ a \end{pmatrix} \end{array}$

3. Which transformation does the matrix $\begin{pmatrix} -\sin 90° & \cos 90° \\ \cos 90° & \sin 90° \end{pmatrix}$ represent?
4. If the matrix $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ maps the quadrilateral $\begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ onto rectangle $\begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$. Find the matrix $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$.
5. A(0,0), B(2,0), C(2,2), and D(0,2) are the vertices of a square ABCD and it is mapped to the parallelogram A'B'C'D' by a 2×2 matrix so that, the vertices of parallelogram are A'(0,0), B'(6,0), C'(8,2) and D'(2,2).
6. Find a 2×2 transformation matrix which transforms a parallelogram PQRS with vertices P(–2,–1), Q(1,0), R(4,3) and S(1,2) into the parallelogram with vertices P'(–7,–2), Q'(2,1), R'(17,4), S'(8,1).

UNIT twelve

Statistics

Estimated Teaching periods : 12

1. Teaching Objectives :

S.N.	Level	Objectives
1.	Knowledge	To define measure of dispersion. To define partition values. To define Q. D. , write formula of Q_1, Q_3 , its coefficient for continuous series. To define M.D. and its coefficient . To define S.D. and its coefficient for continuous series.
2.	Understanding	To tell the meaning of the following formulas. $\rightarrow$ Q.D. $= \frac{1}{2}(Q_3 - Q_1)$ $\rightarrow$ coeff of Q.D. $= \dfrac{Q_3 - Q_1}{Q_3 + Q_1}$ $\rightarrow$ M.D $= \frac{\Sigma \lvert D \rvert}{N}$, M.D.$= \frac{\Sigma f \lvert D \rvert}{N}$ $\sigma = \sqrt{\frac{\Sigma x^2}{N}}$, $\sigma = \sqrt{\frac{\Sigma f x^2}{N}}$ $\sigma = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd^1}{N}\right)^2}$ $\sigma = \sqrt{\frac{\Sigma fd^{1^2}}{N} - \left(\frac{\Sigma fd^1}{N}\right)^2} \times i$
3.	Application	To calculate Q.D., M.D., S.D. and their coefficients by using various formula from given data.
4.	Higher Ability	To solve verbal problems related to Q.D. , M.D. and S.D.

2. Required Teaching Materials:

Chart papers with list of formulae of Q. D. , M. D. , S.D. and their coefficients, calculators.

3. Teaching strategies:

- Discuss concept of measure of dispersion.
- Discuss the concepts of partition values quartiles, median with suitable examples.
- Discuss how to calculate Q_1, Q_3, M_d, M_0, Mean for continuous series.
- Discus to calculate Q. D. and its coefficients for continuous series with formula and

for examples, solve some questions from exercises.

- Discuss mean deviation from mean, median, mode with their formula.
- Define standard deviation and its importance.
- Write the different formulas to calculate standard deviation with its coefficient for continuous series.
- Calculate S.D and its coefficient for examples from the text book continuous series.

Quartile Deviation

Notes:

$\rightarrow Q_1$ class $= \left(\frac{N}{4}\right)^{th}$ term containing class

$\rightarrow Q_3$ class $= \left(\frac{3N}{4}\right)^{th}$ term containing class

Then,

$$Q_1 = L + \frac{\frac{N}{4} - c.f.}{f} \times h \quad , Q_3 = L + \frac{\frac{3N}{4} - c.f.}{f} \times i$$

where, l = lower limit of corresponding quartile class.

f = frequency of corresponding quartile class.

h,i = class size / width of the class of the comparing class.

c.f = cumulative frequency preceding or just before the quartile class.

If the value of the first quartile lies on first class, in this case, the c.f. should be taken as zero because there is no c.f. just before the first class.

– Half of the inter–quartile range is called semi – interquartile range or quartile deviation. it is given by

$$Q.D. = \frac{1}{2}(Q_3 - Q_1)$$

$$\text{Coeff of Q.D.} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

Some solved problems

1. If Q_1 =2.5, Q_3 = 22, find the Q.d. and its coefficient.

Solution

Here, $Q_1 = 2.5$, $Q_3 = 22$

Quartile Deviation (Q.D.) $= \frac{1}{2}(Q_3 - Q_1)$

$= \frac{22 - 2.5}{2}$

$= \frac{19.5}{2}$

$= 9.75$

$$\text{Coefficient of Q.D.} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

$= \frac{22 - 2.5}{22 + 2.5}$

$= \frac{19.5}{24.5}$

$= 0.7959$

2. In a continuous series, if the coefficient of Q.D. is 0.25 and the upper quartile is 60. Find the value of lower quartile.

Solution

Here,coefficient of Q.D. = 0.25

upper quartile $(Q_1) = 60$

Lower quartile $(Q_3) = ?$

By using formula,

coefficient of Q.D. $= \frac{Q_3 - Q_1}{Q_3 + Q_1}$

or, $0.25 = \frac{60 - Q_1}{60 + Q_1}$

or, $15 + 0.25Q_1 = 60 - Q_1$

or, $1.25Q_1 = 45$

$\therefore Q_1 = \frac{45}{1.25} = 36$

3. Calculate the quartile deviation and its coefficient from the following data.

Marks	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90
No. of students:	4	12	16	10	8	6	2

Solution

To calculate Quartile deviation and its coefficient

Marks	No. of students (f)	c.f
20 – 30	4	4
30 – 40	12	16
40 – 50	16	32
50 – 60	10	42
60 – 70	8	50
70 – 80	6	56
80 – 90	2	58
	N = 58	

To find Q_1

$Q_1 = \left(\frac{N}{4}\right)^{th}$ term

$= \left(\frac{58}{4}\right)^{th}$ term

$= 14.5^{th}$ term

In c.f, the c.f just greater than 14.5 is 16 .

