Lecture 5: Midterm Review

http://book.imt-decal.org, Ch. 1, 2

Introduction to Mathematical Thinking

October 3rd, 2018

Suraj Rampure

Announcements

- HW5 won't be collected, but you should do it. There are a lot of proof problems on it, but it's not necessarily comprehensive.
    - Solutions coming soon (before the weekend)
    - Might add a few more problems.
    - On Monday, we'll take up some of these problems.
- Midterm is a week from today, in class, starting at 6:40.
    - Some T/F, ~~some M/C,~~ some short answer.
    - You can bring one 2-sided cheat sheet, handwritten.
    - HWs are a good indication of difficulty; we're not out to get you, we really just want to see how well you've learned the material so far (remember, this class is P/NP).
- **Week after the midterm: Discussion and lecture swapped (disc Wednesday, lec Monday)**

Oct 15

Oct 17

Overview

To put things in context: The course can be thought of as being in two parts.

Part 1

sets of numbers

- Set theory
- Types of functions
- Propositional logic
- Proof techniques

Part 2

$\binom{n}{k}$

- Number theory (modular arithmetic)
- Combinatorics (counting techniques, Pascal's triangle)
- Combinatorics with polynomials (Binomial theorem, Vieta's formulas)

universe

Set Theory (1.1)

De Morgan's Laws for sets

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

A set is a well-defined collection collection of objects.

Set Operations:

- $A^C = \{x : \neg(x \in A)\}$
- $A \cup B : \{x : x \in A \vee x \in B\}$
- $A \cap B : \{x : x \in A \wedge x \in B\}$
- $A - B : \{x : x \in A, \neg(x \in B)\}$
- $A \times B : \{(a, b) : a \in A, b \in B\}$

$A \setminus B$

A is a subset of B: $A \subseteq B$

A can be equal to B

A is a proper subset of B: $A \subset B$

a subset that isn't equal i.e. $A \neq B$

all functions pass VLT
injections also pass HLT

$A = \{1, 2, 3\}$ $\quad f(1) = f(3)$, but $1 \neq 3$ $\therefore$ not inj.

$B = \{4, 5, 6\}$

$R: \{4, 5\}$ $\quad B \neq R$ $\therefore$ not surjective

Functions (1.2)

A function with domain A and codomain B is a subset of $A \times B$ such that there is exactly one ordered pair for each element in A.

$f: \{(1,4), (2,5), (3,4)\}$

yes!

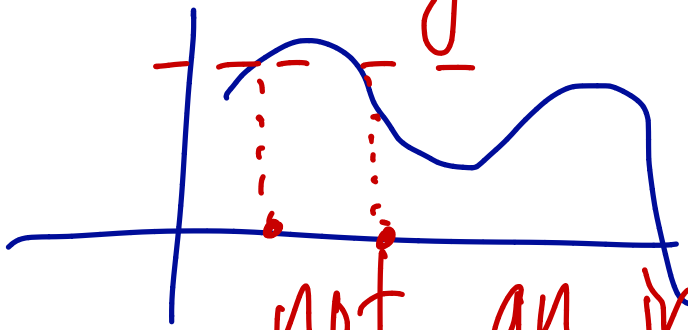
$f(x_1) = f(x_2) \longrightarrow x_1 = x_2$

- Injections (one-to-one)
  - Horizontal line test
  - Strictly increasing, decreasing
- Surjections (onto) → codomain = range
- Bijections (both)

$g: \{(1,4), (1,5), (2,5)\}$

not a function

$\forall\, b \in B, \exists\, a \in A$ s.t. $f(a) = b$



not an injection

dense : between any 2 elements in S, there are infinitely many elements in S

Number Sets (1.3)

$\{1, 2, 3, \dots\}$

$\mathbb{N}_0$: 0 1 2 3 4 5 6 ...

$\mathbb{Z}$: 0 1 −1 2 −2 3 −3 ...

$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

$\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

$\{0, 1, 2, 3, \dots\}$

a set S is countable if there exists a bijection between S and $\mathbb{N}$

Which of these sets are countable? Uncountable?

- What does it mean for an arbitrary set to be countable?
- What was the bijection we showed between $\mathbb{N}_0$ and $\mathbb{Z}$? Between $\mathbb{N}$ and $\mathbb{Q}$?
- *There aren't many questions about this content in the review homework, but make sure to review it*

countable = countably infinite

$\mathbb{N}$ ✓ $\mathbb{N}_0$ ✓ $\mathbb{Z}$ ✓ $\mathbb{Q}$ ✓ $\mathbb{R}$ ✗

dense dense

Propositional Logic (1.4)

Logical Operators

Basic

- Conjunction ($\wedge$) → $\boxed{\text{and}}$
- Disjunction ($\vee$) → $\boxed{\text{OR}}$
- Negation ($\neg$) → $\boxed{\text{NOT}}$

Complex

- Implication $P \Rightarrow Q$, and its equivalent form $\neg P \vee Q$
- Contrapositive $\neg Q \Rightarrow \neg P$
- Converse $Q \Rightarrow P$
- Exclusive OR $P \oplus Q$ → only true when exactly 1 of P, Q is true

Know how to use truth tables.

