Approximation of Integrals with 2D and 3D Surfaces of Absolute Errors using Various Quadrature Methods

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Final project submission for credits towards the course Math 572: Advanced Topics in Numerical Analysis: Numerical Methods & Their Analysis

Southern Illinois University

June 24, 2024

Abstract

Hello

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※ Introduction

Let us suppose we have a function $f \in L^1([a,b])$ in the space of absolutely integrable function with respect to Lebesgue measure. Now consider approximation methods of integrals,

$$I_a^b(f) := \int_a^b f(x)dx$$

This integral can be evaluated using the classical calculus methods. Our study revolves around various methods approximate this integral such that the error is minimized in an ideal setting. The way to address this problem is by making using of Quadrature Methods to eventually figure out the quadrature errors.

Definition 1.1. Let $f : [a,b] \to \mathbb{R}^d$ be an integrable, real-valued function and

$$a \leq x_{1,n} < x_{2,n} ... << x_{k,n} < ... < x_{n,n} = b$$

be a partition of given [a, b]. Then

$$Q_a^b(f,n) := \sum_{k=1}^{n} w_{k,n} f(x_{k,n})$$

is called **quadrature methods** (also **numerical integral of f**) with **weights** $w_{k,n}$, **nodes** $x_{k,n}$ (also **instants, nodal and sample points**), where $n \in \mathbb{N}$ is the number of instances. The related **(quadrature)** error on [a, b] is defined by,

$$\epsilon_n(f,a,b) := \left\| I_a^b(f) - Q_a^b(f,n) \right\|_d$$

The quadrature method $Q_a^b(f,n)$ is said to be **convergent to** $I_a^b(f)$ iff

$$\lim_{n\to+\infty} \epsilon_n(f,a,b) = 0$$

The quadrature method $Q_a^b(f,n)$ is called **convergent to** $I_a^b(f)$ **with order (rate) r > 0 with respect to function class** F iff

$$\forall f \in F \exists\ c = c(f,a,b) \geq 0, \forall n > 0$$

$$\epsilon_n(f,a,b) \leq \frac{c(f,a,b)}{n^r}$$

In our study we will investigate the different quadrature methods as suggested in section of Methodology and check the convergence and error rate based on the theorems discussed in the Math 572 class [1]. The detailed methods will be discussed in the section of Methodology. The Results section will comprise of 3 dimensional error plots the function with different methods. In the Conclusion section we will articulate the theorems and output that we generated in form of error plots. The last section of this study consists of python codes for generating same plots. Our aim of including codes as supplementary for reproducibilty of our work.

※ Methodology

2.1 Quadrature Methods

In this section we will discuss the methods we used to estimate error in detail. Let us begin by considering a **Left Hand Riemann Quadrature Methods (LRMs)**

$$Q_a^b(f,n) := \sum_{k=1}^{n} f(x_{k-1})\Delta x_k$$

along partitions

$$a = x_0 < x_1 < ... < x_k < ... < x_n = b$$

of [a, b] with $\Delta x_k = x_k - x_{k-1}$. This presents an integral-type approximation with kernel (as elementary step function),

$$f_n(x) = I_{(a)}(x)f(x_0) + \sum_{k=1}^{n} I_{(x_{k-1},x_k]}(x)f(x_{k-1})$$

where $I_S(x)$ denotes the indicator function of subscribed set S, that is

$$I_S(x) = \begin{cases} 1 \text{ iff } x \in S \\ 0 \text{ iff } x \notin S \end{cases}$$

Similarly, we can also have **Right Hand Riemann Quadrature Methods (RRMs)** and **Midpoint Riemann Quadrature Methods (MPRMs)**.
The quadrature of right hand riemann quadrature and midpoint riemann quadrature can be give as follows,

$$Q_a^b(f,n) := \sum_{k=1}^{n} f(x_k)\Delta x_k$$

$$Q_a^b(f,n) := \sum_{k=1}^{n} f(\frac{x_k + x_{k-1}}{2})\Delta x_k$$

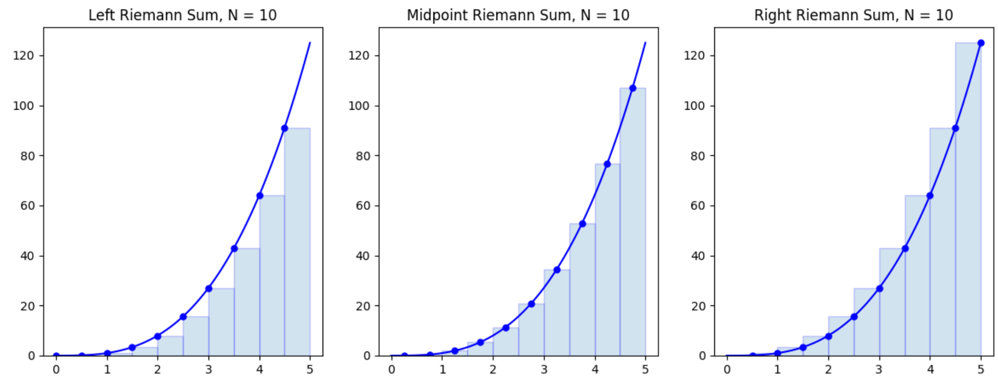


Figure 1: 2D plot of an approximation of a function

Apart from the Riemann Quadrature we also explored the possibility of inclusion of **Trapezoidal Method** and the **Simpsons Method** for our investigation.

The **Trapezoidal Method** is essentially also called as **Outer Midpoint Methods** because it obeys the scheme,

$$TM_a^b(f,n) = \sum_{k=1}^{n} \frac{f(x_k) + f(x_{k-1})}{2} \cdot \Delta x_k$$

whereas **Inner Theta Methods (ITM)** with implicitness constants $\theta \in [0,\ 1]$ are governed by,

$$Q_a^b(f,n) = \sum_{k=1}^{n} f(\theta x_k + (1-\theta)x_{k-1}) \cdot \Delta x_k$$

The scheme we will be using for our quadrature method study is as follows

$$T_a^b(f,n) := \sum_{k=1}^{n} \frac{f(x_k) + f(x_{k-1})}{2} \cdot \Delta x_k \textbf{ as Trapezoidal Method,}$$

$$S_a^b(f,n) := \sum_{k=1}^{n} \frac{f(x_{k-1}) + 4f(\bar{x_k}) + f(x_k)}{6} \cdot \Delta x_k \textbf{ as Simpsons Method,}$$

