Problem Set 5: Midterm Review

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Fall 2018

This homework will not be graded. However, it's a good idea to do as many of these problems as you can, as they will all help you in preparing for our upcoming midterm.

---

1. Determine the truth value of each of the following statements.

   a. If 3 is odd, then 4 = 2 + 2.

   b. If 3 is odd, then 4 = 2 + 3.

   c. If 3 is even, then 4 = 2 + 2.

   d. If 3 is even, then 4 = 2 + 3.

   For the next few problems, assume P is true, Q is false and R is true.

   e. $(P \lor Q) \land R$

   f. $\neg Q \lor P$

   g. $(\neg P) \land (\neg Q) \land R$

   h. $P \iff Q$

   i. $(P \implies Q) \implies \neg R$

   j. $P \oplus Q \oplus R$

   k. $(P \implies Q) \oplus (\neg R)$

   Solution:

   a. True

   b. False

   c. True (3 is not even, therefore this implication is always true)

   d. True (for the same reason as above)

   e. True

   f. True

g. False

h. False

i. True ($P \implies Q$ is false, therefore the second implication is true, regardless of the value of $\neg R$)

j. False ($P \oplus Q$ is true, and "true $\oplus$ true" is false. This holds even if you look at $(Q \oplus R)$ first; an exclusive-or of mulitple statements is true only when an odd number of them are true.)

k. False ("false $\oplus$ false" is false)

2. Use truth tables to prove or disprove each of the following logical equivalences. *(Hint: Recall, logically, "iff" ($\iff$) and "equivalent" ($\equiv$) mean the same thing.*

   a. $P \implies Q \equiv \neg Q \vee P$

   b. $(P \oplus Q) \equiv (P \vee Q) \wedge \neg(P \wedge Q)$

   c. $P \implies Q \equiv \neg Q \implies P$

   d. $(P \vee Q) \wedge R \equiv P \vee (Q \wedge R)$

   e. $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$

   f. $(P \vee (P \wedge Q)) \iff P$ (what does this mean?)

   g. $(P \wedge (P \vee Q)) \iff P$

   h. $P \implies \neg(\neg Q \wedge \neg P) \equiv$ TRUE

3. In each case, determine the value of the provided statement. The universe $\mathbb{U}$ is $\mathbb{Z}$.

   a. $P(17)$, where $P(x) = x \leq 20$

   b. $P(5)$, where $P(x) = (x > 20) \vee (x = 5k, k \in \mathbb{Z})$

   c. $\forall x \, ((x \geq 5) \vee (x < 5))$

   d. $\exists x \, ((x \geq 5) \vee (x < 5))$

   e. $\forall x, y \left(x^2 = y^2 \iff x = y\right)$

   f. $\exists x \exists y \left(x^2 = y^2 \iff x = y\right)$

   g. $\neg \exists x \left(x^2 = 0\right)$

   h. $\forall x \forall y \, (xy \geq x + y)$

   i. $\forall x \exists y (y > x)$

   j. $\forall y \neg(\exists x (y > x))$

   k. $\exists x \forall y (y > x)$

Solution:

a. True

b. True

c. True

d. False

e. False ($(-3)^2 = 3^2$, but $-3 \neq 3$)

f. True (let $x = y = 0$)

g. False (there does exist such an x, namely $x = 0$)

h. False (let $x = 1, y = -1$)

i. True (this says that there is no largest integer)

j. False (with De Morgan's laws, we can rewrite this to be $\forall y \forall x (y \leq x)$; for example, if $x = 1$ and $y = 2$, $y \leq x$ is not true, but it claims to be true for all x and y)

k. False (if such an x existed, it would be the smallest integer, but the integers have no largest or smallest element)

4. Use De Morgan's Laws to rewrite each of the following statements.

a. $\neg(\exists x P(x))$

b. $\neg(\forall x \exists y P(x, y))$

c. $\neg(P \implies Q)$

d. $\neg(\neg Q \vee \neg P)$

e. $\neg(P \oplus \neg Q)$ *(Hint: Re-write $P \oplus \neg Q$, using an identity we saw in lecture and elsewhere on this homework.)*

f. $\neg(\forall x \exists y (P(x) \vee Q(y)))$

g. $\neg((\forall x P(x)) \vee (\exists y Q(y)))$

Solution:

