# Lecture 2: Sets of Numbers, Propositional Logic

http://book.imt-decal.org, Ch. 1.3, 1.4

### Introduction to Mathematical Thinking

### September 12th, 2018

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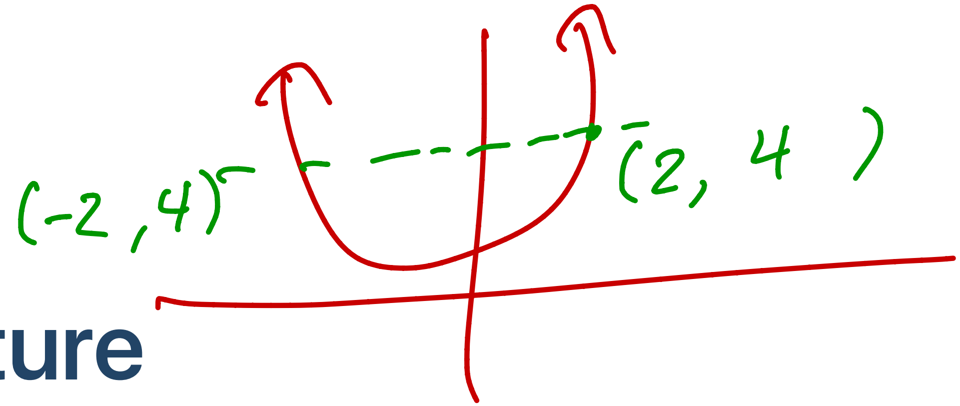
# Announcements

- Midterm date changed to **Wednesday, October 10th** (because of CS 61A midterm). No conflicting dates with 61A/61B/16A/16B.
- Homework 2 will be released after lecture, and is due Monday at 6:30PM.
- Last week's lecture video is up. Today's should be up by soon (done with half of it), along with a walkthrough of the homework problems.
- Office hours are official: Tuesday 2-3PM and Thursday 5-6PM, both in Cory 299.

$2 \neq -2$




# Recap from last lecture

We studied three types of functions.

- **Injections (one-to-one)**: No two elements in the domain map to the same element in the codomain

  $f(x_1) = f(x_2) \implies x_1 = x_2$

- **Surjections (onto)**: Every element in the codomain is mapped to by some element in the domain

  all possible outputs { codomain = range } set of all actual outputs

- **Bijections**: Both injective and surjective

# 1.3: Sets of Numbers

## We've seen the following sets of numbers before:

- Natural numbers
- Whole numbers
- Integers
- Rational numbers
- Irrational numbers
- Real numbers
- Complex numbers

We want to determine the relative sizes of each of these sets.

$|S|$ : # of unique elements in $S$

$|\{1, 2, 2, 8\}| = 3$ $\quad |\{1, 2, 3, 4, 5, \ldots\}| = ???$

Let's extend our definition of cardinality to support sets with infinitely many elements:

> **Definition: Cardinality**
>
> We say two sets have the same cardinality if and only if there exists a bijection between the two sets.

$$A = \{a_1, a_2, a_3\}$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$B = \{b_1, b_2, b_3\}$$

this definition holds for finite sets as well

# Natural Numbers

> **Definition: Natural numbers**
>
> The **natural numbers** (also known as the counting numbers), denoted by $\mathbb{N}$, are the most primitive numbers; ones that occur trivially in nature that can be used to count a (non-zero) number of things.

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

\mathbb{N}

# Whole Numbers

> ### Definition: Whole numbers
>
> The set of **whole numbers**, denoted by $\mathbb{N}_0$, is the union of the set of counting numbers with the number 0.

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, ...\} = \{0\} \cup \mathbb{N}$$

Is there a bijection between the natural numbers and whole numbers?

$\mathbb{N}$ $1, 2, 3, 4, 5, 6, \ldots$

$\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$

$\mathbb{N}_0$ $0, 1, 2, 3, 4, \ldots$

yes bijection $f(x) = x - 1$

Injective

$f(a) = f(b) \Longrightarrow a = b$

$a - 1 = b - 1 \Longrightarrow a = b$

$|\mathbb{N}| = |\mathbb{N}_0|$

Surjective

$f: c+1 \longmapsto c$

every whole number eventually appears

$f: A \longrightarrow B$

$f: x \longmapsto x^2$

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

| $\mathbb{N}$ | $S$ |
|---|---|
| 1 | $s_1$ |
| 2 | $s_2$ |
| 3 | $s_3$ |
| $\vdots$ | $\vdots$ |

### Definition: Countably infinite

We say set $S$ is **countably infinite** if and only if there exists a bijection from the natural numbers (or any other countable set) to $S$. If such a bijection does not exist, we say $S$ is **uncountably infinite**.

