Lecture 2: Sets of Numbers, Propositional Logic

http://book.imt-decal.org, Ch. 1.3, 1.4

Introduction to Mathematical Thinking

September 12th, 2018

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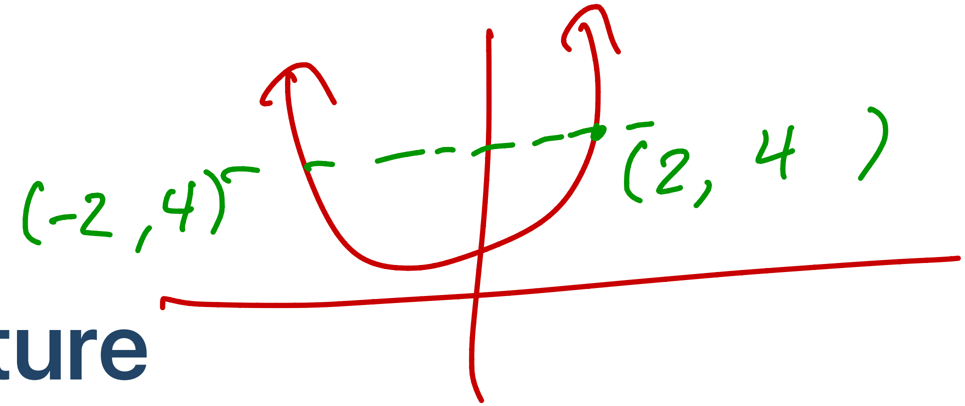
Announcements

- Midterm date changed to **Wednesday, October 10th** (because of CS 61A midterm). No conflicting dates with 61A/61B/16A/16B.
- Homework 2 will be released after lecture, and is due Monday at 6:30PM.
- Last week's lecture video is up. Today's should be up by soon (done with half of it), along with a walkthrough of the homework problems.
- Office hours are official: Tuesday 2-3PM and Thursday 5-6PM, both in Cory 299.

$2 \neq -2$



Recap from last lecture

We studied three types of functions.

- **Injections (one-to-one)**: No two elements in the domain map to the same element in the codomain

  $f(x_1) = f(x_2) \implies x_1 = x_2$

- **Surjections (onto)**: Every element in the codomain is mapped to by some element in the domain

  all possible outputs { codomain = range } set of all actual outputs

- **Bijections**: Both injective and surjective

1.3: Sets of Numbers

We've seen the following sets of numbers before:

- Natural numbers
- Whole numbers
- Integers
- Rational numbers
- Irrational numbers
- Real numbers
- Complex numbers

We want to determine the relative sizes of each of these sets.

$|S|$: # of unique elements in S

$|\{1, 2, 2, 8\}| = 3$ $\quad |\{1, 2, 3, 4, 5, \ldots\}| = ???$

Let's extend our definition of cardinality to support sets with infinitely many elements:

> **Definition: Cardinality**
>
> We say two sets have the same cardinality if and only if there exists a bijection between the two sets.

$$\begin{array}{l} A = \{a_1, \; a_2, \; a_3\} \\ \quad\;\;\;\; \downarrow \quad \downarrow \quad \downarrow \\ B = \{b_1, \; b_2, \; b_3\} \end{array}$$

this definition holds for finite sets as well

Natural Numbers

> ### Definition: Natural numbers
>
> The **natural numbers** (also known as the counting numbers), denoted by $\mathbb{N}$, are the most primitive numbers; ones that occur trivially in nature that can be used to count a (non-zero) number of things.

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

\mathbb{N}

Whole Numbers

> ### Definition: Whole numbers
>
> The set of **whole numbers**, denoted by $\mathbb{N}_0$, is the union of the set of counting numbers with the number 0.

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, ...\} = \{0\} \cup \mathbb{N}$$

Is there a bijection between the natural numbers and whole numbers?

$\mathbb{N}$ $1, 2, 3, 4, 5, 6, \ldots\ldots$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$\mathbb{N}_0$ $0, 1, 2, 3, 4, \ldots\ldots$

<u>yes</u> bijection $f(x) = x-1$

<u>Injective</u>

$f(a) = f(b) \Longrightarrow a = b$

$a-1 = b-1 \Longrightarrow a = b$

$|\mathbb{N}| = |\mathbb{N}_0|$

<u>Surjective</u>

$f: c+1 \longmapsto c$

every whole number eventually appears

$f: A \longrightarrow B$

$f: x \longmapsto x^2$

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$\mathbb{N}$	S
1	s_1
2	s_2
3	s_3
$\vdots$	$\vdots$

> **Definition: Countably infinite**
>
> We say set S is **countably infinite** if and only if there exists a bijection from the natural numbers (or any other countable set) to S. If such a bijection does not exist, we say S is **uncountably infinite**.

