Lecture 7: Counting

http://book.imt-decal.org, Ch. 4 (in progress)

Introduction to Mathematical Thinking

October 24th, 2018

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Counting

How many ways can we arrange the letters in MISSISSIPPI?

How many factors of 1200 are multiples are 24?

How many ways can we select 4 boys and 3 girls from a group of 10 boys and 5 girls?

Today: Establish fundamental counting rules, and introduce the ideas of permutations and combinations.

Next class: Pascal's triangle, and combinatorial proofs. Potentially, introduction to the Binomial Theorem.

$P = \{P_1, P_2\}$

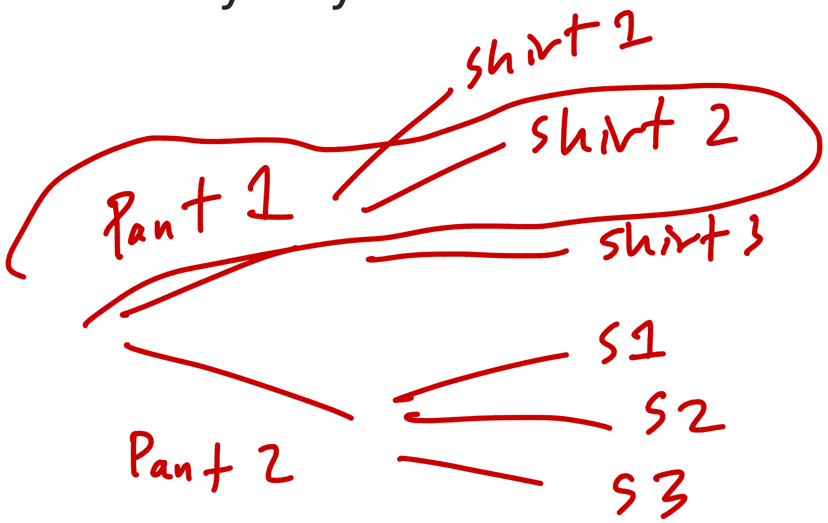
$S = \{S_1, S_2, S_3\}$

$P \times S$

$|P \times S| = |P| \cdot |S|$

Product Rule

Suppose I have 2 pairs of pants and 3 shirts. How many ways can I make an outfit with one pair of pants **and** one shirt?



P1, S2

Pant 1 Shirt 1, Pant 1 Shirt 2, Pant 1 Shirt 3

Pant 2 Shirt 1, Pant 2 Shirt 2, Pant 2 Shirt 3

Total = $(2)(3) = 6$

In general: If we need to make k successive choices, with n_1 options at step 1, n_2 at step 2, ..., n_k at step k: total number of combinations is $n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1} \cdot n_k$

AND means multiply!

Example: How many factors does 1600 have?

$1600 = 16 \cdot 2^2 \cdot 5^2 = 2^6 \cdot 5^2$

$2^a \cdot 5^b$

$a \in \{0, 1, 2, \ldots 6\}$

$b \in \{0, 1, 2\}$

total ways

= (# options for 2)(# of options for 5)

$= (6+1)(2+1) = 7 \cdot 3 = \boxed{21}$

proper factors: $21 - 2 = 19$

Example: How many possible alphanumeric (a-z, 0-9) strings of length 10 can I make?

$\underline{36}\ \underline{36}\ \underline{36}\ \underline{\ \ \cdots\ \ }\ \underline{\ \ \ \ }\ \cdots\ \underline{36} = \boxed{36^{10}}$

Sum Rule

I'm looking for a class to take next semester. I want to choose from a list of 3 stats classes, 4 math classes and 2 chemistry classes. How many different classes can I choose from?

I am choosing 1 of 3 stats classes (S), **or** 1 of 4 math classes (M), **or** 1 of 2 chemistry classes (C). These three sets of classes are all disjoint, so we have

$$|S \cup M \cup C| = |S| + |M| + |C| = 3 + 4 + 2 = 9$$

In general, if we have k disjoint sets of options $n_1, n_2, ...n_k$, the total number of options is $n_1 + n_2 + ... + n_k$.

OR means add!

Principle of Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$50 \qquad 40 \qquad 25$$

We will now use PIE to help us solve counting-style questions.

Suppose there are 50 students at Billy High. Each student is enrolled in at least one of the two math classes the course offers. 40 students are enrolled in calculus, and 25 are enrolled in linear algebra. How many students are in both?

Let x be the number of students in both. Then, from PIE:

$$50 = 40 + 25 - x$$

$$\Rightarrow x = 15$$

Therefore, there are 15 students enrolled in both.