2.2 Experiment

In our study we are interested to investigate a continuous function with different quadrature methods. Let us take any real positive constant b. Consider $x \in [0,b] \longmapsto$ f(x) = x^p. The related absolute error is given as

$$\epsilon_n(b,p) := \left| \int_0^b x^p dx - Q_0^b(f,n) \right|$$

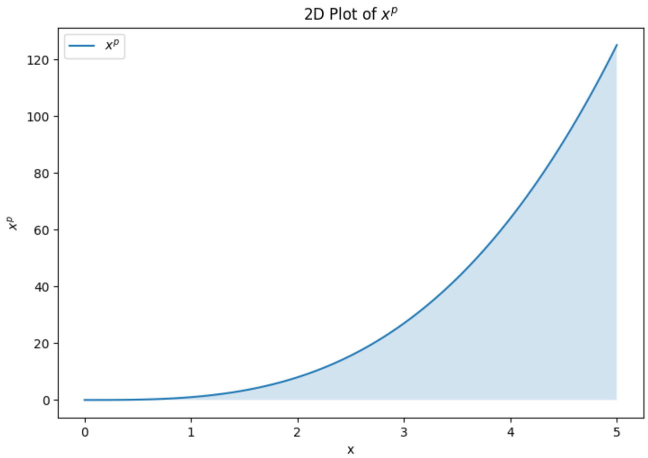


Figure 2: 2D plot of the function x^p

Here $Q_0^b(f, n)$ is the left hand Riemann quadrature (LRM) of $f(x) = x^p$ on [0, b]. If we were to use calculus to compute the integral of the function, we know that,

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1}$$

With this we can compute the exact value of the integral. For example if b = 5, p = 0.5 and n=10, the integral value is 7.4535599249993005.

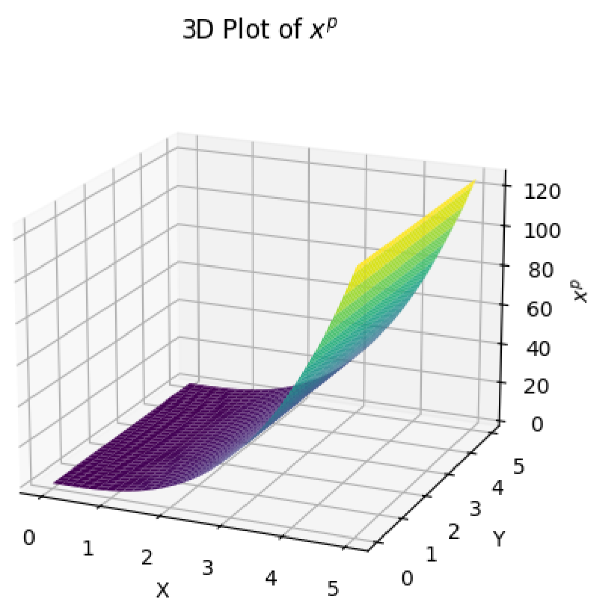


Figure 3: 3D plot of the function x^p

We will try to construct different test cases for plotting the three dimensional surface of absolute error $\epsilon_n(b, p)$ for

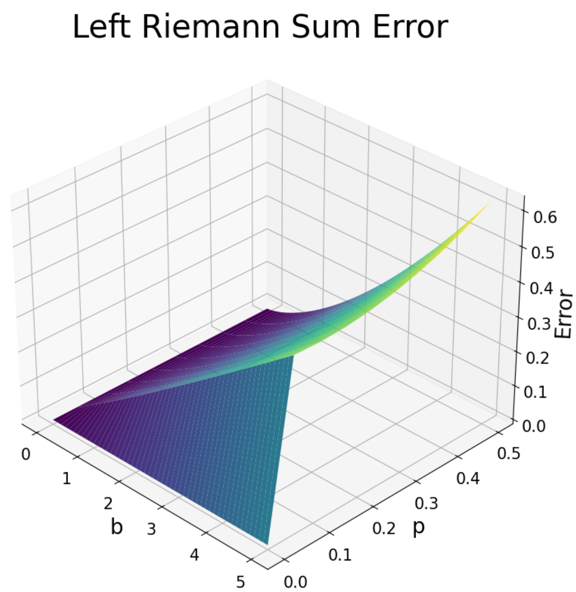
- fixed p = 0.5, 1, 2 versus n $\in \mathbb{N}$ and b $\in \mathbb{R}_+$,
- fixed b = 1, 10, 100, versus n $\in \mathbb{N}$ and $0 < p \leq 3$,
- fixed n = 100, versus b > 0 and p > 0.

※ Results

In this section, we will explore and produce some surface error plots based on the different values of p. b and n.

3.1 Experiment 1

For our first experiment, we will fix p = 0.5, n = 10 and b = 5



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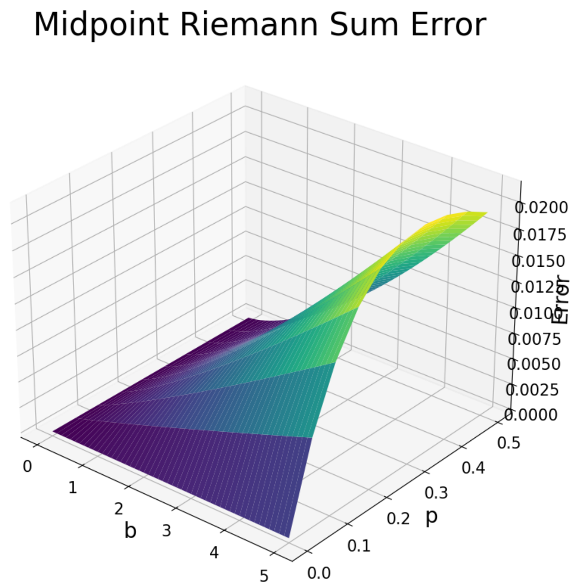

