CS 198-087
Fall 2018

Introduction to Mathematical Thinking

Midterm

This is the midterm examination for Introduction to Mathematical Thinking.

- This exam has 7 questions (including question 0). The exam is out of 72 points.
- The exam will last for exactly 1 hour and 20 minutes, unless you have pre-arranged DSP accomoda-tions.
- You can use the backs of pages for scrap work, but please write your answers only on the fronts of pages.

DO NOT TURN THE PAGE UNTIL INSTRUCTED.

In the meantime, fill out the information on this page.

Name: Esther Lin

@berkeley.edu email: etlin@berkeley.edu

Student ID Number:

Name of student to your left: Kseniya Usovich

Name of student to your right: Sher Shah

0 Preliminary Questions

Points: 3 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam?

4

b) What is your favorite topic so far in this course?

Proofs... kind of

c) Name one of the songs I played in class on Monday.

Bollywood song I don't know the name of

1 True or False

Points: 17 (1 each)

Circle either true or false in each of the below.

a) **True** or **False**: $\forall n \in \mathbb{N}, n^2 + 1 < 0 \implies n^2 + 2 < 0$ 1, 2, 3, 4

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ 1, 3, 6, 10

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$

d) **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd."

e) **True** or **False**: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$

f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3 | n^3 + 2n$ *(Hint: Try and prove or disprove the statement.)*

g) **True** or **False**: The set of all even integers is countably infinite.

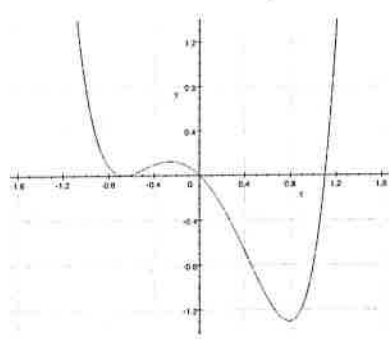
h) **True** or **False**: The union of ten countably infinite sets is uncountably infinite.

i) **True** or **False**: A function is surjective if $\forall a \in A, \exists b \in B : f(a) = b$.

j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective.

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A - B| + |B - A|$.

l) **True** or **False**: The following function is injective.



For the remaining parts, consider sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$.

m) **True** or **False**: There exists a bijection $f : A \to B$.

n) **True** or **False**: The relation $r : \{(1,3), (3,2), (2,2), (4,1), (5,1)\}$ is a function.

o) **True** or **False**: $B \subseteq A$

p) **True** or **False**: $(B \cup \{6\}) \subset A$

q) **True** or **False**: $|A - B| = |A| - |B|$ 2

2 Set Matching

Points: 10 (2 each)

Consider the following sets, where the universe is $\mathbb{N}$:

$A = \{x : x \text{ is prime}\}$

$B = \{x : x = a^3, a \in \mathbb{N}\}$

$C = \{2, 15, 18, 64\}$

$D = \{15, 49, 81\}$ $C\{2, 15, 18,$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set)

$A \cap D$

b) $\{15, 18, 49, 81\}$

$(C \cup D) - \{x : x \text{ is even}, x \in C\}$

c) $\{2\}$

$A \cap C$

d) $\{1,2\}$

e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \land y|x\}$

3 Set Proof

Points: 6

Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$.

4 Choose One

Points: 8 (+2 for completing both)

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational.

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)*

a) $\sqrt{3} = \frac{a}{b}$

$(\sqrt{3})^2 = (\frac{a}{b})^2$

$3 = \frac{a^2}{b^2}$

$3b^2 = a^2$

$3b^2 = (3K)^2$

$3b^2 = 9K^2$

$b^2 = 3K^2$

5 Induction

Points: 16 (8 each)

a) Prove that $8|9^n - 1$, for all $n \in \mathbb{N}$.

① $n=1$ $9^1-1 = 9-1 = 8$

② assume $8 | 9^k - 1$

$n=k+1$ ③ $9^{k+1}-1$

$= (9^k \cdot 9) - 1$

$= 9 \cdot 9^k - 1$

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

① $\sum_{i=0}^{0} 2^0 = 2 - 2^0$

$1 = 1$

② assume $\sum_{i=0}^{k} 2^{-i} = 2 - 2^{-k} = \frac{2 \cdot 2^k - 1}{2^k}$ $\quad \frac{2}{1} - \frac{1}{2^k} = \frac{2 \cdot 2^k}{2^k} - \frac{1}{2^k} = \frac{2 \cdot 2^k - 1}{2^k}$

③ $\sum_{i=0}^{k+1} 2^{-i} = \sum_{i=0}^{k} 2^{-i} + 2^{-(k+1)}$

$n=k+1$ $= 2 - 2^{-k} + 2^{-(k+1)}$

$= \left(2 - \frac{1}{2^k}\right) + \frac{1}{2^{k+1}}$

$= \frac{2 \cdot 2^k - 1}{2^k} + \frac{1}{2^{k+1}}$

$= \frac{4 \cdot 2^k}{2^{k+1}}$

6 Fun with Logic

Points: 12 (a: 2, b: 4, c: 6)

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

$\neg(A \vee B)$

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

$\neg A = (A \downarrow B) \downarrow B$

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*

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Name: Elena Belk

@berkeley.edu email: elenabelk@berkeley.edu

Student ID Number: 3032966991

Name of student to your left: N/A

Name of student to your right: Kseniya Usovich

0 Preliminary Questions

Points: 3 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam? 5

b) What is your favorite topic so far in this course? Propositional logic

c) Name one of the songs I played in class on Monday. I came right at Berkeley time :\
and didn't hear any of them

1 True or False

$(3k+1)^2 = 9k^2+1$

$(3k+2)^2 = 9k^2+4$

Points: 17 (1 each)

Circle either true or false in each of the below.

a) **True** or **False**: $\forall n \in \mathbb{N}, n^2+1<0 \implies n^2+2<0$

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ $\frac{n(n+1)}{2}$

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$

d) **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd."

e) **True** or **False**: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$

$n = 3k$
$n = 3k+1$
$n = 3k+2$

f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3 | n^3+2n$ *(Hint: Try and prove or disprove the statement.)*

$1+2$ ✓ $8+4$ ✓ $9+6$ ✓

$(3k)^3 + 2(3k) = 27k^3 + 6k = 3(9k^3+2k)$ ✓

$(3k+1)^3 + 2(3k+1) = 27k^3 + 9k^2 + 3k + 1 + 6k + 2 = 27k^3 + 9k^2 + 9k + 3$ ✓

$(3k+2)^3 + 2(3k+2) = 27k^3 + 18k^2 + 12k + 8 + 6k + 4$

g) **True** or **False**: The set of all even integers is countably infinite.

h) **True** or **False**: The union of ten countably infinite sets is uncountably infinite.

1 2 3 4 5 6 7 8 9 10
a, b, c, d, …

i) **True** or **False**: A function is surjective if $\forall a \in A, \exists b \in B : f(a) = b$.
every $a \in$ Dom $\to b$

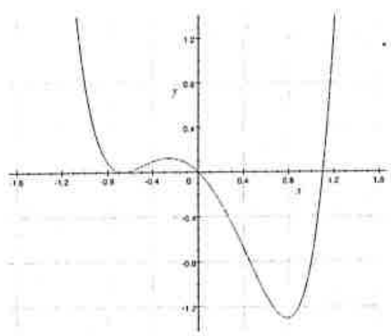
$2x^3 - 15 = 0$
$2x^3 = 15$
$x^3 = \frac{15}{2}$

$\frac{df}{dx} = 8x^2 = 0$, $x = 0$
$\frac{df^2}{dx^2} = 16x = 0$, $x = 0$

j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective.