a. $\forall x \neg P(x)$

b. $\exists x \forall y \neg P(x, y)$

c. $P \wedge \neg Q$

d. $Q \wedge \neg P$

e. $(\neg P \wedge \neg Q) \vee (\neg P \vee \neg Q)$ (to see this, write $P \oplus Q$ as $(P \vee Q) \wedge \neg(P \wedge Q)$)

f. $\exists x \forall y (\neg P(x) \wedge \neg Q(y))$

g. $(\exists x \neg P(x)) \wedge (\forall y \neg Q(y))$

5. Suppose $A = \{j^2 : j \leq 5\}$, $B = \{t : t \text{ is prime}\}$, $C = \{s : s \geq 19\}$, and the universe is $\mathbb{U} = \{t : t \in \mathbb{N}_0, t \leq 25\}$.

   Determine each of the following.

   a. $A \cup B$

   b. $A^C \cup C^C$

   c. $|(A \cup B) \cap B|$

   d. $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cup B \cup C|$

   e. $(A - B) - C$

   f. $(A - B)^C \cup (B - C)^C$

6. Suppose $A = \{a, b, c\}, B = \{0, 1\}$ and $C = \{2\}$.

   a. Find $A \times B$.

   b. Find $A \times B \times C$.

   c. Find $B \times A$.

   d. Prove that if $A \times B = B \times A$, then $A = B$. *(Hint: Remember, giving an example doesn't suffice as a proof. You need to show this rigorously.)*

7. Determine whether each of the following functions is an injection, surjection, bijection, or none.

   a. $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x^2 - 1$

   b. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}, f(x) = \sqrt{x}$

   c. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, f(x) = \sqrt{x}$

   d. $f : \mathbb{R} \to \mathbb{N}, f(x) = 23$

   e. $f : \mathbb{R} \to \mathbb{Z}, f(x) = \lceil x \rceil$ *(Hint: Is it possible for any function on $\mathbb{R} \to \mathbb{Z}$ to be a bijection?)*

   f. $f : \mathbb{N} \to \mathbb{Q}, f(x) = \begin{cases} \frac{1}{4}x + \frac{3}{2} & x \neq 4k, k \in \mathbb{N} \\ -\frac{1}{4}x - 1 & x = 4k, k \in \mathbb{N} \end{cases}$

Solution:

a. *None.* f is not an injection, as $f(-1) = f(1) = 0$, but $-1 \neq 1$. f is not a surjection, as $f(x) = 2$ has no solution (as the inputs to f are only the integers).

b. *Injection only.* f is an injection as square-rooting is unique. f is not a surjection because $\sqrt{x} = -1$ has no solution when the domain is the non-negative real numbers.

c. *Bijection.* Now, when we restrict the codomain to be only the non-negative real numbers, $x \mapsto \sqrt{x}$ is a bijection.

d. *None.* f is not an injection, as all inputs map to the same output. f is not a surjection as the range is simply $\{23\}$, which is not equal to the set of natural numbers.

e. *Surjection only.* f is not an injection, as $\lceil 1.9 \rceil = \lceil 1.91 \rceil$, but $1.9 \neq 1.91$. f is a surjection because for any integer n, $\lceil n \rceil = n$, therefore every integer has a pre-image (i.e. every integer is mapped to).

f. *Injection only.* f is an injection, as no two inputs ever have the same output (write out a few terms of this sequence to see why). f is not a surjection as the rational number 1 is never seen as an output (The only actual outputs are in the set $\{\frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, ...\} \cup \{2, -2, 3, -3, 4, -4, ...\}$. This is similar to a homework problem, but there the domain and codomain were two specific sets, not $\mathbb{N}$ and $\mathbb{Z}$).

8. Suppose $f(x)$ and $g(x)$ are functions.

Prove that if $f(g(x))$ is one-to-one, then $g(x)$ is one-to-one. *(Hint: "one-to-one" is another term for "injective".)*

Solution: While we can do this directly, here we'll use a proof by contraposition. If we're doing a proof by contraposition, we must show that if $g(x)$ is not one-to-one, then $f(g(x))$ is not one-to-one.

Suppose $g(x)$ is not one-to-one.

We can do this by contradiction (remember the form $P \wedge \neg Q$). Let's assume that $f(g(x))$ is one-to-one and that $g(x)$ is not.