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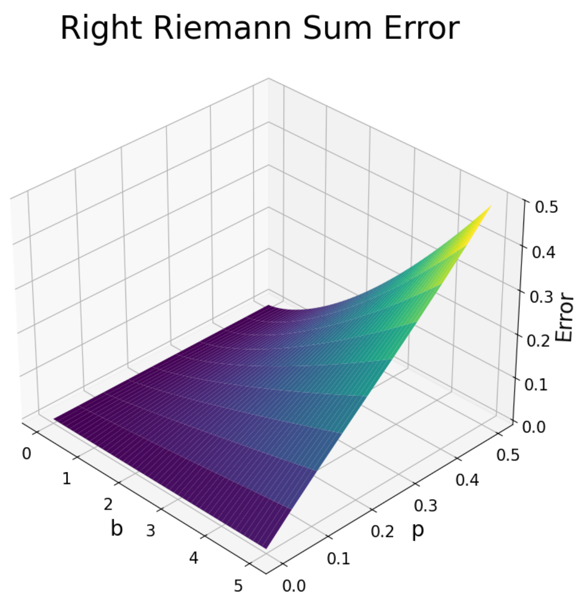

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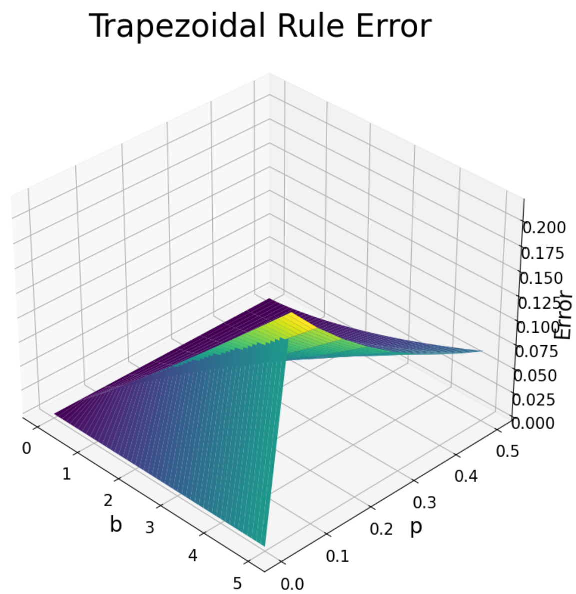

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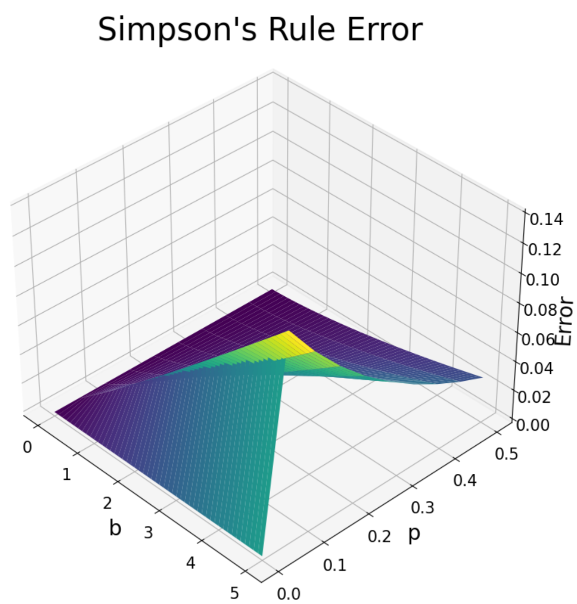
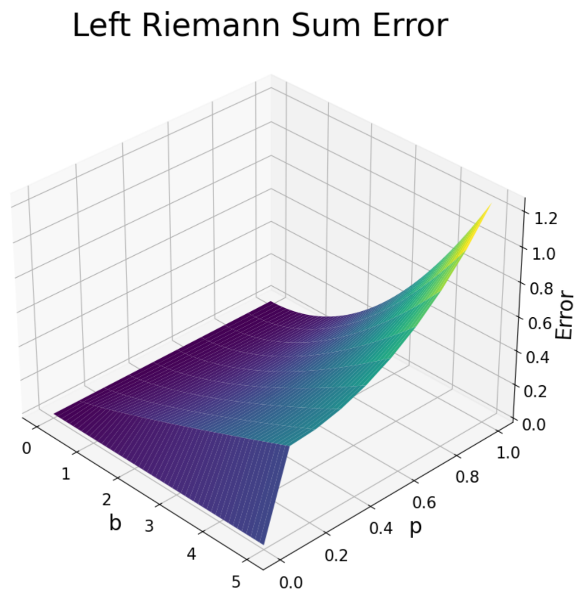


Figure 8

3.2 Experiment 2

we will fix p = 1, n = 10 and b = 5



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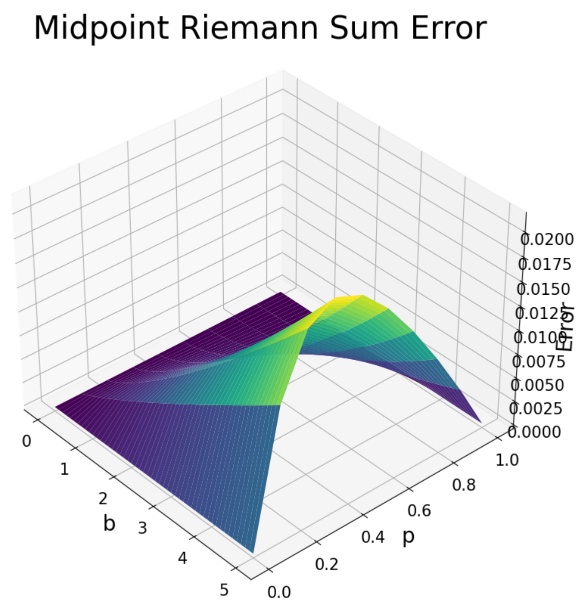

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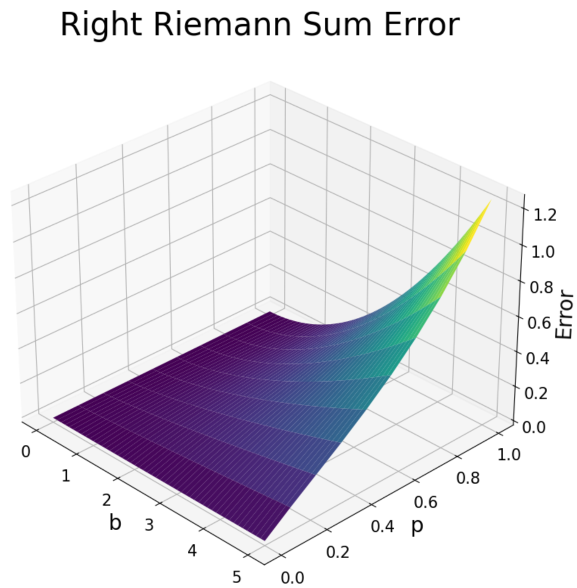