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A - B| + |B - A|$.

l) **True** or **False**: The following function is injective.



For the remaining parts, consider sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$.

m) **True** or **False**: There exists a bijection $f : A \to B$.

n) **True** or **False**: The relation $r : \{(1,3),(3,2),(2,2),(4,1),(5,1)\}$ is a function.

o) **True** or **False**: $B \subseteq A$

p) **True** or **False**: $(B \cup \{6\}) \subset A$

q) **True** or **False**: $|A - B| = |A| - |B|$

$|\{4,5\}| = 5 - 3$
$2 = 2$ ✓

2 Set Matching

Points: 10 (2 each)

Consider the following sets, where the universe is $\mathbb{N}$:

$A = \{x : x \text{ is prime}\}$ 2, 3, 5, 7, 11, 13, ...

$B = \{x : x = a^3, a \in \mathbb{N}\}$ 1, 8, 27, 64, 125, ...

$C = \{2, 15, 18, 64\}$

$D = \{15, 49, 81\}$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set)

$A \cap D$

b) $\{15, 18, 49, 81\}$

$((C \cup D) - A) - B$

c) $\{2\}$

$C \cap A$

d) $\{1, 2\}$

$(A \cap B) \cap (C \cap A)$

e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \wedge y | x\}$ x, y have factor $\neq$ prime

A^c

3 Set Proof

Points: 6

Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$.

If A and B are sets, either they are disjoint, they have some elements in common, or they are the same set.

If A and B are disjoint: $B - A = \{x : x \in B \wedge x \notin A\}$, but B contains none of the same elements as A, so $B - A = B$. Thus $A \cap (B-A) = A \cap B$, but A and B are disjoint, so $A \cap B = \emptyset$ and $A \cap (B-A) = \emptyset$.

If A and B have some common elements: $(B-A)$ is the set of all elements in B that are not in A. Therefore $A \cap (B-A)$ is the intersection of A and a set containing no elements of A, so this intersection is $\emptyset$.

If A and B are the same set: $(B-A) = B - B = \emptyset$

so $A \cap (B-A) = A \cap \emptyset = \emptyset$

Therefore, if A and B are sets, then $A \cap (B-A) = \emptyset$

4 Choose One

Points: 8 (+2 for completing both)

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational.

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)*

a) Proof by contradiction:

Suppose $\sqrt{3}$ is rational. Then, $\sqrt{3} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$

$3 = \frac{a^2}{b^2}$, so a^2 and b^2 must both be odd.

Thus a and b must also be odd.

So $a = 2n+1$ and $b = 2m+1$ for some $n, m \in \mathbb{Z}$

so $3 = \frac{(2n+1)^2}{(2m+1)^2}$

$3b^2 = a^2$

$3(2m+1)^2 = (2n+1)^2$

$3(4m^2+4m+1) = 4n^2+4n+1$

$12m^2+12m+3 = 4n^2+4n+1$

$12m^2+12m+4 = 4n^2+4n+2$

$2(6m^2+6m+2) = 2(2n^2+2n+1)$ so a^2 and b^2 have a common factor of 2 and are both even.

But, a^2 and b^2 must both be odd

$\therefore$ $\sqrt{3}$ is irrational by contradiction.

5 Induction

Points: 16 (8 each)

$8 | 9^n - 1$ means $9^n - 1 = 8m$ for some $m \in \mathbb{N}_0$

a) Prove that $8|9^n - 1$, for all $n \in \mathbb{N}$.

Base case: $n = 0$

$9^0 - 1 = 1 - 1 = 0 = 0 \cdot 8$

$\therefore$ the base case holds

Inductive hypothesis: assume $8 | 9^k - 1$ for some arbitrary $k \in \mathbb{N}_0$,

so $\exists m \in \mathbb{N}_0$ s.t. $9^k - 1 = 8m$

Inductive step: want to show $8 | 9^{k+1} - 1$, meaning $9^{k+1} - 1 = 8p$ for some $p \in \mathbb{N}_0$

have $9^k - 1 = 8m$

$9^k = 8m + 1$

$9 \cdot 9^k = 9(8m+1)$

$9^{k+1} = 9(8m+1)$

$9^{k+1} = 72m + 9$

$9^{k+1} - 1 = 72m + 8$

$9^{k+1} - 1 = 8(9m+1)$ $\therefore$ $8 | 9^{k+1} - 1$

$\therefore$ Induction holds

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

Base case: $n = 0$

$\sum_{i=0}^{0} 2^{-i} = 2^0 = 1 = 1$

$2 - 2^0 = 2 - 1 = 1$ $\therefore$ base case holds

Inductive hypothesis: assume $\sum_{i=0}^{k} 2^{-i} = 2 - 2^{-k}$

$2^0 + 2^{-1} + 2^{-2} + \ldots + 2^{-k} = 2 - 2^{-k}$

Inductive step: want to show $\sum_{i=0}^{k+1} 2^{-i} = 2 - 2^{-(k+1)} = 2 - 2^{-k-1} = 2 - 2^{-k}(2^{-1})$

$2^0 + 2^{-1} + \ldots + 2^{-k} + 2^{-(k+1)} = 2 - 2^{-(k+1)}$

have $2^0 + 2^{-1} + \ldots + 2^{-k} = 2 - 2^{-k} = 2 - 2^{-k}$

$2^0 + 2^{-1} + \ldots + 2^{-k} + 2^{-(k+1)} = 2 - 2^{-k} + 2^{-(k+1)}$

$\sum_{i=0}^{k+1} 2^{-(k+1)} = 2 - 2^{-k} + 2^{-k-1}$

$= 2 + 2^{-k}(-1) + 2^{-k}(2^{-1})$

$= 2 + 2^{-k}(2^{-1} - 1)$

$= 2 + 2^{-k}(\frac{1}{2} - 1)$

$= 2 + 2^{-k}(-\frac{1}{2})$

$= 2 - 2^{-k}(2^{-1})$

$\therefore$ induction holds $\sum_{i=0}^{k+1} 2^{-(k+1)} = 2 - 2^{-k-1} = 2 - 2^{-(k+1)}$

6 Fun with Logic

Points: 12 (a: 2, b: 4, c: 6)

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

A	B	$A \downarrow B$	$\neg(A \vee B)$
1	1	0	0
1	0	0	0
0	1	0	0
0	0	1	1

$A \downarrow B \equiv \neg(A \vee B)$

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

A	$\neg A$	$A \downarrow A$
1	0	0
0	1	1

$\neg A \equiv (A \downarrow A)$

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*

$\neg A \downarrow \neg B$

$A \downarrow B$

A	B	$A \downarrow B$	$(A \downarrow B) \downarrow (A \downarrow B)$
1	1	0	1
1	0	0	1
0	1	0	1
0	0	1	0

$A \vee B \equiv (A \downarrow B) \downarrow (A \downarrow B)$

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Name: Janice Ng

@berkeley.edu email: janiceng

Student ID Number: 3033110056

Name of student to your left: Vandana Ganesh

Name of student to your right: N/A

0 Preliminary Questions

Points: 3 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam?