If $g(x)$ is not one-to-one, there must exist x_1, x_2 such that $g(x_1) = g(x_2)$, but $x_1 \neq x_2$. But, since $f(g(x))$ is one-to-one, $f(g(x_1)) = f(g(x_2)) \implies x_1 = x_2$, contradicting what we just assumed. Therefore, by contradiction, we've shown that if $f(g(x))$ is one-to-one, then $g(x)$ is one-to-one.

9. a. Prove there is no smallest positive rational number.

b. Prove there is no largest prime number. *(Hint: All natural numbers can be written as the product of primes.)*

Solution:

a. Let's proceed by contradiction. Assume there is some smallest positive rational number r. Then, $r' = \frac{r}{2}$ is also positive, and is strictly smaller than r. This means that r is not the smallest positive rational number, which contradicts what we originally assumed. Therefore, by contradiction, we've proved that there is no smallest positive rational number.

b. Once again, let's proceed by contradiction. Assume there is some largest prime number. This must mean that there are only finitely many prime numbers $p_1, p_2, p_3, ...p_n$ (since the smallest prime is $p_1 = 2$), and that the largest prime is p_n.

Then, let's construct the number $P = p_1 \cdot p_2 \cdot ... \cdot p_{n-1} \cdot p_n + 1$. This P is not a multiple of any of the primes $p_1, p_2, ...p_n$. Since any natual number can be written as the product of primes, and none of the primes $p_1, p_2, ..., p_n$ are a factor of P, this must mean that P is prime. However, P is larger than p_n, meaning that p_n is not the largest prime number, contradicting what we originally assumed.

Therefore, by contradiction, we've shown that there is no largest prime number.

10. A perfect number is a positive integer n such that the sum of the factors of n that are less than n, is equal to n. For example, the factors of 6 (that are not equal to 6) are 1, 2, and 3, and 1 + 2 + 3 = 6.

Prove that a prime number cannot be a perfect number.

Solution: Suppose there exists some prime number p that is a perfect number. 1 is only one factor of p less than p. For p to be a perfect number, we would need that $p = 1$, but 1 is neither prime nor composite. This is a contradiction (since we assumed p was prime); therefore, by contradiction, we've shown that a prime number cannot be a perfect number.

11. In base 10, the integer $a_{n-1}a_{n-2}...a_1a_0$ can be written as $10^{n-1}a_{n-1} + 10^{n-2}a_{n-2} + ... + 10^2a_2 + 10^1a_1 + 10^0a_0$. For example, $427 = 4 \cdot 10^2 + 2 \cdot 10^1 + 7 = 400 + 20 + 7$.

Suppose $n \in \mathbb{N}$.

a. Prove that n is divisible by 3 if and only if the sum of the digits of n is divisible by 3.

b. Prove that n is divisible by 9 if and only if the sum of the digits of n is divisible by 9.

(Remember, to prove the statement "A if and only if B", you must prove $A \implies B$ and $B \implies A$.)

Solution: Done in lecture on Wednesday.

12. Prove that there is no integer $n > 3$ such that all of $n, n+2, n+4$ are prime. *(Hint: Break n into three cases – when it is divisible by 3, when it has a remainder of 1 when divided by 3, and when it has a remainder of 2 when divided by 3.)*

Solution:

Let's break n into three cases: $n = 3k$, $n = 3k + 1$ and $n = 3k + 2$.

Case 1: $n = 3k$

When $n = 3k$, $3k$ is clearly not prime, as it's a multiple of 3. Therefore, when $n = 3k$, not all of $n, n+2, n+4$ are prime.

Case 2: $n = 3k+1$

When $n = 3k+1$, $n+2 = 3k+1+2 = 3(k+1)$, which is not prime as it's a multiple of 3. Therefore, when $n = 3k+1$, not all of $n, n+2, n+4$ are prime.

Case 3: $n = 3k+2$

When $n = 3k+2$, $n+4 = 3k+2+4 = 3(k+2)$, which is not prime as it's a multiple of 3. Therefore, when $n = 3k+2$, not all of $n, n+2, n+4$ are prime.

We've shown that in all cases, either n, $n+2$ or $n+4$ is a multiple of 3. Therefore, in no case are all of $n, n+2$ and $n+4$ prime, thus proving the original statement.

13. In any set of n numbers, there is at least one number that is less than or equal to the mean.

    a. Write this statement using propositional logic.

    b. Prove this statement.