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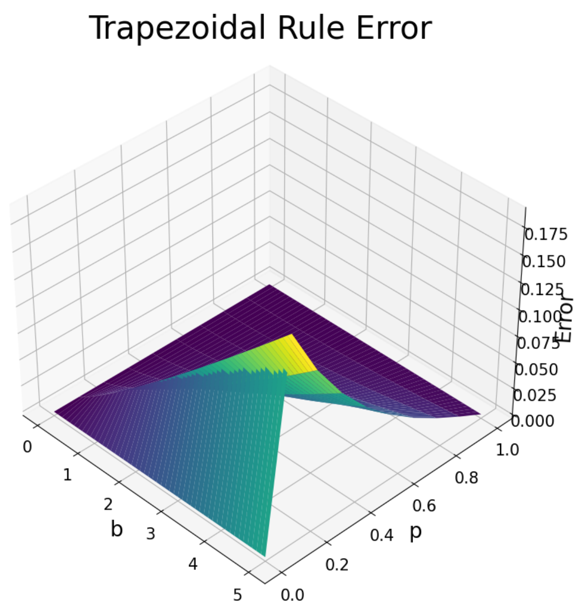

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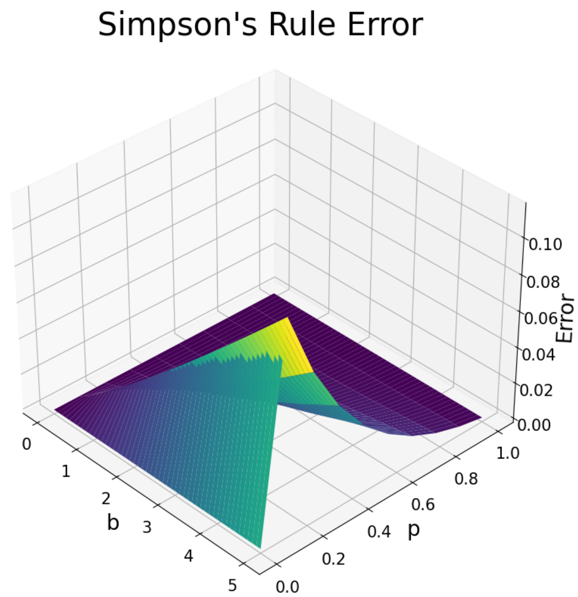
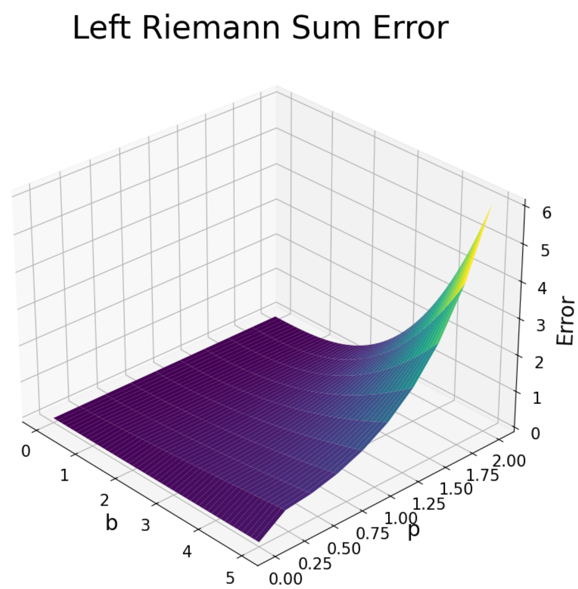


Figure 13

3.3 Experiment 3

we will fix p = 2, n = 10 and b = 5



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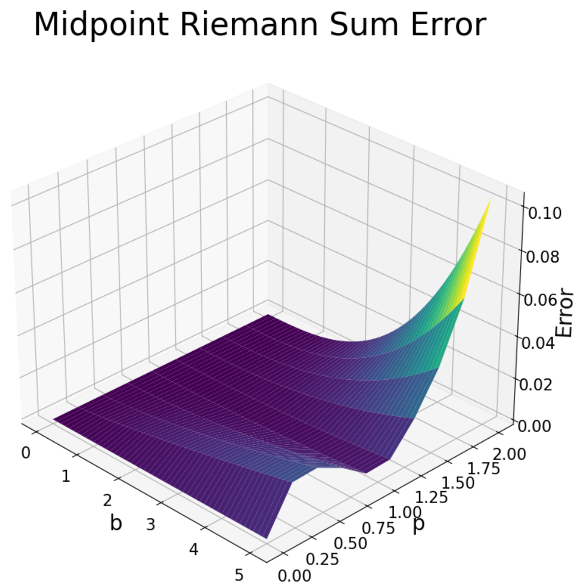

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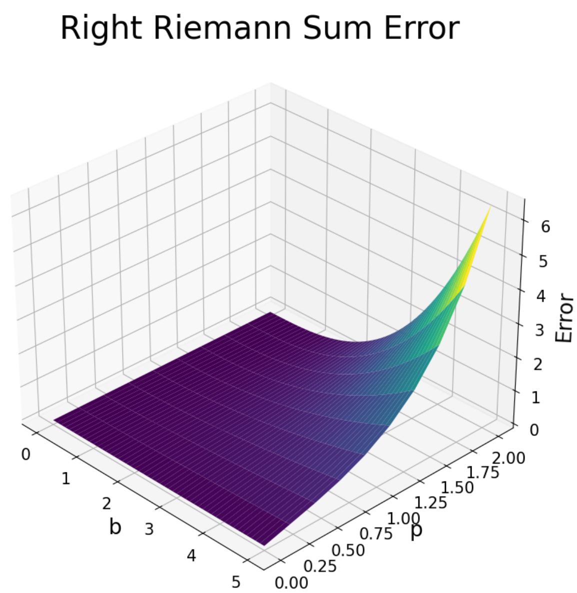