Not sure lol

b) What is your favorite topic so far in this course?

induction

c) Name one of the songs I played in class on Monday.

Some Bollywood song? :) It was catchy though.

1 True or False

Points: 17 (1 each)

$(-n)^2+1 = 2$

Circle either true or false in each of the below.

$(-1)^2$

a) **True** or **False**: $\forall n \in \mathbb{N}, n^2+1<0 \implies n^2+2<0$

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$ $\neg(P \wedge Q)$

d) **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd."

$\neg Q \rightarrow \neg P$ x^2 is even then x is odd

e) **True** or **False**: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$

f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3 | n^3+2n$ *(Hint: Try and prove or disprove the statement.)*

g) **True** or **False**: The set of all even integers is countably infinite.

h) **True** or **False**: The union of ten countably infinite sets is uncountably infinite.

$\forall b \in B, \exists a \in A$ (illustrated)

every R is mapped to

i) **True** or **False**: A function is surjective if $\forall a \in A, \exists b \in B : f(a) = b$.

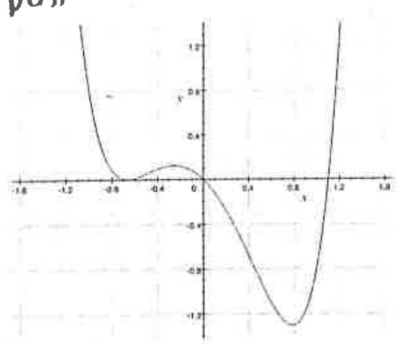
j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective.

or

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A-B| + |B-A|$.

l) **True** or **False**: The following function is injective.

injective posi the vertical + horizontal line test



A = {1, 2, 3, 4, 5}

B = {1, 2, 3}

For the remaining parts, consider sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$.

m) **True** or **False**: There exists a bijection $f : A \to B$.

n) **True** or **False**: The relation $r : \{(1,3),(3,2),(2,2),(4,1),(5,1)\}$ is a function.

vertical line test ~~horizontal li~~

o) **True** or **False**: $B \subseteq A$ no subset, not proper subset

p) **True** or **False**: $(B \cup \{6\}) \subset A$

q) **True** or **False**: $|A-B| = |A| - |B|$ $|A| - |B| =$

b in A but not B.

{4, 5}

2 Set Matching

Points: 10 (2 each)

Consider the following sets, where the universe is $\mathbb{N}$:

$A = \{x : x \text{ is prime}\}$ 1,2,3,5,7,11,13,17,19,23 ..

$B = \{x : x = a^3, a \in \mathbb{N}\}$ 0,1,8,27,72

$C = \{2, 15, 18, 64\}$

$D = \{15, 49, 81\}$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set)

$A \cap D$

b) $\{15, 18, 49, 81\}$

3

$(C \setminus D) \cup D$

(2,18,64)

c) $\{2\}$

$A \cap C$

d) $\{1,2\}$

$(A \cap B) \cup (A \cap C)$

1 2

e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \wedge y|x\}$

x such that for some y subset of Natural numbers, 1 is less than y, y is less than x and x is divisible by y.

3 Set Proof

Points: 6

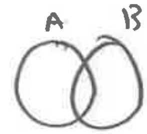
are ~~not in~~ in B, not in A

if A and B are sets, then A and (everything not in A but in B) = empty set

Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$.

$A \cap (B - A) = A \cap (B \setminus A) = A \cap (A^c)$

A B



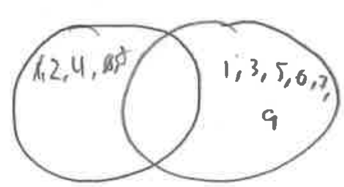
A: {1, 2, 4, 6, 8}

B: {~~1~~, 3, 5, ~~6~~, 7, 9}

→ there is nothing in common between the 2 sets after the similarities are taken away in (B - A)

→ won't have anymore overlap, for example

thus, it is empty.



4 Choose One

Points: 8 (+2 for completing both)

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational.

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)*

a) Proof by contradiction: $\sqrt{3}$ is rational

$a, b \in \mathbb{Z}$

$\sqrt{3} = \frac{a}{b}$

$3 = \frac{a^2}{b^2}$

$3b^2 = a^2$

contradiction: not possible for $3b^2 = a^2$. thus, $\sqrt{3}$ is rational.

b) $x^2 - 3 = 4y$

Proof by cases

① if x = even $(2K)$

$(2K)^2 - 3 = 4y$

$4K^2 - 3 = 4y$

$4K^2 - 4y - 3 = 0$

not possible to find integer

② if x = odd $(2K+1)$

$(2K+1)^2 - 3 = 4y$

$(2K+1)(2K+1) - 3 = 4y$

$4K^2 + 4K + 1 - 3 - 4y = 0$

$4K^2 + 4K - 2 - 4y = 0$

$2(2K^2 + 2K - 2y - 1) = 0$

is not posible to find integer.

5 Induction

$A \cap D$
$A \cap C$

Points: 16 (8 each)

$\frac{9^n-1}{8}$

a) Prove that $8|9^n-1$, for all $n \in \mathbb{N}$.

① Base case $(n=1)$

$\frac{9^1-1}{8} = \frac{9-1}{8} = \frac{8}{8} = 1$ ✓

② Induction Hypothesis
$n=k$
$= \frac{9^k-1}{8}$

③ Induction Step:
$n=k+1$

$\frac{9^{k+1}-1}{8}$ ← want to prove

$\frac{9^k-1}{8} + k+1 = \frac{9^k-1}{8} + \frac{8(k+1)}{8} = \frac{9^k-1}{8} + \frac{8k+8}{8} = \frac{9^k+8k+7}{8}$

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

① Base case $(n=0)$

$\sum_{i=0}^{0} 2^{-0} = 2-2^{-0}$

$\sum_{i=0}^{0} \boxed{1} = 2-1 = \boxed{1}$ ✓

② Induction hypothesis
$n=k$
$\sum_{k=0}^{k} 2^{-i} = \boxed{2-2^{-k}}$

③ Induction Step
want to prove:
$\sum_{i=0}^{k+1} 2^{-i} = 2-2^{-(k+1)} = \boxed{2-2^{-k-1}}$

To prove:
$\sum_{i=0}^{k+1} 2^{-i} = \sum_{i=0}^{k} 2^{i} + (k+1)$

$= 2-2^{-k} + k+1$

$= 3-2^{-k}+k$

6 Fun with Logic

$A \downarrow B$ = True (only when both are F)

= NOR

$\downarrow$

Points: 12 (a: 2, b: 4, c: 6)

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

A	B	$A \downarrow B$	$\neg P \wedge \neg Q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

$\boxed{= \neg P \wedge \neg Q}$

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

not A

A	$\neg A$	$A \downarrow A$
T	F	F
F	T	T

$\neg A \quad (A \downarrow A)$

$= \boxed{A \downarrow A}$

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*

or

B nor B

A	B	$A \vee B$	$(A \downarrow B) \downarrow (A \downarrow A)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$\boxed{(A \downarrow B) \downarrow (A \downarrow A)}$

CS 198-087 Introduction to Mathematical Thinking
Fall 2018

Midterm

This is the midterm examination for Introduction to Mathematical Thinking.

