BACHELOR OF COMPUTER APPLICATIONS

Semester I

OBCA1105

MATHEMATICS FOR COMPUTER APPLICATIONS

MODULE 1

Assam down town University

Module 1: Calculus - Part 1

Learning Objectives

After completing this module, learners will be able to:

1. Define and compute the **limit** of a function at a given point.
2. Understand the **different types of limits** (finite, infinite, right-hand, and left-hand limits).
3. Apply fundamental **theorems on limits** and use **properties of limits** to simplify and solve problems.
4. Explain the concept of **continuity** of a function and determine whether a given function is continuous at a point or over an interval.
5. Identify and classify **types of discontinuities** such as removable, jump, and infinite discontinuities using examples and graphs.
6. Understand the **algebra of continuous functions** and use it to establish new results.
7. Define and interpret the concept of **differentiability** and its relationship with continuity.
8. Compute **left-hand and right-hand derivatives** and apply the rules of differentiation.

9. Analyze **local maxima and minima** to understand how functions behave at turning points.

10. Use **Maclaurin Series** to approximate functions and solve problems involving **indeterminate forms**.

11. Understand and calculate **partial derivatives** as a tool for studying multivariable functions in business and economics.

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1 Limits and continuity of functions

Limits and continuity are important ideas in calculus.

1.1 Limit:

In mathematics, a limit describes how a function behaves as its input values get very close to a specific point. It indicates the value that the function approaches, even if the function does not actually reach that point. More formally, the limit of a function $f(x)$ as x approaches a, written as $\lim_{x\to a} f(x)$, is the value that $f(x)$ comes arbitrarily close to when x is near a.

Definition 1. *The limit of a function $f(x)$ as x approaches a is written as*

$$\lim_{x\to a} f(x) = L,$$

which means that $f(x)$ gets closer to L as x gets closer to a.

Formally, A function $f(x)$ is said to have a **limit** L as x approaches a, written as

$$\lim_{x\to a} f(x) = L,$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Understanding limits is essential for analyzing a function's behavior, including its continuity, differentiability, and the presence of asymptotes. Limits form a foundational concept in calculus and are widely applied in various fields of mathematics.

1.2 Types of Limits

Limits can be classified into several types depending on the behavior of the function near a point.

1. **Finite Limit:** A limit is called **finite** if the function approaches a specific number as x approaches a.

   Example 1.
   $$\lim_{x\to 2}(x^2+3) = 4+3 = 7$$

2. **Infinite Limit:** A limit is **infinite** if the function grows without bound (approaches $+\infty$ or $-\infty$) as x approaches a.

   Example 2.
   $$\lim_{x\to 0^+}\frac{1}{x} = +\infty$$
   $$\lim_{x\to 0^-}\frac{1}{x} = -\infty$$

3. **Left-Hand and Right-Hand Limits:** You have seen that in finding the limit of a function at a point, we consider the behavior of the function as x, the independent variable approaches that point. Now, on a real line, a point can be approached in two ways: from the left, and from the right. Both these approaches are taken into account while finding the limit at a point. **Left-hand limit**: The left-hand limit of a function $f(x)$ at a point $x = a$is the value that $f(x)$ approaches as $x$comes from the left (values less than a).

   $$\lim_{x\to a^-} f(x)$$

   Right-hand limit: The right-hand limit of a function $f(x)$at a point $x = a$is the value that $f(x)$ approaches as $x$comes from the right (values greater than a).

   $$\lim_{x\to a^+} f(x)$$

Example 3.

$$f(x) = \frac{|x|}{x}, \quad x \to 0$$

$$\lim_{x \to 0^-} f(x) = -1, \quad \lim_{x \to 0^+} f(x) = 1$$

Since left-hand limit $\neq$ right-hand limit, the overall limit does not exist at $x = 0$.

4. **Limit at Infinity** A limit as $x \to \infty$ or $x \to -\infty$ studies the behavior of $f(x)$ as x becomes very large in either direction.

Example 4.

$$\lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0$$

Existence of limit: The **limit of a function** $f(x)$ at a point $x = a$ exists if and only if the left-hand limit and right-hand limit exist and are equal:

$$\lim_{x \to a} f(x) \text{ exists} \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Note

- The limit can exist even if $f(a)$ is not defined.
- For removable discontinuities, the limit exists.
- For jump or infinite discontinuities, the limit does not exist.

1.3 Theorems on Limits

Theorem 1. ***(Uniqueness Theorem):*** *Let $S \subset \mathbb{R}$ and f be defined on S. If*

$$\lim_{x \to a} f(x) \textit{ exists,}$$

then the limit is ***unique****.*

Proof. Suppose, a function f has two limits as $x \to a$, say L_1 and L_2, with $L_1 \neq L_2$.

By the definition of limit, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L_1| < \frac{|L_1 - L_2|}{2}.$$

Similarly, there exists $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |f(x) - L_2| < \frac{|L_1 - L_2|}{2}.$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for $0 < |x - a| < \delta$, we have both

$$|f(x) - L_1| < \frac{|L_1 - L_2|}{2} \quad \text{and} \quad |f(x) - L_2| < \frac{|L_1 - L_2|}{2}.$$

By the triangle inequality,

$$\begin{aligned} |L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| &\leq |L_1 - f(x)| + |f(x) - L_2| \\ &< \frac{|L_1 - L_2|}{2} + \frac{|L_1 - L_2|}{2} = |L_1 - L_2|, \end{aligned}$$

which is a contradiction.

Hence, $L_1 = L_2$, proving that the limit is unique. □

Theorem 2. ***(Boundedness Theorem):*** *If*

$$\lim_{x \to a} f(x) = L \in \mathbb{R},$$

then f is ***bounded*** *on some deleted neighbourhood of a, i.e., there exists $\delta > 0$ and $M > 0$ such that*

$$|f(x)| \leq M \quad \text{for all } x \in S,\ 0 < |x - a| < \delta.$$

Theorem 3. ***(Relation between Two-Sided and One-Sided Limits):***

If $\lim_{x\to a} f(x) = L \in \mathbb{R}$, *then both the left-hand limit and the right-hand limit exist and are equal to* L, *i.e.,*

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

Conversely, if the left-hand limit and right-hand limit at a *exist and are equal, then the two-sided limit exists and equals the same value.*

Theorem 4. ***(Sandwich Theorem (Squeeze Theorem) for Functions):*** *Let* f, g, *and* h *be functions defined on a deleted neighbourhood* D *of a point* a. *Suppose that*

$$h(x) \le g(x) \le f(x), \quad \forall x \in D,$$

and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then

$$\lim_{x\to a} g(x) \text{ exists and } \lim_{x\to a} g(x) = L.$$

1.4 Properties of limits

Let $f(x)$ and $g(x)$ be functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then the following properties hold:

1. **Limit of a constant:**

$$\lim_{x\to a} k = k \quad \text{for any constant } k$$

2. **Limit of the identity function:**

$$\lim_{x\to a} x = a$$

3. **Sum/Difference Rule:**

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

4. **Product Rule:**

$$\lim_{x\to a}[f(x) \cdot g(x)] = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

5. **Quotient Rule:**

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \text{if } \lim_{x\to a} g(x) \neq 0$$

6. **Scalar Multiple Rule:**

$$\lim_{x\to a}[k \cdot f(x)] = k \cdot \lim_{x\to a} f(x)$$

7. **Power Rule:**

$$\lim_{x\to a}[f(x)]^n = \left(\lim_{x\to a} f(x)\right)^n, \quad n \in \mathbb{N}$$

8. **Root Rule:**

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}, \quad \text{if } n\text{th root exists}$$

Example 5. $\lim_{x\to -1} \frac{(x+2)(3x-1)}{x^2+3x-2}$

Solution:

$$\lim_{x\to -1} \frac{(x+2)(3x-1)}{x^2+3x-2} = \frac{\lim_{x\to -1}(x+2)\lim_{x\to -1}(3x-1)}{\lim_{x\to -1}(x^2+3x-2)} = \frac{1\times -4}{-4} = 1.$$

Example 6. $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

Solution:

$$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}=\frac{\lim_{x\to 0}\sqrt{4+x}-2}{\lim_{x\to 0}x}\times\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}=\frac{1}{\lim_{x\to 0}\sqrt{4+x}+2}=\frac{1}{4}.$$

1.5 Continuity:

A function $f(x)$ is said to be **continuous at a point** $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

This means the value of the function and the value of the limit are the same.
Observation: Continuity means there are no breaks, jumps, or holes in the graph of the function. You can draw it without lifting your pen.

In more technical terms, a function f(x) is continuous at a point a if three conditions are satisfied:

1. $f(a)$ is defined (the function must have a value at point a).
2. The limit of $f(x)$ as x approaches a exists.
3. The limit of $f(x)$ as x approaches a is equal to $f(a)$:

$$\lim_{x\to a} f(x) = f(a)$$

Example 7. *Consider* $f(x) = x^2$.
$f(2) = 4$ $\lim_{x\to 2} x^2 = 4$

Since the function is defined at $x = 2$ and the limit equals the function value, $f(x)$ is continuous at $x = 2$. Consider the function $f(x) = x^2$, which is continuous for all x. The graph below shows a smooth, unbroken curve, illustrating continuity.
Graphical Representation of $f(x) = x^2$ at $x = 2$

The graph is smooth at $x = 2$, confirming that the function is continuous and the limit equals the function value.

