DIAGNOSTIC ASSESSMENT

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
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The purpose of this assessment is to determine whether or not you will get value out of this course. If you can already attempt or answer at least half of these problems, this course may not be for you. One thing to keep in mind — these problems all have relatively short solutions. If you have any questions, don't hesitate to email us at **imt-decal@berkeley.edu**.

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1. What is the last digit of $3^{27} + 4^{27} + 7^{27}$?
2. How many factors does 2400 have? How many factors does it have that are multiples of 24?
3. Prove, using induction, that $\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3$ for $n \in \mathbb{Z}^+$.
4. Find the sum of the coefficients in the expansion of $(xy + 1)^5$.
5. What is the sum of the roots of the polynomial $p(x) = x^{10} - 10x^9 + 7x^2 - 11x + 19$?
6. Prove Pascal's identity; that is, show that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.
7. Suppose I have a family of 7 adults and 5 children, and I can fit 6 other people in my car. How many ways can I pick the 6 people to come in my car, if I have to have at least 2 children come with me?
8. Consider the function $f(x) = x^2$. Give a domain and codomain such that $f(x)$ is
   a. a surjection, but not an injection
   b. a bijection
9. Determine the number of subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ that are not subsets of $\{2, 4\}$.
10. Suppose $A = \{t^2 : t \in \mathbb{N}_0, t \text{ is prime}, t < 5\}$, $B = \{2s : s \in \mathbb{N}_0, s < 10\}$ and $U = \{0, 1, 2, ...20\}$. Determine the following numbers or sets (where A^C represents the complement of set A):
    a. $|A \cap B|$
    b. $(A \cup B)^C$
    c. $(A^C \cap B)^C$

Answers

These solutions are intentionally not comprehensive. If you have no idea as to how we arrived at these answers, don't fret — we will cover these types of problems in-depth and extensively in the course.

1. **4**. The last digit of 3^{27} is 7, the last digit of 4^{27} is 4 and the last digit of 7^{27} is 3, and $7+4+3 = 14$, which has a final digit of 4.
2. **36 and 9, respectively.** The prime factorization of 2400 is $2^5 \cdot 5^2 \cdot 3$, yielding our answer as $(5+1)(2+1)(1+1) = 36$. It has $(2+1)(2+1) = 9$ factors that are multiples of 24, as the prime factorization of $2400/24 = 100$ is $2^2 \cdot 5^2$.
3. Remember that $\sum_{i=1}^{n} = \frac{n(n+1)}{2}$.
   Base case ($k = 1$): $1^2 = 1^3$ clearly holds.
   Induction hypothesis: Assume $\left(sum_{i=1}^{k} i\right)^2 = \sum_{i=1}^{k} i^3$

   Induction step:

$$\sum_{i=1}^{k+1} i^3$$

$$\begin{aligned}
&= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \left(\sum_{i=1}^{k} i\right)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2+4(k+1))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\sum_{i=1}^{k+1} i\right)^2
\end{aligned}$$

   as required.
4. **32**: Substituting $x = 1$ and $y = 1$ yields $(1+1)^5 = 2^5 = 32$.
5. **10**: By Vieta's formulas, the sum of the roots of $p(x)$ is given by the negative of the coefficient on x^9, as $p(x)$ has degree 10.
6. We could also do this with a combinatorial proof, but for the sake of algebraic concreteness:

$$
\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} \\
&= n!\frac{(k+1)!(n-k-1)! + k!(n-k)!}{k!(n-k)!(k+1)!(n-k-1)!} \\
&= n!k!(n-k-1)!\frac{k+1+n-k}{k!(n-k)!(k+1)!(n-k-1)!} \\
&= n!\frac{n+1}{(k+1)!(n-k)!} \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \binom{n+1}{k+1}
\end{aligned}
$$

7. $\binom{7}{4}\binom{5}{2} + \binom{7}{3}\binom{5}{3} + \binom{7}{2}\binom{5}{4} + \binom{7}{1}\binom{5}{5}$: Either we take 2 children and 4 adults, or 3 children and 3 adults, or 4 children and 2 adults, or 5 children and 1 adult.

8. (a) Domain: $\mathbb{R}$, Codomain: $\mathbb{R}_{\geq 0}$

   (b) Domain: $\mathbb{R}_{\geq 0}$, Codomain: $\mathbb{R}_{\geq 0}$

9. **123**: Total subsets - subsets of $\{2, 4\} = 2^7 - 2^2 = 123$.

10. $A = \{4, 9\}, B = \{0, 2, 4, 6, 8, ...20\}$

    (a) **1**: $A \cap B = \{4\}$

    (b) $\{1, 3, 5, 7, 11, 13, 15, 17, 19\}$

    (c) $\{1, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19\}$