Lecture 5: Midterm Review

http://book.imt-decal.org, Ch. 1, 2

Introduction to Mathematical Thinking

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Announcements

- HW5 won't be collected, but you should do it. There are a lot of proof problems on it, but it's not necessarily comprehensive.
  - Solutions coming soon (before the weekend)
  - Might add a few more problems.
  - On Monday, we'll take up some of these problems.
- Midterm is a week from today, in class, starting at 6:40.
  - Some T/F, some M/C, some short answer.
  - You can bring one 2-sided cheat sheet, handwritten.
  - HWs are a good indication of difficulty; we're not out to get you, we really just want to see how well you've learned the material so far (remember, this class is P/NP).
- **Week after the midterm: Discussion and lecture swapped (disc Wednesday, lec Monday)**

Overview

To put things in context: The course can be thought of as being in two parts.

Part 1

- Set theory
- Types of functions
- Propositional logic
- Proof techniques

Part 2

- Number theory (modular arithmetic)
- Combinatorics (counting techniques, Pascal's triangle)
- Combinatorics with polynomials (Binomial theorem, Vieta's formulas)

Set Theory (1.1)

A set is a well-defined collection collection of objects.

Set Operations:

- $A^C = \{x : \neg(x \in A)\}$
- $A \cup B : \{x : x \in A \lor x \in B\}$
- $A \cap B : \{x : x \in A \land x \in B\}$
- $A - B : \{x : x \in A, \neg(x \in B)\}$
- $A \times B : \{(a, b) : a \in A, b \in B\}$

A is a subset of B: $A \subseteq B$
A is a proper subset of B: $A \subset B$

Functions (1.2)

A function with domain A and codomain B is a subset of $A \times B$ such that there is exactly one ordered pair for each element in A.

- Injections (one-to-one)
  - Horizontal line test
  - Strictly increasing, decreasing
- Surjections (onto)
- Bijections (both)

Number Sets (1.3)

$$\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$$

Which of these sets are countable? Uncountable?

- What does it mean for an arbitrary set to be countable?
- What was the bijection we showed between $\mathbb{N}_0$ and $\mathbb{Z}$? Between $\mathbb{N}$ and $\mathbb{Q}$?
- *There aren't many questions about this content in the review homework, but make sure to review it*

Propositional Logic (1.4)

Logical Operators

Basic

- Conjunction ($\land$)
- Disjunction ($\lor$)
- Negation ($\neg$)

Complex

- Implication $P \Rightarrow Q$, and its equivalent form $\neg P \lor Q$
- Contrapositive $\neg Q \Rightarrow \neg P$
- Converse $Q \Rightarrow P$
- Exclusive OR $P \oplus Q$

Know how to use truth tables.

Existential Quantifiers

- "for all" ($\forall$)
- "there exists" ($\exists$)

De Morgan's Laws:

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$$

$$\neg(\exists x P(x)) \equiv \forall x \neg P(x)$$

What is the negation of $\forall x \exists y (P(x, y) \vee Q(y))$?

Also should know how to convert statements in English to statements using propositional logic (many examples in book).

Proof Techniques (2.1, 2.2)

- Direct Proof
- Proof by Contradiction
- Proof by Contraposition
- Vacuous Proofs
- Proof by Cases
- Counterexamples (not a proof technique!)
- Proof by Induction

Attendance

tinyurl.com/KDisleaving

Rest of today: Walking through examples.

Example 1: Prove that if p is a prime greater than 3, then $24 \mid p^2 - 1$.

Example 2: Prove that $3|n^3 - n$, *for all* $n \in \mathbb{N}_0$, *using (1) induction and (2) a direct proof.*

Example 3: Suppose A, B *are two countable sets. Prove that* $A \cup B$ *is also countable.*

Example 4: Prove that n *is a multiple of 3 if and only if the sum of the digits of n is a multiple of 3.*