Problem Set 4: Proof Techniques, Mathematical Induction

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Fall 2018

This homework is due on Monday, October 1st, at 6:30PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. a. Prove, using induction, that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

   b. Prove, using induction, that $\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3$. *(Hint: In lecture, we showed that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.)*

Solution:

a. *Base Case* $n = 1$:

$$\sum_{i=1}^{1} i^2 = 1^2 = 1; \frac{1(2)(3)}{6} = 1$$

$\therefore$, the base case holds.

Induction Hypothesis : Assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$, for some arbitrary $k \in \mathbb{N}$.

Induction Step : Now, we need to show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$, using the information we assumed in the hypothesis.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

as required. Therefore, induction holds, and the statement holds true.

b. *Base Case* $n = 1$:

$$(\sum_{i=1}^{1} i)^2 = 1^2 = 1; \sum_{i=1}^{1} i^3 = 1$$

$\therefore$, the base case holds.

Induction Hypothesis : Assume $\left(\sum_{i=1}^{k} i\right)^2 = \sum_{i=1}^{k} i^3$, for some arbitrary $k \in \mathbb{N}$.

Induction Step : We need to show that $\left(\sum_{i=1}^{k+1} i\right)^2 = \sum_{i=1}^{k+1} i^3$, using the information we assumed in the hypothesis. We will rely on the identity given in the hint.

This time, we will start with the right hand side and work towards the left hand side.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \left(\sum_{i=1}^{k} i\right)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= \left(\sum_{i=1}^{k+1} i\right)^2
\end{aligned}$$

as required. Therefore, induction holds, and the statement holds true.

2. Recall the power rule for derivatives:

$$\frac{d}{dx} x^n = n x^{n-1}$$

Prove this rule for all $n \in \mathbb{N}_0$ using induction. *(Hint: You will need to use the product rule for derivatives.)*

Solution: *Base Case* $n = 0$:

$$\frac{d}{dx}x^0 = 0x^{-1} = 0$$

as required.

Induction Hypothesis $n = k$: Assume that $\frac{d}{dx}x^k = kx^{k-1}$ for some arbitrary k.

Induction Step: We will use the product rule for derivatives, as indicated in the hint.

$$\begin{aligned}
\frac{d}{dx}x^{k+1} &= \frac{d}{dx}x^k \cdot x \\
&= \left(\frac{d}{dx}x^k\right) \cdot x + x^k \cdot \left(\frac{d}{dx}x\right) \\
&= kx^{k-1} \cdot x + x^k \cdot 1 \\
&= kx^k + x^k \\
&= (k+1)x^k
\end{aligned}$$

as required. Therefore, by induction, we've proved that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{N}_0$.

3. De Moivre's theorem gives us a way to exponentiate complex numbers of the form $R(\cos t + i \sin t)$:

$$(R(\cos t + i \sin t))^n = R^n(\cos nt + i \sin nt)$$

We can prove this using Euler's theorem, which tells us that $\cos t + i \sin t = e^{it}$.

$$(R(\cos t + i \sin t))^n = (Re^{it})^n = R^n e^{int} = R^n(\cos nt + i \sin nt)$$

Prove De Moivre's theorem for $n \in \mathbb{N}_0$, using induction. *(Hint: You will need the trignometric identities* $\cos(a + b) = \cos a \cos b - \sin a \sin b$ *and* $\sin(a + b) = \sin a \cos b + \cos a \sin b$*.)*

Solution:

Base Case $n = 0$ is trivially true:

$$(R(\cos t + i \sin t))^0 = R^0(\cos 0t + i \sin 0t) = 1$$

Induction Hypothesis : Assume $(R(\cos t + i \sin t))^k = R^k(\cos kt + i \sin kt)$, for some arbitrary $k \in \mathbb{N}$.

