Lecture 2: Sets and Functions

Introduction to Mathematical Thinking

January 31st, 2019

Suraj Rampure

Announcements

- Permission codes will be sent out tomorrow morning. The deadline to add units without a fee is tomorrow, so make sure to add the units ASAP!
    - Along with your permission code, you will get invites to Piazza and Gradescope.
    - **If you're also in EE 16A, there shouldn't be an issue with regards to the time conflict. However, if you get some sort of error, please email me with your information and I'll forward it to the enrollment office who will manually enroll you.**
- Homework 1 will be out late Friday or early Saturday, and is due next Friday at 11:59PM
- Office hours start next week!
- Content from last lecture is available on the site.
    - Generally, annotated slides will be posted almost immediately after lecture, while the video may take a day.
- **Based on the votes from last class, the first quiz will be on Thursday, Feb. 14**. Note that this date is after the drop deadline.

Quick Primer on Notation

We're about to introduce a lot of new notation. I'll explain this notation as it appears, but just so that you are aware:

- $\forall$ means "for all"
- $\exists$ means "there exists"
- $:$, $|$, and $s.t.$ all mean "such that"
- $\Rightarrow$ means "implies"
  - We will talk significantly more about the implication in a few lectures, but this will suffice for now

Chapter 1.5 in our book contains a nice cheat sheet regarding all of this notation.

Sets and Set Operations

Corresponds to 1.1 in book.imt-decal.org

Sets

In mathematics, a **set** is a well-defined collection of objects.

By well-defined, we mean that for any object x, we can easily determine whether or not x is an element (also referred to as a "member") of the set.

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

$x \in S$

"element of"

"is in"

$-3 \in \mathbb{Z}$

$\pi \notin \mathbb{Z}$

"is not an element of"

Describing Sets

We define sets in one of two ways:

- By listing out all of the elements in the set, e.g. $A_1 = \{1, 3.14, -15, \pi + e\}$, $A_2 = \{..., -3, -2, -1, 0, 1, 2, ...\}$ or $A_3 = \{\text{Lebron, Jordan, Kobe}\}$
- By stating a property that determines the set, e.g. $A_4 = \{x : x + 3 \text{ is prime}\}$ or $A_5 = \{t : t^2 - 5t + 6 = 0\}$

such that

set-builder notation

$B = \{ x^2 : 0 \le x \le 10, x \in \mathbb{Z} \}$

all x^2 such that x is an integer between 0, 10 (inclusive)

[x ** 2 for x in range(11)]   list comprehension!

Cardinality

The **cardinality** $|S|$ of a finite set S is the number of elements in S.

$A = \{0, 1, 2, 3, 4, 5\}$

$|A| = 6$

sets only consist of unique elements!

$B = \{-1, -1, 2, 2, 3, 4\}$

$|B| = 4 \quad (-1, 2, 3, 4)$

$\mathbb{Z} = \{, \ldots, -3, -2, -1, 0, 1, 2, \ldots\} \quad |\mathbb{Z}| = ?$

$3 < 4$

$\rightarrow 3 \leq 4$

$\rightarrow$ and $3 \neq 4$

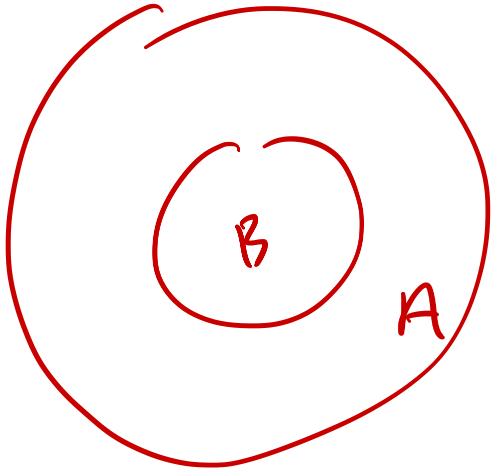
$\leq$

equality allowed

$<$

Subsets

Given some set A, another set B is a **subset** of A if and only if every element in B is also an element of A. Equivalently, we can say A is a **superset** of B.



superset: $A \supseteq B$ subset: $B \subseteq A$ "B is a subset of A"

$A \supset B$ $B \subset A$ "B is a proper subset of A"

$\rightarrow B \subseteq A$ AND $B \neq A$

i.e. a subset that is not equal

$B = \{\text{LeBron}, \text{Kobe}\}$

$A = \{\text{LeBron}, \text{Kobe}, \text{Jordan}\}$

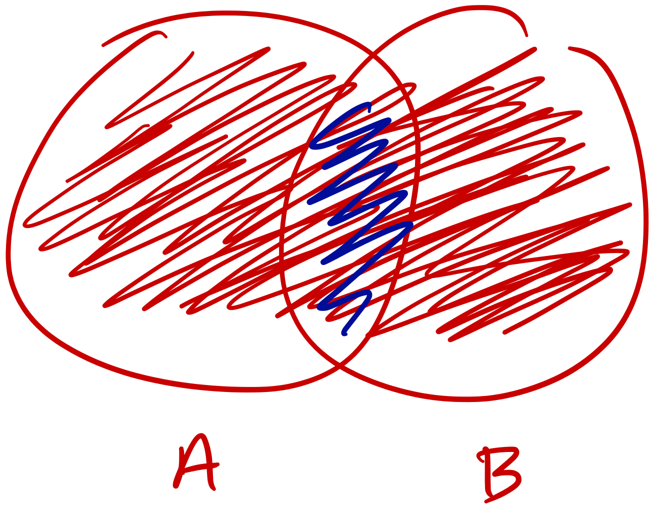
$B \subseteq A$ $B \subset A$

Set Operations

We will now look at operations that act on two (or more) sets.