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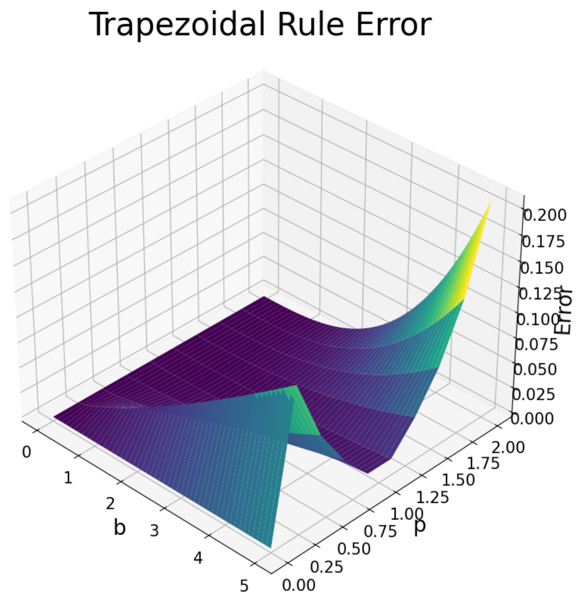

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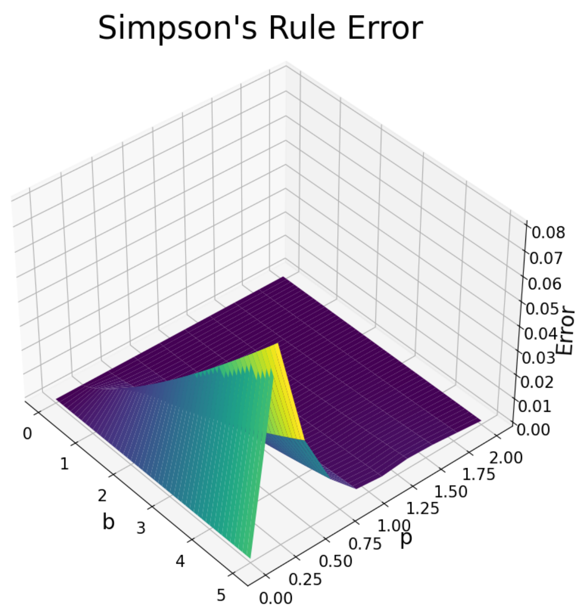
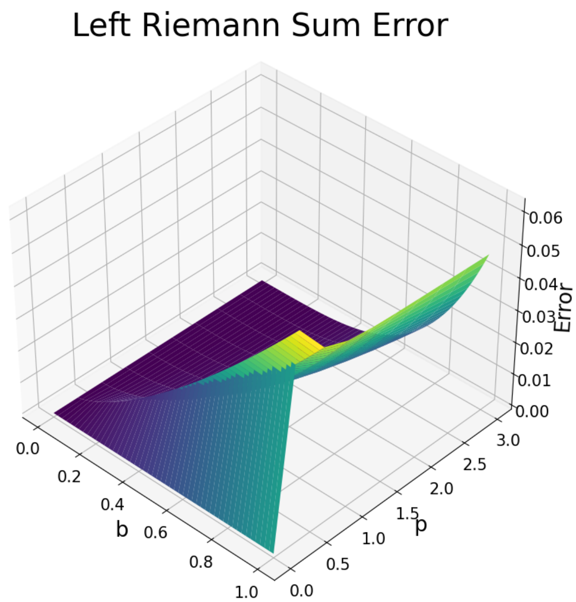


Figure 18

3.4 Experiment 4

we will fix p = 3, n = 10 and b = 1



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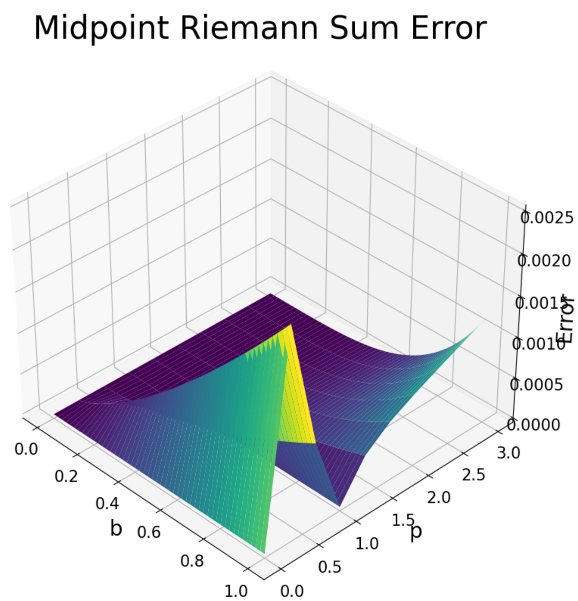

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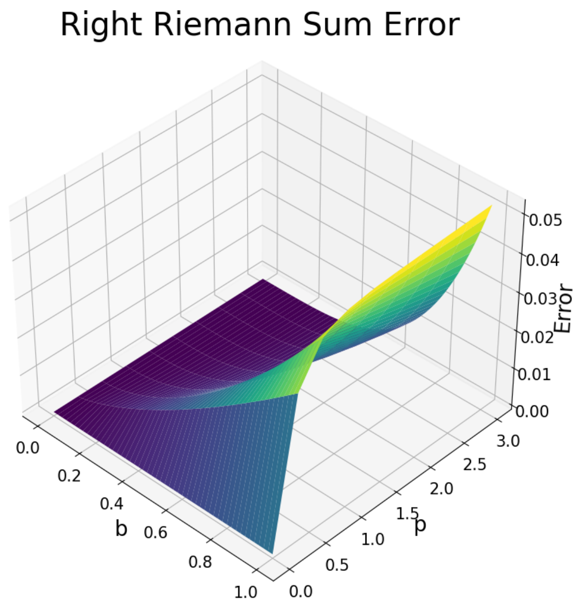

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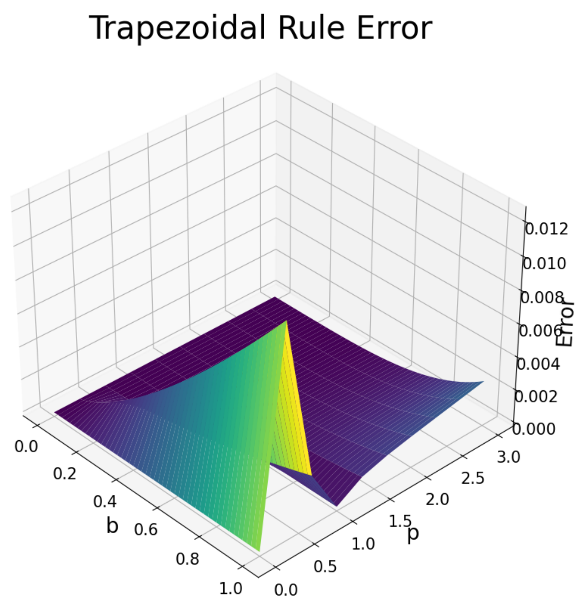