So Q_1 lies in the class 30 – 40. ie. (30 – 40) is the first quartile class.

L = 30

c.f = 4

i = 10

Now, $Q_1 = L + \dfrac{\frac{N}{4} - c.f.}{f} \times i$

$= 30 + \dfrac{14.5 - 4}{12} \times 10$

$= 30 + 8.75$

$\therefore Q_1 = 38.75$

Again, To find Q_3

$Q_3 = \dfrac{3N}{3} = \dfrac{3 \times 58}{4} = 43.5$

c.f just greater than 43.5 is 50. So, Q_3 lies in the class 60 – 70 i.e. (60 – 70) is the third quartile class

L = 60

c.f = 42

f = 8

i = 10

Now,

$Q_3 = L + \dfrac{\frac{3N}{4} - c.f.}{f} \times i$

$= 60 + \dfrac{43.5 - 42}{8} \times 10$

$= 60 + \dfrac{1.5}{3} \times 10$

$= 60 + 1.875$

$= 61.875$

Quartile Deviation $= \dfrac{Q_3 - Q_1}{2} = \dfrac{61.875 - 38.75}{2}$

$= \dfrac{23.125}{2}$

$= 11.5625$

Coefficient of Quartile Deviation $= \dfrac{Q_3 - Q_1}{Q_3 + Q_1} = \dfrac{61.875 - 38.75}{61.875 + 38.75} = 0.23$

4. Calculate the quartile deviation and its coefficient from the given data.

Marks	less than 30	30 – 40	40 – 50	50 – 60	60 – 70	70 and above
No. of students:	4	12	16	10	8	6

Solution

To calculate quartile deviation and its coefficient

Marks	No. of students (f)	c.f
less than 30	3	3
30 – 40	6	9
40 – 50	9	18
50 – 60	5	23
60 – 70	4	27
70 and above	2	29
	N = 29	

To find Q_1

$$\frac{N}{4} = \frac{29}{4} = 7.25$$

c.f, just greater than 7.25 is 9.

So Q_1 lies in the class 30 – 40. ie. 30 – 40 is the 1^{st} quartile class.

L = 30

c.f = 4

f = 6

i = 10

$$\text{Now, } Q_1 = L + \frac{\frac{N}{4} - \text{c.f.}}{f} \times i$$

$$= 30 + \frac{7.25 - 3}{6} \times 10$$

$= 37.08$

To find Q_3

$$\frac{3N}{4} = \frac{3 \times 29}{4} = 21.75$$

c.f just greater than 21.75 is 23. and its corresponding class is 50 – 60. So 50 – 60 is 3^{rd} quartile class

L = 50

c.f = 18

f = 5

i = 10

Now,

$$Q_3 = L + \frac{\frac{3N}{4} - \text{c.f.}}{f} \times i$$

$= 50 + \frac{21.75 - 18}{5} \times 10$

$= 50 + 7.5$

$= 57.5$

Now,

Quartile Deviation $= \frac{Q_3 - Q_1}{2} = \frac{57.5 - 37.08}{2}$

$= \frac{23.125}{2}$

$= 10.21$

Again,

coefficient of Quartile Deviation $= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{57.5 - 37.08}{57.5 + 37.08} = 0.215$

(c)

Solution

To calculate quartile deviation and its coefficient

Marks	No. of students (f)	c.f.
Below 30	2	2
30 – 40	4	6
40 – 50	6	12
50 – 60	8	20
60 – 70	6	26
70 and above	4	30
	N = 29	

To find Q_1

$\frac{N}{4} = \frac{30}{4} = 7.5$

c.f, just greater than 7.5 is 12.

So Q_1 lies in the class 40 – 50. i.e. 40 – 50 is 1st quartile class.

L = 40

c.f = 6

f = 6

i = 10

Now, $Q_1 = L + \frac{\frac{N}{4} - c.f.}{f} \times i$

$= 40 + \frac{7.5 - 6}{6} \times 10$

$= 42.5$

To find Q_3

$\frac{3N}{4} = \frac{3 \times 30}{4} = 22.5$

c.f just greater than 22.5 is 26 and its corresponding class is 60 – 70. So 60 – 70 is 3rd quartile class.

L = 6

c.f = 20

f = 60

i = 10

Now,

$$Q_3 = L + \frac{\frac{3N}{4} - c.f.}{f} \times i$$

$$= 60 + \frac{22.5 - 20}{6} \times 10$$

$$= 60 + 4.17$$

$$= 64.17$$

Quartile Deviation $= \dfrac{Q_3 - Q_1}{2} = \dfrac{64.17 - 42.5}{2}$

$= 10.835$

Again,

coefficient of Quartile Deviation $= \dfrac{Q_3 - Q_1}{Q_3 + Q_1} = \dfrac{64.17 - 42.5}{64.17 + 42.5} = 0.203$

4. Calculate the quartile deviation and its coefficient from the given data.

C.I	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79
Frequency	100	80	75	25	70	40

Solution

Correction factor $= \dfrac{30 - 29}{2} = \dfrac{1}{2} = 0.5.$

Class Interval	frequency	c.f
20 – 29.5	100	100
29.5 – 39.5	80	180
39.5 – 49.5	75	255
49.5 – 59.5	95	350
59.5 – 69.5	70	420
69.5 – 75.5	40	460
	N = 460	

To find Q_1

$$\frac{N}{4} = \frac{460}{4} = 115$$

c.f, just greater than 115 is 180 .

