Lecture 2.5: Propositional Logic

(overflow from last week)

http://book.imt-decal.org, 1.4

Introduction to Mathematical Thinking

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Announcements

- Office hours are official: Tuesday 2-3PM and Thursday 5-6PM, both in Cory 299.
- Reminder: Midterm date is now Oct. 10 (3 weeks from this Wednesday)
- HW1 grades are out, HW2 was just due.
- Will start recording lecture videos live (webcasting) instead of recording beforehand. Allows me to get the videos out immediately after class.

Today's class: Lecture for ~20 minutes, then discussion. Discussion will be Guerilla-section style from now on.

1.4: Propositional Logic

Recap:

A proposition is a statement that is has a definitive value - either true or false. $P(x)$ usually represents a prop. that depends on variable x.

Three logical operators:

1. **Conjunction**: $A \wedge B$, read "A and B"
   $$A \cap B = \{x : x \in A \underset{\text{and}}{\wedge} x \in B\}$$
2. **Disjunction**: $A \vee B$, read "A or B"
   $$A \cup B = \{x : x \in A \vee x \in B\}$$
3. **Negation**: $\neg A$, read "not A"
   $$A^C = \{x : \neg(x \in A)\}$$

e.g. $P(t)$: "t is prime", $E(t)$: "t is even"
What does $P(t) \vee \neg E(t)$ mean?

t is prime or not even

Implications, Truth Tables

$P \to Q$, read "P implies Q", says that if P is true, then Q must be true; if P is false, it says nothing about Q (Q could either be true or false). e.g., if all we know about today's date is that it's Christmas, we also know that the current month is December. However, if we don't know that it's Christmas, then it may or may not be December.

Truth tables: How we prove **logical equivalences.**

P	Q	$P \to Q$	$\neg P \lor Q$
True	True	True	True
True	False	False	False
False	True	True	True
False	False	True	True

$P, Q, R \to 8$

$$\boxed{P \to Q \underbrace{\equiv}_{\text{equivalent}} \neg P \vee Q}$$

$(A \vee B) \wedge \neg(A \wedge B)$

$(\neg A \wedge B) \vee (A \wedge \neg B)$

Exclusive OR

Our regular **OR** operation $A \vee B$ is true when only A is true, when only B is true, and when when both are true.

A	B	$A \vee B$
True	True	True
True	False	True
False	True	True
False	False	False

A	B	$A \oplus B$
T	T	F
T	F	T
F	T	T
F	F	F

$A \oplus B$, read "A xor B", is true when exactly one of A, B is true. What does its truth table look like?

\oplus

Claim: $A \oplus B \equiv (A \vee B) \wedge \neg(A \wedge B)$.

Proof:

$T \wedge \neg(T) \to F$

A	B	$A \oplus B$	$(A \vee B) \wedge \neg(A \wedge B)$
True	True	False	False
True	False	True	True
False	True	True	True
False	False	False	False

$(T \vee F) \wedge \neg(T \wedge F)$

$\equiv T \wedge \neg(F)$

$\equiv T \wedge T$

$\equiv T$

Two extensions of the implication $P \to Q$:

- Contrapositive: $\neg Q \to \neg P$
- Converse: $Q \to P$

Contrapositive

The contrapositive, $\neg Q \rightarrow \neg P$, of an implication is actually logically equivalent to the implication itself!

$$\neg Q \rightarrow \neg P \equiv \neg(\neg Q) \vee (\neg P)$$

$$\equiv Q \vee \neg P$$

$$\equiv \neg P \vee Q$$

$$\equiv P \rightarrow Q$$

$P \rightarrow Q \equiv \neg P \vee Q$

$\neg Q \Rightarrow \neg P \equiv P \Rightarrow Q$

P

Q

Example: The contrapositive of the statement "if it is sunny outside, I will wear my sunglasses" is "if I don't wear my sunglasses, it is not sunny outside."

$\neg Q$

$\neg P$

$P \to Q$

Converse

The converse, $Q \rightarrow P$ of an implication, unlike the contrapositive, is not equivalent to the original implication. (unlike contrapositive)

If today is Christmas (P), then it implies that the current month is December (Q), however it being December (Q) doesn't mean that today is Christmas (P).

$A \leftrightarrow B$

$A \Longleftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$

$\equiv (A \equiv B)$

If and only if

$A \longleftrightarrow B$, read "A if and only if B", says that A is true only when B is true, and A is false only when B is false - **in other words, that two statements are equivalent** ($A \equiv B$).

$A \to B$
$B \to A$ $\to$ $A \leftrightarrow B$

"it is $\overbrace{\text{Christmas}}^{A}$ if and only if $\overbrace{\text{it is December 25th}}^{B}$" decomposes into:

1. "it is Christmas if it is December 25th" $A \to B$
2. "it is December 25th if it is Christmas" $B \to A$

This is just a fancy way of saying "Christmas is on December 25th".

Consider the following truth table:

A	B	$A \leftrightarrow B$	$(A \to B) \land (B \to A)$
True	True	True	True
True	False	False	False
False	True	False	False
False	False	True	True

e.g., Today being Christmas implies that today is December 25th, and today being December 25th implies that today is Christmas - since these two propositions imply one another, we can say they're equivalent.

Important for proof techniques!

Existential Quantifiers

- **universal quantifier** $\forall$, read "for all"
- **existential quantifier** $\exists$, read "there exists"

Recall: $f : A \to B$ is surjective when $\forall b \in B, \exists a \in A : b = f(a)$

for all b in B, there exists an a in A

e.g. Suppose $E(x)$: "x is even" and $U(x)$: "x is odd." What do the following statements mean?

1) $$\forall x \in \mathbb{N}, E(x) \vee U(x)$$

for all naturals x,
x is even or
x is odd :
$\boxed{\text{TRUE}}$

2) $$\forall x \in \mathbb{N}, E(x) \wedge U(x)$$

all natural numbers are both even
and odd $\boxed{\text{FALSE}}$

De Morgan's Laws for Existential Quantifiers

Allows us to change a universal quantifier into an existential quantifier.

$$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x) \tag{1}$$

$$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x) \tag{2}$$

e.g $P(x)$: "x is prime", $\mathbb{U} = \mathbb{Z}^+$.

1. "If it is not true that x is prime for all x, then there must exist some x that is not prime."
2. "If it is not true that there exists an x that is prime, then for all x, x is not prime."