One way to think of this is to give each number a waiting number in an infinitely long line!

there exists an ordering of the set

$f: \mathbb{N} \to S$

$f: \mathbb{N}_0 \to S$

OR $f: S \to \mathbb{N}$, $S \to \mathbb{N}_0$

$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z}$

## Integers

> ### Definition: Integers
>
> The set of integers, denoted by $\mathbb{Z}$, is the union of the whole numbers with their negatives

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

Are the integers countably infinite, or uncountably infinite?

yes!

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\mathbb{N}_0$ | 0, | 1, | 2, | 3, | 4, | 5, | 6, | ... |
| $\mathbb{Z}$ | 0, | 1, | $-1$, | 2, | $-2$, | 3, | $-3$, | ... |

$$f(x) = \begin{cases} \dfrac{-x}{2} & x \text{ is even} \\[2ex] \dfrac{x+1}{2} & x \text{ is odd} \end{cases}$$

eventually will get to any integer I want

BIJECTION

$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}$

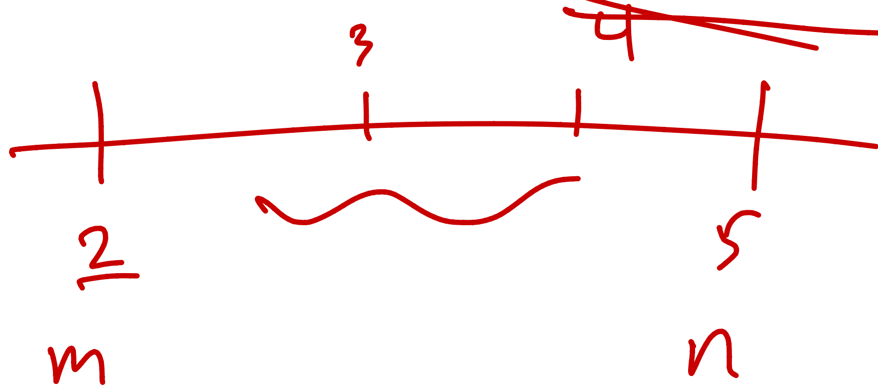
## Rational Numbers

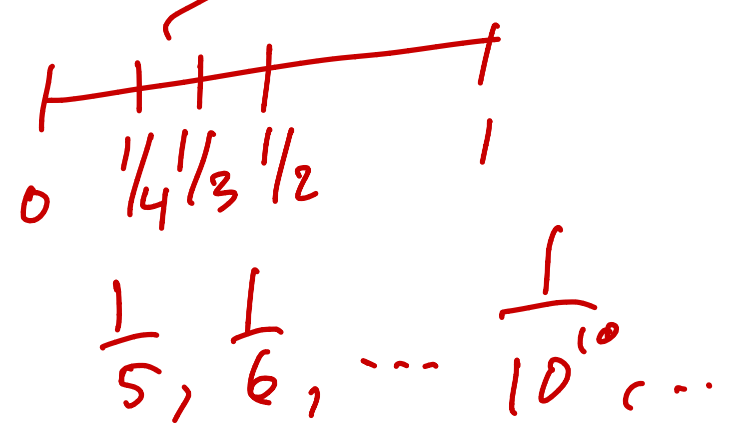
> ### Definition: Rational numbers
>
> The set of rational numbers, denoted by $\mathbb{Q}$, is the set of all possible combinations of one integer divided by another, with the latter integer being non-zero.

rationals are $\boxed{\text{DENSE}}$

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$$

Are the rational numbers countably infinite, or uncountably infinite?



1) $f : \mathbb{N} \to \mathbb{Q}$ $\qquad f : x \mapsto x$

This is an injection!