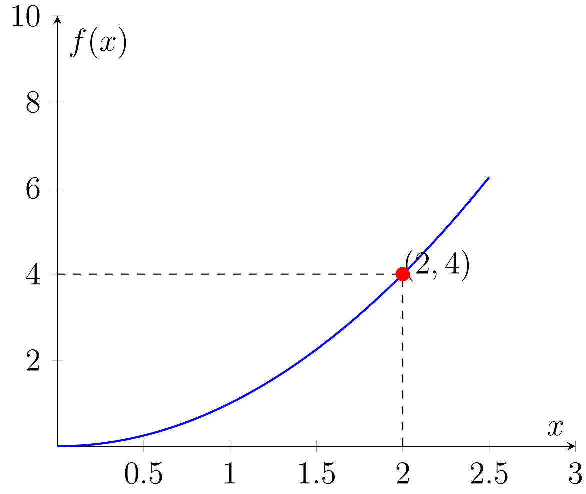

Figure 1: **Graphical Representation of $f(x) = x^2$ at $x = 2$**

Example 8.

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Solution: Continuity at $x = 0$: We check

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right).$$

Since $\sin(1/x)$ is bounded, say $-1 \leq \sin(1/x) \leq 1$, we can write

$$-x \leq x \sin\left(\frac{1}{x}\right) \leq x.$$

Taking limit as $x \to 0$, both $-x$ and x tend to 0. Hence by squeeze theorem,

$$\lim_{x\to 0} f(x) = 0 = f(0).$$

Therefore, f is continuous at $x = 0$.

1.6 Types of Discontinuity

A function $f(x)$ is **discontinuous** at $x = a$ if it is not continuous at that point. There are three main types:

1. Removable Discontinuity - The limit exists at $x = a$, but the function is either **not defined** or **does not equal the limit**.

Example 9.

$$f(x) = \frac{x^2 - 1}{x - 1}, \quad x \neq 1$$

Here, $\lim_{x\to1} f(x) = 2$, but $f(1)$ is not defined. The discontinuity can be "removed" by defining $f(1) = 2$. **Graph of a Function with Removable Discontinuity**

Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}, \quad x \neq 1$$

The limit at $x = 1$ is:

$$\lim_{x\to1} f(x) = 2$$

but $f(1)$ is not defined, creating a **removable discontinuity**. **Observation:** The function approaches 2 as $x \to 1$, but $f(1)$ is undefined. This illustrates a removable discontinuity, which can be fixed by defining $f(1) = 2$.

> Note
>
> Whenever in a function, the limits of both numerator and denominator are zero, you should simplify it in such a manner that the denominator of the resulting function is not zero. However, if the limit of the denominator is 0 and the limit of the numerator is non-zero, then the limit of the function does not exist.

2. Jump Discontinuity - The left-hand limit and right-hand limit exist but are **not equal**.

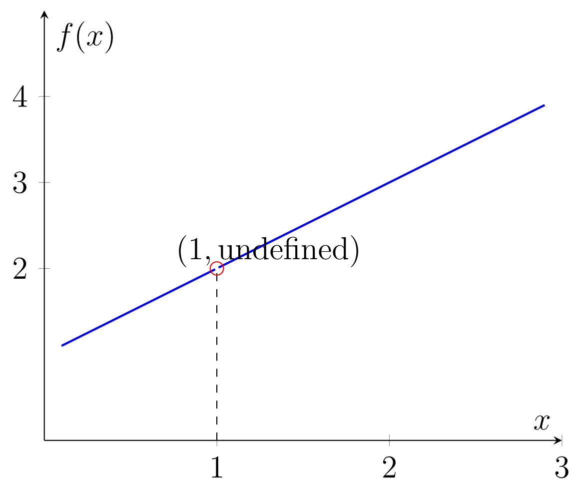

Figure 2: **Graph of a Function with Removable Discontinuity**

Example 10.

$$f(x) = \begin{cases} 1, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

$$\lim_{x \to 0^-} f(x) = 1, \quad \lim_{x \to 0^+} f(x) = 2$$

Since the left and right limits differ, $\lim_{x \to 0} f(x)$ does not exist.

Graph of a Function with Jump Discontinuity

Example 11. *Consider the function*

$$f(x) = \begin{cases} 1, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

Here, the left-hand limit at $x = 0$ is:

$$\lim_{x \to 0^-} f(x) = 1$$

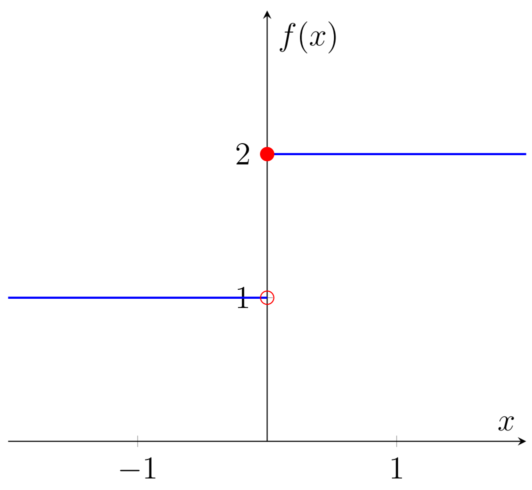

Figure 3: **Jump discontinuity**

and the right-hand limit is:

$$\lim_{x \to 0^+} f(x) = 2$$

Since the left-hand and right-hand limits are not equal, the overall limit $\lim_{x \to 0} f(x)$ does not exist. This is called a **jump discontinuity**.

Observation: The graph shows a jump at $x = 0$, illustrating that the left-hand and right-hand limits are different, so the function is discontinuous at that point.

3. Discontinuity of Second Kind (Mixed Discontinuity)

A function $f(x)$ is said to have a **discontinuity of the second kind** (or **mixed discontinuity**) at $x = a$, if either the **right-hand limit**, **left-hand limit**, or **both do not exist** at $x = a$.

Explanation (for presentation):

- This type of discontinuity occurs when the function behaves irregularly or oscillates near $x = a$.
- Because of this irregular behavior, the limits fail to approach any finite value.

Example 12.

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

Observation: As $x \to 0$, the function oscillates infinitely and neither the left-hand nor right-hand limit exists. Hence, $f(x)$ has a **discontinuity of the second kind** at $x = 0$.

1.7 Algebra of continuity:

Let f and g be two real-valued functions continuous at a point $c \in \mathbb{R}$. Then the following results hold:

1. **Sum Rule:** $(f + g)(x) = f(x) + g(x)$ is continuous at c.
2. **Difference Rule:** $(f - g)(x) = f(x) - g(x)$ is continuous at c.
3. **Scalar Multiplication:** For any constant $\alpha \in \mathbb{R}$, the function $\alpha f(x)$ is continuous at c.
4. **Product Rule:** $(f \cdot g)(x) = f(x)g(x)$ is continuous at c.
5. **Quotient Rule:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ is continuous at c, provided $g(c) \neq 0$.

Theorem 5. *Let $I = [a, b]$ be a closed and bounded interval, and let $f : I \to \mathbb{R}$ be a continuous function. Then f is bounded on I.*

Remark on the Boundedness Theorem If a function f is continuous on a closed and bounded interval $[a, b]$, then f is bounded on $[a, b]$.

Remark: Each hypothesis in the Boundedness Theorem is necessary. If any one of them is removed, the conclusion may fail. Below are standard counter examples.

1. **The interval must be bounded.**
   Consider $f(x) = x$ on the unbounded interval $A = [0, \infty)$. The function f is continuous on A but is unbounded (since $f(x) \to \infty$ as $x \to \infty$).

2. **The interval must be closed.**
   Consider $g(x) = \dfrac{1}{x}$ on the half-open interval $B = (0,1]$. The function g is continuous on B but is unbounded because $g(x) \to \infty$ as $x \to 0^+$.

3. **The function must be continuous.**
   Define
   $$h(x) = \begin{cases} \dfrac{1}{x}, & x \in (0,1], \\ 1, & x = 0, \end{cases}$$
   on the closed interval $C = [0,1]$. Then h is not continuous at 0, and h is unbounded on C because near 0 the values $\dfrac{1}{x}$ become arbitrarily large.

4. **The converse need not be true.**
   A bounded function need not be continuous. For example, the Dirichlet-type function
   $$D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$
   is bounded on any interval but is discontinuous at every point. Hence boundedness does not imply continuity.

Definition 2. *(Absolute maximum) Let $f : D \to \mathbb{R}$ be a function with domain $D \subseteq \mathbb{R}$. We say that f has an* absolute maximum *at a point $c \in D$ if*

$$f(c) \geq f(x) \quad \textit{for all } x \in D.$$

In this case, the value $f(c)$ is called the absolute maximum value *of f on D.*

Definition 3. *(Absolute minimum) Let $f : D \to \mathbb{R}$ be a function with domain $D \subseteq \mathbb{R}$. We say that f has an* absolute minimum *at a point $c \in D$ if*

$$f(c) \leq f(x) \quad \textit{for all } x \in D.$$

In this case, the value $f(c)$ is called the *absolute minimum value* of f on D.