Solution:

a. Suppose $x_1, x_2, ...x_n$ is some sequence of numbers. Then:

$$\forall n \in \mathbb{N}, \exists\, i : x_i \leq \frac{x_1 + x_2 + ... + x_n}{n}$$

b. Let's do this as a proof by contradiction. Let's assume that there does not exist a single number that is less than or equal to the mean. This implies that each $x_1, x_2, ..., x_n$ are greater than the mean.

$$\begin{aligned} x_1 &> \frac{x_1 + x_2 + ... + x_n}{n} \\ x_2 &> \frac{x_1 + x_2 + ... + x_n}{n} \\ &\vdots \\ x_n &> \frac{x_1 + x_2 + ... + x_n}{n} \end{aligned}$$

Adding the above equations:

$$x_1 + x_2 + ... + x_n > n \cdot \frac{x_1 + x_2 + ... + x_n}{n} \implies x_1 + x_2 + ... + x_n > x_1 + x_2 + ... + x_n$$

This is a contradiction, as the quantity $\sum_{i=1}^{n} x_i$ cannot be greater than itself (no number can be greater than itself). Therefore, by contradiction, we've proven the original statement.

14. Consider the series defined by $t_0 = 1$, $t_n = 2t_{n-1} + 7, \forall n \in \mathbb{N}_0$. Use induction to prove that $t_n \leq 2^{n+3} - 7$.

Solution:

Base Case: $n = 0$

$t_0 = 1$, and $2^{0+3} - 7 = 1 \geq 0$, therefore the base case holds.

Induction Hypothesis:

Assume that $t_k \leq 2^{k+3} - 7$ for some arbitrary integer k.

Induction Step:

$$\begin{aligned} t_{k+1} &= 2t_k + 7 \\ &\leq 2(2^{n+3} - 7) + 7 \\ &= 2^{n+4} - 14 + 7 \\ &= 2^{n+4} - 7 \end{aligned}$$

as required. Therefore, by induction, the statement $t_n \leq 2^{n+3} - 7$ holds.

15. The harmonic series $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ is known to be unbounded as $n \to \infty$. In this problem, we will use induction to prove that the harmonic series is unbounded.

Using induction, prove that $\forall n \in \mathbb{N}, H_{2^n} \geq 1 + \frac{n}{2}$. Why does this prove that the harmonic series is unbounded?

Solution: *Base Case:* $n = 1$

$H_{2^1} = 1 + \frac{1}{2} \geq 1 + \frac{1}{2}$, as required, therefore the base case holds.

Induction Hypothesis: Assume that $H_{2^k} \geq 1 + \frac{k}{2}$ for some arbitrary integer k.

Induction Step:

$$
\begin{aligned}
H_{2^{k+1}} &= H_k + \frac{1}{2^k+1} + \frac{1}{2^k+2} + ... + \frac{1}{2^{k+1}-1} + \frac{1}{2^{k+1}} \\
&\geq H_k + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + ... + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} \\
&= H_k + \frac{2^k}{2^{k+1}} \\
&\geq 1 + \frac{k}{2} + \frac{1}{2} \\
&= 1 + \frac{k+1}{2}
\end{aligned}
$$

The inequality in the second line holds because each term $\frac{1}{2^k+1}, \frac{1}{2^k+2}, \cdots \frac{1}{2^{k+1}-1}, \frac{1}{2^{k+1}}$ is greater than or equal to the term $\frac{1}{2^{k+1}}$ (since the denominators are increasing, the fractions are decreasing, so 2^{k+1} is the largest denominator we have and thus $\frac{1}{2^{n+1}}$ is the smallest number we have).

The equality in the third line comes from the fact that there are 2^k terms of the form $\frac{1}{2^{k+1}}$. (Remember, $2^k + 2^k = 2(2^k) = 2^{k+1}$).

Therefore, by induction, we have that $H_{2^n} \geq 1 + \frac{n}{2}$.

Since there is no largest integer, there is no largest value of $1 + \frac{n}{2}$. Since $H_{2^n} \geq 1 + \frac{n}{2}$, this means that there is no largest value of H_{2^n}, meaning that the sum H_{2^n} (and also the sum H_n) does not approach a finite value as $n \to \infty$.

16. In this problem, f_i will refer to the Fibonnaci sequence. This sequence is defined by $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, \forall n \geq 2, n \in \mathbb{N}$.