2) $f : \mathbb{Q} \to \mathbb{N}$ : next slide

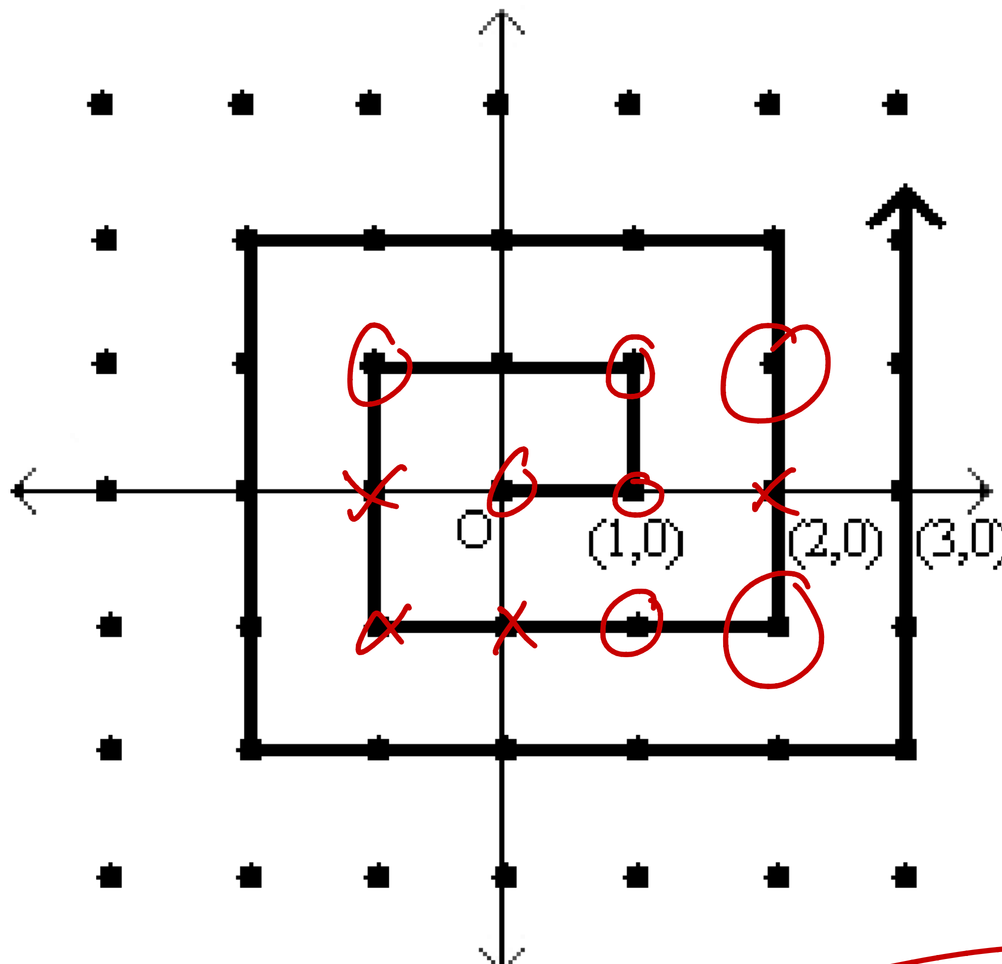
$$A = \{a_1, a_2, a_3\}$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$B = \{b_1, b_2, b_3, b_4\}$$

since $|A| \le |B|$,
we can map each element in $A$ to a diff. element in $B$

position $(a, b) \to \frac{b}{a}$

Injection from $\mathbb{Q} \to \mathbb{N}$:

| $\mathbb{Q}$ | $\mathbb{N}$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| $-1$ | 3 |
| $-1/2$ | 4 |
| $1/2$ | 5 |
| | 6 |
| | $\vdots$ |

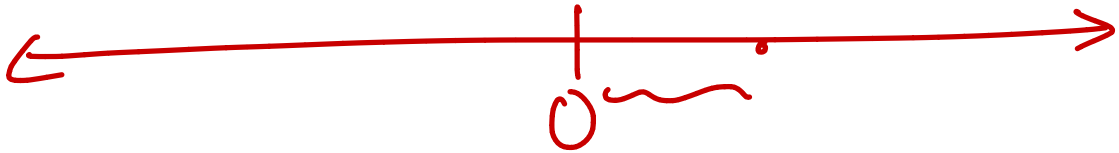



injection $\mathbb{N} \to \mathbb{Q}$

injection $\mathbb{Q} \to \mathbb{N}$

$\therefore$ bijection $\mathbb{N} \to \mathbb{Q}$

$\therefore$ $\mathbb{Q}$ countably infinite




## Definition: Real numbers

The set of real numbers, denoted by $\mathbb{R}$, is the set of all possible distances from 0 on a number line

