MATH 507: Partial Differential Equations
Lecture Notes

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Contents

1 Review

1.1 Directional derivative

Let $\mathbb{R}^n : \bar{x} = (x_1, x_2, x_3, ..., x_n)$ and let f be a function mapping from $\mathbb{R}^n$ to $\mathbb{R}$, that is, $f : \mathbb{R}^n \to \mathbb{R}$ such that,

$$f(\bar{x}) = f(x_1, x_2, x_3, ..., x_n))$$

Given a vector $\bar{v} = (v_1, v_2, v_3, ..., v_n) \in \mathbb{R}^n$, then the direction derivative of $f(\bar{x})$ along with $\bar{x}$ is defined as,

$$\frac{\partial f}{\partial \bar{v}} = \nabla_{\bar{x}} f(\bar{x}) = lim_{h\to 0}\frac{f(\bar{x}+h\bar{v}) - f(\bar{x})}{h} \tag{1}$$

If $f \in C^1$ then,

$$\frac{\partial f}{\partial \bar{v}} = Df \cdot \bar{v} \tag{2}$$

Here, $Df(\bar{x}) = (f_{x_1}, f_{x_2}, f_{x_3}, ..., f_{x_n})$ is the gradient.
When $\bar{v} = \bar{e_1} = (1, 0, 0, ..., 0)$, then

$$\begin{aligned}
\frac{\partial f}{\partial e_1} &= (f_{x_1}, f_{x_2}, f_{x_3}, ..., f_{x_n}) \cdot (1, 0, 0, ..., 0) \\
\frac{\partial f}{\partial e_1} &= f_{x_1} = \frac{\partial f}{\partial x_1} \\
... &= ... \\
\frac{\partial f}{\partial e_j} &= f_{x_j} = \frac{\partial f}{\partial x_j}
\end{aligned}$$

If $\frac{\partial f}{\partial \bar{x}} = 0$ for any $\bar{x} \in \mathbb{R}^n$, then $f(\bar{x} + s\bar{v}) =$ constant for some s $\in \mathbb{R}$.

$$\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{d}{ds}(x_1 + sv_1, x_2 + sv_2, x_3 + sv_3, ..., x_n + sv_n) \\
&= \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial s}(x_1 + sv_1) + \frac{\partial f}{\partial x_2}\frac{\partial f}{\partial s}(x_2 + sv_2) + \frac{\partial f}{\partial x_3}\frac{\partial f}{\partial s}(x_3 + sv_3) + ... + \frac{\partial f}{\partial x_n}\frac{\partial f}{\partial s}(x_n + sv_n) \\
&= Df \cdot \bar{v} \\
&= 0
\end{aligned}$$