Lecture 9: Binomial Theorem Cont'd, Vieta's Formulas

http://book.imt-decal.org, Ch. 5 (in progress)

Introduction to Mathematical Thinking

November 7, 2018

Suraj Rampure

Announcements

- HW 9 out today, due Wednesday, November 14th
- Extra credit assignment for those at risk of NPing the course... more details coming soon
    - Will be released after the final, due end of dead week
    - Will be graded on *correctness*, not just completion
    - Before the final, we will let you know the overall number of points you have in the course

Recap: Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

General Term

$$t_k = \binom{n}{k} x^{n-k} y^k, 0 \leq k \leq n$$

Trinomial Theorem?

Suppose we want to expand $(x+y+z)^n$. We *could* treat $x+y$ as a single term and use the binomial expansion...

$$(x+y+z)^n = ((x+y)+z)^n$$

$$= \binom{n}{0}(x+y)^n + \binom{n}{1}(x+y)^{n-1}z + \ldots + \binom{n}{n-1}(x+y)z^{n-1} + \binom{n}{n}z^n$$

However, we would then need to expand each term $(x+y)^i$ again with the binomial theorem... that's messy.

Suppose a general term in the expansion of $(x+y+z)^n$ contains a xs, b ys and c zs. Then, we must have that $a+b+c=n$, since the total number of parentheses we choose from in the expansion must be exactly n. Then:

$$t_{a,b,c} = \frac{n!}{a!b!c!}x^a y^b z^c$$

The coefficient $\frac{n!}{a!b!c!}$ comes from the number of ways to arrange a xs, b ys and c zs (think back to counting the number of permutations of MISSISSIPPI).

$$\binom{N}{a}\binom{N-a}{b}\binom{N-a-b}{c} = \frac{N!}{a!(N-a)!}\cdot\frac{(N-a)!}{b!(N-a-b)!}\cdot\frac{(N-a-b)!}{c!(N-a-b-c)!}$$

$$= \frac{N!}{a!b!c!}$$

Example: Calculate the coefficient of x^4 in the expansion of $(x - 3x^{-2} + 4)^8$.

$$
\begin{aligned}
t_{a,b,c} &= \frac{8!}{a!b!c!} x^a (-3x^{-2})^b (4)^c \\
&= (-1)^b \frac{8!}{a!b!c!} 3^b 4^c x^{a-2b}
\end{aligned}
$$

We need $a - 2b = 4$, with the constraints $0 \leq a, b, c \leq 8$ and $a + b + c = 8$. With some trial and error, we can identify the only two solutions, $(4, 0, 4)$ and $(6, 1, 1)$.

Then:

$$
\begin{aligned}
t_{4,0,4} &= (-1)^0 \frac{8!}{4!0!4!} 3^0 4^4 x^4 = 17920x^4 \\
t_{6,1,1} &= (-1)^1 \frac{8!}{6!1!1!} 3^1 4^1 x^4 = -336x^4
\end{aligned}
$$

Thus, the coefficient on x^4 is $17920 - 336 = 17584$.

Generalization of the "Trinomial Theorem"

$$(x+y+z)^n = \sum_{a,b,c:a+b+c=n} \frac{n!}{a!b!c!} x^a y^b z^c$$

This is similar to the way we can represent the binomial theorem:

$$(x+y)^n = \sum_{a,b:a+b=n} \frac{n!}{a!b!} x^a y^b$$

However, this expression of the "trinomial" theorem is less meaningful, as there's no easy way to express this sum any simpler.

Multinomial Theorem

We can further define the "multinomial" coefficient:

$$\binom{n}{k_1, k_2, ...k_m} = \frac{n!}{k_1!k_2!...k_m!}$$

Under the assumption $k_1 + k_2 + ... + k_m = n$, this term is the number of ways to select one subset of size k_1, one subset of size k_2, ... and one subset of size k_m from a group of n items.

$$\binom{n}{k_1} \cdot \binom{n - k_1}{k_2} \cdot \binom{n - k_1 - k_2}{k_3} \cdot ... \cdot \binom{n - k_1 - k_2 - ... - k_{m-1}}{k_m}$$

$$= \frac{n!}{k_1!k_2!k_3! \cdot ... \cdot k_m!}$$

For example, $\binom{11}{1,4,4,2}$ is the number of permutations of MISSISSIPPI (we choose 1 character to be an M, 4 to be an I, 4 to be an S and 2 to be a P).