- This exam has 7 questions (including question 0). The exam is out of 72 points.
- The exam will last for exactly 1 hour and 20 minutes, unless you have pre-arranged DSP accomodations.
- You can use the backs of pages for scrap work, but please write your answers only on the fronts of pages.

DO NOT TURN THE PAGE UNTIL INSTRUCTED.

In the meantime, fill out the information on this page.

Name: Ahmed Shehata

@berkeley.edu email: shehata_ahmed@berkeley.edu

Student ID Number: 3033933387

Name of student to your left:

Name of student to your right:

0 Preliminary Questions

Points: 3 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam?

5

b) What is your favorite topic so far in this course? proofs

c) Name one of the songs I played in class on Monday.

I forgot, it was indian song i think

$1+2=3$

1 True or False

Points: 17 (1 each)

Circle either true or false in each of the below. $F \Rightarrow F : T$

a) **True** or **False**: $\forall n \in \mathbb{N}, \overbrace{n^2+1<0}^{F} \implies \overbrace{n^2+2<0}^{F}$ T

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ T

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$ F

d) **True** or **False**: The contrapositive of the statement "if $\underbrace{x \text{ is even}}_{P}$, then $\underbrace{x^2 \text{ is odd}}_{Q}$" is "if x^2 is even, then x is odd." T

e) **True** or **False**: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$ T

f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3 | n^3 + 2n$ *(Hint: Try and prove or disprove the statement.)* $n(n^2+2) = 3K \Rightarrow T$

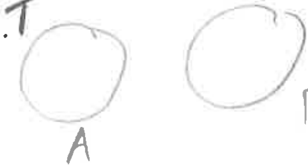
g) **True** or **False**: The set of all even integers is countably infinite. T

h) **True** or **False**: The union of ten countably infinite sets is uncountably infinite. F

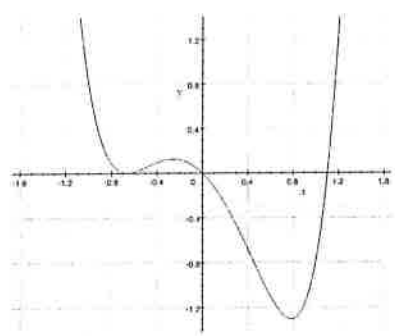
i) **True** or **False**: A function is surjective if $\forall a \in A, \exists b \in B : f(a) = b$. F

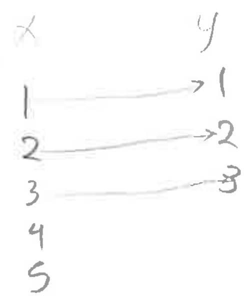
j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective. T

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A - B| + |B - A|$. T



l) **True** or **False**: The following function is injective. F





$\{4,5\}$

For the remaining parts, consider sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$.

m) **True** or **False**: There exists a bijection $f : A \to B$. F

n) **True** or **False**: The relation $r : \{(1,3),(3,2),(2,2),(4,1),(5,1)\}$ is a function. T

o) **True** or **False**: $B \subseteq A$ T

p) **True** or **False**: $(B \cup \{6\}) \subset A$ F

q) **True** or **False**: $|A - B| = |A| - |B|$ T

2 Set Matching

Points: 10 (2 each)

Consider the following sets, where the universe is $\mathbb{N}$:

$A = \{x : x \text{ is prime}\}$

$B = \{x : x = a^3, a \in \mathbb{N}\}$

$C = \{2, 15, 18, 64\}$

$D = \{15, 49, 81\}$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set) $A \cap D$

$\{2, 15, 18, 49, 81, 64\}$

b) $\{15, 18, 49, 81\}$ $D \cup \{18\}$ or $(C \cup D) - \{\underbrace{(A \cap C)}_{\{2\}} + \underbrace{(B \cap C)}_{\{64\}}\}$

c) $\{2\}$ $A \cap C$

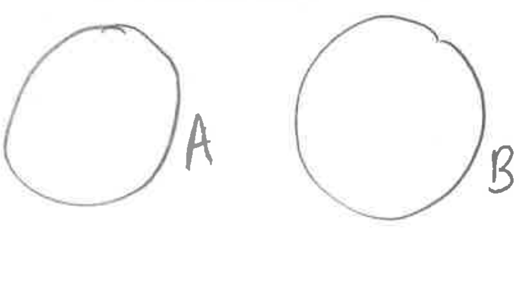
d) $\{1, 2\}$ $(\underbrace{A \cap C}_{\{2\}}) \cup \{1\}$

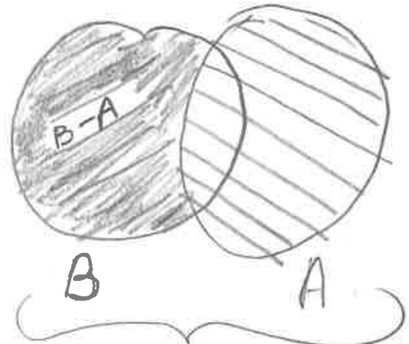
e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \wedge y | x\}$ $\rightarrow A^c$

3 Set Proof

Points: 6

Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$. $\rightarrow$ by definiton of B-A





$A \cap (B - A) = \{\ \}$

4 Choose One

Points: 8 (+2 for completing both)

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational.

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)*

a) proof by contradiction:

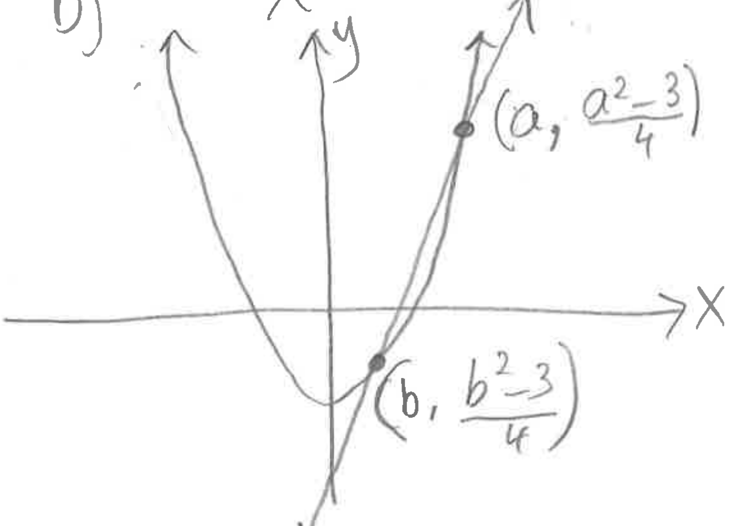
pretend/assume $\sqrt{3}$ is rational $\Rightarrow \sqrt{3} = \frac{p}{q}$

$$3 = \frac{p^2}{q^2} \Rightarrow 3q^2 = p^2$$

p^2 has even number of prime factors
$3q^2$ has odd number of prime factors
$\}$ There for a contradiction.

conclusion: $\sqrt{3}$ is not rational $\Rightarrow \sqrt{3}$ is irrational.