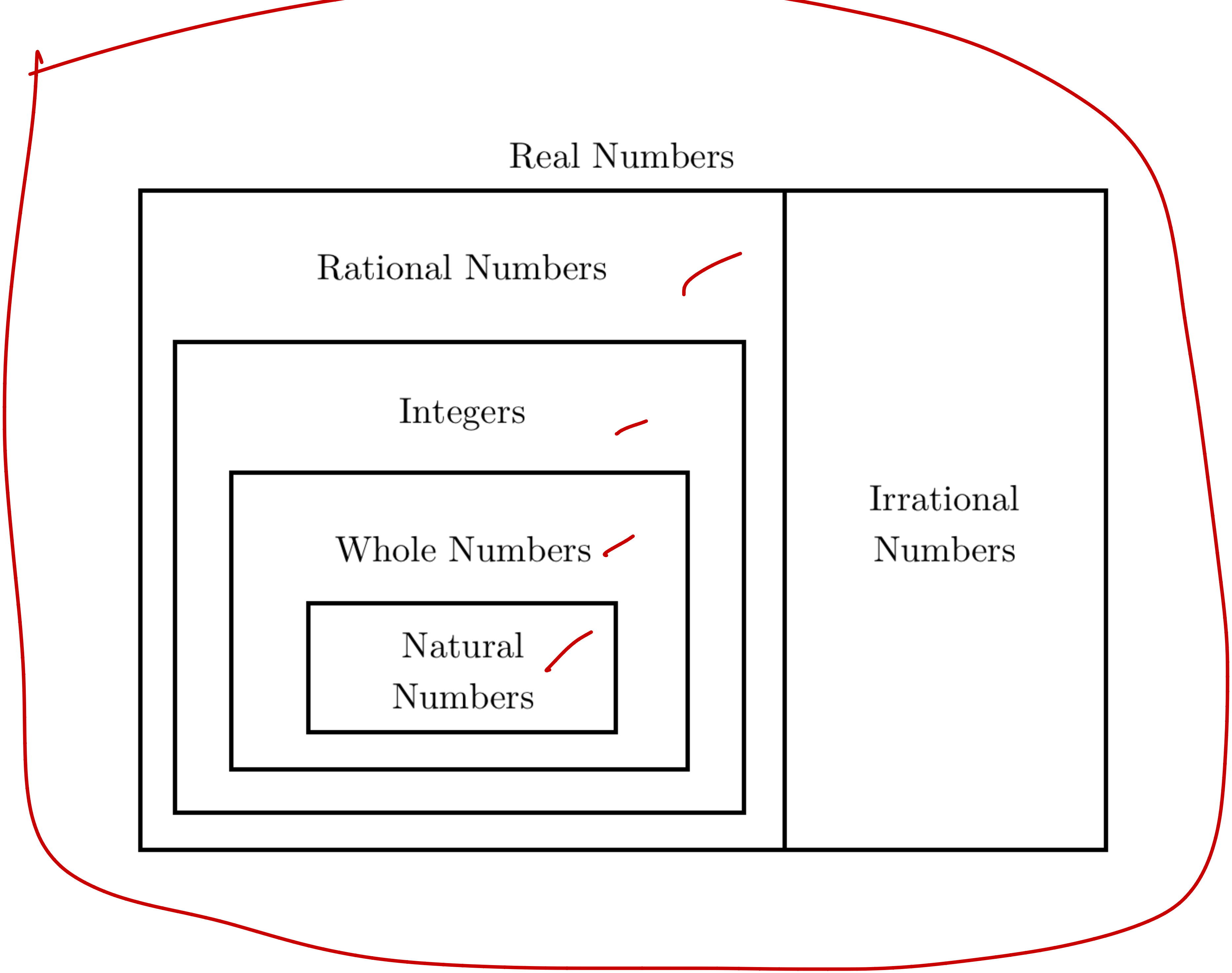
$$\mathbb{R} = \{3, \pi, -\sqrt{63}, 0.1224, \frac{2}{3}, \ldots\}$$

## Definition: Irrational numbers

The set of irrational numbers, denoted by $\mathbb{R} - \mathbb{Q}$, is the set of real numbers that are not rational. That is, they are real numbers that cannot be written as an integer divided by another integer.

$$\mathbb{R} - \mathbb{Q} = \{\pi, -e, \sqrt{5}, \ldots\}$$

# Cantor's Diagonalization


Real Numbers
Rational Numbers
Integers
Whole Numbers
Natural Numbers
Irrational Numbers

# 1.4: Propositional Logic

2 + 2 not a proposition

> ### Definition: Proposition
>
> A proposition is a statement that is has a definitive value - either true or false.

Are the following statements propositions?

✓ - "13 is prime"
✓ - "$x$ is prime" → $P(x)$ proposition that depends on $x$
✓ - "it is 93 degrees outside right now"
✗ - "LeBron James is the greatest basketball player of all time"

I think its true but not a proposition

## Logical Operators

Lots of parallels to set theory!

1. **Conjunction**: $A \wedge B$, read "$A$ and $B$"

   similar to intersection

   $$A \cap B = \{ x : x \in A \wedge x \in B \}$$

2. **Disjunction**: $A \vee B$, read "$A$ or $B$"

   similar to union

   $$A \cup B = \{ x : x \in A \vee x \in B \}$$

3. **Negation**: $\neg A$, read "not $A$"

   similar to complement

   $$A^c = \{ x : \neg (x \in A) \}$$

We can use conjunctions, disjunctions and negations to create more complicated logical statements.