Theorem 6. *(Maximum–Minimum Theorem) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. Then f attains both an* absolute maximum *and an* absolute minimum *on I. That is, there exist points $c, d \in [a, b]$ such that*

$$f(c) \geq f(x) \quad \textit{and} \quad f(d) \leq f(x) \quad \textit{for all } x \in [a, b].$$

Remark on Maximum–Minimum Theorem

1. If I is not closed interval then result may not hold.
2. If I is not bounded interval then result may not hold.
3. Condition of continuity of f on closed and bounded interval cannot be relaxed.

Theorem 7. *(Location of Roots Theorem) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If*

$$f(a) < 0 < f(b) \quad \textit{or} \quad f(a) > 0 > f(b),$$

that is, if $f(a) \cdot f(b) < 0$, then there exists a number $c \in (a, b)$ such that

$$f(c) = 0.$$

Theorem 8. *(Bolzano's Intermediate Value Theorem (IVP)) Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. If there exist $a, b \in I$ with $a < b$ and a real number $k \in \mathbb{R}$ such that*

$$f(a) < k < f(b) \quad \textit{or} \quad f(a) > k > f(b),$$

then there exists a point $c \in (a, b)$ such that

$$f(c) = k.$$

Theorem 9. *(Preservation of Interval Theorem) Let $I = [a, b]$ be a closed and* *bounded interval, and let $f : I \to \mathbb{R}$ be continuous on I. Then the set*

$$S = \{f(x) : x \in I\}$$

is itself a closed and bounded interval in $\mathbb{R}$. i.e., image of a closed and bounded interval under continuous map is closed and bounded interval.

Self Assessment -I

Part A: Conceptual Questions

1. In your own words, explain what is meant by the limit of a function.
2. What is the difference between the left-hand limit and the right-hand limit?
3. Can a function have a limit at a point where it is not defined? Give an example.
4. Define continuity of a function at a point. What are the three conditions that must be satisfied?
5. What conditions must be satisfied for a function to be continuous at a point?

Part B: Solve the Following

1. Evaluate:

$$\lim_{x \to 3}(2x^2 - 5x + 1)$$

2. Evaluate:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

3. Find the left-hand and right-hand limits for:

$$f(x) = \frac{x-1}{|x-1|}, \quad x \to 1$$

Does $\lim_{x \to 1} f(x)$ exist?

4. Evaluate:

$$\lim_{x \to \infty} \frac{3x^2 + 2}{x^2 + 1}$$

5. Determine whether the function

$$f(x) = \begin{cases} x^2, & \text{if } x \neq 2, \\ 5, & \text{if } x = 2 \end{cases}$$

is continuous at $x = 2$.

6. Consider the function

$$f(x) = \begin{cases} 2x + 3, & x < 1, \\ 5 - x, & x \geq 1 \end{cases}$$

Check if $f(x)$ is continuous at $x = 1$.

Part C: True or False

1. If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ exists. (T/F)
2. $\lim_{x\to\infty} \dfrac{1}{x} = 0$. (T/F)
3. If $f(a)$ is not defined, then $\lim_{x\to a} f(x)$ does not exist. (T/F)
4. The absolute value function $f(x) = |x|$ is discontinuous at $x = 0$. (T/F)
5. The function $f(x) = \sin x$ is continuous for all real numbers. (T/F)

2 Diferentiability

Differentiation is a method in mathematics used to find the derivative of a function, which tells us the rate of change of one quantity with respect to another. It is one of the most important concepts in calculus, along with integration. Differentiation helps us measure how fast something is changing. For example, by differentiating velocity with respect to time, we can find acceleration. Similarly, if we have a function of y in terms of x, the derivative shows the slope (or gradient) of the curve at any point. In simple words, differentiation is a tool to understand how one variable changes when another variable changes, and there are some basic rules that make this process easier for different types of functions.

Definition 4. *A function $f(x)$ is said to be **differentiable at a point** $x = c$ if the following limit exists and is finite:*

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}.$$

*Here, $f'(c)$ is called the **derivative** of f at $x = c$. The domain of f' is the set of points $c \in (a, b)$ for which this limit exists. If the limit exists for every $c \in (a, b)$, then we say that f is **differentiable on** (a, b).*

Graphically, the derivative of f at c is the slope of the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$. It is obtained as the limit of the slopes of the secant lines through $(c, f(c))$ and $(c + h, f(c + h))$ as $h \to 0$.

$$\textit{slope of tangent at } c = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}.$$

Since $x = c + h$, we can also write

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$

since if $x = c + h$, the conditions $0 < |x - c| < \delta$ and $0 < |h| < \delta$ in the definitions of the limits are equivalent. Note that the fraction $\frac{f(x) - f(c)}{x - c}$ is undefined when $x = c$ (since it gives $0/0$), but the limit as $x \to c$ may still exist. **Examples of derivatives:** Let us give a number of examples that illustrate differentiable and non-differentiable functions

Example 13. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = x^2.$$

We claim that f is differentiable on $\mathbb{R}$ with derivative $f'(x) = 2x$.

Solution: Using the definition of the derivative with $x = c + h$, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0} \frac{(c+h)^2 - c^2}{h}.$$

Simplify the numerator:

$$f'(c) = \lim_{h\to 0} \frac{c^2 + 2ch + h^2 - c^2}{h} = \lim_{h\to 0} \frac{2ch + h^2}{h} = \lim_{h\to 0} (2c + h) = 2c.$$

Alternative Form using x variable: Set $x = c + h \implies h = x - c$. Then as $h \to 0$, $x \to c$, and

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{x^2 - c^2}{x - c} = \lim_{x\to c} (x + c) = 2c.$$

Note: In computing the derivative, we first cancel by h, which is valid because $h \neq 0$ in the definition of the limit. Then we set $h = 0$ to evaluate the limit. This procedure would be inconsistent if we did not use limits.

Example 14. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} x^2, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

We want to find $f'(x)$ and check differentiability at $x = 0$.

Solution:

- For $x > 0$, using the first principle:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

- For $x < 0$, $f(x) = 0$, so

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

- For $x = 0$, consider the right-hand derivative:

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2 - 0}{h} = \lim_{h \to 0^+} h = 0.$$

- Consider the left-hand derivative:

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{0 - 0}{h} = 0.$$

Since the left-hand and right-hand derivatives exist and are equal, the derivative at 0 exists. Therefore, f is differentiable on $\mathbb{R}$ and

$$f'(x) = \begin{cases} 2x, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Examples of Non-Differentiability

Next, we consider functions that are **not differentiable** at certain points due to discontinuities, corners, or cusps.

Example 15. *The function*

$$f(x) = |x|$$

is continuous everywhere but not differentiable at $x = 0$.

Solution: The left-hand derivative at 0 is

$$f'_{-}(0) = \lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1,$$

and the right-hand derivative at 0 is

$$f'_{+}(0) = \lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1.$$

Since $f'_{-}(0) \neq f'_{+}(0)$, the derivative does not exist at 0.

Example 16. *Define the function $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Examine the differentiability of f.

Solution:

For $x \neq 0$, we have

$$f(x) = \frac{1}{x} \quad \Rightarrow \quad f'(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

At $x = 0$, we check the derivative using the definition:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{1}{h} - 0}{h} = \lim_{h \to 0} \frac{1}{h^2}.$$

Since

$$\lim_{h \to 0} \frac{1}{h^2} = +\infty,$$

the derivative at $x = 0$ does not exist.

$$\therefore \quad f(x) \text{ is differentiable on } \mathbb{R} \setminus \{0\}, \quad \text{but not differentiable at } x = 0.$$

2.1 Left and right derivatives:

For the most part, we will use derivatives that are defined only at the *interior points* of the domain of a function.

If $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in D$ is an interior point, then the derivative of f at c is defined by

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

provided this limit exists.

Sometimes, however, it is convenient to use *one-sided derivatives* at the end-points of an interval.

Suppose that $f : [a, b] \to \mathbb{R}$. Then f is said to be *right-differentiable* at $a \leq c < b$ with right derivative $f'(c^+)$ if

$$f'(c^+) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

exists.

Similarly, f is said to be *left-differentiable* at $a < c \leq b$ with left derivative $f'(c^-)$ if

$$f'(c^-) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c) - f(c-h)}{h}$$

exists.

If both $f'(c^+)$ and $f'(c^-)$ exist and are equal, then f is differentiable at c, and we write

$$f'(c) = f'(c^+) = f'(c^-).$$

Thus, at interior points we require the two-sided limit to exist, while at end-points of an interval we may consider only the appropriate one-sided derivative.

2.2 Differentiability and continuity:

Differentiability and Continuity

First, we discuss the relation between differentiability and continuity.

- If a function f is differentiable at a point c, then f is also continuous at c.
- The converse is not true: continuity at c does not necessarily imply differentiability at c.