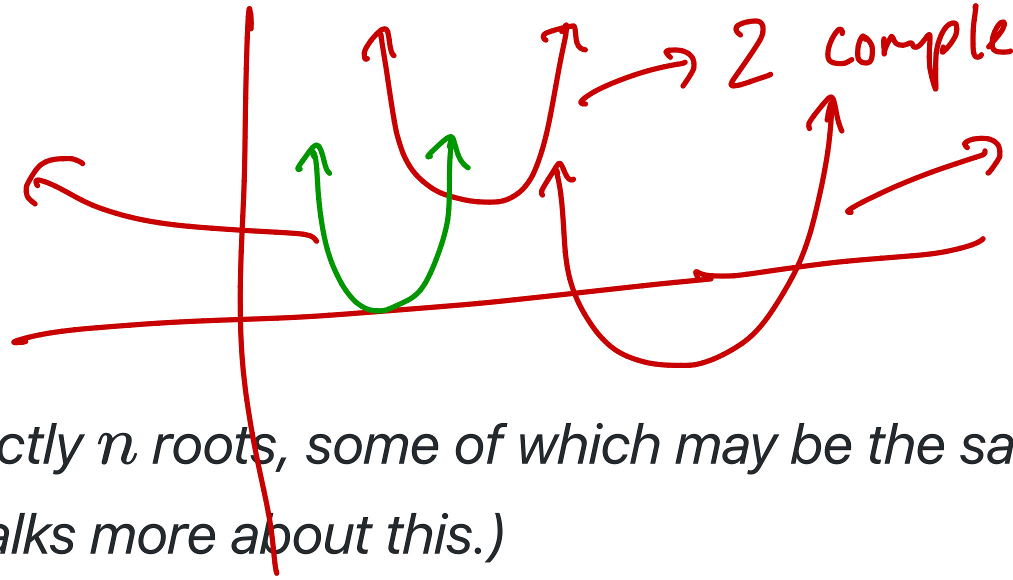
Then:

$$(x_1 + x_2 + ... + x_m)^n = \sum_{k_1+k_2+...+k_m=n} \binom{n}{k_1, k_2, ...k_m} \prod_{i=1}^{m} x_i^{k_i}$$

This last expansion is that of the "multinomial" theorem!

Question: What is the sum of all multinomial coefficients of m terms? (Hint: With $m = 2$, what is this quantity?)

2 real roots,
repeated



2 complex roots

2 real roots

Vieta's Formulas

(Recall: A polynomial of degree n has exactly n roots, some of which may be the same, and some of which may be complex. The textbook talks more about this.)

Vieta's formulas give us a way to interpret a polynomial in standard form, e.g.
$p(x) = ax^2 + bx + c$, in terms of its roots, without having to find the roots specifically.

In the above $p(x)$: what is the sum of the roots? The product?

One way to determine: Use the quadratic formula to solve for both roots, and simplify.

complex: $a + bi$, $a, b \in \mathbb{R}$

$p(x) = ax^2 + bx + c$

Using the quadratic formula:

$$r_1, r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Then:

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$$

$$r_1 r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

It works! How would we extend this to cubic polynomials, though?

There's a simpler way to look at this.

$p(x) = a(x-r_1)(x-r_2)$

Suppose $p(x) = ax^2 + bx + c$ has two roots, r_1 and r_2. Then:

$p(x) = ax^2 + bx + c$

$$p(x) = a(x - r_1)(x - r_2) = ax^2 - a(r_1 + r_2)x + ar_1r_2$$

By comparison, we can see $b = -a(r_1 + r_2)$ and $c = ar_1r_2$, i.e.

$ax^2 - a(r_1+r_2)x + ar_1r_2$

$ax^2 + b\ x + c$

$$r_1 + r_2 = -\frac{b}{a}$$

$$r_1 r_2 = \frac{c}{a}$$

This is the same result we found before, using the quadratic formula.

These are **Vieta's formulas for degree-2 polynomials**.

