Lecture 7.5: Stars and Bars

http://book.imt-decal.org, Ch. 4 (in progress)

Introduction to Mathematical Thinking

October 29th, 2018

Suraj Rampure

Announcements

- Homework 7 due Wednesday
- Last Wednesday's video didn't record audio for some reason... sorry, will make sure this doesn't happen again

Stars and Bars

How many ways can I distribute 12 pieces of candy to 3 of my friends?

- 12 "stars", or items
- 3 1 = 2 "bars", or separators

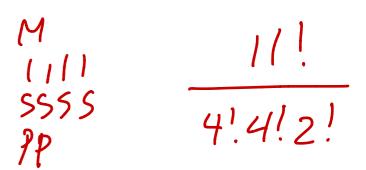
For example:

This setup represents friend 1 getting 2 pieces, friend 2 getting 6 pieces and friend 3 getting 4 pieces.

This setup represents friend 1 getting 1 piece, friend 2 getting 0 pieces and friend 3 getting 11.

This problem boils down to finding the number of permutations of

In our case, this is
$$\frac{14!}{12!2!}=\binom{14}{2}=\binom{14}{2}=\binom{14}{12}=\binom{14}{2}$$



In general, the number of ways to arrange n indistinguishable (i.e. identical) items into k

distinguishable bins is

$$\begin{pmatrix} a+b \end{pmatrix} = \frac{(a+b)!}{a!b!} = \begin{pmatrix} a+b \\ b \end{pmatrix}$$

Since k bins corresponds to k-1 bars, we can also write this number as

$$\begin{pmatrix} n \\ K \end{pmatrix} = \begin{pmatrix} n \\ n - K \end{pmatrix}$$

$$\begin{pmatrix}
stars + bars \\
bars
\end{pmatrix} = \begin{pmatrix}
stars + bars \\
stars
\end{pmatrix} = \begin{pmatrix}
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\end{pmatrix} = \begin{pmatrix}
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Example: Determine the number of non-negative integer solutions to the equation

$$x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto x_5 = 25$$

This fits the same model. We have 25 "stars", and 4 "bars", meaning there exist

$$\begin{pmatrix} 29 \\ 4 \end{pmatrix}$$
 $\begin{pmatrix} 7544 \\ 4 \end{pmatrix}$

solutions to this equation.

Followup: How many positive integer solutions exist to this equation?

Previously, we had that $x_1, x_2, x_3, x_4, x_5 \geq 0$. We now want $x_i > 0$, or equivalently, $x_i \geq 1$.

To account for this, we can define $x_i' = x_i - 1$. Now, the only constraint we have is $x_i' \geq 0$, which we already know how to solve (from the last slide).

The number of positive integer solutions to

$$(x_1'-1) + (x_2'-1) + (x_3+x_4+x_5=25)$$
 $(x_5'-1) = 25-5$

is the same as the number of non-negative integer solutions to

$$x_1' + x_2' + x_3' + x_4' + x_5' = 25 - 5 \neq 20$$

which is $\binom{24}{4}$.

How many ways can I distribute 12 pieces of candy to 3 of my friends, such that they all get at least one piece?

This is the same problem as finding the number of positive integer solutions to $x_1+x_2+x_3=12$ which from the previous slide we have as

$$\binom{12-1}{3-1} = \binom{11}{2} = 55$$

Attendance