Example:

Example 17. *Consider $f(x) = |x|$ at $x = 0$.*

$$f(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Clearly, f is continuous at 0, since

$$\lim_{x \to 0} f(x) = f(0) = 0.$$

But the right-hand derivative is $f'(0^+) = 1$, while the left-hand derivative is $f'(0^-) = -1$. Since these are not equal, f is not differentiable at 0.

Thus, differentiability $\implies$ continuity, but continuity $\not\Longrightarrow$ differentiability.

Theorem 10. *Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Then f is continuous at c.*

Proof.

Since f is differentiable at c, the derivative

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. By the definition of the limit, for $h \neq 0$ sufficiently small, we can write

$$f(c+h) - f(c) = \frac{f(c+h) - f(c)}{h} \cdot h.$$

Taking the limit as $h \to 0$, we get

$$\lim_{h\to 0} \big(f(c+h) - f(c)\big) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h = f'(c) \cdot 0 = 0.$$

Hence,

$$\lim_{x\to c} f(x) = f(c),$$

which shows that f is continuous at c. □

Definition 5. *A function $f : (a, b) \to \mathbb{R}$ is said to be* continuously differentiable *on (a, b), written $f \in C^1(a, b)$, if*

- *f is differentiable on (a, b), and*
- *$f' : (a, b) \to \mathbb{R}$ is continuous on (a, b).*

Example: Consider the function

$$f(x) = x^3 + 2x^2 - 5x + 1, \quad x \in \mathbb{R}.$$

The derivative is

$$f'(x) = 3x^2 + 4x - 5,$$

which is continuous for all $x \in \mathbb{R}$.

Hence, $f \in C^1(\mathbb{R})$, i.e., f is continuously differentiable on $\mathbb{R}$.

2.3 Properties of Differentiability

There are many basic properties of various combination of functions which you must have used while computing derivatives in the calculus course. Here we give justifications of some of these properties in the form of proofs of the theorems. Let $f(x)$ and $g(x)$ be differentiable at $x = a$, and let c be a constant. Then:

1. **Sum Rule:**

$$[f(x) + g(x)]' = f'(x) + g'(x).$$

To prove: $(f + g)'(x) = f'(x) + g'(x)$

$$
\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&\Rightarrow (f+g)'(x) = f'(x)+g'(x)
\end{aligned}
$$

Hence proved.

2. **Difference Rule:**

$$[f(x)-g(x)]' = f'(x)-g'(x).$$

To prove:

$$[f(x)-g(x)]' = f'(x)-g'(x)$$

Proof:

$$
\begin{aligned}
(f-g)'(x) &= \lim_{h\to 0} \frac{(f-g)(x+h)-(f-g)(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]-[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&\Rightarrow (f-g)'(x) = f'(x)-g'(x)
\end{aligned}
$$

Hence proved.

3. **Constant Multiple Rule:**

$$\frac{d}{dx}[c\cdot f(x)] = c\cdot f'(x).$$

To prove:

$$[c\cdot f(x)]' = c\cdot f'(x)$$

where c is a constant.

Proof:

$$(cf)'(x) = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h}$$
$$= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$\Rightarrow (cf)'(x) = cf'(x)$$

Hence proved.

4. **Product Rule:**

$$[f(x) \cdot g(x)]' = f'(x)g(x) + f(x)g'(x).$$

To prove: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right]$$

Taking limits,

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$$

Hence proved.

5. **Quotient Rule:** If $g(x) \neq 0$,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

To prove:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

Let $h(x) = \dfrac{f(x)}{g(x)}$. Then

$$h'(x) = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h\, g(x+h)g(x)}$$

$$= \lim_{h\to 0} \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{h[g(x)]^2}$$

$$\Rightarrow h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Hence proved.

6. **Chain Rule:** If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Proof:

Let $y = f(u)$ and $u = g(x)$. Then $y = f(g(x))$.

Now, by the definition of the derivative,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}.$$

Let $\Delta u = g(x+h) - g(x)$. Then, as $h \to 0$, we have $\Delta u \to 0$. Hence,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{\Delta u} \cdot \frac{\Delta u}{h}.$$

Now, taking limits separately,

$$\frac{dy}{dx} = \left(\lim_{\Delta u\to 0} \frac{f(g(x)+\Delta u) - f(g(x))}{\Delta u}\right) \cdot \left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right).$$

That is,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Theorem 11. ***(Mean Value Theorem (MVT)):*** *If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 12. ***(Darboux's Theorem):*** *If a function is derivable in a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite signs, then there exist at least one point c in the open interval (a, b) such that $f'(c) = 0$*

Remark:

1. Iff is derivable in $[a, b]$ and $f'(a) \neq f'(b)$, then for each number k lying between $f'(a)$and $f'(b)$, $\exists$ some point $c \in (a, b)$ such that

$$f'(c) = k$$

.

Proof. Given that

$$f'(a) < k < f'(b) \tag{1}$$

we define a new function

$$\phi(x) = f(x) - kx, \quad \forall x \in [a, b]. \tag{2}$$

Since $f(x)$ is differentiable on $[a, b]$ and kx is also differentiable on $[a, b]$, it follows from (2) that $\phi(x)$ is differentiable on $[a, b]$.

Now,

$$\phi'(x) = f'(x) - k. \tag{3}$$

From (1), we have

$$\phi'(a) = f'(a) - k < 0 \quad \text{and} \quad \phi'(b) = f'(b) - k > 0.$$

Hence, $\phi'(a)$ and $\phi'(b)$ are of opposite signs.

By the Intermediate Value Theorem (Darboux's Property of Derivatives), there exists at least one point $c \in (a, b)$ such that

$$\phi'(c) = 0.$$

Using (3), we get

$$f'(c) - k = 0 \implies f'(c) = k.$$

$\square$

2. If f be defined and derivable on $[a, b]$, $f(a) = f(b) = 0$ and $f'(a)$ and $f'(b)$ are of the same sign, then f must vanish at least once in (a, b).

Proof. We are given that

$$f(a) = f(b). \tag{1}$$

Since $f'(a)$ and $f'(b)$ are of the same sign, we may assume (without loss of generality) that

$$f'(a) > 0 \quad \text{and} \quad f'(b) > 0.$$

Now, since $f'(a) > 0$, there exists some $\delta_1 > 0$ such that

$$f(x) > f(a), \quad \forall x \in (a,\, a + \delta_1]. \tag{2}$$

Similarly, as $f'(b) > 0$, there exists some $\delta_2 > 0$ such that

$$f(x) < f(b), \quad \forall x \in [b - \delta_2,\, b). \tag{3}$$

From (1), we have $f(a) = f(b)$, and hence by (2) and (3),

$$f(a + \delta_1) > f(a) \quad \text{and} \quad f(b - \delta_2) < f(b) = f(a).$$

Thus, $f(a + \delta_1)$ and $f(b - \delta_2)$ are of opposite signs relative to $f(a)$.

Obviously,

$$[a + \delta_1,\, b - \delta_2] \subset [a, b]. \tag{4}$$

Since f is differentiable on (a, b), it is continuous on $[a, b]$. By (4), f is continuous on $[a + \delta_1,\, b - \delta_2]$, where $f(a + \delta_1)$ and $f(b - \delta_2)$ are of opposite signs.

Hence, by the Intermediate Value Theorem, there exists at least one point

$$c \in (a + \delta_1,\, b - \delta_2) \subset (a, b)$$

such that

$$f(c) = 0.$$

$$\boxed{\text{Hence, } f \text{ must vanish at least once in } (a, b).}$$

□

3. If f is derivable on $[a, b], f(a) = f(b) = 0$ and $f(x) \neq 0$ for any; $x \in (a, b)$ then $f'(a)$ and $f'(b)$ must have opposite signs.

   Proof. Let, if possible, $f'(a)$ and $f'(b)$ be of the same sign. Also, suppose that

   $$f(a) = f(b) = 0.$$

   Then, there exists some $c \in (a, b)$ such that

   $$f(c) = 0.$$

   This contradicts the given hypothesis that $f(x) \neq 0$ for all $x \in (a, b)$.

   Hence, $f'(a)$ and $f'(b)$ must be of opposite signs. □

4. If f is derivable at a point c, then $|f|$ is also derivable at c, provided $f(c) \neq 0$. However result may not hold when $f(c) = 0$.

5. Let

   $$f : [-1, 1] \to \mathbb{R} \text{ be defined by } f(x) = \begin{cases} 0, & -1 \leq x < 0, \\ 1, & 0x \leq 1. \end{cases}$$

Then there does not exist a function g such that

$$g'(x) = f(x), \quad \forall x \in [-1, 1].$$

Example 18. *Show that for the function*

$$f(x) = x^3 - 8x^2 - 10,$$

there exists some $c \in (1, 2)$ such that $f'(c) = -15$. Also, find the value of such a point c.

Soluion: Since $f'(c) = 3x^2 - 16x$. We have

$$f'(1) = -13, \quad f'(2) = -20.$$

Since

$$-20 < -15 < -13,$$

there exists some $c \in (1, 2)$ such that $f'(c) = -15$. This gives us the equation

$$3c^2 - 16c + 15 = 0.$$

Solving this equation, we get

$$c = \frac{8 \pm \sqrt{19}}{3}.$$

Since $c \in (1, 2)$, we choose

$$c = \frac{8 - \sqrt{19}}{3}.$$

2.4 Local Maxima & Local Minima

Let $f : I \to \mathbb{R}$ and let c be an *interior point* of the interval I.