Union

The **union** of two sets A, B is the set of everything contained by either A or B.



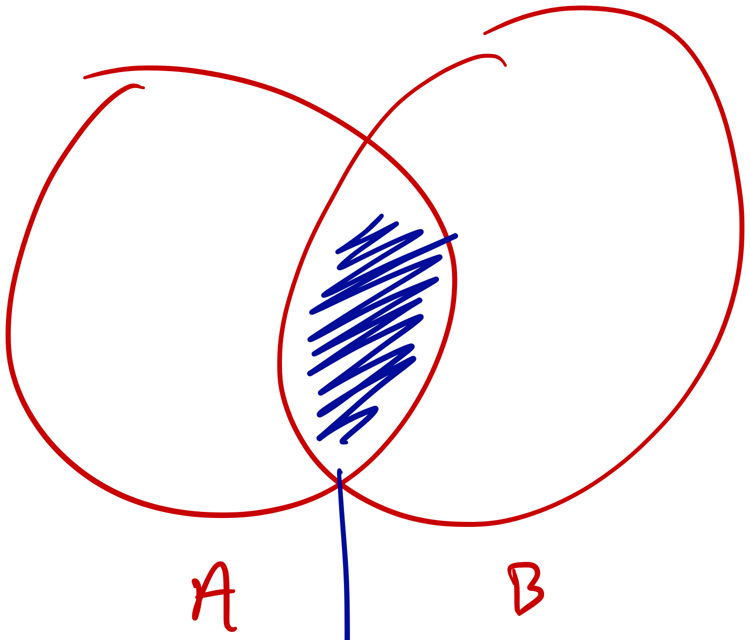
$$\boxed{A \cup B} = \{x : x \in A \text{ or } x \in B\}$$

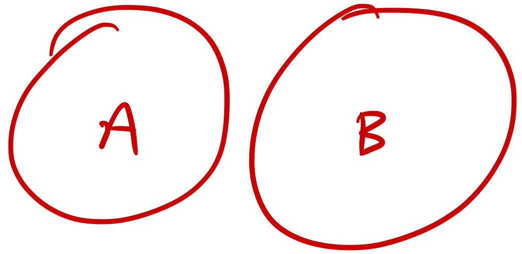
$$\{x : x \in A \underbrace{\vee}_{\text{"or"}} x \in B\}$$

"disjunction"

Intersection

The **intersection** of two sets A, B is the set of everything contained in both A and B.





DISJOINT

when $A \cap B = \underbrace{\varnothing}_{\text{empty set}} = \{\}$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$$= \{x : x \in A \wedge x \in B\}$$

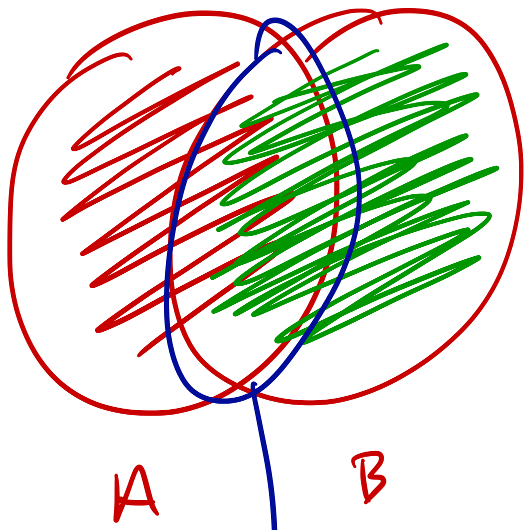
"conjunction" : and

Cardinality of the Union

$|A|, |B| < \infty$ also known

Suppose A, B are two sets, and $|A \cap B|$ is a known quantity. How can we determine $|A \cup B|$?

"cardinality of the intersection"



double counted!

$$|A \cup B| = |A| + |B| - |A \cap B|$$

"Principle of Inclusion - Exclusion"