One way to think of this is to give each number a waiting number in an infinitely long line!

there exists an ordering of the set

$f: \mathbb{N} \to S$

$f: \mathbb{N}_0 \to S$

OR $f: S \to \mathbb{N}$, $S \to \mathbb{N}_0$

$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z}$

Integers

> ### Definition: Integers
>
> The set of integers, denoted by $\mathbb{Z}$, is the union of the whole numbers with their negatives

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

Are the integers countably infinite, or uncountably infinite?

yes!

$\mathbb{N}_0$	0,	1,	2,	3,	4,	5,	6,	...
$\mathbb{Z}$	0,	1,	-1,	2,	-2,	3,	-3,	...

$$f(x) = \begin{cases} \dfrac{-x}{2} & x \text{ is even} \\[2ex] \dfrac{x+1}{2} & x \text{ is odd} \end{cases}$$

BIJECTION

eventually will get to any integer I want

Rational Numbers

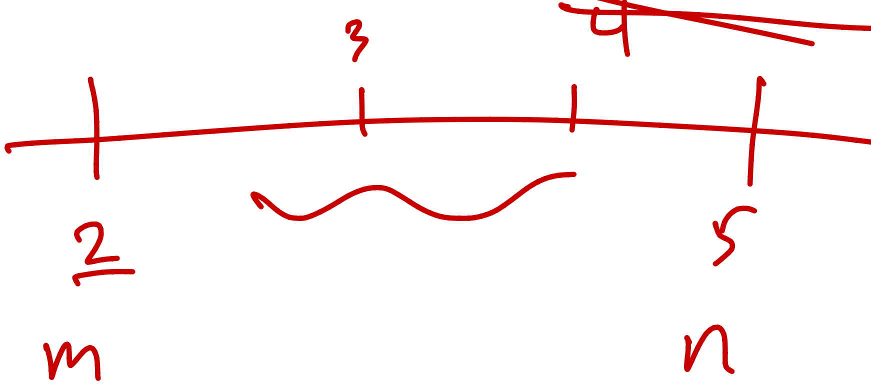
$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}$

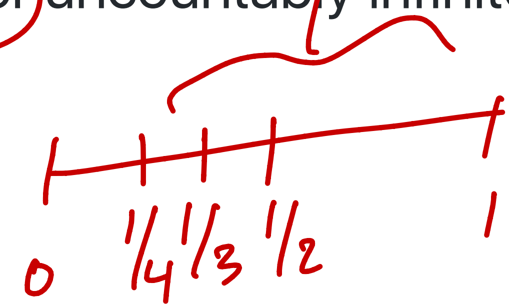
> **Definition: Rational numbers**
>
> The set of rational numbers, denoted by $\mathbb{Q}$, is the set of all possible combinations of one integer divided by another, with the latter integer being non-zero.

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$$

rationals are $\boxed{\text{DENSE}}$

Are the rational numbers countably infinite, or uncountably infinite?





$\frac{1}{5}, \frac{1}{6}, \ldots \frac{1}{10^{10}}, \ldots$

1) $f : \mathbb{N} \to \mathbb{Q}$ $\quad$ $f : x \mapsto x$

This is an injection!