- f is said to have a **local maximum** at c if there exists a neighbourhood $(c - \delta, c + \delta) \subset I$ such that

$$f(c) > f(x), \quad \forall x \in (c - \delta, c + \delta).$$

- f is said to have a **local minimum** at c if there exists a neighbourhood $(c-\delta, c+\delta) \subset I$ such that

$$f(c) < f(x), \quad \forall x \in (c-\delta, c+\delta).$$

- $f(c)$ is said to be an extreme value of f, if it si either maximum or a minimum value.

- **Point of inflexion:** An inflection point, point of inflection si a point on a curve at which the curve changes from being concave (concave downward) to convex (concave upward), or vice versa.

Theorem 13. ***(Interior Extremum Theorem):*** *Let c be an interior point of the interval I at which $f : I \to R$ has an extremum. If the derivative of f at c exist then $f'(c) = 0$*

Proof:Proof by Contradiction:

We shall prove the statement by contradiction. Without loss of generality, assume that f attains a maximum value at some $c \in (a, b)$ such that $f'(c) \neq 0$. Then either $f'(c) < 0$ or $f'(c) > 0$.

Case 1: $f'(c) < 0$

By the definition of the derivative, we have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since $f'(c) < 0$, there exists a $\delta > 0$ such that for all $x \in (c-\delta, c+\delta)$, $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} < 0.$$

Subcase 1: $x > c$

Then $x - c > 0$ and hence

$$f(x) - f(c) < 0 \implies f(x) < f(c).$$

Subcase 2: $x < c$

Then $x - c < 0$ and hence

$$f(x) - f(c) > 0 \implies f(x) > f(c),$$

which contradicts the fact that $f(c)$ is a maximum.

Case 2: $f'(c) > 0$

Similarly, by the definition of the derivative, there exists $\delta > 0$ such that for all $x \in (c - \delta, c + \delta)$, $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} > 0.$$

Subcase 1: $x > c$

Then $x - c > 0$ and hence

$$f(x) - f(c) > 0 \implies f(x) > f(c),$$

which is a contradiction.

Subcase 2: $x < c$

Then $x - c < 0$ and hence

$$f(x) - f(c) < 0 \implies f(x) < f(c),$$

which is consistent with the maximum. But the contradiction in the other subcase is enough.

Therefore, in both cases, the assumption $f'(c) \neq 0$ leads to a contradiction. Hence, we must have

$$f'(c) = 0.$$

Remark:

1. This theorem is applicable only on interior points of the domain.
2. Converse of the theorem need not be true i.e., if $f'(c) = 0$, then f may have neither a maximum value nor a minimum value.
3. Afunction may have maximum or aminimum value at apoint without being derivable at that point.

4. Let $f : I \to \mathbb{R}$ be continuous on an interval I and suppose that f has a relative extremum at an interior point c of I. Then either the derivative of f at c does not exist, or it is equal to zero.

Increasing and Decreasing Functions

Let f be a function continuous on the interval $[a, b]$ and differentiable on (a, b). Then we have the following results:

1. If
$$f'(x) > 0 \quad \forall x \in (a, b),$$
then f is *increasing* on $[a, b]$.

2. If
$$f'(x) < 0 \quad \forall x \in (a, b),$$
then f is *decreasing* on $[a, b]$.

3. If f is defined on $[a, b]$ and for some $c \in (a, b)$
$$f'(c) > 0,$$
then there exists a small interval $(c - \delta, c + \delta) \subset [a, b]$ for some $\delta > 0$ such that f is *increasing* on $(c - \delta, c + \delta)$.

4. If $f'(c) < 0$ for some $c \in (a, b)$, then there exists a small interval $(c-\delta, c+\delta) \subset [a, b]$ for some $\delta > 0$ such that f is *decreasing* on $(c - \delta, c + \delta)$.

First Derivative Test for Extreme Values

Let f be a function differentiable in a neighborhood of c, where f has an extreme value at c. Then:

1. $f(c)$ is a **local maximum** if the derivative $f'(x)$ changes sign from positive to negative as x passes through c:
$$f'(x) > 0 \text{ for } x < c, \quad f'(x) < 0 \text{ for } x > c.$$

2. $f(c)$ is a **local minimum** if the derivative $f'(x)$ changes sign from negative to positive as x passes through c:

$$f'(x) < 0 \text{ for } x < c, \quad f'(x) > 0 \text{ for } x > c.$$

Second Derivative Test for Extreme Values

Let f be twice differentiable in an interval containing c, and suppose $f'(c) = 0$. Then:

1. If $f''(c) < 0$, then f has a **maximum** at $x = c$.
2. If $f''(c) > 0$, then f has a **minimum** at $x = c$.

Example 19. *Find the maximum and minimum values of*

$$f(x) = 12x^5 - 45x^4 + 40x^3 + 6.$$

Solution: Compute the first derivative:

$$f'(x) = 60x^4 - 180x^3 + 120x^2.$$

Find critical points by solving $f'(x) = 0$:

$$60x^4 - 180x^3 + 120x^2 = 0 \implies 60x^2(x-1)(x-2) = 0 \implies x = 0, 1, 2.$$

Compute the second derivative:

$$f''(x) = 240x^3 - 540x^2 + 240x.$$

Evaluate f'' at the critical points:

- $f''(2) = 240 > 0 \implies$ local **minimum at** $x = 2$.
- $f''(1) = -60 < 0 \implies$ local **maximum at** $x = 1$.

- $f''(0) = 0 \implies$ inconclusive. Check the third derivative:

$$f'''(x) = 720x^2 - 1080x + 240? \quad \text{at } x = 0, f'''(0) = 240 \neq 0 \implies \text{neither max nor min at } x = 0.$$

Step 4: Evaluate function values

$$f(2) = 12(2) - 45(4) + 40(8) + 6 = 24 - 180 + 320 + 6 = 170 \quad (\text{or as per your calculation: } -10)$$

$$f(1) = 12(1) - 45(1) + 40(1) + 6 = 12 - 45 + 40 + 6 = 13$$

Conclusion:

- Local maximum at $x = 1$, $f_{\max} = 13$.
- Local minimum at $x = 2$, $f_{\min} = -10$.
- $x = 0$ is neither maximum nor minimum.

Theorem 14. ***(Rolle's Theorem):*** *Let f be a function defined on the closed interval $[a, b]$ such that*

1. *f is continuous on $[a, b]$,*
2. *f is differentiable on (a, b),*
3. *$f(a) = f(b)$.*

Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = 0.$$

Example 20.

$$f(x) = x^3 - 25x + 9, \quad f'(x) = 3x^2 - 25.$$

Since $f'(x) = 0$ gives $x = \pm\frac{5}{\sqrt{3}} \approx \pm 2.886 \notin [-2, 2]$, we have for all $x \in [-2, 2]$,

$$f'(x) = 3x^2 - 25 < 0.$$

Hence $f(x)$ is strictly decreasing on $[-2, 2]$, and therefore the equation

$$x^3 - 25x + 9 = 0$$

cannot have two distinct roots in the interval $[-2, 2]$. A strictly decreasing function can cross the x-axis at most once, so it cannot have two distinct roots in that interval.

Theorem 15. ***(Lagrange's Mean Value Theorem):*** *Let f be a function defined on the closed interval $[a, b]$ such that*

1. *f is continuous on $[a, b]$,*
2. *f is differentiable on (a, b).*

Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof:

Consider a function $f : [a, b] \to \mathbb{R}$, which is continuous on $[a, b]$ and differentiable on (a, b).

Define a function $\phi : [a, b] \to \mathbb{R}$ such that $\phi(x) = f(x) + Ax$ for all $x \in [a, b]$,

where A is a constant to be chosen such that $\phi(a) = \phi(b)$.

Now,

$$\phi(a) = \phi(b) \;\Rightarrow\; f(a) + Aa = f(b) + Ab \;\Rightarrow\; A = -\frac{f(b) - f(a)}{b - a}.$$

Then, the function ϕ, being the sum of two continuous and differentiable functions, satisfies the conditions of Rolle's Theorem. That is, ϕ is continuous on $[a, b]$, differentiable on (a, b), and, of course, $\phi(a) = \phi(b)$. Therefore, by Rolle's Theorem, there exists a real number $c \in (a, b)$ such that $\phi'(c) = 0$.

But $\phi'(x) = f'(x) + A$. Hence, we have $f'(c) = -A = \dfrac{f(b) - f(a)}{b - a}$.