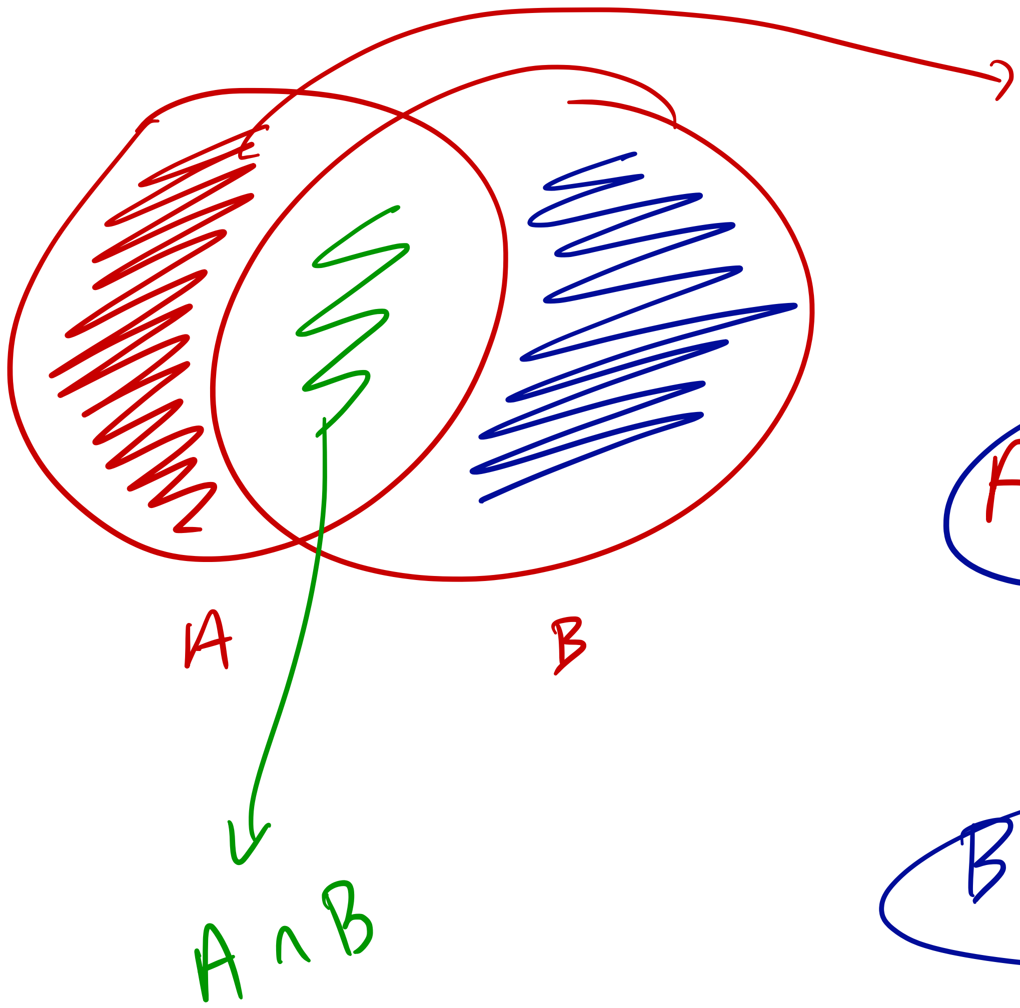
$A = \{1, 2, 3, 4\}$

$B = \{3, 4\}$

$A \setminus B = \{1, 2\}$

Set Difference

The **difference** of two sets A, B is the set of everything contained in A but not contained in B.



set difference : $A - B$

$A \setminus B$

$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

$B \setminus A = \{x : x \in B \text{ and } x \notin A\}$

$$B \times A = \{ (x,1), (y,1), (z,1) \quad \cdots \quad (z,3) \}$$

Cartesian Product

The **Cartesian product** of two sets A, B is the set of all possible combinations of one element in A and one element in B.

$$A \times B \neq B \times A$$

$$A \times B = \{ (x, y) \mid x \in A \text{ and } y \in B \}$$

$$A = \{1, 2, 3\}$$

$$B = \{x, y, z\}$$

$$A \times B = \{ (1,x), (1,y), (1,z), (2,x), \ldots (2,z), \ldots (3,z) \}$$

$$= \{ (1,x), (2,x), (3,x), (1,y) \ldots (3,z) \}$$

$$|A \times B| = |A| \cdot |B|$$

Cartesian plane vectors $\quad \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $\mathbb{R}^n$: n length vectors

Universes and Complements

We usually deal with a **universe**, $\mathbb{U}$, that represents the set of all objects under consideration. For example, if we're looking at different groups of undergraduates at Berkeley, our "universe" could be the set of *all* undergraduates at Berkeley.

This enables us to talk about the notion of a **complement**.

$\mathbb{U} = \{1, 2, 3, \ldots 10\}$

$A^c = \mathbb{U} \setminus A$

$= \{x : x \in \mathbb{U} \text{ and } x \notin A\}$

$A = \{2, 4, 6, 8\}$

complement of A : $\{1, 3, 5, 7, 9, 10\}$

A^c

A', $\bar{A}$

De Morgan's Laws

1 $(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

This is not a proof!!!

$\mathcal{U} = \{1, 2, 3, \dots 10\}$

$A = \{1, 2, 3, 5, 6, 10\}$

$B = \{2, 4, 6, 8, 10\}$

1 $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$

$\rightarrow \boxed{(A \cup B)^c = \{7, 9\}}$

$A^c = \{4, 7, 8, 9\}$

$B^c = \{1, 3, 5, 7, 9\}$

$\rightarrow \boxed{A^c \cap B^c = \{7, 9\}}$

Functions

Corresponds to 1.2 in book.imt-decal.org

Relations

A relation with *domain* A and *codomain* B is a subset of $A \times B$.

relation is a set!

$A = \{1, 2, 3, 4\}$

$B = \{1, 5, 7\}$

$r: \{(1,1), (1,7), (3,5), (4,1), (4,5)\}$

$A \times B = \{(1,1), (2,1), \ldots (4,5), (4,7)\}$

r is a relation with domain A and codomain B

→ tells you input and output are related in some way "inputs" "outputs"

We say f is a **function** with *domain* A and *codomain* B if $f \subset A \times B$ such that for every element in A, there is exactly one element in B. Symbolically, we represent this as $f : A \to B$.

for every input, exactly one output

$A = \{1, 2, 3, 4\}$

$B = \{1, 5, 7\}$

$A \times B = \{(1,1), (2,1), \ldots (4,5), (4,7)\}$

$f : \{(1, \underline{1}), (2, \underline{1}), (3, \underline{5}), (4, \underline{1})\}$

each $a \in A$ is mapped to exactly one $b \in B$

Vertical line test!

e.g. $A = \{\text{"Hi"}, \text{"Hey"}\}$, $B = \{5, 10\}$

$A \times B = \{(\text{"Hi"}, 5), (\text{"Hi"}, 10), (\text{"Hey"}, 5), (\text{"Hey"}, 10)\}$

Are the following functions?