For parts b and c of this problem, you will need to use **strong induction**. In regular mathematical induction, in the induction hypothesis we assume that $P(k)$ holds, for some arbitrary value of k. In strong induction, instead of assuming just $P(k)$, we assume $P(0) \wedge P(1) \wedge ... P(k-1) \wedge P(k)$, i.e. that the proposition holds for all non-negative integers up to and including k. This is useful if, in our induction step, we need to assume more than just $P(k)$.

a. Prove that $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$.

b. Prove that $f_n > 2n$, for $n \geq 8$.

c. Prove that $f_n \leq 2^n$.

Solution:

a. *Base Case:* $n = 1$ $\sum_{i=1}^{1} f_i^2 = 1^2 = 1$. Also, $f_1 f_2 = 1 \cdot 1 = 1$, therefore the base case holds.

Induction Hypothesis: Assume that $\sum_{i=1}^{k} f_i^2 = f_k f_{k+1}$ for some arbitrary value of k.

Induction Step:

$$\begin{aligned}\sum_{i=1}^{k+1} f_i^2 &= \sum_{i=1}^{k} f_i^2 + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2}\end{aligned}$$

Therefore, by induction, the statement holds.

b. This problem requires us to use strong induction, as in the induction step we will expand f_{k+1} to $f_k + f_{k-1}$ and will need to use the hypothesis for both f_k and f_{k-1}.

Since our base case here is $n = 8$, we should look at $f_1, ..., f_7$. $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, f_8 = 21$.

Base Case: $n = 8$

$f_n = 21 > 2 \cdot 8$, therefore the base case holds.

Induction Hypothesis: Assume $f_i > 2i$ for all integers $i \in \{1, 2, 3, 4, ...k\}$, for some arbitrary integer k (this is where induction and strong induction differ).

Induction Step:

$$\begin{aligned}f_{k+1} &= f_k + f_{k-1} \\ &> 2k + 2(k-1) \\ &> 2k + 2 \\ &> 2(k+1)\end{aligned}$$

as required.

c. *Base Case:* $n = 1$

$f_1 = 1 \leq 2^1 = 2$, therefore the base case holds.

Induction Hypothesis: Assume that $f_i \leq 2^i$ for all integers $1 \leq i \leq k$, for some arbitrary integer k.

Induction Step:

$$
\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
&\leq 2^k + 2^{k-1} \\
&\leq 2^k + 2^k \\
&\leq 2(2^k) \\
&\leq 2^{k+1}
\end{aligned}
$$

as required. Therefore, by induction, the statement holds.

17. In this problem, we will prove that $3|n^3 - n$ (i.e. that $n^3 - n$ is divisible by 3) for all $n \in \mathbb{N}_0$.
    a. Prove this directly.
    b. Prove this using induction.

Solution:

a. We can factor $n^3 - n$ into $n(n^2 - 1)$, and further into $n(n-1)(n+1)$. This represents the product of three consecutive integers, and in any three consecutive integers, we know that exactly one is a multiple of three (to see this, write out the sequence of positive integers). Therefore, either $n - 1$, n or $n + 1$ will have a factor of three in it, and therefore $n^3 - 3$ will be divisible by 3.

   It is also possible to do this with casework, but that's significantly more messy.

b. *Base Case:* $n = 0$: 3 does indeed divide 0, as we can find some integer j such that $0 = 3j$ (namely, $j = 0$).

   Induction Hypothesis: Assume that $3|k^3 - k$ for some arbitrary integer k.

   Induction Step:

   We now want to show that $3|(k+1)^3 - (k+1)$, i.e. that we can write $(k+1)^3 - (k+1)$ as $3 \cdot$ (some integer).

   We know that we can write $k^3 - k = 3j$, for some integer j, from the induction hypothesis. Then:

$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\
&= k^3 - k + 3k^2 + 3k \\
&= 3j + 3(k^2 + k) \\
&= 3(j + k^2 + k)
\end{aligned}
$$

   Therefore, by induction, the statement holds true.