b) $x^2 - 3 = 4y \Rightarrow 4 \mid x^2 - 3$



$$\text{slope} = \frac{\frac{a^2 \cancel{-3} - b^2 \cancel{+3}}{4}}{a-b} = \frac{a^2-b^2}{a-b} \cdot \frac{1}{4}$$

$$= \frac{1}{4} \cdot (a+b)$$

$$= \frac{a+b}{4}$$

5 Induction

Points: 16 (8 each)

a) Prove that $8 | 9^n - 1$, for all $n \in \mathbb{N}$.

base case: $n=1$

$8 | 9-1 \Rightarrow 8|8$ ✓

inductive hypothesis:

assume $8 | 9^k - 1 \Rightarrow 9^k - 1 = 8c$

proof for $k+1$:

$$9^{k+1} - 1 = 9^k \cdot 9 - 1 = 9^k(8+1) - 1$$

$$= \underbrace{8 \cdot 9^k}_{\text{divisible by } 8} + \underbrace{9^k - 1}_{\text{divisible by 8 form the hypothesis}}$$ ✓

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

for $n=1$ $\sum_{i=0}^{1} 2^{-i} = 2^0 + 2^{-1}$

$$= 1 + \frac{1}{2} = \frac{3}{2} = 2 - 2^{-1}$$ ✓

for $n=k$:

assume $\sum_{i=0}^{k} 2^{-i} = 2 - 2^{-k}$

proof for $k+1$:

$$\sum_{i=0}^{k+1} 2^{-i} = \underbrace{\sum_{i=0}^{k} 2^{-i}}_{2 - 2^{-k}} + \underbrace{\sum_{k}^{k+1} 2^{-i}}_{2^{-k} + 2^{-(k+1)}} - 2^{-k}$$

$$2 - 2^{-k} + 2^{-k} + 2^{-(k+1)} - 2^{-k}$$

$$= 2 - 2^{-k-1} = 2 - 2^{-(k+1)}$$ ✓

A A

A	A	Gr
T	T	T
F	F	F

6 Fun with Logic

Points: 12 (a: 2, b: 4, c: 6)

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

$$(\sim A \wedge \sim B) = \sim(A \vee B) = A \downarrow B$$

A	B	Nor
T	T	F
T	F	F
F	T	F
F	F	T

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

$$\sim A = \sim(A \vee A)$$
$$= A \downarrow A$$

A	$\sim A$	Result
T	F	T
F	T	F

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*

$$A \downarrow B = \sim(A \vee B)$$

$$\sim(A \downarrow B) = (A \vee B)$$

$$(A \downarrow B) \downarrow (A \downarrow B) = A \vee B$$

CS 198-087
Fall 2018

Introduction to Mathematical Thinking

Midterm

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In the meantime, fill out the information on this page.

Name: Hunter Gissinger

@berkeley.edu email: huntergissinger@berkeley.edu

Student ID Number: 3032964612

Name of student to your left: Ahmad Maarouf

Name of student to your right: N/A

0 Preliminary Questions

Points: 3 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam?

7-8

b) What is your favorite topic so far in this course?

Induction / Proofs

c) Name one of the songs I played in class on Monday.

Bollywood Baby! / Eminem / Drake

Song request: Lebron James / Kevin Durant's Song on soundcloud

multiples of 4: 8, 16, etc

$P \to Q$ False if Q False, P true

1 True or False

Points: 17 (1 each)

Circle either true or false in each of the below.

F F

a) **True** or **False**: $\forall n \in \mathbb{N}, n^2+1<0 \implies n^2+2<0$

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$

Q

d) **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd." $\neg P$ P $\neg Q$

e) **True** or **False**: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$

f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3 | n^3+2n$ *(Hint: Try and prove or disprove the statement.)* n^3+2n is a multiple of 3

g) **True** or **False**: The set of all even integers is countably infinite.

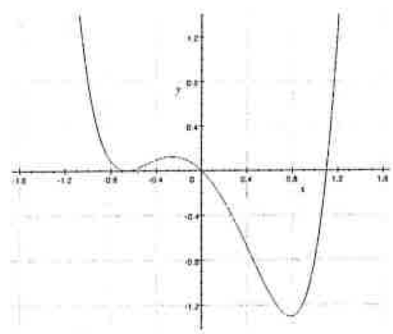
h) **True** or **False**: The union of ten countably infinite sets is uncountably infinite.

i) **True** or **False**: A function is surjective if $\forall a \in A, \exists b \in B : f(a) = b$.

j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective.

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A - B| + |B - A|$.

l) **True** or **False**: The following function is injective.



For the remaining parts, consider sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$.

m) **True** or **False**: There exists a bijection $f : A \to B$.

n) **True** or **False**: The relation $r : \{(1,3),(3,2),(2,2),(4,1),(5,1)\}$ is a function.

o) **True** or **False**: $B \subseteq A$

p) **True** or **False**: $(B \cup \{6\}) \subset A$

q) **True** or **False**: $|A - B| = |A| - |B|$

2 Set Matching

Points: 10 (2 each)

Consider the following sets, where the universe is $\mathbb{N}$:

$A = \{x : x \text{ is prime}\}$

$B = \{x : x = a^3, a \in \mathbb{N}\}$

$C = \{2, 15, 18, 64\}$

$D = \{15, 49, 81\}$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set)

$D \cap A$

b) $\{15, 18, 49, 81\}$

$\{x : x \% 3 = 0, D \cap C\}$

c) $\{2\}$

$\{x : x < 3, C\}$

d) $\{1, 2\}$

$\{x : x \leq 1, B\} \cup \{x : x \leq 2, C\}$

e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \wedge y | x\}$

$\{x : x \% 2 = 0, C\}$

3 Set Proof

Points: 6

any two sets

Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$.

Assume $\neg S$; where S is $A \cap (B - A) = \emptyset$

$A \cap (B - A)$

$= A \cap B'$ where $B' \subseteq A^c$

$\approx A \cap A^c \rightarrow$ contradiction, this $= \emptyset$

4 Choose One

Points: 8 (+2 for completing both)

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational.

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)*

(a) Prove that $\sqrt{3}$ is irrational

Assume $\neg S$, so $\sqrt{3}$ is rational and can be represented by $\frac{x}{y}$ such that $x, y \in \mathbb{Z}$ and $\gcd(x,y) = 1$

$\sqrt{3} = \frac{x}{y}$

$3 = \frac{x^2}{y^2}$

$x^2 = 3y^2 \rightarrow$ x is a multiple of 3

$(3k)^2 = 3y^2$

$9k^2 = 3y^2$

$y^2 = 3k^2 \rightarrow$ y is a multiple of 3 so $\gcd(x,y) \geq 3$

$\therefore$ contradiction

5 Induction

Points: 16 (8 each)

$9^n - 1$ is a multiple of 8

a) Prove that $8|9^n - 1$, for all $n \in \mathbb{N}$.