$f = \{(\text{"Hi"}, 10), (\text{"Hey"}, 5)\}$ ✓

$g = \{(\text{"Hi"}, 5), (\text{"Hey"}, 5)\}$ ✓

$h = \{(\text{"Hi"}, 10), (\text{"Hi"}, 5)\}$ ✗

- 2 outputs for Hi
- 0 outputs for Hey

f, g, h all relations

f, g functions

$f, g \subset A \times B$

$f = \{(\text{"Hi"}, 10), (\text{"Hey"}, 5)\}$
$g = \{(\text{"Hi"}, 5), (\text{"Hey"}, 5)\}$

$f : A \longrightarrow B$

$f(A) = B$

set A set B

A is the **domain**, as it represents all possible inputs.
B is the **codomain**, as it represents all possible outputs; as a result, we say that $f(A) = B$.

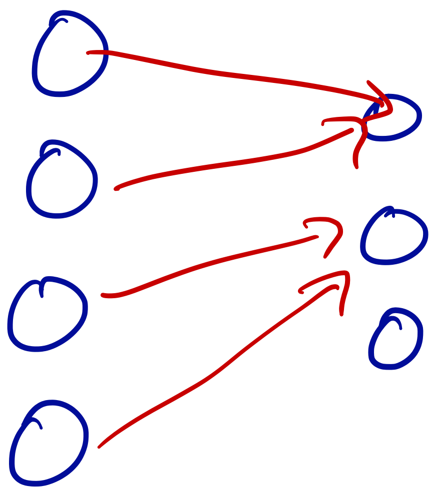
The **image**, or **range**, is the set of all actual outputs of a function.

$f(x,y) = e^{xy} + x^2$
$f : \mathbb{R}^2 \to \mathbb{R}^2$

f: codomain: B $\{5, 10\}$
range: $\{5, 10\}$

g: codomain: $\{5, 10\}$
range: $\{5\}$

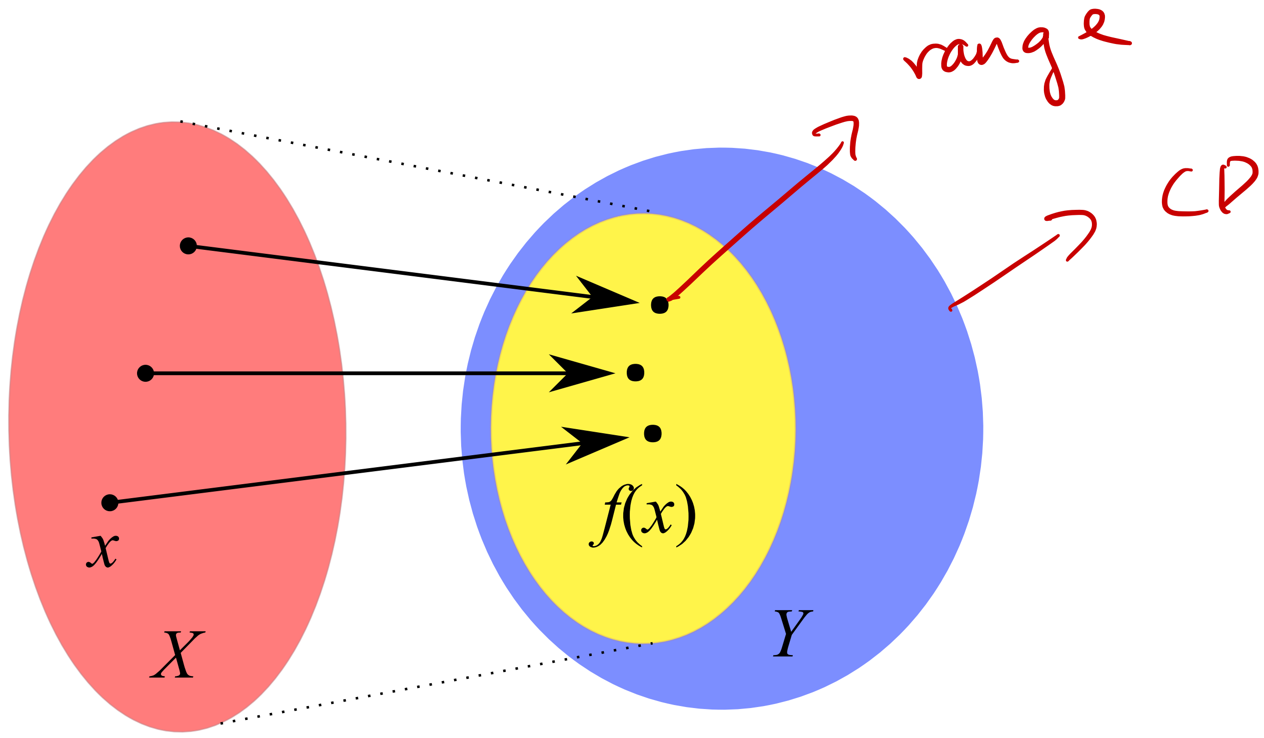
exactly one arrow



Domain CD

range $\subseteq$ codomain

In the following image, the red area represents the domain, the blue the codomain, and the yellow the range.