So Q_1 lies in the class 29.5 – 39.5 ie. 29.5 – 39.5 is the 1st quartile class.

L = 29.5

c.f = 100

f = 80

i = 10

$$\text{Now, } Q_1 = L + \frac{\frac{N}{4} - \text{c.f.}}{f} \times i$$

$$= 29.5 + \frac{115 - 100}{80} \times 10$$

= 31.375

To find Q_3

$$\frac{3N}{4} = \frac{3 \times 460}{4} = 345$$

c.f just greater than 345 is 350 and its corresponding class is 49.5 – 59.5. So 49.5 – 59.5 is 3rd quartile class

L = 49.550

c.f = 255

f = 95

i = 10

Now,

$$Q_3 = L + \frac{\frac{3N}{4} - \text{c.f.}}{f} \times i$$

$$= 49.5 + \frac{345 - 255}{95} \times 10$$

= 58.97

Now,

$$\text{Quartile Deviation} = \frac{Q_3 - Q_1}{2} = \frac{58.97 - 31.375}{2}$$

= 13.79

Again,

$$\text{coefficient of Quartile Deviation} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{58.97 - 31.375}{58.97 + 31.375} = 0.30$$

6(a) (text book Q.8(a)

Solution

To calculate semi – interquartile range and its coefficient. we tabulate the given data as follows:

Class	Tally marks	frequency	c.f
10 – 20	||	2	2
20 – 30	|||	3	5
30 – 40	~~||||~~ ||	7	12
40 – 50	~~||||~~ ~~||||~~ |||	13	25

50 – 60	𝍸 𝍸 |||	13	38
60 – 70	𝍸 ||||	9	47
70 – 80	||	2	49
80 – 90	|	1	50
		N = 29	

For the first quartile(Q_1),

$$= \frac{N}{4} = \frac{50}{4} = 12.5$$

c.f, just greater 12.5 is 25 whose class is (40 – 50).

Hence Q_1 lies in the class (40 – 50).

L = 40

c.f = 12

f = 13

i = 10

$$Q_1 = L + \frac{\frac{N}{4} - \text{c.f.}}{f} \times i$$

$$= 40 + \frac{12.5 - 12}{13} \times 10$$

$$= 40.38$$

For the upper quartile (Q_3),

$$\frac{3N}{4} = \frac{3 \times 50}{4} = 37.5$$

c.f just greater 37.5 is 38 whose class is (50 – 60).

Hence Q_3 lies in (50 – 60).

L = 50

c.f = 13

f = 25

i = 10

Now,

$$Q_3 = L + \frac{\frac{3N}{4} - \text{c.f.}}{f} \times i$$

$$= 50 + \frac{37.5 - 25}{13} \times 10$$

$$= 59.62$$

Now,

$$\text{Semi – interquartile range} = \frac{Q_3 - Q_1}{2} = \frac{59.62 - 40.38}{2}$$

$$= \frac{23.125}{2}$$

$= 9.62$

Again,

coefficient of Quartile Deviation $= \dfrac{Q_3 - Q_1}{Q_3 + Q_1}$

$$\frac{59.62 - 40.38}{59.62 + 40.38} = \frac{19.24}{100} = 0.1924$$

tex book a.8(b)

6(b)To calculate the quartile deviation of given data.

Solution

To calculate quartile deviation and its coefficient

Class	Tally marks	frequency	c.f
0 – 4	𝍸	5	5
4 – 8	𝍸 |||	8	13
8 – 12	𝍸 |	6	19
12 – 16	||||	4	23
16 – 20	𝍸 |	6	29
		N = 29	

For the first quartile(Q_1),

$$= \frac{N}{4} = \frac{29}{4} = 7.25$$

c.f, just greater 7.25 is 13 hence Q_1 lies in (4–8).

$L = 4$

$c.f = 5$

$f = 8$

$i = 10$

$$Q_1 = L + \frac{\frac{N}{4} - c.f.}{f} \times i$$

$$= 4 + \frac{7.25 - 5}{8} \times 10$$

$= 6.81$

For (Q_3),

$$\frac{3N}{4} = \frac{3 \times 29}{4} = 21.75$$

c.f just greater 21.75 is 23

Hence Q_3 lies in (12 – 16).

$L = 12$

$c.f = 19$

$f = 4$

$i = 4$

Now,

$$Q_3 = L + \frac{\frac{3N}{4} - c.f.}{f} \times i$$

$$= 12 + \frac{21.75 - 19}{4} \times 4$$

$= 14.75$

Now,

$$Q.D. = \frac{Q_3 - Q_1}{2}$$

$$= \frac{14.75 - 6.81}{2}$$

$= 3.97$

coefficient of Quartile Deviation $= \dfrac{Q_3 - Q_1}{Q_3 + Q_1}$

$$\frac{14.75 - 6.81}{14.75 + 6.81} = \frac{7.94}{21.56} = 0.3668$$

Mean Deviation (or Average Deviation)

Notes:

1. Mean Deviation from mean.

For continuous series or grouped data

$$M.D. = \frac{\Sigma f|m - \overline{x}|}{N}$$

where m = mid – value of class interval

$$\text{coefficient of M.D. from mean} = \frac{\text{M.D. from mean}}{\text{Mean}}$$

2. Mean Deviation from median:

For continuous series or grouped data.