Base Case:

$n = 0$

$9^0 - 1 = 8$ ✓

$\therefore$ Base case Holds

IH: Assume $8 | 9^K - 1$

IS (Want to show $8 | 9^{K+1} - 1$)

$9^K - 1 \rightarrow$ divisible by 8, so can be written as $8(x)$ ← Factor

$9 \cdot (9^K - 1) + 8 = (9^{K+1} - 1)$

$= 9(8(x)) + 8$

$= 72(x) + 8 \rightarrow$ must sum to factor of 8

$\therefore$ Induction holds

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

Base Case:

$n = 0$

LH	RH
$\sum_{i=0}^{0} \frac{1}{2^0} = 1$	$2 - 1 = 1$

$\therefore$ Base Case Holds

IH: Assume $\frac{1}{2^0} + \frac{1}{2^1} + \dots \frac{1}{2^K} = 2 - \frac{1}{2^K}$

IS: (Want to prove: $\frac{1}{2} + \frac{1}{2^1} \dots + \frac{1}{2^K} + \frac{1}{2^{K+1}} = 2 - \frac{1}{2^{K+1}}$)

$\frac{1}{2^0} + \frac{1}{2^1} + \dots \frac{1}{2^K} = 2 - \frac{1}{2^K}$

$\frac{1}{2^0} + \frac{1}{2^1} + \dots \frac{1}{2^K} + \left(\frac{1}{2^{K+1}}\right) = 2 - \frac{1}{2^K} + \left(\frac{1}{2^{K+1}}\right)$

$= 2 - \left(\frac{1}{2^K} + \frac{1}{2 \cdot 2^K}\right)$

$= 2 - \left(\frac{1}{2 \cdot 2^{2K}}\right)$

$= 2 - \left(\frac{1}{2^{2(K+1)}}\right)$

$\therefore$ Induction Holds

$A \downarrow B$

$F \quad F \to T$

6 Fun with Logic

Points: 12 (a: 2, b: 4, c: 6)

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

pred: $(\neg A \wedge \neg B) \wedge (\neg A \vee B) \wedge (A \cup \neg B)$

A	B	Pred
T	T	False
T	F	False
F	T	False
F	F	True

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

pred: $(A \downarrow A)$

A	Pred
T	F
F	T

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*

$\neg A \downarrow \neg B$

$(\neg A \downarrow B \quad \downarrow A \downarrow \neg B) \downarrow A \downarrow B$

F F T

T T F

F

pred: $[((A \downarrow A) \downarrow B) \downarrow (A \downarrow (B \downarrow B))] \downarrow (A \downarrow B)$

A	B	pred
T	T	T
T	F	T
F	T	T
F	F	F

CS 198-087 Introduction to Mathematical Thinking
Fall 2018

Midterm

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In the meantime, fill out the information on this page.

Name: Michael Devinsky

@berkeley.edu email: Michael. Devinsky @ " "

Student ID Number: 30333858 5 1

Name of student to your left: David Pan

Name of student to your right:

None

0 Preliminary Questions

Points: 3 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam?

6

b) What is your favorite topic so far in this course?

Proofs because I need to get better.

c) Name one of the songs I played in class on Monday.

The Bollywood music (don't think you told us name)

1 True or False

Points: 17 (1 each)

$n^2 = -1$

Circle either true or false in each of the below.

a) **True** or **False**: $\forall n \in \mathbb{N}, n^2+1<0 \implies n^2+2<0$ <1

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ $\sqrt[3]{135}$

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$

d) **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd."

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f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3|n^3+2n$ *(Hint: Try and prove or disprove the statement.)*

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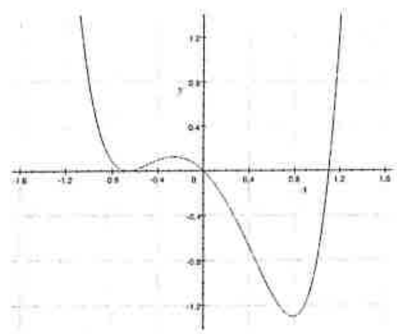
j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective.

10 5 5 10

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A-B| + |B-A|$.

$|A| + |B| - |A \cap B|$

l) **True** or **False**: The following function is injective.



6

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m) **True** or **False**: There exists a bijection $f : A \to B$.

n) **True** or **False**: The relation $r : \{(1,3),(3,2),(2,2),(4,1),(5,1)\}$ is a function.

o) **True** or **False**: $B \subseteq A$

p) **True** or **False**: $(B \cup \{6\}) \subset A$

q) **True** or **False**: $|A-B| = |A| - |B|$

2 Set Matching

Points: 10 (2 each)

Consider the following sets, where the universe is $\mathbb{N}$:

$A = \{x : x \text{ is prime}\}$

$B = \{x : x = a^3, a \in \mathbb{N}\}$ ~~ε~~ $64 = 4^3$

$C = \{2, 15, 18, 64\}$

$D = \{15, 49, 81\}$ $\{2, 15, 18, 64, 49, 81\}$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set)

$C \cap D, \cap A$

b) $\{15, 18, 49, 81\}$ $\{15, 49$

$(C \cap D) \cup (A \cap D)$

c) $\{2\}$

$A \cap C$

d) $\{1,2\}$

$(|C| - |D|) \cup A \cap C$ 2 4

e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \wedge y|x\}$

There exists y within Nat such that $1 < y < x$ and x divides y

A

3 Set Proof

Points: 6

Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$.

$B - A$ = whatever is in B but "not in A."

(Then if you take intersection of everything "not in A" with A, you'll have no overlap and therefore $\emptyset$

4 Choose One

Points: 8 (+2 for completing both)

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational.

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)*

a) Proof by contradiction.

will assume $\sqrt{3}$ is rational

there exists some $p \in \mathbb{Z}^+$ and $q \in \mathbb{Z}$ such that $\sqrt{3} = \frac{p}{q}$.

$q\sqrt{3} = p$. We know that p has to $\in \mathbb{Z}^+$

$$\frac{q\sqrt{3}}{p} = 1$$ there is some contradiction in here, but my math skills aren't strong enough to find it.

5 Induction

Points: 16 (8 each)

a) Prove that $8|9^n - 1$, for all $n \in \mathbb{N}$.

Proof by Contrapositive.

There exists an $n \in \mathbb{N}$ such that $8 \mid 9^n - 1$

$n=1$. $8 \mid 9^1 - 1$ $\therefore$ it is proven

$\underset{1}{\overset{\shortparallel}{}}$

$\frac{1}{2^n}$ 2^{-2} $\frac{1}{4}$

2^{-1} $\frac{1}{2}$

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

Proof by induction.

1) Bc: $n=0$, $\sum_{i=0}^{0} 2^{-0} = 2 - 2^{-0} = 1$ in both cases ✓

2^{-1} $\frac{1}{2}$

2) Assume this is true for $n=k$ $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$

3) Show it holds for $n=h+1$. $\sum_{i=0}^{n+1} 2^{-i} = 2 - 2^{-(n+1)} = 1 + 2 + \ldots 2^{-n} + 2^{-(n+1)}$

$\therefore$ ($\frac{n}{2}$ times both sides)

$= \frac{n}{2} + n + \ldots + 2 + 1$

Somehow eventually this will equal our $2 - 2^{-(n+1)}$ and then induction will hold

6 Fun with Logic

Points: 12 (a: 2, b: 4, c: 6)

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

$\neg A \wedge \neg B$

A	B	$A \downarrow B$
T	T	F
T	F	F
F	T	F
F	F	T

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

$A \downarrow A$

A	$A \downarrow A$	$\neg A$
T	F	F
T	F	F
F	T	T
F	T	T

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*

A	B	$A \vee B$	$(A \downarrow B) \downarrow (A \downarrow B)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

CS 198-087 Introduction to Mathematical Thinking
Fall 2018

Midterm

This is the midterm examination for Introduction to Mathematical Thinking.