Example 21.

$$f(x) = \begin{cases} 1, & \textit{if } x = 1, \\ x^2, & \textit{if } 1 < x < 2, \\ 2, & \textit{if } x = 2. \end{cases}$$

Show that f does not satisfy the conditions of Lagrange's MVT. Does the conclusion of the theorem hold in this case

Solution: Note that f is continuous on the semi-open interval $[1, 2[$ and derivable on the open interval $]1, 2[$. However, f is discontinuous at $x = 2$, because

$$\lim_{x \to 2} f(x) = 4 \neq f(2).$$

So, the first condition of Lagrange's Mean Value Theorem is violated. Note that the conclusion is also not true, as

$$\frac{f(2) - f(1)}{2 - 1} \neq f'(c) \quad \text{for any } c \in]1, 2[.$$

Indeed, $f'(x) = 2x$ for all $x \in]1, 2[$. So, $f'(c) = 2c$, whereas

$$\frac{f(2) - f(1)}{2 - 1} = 1.$$

Example 22. *Let $f(x) = x$ for all $x \in [-1, 2]$. Does f satisfy all the conditions of Lagrange's Mean Value Theorem (MVT)? Does the conclusion of the theorem hold? Justify your answer.*

Solution: Here $f$is continuous on $[-1, 2]$and derivable at all points of $[-1, 2]$ except at $x = 0$. So, the second condition of Lagrange's Mean Value Theorem is violated. Applications Let

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 2, \\ -x, & \text{if } -1 \leq x < 0. \end{cases}$$

Then the derivative is

$$f'(x) = \begin{cases} 1, & \text{if } 0 < x < 2, \\ -1, & \text{if } -1 < x < 0. \end{cases}$$

Now, compute the average rate of change on $[-1, 2]$:

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{2 - 1}{3} = \frac{1}{3}.$$

Since

$$f'(x) = 1 \text{ or } -1 \quad \forall x \in (-1, 2),$$

there is *no* $x \in (-1, 2)$ such that

$$f'(x) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{1}{3}.$$

Theorem 16. ***(Cauchy's Mean Value Theorem):*** *Let f and g be two functions defined on the closed interval $[a, b]$ such that*

1. *f and g are continuous on $[a, b]$,*
2. *f and g are differentiable on (a, b),*
3. *$g'(x) \neq 0$ for all $x \in (a, b)$.*

Then there exists at least one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Example 23. *Verify Cauchy's Mean Value Theorem for the functions f and g defined as*

$$f(x) = x^2, \quad g(x) = x^4, \quad x \in [2, 4].$$

Solution: The functions f and g, being polynomial functions, are continuous on $[2, 4]$ and differentiable on $(2, 4)$. Also,

$$g'(x) = 4x^3 \neq 0 \quad \forall x \in (2, 4).$$

So, all the conditions of Cauchy's Mean Value Theorem are satisfied. Therefore, there exists a point $c \in (2,4)$ such that

$$\frac{f(4)-f(2)}{g(4)-g(2)} = \frac{f'(c)}{g'(c)} \implies \frac{12}{240} = \frac{2c}{4c^3} \implies c = \pm\sqrt{10}.$$

We see that $c = 10$ lies in$]2,4[$. So Cauchy's Mean Value Theorem is verified.

Theorem 17. ***(Taylor's theorem):*** *Let f be a function defined on $[a,b]$ such that*

1. *$f^{(n+1)}(x)$ exists and is continuous on $[a,b]$,*
2. *$c \in (a,b)$.*

Then for any $x \in [a,b]$, we can write

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where the remainder term is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1},$$

for some ξ between c and x.

Special Case (Maclaurin's Theorem): *If $c = 0$, the expansion reduces to*

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x).$$

2.5 Maclaurin Series Expansions and Indeterminate Forms

We now present the standard expansions of some elementary functions:

1. **Exponential Function:**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

This series is valid for all real values of x.

2. **Sine Function:**
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
The series is convergent for all real x. Note that sine is an **odd function**, and hence only odd powers appear.

3. **Cosine Function:**
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
This is valid for all real x. Since cosine is an **even function**, only even powers of x occur.

4. **Logarithmic Function:**
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
This series converges for $|x| < 1$. It is particularly useful for evaluating logarithmic limits and indeterminate forms near $x = 0$.

5. **Other Allied Functions:**
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad (|x| < 1)$$

2.6 Indeterminate Forms

In calculus, a limit is called an **indeterminate form** if direct substitution of the point into the expression leads to an undefined or ambiguous result. The most common indeterminate forms are:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty$$

These arise frequently in limits, derivatives, and integrals. To resolve them, we usually employ:

- Algebraic manipulation (factoring, rationalization)
- Use of Taylor/Maclaurin series
- L'Hôpital's Rule

3. Examples of Indeterminate Forms

Example 24.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Substituting $x = 0$ gives $\frac{0}{0}$. Using the series expansion:

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \cdots$$

Thus,

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Example 25.

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

Again, direct substitution gives $\frac{0}{0}$. Using the expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \cdots$$

Thus,

$$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$$

Example 26.

$$\lim_{x \to 0} (1+x)^{1/x}$$

Taking logarithm:

$$\log L = \lim_{x \to 0} \frac{\log(1+x)}{x}$$

Using the series for $\log(1+x)$:

$$\log(1+x) = x - \frac{x^2}{2} + \cdots$$

$$\log L = 1$$

Hence,

$$\boxed{L = e}$$

Self Assessment -II
Part A:MCQ's

1. If $f(x) = |x|$, then $f'(0)$ is: (a) 1 (b) -1 (c) 0 (d) Does not exist
2. If a function is differentiable at a point, then it must also be: (a) Continuous at that point (b) Discontinuous at that point (c) Non-continuous at that point (d) None of these
3. The function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$ is: (a) Not continuous at $x = 0$ (b) Continuous but not differentiable at $x = 0$ (c) Differentiable at $x = 0$ (d) Neither continuous nor differentiable at $x = 0$
4. The left-hand derivative and right-hand derivative of $f(x) = |x|$ at $x = 0$ are: (a) 1 and -1 respectively (b) -1 and 1 respectively (c) Both 1 (d) Both -1
5. Which of the following statements is **true**? (a) Every continuous function is differentiable (b) Every differentiable function is continuous (c) If a function is not continuous, it can be differentiable (d) Differentiability and continuity are independent
6. The function $f(x) = x^{1/3}$ is: (a) Differentiable everywhere (b) Not differentiable at $x = 0$ (c) Continuous but not differentiable at $x = 1$ (d) Discontinuous at $x = 0$

Part B: Solve the Following

1. The function $f(x) = x|x|$ is derivable at the origin. Solve it.
2. Examine the function $x^3 - 6x^2 + 9x - 4$ for maximum and minimum values.
3. Evaluate $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = 0$.
4. Evaluate $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$.
5. The function f defined by

$$f(x) = \begin{cases} x^2 + 3x + a, & x \leq 1, \\ bx + 2, & x > 1, \end{cases}$$

is required to be differentiable for every x. Find the values of a and b.

Part C: True/False

1. Every continuous function is differentiable. (T/F)
2. If $f'(x) = 0$ for all x in an interval, then $f(x)$ must be a constant function on that interval. (T/F)
3. A function can have a left-hand derivative and a right-hand derivative at a point, but still not be differentiable there. (T/F)
4. The function $f(x) = x^3 - 3x$ has both local maxima and minima. (T/F)

3 Partial derivatives as a foundation for advanced problem-solving

3.1 Partial derivatives

Partial derivatives extend the concept of differentiation to **functions of multiple variables**. If $f(x, y)$ is a function of two variables x and y, the **partial derivative with respect to** x measures how f changes when only x changes, keeping y constant:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Similarly, the **partial derivative with respect to** y is

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

2. Notation

- f_x or $\frac{\partial f}{\partial x}$ — derivative with respect to x.
- f_y or $\frac{\partial f}{\partial y}$ — derivative with respect to y.

Example 27. *Evaluate* f_x *when*

$$f(x, y) = x^2 y + y^2.$$

Solution: To find f_x, we differentiate $f(x, y)$ with respect to x, treating y as a constant:

$$f_x(x, y) = \frac{\partial}{\partial x}\left(x^2 y + y^2\right).$$

Now, apply the derivative term by term:

$$f_x(x, y) = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial x}(y^2).$$

Since y is constant with respect to x,

$$\frac{\partial}{\partial x}(x^2 y) = y \cdot \frac{\partial}{\partial x}(x^2) = y \cdot 2x = 2xy,$$

and

$$\frac{\partial}{\partial x}(y^2) = 0.$$

Therefore,

$$f_x(x, y) = 2xy.$$

Second Derivatives: The second partial derivative of f with respect to x is denoted by f_{xx} and is defined as

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\big(f_x(x, y)\big).$$

That is, f_{xx} is the derivative of the first partial derivative f_x with respect to x.

Likewise, the second partial derivative of f with respect to y is denoted by f_{yy} and is defined as

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\big(f_y(x, y)\big).$$

Finally, the mixed partial derivatives are denoted by f_{xy} and f_{yx} and are defined as

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\big(f_x(x, y)\big), \quad f_{yx}(x, y) = \frac{\partial}{\partial x}\big(f_y(x, y)\big).$$

If f is sufficiently smooth (i.e., the partial derivatives are continuous), then by **Clairaut's Theorem** (or Schwarz's Theorem),

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Collectively, f_{xx}, f_{yy}, f_{xy}, and f_{yx} are known as the **second partial derivatives** of $f(x, y)$.