$f : X \to Y$

In order to define a function f, we need to specify the domain, codomain, and how to **map** elements of the domain to the codomain. We can do this in two ways:

- List out the pairs (as we did for the previous examples)
- Specify a rule that describes how to get from input to corresponding output

$f : A \longrightarrow B$

$f : x \longmapsto e^{x^2}$

There are two different notations we can use to specify a rule:

$$f(x) = x^2$$

$$f : x \mapsto x^2$$

Note the difference between $\longrightarrow$ and $\mapsto$.

Function definitions don't necessarily need to be algebraic!

$$f(x) = \text{the } x^{th} \text{ smallest prime number}$$

Injections, Surjections and Bijections

We'll now look at a few different (related) types of functions.

Injections

We say a function $f : A \to B$ is **injective**, or **one-to-one**, if no two elements in the input have the same output.

implies

In formal mathematical notation, we'd say

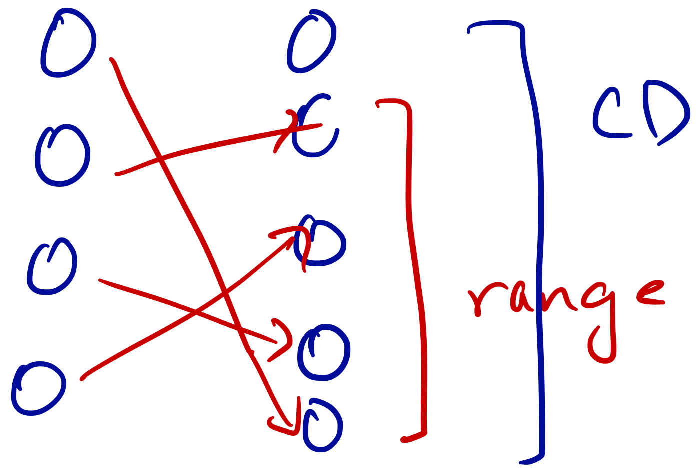
for all x, y

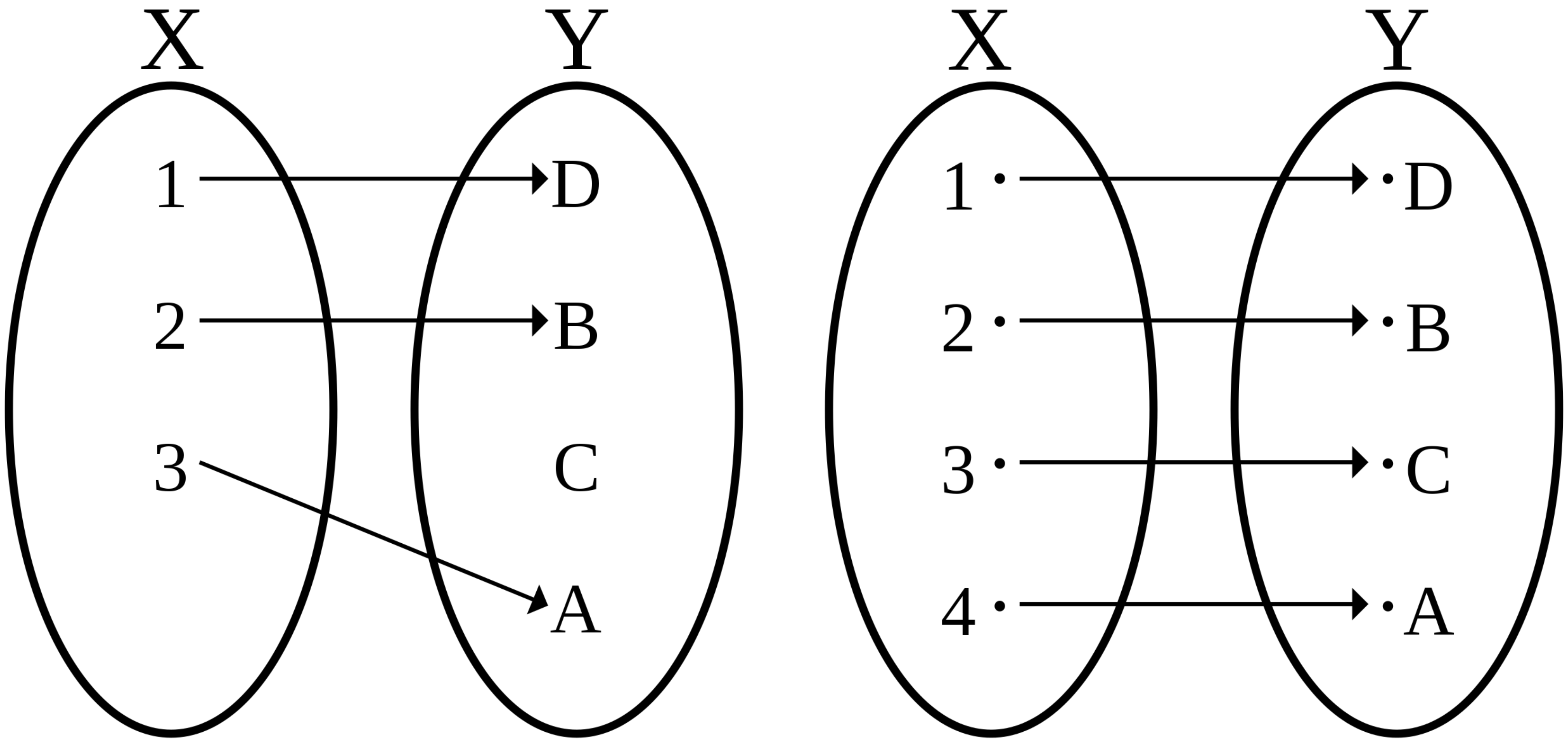
$$\forall x, y \in A, f(x) = f(y) \Rightarrow x = y$$