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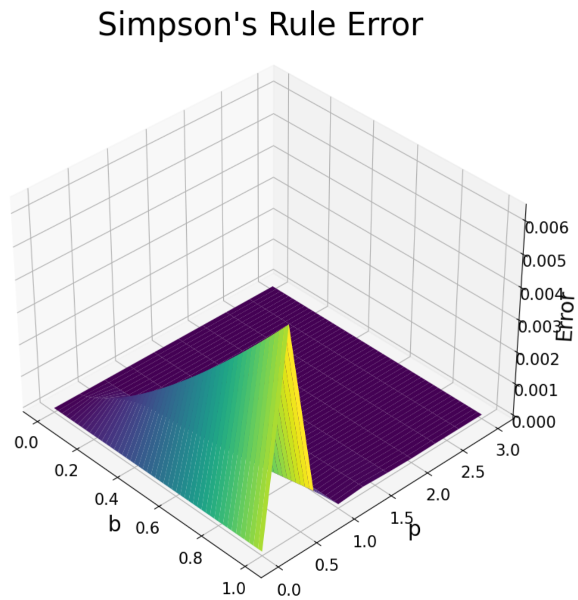
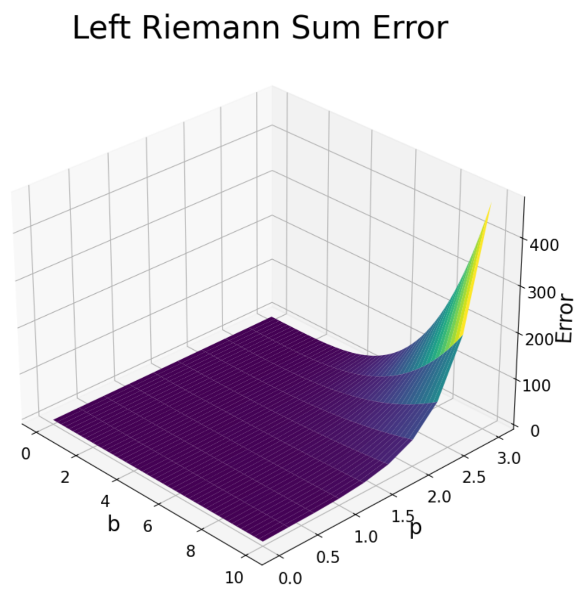


Figure 23

3.5 Experiment 5

we will fix p = 3, n = 10 and b = 10



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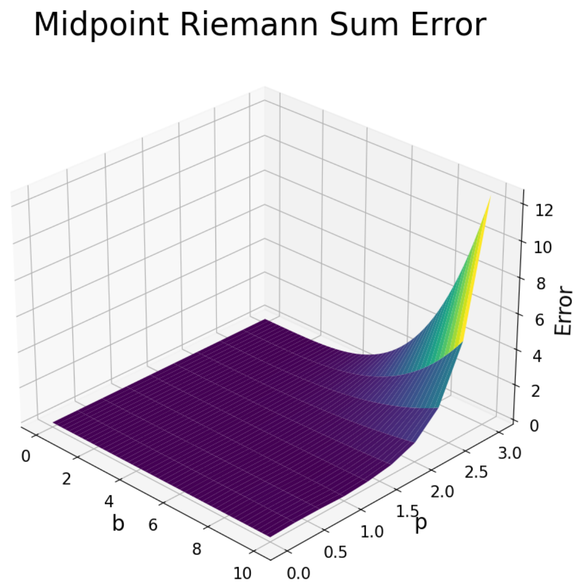

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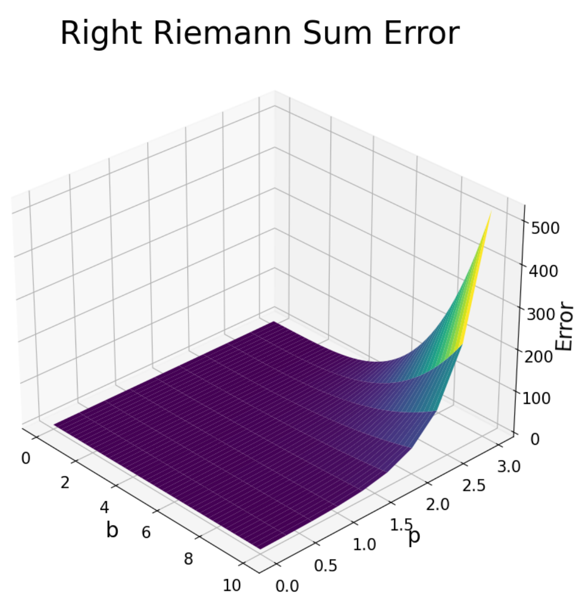

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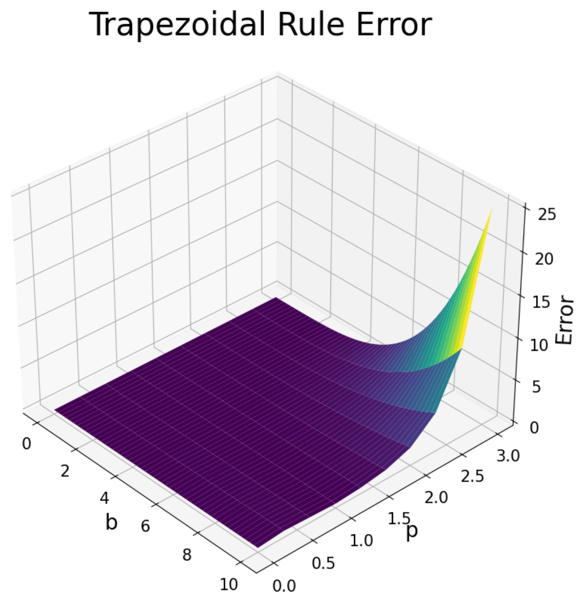