In $p(x) = 4x^2 + 3x + 3$, we see that the sum of the roots is $-\frac{3}{4}$ and product is $\frac{3}{4}$.

$p(x) = a_3\left(x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3\right)$

What about for cubic polynomials, of the form $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$? Let's assume $p(x)$ has roots r_1, r_2, r_3. Then:

$$a_3x^3 + a_2x^2 + a_1x + a_0 = a_3(x - r_1)(x - r_2)(x - r_3)$$

We have the following choices in this expansion:

- We can choose 3 xs and no roots, yielding x^3 ✓
- We can choose 2 xs and one root, yielding $(-r_1 - r_2 - r_3)x^2 = -(r_1 + r_2 + r_3)x^2$
- We can choose 1 x and two roots, yielding $((-r_1) \cdot (-r_2) + (-r_1) \cdot (-r_3) + (-r_2) \cdot (-r_3))x = (r_1r_2 + r_1r_3 + r_2r_3)x$
- We can choose no xs and three roots, yielding $((-r_1) \cdot (-r_2) \cdot (-r_3)) = -r_1r_2r_3$

$(-r_1 - r_2 - r_3)x^2$

This gives $r_1 + r_2 + r_3 = -\frac{a_2}{a_3}$, $r_1r_2 + r_1r_3 + r_2r_3 = \frac{a_1}{a_3}$, and $r_1r_2r_3 = -\frac{a_0}{a_3}$. Note the alternating signs.

$-a_3(r_1 + r_2 + r_3) = a_2$

symmetric sums

$p(x) = a(x-r_1)(x-r_2)(x-r_3)(x-r_4)$

Each successive term is a **sum of products of roots, taken in different quantities at a time**.

- $-\frac{a_2}{a_3} = r_1 + r_2 + r_3$ is the sum of the products of the roots, taken one at a time, since multiplying a constant by nothing is the constant itself.
  - There are $\binom{3}{1} = 3$ terms in this sum
- $\frac{a_1}{a_3} = r_1 r_2 + r_1 r_3 + r_2 r_3$ is the sum of the product of the roots, taken two at a time â€" it features all 3 possible combinations of two different roots multiplied together.
  - There are $\binom{3}{2} = 3$ terms in this sum
- $-\frac{a_0}{a_3} = r_1 r_2 r_3$ is the sum of the product of the roots, taken three at a time â€" there is only one way to take three items at once, and this is that one way.
  - There are $\binom{3}{3} = 1$ terms in this sum

Exercise: Without manual expansion, determine Vieta's formulas for polynomials of degree 4.

$f(x) = 5x^4 + 3x^3 + \quad$ sum of roots : $-\frac{3}{5}$

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = a_4(x-r_1)(x-r_2)(x-r_3)(x-r_4)$$

3 x's, 1 r

$$-\frac{a_3}{a_4} = r_1 + r_2 + r_3 + r_4$$

$\binom{4}{3} = 4$

2 x's, 2 r

$$\frac{a_2}{a_4} = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4$$

$\binom{4}{2} = 6$

1 x, 3 r's

$$-\frac{a_1}{a_4} = r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4$$

$\binom{4}{1} = 4$

0 x, 4 r's

$$\frac{a_0}{a_4} = r_1r_2r_3r_4$$

$\binom{4}{0} = 1$

We can generalize this to n-degree polynomials!

$$p(x) = a_4\left(x^4 - (r_1 + r_2 + r_3 + r_4)x^3 + (r_1r_2 + r_1r_3 + \cdots)x^2 - (\quad)x + r_1r_2r_3r_4\right)$$

$p(x) = a_2 x^2 + a_1 x + a_0$

Generalized Vieta's Formulas

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

$$= a_n \sum_{k=0}^{n} (-1)^k \big(\text{sum of the products of the roots of } p(x) \text{, taken } k \text{ at a time}\big) x^{n-k}$$

$= a_2 \left((-1)^0 (1)\, x^{2-0} + (-1)^1 (r_1 + r_2)\, x^1 + (-1)^2 (r_1 r_2) \right)$

The algebraic definition isn't as important. What's more important is identifying this pattern.