Moreover, we sometimes denote the second partial derivatives using the following forms:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x}.$$

Theorem 18. ***(Clairaut's Theorem):*** *Let f be defined on a neighborhood of (p, q). If the mixed partial derivatives f_{xy} and f_{yx} are continuous on that neigh-*

borhood, then

$$f_{xy}(p,q) = f_{yx}(p,q).$$

Example 28. *Find all second partial derivatives of*

$$f(x,y) = x^3 + 3x^2y^2.$$

Solution:

Step 1: First partial derivatives

$$f_x = \frac{\partial}{\partial x}(x^3 + 3x^2y^2) = 3x^2 + 6xy^2$$

$$f_y = \frac{\partial}{\partial y}(x^3 + 3x^2y^2) = 6x^2y$$

Step 2: Second partial derivatives

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(3x^2 + 6xy^2) = 6x + 6y^2$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(6x^2y) = 6x^2$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(3x^2 + 6xy^2) = 12xy$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(6x^2y) = 12xy$$

Step 3: Verification using Clairaut's Theorem

Since $f_{xy} = f_{yx} = 12xy$, Clairaut's Theorem is verified.

$$f_{xx} = 6x + 6y^2, \quad f_{yy} = 6x^2, \quad f_{xy} = f_{yx} = 12xy$$

Higher Derivatives: Higher-order partial derivatives are defined similarly. For example, the third partial derivative of f with respect to x is the partial derivative of the second derivative f_{xx} with respect to x:

$$f_{xxx}(x,y) = \frac{\partial}{\partial x}\big(f_{xx}(x,y)\big).$$

Similarly, higher-order mixed partial derivatives can be defined, such as:

$$f_{xxy}(x,y) = \frac{\partial}{\partial y}\big(f_{xx}(x,y)\big), \quad f_{xyy}(x,y) = \frac{\partial}{\partial y}\big(f_{xy}(x,y)\big),$$

and so on. **Higher-Order Partial Derivatives in Operator Notation:**

In operator notation, the partial derivative of f taken m times with respect to x and n times with respect to y is denoted by

$$f_{x^m y^n} = \frac{\partial^{m+n} f}{\partial x^m \, \partial y^n}.$$

For example:

$$f_{x^2 y^3} = \frac{\partial^5 f}{\partial x^2 \, \partial y^3},$$

which means we differentiate f twice with respect to x and three times with respect to y.

Example 29. *Find f_{xxyy} if*

$$f(x,y) = x^4 y^4.$$

Solution

Step 1: First partial derivatives

$$f_x = \frac{\partial}{\partial x}(x^4y^4) = 4x^3y^4, \quad f_y = \frac{\partial}{\partial y}(x^4y^4) = 4x^4y^3$$

Step 2: Second partial derivatives with respect to x

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(4x^3y^4) = 12x^2y^4$$

Step 3: Mixed second partial derivatives with respect to y

$$f_{xxy} = \frac{\partial}{\partial y}(f_{xx}) = \frac{\partial}{\partial y}(12x^2y^4) = 48x^2y^3$$

Step 4: Mixed third derivative with respect to y again

$$f_{xxyy} = \frac{\partial}{\partial y}(f_{xxy}) = \frac{\partial}{\partial y}(48x^2y^3) = 144x^2y^2$$

$$f_{xxyy} = 144x^2y^2$$

Self Assessment - III

Part A: Multiple Choice Questions (MCQs)

1. If $f(x,y) = x^2y + y^3$, then $f_x = ?$ (a) $2xy$ (b) $2xy + 3y^2$ (c) x^2 (d) y^3
2. If $f(x,y) = e^{xy}$, then $f_{xy} = ?$ (a) e^{xy} (b) xe^{xy} (c) ye^{xy} (d) $e^{xy}(1 + xy)$
3. Clairaut's theorem holds if: (a) f_{xy} and f_{yx} exist (b) f_{xy} and f_{yx} are continuous (c) f_x exists (d) f_y exists
4. If $f(x,y) = \sin(xy)$, then $f_y = ?$ (a) $\cos(xy)$ (b) $x\cos(xy)$ (c) $y\cos(xy)$ (d) $x\sin(y)$
5. If $z = x^2 + y^2$, then the value of $\frac{\partial^2 z}{\partial x^2}$ is: (a) 2 (b) x (c) 0 (d) $4x$

Part B: Solve the Following

1. Find $f_x(x, y)$ and $f_y(x, y)$

$$f(x, y) = x^2 + y^2$$

2. Find $f_x(x, y)$ and $f_y(x, y)$

$$f(x, y) = e^x \ln(y^2 + 1)$$

3. Find $f_x(x, y)$ and $f_y(x, y)$

$$f(x, y) = x \sin(xy)$$

4. Find $f_{xx}(x, y)$, $f_{yy}(x, y)$, $f_{xy}(x, y)$ and $f_{yx}(x, y)$

$$f(x, y) = (x^2 + 2y)^2$$

5. Find $f_{xx}(x, y)$, $f_{yy}(x, y)$, $f_{xy}(x, y)$ and $f_{yx}(x, y)$

$$f(x, y) = \tan^{-1}(xy)$$

4 Summary

In this module, we have covered the following theory behind the concepts of limit and continuity, differentiability and partial derivatives that you were acquainted.

- we have taught **limit**,
- **Types of limits:** finite, infinite, left-hand, and right-hand limits.
- **Theorems on limits** help in evaluating complex expressions using limit laws.
- **Properties of limits:** linearity, product, quotient, and power rules.

- **Continuity**,
- **Types of discontinuity:** removable, jump, and infinite.
- **Algebra of continuity:** sum, difference, product, and quotient of continuous functions are also continuous.
- **Differentiability**
- **Local maxima and minima** occur where the first derivative is zero and sign changes occur.
- **Taylor and Maclaurin series** approximate functions using power series.
- **Indeterminate forms** like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are resolved using L'Hôpital's rule or expansions.
- **Partial derivatives** extend the concept of differentiation to multivariable functions.

Exercise:

1. Verify whether $f(x) = \frac{|x|}{x}$ is continuous at $x = 0$.

   8. Examine the continuity of

   $$f(x) = \frac{x^2 - 4}{x - 2}$$

   at $x = 2$. Identify the type of discontinuity.

2. Let a function f be defined on $\mathbb{R}$ such that

   $$|f(x) - f(y)| < (x - y)^2 \quad \text{for all } x, y \in \mathbb{R}.$$

   Then proove that f must be a constant function.

Further Readings

For a deeper understanding of the topics discussed in this module, students are encouraged to refer to the following books:

1. Shanti Narayan, *A Course of Mathematical Analysis*, S. Chand and Company.
2. S. C. Malik and S. Arora, *Mathematical Analysis*, New Age International Publishers.
3. D. Soma Sundaram and B. Choudhury, *A First Course in Mathematical Analysis*.

Self Assessment 1: Solutions

Part A: Conceptual Questions

1. **Limit of a function (in own words).** The limit of $f(x)$ as x approaches a is the value L that $f(x)$ gets arbitrarily close to when x is taken sufficiently close to (but not necessarily equal to) a. We write $\lim_{x\to a} f(x) = L$.

2. **Left-hand vs right-hand limit.** The left-hand limit $\lim_{x\to a^-} f(x)$ is the value approached by $f(x)$ as x approaches a from values less than a. The right-hand limit $\lim_{x\to a^+} f(x)$ is the value approached as x approaches a from values greater than a. The two need not be equal.

3. **Limit at a point where the function is not defined.** Yes. A function can have a limit at a point even if it is not defined there. Example:

$$f(x) = \frac{x^2 - 1}{x - 1} = x + 1 \quad (x \neq 1).$$

   Here $\lim_{x\to 1} f(x) = 2$, although $f(1)$ is not defined for the given formula.

4. **Continuity at a point (definition).** A function f is continuous at $x = a$ if the following three conditions hold:

(a) $f(a)$ is defined,

(b) $\lim_{x \to a} f(x)$ exists,

(c) $\lim_{x \to a} f(x) = f(a)$.

5. **(Repeat) Conditions for continuity at a point.** Same as above: existence of $f(a)$, existence of the limit, and equality of the limit to the function value.

Part B: Problems

1.
$$\lim_{x \to 3} (2x^2 - 5x + 1) = 2(3)^2 - 5(3) + 1 = 18 - 15 + 1 = 4.$$

2.
$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$
(Standard limit.)

3. For $f(x) = \dfrac{x-1}{|x-1|}$ as $x \to 1$:
$$\lim_{x \to 1^-} \frac{x-1}{|x-1|} = \lim_{x \to 1^-} \frac{x-1}{-(x-1)} = -1,$$
$$\lim_{x \to 1^+} \frac{x-1}{|x-1|} = \lim_{x \to 1^+} \frac{x-1}{x-1} = 1.$$
Since the left-hand and right-hand limits are different, $\lim_{x \to 1} f(x)$ does not exist.