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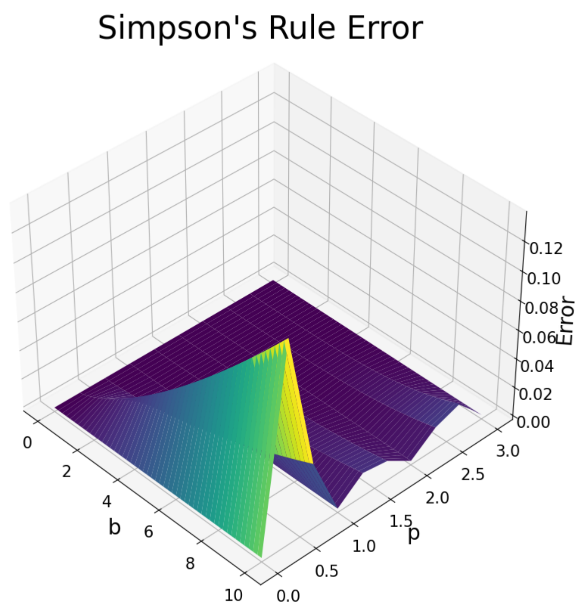
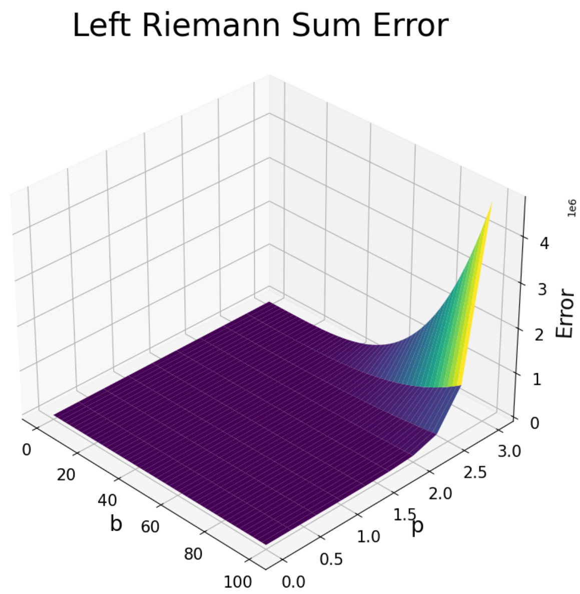


Figure 28

3.6 Experiment 6

we will fix p = 3, n = 10 and b = 100



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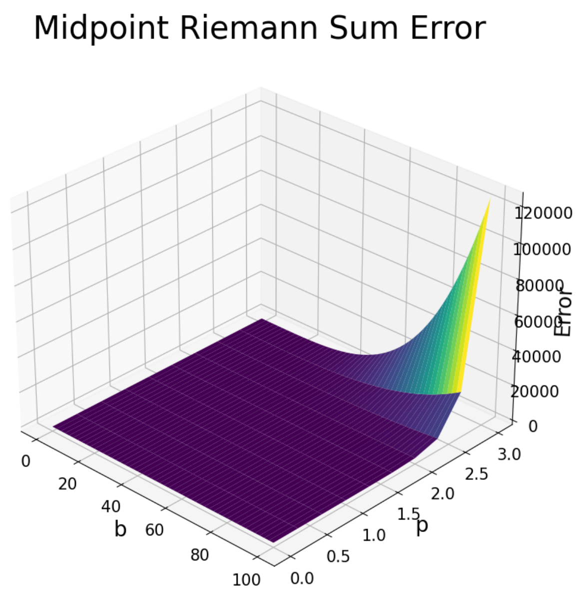

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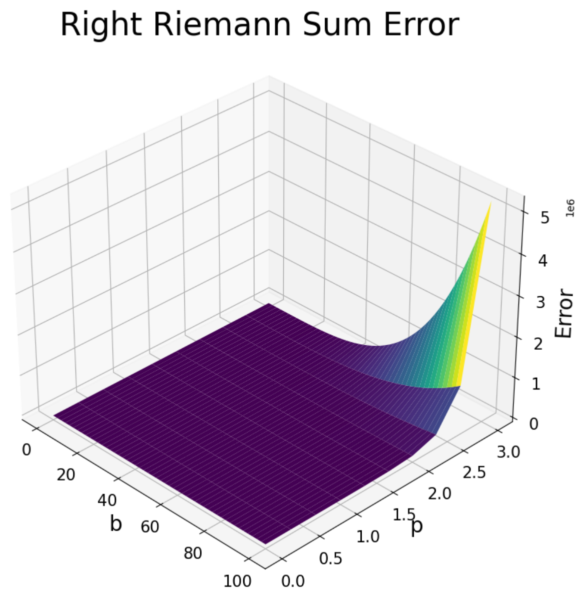

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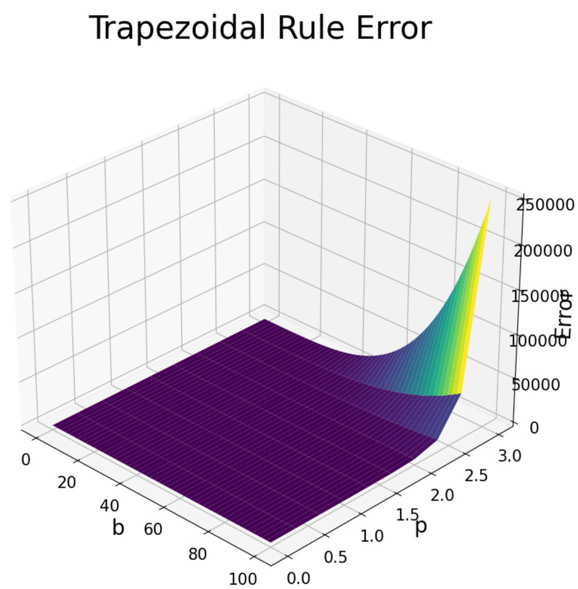

