Assignment 14

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***Abstract*—This document shows some properties of matrices, their inverse and determinant.**

1 Problem

The matrix $\mathbf{A} = \begin{pmatrix} 5 & 9 & 8 \\ 1 & 8 & 2 \\ 9 & 1 & 0 \end{pmatrix}$ satisfies:

1) $\mathbf{A}$ is invertible and the inverse has all integer entries.
2) $\det(\mathbf{A})$ is odd.
3) $\det(\mathbf{A})$ is divisible by 13
4) $\det(\mathbf{A})$ has atleast two prime divisors.

2 Solution

Performing some elementary row operations on the given matrix,

$$\begin{pmatrix} 5 & 9 & 8 \\ 1 & 8 & 2 \\ 9 & 1 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{9}R_1]{R_2 \leftarrow R_2 - \frac{1}{5}R_1} \begin{pmatrix} 5 & 9 & 8 \\ 0 & \frac{31}{5} & \frac{2}{5} \\ 0 & \frac{-76}{5} & \frac{-72}{5} \end{pmatrix} \tag{2.0.1}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + \frac{76}{31}R_2} \begin{pmatrix} 5 & 9 & 8 \\ 0 & \frac{31}{5} & \frac{2}{5} \\ 0 & 0 & \frac{-416}{31} \end{pmatrix} \tag{2.0.2}$$

After obtaining a triangular form of the matrix, we can say

$$\left|\mathbf{A}\right| = \text{product of diagonal entries of the triangular matrix} \tag{2.0.3}$$

$$= 5 \times \frac{31}{5} \times \frac{-416}{31} = -416 \tag{2.0.4}$$

1. $\mathbf{A}$ is invertible and the inverse has all integer entries	$det(\mathbf{A}) \neq 0$, hence $\mathbf{A}$ is invertible. $\mathbf{A}$ is an integer matrix and has all integer entries. $\therefore\ det(\mathbf{A})$ is an integer If $\mathbf{A}^{-1}$ is an integer matrix and has all integer entries, then $det(\mathbf{A}^{-1})$ will be an integer. $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_{3\times 3}$ $det(\mathbf{A}\mathbf{A}^{-1}) = det(\mathbf{I}) = 1$ $det(\mathbf{A})det(\mathbf{A}^{-1}) = 1$ This is possible only when $det(\mathbf{A}) = \pm 1$

	But we have seen that $det(\mathbf{A}) = -416$, hence $\mathbf{A}^{-1}$ does not have all integer entries. Given option is false.
2. det($\mathbf{A}$) is odd	False, as seen from (2.0.4)
3. det($\mathbf{A}$) is divisible by 13	True. Since $det(\mathbf{A}) = -416$, which is divisible by 13.
4. det($\mathbf{A}$) has atleast two prime divisors	True. $\mathbf{A} \in M_{n\times n}(\mathbb{Z})$ is an $n \times n$ nonzero matrix of integers Then we can find unimodular matrices (i.e. square integer invertible matrices having determinant as $\pm$ 1) $\mathbf{S}_{n\times n}, \mathbf{T}_{n\times n} \in GL_n(\mathbb{Z})$ such that $\mathbf{A}' = \mathbf{SAT}$ $\mathbf{A}'$ is a diagonal matrix and this represents the Smith Normal form of $\mathbf{A}$ $\mathbf{A}' = \begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 \\ 0 & \alpha_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ \vdots & 0 & 0 & \alpha_r & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$ where $\alpha_i \mid \alpha_{i+1} \quad \forall 1 \leq i < r$ and the elements α_i are called elementary divisors, given by $\alpha_i = \frac{d_i(\mathbf{A})}{d_{i-1}(\mathbf{A})}$ where $d_i(\mathbf{A})$ = gcd of all $i \times i$ minors of the matrix $\mathbf{A}$ Now, for the given matrix $\mathbf{A}$, $d(\mathbf{A})$ = greatest positive number dividing all co-efficients $\therefore d(\mathbf{A}) = \gcd\{a_{ij} : a_{ij} \in \mathbf{A}, 1 \leq i, j \leq n\} = 1$ Then, we compute $\mathbf{S}$ and $\mathbf{T}$ by starting with $\mathbf{I}_{3\times 3}$ and then, modifying $\mathbf{S}$ by performing a corresponding invertible column operation for each row operation in $\mathbf{A}$ and modifying $\mathbf{T}$ by performing a corresponding invertible row operation for each column operation in $\mathbf{A}$ while doing Eucledian Division algorithm on $\mathbf{A}$ The Smith normal form obtained is: $\mathbf{A}' = \begin{pmatrix} 14 & -15 & -6 \\ 16 & -17 & -7 \\ 71 & -76 & -31 \end{pmatrix} \begin{pmatrix} 5 & 9 & 8 \\ 1 & 8 & 2 \\ 9 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -82 \\ 0 & 1 & -94 \\ 0 & 0 & 1 \end{pmatrix}$

	$\implies \mathbf{A'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 416 \end{pmatrix}$ where the diagonal elements are called elementary divisors. $\mathbf{A}$ and $\mathbf{A'}$ are equivalent matrices, since we expressed $\mathbf{A'} = \mathbf{SAT}$ with $\mathbf{S}$ and $\mathbf{T}$ being invertible.

TABLE 1: Verifying with given options