$$M.D. = \frac{\Sigma f|m - M_d|}{N}$$

where m = mid – value of class interval

$$\text{coefficient M.D. from median} = \frac{\text{M.D. from mean}}{\text{Median}}$$

Some solved problems

1. In a continuous series of data, of $= \sum f\,|m - M_d|$ $= 250$, $N = 25$, find M.D. and its coefficient.

Solution

Here, $\sum f\,|m - M_d| = 250$,

Median (M_d) = 22, N = 25

M.D. from median $= \frac{\Sigma f|m - M_d|}{N} = \frac{250}{25} = 10$

coefficient of M.D. from mean $= \frac{\text{M.D. from mean}}{\text{Mean}}$

$= \frac{10}{22}$

$= 0.45$

2. Calculate the mean deviation from

i) mean ii) median for the following data. Also compute their corresponding coefficients.

class interval	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
Frequency	10	12	25	35	40	50

Solution :

To calculate mean deviation from mean and its coefficient we have the following table.

Class interval	Mid – value(m)	Frequency(f)	c.f.	$\lvert m - \bar{x} \rvert = \lvert m - 38.55 \rvert$	$f\lvert m - \bar{x} \rvert$
0 – 10	5	10	50	33.55	335.5
10 – 20	15	12	180	23.55	252.6
20 – 30	25	25	625	13.55	338.75
30 – 40	35	35	1225	3.55	124.25
40 – 50	45	40	1800	6.45	258.00
50 – 60	55	50	2750	16.45	822.5
Total 60 –70		N = 172	6630		

Mean ($\bar{x}$) $= \frac{\Sigma fm}{N} = \frac{6630}{172} = 38.55$

Mean deviation from mean $= \frac{\Sigma f|m - \bar{x}|}{N}$

$= \frac{2161.6}{172}$

$= 12.57$

Coefficient of M.D. from mean $= \frac{\text{M.D. from mean}}{\text{Mean}}$

$= \frac{12.57}{38.55}$

$= 0.3260$

To calculate mean deviation from median.

class interval	Mid – value(m)	Frequency(f)	c.f	$\lvert m - \bar{x} \rvert = \lvert m - 41 \rvert$	$f\lvert m - M \rvert$
0 – 10	5	10	10	36	360
10 – 20	15	12	22	26	312
20 – 30	25	25	47	16	400
30 – 40	35	35	82	6	210

40 – 50	45	40	122	4	160
50 – 60	55	50	172	14	700
		N = 172			2142

Median $(M_d) = L + \dfrac{\frac{N}{2} - c.f.}{f} \times i$

$\dfrac{N}{2} = \dfrac{172}{2} = 86$

c.f. just greater 86 is 122 whose corresponding class in 40 – 50

c.f = 82, N = 172, L = 40, i = 10

$M_d = L + \dfrac{\frac{N}{2} - c.f.}{f} \times i$

$= 40 + \dfrac{86 - 82}{40} \times 10$

$= 40 + \dfrac{4}{4}$

$= 41$

Mean deviation from median $= \dfrac{\Sigma f\,|m - M_d|}{N}$

$= \dfrac{2142}{172} = 12.45$

$= 12.542$coefficient of M.D. from median $= \dfrac{\text{M.D. from mean}}{\text{Median}}$

$= \dfrac{12.45}{41}$

$= 0.3036$

3. Compute mean deviation and its coefficient from mean of the following data.

mid – value	5	15	25	35	45	55
No. fo students	2	4	6	8	6	4

Solution

To compute M.D from mean and its coefficient

Mid – value (m)	No. of students(f)	fm	$\|m - \bar{x}\|$	$f\|m - \bar{x}\|$
5	2	10	28	56
15	4	60	18	72
25	6	150	8	48
35	8	280	2	16
45	6	270	12	72
55	4	220	22	88
	30	990		352

Mean $(\bar{x}) = \dfrac{\Sigma fm}{N} = \dfrac{990}{30} = 33$

M.D. from means $= \frac{\Sigma f|m - \bar{x}|}{N}$

$= \frac{352}{30}$

$= 11.73$

coefficient of M.D. from mean $= \frac{\text{M.D. from mean}}{\text{Mean}}$

$= \frac{11.73}{33}$

$= 0.3553$

4. Construct a frequency distribution table taking a class interval of 10 and calculate the M.D. from median.

28, 49, 35, 5, 18, 14, 24, 7, 38, 46, 30, 21, 16, 31, 45, 27, 10, 4, 17, 29, 35, 36, 41, 47, 44, 33, 34, 17, 18, 20

Solution

To construct 6 frequency table and calculate mean deviation from median.

To compute M.D from mean and its coefficient

class	Tally marks	Frequency (f)	c.f	mid – value (m)	$\lvert m - d\rvert = \lvert m - 20\rvert$	$f\lvert m - M_d\rvert$
0 – 10	|||	3	3	5	15	45
10 – 30	~~||||~~ ||	7	10	15	5	35
20 – 30	~~||||~~ |	6	16	25	5	30
30 – 40	~~||||~~ |||	8	24	35	15	120
40 – 50	~~||||~~	6	30	45	25	150
		N = 30				380

Here, $= \frac{N}{2} = = \frac{30}{2} = 15$

c.f just greater 15 is 16 whose corresponding class is 20 – 30.