Induction Step : Now, we need to show $(R(\cos t + i \sin t))^{k+1} = R^{k+1}(\cos(k+1)t + i \sin(k+1)t)$, using the information we assumed in the hypothesis.

$$
\begin{aligned}
(R(\cos t + i\sin t))^{k+1} &= (R(\cos t + i\sin t))^k (R(\cos t + i\sin t)) \\
&= R^k(\cos kt + i\sin kt) R(\cos t + i\sin t) \\
&= R^{k+1}(\cos kt\cos t + i\cos kt\sin t + i\sin kt\cos t + i^2\sin kt\sin t) \\
&= R^{k+1}(\cos kt\cos t - \sin kt\sin t + i(\cos kt\sin t + \sin kt\cos t))
\end{aligned}
$$

Now, we'll need the identities given in the hint:

$$
\begin{aligned}
\cos kt\cos t - \sin kt\sin t &= \cos(kt + t) = \cos(k+1)t \\
\cos kt\sin t + \sin kt\cos t &= \sin(kt + t) = \sin(k+1)t
\end{aligned}
$$

Then:

$$R^{k+1}(\cos kt\cos t - \sin kt\sin t + i(\cos kt\sin t + \sin kt\cos t)) = R^{k+1}(\cos(k+1)t + i\sin(k+1)t)$$

as required. Therefore, induction holds, and the statement holds true.

4. Consider the following inequality:

$$|\bigcup_{i=1}^{n} A_i| \leq \sum_{i=1}^{n} |A_i|$$

In other words, $|A_1 \cup A_2 \cup ... \cup A_n| \leq |A_1| + |A_2| + ... + |A_n|$.

a. Draw a picture and reason about why this is true intuitively.

b. Prove this using induction.

Solution:

a. ...

b. *Base Case* $n = 1$: $|A_1| \leq |A_1|$, therefore the base case holds.

Induction Hypothesis $n = k$ Assume that $|A_1 \cup A_2 \cup ... \cup A_k| \leq |A_1| + |A_2| + ... + |A_k|$, for some arbitrary k.

Induction Step Let $A^k = A_1 \cup A_2 \cup ... \cup A_k$. Then:

$$
\begin{aligned}
|A_1 \cup A_2 \cup ... \cup A_k \cup A_{k+1}| &= |A^k \cup A_{k+1}| \\
&= |A^k| + |A_{k+1}| - |A^k \cap A_{k+1}| \quad \textit{(by the Principle of Inclusion-Exclusion)} \\
&\leq |A^k| + |A_{k+1}| \quad \textit{(since all cardinalities are non-negative)} \\
&\leq |A_1| + |A_2| + ... + |A_k| + |A_{k+1}| \quad \textit{(from the Induction Hypothesis)}
\end{aligned}
$$

as required. Therefore, by induction, the original statement holds.

5. Prove that for $n \geq 3$, the sum of the interior angles of a polygon with n vertices is $180(n-2)$. *(Attempt this problem, but don't spend too long on it, and don't worry if you don't get it. Hint: Drawing pictures will help.)*

Solution: *Base Case* $n = 3$: In a triangle, by definition, the sum of the interior angles is 180.

Induction Hypothesis $n = k$: Assume that for a polygon with k vertices, the sum of the interior angles is $180(k-2)$.

Induction Step Consider a polygon with $k+1$ vertices. Suppose the vertices are labelled $A_1, A_2, ...A_k, A_{k+1}$. We can draw a line segment between A_1 and A_3 (or really, any A_i and A_{i+2}). This splits our figure into two sub-polygons:

- Triangle A_1, A_2, A_3, with interior angles summing to 180
- k-vertex polygon $A_3, A_4, ...A_k, A_{k+1}, A_1$, with interior angles summing to $180(k-2)$ (from the induction hypothesis)

Then, the sum of the interior angles in the entire figure is the sum of the interior angles in the two sub-figures, which is $180+180(k-2) = 180(k-1)$, which is precisely $180((k+1)-2)$, as required.

Therefore, by induction, the statement holds.

6. *Note: This problem may seem very long, but most of it is reading. The work you have to do is relatively little.*

In this problem, we will look at a new way to derive formulas for sums of the form $1^k + 2^k + ... + n^k$ for any $k \in \mathbb{Z}^+$. You already proved these formulas in the first problem.

Consider the sum

$$(2-1) + (3-2) + (4-3) + (5-4) + ... + (100-99)$$

We can rewrite it as

$$-1 + (2-2) + (3-3) + (4-4) + ... + (99-99) + 100$$

Then, all of the terms other than -1 and 100 cancel out, giving us that the value of this sum is $-1+100=99$. Such a sum, where part of term i cancels out with part of term $i+1$ and $i-1$, is known as a **telescoping sum**. The act of cancelling these terms is called the **Method of Differences**.