18. Recall, in lecture we showed that $1 + 2 + ... + n = \frac{n(n+1)}{2}$ as follows:

$$\begin{aligned} S_n &= 1 + 2 + 3 + ... + (n-1) \\ S_n &= (n-1) + (n-2) + ... + 1 \\ 2S_n &= n(n+1) \\ S_n &= \frac{n(n+1)}{2} \end{aligned}$$

a. An arithmetic sequence with initial term a_0 and common difference d is defined by $a_n = a_0 + (n-1)d$ for $n \in \mathbb{N}$. Prove that $\sum_{i=1}^{n} a_i = n\frac{2a_0+(n-1)d}{2}$, using (i) induction and (ii) a direct proof similar to the one above.

b. A geometric series with initial term a and common ratio r is defined by $a_n = ar^{n-1}$ for $n \in \mathbb{N}$. Prove that $\sum_{i=1}^{n} a_i = \frac{a_0(r^n-1)}{r-1}$ using (i) induction and (ii) a direct proof similar to the one above.

Solution:

a. *(i) Induction*

Base Case: $n = 1$

We know the first term $a_1 = a_0 + (1-1)d = a_0$. Also, $\sum_{i=1}^{1} a_i = 1 \cdot \frac{2a_0+(1-1)d}{2} = a_0$. Therefore, the base case holds.

Induction Hypothesis: Assume that $\sum_{i=1}^{k} a_i = k\frac{2a_0+(k-1)d}{2}$.

Induction Step:

$$\begin{aligned} \sum_{i=1}^{k+1} a_i &= \sum_{i=1}^{k} a_i + a_{k+1} \\ &= k\frac{2a_0+(k-1)d}{2} + a_0 + kd \\ &= \frac{2ka_0 + k(k-1)d}{2} + \frac{2(a_0+kd)}{2} \\ &= \frac{2ka_0 + 2a_0 + k^2d - kd + 2kd}{2} \\ &= (k+1)\frac{2a_0+kd}{2} \end{aligned}$$

as required. Therefore, by induction, the statement holds.

(ii) Direct Proof

We will proceed by writing S_n twice, once in the forward direction and once reversed. We will then notice that corresponding terms each sum to the same quantity, $2a_0 + (n-1)d)$. This is exactly what we did in lecture and above.

$$
\begin{aligned}
S_n &= a_0 + (a_0 + d) + (a_0 + 2d) + ... + (a_0 + (n-2)d) + (a_0 + (n-1)d) \\
S_n &= (a_0 + (n-1)d) + (a_0 + (n-2)d) + ... + (a_0 + 2d) + (a_0 + d) + a_0 \\
2S_n &= n \cdot (2a_0 + (n-1)d) \\
S_n &= \frac{n \cdot (2a_0 + (n-1)d)}{2}
\end{aligned}
$$

as required.

b. *(i) Induction*

Base Case: $n = 1$ We know the first term is $a_1 = a_0 r^{1-1} = a_0$. Also, $\sum_{i=1}^{1} a_i = a_0 \frac{r^1 - 1}{r-1} = a_0$. Therefore, the base case holds.

Induction Hypothesis: Assume that $\sum_{i=1}^{k} a_i = \frac{a_0(r^k - 1)}{r-1}$ for some arbitrary integer k.

Induction Step:

$$
\begin{aligned}
\sum_{i=1}^{k+1} a_i &= \sum_{i=1}^{k} a_i + a_{k+1} \\
&= \frac{a_0(r^k - 1)}{r-1} + a_0 r^k \\
&= a_0 \left(\frac{r^k - 1}{r-1} + \frac{r^k(r-1)}{r-1} \right) \\
&= a_0 \frac{r^k - 1 + r^{k+1} - r^k}{r-1} \\
&= \frac{a_0(r^{k+1} - 1)}{r-1}
\end{aligned}
$$

as required. Therefore, by induction, the statement holds.

(ii) Direct Proof

We will proceed by looking at the expansions of S_n and rS_n, and subtracting.

$$
\begin{aligned}
S_n &= a_0 + a_0 r + a_0 r^2 + ... + a_0 r^{n-2} + a_0 r^{n-1} \\
rS_n &= a_0 r + a_0 r^2 + a_0 r^3 + ... + a_0 r^{n-1} + a_0 r^n \\
rS_n - S_n &= a_0 r^n - a_0 \\
(r-1)S_n &= a_0(r^n - 1) \\
S_n &= \frac{a_0(r^n - 1)}{r-1}
\end{aligned}
$$

as required.

19. Prove that $0.9999999.... = 1$.

Solution:

$$\begin{aligned} x &= 0.99999... \\ 10x &= 9.99999... \\ \implies 9x &= 9 \\ \implies x &= 1 \end{aligned}$$