PROBLEM SET 1: SETS AND FUNCTIONS

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
SPRING 2019

This homework is due on Friday, February 8th at 11:59PM on Gradescope. As usual, this homework is graded on effort, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. Fill out the following student information form: https://goo.gl/forms/Kv5og6iAcKrO7jBj1.

2. A partition of a set A is defined as a collection of subsets $A_1, A_2, ..., A_n$ such that:

   - $A_i \subseteq A$
   - $A_i \cap A_j = \emptyset, \forall i \neq j$
   - $A_1 \cup A_2 \cup ... \cup A_n = A$

   For any other set $B \subset A$, show that

   $$|B| = |B \cap A_1| + |B \cap A_2| + ... + |B \cap A_n|$$

   You don't need to do anything rigorous — draw a picture, and justify to yourself why this is true. *As an aside: In mathematics, we sometimes use the $\coprod$ symbol to denote a union of disjoint sets. In our case, we could say $A_1 \coprod A_2 \coprod ... \coprod A_n = A$, since each A_i is disjoint from one another.*

   Solution: Since $A_1 \cup A_2 \cup ... \cup A_n = \mathbb{A}$, and that all A_i are disjoint, we know that B is divided into n parts, each of which overlaps with exactly one A_i. As a result, we have that $|B| = |B \cap A_1| + |B \cap A_2| + ... + |B \cap A_n|$.

3. In this question, we will introduce the Principle of Inclusion-Exclusion, which allows us to measure the size of the union of two sets. We will study this more when we learn counting, as there are significant implications of PIE in combinatorics.

   a. The Principle of Inclusion-Exclusion for two sets states that $|A \cup B| = |A| + |B| - |A \cap B|$. Derive this identity. *(Hint: Draw a picture.)*

   b. The Principle of Inclusion-Exclusion for three sets states that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Derive this identity.

   c. (Optional) Generalize the Principle of Inclusion-Exclusion for any number of n sets. *(Hint: It may help to first derive the expression for four sets. Do you notice a pattern?)*

Solution:

a. First, we count every item in A and B individually, yielding $|A| + |B|$. We then see that the intersection $A \cap B$ has been counted twice once in $|A|$, and once in $|B|$. By subtracting $|A \cap B|$ we yield $|A \cup B| = |A| + |B| - |A \cap B|$ as required. (Derived in lecture, see video in textbook.)

b. Again, we start by counting each set individually, giving us $|A| + |B| + |C|$. We now notice that each pairwise overlap has been counted twice $|A \cap B|$ was counted in both $|A|$ and $|B|$, $|A \cap C|$ was counted in both $|A|$ and $|C|$, and $|B \cap C|$ was counted in both $|B|$ and $|C|$; additionally, the triple intersection $|A \cap B \cap C|$ is counted three times. By subtracting $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$, we have subtracted the triple overlap $|A \cap B \cap C|$ three times (as it is part of each pairwise intersection). Since it was originally counted three times, we need to add it back once. Thus, our final relation yields $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

c. In general, we sum all individual cardinalities, subtract pairwise intersections, add back intersections of triplets, subtract intersections of each combination of four sets, and so on and so forth.

4. Let A and B be sets. Determine $|A - B| + |B - A|$ (that is, the size of the set of elements that are either in A, or in B, but not both) in terms of $|A|, |B|$ and $|A \cap B|$.

Solution: By the principle of inclusion-exclusion, we know the number of elements in A or B (including the overlap) is $|A| + |B| - |A \cap B|$. However, we want to exclude the elements that are in the overlap, so we subtract by $|A \cap B|$ again, yielding $|A| + |B| - 2|A \cap B|$.

5. The following information may be useful in the remaining questions.

- $\mathbb{N} = \{1, 2, 3, 4, ...\}$
- $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$
- $\mathbb{R}_{\geq 0}$ = the set of all non-negative real numbers

Sets A, B, C are defined over a universe $\mathbb{U} = \{z : z \in \mathbb{N}_0, z \leq 25\}$ as follows:

- $A = \{x : x \text{ is prime}, x \leq 25\}$
- $B = \{2k : k \in \mathbb{N}_0, k \leq 12\}$
- $C = \{t^2 : t \in \mathbb{N}_0, t \leq 5\}$

Determine the sets that result after each of these set operations.

a. $A \cap B$

b. $A^C \cup B^C$ *(Hint: How can you use De Morgan's Laws to re-use your result from part a?)*

c. $(A \cup B) \cap C$

d. $B - C$

e. $A \setminus B^C$

f. $A^C \cap B^C \cap C^C$

Solution:

It may help to first identify A, B, and C.

