Lecture 8: Pascal's Triangle, Combinatorial Proofs, Binomial Theorem

http://book.imt-decal.org, Ch. 4 (in progress)

Introduction to Mathematical Thinking

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Announcements

- Happy Halloween!
    - My costume: https://www.youtube.com/watch?v=zWRcSfey8gw
- HW 8 out by tomorrow, due Wednesday
- Textbook hasn't been updated for Modular Arithmetic and Counting, but starting with the Binomial Theorem it is, so would highly recommend looking at it

Pascal's Triangle

0th 1
1st 1 1
2nd 1 2 1
3rd 1 3 3 1
4th 1 4 6 4 1
5th 1 5 10 10 5 1
1 6 15 20 15 6 1
7
8

2nd diagonal :
1, 2, 3, 4, 5, 6, . . .

n+1 numbers in nth row

each row symmetric

$\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \ldots$

$1, 2, 3, \ldots$

$\binom{r}{0}$ $\quad$ $\binom{r}{1}$ $\quad$ $\binom{r}{2}$

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}$$

n^{th} row: $\binom{n}{0}, \binom{n}{1}, \ldots$

n^{th} row, k^{th} col : $\binom{n}{k}$

Combinatorial Proofs

$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$

Instead of proving a statement algebraically, we can prove statements *combinatorially*, by showing that both sides of the equals sign count the same quantity.

$S = \{1, 2, 3\}$

0 elements

1 element

2 elements

3 elements

For example: We know that the n-th row of Pascal's Triangle is given by $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$. Let's give a combinatorial proof of the fact that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.

LHS RHS

LHS: Count the number of subsets of a set of size n. We can either choose 0 elements, or 1 element, or 2 elements, ... , or n elements. We can choose k elements in $\binom{n}{k}$ ways.

RHS: For each of the n elements, we have two choices - it is either included in our subset, or not. This yields 2^n total options.

Thus, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$, and the sum of the n-th row of Pascal's Triangle is 2^n.

m n

Example

LHS RHS

For example, let's prove $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$.

Suppose we have m Warriors fans and n Lakers fans.

LHS: Number of ways to select 2 basketball fans from the set of $m + n$.

RHS: To select 2 basketball fans from our set of $m + n$, we either take

- 2 Warriors fans, $\binom{m}{2}$ or
- 2 Lakers fans, $\binom{n}{2}$ or
- 1 Warriors fan and 1 Lakers fan, $\binom{m}{1}\binom{n}{1} = mn$

$+$

Our total is then $\binom{m}{2} + \binom{n}{2} + mn$, which must be equal to $\binom{m+n}{2}$.

Pascal's Identity

Formalization of the fact that the sum of two adjacent numbers in the triangle is the number directly below them.

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Algebraic proof: Was on this week's homework.

Can you think of a combinatorial proof for this?

Algebraic Proof

$$\frac{(k+1)}{(k+1)} \qquad\qquad \frac{(n-k)}{(n-k)}$$

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}$$

$$= \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!}$$

$$= \frac{n!(k+1+n-k)}{(k+1)!(n+k)!} = \frac{(n+1)!}{(k+1)!(n+k)!}$$

$$= \binom{n+1}{k+1}$$

$\{ \_, \_, \_ \}$

p_1, p_2, p_3, p_4, p_5

Combinatorial Proof

$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$

Case 1: include p_1

2: don't include p_1

RHS: Number of ways to choose $k+1$ people from a group of $n+1$.

LHS: Suppose we want to choose $k+1$ people from a group of $n+1$. Suppose the people are numbered $p_1, p_2, ...p_{n+1}$. Consider the very first person: either we include them in our subset or do not include them.

p_1

$\binom{1}{1}$

- If we include them, there are n people remaining and we need to choose k of them: $\binom{n}{k}$
- If we do not include them, there are n people remaining and we need to choose $k+1$ of them: $\binom{n}{k+1}$

Thus, the total number of ways to choose $k+1$ people from a group of $n+1$ is $\binom{n}{k} + \binom{n}{k+1}$. We've already shown this quantity is $\binom{n+1}{k+1}$, though, so these expressions both must be the same!

Example: Give a combinatorial proof of Vandermonde's Identity, that is:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\underline{\binom{n}{n-k}}$$

$$\underbrace{\binom{m+n}{r}} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

LHS: Number of ways to choose r basketball fans from m Warriors fans and n Lakers fans

RHS: Suppose we want to choose r basketball fans from m Warriors fans and n Lakers fans. If we choose k Warriors fans, we need to choose $r-k$ Lakers fans. The total number of fans we choose must always be r, and this value of k can be anything from 0 to r.