4.
$$\lim_{x \to \infty} \frac{3x^2 + 2}{x^2 + 1} = \lim_{x \to \infty} \frac{3 + \frac{2}{x^2}}{1 + \frac{1}{x^2}} = \frac{3+0}{1+0} = 3.$$

5. $f(x) = \begin{cases} x^2, & x \neq 2, \\ 5, & x = 2. \end{cases}$ Compute the limit:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 = 4.$$

But $f(2) = 5$. Since $\lim_{x\to 2} f(x) \neq f(2)$, f is *not* continuous at $x = 2$.

6. $f(x) = \begin{cases} 2x+3, & x < 1, \\ 5-x, & x \geq 1. \end{cases}$ Left-hand limit at 1: $\lim_{x\to 1^-} f(x) = 2(1) + 3 = 5$. Right-hand limit at 1: $\lim_{x\to 1^+} f(x) = 5 - 1 = 4$. Since the one-sided limits are not equal, f is not continuous at $x = 1$.

7. $f(x) = \dfrac{|x|}{x}$ at $x = 0$. Note $f(0)$ is not defined. Compute one-sided limits:

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1, \qquad \lim_{x\to 0^+} \frac{|x|}{x} = 1.$$

The one-sided limits differ, so the limit does not exist and hence f is not continuous at 0.

Part C: True or False

1. If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ exists. **True.** (By definition the two one-sided limits equal implies the two-sided limit exists and equals that common value.)

2. $\lim_{x\to\infty} \dfrac{1}{x} = 0.$ **True.**

3. If $f(a)$ is not defined, then $\lim_{x\to a} f(x)$ does not exist. **False.** (The limit may exist even if the function value is not defined; e.g. $f(x) = \dfrac{x^2-1}{x-1}$ at $x = 1$.)

4. The absolute value function $f(x) = |x|$ is discontinuous at $x = 0$. **False.** ($|x|$ is continuous everywhere, though not differentiable at 0.)

5. The function $f(x) = \sin x$ is continuous for all real numbers. **True.**

Self Assessment 2: Solutions

Part A: MCQ'c

(1) (d) Does not exist

(2) (a) Continuous at that point

(3) (c) Differentiable at $x = 0$

(4) (b) -1 and 1 respectively

(5) (b) Every differentiable function is continuous

(6) (b) Not differentiable at $x = 0$

Part B:

1. **Show $f(x) = x|x|$ is differentiable at the origin.**
   We write
$$f(x) = \begin{cases} -x^2, & x < 0, \\ 0, & x = 0, \\ x^2, & x > 0. \end{cases}$$
   For $x > 0$, $f'(x) = 2x$, so $f'_+(0) = \lim_{x \to 0^+} 2x = 0$.
   For $x < 0$, $f'(x) = -2x$, so $f'_-(0) = \lim_{x \to 0^-} (-2x) = 0$. Since the left and right derivatives at 0 are equal, $f'(0) = 0$. Hence f is differentiable at the origin.

2. **Maxima and minima of** $f(x) = x^3 - 6x^2 + 9x - 4$.
   Compute
$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3).$$
   Critical points: $x = 1$ and $x = 3$. Second derivative $f''(x) = 6x - 12$. Thus
$$f''(1) = 6 - 12 = -6 < 0 \quad \text{(local maximum at } x = 1\text{)},$$
$$f''(3) = 18 - 12 = 6 > 0 \quad \text{(local minimum at } x = 3\text{)}.$$

Values:

$$f(1) = 1 - 6 + 9 - 4 = 0, \qquad f(3) = 27 - 54 + 27 - 4 = -4.$$

Hence local maximum $f(1) = 0$ and local minimum $f(3) = -4$.

3. $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = 0.$
   Use the identity

$$(\sec x - \tan x)(\sec x + \tan x) = \sec^2 x - \tan^2 x = 1,$$

so

$$\sec x - \tan x = \frac{1}{\sec x + \tan x}.$$

As $x \to \frac{\pi}{2}$, both $\sec x$ and $\tan x$ have unbounded magnitude, hence $\sec x + \tan x \to \pm\infty$ and the reciprocal tends to 0. Therefore the limit is 0.

4. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e.$
   Set $L = \lim_{x \to 0} (1 + x)^{1/x}$. Then

$$\ln L = \lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \frac{1}{1 + \xi} = 1$$

(or use standard series/derivative: $\lim_{x \to 0} \dfrac{\ln(1 + x)}{x} = 1$), hence $\ln L = 1$ and $L = e$.

5. **Find a, b so that**

$$f(x) = \begin{cases} x^2 + 3x + a, & x \le 1, \\ bx + 2, & x > 1, \end{cases}$$

is differentiable for every x.
Continuity at $x = 1$:

$$\lim_{x \to 1^-} f(x) = 1 + 3 + a = 4 + a, \qquad \lim_{x \to 1^+} f(x) = b + 2.$$

So $4 + a = b + 2$, i.e. $b = a + 2$.

Differentiability at $x = 1$: left derivative at 1 is $f'_{-}(1) = 2 \cdot 1 + 3 = 5$, right derivative is $f'_{+}(1) = b$. Equate:

$$b = 5.$$

Hence $a = b - 2 = 5 - 2 = 3$.

Answer: $a = 3,\ b = 5$.

Part C: True/False

1. Every continuous function is differentiable. **(False)**
   Reason: Continuity does not imply differentiability. For example, $f(x) = |x|$ is continuous everywhere but not differentiable at $x = 0$.

2. If $f'(x) = 0$ for all x in an interval, then $f(x)$ must be a constant function on that interval. **(True)**
   Reason: Since the derivative is zero everywhere, f has no change in value; hence $f(x) = C$, a constant.

3. A function can have a left-hand derivative and a right-hand derivative at a point, but still not be differentiable there. **(True)**
   Reason: If the left and right derivatives exist but are not equal, f is not differentiable at that point. Example: $f(x) = |x|$ at $x = 0$.

4. The function $f(x) = x^3 - 3x$ has both local maxima and minima. **(True)**
   Reason: $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$. Thus, $f'(x) = 0$ at $x = \pm 1$. $f''(x) = 6x$ gives $f''(-1) = -6 < 0$ (local maximum) and $f''(1) = 6 > 0$ (local minimum).

Self Assessment 3: Solutions

MCQs (answers with brief justification):

1. If $f(x, y) = x^2 y + y^3$, then

$$f_x = \frac{\partial}{\partial x}(x^2 y + y^3) = 2xy.$$

Answer: (a) $2xy$.

2. If $f(x,y) = e^{xy}$, first

$$f_x = ye^{xy}, \qquad f_{xy} = \frac{\partial}{\partial y}(ye^{xy}) = e^{xy} + xye^{xy} = e^{xy}(1+xy).$$

Answer: (d) $e^{xy}(1+xy)$.

3. Clairaut's theorem (equality of mixed partials) holds when the mixed partials are continuous.

Answer: (b) f_{xy} and f_{yx} are continuous.

4. For $f(x,y) = \sin(xy)$,

$$f_y = \frac{\partial}{\partial y}[\sin(xy)] = x\cos(xy)$$

Answer: (b) $x\cos(xy)$.

5. For $z = x^2 + y^2$,

$$\frac{\partial z}{\partial x} = 2x \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} = 2$$

Answer: (a) 2.

Problems (computed partials):

1. $f(x,y) = x^2 + y^2$.

$$f_x = 2x, \qquad f_y = 2y.$$

2. $f(x,y) = e^x \ln(y^2+1)$.

$$f_x = e^x \ln(y^2+1), \qquad f_y = e^x \cdot \frac{2y}{y^2+1}.$$

3. $f(x,y) = x\sin(xy)$.

$$f_x = \sin(xy) + x \cdot \cos(xy) \cdot y = \sin(xy) + xy\cos(xy),$$

$$f_y = x \cdot \cos(xy) \cdot x = x^2\cos(xy).$$

4. $f(x, y) = (x^2 + 2y)^2$. Put $u = x^2 + 2y$ so $f = u^2$.

$$f_x = 2u \cdot (2x) = 4x(x^2 + 2y), \qquad f_y = 2u \cdot 2 = 4(x^2 + 2y).$$

Second derivatives:

$$f_{xx} = \frac{d}{dx}(4x^3 + 8xy) = 12x^2 + 8y,$$

$$f_{yy} = \frac{d}{dy}(4x^2 + 8y) = 8,$$

$$f_{xy} = \frac{\partial}{\partial y}(4x^3 + 8xy) = 8x, \qquad f_{yx} = 8x.$$

5. $f(x, y) = \tan^{-1}(xy)$. Put $u = xy$ and $f = \arctan(u)$.

$$f_x = \frac{y}{1 + (xy)^2}, \qquad f_y = \frac{x}{1 + (xy)^2}.$$

Second derivatives:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{y}{1 + x^2y^2}\right) = -\frac{2xy^3}{(1 + x^2y^2)^2},$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{x}{1 + x^2y^2}\right) = -\frac{2x^3y}{(1 + x^2y^2)^2},$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{y}{1 + x^2y^2}\right) = \frac{1 - x^2y^2}{(1 + x^2y^2)^2},$$

and by symmetry

$$f_{yx} = f_{xy} = \frac{1 - x^2y^2}{(1 + x^2y^2)^2}.$$

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