This translates to "for all x, y in the domain, if x's output is equal to y's output, then x and y must be the same element.

$$\forall x, y \in A, \quad x \neq y \Longrightarrow f(x) \neq f(y)$$

no two arrows point to same circle





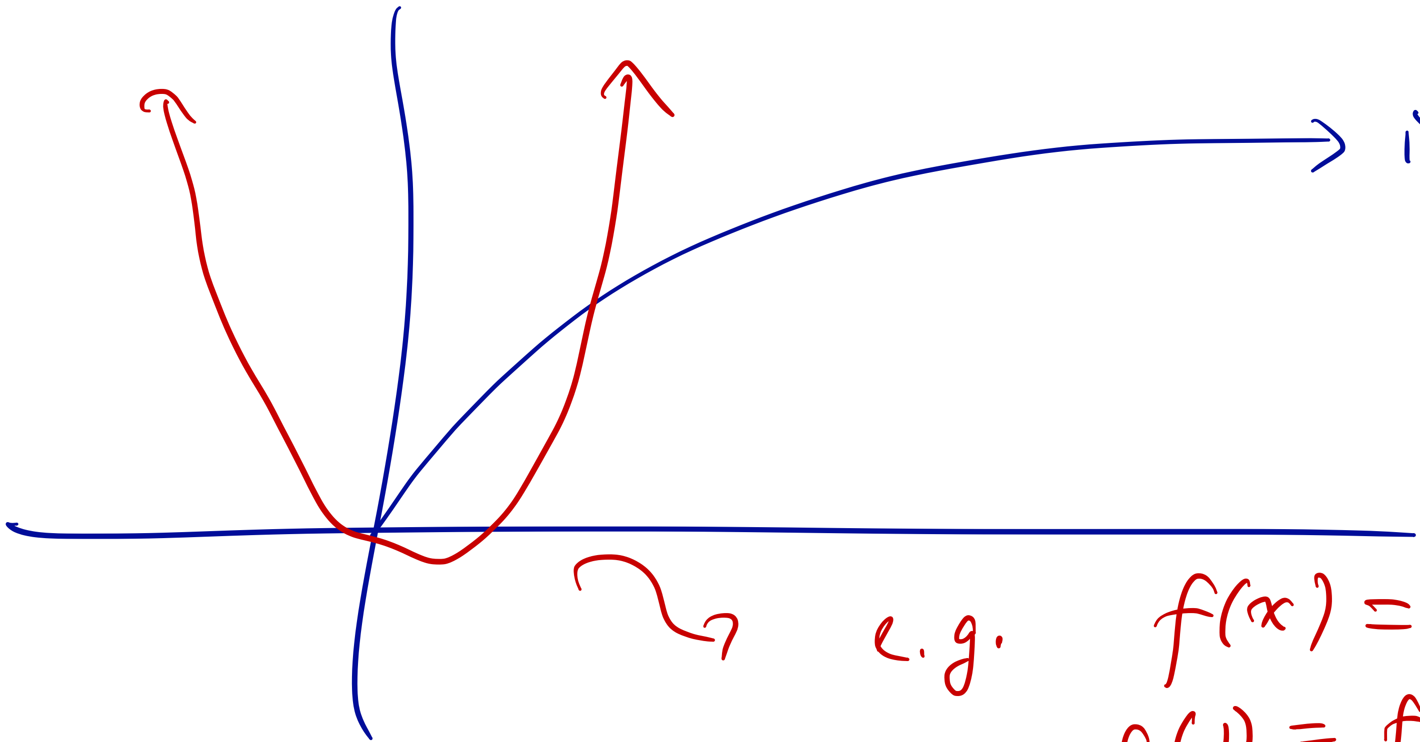
Both of these functions are examples of injections.

Are the following functions injections?

Your answer may change depending on the domain and codomain!

- $f(x) = e^x$
- $g : x \mapsto \sqrt{x^2}$
- $h(x) = x^3 + 3x$

vertical line test: check if function



injection?

e.g. $f(x) = x^2$

$f(1) = f(-1) = 1$ BUT $1 \neq -1$

The "Injection" Test

We know that the derivative $f'(x)$ of a function $f(x)$ describes the rate of change of f. Specifically, $f'(a)$ tells us the rate at which $f(x)$ is changing at $x = a$.

It turns out, if a function's derivative is always greater than 0 or always less than 0, then the function is injective.

strictly increasing

strictly decreasing

$h(x) = x^3 + 3x$

$h'(x) = 3x^2 + 3 \geq 3 > 0$: strictly increasing

$\rightarrow$ injection!

Injections and Cardinality

Suppose that $h : A \to B$ is an injection. What is the relationship between $|A|$ and $|B|$?

Sample Problem

Prove that the *composition* of two injective functions is also injective.