$A = \{2, 3, 7, 11, 13, 17, 19, 23\}$

$B = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24\}$

$C = \{0, 1, 4, 9, 16, 25\}$

a. $\{2\}$

b. Recall, in lecture we saw that De Morgan's Laws state $(A \cap B)^C = A^C \cup B^C$. We already found $A \cap B$ in part a, and so we are simply looking for the complement of this result. Our result for $A\cap$ was the set $\{2\}$, so our result here is the set of all whole numbers less than 25, not including 2. In other words, $\{n : n \in \mathbb{N}_0, n \leq 25, n \neq 2$

c. $\{2, 6, 8, 10, 12, 14, 18, 20, 22, 24\}$

d. $\{2\}$

e. $\{5, 15, 21\}$

6. In lecture, we showed that the composition of two injective functions is also injective, as follows:

- Assume f, g are both one-to-one functions.
- Consider $f(g(x_1)) = f(g(x_2))$. Since $f(\cdot)$ is injective, we have that $g(x_1) = g(x_2)$.
- Since $g(\cdot)$ is injective, we have that $x_1 = x_2$.
- Therefore, we have that $f(g(x_1)) = f(g(x_2))$ implies that $x_1 = x_2$, meaning that the function $f(g(x))$ is injective.

Use a similar argument to show that the composition of two surjective functions is also surjective.

Solution: Suppose $g : A \to B$ and $f : B \to C$. Notice this means that $f(g(\cdot)) : A \to C$. To prove that the composition of two injective functions is injective, we showed that $f(g(x_1)) = f(g(x_2)) \implies x_1 = x_2$. To show that the composition of two surjective functions is also surjective, we need to show that $\forall c \in C, \exists\, a \in A : f(g(a)) = c$.

Since $f : B \to C$ is surjective, $\forall c \in C, \exists\, b \in B : f(b) = c$.

Then, since $g : A \to B$ is surjective, $\forall b \in B, \exists\, a \in A : g(a) = b$.

Putting these two statements together, we have that $\forall a \in A, \exists\, c \in C : f(g(a)) = c$.

$\forall$ translates to "for all", and $\exists$ translates to "there exists". Refer to the textbook if you are unsure of the notation.

7. Determine whether each of the following functions is injective, surjective, both (bijective) or none.

   a. $f : \{2, 3, 4\} \to \{2, 3, 4\}, \{(2, 2), (3, 2), (4, 4)\}$

   b. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$

   c. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 3x^2 - 7x - 2$ *(Hint: Think about the derivative of $f(x)$, and the test for injections we saw in lecture).*

   d. $f : \mathbb{R}_{\geq 0} \to \mathbb{N}, f(x) = \lceil x \rceil$ *(Hint: This is the ceiling function.)*

   e. $f : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}, f(x, y) = x^2 + y^2$

   f. $f : \mathbb{N} \to \{t : t \in \mathbb{N}, t \text{ is prime}\}, f(x) =$ the x^{th} prime number

Solution:

a. f is neither injective nor surjective. f is not injective, as $f(2) = f(3) = 2$. f is not surjective as 3 has no pre-image (that is, there is no x such that $f(x) = 3$).

b. f is a bijection. f is injective, as $a^3 = b^3 \implies a = b, \forall a, b \in \mathbb{R}$ (note, if we allow the domain to span the set of complex numbers, this is no longer true.) f is surjective since $\forall y \in \mathbb{R}, \exists\, x \in \mathbb{R} : y = x^3$. Specifically, every real number has a cubed root.

c. f is a surjection, but not an injection. f is not an injection, as it has one local maximum and one local minimum, implying that it changes directions twice, meaning it does not pass the horizontal line test required for functions to be injections. f is a surjection as there is always a solution to $f(x) = c, \forall c \in \mathbb{R}$. (Think about how $f(x)$ extends infinitely in the positive and negative directions.)

d. f is a surjection, but not an injection. f is not an injection as $\lceil 1.5 \rceil = \lceil 1.6 \rceil = 2$, but $1.5 \neq 1.6$. f is surjective, as for every positive integer n, there exists at least one a such that $\lceil a \rceil = n$. For example, $\lceil n \rceil = n$.

e. f is a surjection, but not an injection. f is not an injection, as $f(0, 5) = f(5, 0) = 25$, but $(0, 5) \neq (5, 0)$. f is a surjection, as for all non-negative real numbers r, there exists at least one ordered pair (x, y) such that $x^2 + y^2 = r$. One such ordered pair is $(\sqrt{r}, 0)$. *Note: f is a function, of multiple variables.*

f. f is a bijection. f is an injection, as the i^{th} prime number and j^{th} prime number are different, by nature, for all positive integers i, j. f is a surjection by definition; our codomain is the set of all prime numbers, and the outputs of $f(\cdot)$ are precisely the prime numbers, in order – $f(1)$ is the first prime number, $f(2)$ is the second prime number, and so on.