2) $f : \mathbb{Q} \to \mathbb{N}$: next slide

$A = \{a_1, a_2, a_3\}$

$B = \{b_1, b_2, b_3, b_4\}$

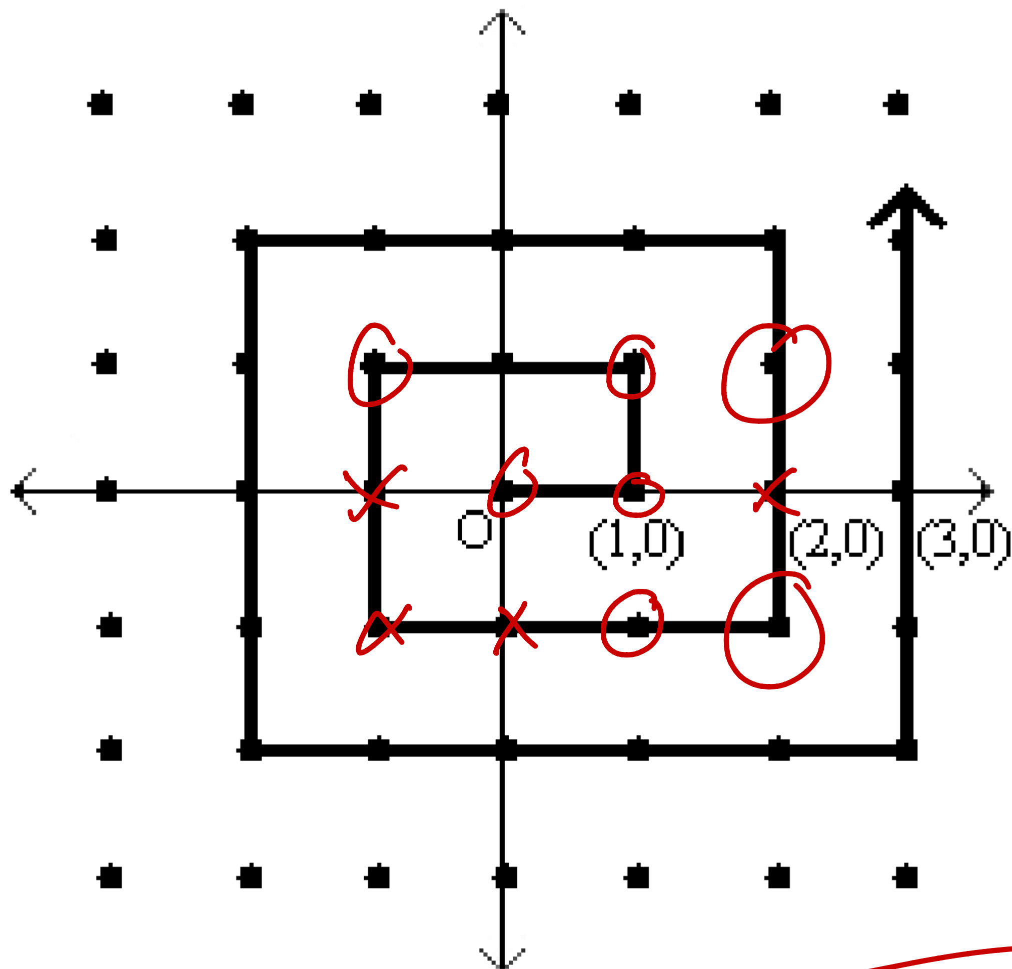
since $|A| \le |B|$, we can map each element in A to a diff. element in B

position

$(a, b) \to \frac{b}{a}$

Injection from $\mathbb{Q} \to \mathbb{N}$:

$\mathbb{Q}$	$\mathbb{N}$
0	1
1	2
-1	3
$-1/2$	4
$1/2$	5
	6
	⋮



injection

$\mathbb{N} \to \mathbb{Q}$

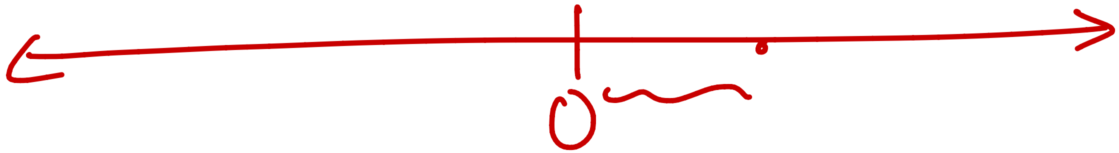
injection

$\mathbb{Q} \to \mathbb{N}$

$\therefore$ bijection

$\mathbb{N} \to \mathbb{Q}$

$\therefore$ $\mathbb{Q}$ countably infinite



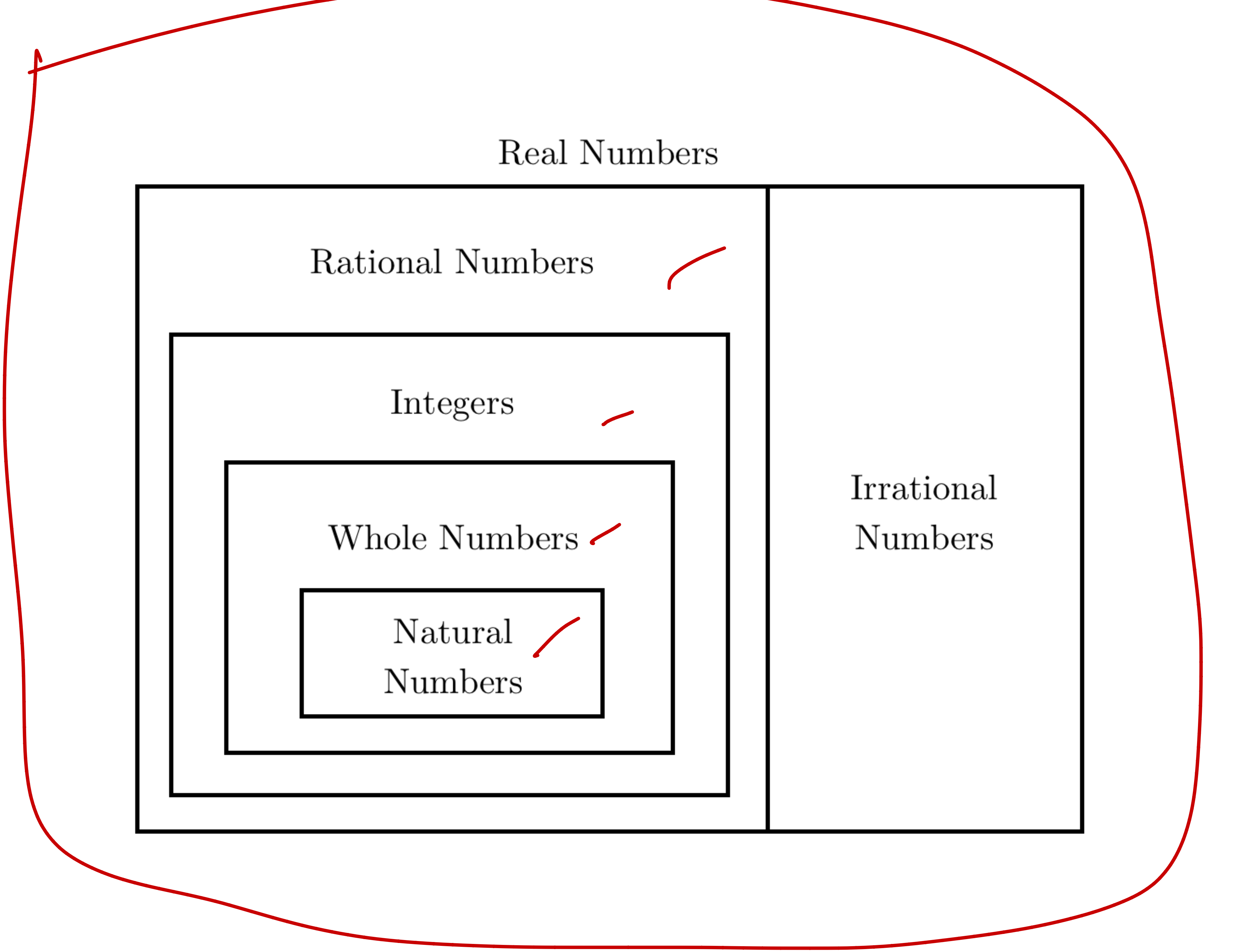
> ### Definition: Real numbers
>
> The set of real numbers, denoted by $\mathbb{R}$, is the set of all possible distances from 0 on a number line

$$\mathbb{R} = \{3, \pi, -\sqrt{63}, 0.1224, \frac{2}{3}, \ldots\}$$

> ### Definition: Irrational numbers
>
> The set of irrational numbers, denoted by $\mathbb{R} - \mathbb{Q}$, is the set of real numbers that are not rational. That is, they are real numbers that cannot be written as an integer divided by another integer.

$$\mathbb{R} - \mathbb{Q} = \{\pi, -e, \sqrt{5}, \ldots\}$$

Cantor's Diagonalization


Real Numbers
Rational Numbers
Integers
Whole Numbers
Natural Numbers
Irrational Numbers

1.4: Propositional Logic

2 + 2 not a proposition

> ### Definition: Proposition
>
> A proposition is a statement that is has a definitive value - either true or false.

Are the following statements propositions?

- ✓ "13 is prime"
- ✓ "x is prime" → $P(x)$ proposition that depends on x
- ✓ "it is 93 degrees outside right now"
- ✗ "LeBron James is the greatest basketball player of all time"

I think its true but not a proposition

Logical Operators

Lots of parallels to set theory!

1. **Conjunction**: $A \wedge B$, read "A and B"

   similar to intersection

   $$A \cap B = \{ x : x \in A \wedge x \in B \}$$

2. **Disjunction**: $A \vee B$, read "A or B"

   similar to union

   $$A \cup B = \{ x : x \in A \vee x \in B \}$$

3. **Negation**: $\neg A$, read "not A"

   similar to complement

   $$A^c = \{ x : \neg (x \in A) \}$$

We can use conjunctions, disjunctions and negations to create more complicated logical statements.

$U(x)$: $x < 100$

$E(x)$: x is even

$P(x)$: x is prime

$$\underbrace{U(x)}_{x < 100} \underset{\text{and}}{\wedge} \underbrace{\left(E(x) \vee P(x) \right)}_{x \text{ is even or prime}}$$

Implications

$P \Rightarrow Q$, read "P implies Q", says that if P is true, then Q must be true; if P is false, it says nothing about Q (Q could either be true or false).

P

For example, if all we know about today's date is that it's Christmas, we also know that the current month is December. However, if we don't know that it's Christmas, then it may or may not be December.

"equivalent"

→ they have the same value

Truth Tables

We claim that $\boxed{P \to Q \equiv \neg P \lor Q}$.

P	Q	$P \to Q$	$\neg P \lor Q$
True	True	True	True
True	False	False	False
False	True	True	True
False	False	True	True

This suffices as a proof!