L = 20, i = 10, f = 6, N = 30, c.f. = 15

Median $(M_d) = L + \frac{\frac{N}{2} - c.f.}{f} \times i$

$= 20 + \frac{15 - 15}{6} \times 10$

$= 20$

M.D. from means $= \frac{\Sigma f|m - \bar{x}|}{N}$

$= \frac{380}{30}$

$= 12.67$

5. (a) Compute the mean deviation and its coefficient from i) mean and ii) median from the given data.

Age years	Number of people
less than 10	10
less than 20	25
less than 30	40
less than 40	45
less than 50	70
less than 60	85
less than 70	100
less than 80	110
less than 90	120

Solution

Given table can be written as continuous class as followes.

To compute M.D and its coefficient from mean

class nterval(A)	frequency(f)	Mid – value(m)	fm	$\lvert m-\bar{x}\rvert$	$f\lvert m-\bar{x}\rvert$
0 – 10	10	5	50	39.58	395.8
10 – 20	25 – 10 =15	15	225	29.58	443.7
20 – 30	45 – 25 =15	25	375	19.58	293.7
30 – 40	45 – 40 =5	35	175	9.58	47.9
40 – 50	70 – 45 =25	45	1125	9.42	10.5
50 – 60	85 – 70 =15	55	825	10.42	156.3
60 – 70	100 – 85 =15	65	975	20.42	306.3
70 – 80	110 – 100 =10	75	750	30.42	304.2
80 – 90	120 – 110 =10	85	850		1958.4
	N = 120		5350		

Mean $(\bar{x}) = \frac{\Sigma fm}{N} = \frac{5350}{120} = 44.58$

M.D. from means $= \frac{\Sigma f\lvert m-\bar{x}\rvert}{N}$

$= \frac{1958.4}{120}$

$= 16.32$

coefficient of M.D. from mean $= \frac{\text{M.D. from mean}}{\text{Mean}}$

$= \frac{16.32}{44.58}$

$=0.3661$

Again, to calculate M.D. from median and its coefficient we can write above data as follows.

(b)

Class Interval(Age)	Frequency(f)	c.f.	Mid – value(m)	$\lvert m-\bar{x}\rvert$	$f\lvert m-\bar{x}\rvert$
0 – 10	10	10	5	41	410
10 – 20	15	25	15	31	465
20 – 30	15	40	25	21	315

30 – 40	5	45	35	11	55
40 – 50	25	70	45	1	25
50 – 60	15	85	55	9	135
60 – 70	15	100	65	19	285
70 – 80	10	110	75	29	290
80 – 90	10	120	85	39	390
	N = 120				2370

Here, $= \frac{N}{2} = = \frac{120}{2} = 60$

c.f just greater 60 is 70 whose corresponding class is (40 – 70).

L = 40, i = 10, f = 25, c.f. = 45

$$\text{Median } (M_d) = L + \frac{\frac{N}{2} - c.f.}{f} \times i$$

$$= 40 + \frac{60 - 45}{25} \times 10$$

$$= 40 + \frac{15}{5} \times 2$$

$$= 46$$

$$\text{M.D. from means} = \frac{\Sigma f\,|m - \bar{x}|}{N}$$

$$= \frac{2370}{120}$$

$$= 19.75$$

$$\text{coefficient of M.D. from median} = \frac{\text{M.D. from median}}{M_d}$$

$$= \frac{19.75}{46}$$

$$= 0.4293$$

Standard Deviation (S.D)

Standard Deviation when a grouped data or continuous series is given.

1. By direct method,

where m = mid – value of class – interval.

2. By actual mean method, S.D.(σ)= $\sqrt{\frac{\Sigma f(m - \bar{x})^2}{N}}$

where $\bar{x}$ = arithmetic mean

3. By assumed mean method or short – cut method

$$\text{S.D.}(\sigma) = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2}$$

where d = m – a

a = assumed mean

m = mid – value of each class

$$\bar{x} = a + \frac{\Sigma fd}{N}$$

4. By step – deviation method:

$$\text{S.D.}(\sigma) = \sqrt{\frac{\Sigma fd^{1^2}}{N} - \left(\frac{\Sigma fd^1}{N}\right)^2} \times i$$

where, $d^1 = \frac{m - a}{i}$

i = common factor

$\overline{x} = a + \frac{\Sigma fd^1}{N} \times i$

5. Variance = $(\text{S.D.})^2 = \sigma^2$

6. Coefficient of standard deviation = $\frac{6}{\overline{x}}$

7. Coefficient of variation (C.V.)= $\frac{6}{\overline{x}} \times 100\%$

Some solved problems

1. Calculate S.D. and its coefficient for given continuous series.

(a) $\Sigma fm^2 = 566500$, N = 100, $\Sigma fm = 7400$, $\overline{x} = 31$

Solution

Standard deviation $(\sigma) = \sqrt{\frac{\Sigma fm^2}{N} - \left(\frac{\Sigma fm}{N}\right)^2}$

$= \sqrt{\frac{566500}{100} - \left(\frac{7400}{100}\right)^2}$

$= \sqrt{5665 - (74)^2}$

$= \sqrt{5665 - 5476}$

$= \sqrt{189} = 13.75$

Coefficient of S.D. $= \frac{6}{\overline{x}}$

$= \frac{13.75}{31} = 0.4435$

(a) $\Sigma fd^1 = -4$, $\Sigma fd^{1\,2} = 28$, N = 29, $\overline{x} = 24.5$, i = 10

Solution

Standard deviation $(\sigma) = \sqrt{\frac{\Sigma fd^{1^2}}{N} - \left(\frac{\Sigma fd^1}{N}\right)^2} \times 10$