$U(x)$ : $x < 100$

$E(x)$ : $x$ is even

$P(x)$ : $x$ is prime

$$\underbrace{U(x)}_{x < 100} \underset{\text{and}}{\wedge} \underbrace{(E(x) \vee P(x))}_{x \text{ is even or prime}}$$

## Implications

$P \Rightarrow Q$, read "$P$ implies $Q$", says that if $P$ is true, then $Q$ must be true; if $P$ is false, it says nothing about $Q$ ($Q$ could either be true or false).

P

For example, if all we know about today's date is that it's Christmas, we also know that the current month is December. However, if we don't know that it's Christmas, then it may or may not be December.

Q

"equivalent"
→ they have the same value

## Truth Tables

We claim that $\boxed{P \to Q \equiv \neg P \lor Q.}$

| $P$ | $Q$ | $P \to Q$ | $\neg P \lor Q$ |
|---|---|---|---|
| True | True | True | True |
| True | False | False | False |
| False | True | True | True |
| False | False | True | True |

This suffices as a proof!

## Exclusive OR

Consider our regular "OR" operation:

| $A$ | $B$ | $A \lor B$ |
|---|---|---|
| True | True | True |
| True | False | True |
| False | True | True |
| False | False | False |

$A \oplus B$, read "$A$ xor $B$", is true when exactly one of $A, B$ is true.

Claim: $A \oplus B \equiv (A \vee B) \wedge \neg(A \wedge B)$.

Proof:

| $A$ | $B$ | $A \oplus B$ | $(A \vee B) \wedge \neg(A \wedge B)$ |
|---|---|---|---|
| True | True | False | False |
| True | False | True | True |
| False | True | True | True |
| False | False | False | False |

## Contrapositive

The contrapositive, $\neg Q \rightarrow \neg P$, of an implication is actually logically equivalent to the implication itself!

$$\neg Q \rightarrow \neg P \equiv \neg(\neg Q) \vee (\neg P)$$

$$\equiv Q \vee \neg P$$

$$\equiv \neg P \vee Q$$

$$\equiv P \rightarrow Q$$

Example: The contrapositive of the statement "if it is sunny outside, I will wear my sunglasses" is "if I don't wear my sunglasses, it is not sunny outside."

## Converse

The converse, $Q \to P$ of an implication, unlike the contrapositive, is not equivalent to the original implication.

If today is Christmas ($P$), then it implies that the current month is December ($Q$), however it being December ($Q$) doesn't mean that today is Christmas ($P$).

## If and only if

$A \qquad B$, read "$A$ if and only if $B$", says that $A$ is true only when $B$ is true, and $A$ is false only when $B$ is false - in other words, that two statements are equivalent.

"it is Christmas if and only if it is December 25th" decomposes into:

1. "it is Christmas if it is December 25th"
2. "it is December 25th if it is Christmas"

This is just a fancy way of saying "Christmas is on December 25th".

Consider the following truth table:

| $A$ | $B$ | $A \qquad B$ | $(A \to B) \land (B \to A)$ |
|---|---|---|---|
| True | True | True | True |
| True | False | False | False |
| False | True | False | False |
| False | False | True | True |

e.g., Today being Christmas implies that today is December 25th, and today being December 25th implies that today is Christmas - since these two propositions imply one another, we can say they're equivalent.

**Important for proof techniques!**

## Existential Quantifiers

- **universal quantifier** $\forall$, read "for all"
- **existential quantifier** $\exists$, read "there exists"

Recall: $f : A \to B$ is surjective when $\forall b \in B, \exists a \in A : b = f(a)$

e.g. Suppose $E(x)$: "$x$ is even" and $U(x)$: "$x$ is odd." What do the following statements mean?

$$\forall x \in \mathbb{N}, E(x) \vee U(x)$$

$$\forall x \in \mathbb{N}, E(x) \wedge U(x)$$

## De Morgan's Laws for Existential Quantifiers

Allows us to change a universal quantifier into an existential quantifier.

$$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x) \tag{1}$$

$$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x) \tag{2}$$

e.g $P(x)$: "$x$ is prime", $\mathbb{U} = \mathbb{Z}^+$.

1. "If it is not true that $x$ is prime for all $x$, then there must exist some $x$ that is not prime."
2. "If it is not true that there exists an $x$ that is prime, then for all $x$, $x$ is not prime."