Another example:

$$\begin{aligned}\sum_{k=1}^{100}\big(\cos(k)-\cos(k-1)\big) &= (\cos 1-\cos 0)+(\cos 2-\cos 1)+...+(\cos 100-\cos 99)\\ &= -\cos 0+(\cos 1-\cos 1)+(\cos 2-\cos 2)+...+(\cos 99-\cos 99)+\cos 100\\ &= \cos 100-\cos 0\end{aligned}$$

How can we use this to get formulas for sums of the form $\sum_{i=1}^{n} i^k$?

Consider $(i+1)^2 = i^2+2i+1$. We can rearrange this to have $(i+1)^2-i^2 = 2i+1$. Then, on the left hand side we have something that resembles a telescoping sum. Since both the left hand side and right hand side are equal, if I sum both sides from $i=1$ to $i=n$, the results should be the same.

$$\begin{aligned}\sum_{i=1}^{n}\big((i+1)^2-i^2\big) &= \sum_{i=1}^{n}\big(2i+1\big)\\ (2^2-1^2)+(3^2-2^2)+...+((n+1)^2-n^2) &= 2\sum_{i=1}^{n}+\sum_{i=1}^{n}1\\ (n+1)^2-1^2 &= 2\sum_{i=1}^{n}+n\end{aligned}$$

We used the fact that $\sum 2x = 2\sum x$ and $\sum_{i=1}^{n} 1 = 1+1+...+1 = n$.

Now, we can simply rearrange for $\sum_{i=1}^{n} i$:

$$\begin{aligned}(n+1)^2-1^2 &= 2\sum_{i=1}^{n} i+n\\ 2\sum_{i=1}^{n} i &= (n+1)^2-n-1\\ \sum_{i=1}^{n} i &= \frac{n^2+2n+1-n-1}{2}\\ &= \frac{n(n+1)}{2}\end{aligned}$$

as we've seen before.

Suppose we wanted to use this technique to find an expression for $\sum_{i=1}^{n} i^2$. We would have to start with the expansion of $(i+1)^3 = i^3+3i^2+3i+1$, rearrange to have $(i+1)^3-i^3 = 3i^2+3i+1$, and take the sum on both sides. This will yield

$$\sum_{i=1}^{n}\left((i+1)^3 - i^3\right) = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

The left hand side will evaluate to be $-1^3 + (n+1)^3$. The right hand side has both $\sum_{i=1}^{n} i^2$, which we are trying to find, and also $\sum_{i=1}^{n} i$, which we already know. To proceed, you will have to substitute the value $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and rearrange to find $\sum_{i=1}^{n} i^2$.

a. Determine the value of $\sum_{i=1}^{n} \frac{1}{i(i+1)}$. *(Hint: How can you rewrite $\frac{1}{i(i+1)}$ as the difference of two terms?)*

b. Derive $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ using a telescoping sum. Most of this has already been done for you!

c. Derive $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ using a telescoping sum.

Solution:

a. Notice that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$. Then:

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+1}\right) &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}\end{aligned}$$

b.

$$\begin{aligned}
\sum_{i=1}^{n}\left((i+1)^3 - i^3\right) &= 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\
(n+1)^3 - 1 &= 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n \\
(n+1)^3 - 1 - 3\frac{n(n+1)}{2} - n &= 3\sum_{i=1}^{n} i^2 \\
\frac{2(n+1)^3 - 2 - 3n(n+1) - 2n}{2} &= 3\sum_{i=1}^{n} i^2 \\
\frac{(n+1)(2(n+1)^2 - 3n - 2)}{6} &= \sum_{i=1}^{n} i^2 \\
\frac{n(n+1)(2n+1)}{6} &= \sum_{i=1}^{n} i^2
\end{aligned}$$

as required.

c.

$$\begin{aligned}
\sum_{i=1}^{n}((i+1)^4 - i^4) &= 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} i \\
(n+1)^4 - 1 &= 4\sum_{i=1}^{n} i^3 + 6\left(\frac{n(n+1)(2n+1)}{6}\right) + 4\left(\frac{n(n+1)}{2}\right) + n
\end{aligned}$$

Rearranging and solving for $\sum_{i=1}^{n} i^3$ will yield the answer required.