- This exam has ~~7~~ 6 questions (including question 0). The exam is out of 72 points.
- The exam will last for exactly 1 hour and 20 minutes, unless you have pre-arranged DSP accomoda-tions.
- You can use the backs of pages for scrap work, but please write your answers only on the fronts of pages.

DO NOT TURN THE PAGE UNTIL INSTRUCTED.

In the meantime, fill out the information on this page.

Name: DIVYA MOHAN

@berkeley.edu email: 21 dmohan

Student ID Number: 3032734499

Name of student to your left: ——

Name of student to your right: Vandana Ganesh

0 Preliminary Questions

Points: 3 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam?

7

b) What is your favorite topic so far in this course?

proving math concepts :)

c) Name one of the songs I played in class on Monday.

That one Indian song

1 True or False

Points: 17 (1 each)

Circle either true or false in each of the below.

$n^2 > 0$

a) **True** or **False**: $\forall n \in \mathbb{N}, n^2+1 < 0 \implies n^2+2 < 0$

F

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ $\qquad \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$ $\qquad \neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q$

d) **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd."

$3K$ ✓ $\qquad n(n^2+2)$

$(3K+1)^2 + 2 = 9K^2 + 6K + 1 + 2$ ✓

e) **True** or **False**: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$

f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3 | n^3 + 2n$ *(Hint: Try and prove or disprove the statement.)*

$(3K+2)^2 + 2 = 9K^2 + 12K + 4 +$ ⊗

g) **True** or **False**: The set of all even integers is countably infinite.

h) **True** or **False**: The union of ten countably infinite sets is uncountably infinite.

i) **True** or **False**: A function is surjective if $\forall a \in A, \exists b \in B : f(a) = b$.

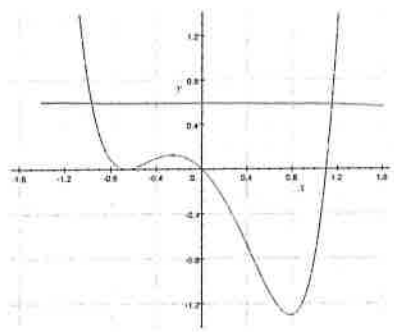
$5 \cdot (25+2)$

j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective.

$5 \cdot 27$

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A - B| + |B - A|$. ✓

l) **True** or **False**: The following function is injective.



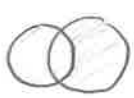
For the remaining parts, consider sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$.

m) **True** or **False**: There exists a bijection $f : A \to B$.

n) **True** or **False**: The relation $r : \{(1,3),(3,2),(2,2),(4,1),(5,1)\}$ is a function.

o) **True** or **False**: $B \subseteq A$

p) **True** or **False**: $(B \cup \{6\}) \subset A$



q) **True** or **False**: $|A - B| = |A| - |B|$

2 Set Matching

Points: 10 (2 each)

Consider the following sets, where the universe is $\mathbb{N}$: $81 = 9 \times 9 = 3^4$

$A = \{x : x \text{ is prime}\}$ prime #

$B = \{x : x = a^3, a \in \mathbb{N}\}$ cubes

$C = \{2, 15, 18, 64\}$ $64 = 4^3$

$D = \{15, 49, 81\}$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set) $D \cap B$

b) $\{15, 18, 49, 81\}$ $((C \cup D) - A) - B$

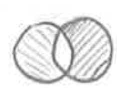
c) $\{2\}$ $[C \cup D - (C \cap D)] \cap A$

d) $\{1,2\}$ $(A \cap B) \cup [[C \cup D - (C \cap D)] \cap A]$

e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \land y | x\}$ A

3 Set Proof

Points: 6



Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$.

$A \cap (B - A) = \{x : x \in A \land (x \in B \land x \notin A)\}$

however, this condition is impossible since x cannot be an element of A and NOT be an element of A. Hence, this set is empty.

4 Choose One

Points: 8 (+2 for completing both)

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational.

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)*

Proof by cases

case 1 : $x = 4K$, $K \in N$

$$y = \frac{x^2 - 3}{4} = \frac{(4K)^2 - 3}{4} = 4K^2 - 3/4$$

=> y cannot be an integer

case 2: $x = 4K + 1$, $K \in N$

$$y = \frac{(4K+1)^2 - 3}{4} = \frac{16K^2 + 8K + 1 - 3}{4} = 4K^2 + 2K - 2/4$$

=> y cannot be an integer

case 3: $x = 4K + 2$, $K \in N$

$$y = \frac{(4K+2)^2 - 3}{4} = \frac{16K^2 + 16K + 4 - 3}{4} = 4K^2 + 4K + 1/4$$

=> y cannot be an integer

case 4: $x = 4K + 3$, $K \in N$

$$y = \frac{(4K+3)^2 - 3}{4} = \frac{16K^2 + 24K + 9 - 3}{4} = 4K^2 + 6K + 6/4$$

=> y cannot be an integer

x must be one of the cases listed above, but none of the cases allow y to be an integer solution.

5 Induction

Points: 16 (8 each)

$9^n - 1 = 8c,\ c \in N$

a) Prove that $8 | 9^n - 1$, for all $n \in \mathbb{N}$.

Base Case $(n = 1)$

$9^1 - 1 = 8 = 8c,\ c = 1 \in N$ ✓

Induction Hypothesis $(n = k)$

assume $9^k - 1 = 8c,\ c \in N$

Induction Step $(n = k+1)$

$$9^{k+1} - 1 = 9 \cdot 9^k - 1$$
$$= 9 \cdot (8c_1 + 1) - 1 \qquad (9^k = 8c + 1 \text{ (from IH)})$$
$$= 72c_1 + 9 - 1$$
$$= 72c_1 + 8 = 8c_2$$
$$\Rightarrow 9c_1 + 1 = c_2 \in \mathbb{N}$$ ✓

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

Base Case $(n = 0)$

$$\sum_{i=0}^{0} 2^{-i} = 2^0 = 1$$ ✓
$$2 - 2^0 = 2 - 1 = 1$$

Induction Hypothesis $(n = k)$

assume $\sum_{i=0}^{k} 2^{-i} = 2 - 2^{-k}$

Induction Step $(n = k+1)$

$$\sum_{i=0}^{k+1} 2^{-i} = 2^0 + 2^{-1} + \ldots 2^{-k} + 2^{-k-1}$$
$$= (2 - 2^{-k}) + 2^{-k-1} \qquad \text{(from IH)}$$
$$= 2 - 2^{-k} + \frac{2^{-k}}{2}$$
$$= 2 + 2^{-k}(-1 + 1/2) = 2 + 2^{-k}(-1/2)$$
$$= 2 - 2^{-k-1}$$ ✓

6 Fun with Logic

Points: 12 (a: 2, b: 4, c: 6)

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

A	B	$A \downarrow B$	$\neg A \wedge \neg B$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

~~A~~	~~F~~	~~$\neg A$~~	~~$A \downarrow F$~~
~~T~~	~~F~~	~~F~~	~~F~~
~~F~~	~~F~~	~~T~~	~~T~~

A	A	$\neg A$	$A \downarrow A$
T	T	F	F
F	F	T	T

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*

A	B	$A \vee B$	$(A \downarrow B) \downarrow (A \downarrow B)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

$$A \vee B \equiv \neg (A \downarrow B)$$
$$\equiv (A \downarrow B) \downarrow (A \downarrow B)$$ (using part b & c)

CS 198-087
Fall 2018

Introduction to Mathematical Thinking

Midterm

This is the midterm examination for Introduction to Mathematical Thinking.