$= \sqrt{\frac{28}{29} - \left(\frac{-4}{29}\right)^2} \times 10$

$= \sqrt{0.9655 - 0.0190} \times 10$

$= \sqrt{0.9965} \times 10$

$= 9.73$

Coefficient of S.D. $= \frac{6}{\overline{x}}$

$= \frac{9.73}{24.5} = 0.3971.$

2. Compute standard deviation and its coefficients from the following method.

i) direct method ii) short – cut method iii) step – deviation

class interval	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70
Frequency	10	8	32	40	22	18

Solution

i) To compute S.D. by direct method

Class– interval	Mid – value(m)	Frequency(f)	fm	fm^2
10 – 20	15	10	150	2250
20 – 30	25	8	200	5000
30 – 40	35	32	1120	39200
40 – 50	45	40	1800	8100
50 – 60	55	22	1210	66550
60 – 70	65	18	1170	76050
		N = 130	$\sum fm = 5650$	270050

From table, $N = 130$, $\sum fm = 5650$, $\sum fm^2 = 270050$,

Solution

Standard deviation (σ) = S.D.$(\sigma) = \sqrt{\frac{\Sigma fm^2}{N} - \left(\frac{\Sigma fm}{N}\right)^2}$

$= \sqrt{\frac{270050}{130} - \left(\frac{5650}{130}\right)^2}$

$= \sqrt{2077.3076 - 1888.9053}$

$= \sqrt{188.4023}$

$= 13.72$

Mean$(\bar{x}) = \frac{\Sigma fm}{N} = \frac{5650}{130} = 43.46$

Coefficient of S.D. $= \frac{6}{\bar{x}}$

$= \frac{13.72}{43.46} = 0.3157$

ii) To compute standard deviation by short – cut method or assumed mean method.

Let $a = 45$

Class– interval	mid – value(m)	frequency(f)	d= m – a	fm	fd^2
10 – 20	15	10	– 30	– 300	900
20 – 30	25	8	– 20	– 160	3200
30 – 40	35	32	– 10	– 320	3200
40 – 50	45	40	0	0	0
50 – 60	55	22	10	220	2200
60 – 70	65	18	20	360	7200
		N = 130		– 200	24800

Now, standard deviation is given by,

$$\sigma = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2}$$

$$= \sqrt{\frac{24800}{130} - \left(\frac{-200}{130}\right)^2}$$

$$= \sqrt{190.7652 - 2.3667}$$

$$= \sqrt{188.4025}$$

$$= 13.72$$

$$\text{Mean}(\bar{x}) = a + \frac{\Sigma fm}{N}$$

$$= 40 + \frac{-200}{130} = 45.46$$

$$\text{coefficient of S.D.} = \frac{6}{\bar{x}}$$

$$= \frac{13.72}{43.46} = 0.3157$$

iii) To calculate standard deviation by step deviation method method

Class– interval	mid – value(m)	frequency(f)	$d_1 = \frac{m - a}{i}$	fd^1	$fd^{1\,2}$
10 – 20	15	10	– 3	– 30	90
20 – 30	25	8	– 20	– 16	32
30 – 40	35	32	– 1	– 32	32
40 – 50	45	40	0	0	0
50 – 60	55	22	1	22	22
60 – 70	65	18	2	36	72
		N = 130		– 20	248

$$\text{Standard deviation (S.D.)} = \sigma = \sqrt{\frac{\Sigma fd^{1\,2}}{N} - \left(\frac{\Sigma fd^1}{N}\right)^2} \times i$$

$$= \sqrt{\frac{248}{130} - \left(\frac{(-20)}{130}\right)^2} \times 10$$

$$= \sqrt{1.9077 - 0.0237} \times 10$$

$$= \sqrt{1.88} \times 10$$

$$= 13.72$$

$$\text{Mean}(\bar{x}) = a + \frac{\Sigma fd^1}{N} \times 10$$

$$= 40 + \frac{-20}{130} \times 10 = 43.46$$

$$\text{Coefficient of S.D.} = \frac{6}{\bar{x}}$$

$$= \frac{13.72}{43.46} = 0.3157$$

3. Calculate the standard deviation and its coefficient of variation from the following.

x	$0 \le x \le 10$	$10 \le x \le 20$	$20 \le x \le 30$	$30 \le x \le 40$	$40 \le x \le 50$
f	7	10	14	12	6

Solution

Here, $0 \le x \le 10$ means value of x is from 0 to 10 exclusively, we can write x belongs to class 0 – 10 etc.

To calculate standard deviation of given data by using by step deviation method method

Let a = 25, then $d^1 = \frac{m-a}{i} = \frac{m-25}{i}$

Class	frequency(f)	Mid – value (m)	$d_1 = \frac{m-a}{i}$	fd^1	fd^{1^2}
0 – 10	7	5	– 2	– 14	28
10 – 20	10	15	– 1	– 10	10
20 – 30	14	25	0	0	0
30 – 40	12	35	1	12	12
40 – 50	45	45	2	12	24
	N = 49			0	70

Standard deviation = $\sigma = \sqrt{\frac{\Sigma fd^{1^2}}{N} - \left(\frac{\Sigma fd^1}{N}\right)^2} \times i$

$= \sqrt{\frac{74}{49} - \left(\frac{0}{49}\right)^2} \times 10$

$= \sqrt{1.5210} \times 10$

$= 12.33$

Mean$(\overline{x}) = a + \frac{\Sigma fd^1}{N} \times 10$

$= 25 + \frac{0}{49} \times 10 = 25$

Coefficient of S.D. $= \frac{6}{\overline{x}}$

$= \frac{12.33}{25} = 0.4932$

4. Calculate the standard deviation and its coefficient of variation from the following.

(a)

x	less than 10	less than 20	thess than 30	less than 40	less than 50
f	12	19	24	33	40

Solution

Given less than frequency table can be written in the following continuous class.