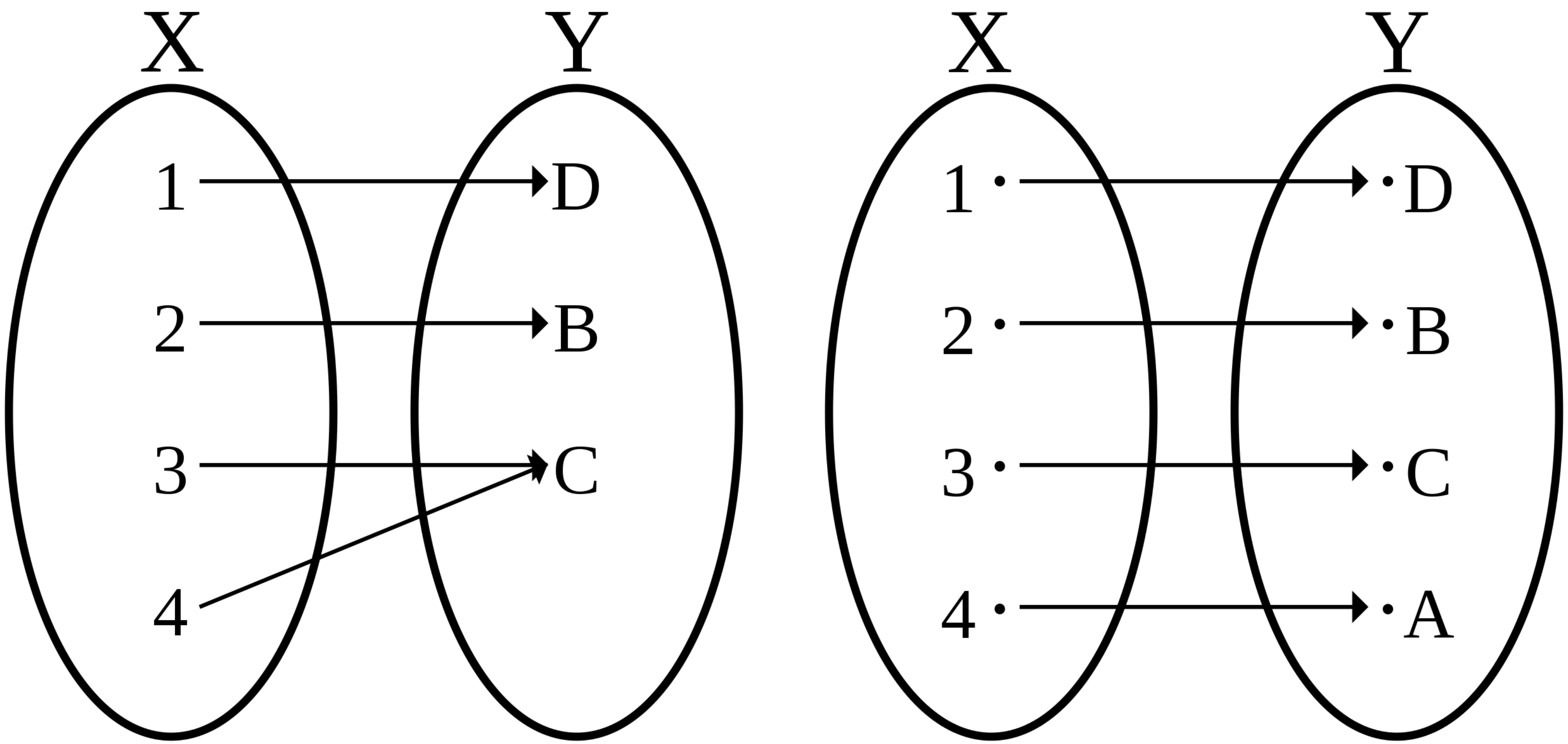
Surjections

We say a function $f : A \to B$ is surjective, or **onto**, if every element in B is mapped to by an element in A, i.e. when the codomain and range are the same set (that is, all possible outputs are actually seen as outputs).

Mathematically, we say

$$\forall\, b \in B, \exists\, a \in A : f(a) = b$$

This is read, "for all b in the codomain, there exists some a in the domain such that f maps a to b.



Both of these functions are examples of surjections.

Anything look familiar?

What domain and codomain make $f : x \mapsto x^2$ a surjection? Not a surjection?