$P \to Q \equiv \neg Q \to \neg P$

$P \to Q \cancel{\equiv} Q \to P$

not true in general

$\boxed{P \text{ iff } Q : P \to Q \wedge Q \to P}$

If and only if $\longleftrightarrow$

Existential Quantifiers

$$\boxed{\forall b \in B, \exists a \in A : f(a) = b}$$

- "for all" (∀)
- "there exists" (∃)

for all

there exists

De Morgan's Laws:

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$$

$$\neg(\exists x P(x)) \equiv \forall x \neg P(x)$$

What is the negation of $\forall x \exists y (P(x, y) \vee Q(y))$?

$\exists x \forall y (\neg P(x,y) \wedge \neg Q(y))$

Also should know how to convert statements in English to statements using propositional logic (many examples in book).

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Proof Techniques (2.1, 2.2)

- Direct Proof
- Proof by Contradiction
- Proof by Contraposition
- Vacuous Proofs
- Proof by Cases
- Counterexamples (not a proof technique!)
- Proof by Induction

assume $\neg S$
show a contradiction

$P \to Q$ by showing $\neg Q \to \neg P$

P is false! $\therefore$ $P \to Q$ always true

if the Warriors didn't blow a 3-1 lead, LeBron is the Goat

Attendance

tinyurl.com/KDisleaving

LA 2019

Rest of today: Walking through examples.

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, ...

Example 1: Prove that if p is a prime greater than 3, then $24 \mid p^2 - 1$.

i.e. p^2-1 is a multiple of 24

$$p^2-1 = (p-1)(p+1)$$

$24 = 2^3 \cdot 3$

$= 8 \cdot 3$

→ both $p-1$, $p+1$ are even

→ at least 1 is multiple of 4

$\therefore$ p^2-1 is multiple of $2 \cdot 4 = 8$

→ any 3 consecutive naturals: 1 will be multiple of 3

$\therefore$ either $p-1$ or $p+1$ has factor of 3

→ $\therefore$ p^2-1 has a factor of 24, as required.

$8|24 \rightarrow 24 = 8c$ wholes $3|k^3-k \rightarrow k^3-k=3c$

Example 2: Prove that $3|n^3-n$, for all $n \in \mathbb{N}_0$, using (1) induction and (2) a direct proof.

n^3-n multiple of 3

Induction

Base Case: $n=0$

$n^3-n=0^3-0=0$, indeed a multiple of 3

$0=3k$
$k=0$

IH

Assume $3|k^3-k$ for arbitrary k

$k^3-k=3j$

IS

$(k+1)^3-(k+1)$

$= k^3+3k^2+3k+\cancel{1}-k-\cancel{1}$

$= k^3-k+3(k^2+k)$

$= \boxed{3(j+k^2+k)}$

3·(some int)

$\therefore$ by induction, the statement holds.

2) Direct

$n^3-n=n(n^2-1)$

$=(n-1)(n)(n+1)$

3 consecutive ints, 1 will be multiple of 3

$\therefore n^3-n$ is multiple of 3

Example 3: Suppose A, B are two countable sets. Prove that $A \cup B$ is also countable.

countably infinite

$$\begin{array}{lcccccccccccc} A \cup B : & a_1, & b_1, & a_2, & b_2, & a_3, & b_3, & \cdots\cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbb{N} : & 1, & 2, & 3, & 4, & 5, & 6 & \end{array}$$

only care about unique elements;

if some element in A = some element in B,

we only list it the first time

$N = a_{k-1} a_{k-2} \dots a_2 a_1 a_0$

Example 4: Prove that n is a multiple of 3 if and only if the sum of the digits of n is a multiple of 3.

$N = a_3 a_2 a_1 a_0$ (digits)

$= 10^3 \cdot a_3 + 10^2 \cdot a_2 + 10^1 \cdot a_1 + 10^0 \cdot a_0$

$= \boxed{(10^3 - 1) a_3 + (10^2 - 1) a_2 + (10^1 - 1) a_1 + (10^0 - 1) a_0}$ always multiple of 3

$+ (a_3 + a_2 + a_1 + a_0)$

$10^3 - 1 = 999$

$10^2 - 1 = 99$

$\vdots$

$\therefore$ if $N = 3k$, then $a_3 + a_2 + a_1 + a_0 = 3j$

and if $a_3 + a_2 + a_1 + a_0 = 3l$, then $N = 3m$