Let i = 25, then $d^1 = \frac{m-a}{i}$

Class	Mid – value (m)	frequency(f)	$d^1 = \frac{m-a}{i}$	fd^1	$fd^{1\,2}$
0 – 10	5	12	– 2	– 24	48
10 – 20	15	19 – 12 = 7	– 1	– 7	7
20 – 30	25	24 – 19 = 5	0	0	0
30 – 40	35	33 – 24 = 9	1	9	9
40 – 50	45	40 – 33 = 7	2	14	28
		N = 40		– 8	92

Standard deviation (σ) = $\sqrt{\frac{\Sigma fd^{1 2}}{N} - \left(\frac{\Sigma fd^1}{N}\right)^2} \times i$

$= \sqrt{\frac{92}{40} - \left(\frac{-8}{40}\right)^2} \times 10$

$= \sqrt{2.3 - 0.04} \times 10$

$= \sqrt{2.26} \times 10$

= 15.03

Mean($\bar{x}$) = $a + \frac{\Sigma fd^1}{N} \times i$

$= 25 + \left(\frac{-8}{40}\right) \times 10$

= 23

Coefficient of variatin = $\frac{6}{\bar{x}} \times 100\% = \frac{15.03}{23}$

= 65.35%

b)

x	above 20	above 40	above 60	above 80	above 100 and less than 120
f	50	42	30	18	7

Solution

Given more than cumulative frequency table can be written in the following continuous.

Let a = 70, then $d_1 = \frac{m-a}{i} = \frac{m-70}{10}$

Class	Mid – value (m)	frequency(f)	$d^1 = \frac{m-a}{i}$	fd^1	$fd^{1\,2}$
20 – 40	30	50 – 48 = 8	– 2	– 16	32
40 – 60	50	12	– 1	– 12	12
60 – 80	70	18	0	0	0
80 – 100	90	11	1	11	11
100 – 120	110	7	2	14	28
		N = 50		– 3	83

Standard deviation (σ) = $\sqrt{\frac{\Sigma fd^{1 2}}{N} - \left(\frac{\Sigma fd^1}{N}\right)^2} \times i$

$$=\sqrt{\frac{83}{50}-\left(\frac{83}{50}\right)^2}\times 20$$

$$=\sqrt{1.66-0.0036}\times 20$$

$$=\sqrt{1.6564}\times 20$$

$= 25.74$

$\text{Mean}(\bar{x}) = a + \frac{\Sigma fd^1}{N}\times i$

$= 70 + \left(\frac{-3}{50}\right)\times 20$

$= 68.8$

Coefficient of variation (c.v) $= \frac{6}{\bar{x}}\times 100\%$

$= \frac{25.74}{68.8}\times 100\%$

$= 37.41\%$

c)

Marks	0 – 10	0 – 20	0 – 30	0 – 40	0 –50
No. of students	7	18	30	42	50

Solution

This is less than cumulative frequency table. We can change it as continuous frequency table as follows.

Let a = 25, then $d^1 = \frac{m-a}{i} = \frac{m-25}{10}$

Class	frequency(f)	Mid – value (m)	$d_1 = \frac{m-a}{i}$	fd^1	$fd^{1\,2}$
0 – 10	7	5	– 2	– 14	28
10 – 20	18 – 7 =11	15	– 1	– 11	11
20 – 30	30 – 18 = 12	25	0	0	0
30 – 40	42 – 30 = 12	35	1	12	12
40 – 50	50 – 42 = 8	45	2	16	32
	N = 50		0	3	83

Standard deviation $= \sigma = \sqrt{\frac{\Sigma fd^{1\,2}}{N}-\left(\frac{\Sigma fd^1}{N}\right)^2}\times i$

$$=\sqrt{\frac{83}{50}-\left(\frac{3}{50}\right)^2}\times 10$$

$$=\sqrt{1.66-0.0036}\times 10$$

$= 12.87$

$\text{Mean}(\bar{x}) = a + \frac{\Sigma fd^1}{N}\times i$

$= 25 + \frac{3}{50}\times 10$

$= 25 + 0.6$

$= 25.6$

$c.v = \frac{6}{x} \times 100\%$

$= \frac{12.87}{25.6} \times 100\ \% = 50.27\ \%$

5. From the given data which series is more variable (inconsistent).

Variable		10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70
Frequency	section A	10	18	32	40	22	18
	section B	18	22	40	32	20	10

Solution

We calculate coefficient of variation two compare variability of two givens data. More c.v. more will be variability.

class		Series A				B			
	m	fA	$d_A{}^1 = \frac{m-45}{10}$	$f_Ad_B{}^1$	$f_Ad_A{}^{1\,2}$	f_B	$d_B{}^1 = \frac{m-45}{10}$	$f_Bd_B{}^1$	$f_Bd_B{}^1$
10 – 20	15	10	– 3	– 30	90	18	– 3	– 54	162
20 – 30	25	18	– 2	– 36	72	22	– 2	– 44	88
30 – 40	35	32	– 1	– 32	32	40	– 1	– 40	40
40 – 50	45	40	0	0	0	32	0	0	0
50 – 60	55	22	1	22	22	20	1	20	21
60 – 70	65	18	2	36	72	10	2	20	40
		$N_A =$ 140		– 40	288	142 = N_B		– 98	351