Example: How many numbers integers between 1 and 1000 (inclusive) are multiples of 3 or 5?

mult of 3: $3 \cdot 1, 3 \cdot 2, \ldots \quad 3 \cdot 333 : \boxed{333}$

mult of 5: $5 \cdot 1, 5 \cdot 2, \ldots \quad 5 \cdot 200 : \boxed{200}$

mult of 15: $15 \cdot 1, 15 \cdot 2, \ldots \quad \underbrace{15 \cdot 66}_{990} : \boxed{66}$

Total: $333 + 200 - 66 = \boxed{467}$

Permutations

Suppose I have 5 people, named Alpha, Beta, Charlie, David and Edgar.

How many ways can I arrange them in a line?

$\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 5!$

How many ways can I arrange 3 of them in a line?

$\underline{5}\ \underline{4}\ \underline{3} = 60$

BAC ⇒ diff
CBA

Recall, $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$

The number of ways to arrange k items, selected from a group of n, where **order matters** is

$$P(n,k) = \frac{n!}{(n-k)!}$$

From the second part of the previous example, we have $\underline{5 \cdot 4 \cdot 3} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!}$. $\frac{5!}{(5-3)!}$

The number of permutations of a set of n *distinct* items is $n!$. This matches the formula $(P(n,n) = \frac{n!}{0!} = n!)$

Repeated Characters and Arrangements

How many permutations of DAD are there?

Is it $3! = 6$?

DAD, ADD, DDA

No – there are only 3. We need to account for the repeated D character. "divide the movement"

$D_1 A D_2$ $D_2 A D_1$

$D_1 D_2 A$ $D_2 D_1 A$

$A D_1 D_2$

$A D_2 D_1$

how many permutations of $D_1 D_2$: $2!$

$D_1 A D_2 D_3$ ......

DADD : $\dfrac{4!}{3!}$

Example: How many permutations are there of MISSISSIPPI? *(Hint: First, determine the number of each letter.)*

M : 1

IIII : 4

SSSS : 4

PP : 2

$$\frac{11!}{(4!)(4!)(2!)}$$

repeated Is

repeated Ss

repeated Ps

Example: How many times does the substring "DOG" appear in all permutations of "BABYDOG"?

$\boxed{DOG}$

B B

A

Y

$$\boxed{\frac{5!}{2!}}$$

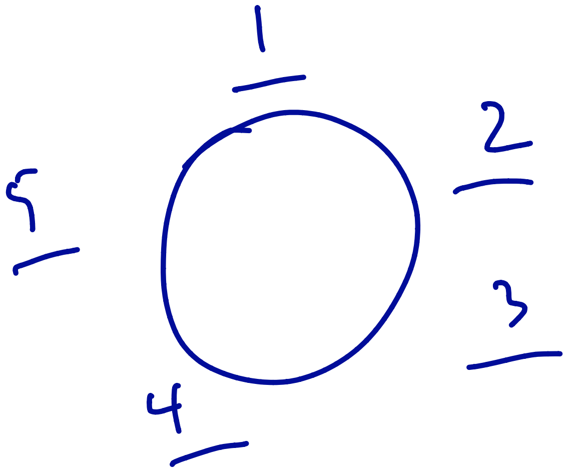
How many times do the characters of DOG appear next to each other, any order:

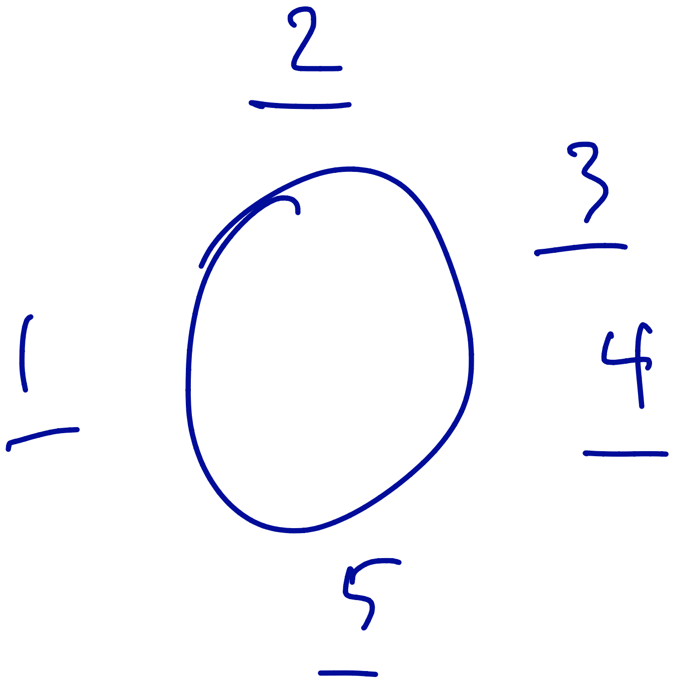
DOG GOD ODG

DGO GDO OGD

$$\boxed{\frac{5!}{2!} \cdot 3!}$$

Example: How many ways can I seat 5 people around a circular dinner table?