We could choose 0 Warriors fans and r Lakers fans, or 1 and $r-1$, or 2 and $r-2$, ..., or $r-1$ and 1, or r and 0, giving us $\underline{\binom{m}{0}\binom{n}{r}} + \underline{\binom{m}{1}\binom{n}{r-1}} + ... + \underline{\binom{m}{r}\binom{n}{0}} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$, as required.

let $m=r=n$

$$\binom{n}{k} = \binom{n}{n-k}$$

Corollary: $\boxed{\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2}$

$\binom{n}{k}\binom{n}{k}$

Attendance

http://tinyurl.com/suitshorts

x+y → 2

x+x^2 → 1

Binomial Theorem

similar to Cartesian product!

Binomial: A polynomial with two terms, joined by addition.

$$(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd$$

When multiplying two binomials, the result is every combination of one term in the first binomial multiplied by one term in the second binomial.

$$(x+y)^2 = (x+y)(x+y) = xx + xy + yx + yy = x^2 + 2xy + y^2$$

$$(x+y)^2 = (x+y)(x+y) = xx + xy + yx + yy = x^2 + 2xy + y^2$$

$x^2y^0 \quad x^1y^1 \quad x^0y^2$

Either we choose...

- 2 xs and 0 ys: $\binom{2}{0}$ $\binom{2}{2}$
- 1 x and 1 y: $\binom{2}{1}$ $\binom{2}{1}$
- 0 xs and 2 ys: $\binom{2}{2}$ $\binom{2}{0}$

Now, we have $\binom{2}{0}$ terms of the form x^2, $\binom{2}{1}$ terms of the form xy and $\binom{2}{2}$ terms of the form y^2:

$$(x+y)^2 = \underbrace{\binom{2}{0}}_{1} x^2 + \underbrace{\binom{2}{1}}_{2} xy + \underbrace{\binom{2}{2}}_{1} y^2$$

To generalize: Each term in the expansion of $(x+y)^n$ has k xs and $n-k$ ys, for $k = 0, 1, \ldots n$.

Formalization of the Binomial Theorem

The binomial theorem states

choosing the exponent on y

$$\binom{n}{n-k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^n = \binom{n}{0} x^k y^0 + \binom{n}{1} x^{k-1} y + \binom{n}{2} x^{k-2} y^2$$

$$+ \cdots\cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} y^n$$

For example, let's expand $(3a^2 - 2b)^5$.

$x = 3a^2$

$y = -2b$

$$(3a^2-2b)^5 = \binom{5}{0}(3a^2)^5 + \binom{5}{1}(3a^2)^4(-2b) + \binom{5}{2}(3a^2)^3(-2b)^2 + \binom{5}{3}(3a^2)^2(-2b)^3$$

$$10 \quad 27 \quad -4 \qquad + \cdots$$

Example: What is the sum of the coefficients of $(3x^2 - 4x)^{\overset{n}{\cancel{12}}}$?

$$(3x^2-4x)^{12} = \binom{12}{0}(3x^2)^{12} + \binom{12}{1}(3x^2)^{11}(-4x) + \binom{12}{2}(3x^2)^{10}(-4x)^2 + \cdots$$

substitute $x = 1$:

$$(3\cdot 1 - 4\cdot 1)^{12} = (-1)^{12} = 1$$

$$p(x,y) = 4x^2 + 13xy - 17y^2$$

$$p(1,1) = 4(1) + 13(1)(1) - 17(1) = 4 + 13 - 17 = 0$$

General Term

We define the k-th term in the expansion of a binomial as

$$t_k = \binom{n}{k} x^{n-k} y^k$$

with $k \in \{0, 1, 2, ..., n\}$.

General term of $(3a^2 - 2b)^5$:

$$\begin{aligned} t_k &= \binom{5}{k} (3a^2)^{5-k} (-2b)^k \\ &= \binom{5}{k} 3^{5-k} a^{10-2k} (-1)^k 2^k b^k \\ &= \boxed{(-1)^k \binom{5}{k} 2^k 3^{5-k}} a^{10-2k} b^k \end{aligned}$$

$t_3 = \ldots\ldots a^4 b^3$

$$(x^5 - x^{-2})^7$$

Example: What is the general term of $(x^5 - \frac{1}{x^2})^7$?

$$t_k = \binom{7}{k} (x^5)^{7-k} (-x^{-2})^k$$

$$= (-1)^k \binom{7}{k} x^{35-5k} x^{-2k}$$

$$= (-1)^k \binom{7}{k} x^{35-7k}$$

Highly recommend looking at

http://book.imt-decal.org/5. Polynomials/5.4 The Binomial Theorem.html

$$t_k = (-1)^k \binom{7}{k} x^{35-7k}$$

coefficient on x^{14} : $35 - 7k = 14$

$k = 3$ $\quad t_3$

coefficient on x^{30} : $35 - 7k = 30$

$k = \frac{5}{7}$

$f(x, y) = (3x - y^2)^7$

$g(x, y) = (4x^2 - 3y)^8$

General term of expansion of $\underline{f(x, y)} \cdot \underline{g(x, y)}$?

$t_{k, j}$

$f(x) = (3x - 4y + (4y^2))^7 \rightarrow$ Trinomial ?

$\square$

$\rightarrow (\square + 14y^2)^7$