For series A,

$$\text{Standard deviation } (\sigma_A) = \sqrt{\frac{\Sigma f_Ad_A{}^{1\,2}}{N} - \left(\frac{\Sigma f_Ad_A{}^1}{N}\right)^2} \times i$$

$$= \sqrt{\frac{288}{140} - \left(\frac{-40}{140}\right)^2} \times 10$$

$$= \sqrt{2.0571 - 0.0816} \times 10$$

$$= \sqrt{1.9755} \times 10$$

$= 14.05$

$$\text{Mean}(\overline{x}_A) = a + \frac{\Sigma f_Ad_A{}^1}{N_A} \times i$$

$$= 45 + \left(\frac{-40}{140}\right)^2 \times 10$$

$= 42.14$

$$\text{c.v. (A)} = \frac{6_A}{\overline{x}_A} \times 100\%$$

$= \frac{14.05}{42.14} \times 100\ \% = 33.34\ \%$

Again for series B,

Standard deviation (σ_B)= $\sqrt{\frac{\Sigma f_B {d_B^1}^2}{N_B} - \left(\frac{\Sigma f_B d_B^1}{N_B}\right)^2} \times i$

$= \sqrt{\frac{351}{142} - \left(\frac{-98}{142}\right)^2} \times 10$

$= \sqrt{2.4718 - 0.4763} \times 10$

$= \sqrt{1.9955} \times 10$

$= 1.413$

Mean($\bar{x}_B$)= $a + \frac{\Sigma f_B d_B^1}{N_B} \times i$

$= 45 + \left(\frac{-98}{142}\right)^2 \times 10$

$= 38.09$

c.v. (B) $= \frac{6_B}{\bar{x}_B} \times 100\%$

$= \frac{14.13}{38.09} \times 100\ \% = 37.01\ \%$

Since c.v. (B) > c.v. (A), the series B is more variable or more inconsistent.

Questions for practice

1. If $Q_1 = 45$, Q.D = 20, find the third quartile and coefficient of Quartile deviation.
2. If coefficient of Q.D is 0.5, third quartile is 25, find the first quartile.
3. Find the Quartile Deviation and its coefficient for the following given data.

(a)

x	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90
f	2	4	6	12	15	12	6	4	3

(b)

x	less than 20	less than 30	less than 40	less than 50
f	5	10	20	25

(c)

x	more than 10	more than 20	more than 30	more than 40
f	40	30	20 10	5

(d)

Marks	0 –10	0 – 20	0 – 30	0 – 40	0 – 50
No. of students	5	15	25	40	50

(e)

Marks	0 – 50	0 – 40	0 – 30	0 – 20	0 – 10
No. of students	40	30	20	10	5

4. Taking class interval 10 find the quartile deviation from the given data

40, 50, 60, 70, 50, 80, 70, 90, 13, 22

70, 80, 50, 60, 70, 85, 95, 65, 50, 45

22, 45, 60, 70, 85, 70, 90, 72, 55, 49

5. If Q_1 = 32, find the value of x and then quartile deviation from given date.

x	0 – 20	0 – 40	0 – 60	0 – 80	0 – 100
f	10	20	30 + x	40 + x	50 + x

Mean Deviation

1. **Find the coefficient of mean deviation of a continuous series having 20 samples whose mean is 40 and $\sum f|d| = 240$.**
2. **Find the coefficient of mean deviation from median whose median is 35 and number of item is 40 and $\sum f|d| = 440$.**

3. Find the mean deviation from mean and its coefficients.

(a)

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
No. of students	4	10	7	6	3

(b)

x	0 – 10	0 – 20	0 – 30	0 – 40	0 – 50
f	5	13	25	40	50

4. Find the mean deviation from median and its coefficient.

(a)

x	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
y	5	10	15	10	5

(b)

x	0 – 10	0 – 20	0 – 30	0 – 40	0 – 50
f	2	5	10	15	20

(c)

Marks	10 – 20	20 – 30	20 – 40	40 – 50
No. of students	5	7	15	3

Standard Deviation:

1. **In a continuous series, N = 25, $\sum fd = 480$, $\sum fd^2 = 3240$, $d=(m - \overline{x})$ find the coefficient of standard deviation.**

2. **In a continuous series N = 40, $\sum fm = 100$, $\sum fd^2 = 4060$ find the coefficient of standard deviation.**

2. **In a continuous series N = 40, $\sum fm = 100$, $\sum fd^2 = 4060$ find the coefficient of standard deviation.**

3. **In a continuous series N = 40, $\sum fd^1 = 7$, $\sum fd^{1^2} = 75$, i = 10, assumed mean a = 20, find the standard deviation and its coefficient**

4. **Find the standard deviation and its coefficient from the following data :**

(a)

x	10 – 20	20 – 30	20 – 40	40 – 50	50 – 60
y	10	15	20	10	5

(b)

x	0 –6	0 – 12	0 – 18	0 – 24	0 – 30
y	5	10	15	20	25

(c)

marks	less than 10	less than 20	less than 30	less than 40	less than 50
No. of students	5	9	12	20	25

5. **Prepare a frequency distribution table taking class interval 10. Calculate the standard deviation and its coefficient.**

20, 22, 24, 25, 28, 10, 22, 70, 80, 45

33, 45, 37, 80, 75, 95, 80, 75, 78, 88

60, 66, 65, 68, 78, 90, 88, 78, 79, 90