Surjections and Cardinality

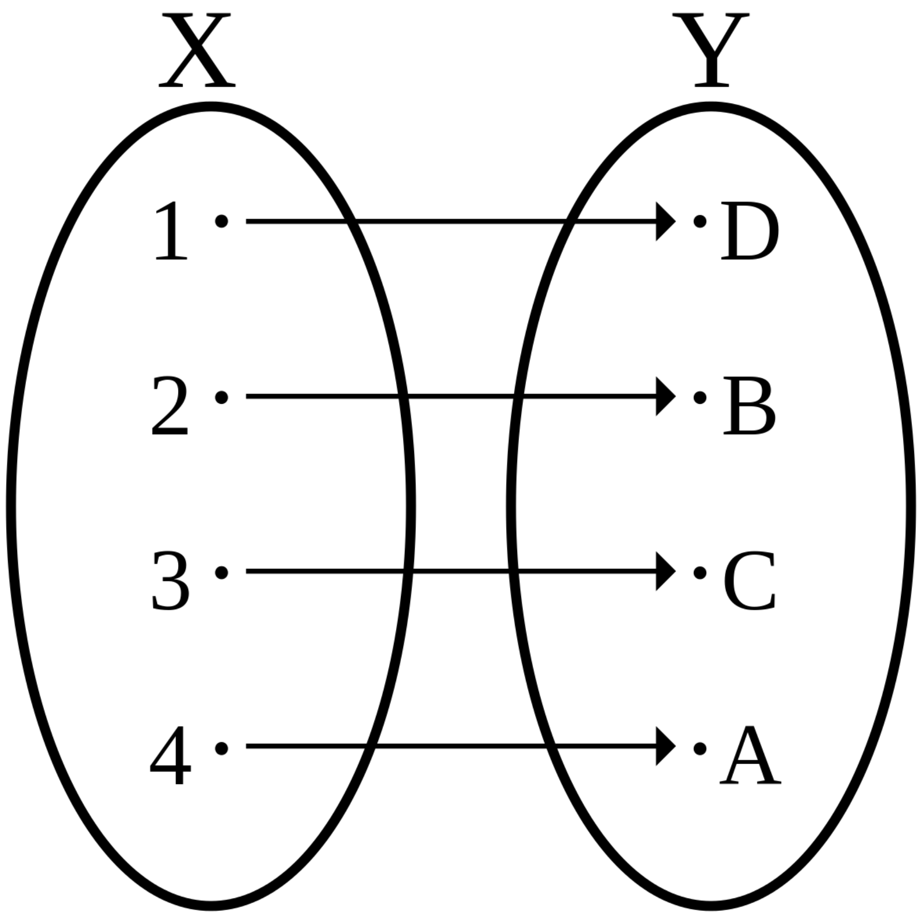
Suppose that $h : A \to B$ is a surjection. What is the relationship between $|A|$ and $|B|$?

Bijections

A function f is bijective **if and only if** it is both injective and surjective.

That is, a function is a bijection if and only if:

- no two elements in the domain map to the same element in the range
- every element in the codomain has something mapping to it from the domain.



This function is an example of a bijection.

Are the following functions bijections?

- $f(x) = x^3$, with domain $\mathbb{R}$ and codomain $\mathbb{R}$
- $d(x, y) = x^2 + y^2$, with domain $\mathbb{R}^2$ and codomain $\mathbb{R}$
- $t \mapsto \text{the } t^{th} \text{ prime number}$, with domain $\{1, 2, 3, 4, ...\}$ and codomain $\{t : \text{t is prime}\}$

Bijections and Cardinality

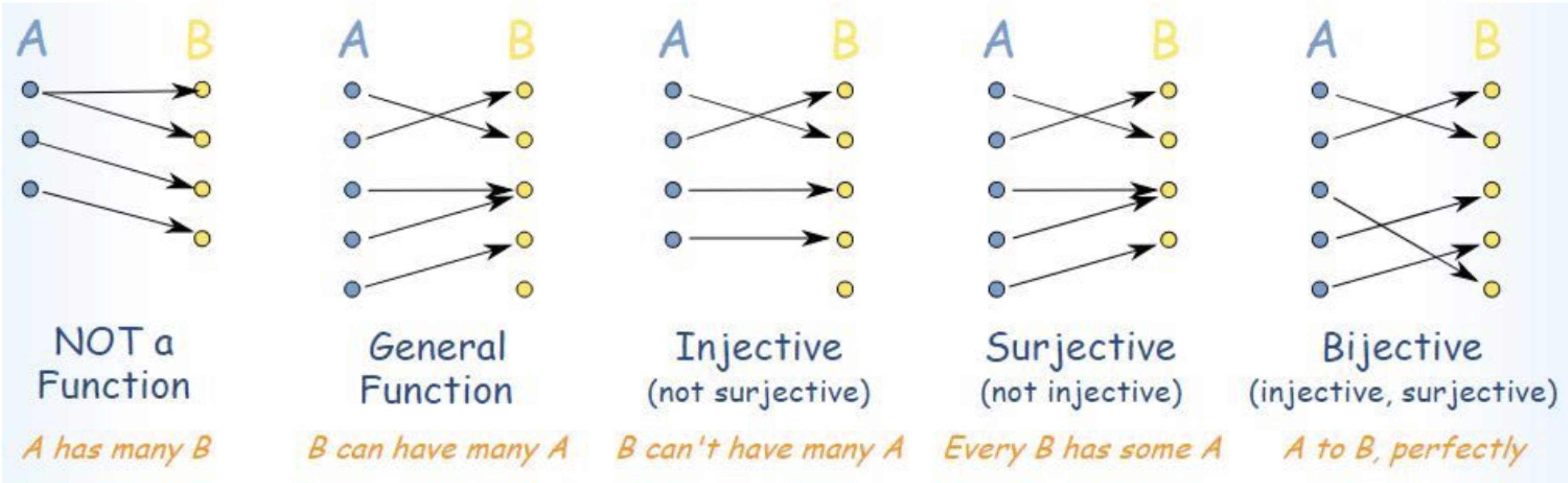
Suppose that $h : A \to B$ is a bijection. What is the relationship between $|A|$ and $|B|$?

Another Definition of Cardinality

We can say that two sets A, B have the same cardinality if there exists a bijection $f : A \to B$ between them.

Why does this definition matter?

Summary of Types of Relations



Next Time...

- We will use our more robust version of cardinality to discuss the relationships number sets you've seen before (natural numbers, whole numbers, integers, rationals, reals, etc.)
- Time permitting, will also begin talking about propositional logic
- Textbook sections are being edited as we go

Image Sources

- Wikipedia: source
- Math is Fun: source