- This exam has 7 questions (including question 0). The exam is out of 72 points.
- The exam will last for exactly 1 hour and 20 minutes, unless you have pre-arranged DSP accomoda-tions.
- You can use the backs of pages for scrap work, but please write your answers only on the fronts of pages.

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In the meantime, fill out the information on this page.

Name: Vandana Ganesh

@berkeley.edu email: vandanag@berkeley.edu

Student ID Number: 3033664508

Name of student to your left: Divya Mohan

Name of student to your right: Janice Ng

0 Preliminary Questions

Points: 3 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam?

5

b) What is your favorite topic so far in this course?

proofs (minus induction step :( )

c) Name one of the songs I played in class on Monday.

Chammak Chello

1 True or False

Points: 17 (1 each)

Circle either true or false in each of the below.

always false

a) **True** or **False**: $\forall n \in \mathbb{N}, n^2+1 < 0 \implies n^2+2 < 0$

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ $\frac{n(n+1)}{2}$

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$ one or the other, not both

should be $\vee$

d) **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd." $\neg Q \to \neg P$

e) **True** or **False**: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$

f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3 | n^3 + 2n$ *(Hint: Try and prove or disprove the statement.)* $\frac{n(n^2+2)}{3}$

pos ints

g) **True** or **False**: The set of all even integers is countably infinite.

h) **True** or **False**: The union of ten countably infinite sets is uncountably infinite.

i) **True** or **False**: A function is surjective if $\forall a \in A, \exists b \in B : f(a) = b$. for all ~~input~~ out, exists ~~output~~ in

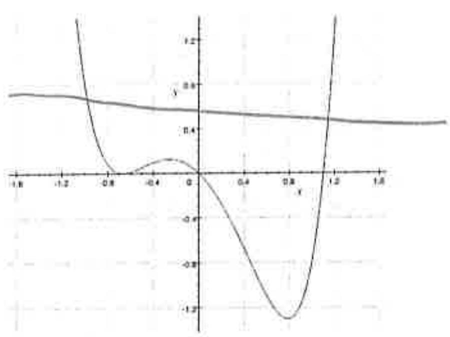
j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective. continuous so should have all outputs that are $\mathbb{R}$?

or A only b only

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A - B| + |B - A|$.

no common

l) **True** or **False**: The following function is injective.



For the remaining parts, consider sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$.

m) **True** or **False**: There exists a bijection $f : A \to B$.

n) **True** or **False**: The relation r: $\{(1,3),(3,2),(2,2),(4,1),(5,1)\}$ is a function.

proper subset! proper

o) **True** or **False**: $B \subseteq A$ * should be $\subset$ i think a proper

subset only

p) **True** or **False**: $(B \cup \{6\}) \subset A$ $\subseteq$ = subset ss is an ss

q) **True** or **False**: $|A - B| = |A| - |B|$ but an ss

should be $A \cap B$ isn't always a proper ss...?

minus that only

2 Set Matching

Points: 10 (2 each)

Consider the following sets, where the universe is $\mathbb{N}$:

$A = \{x : x \text{ is prime}\}$

$B = \{x : x = a^3, a \in \mathbb{N}\}$

$C = \{2, 15, 18, 64\}$

$D = \{15, 49, 81\}$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set)

N^c

b) $\{15, 18, 49, 81\}$

$D \cup (C - A \cap C - B)$

c) $\{2\}$

$((C \cap A) - D)$

~~d)~~ $\{\cancel{1}, 2\}$

$(A \cap B) \cup ((C - D) \cap A)$

e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \land y | x\}$ x such that y such that $1 < y < x$ and x is divisible by y

$y : \{C \cap A\}$

$x : \{C \cap B\}$

$\{C \cap B\} \times \{C \cap A\}$

3 Set Proof

Points: 6

Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$.

B - A indicates to take the difference of B and A, and only include values that are in B and not in A. So, A and the set with no terms in A would not have any common values except the zero set.

b) $x^2 - 3 = 4y$

$x = 4y+3 \nearrow 16y^2 + 4y + 6 = 0$

$8y^2 + 2y + 3 = 0$ no integer sol

$x = 0$ $-3 = 4y$ $y = \frac{-3}{4} \Rightarrow$ not integer

$x \neq 0$ $\frac{x^2 - 3}{4} = y$

$x = \sqrt{3}$ $0 = 4y$ but $\sqrt{3}$ is irrational

4 Choose One

Points: 8 (+2 for completing both)

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational. – answer

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)* – incomplete

a) assume $\sqrt{3}$ is rational

$\sqrt{3} = \frac{a}{b}$ gcf$(a, b) \to 1$ * reduced form!

$\sqrt{3}\, b = a$

so, a must be divisible by $\sqrt{3}$.

$\sqrt{3} = \frac{a}{b}$

$\sqrt{3}\, b = a$

$b = \frac{a}{\sqrt{3}}$

$b^2 = \frac{a^2}{3}$

$b = a\left(\frac{1}{\sqrt{3}}\right)$

$b = a\frac{\sqrt{3}}{3}$

so, b must be divisible by $\sqrt{3}$

a, b both are able to be reduced by $\sqrt{3}$ so the gcf is not 1

$\therefore$ by contradiction $\sqrt{3}$ is irrational

5 Induction

Points: 16 (8 each)

a) Prove that $8|9^n - 1$, for all $n \in \mathbb{N}$.

$(9^n - 1) = 8x$

$9 \cdot 9^{n-1} - 1 = 8x$

$\underbrace{9(1 \cdot 9^{n-2})}_{\text{divisible by } 9} - 1 = 8x$

a number divisible by 9 − 1 must be divisible by 8

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

base case:

$n = 0$

LH	RH
2^0	$2 - 2^0$
1	1

holds true

ih:

assume

$\sum_{i=0}^{k} 2^{-i} = 2 - 2^{-k}$

for an arbitrary k

is:

$\sum_{i=0}^{k+1} 2^{-i} = \sum_{i=0}^{k} 2^{-i} + 2^{-(k+1)}$

$\cancel{2} - 2^{-k-1} = \cancel{2} - 2^{-k} + 2^{-k-1}$

$-2^{-k-1} = 2(-2^{-k-1} + 2^{-k-2})$

6 Fun with Logic

Points: 12 (a: 2, b: 4, c: 6)

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

$A \downarrow B = \neg A \wedge \neg B$

A	B	$A \downarrow B$	$\neg A \wedge \neg B$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

$A \downarrow A$

A	$\neg A$	$A \downarrow A$
T	F	F
T	F	F
F	T	T
F	T	T

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*

A or B

! both true and ! both false

$(A \downarrow B) \downarrow ((A \downarrow A) \downarrow (B \downarrow B))$

A	B	$A \vee B$	$(A \downarrow B) \downarrow ((A \downarrow A) \downarrow (B \downarrow B))$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F