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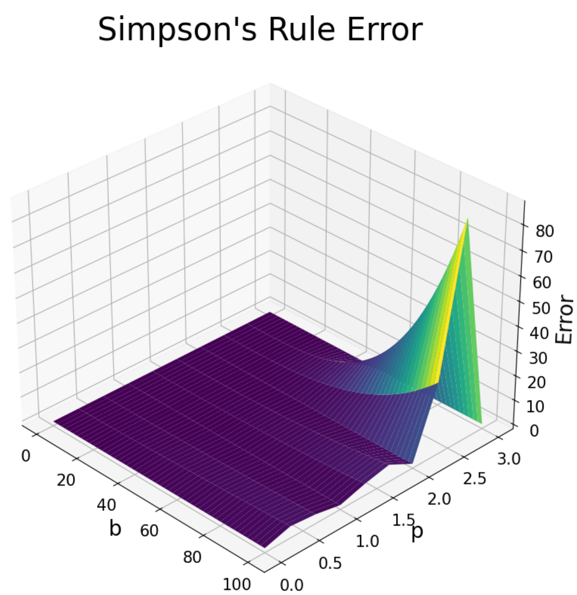
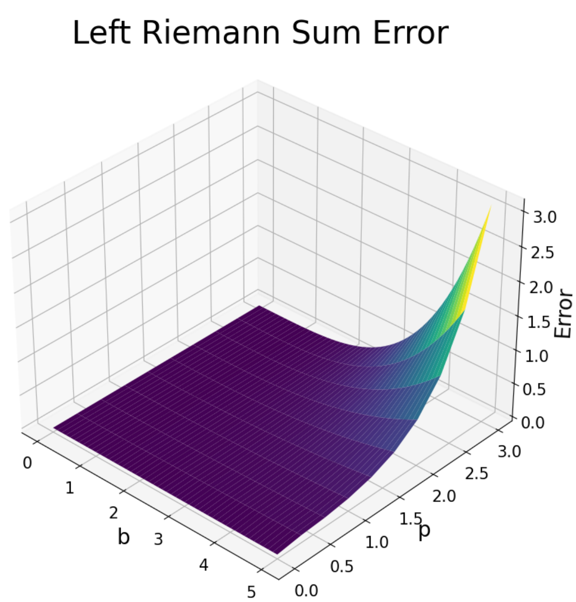


Figure 33

3.7 Experiment 7

we will fix p = 3, n = 100 and b = 5



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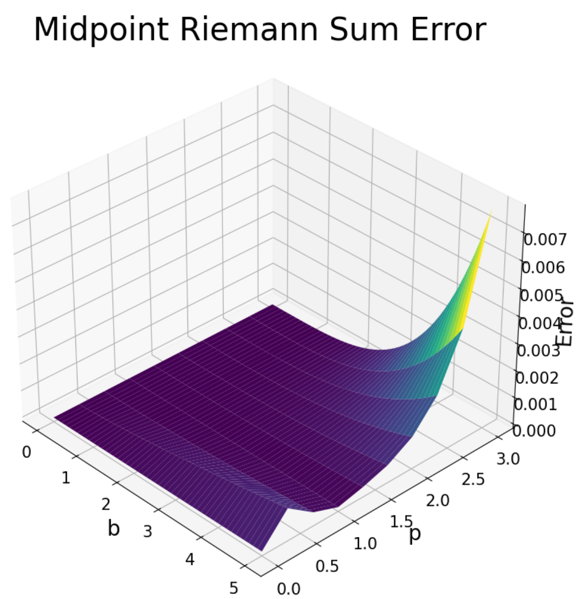

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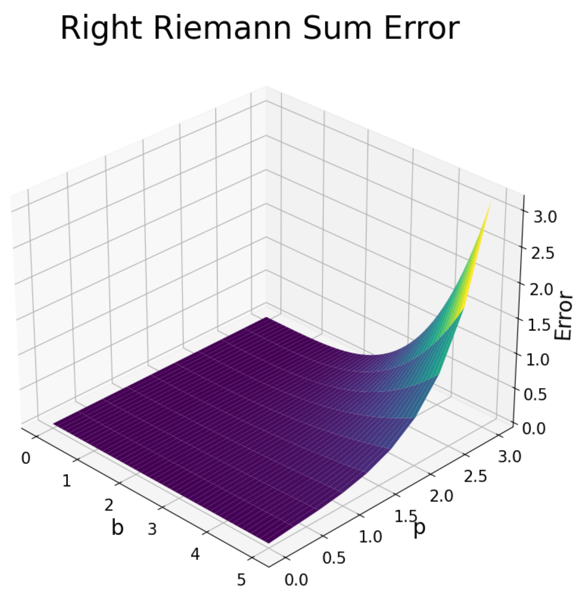

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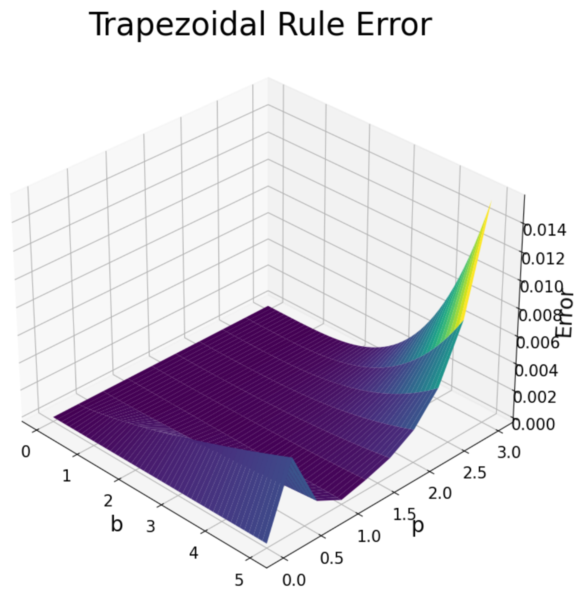

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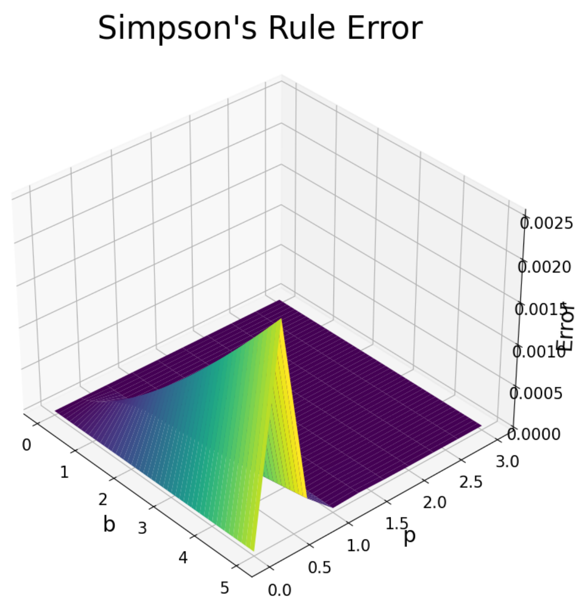

Figure 38

※ Conclusion

※ References

References

[1] H. Schurz. *Lecture Notes of Approximation Theory.* SIU, Carbondale, 2016

[2] LastName, FirstName. “Webpage Title”. WebsiteName, OrganizationName. Online; accessed Month Date, Year.
`www.URLhere.com`

※ Supplementary: Python programs