Lecture 4: Mathematical Induction (Proofs, cont'd.)

http://book.imt-decal.org, Ch. 2.2

Introduction to Mathematical Thinking

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Announcements

- HW4 is already released, will be due Monday at 6:30PM
- Midterm is in two weeks from today *in class*; HW5, which will be released next week, will be mostly review
  - Let us know ASAP if you can't make it
  - Discussion on Monday, Oct. 8 will be review
  - We will hold DSP accomodations
- HW3 solutions will be posted soon (sorry for the delay...)

Recap

Types of Proofs:

- Direct Proof
- Proof by Contradiction
- Proof by Contraposition
- Vacuous Proofs
- Proof by Cases
- Counterexamples
- **Proof by Induction**

Will learn best by doing examples!

assume $\neg S$, and show some contradiction, implying $\neg S$ is false, i.e. S is true

show $P \to Q$ by showing $\neg Q \to \neg P$

RECURSION

Motivation

Suppose you're sitting in a massive lecture hall, and want to find out how many rows you're sitting from the front of the room. You *could* sit there and count, but consider this basic principle:

- The person sitting in the first row knows their row number *by default*: they're in the first row! "Base Case"
- If one knows the row number of the person in front of them, they add 1 to get their own row number

Mathematical Induction

We use induction to prove properties about **all natural numbers**. Induction has three steps:

1. Base Case: Establish that the statement holds for $n = 0$ or $n = 1$ (or whatever makes the most sense in the situation)

2. Induction Hypothesis: Assume that the statement holds true for $n = k$, for some arbitrary k

3. Induction Step: Given the fact that the statement holds true for $n = k$, show that it holds for $n = k + 1$

→ most of the work is here

What does this remind you of from 61A? What are the parallels?

Example 1

$P(n)$: the sum from 1 to n

Prove that $\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

Base Case $n = 1$

LS	RS
$\sum_{i=1}^{1} i = 1$	$\frac{1(2)}{2}$
	$= 1$

$\therefore$ Base case holds $P(1)$

Induction Hypothesis

Assume $1+2+\ldots+k = \frac{k(k+1)}{2}$.

assume $P(k)$

Induction Step

$P(k) \longrightarrow P(k+1)$

(want to show $1+\ldots+k+1 = \frac{(k+1)(k+2)}{2}$

$$1+2+\ldots+k = \frac{k(k+1)}{2}$$

$\downarrow$ add $k+1$ to both sides

$$1+2+\ldots+k+k+1 = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$\therefore$ induction holds

In terms of propositional logic:

could be $P(1)$ (depends on proof!)

Base Case: Show $P(0)$ holds true

Induction Hypothesis: Assume $P(k)$ holds true for some arbitrary $k \in \mathbb{N}_0$

Induction Step: Show $P(k) \Rightarrow P(k+1), \forall k$.

$\forall n \in \mathbb{N}_0$

$$\forall P\Big(P(0) \land \forall k\big(P(k) \Rightarrow P(k+1)\big) \Rightarrow \cancel{\forall n}\big(P(n)\big)\Big)$$

where $P(\cdot)$ represents the statement we are trying to prove.

$P(k) \rightarrow P(k+1)$

More explicitly:

$$P(0) \Rightarrow P(1) \Rightarrow P(2) \Rightarrow \ldots$$

A little more on Example 1: How can we derive this?

$1^2+2^2+\ldots+n^2$

$1^3+2^3+\ldots+n^3$

$$\boxed{\sum_{i=1}^{n} i = 1 + 2 + \ldots + n =} \frac{n(n+1)}{2}$$

pretend we didn't know RHS

$$S = 1 + 2 + 3 + \ldots + (n-1) + n$$
$$S = n + (n-1) + (n-2) + \ldots + 2 + 1$$
$+$

$$2S = (n+1) + (n+1) + \ldots + (n+1)$$

$$2S = n(n+1) \longrightarrow \boxed{S = \frac{n(n+1)}{2}}$$

direct proof!

Example 2

not what we want to prove

The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is defined by $F_1 = 1, F_2 = 1, \boxed{F_n = F_{n-2} + F_{n-1}}$

Prove that $\sum_{i=1}^{n} F_i = F_1 + F_2 + \ldots + F_n = F_{n+2} - 1.$

Base Case $n = 1$

LS	RS
$F_1 = 1$	$F_3 - 1 = 2 - 1$
	$= \boxed{1}$

show $P(1)$

$\therefore$ base case holds.

IH

assume $P(k)$

Assume $\sum_{i=1}^{k} F_i = F_{k+2} - 1$

Induction Step

want to show that

$F_1 + F_2 + \ldots + F_k + F_{k+1} = F_{k+3} - 1$

show $P(k) \to P(k+1)$

start with IH: $\boxed{F_1 + F_2 + \ldots + F_k = F_{k+2} - 1}$

add F_{k+1} to BS

$F_1 + F_2 + \ldots + F_{k+1} = F_{k+1} + F_{k+2} - 1$

$= F_{k+3} - 1$

$\therefore$ induction holds.

Example 3

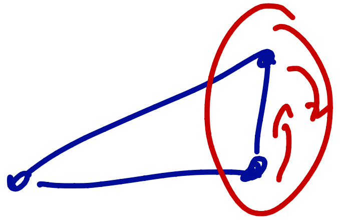
note: we don't assume statement is true for 2k+3 airports. just assuming that we have 2k+3.

Suppose that there are $2n + 1$ airports where n is a positive integer. The distances between any two airports are all different. For each airport, there is exactly one airplane departing from it, and heading towards the closest airport. Prove by induction that there is an airport which none of the airplanes are heading towards.

Base Case: $n = 1$

3 airports

- consider scalene triangle



has no incoming planes

IH $n = k$

Holds true for $2k+1$ airports

IS show True for $n = k+1$, i.e. $2k+3$ airports

→ start with $2k+3$ airports

→ remove the 2 airports that are closest together, left with only $2k+1$, which is true by IH.

$\therefore$ induction holds