$p(x) = 1x^{100} - 100x^{99} + 1 \rightarrow$ sum of roots: $-\frac{(-100)}{1} = 100$

product of roots: $\frac{2}{1} = 2$

The binomial theorem is actually just a special case of Vieta's formulas, when all roots are the same! For example, suppose $n = 4$, $r_i = c$ for all i and the leading coefficient is 1. Then:

$$p(x) = (x - c)^4 = \binom{4}{0}x^4 - \binom{4}{1}x^3c + \binom{4}{2}x^2c^2 - \binom{4}{3}xc^3 + \binom{4}{4}c^4$$

Using Vieta's formulas for $n = 4$:

$$a_3 = -(r_1 + r_2 + r_3 + r_4) = -4c$$

$$a_2 = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = 6c^2$$

$$a_1 = -(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4) = -4c^3$$

$$a_0 = r_1r_2r_3r_4 = c^4$$

$$x^2-18x+18=0$$

Example: Suppose a, b satisfy $x^2 - 18x + 18$. Determine $a^2 + b^2$.

Example: Suppose a, b satisfy $x^2 - 18x + 18$ $= 0$. Determine $a^2 + b^2$.

Solution:

From Vieta's, we know $a + b = 18$ and $ab = 18$. Then:

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$\Rightarrow \underline{a^2 + b^2} = (a+b)^2 - 2ab = 18^2 - 2 \cdot 18 = 288$$

$+0x^2$

Example: $p(x) = x^3 - Ax + 15$ has three real roots, two of which sum to 5. What is $|A|$?

Example: $p(x) = x^3 - Ax + 15$ has three real roots, two of which sum to 5. What is $|A|$?

Solution: Let r_1, r_2 be the roots that sum to 5. This must mean $r_3 = -5$, since $r_1 + r_2 + r_3 = 5 - 5 = 0$ (there is no x^2 term).

$$r_1 r_2 = \frac{r_1 r_2 r_3}{r_3}$$

We also know $r_1 r_2 r_3 = -15$. Then,

$$-A = r_1 r_2 + r_1 r_3 + r_2 r_3 = r_3(r_1 + r_2) + r_1 r_2$$

$$= r_3(r_1 + r_2) + \frac{r_1 r_2 r_3}{r_3}$$

$$= -5(5) + \frac{-15}{-5} = -25 + 3 = -22$$

$r_1 + r_2 = 5$

$r_3 = -5$

$r_1 r_2 r_3 = -15$

Thus, $|A| = 22$.

Example: Let $f(x) = (x^2 + 6x + 9)^{50} - 4x + 3$, and suppose $f(x)$ has 100 roots, $r_1, r_2, r_3, \ldots, r_{100}$. Determine $(r_1 + 3)^{100} + (r_2 + 3)^{100} + \ldots + (r_{100} + 3)^{100}$.

$$f(x) = (x+3)^{100} - 4x + 3$$

Example: Let $f(x) = (x^2 + 6x + 9)^{50} - 4x + 3$, and suppose $f(x)$ has 100 roots, $r_1, r_2, r_3, \ldots, r_{100}$. Determine $(r_1 + 3)^{100} + (r_2 + 3)^{100} + \ldots + (r_{100} + 3)^{100}$.

Solution: All roots r_i must satisfy $(x^2 + 6x + 9)^{50} - 4x + 3 = 0$, i.e. $(x + 3)^{100} = 4x - 3$. Taking the sum over all roots r_i on both sides:

$$\boxed{\sum_{i=1}^{100} (r_i + 3)^{100}} = \sum_{i=1}^{100} (4r_i - 3) = \boxed{4 \sum_{i=1}^{100} r_i - 300}$$

Now, we just need to find the sum of the roots r_i. Using the *binomial theorem*, we find the coefficient of the second term of $(x + 3)^{100}$ to be $\binom{100}{1} 3 = 300$, meaning the sum of the roots is -300. Then:

sum of roots of $(x+3)^{100} = -300$

$$\sum_{i=1}^{100} (r_i + 3)^{100} = 4 \cdot (-300) - 300 = -1500$$

$(x+3)^{100} = x^{100} + \binom{100}{1} x^{99} \cdot 3 + \ldots$