→ rotated, but arrangement is same

$$\frac{5!}{5} = 4!$$

Combinations

Now, suppose we have the same 5 people (A, B, C, D, E). How many ways can I select 3 of them, where order no longer matters? (i.e., BED and DBE mean the same thing)

From before: $\frac{5!}{2!}$ gave us the number of ways we could arrange 3 of them in a line, i.e. selecting 3 of them where order mattered.

When order mattered, the same 3 people appeared in the line $3! = 6$ times:

ABC, ACB, BAC, BCA, CAB, CBA

Now that order doesn't matter, we need to divide our previous result $\frac{5!}{2!}$ by $3!$, since it counted each group of three $3!$ times, when they only needed to be counted once. This yields

$$\frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3!}{2 \cdot 1 \cdot 3!} = 10$$

In general, the number of ways to select k items from a group of n, where **order does not matter** is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

This is read "n choose k", and is sometimes referred to as the binomial coefficient.

Note: $\binom{n}{k} = \binom{n}{n-k}$. This is because choosing k items to include is the same as choosing $n-k$ items to discard. In the previous example, instead of counting the number of ways we can select 3 people from $\{A, B, C, D, E\}$, we could count the number of ways we can discard 2 people.

$$\binom{5}{2} = \binom{5}{3}$$

$$\binom{n}{n} = \binom{n}{0}$$

Example: Suppose I want to select 8 children from a group of 8 boys and 9 girls. How many ways can I select 3 boys and 5 girls to form this group?

Number of ways to select boys: $\binom{8}{3}$
Number of ways to select girls: $\binom{9}{5}$

Total number of ways:

$$\binom{8}{3}\binom{9}{5}$$

Followup: How many ways can I select at least 4 boys to form this group?

Choosing at least 4 boys is the same as choosing 4 boys OR 5 boys OR 6 boys OR 7 boys OR 8 boys. In each case, when we choose k boys, we need to also choose $8-k$ girls.

$$ways(\text{at least 4 b}) = ways(\text{4 b}) + ways(\text{5 b}) + ways(\text{6 b}) + ways(\text{7 b}) + ways(\text{8 b})$$

$$= \binom{8}{4}\binom{9}{4} + \binom{8}{5}\binom{9}{3} + \binom{8}{6}\binom{9}{2} + \binom{8}{7}\binom{9}{1} + \binom{8}{8}\binom{9}{0}$$

Example: Deck of cards

Recall, in a standard deck of cards, there are 52 cards. Each card has 1 of 4 suits (Spades, Clubs, Hearts, Diamonds) and 1 of 13 values (Ace, 2, 3, ... 10, Jack, Queen, King). In a hand of cards, the order does not matter.

1. How many 5 card hands are there in Poker?

$$\boxed{\binom{52}{5}}$$

2. How many 5 card hands are there that include exactly one pair (values aabcd, e.g. 2 3s, or 2 5s, etc.)?

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3$$

3. How many 5 card hands are there that include a four-of-a-kind (values aaaab, e.g. 4 3s and a 5)

$$\left(\underset{\text{\# of face values for aaaa}}{13}\cdot\right)\left(\underset{\text{\# of face values for b}}{12}\cdot\right)\left(\underset{\text{\# of suits for b}}{4}\right) = \boxed{13 \cdot 48}$$

A - 5 4 - 8 7 - J
2 - 6 5 - 9 8 - Q
3 - 7 6 - 10 9 - K

4. How many 5 card hands are there that have a straight, i.e. where all card values are consecutive? (e.g. 3, 4, 5, 6, 7, but the suits don't matter)

$$9 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = \boxed{9 \cdot 4^5}$$

5. How many 5 card hands are there that are a straight flush, i.e. where all card values are consecutive and all cards are of the same suit? (e.g. 3, 4, 5, 6, 7, where all cards are diamonds)

$$9 \cdot \binom{4}{1} = \boxed{9 \cdot 4}$$

6. How many 5 card hands are there where all cards are of the same suit?

$$\boxed{\binom{4}{1}\binom{13}{5}}$$

$$1+2+\cdots+n = \binom{n+1}{2}$$

$$S = 1+2+\cdots+n$$
$$S = n+(n-1)+\cdots+1$$

combinatorial proof

Show LHS and RHS count same quantity

ABCDE

$\binom{5}{2}$	
AB, AC, AD, AE	4
BC, BD, BE	3
CD, CE	2
DE	1