QUIZ 1

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
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You will have 30 minutes to work on the quiz. Please fit all of your answers in the space provided. You are not allowed to consult any notes or use any electronics.

Total points: 7 + 9 + 9 = 25

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1. *True or False (7 pts, 1 pt each)*

   Circle either true or false in each of the below. No justification is needed.

   a. **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd."

   b. **True** or **False**: The set of all positive multiples of 37 is uncountably infinite.

   c. **True** or **False**: The function $f(x) = x^4 - x^2$ with domain and codomain $\mathbb{R}$ is injective.

   d. **True** or **False**: For any two finite sets A and B, $|A - B| = |A| - |A \cap B|$.

   Consider sets $S = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ and $T = \{1, 0, -1, \frac{1}{2}\}$.

   e. **True** or **False**: The mapping $w \mapsto \cos(w)$ is a bijection from S to T.

   f. **True** or **False**: There exists a bijection $f : S \to T$.

   g. **True** or **False**: Suppose $g : S \to T$ is a mapping. It is possible for g to be an injection, but not a surjection.

Solution:

a. **True**.

b. **False**. This set is countably infinite.

c. **False**. $f(0) = f(1)$, but $0 \neq 1$.

d. **True**. This is evident by drawing a picture.

e. **False**. This mapping sends both $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ to 0. Thus, this mapping is not an injection, and so it is not a surjection.

f. **True**. $|S| = |T|$, thus there exists a bijection between S and T.

g. **False**. If g is an injection, then no $s \in S$ map to the same $t \in T$. But since there are only four elements in both S and T, and each s maps to a unique t, every element in t is being mapped to by some s, and thus g must also be a surjection.

2. *Set Operations (9 pts, 3 pts each)*

   Consider the following sets, where the universe is $\mathbb{N}$:

   - $A = \{x : x \text{ is prime}\}$
   - $B = \{2k : k \in \mathbb{N}\}$
   - $C = \{x : (x \leq 30) \wedge (x = 6k, k \in \mathbb{N})\}$
   - $D = \{18, 24, 73, 4\}$

   Using **only** the above sets and any set operations (union, intersection, difference, Cartesian product, and complement), construct the following sets. For example, to create the set $\{73\}$, we can perform the operation $A \cap D$ or $D - B$. *(There may be more than one potential answer, but you only need to identify one.)*

   a. $\{18, 24, 4\}$

   b. $\emptyset$ *(the empty set)*

   c. $\{2, 6, 12, 30\}$

Solution:

a. $\boxed{D - A}$. We want all elements in D except for 73, which is prime. By taking the difference of D and A, we get all elements in D that are not in A.

b. $\boxed{A \cap C}$. Notice, A is the set of all primes, and C is the set of multiples of 6 that are less than or equal to 30. No multiple of 6 is prime.

c. $\boxed{(A \cap B) \cup (C - D)}$. First, we notice that $6, 12, 30$ make up the set of elements in C that are not in D (i.e. they are the multiples of 6, less than 30, that are not 18 or 24). Then, to isolate 2, we intersect the set of primes and the set of even numbers. We then take the union of these two sets to get our result.

Of the following two questions, complete one.

3. *Fun at the Zoo (9 pts, 3 pts each)*

   Consider the statement, ”if pigs can fly, then dogs can run or gorillas are not humans.”

   a. Write this statement using propositional logic. Define any variables that you are using (e.g. P: ”pigs can fly”).

   b. Determine the contrapositive of this statement, written in English.

   c. Determine the negation of this statement, written in English.

Solution:

a. Let P: pigs can fly, D: dogs can run, G: gorillas are humans. Note that P, D, G are all propositions. Then, we have

$$\boxed{P \implies (D \vee \neg G)}$$

b. We’ll first find the contrapositive in logic, then re-write it in English. Recall, contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$. Then,

$$\begin{aligned} &\neg(D \vee \neg G) \implies \neg P \\ \equiv\ &(\neg D \wedge G) \implies \neg P \end{aligned}$$

In English, this reads ”if dogs can’t run and gorillas are humans, then pigs can’t fly.”

c. The negation of a proposition $P \implies Q$ is $P \wedge \neg Q$. Then, we have

$$\begin{aligned} \neg(P \implies (D \vee \neg G)) &\equiv P \wedge \neg(D \vee \neg G) \\ &\equiv P \wedge (\neg D \wedge G) \end{aligned}$$

In English, this reads ”pigs can fly and dogs can’t run and gorillas are humans.”

4. *NAND Gates (9 pts, 3 pts each)*

We define the NAND operation, $A \uparrow B$, to be false only when both A and B are true, and true in all other cases.

a. Re-write $A \uparrow B$ using just $\vee$, $\wedge$, and $\neg$. Prove your result using a truth table.

b. Using just the NAND operation ($\uparrow$), create an expression that is equivalent to $\neg A$.

c. Using just the NAND operation ($\uparrow$), create an expression that is equivalent to $A \vee B$. *(Hint: Re-write your result from part a using De Morgan's Laws, then use your result from part b.)*

Solution:

a. We notice that NAND seems to be the reverse of the standard AND operation, i.e $A \uparrow B \equiv \boxed{\neg(A \wedge B)}$. We use a truth table to prove our result.

A	B	$A \uparrow B$	$\neg(A \wedge B)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

b. Notice, when A is true, $A \wedge A$ is true, and when A is false, $A \wedge A$ is false. Thus, since $A \uparrow B \equiv \neg(A \wedge B)$, we have that $\neg A \equiv \boxed{A \uparrow A}$.

c. Notice, with De Morgan's Laws, we have that $A \uparrow B \equiv \neg(A \vee B) \equiv \neg A \wedge \neg B$. If we replace A with $\neg A$ and B with $\neg B$ on both sides, we see that

$$\neg A \uparrow \neg B \equiv A \wedge B$$

In part b, we found that $\neg A \equiv A \uparrow A$. This allows us to rewrite $A \wedge B$ as

$$A \vee B \equiv \boxed{(A \uparrow A) \uparrow (B \uparrow B)}$$