Challenging Problem 1

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Abstract—**This document shows the method to find the closest points on two skew lines in 3-Dimension.**

Download all python codes from

https://github.com/surbhi0912/EE5609/tree/master/challenging_problem/challenging1/codes

and latex-tikz codes from

https://github.com/surbhi0912/EE5609/tree/master/challenging_problem/challenging1

1 Problem

In 3-Dimensional Space, find the points on the two skew lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{1.0.2}$$

such that the points are closest to each other

2 Solution

In the given problem,

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{2.0.1}$$

where L_1 is passing through the point $A_1(1, 1, 0)$ and direction vector $\mathbf{m}_1$,
And L_2 is passing through the point $A_2(2, 1, -1)$ and direction vector $\mathbf{m}_2$

Let us take a point $\mathbf{E}$ on Line L_1 and $\mathbf{F}$ on Line L_2 such that they are closest to each other.

Then $\mathbf{E}$ and $\mathbf{F}$ can be expressed using Equation 1.0.1 and 1.0.2 respectively as follows :

$$\mathbf{E} = \mathbf{A}_1 + \lambda_1 \mathbf{m}_1 \tag{2.0.2}$$

$$\mathbf{F} = \mathbf{A}_2 + \lambda_2 \mathbf{m}_2 \tag{2.0.3}$$

Now, the position vector from $\mathbf{E}$ to $\mathbf{F}$, ie $\mathbf{F} - \mathbf{E}$ is given as,

$$\mathbf{F} - \mathbf{E} = (\mathbf{A}_2 + \lambda_2 \mathbf{m}_2) - (\mathbf{A}_1 + \lambda_1 \mathbf{m}_1) \tag{2.0.4}$$

$$= (\mathbf{A}_2 - \mathbf{A}_1) + (\lambda_2 \mathbf{m}_2 - \lambda_1 \mathbf{m}_1) \tag{2.0.5}$$

$$= (\mathbf{A}_2 - \mathbf{A}_1) + \begin{pmatrix} \mathbf{m}_2 & \mathbf{m}_1 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} \tag{2.0.6}$$

Since the points $\mathbf{E}$ and $\mathbf{F}$ are closest to each other, position vector $\mathbf{F} - \mathbf{E}$ is perpendicular to the skew lines L_1 and L_2, thus we can say that $\mathbf{F} - \mathbf{E}$ is perpendicular to the direction vectors of these lines, ie $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively. Therefore,

$$\mathbf{m}_1^T (\mathbf{F} - \mathbf{E}) = 0 \tag{2.0.7}$$

$$\mathbf{m}_2^T (\mathbf{F} - \mathbf{E}) = 0 \tag{2.0.8}$$

Using the values of $\mathbf{F} - \mathbf{E}$ from Equation 2.0.6 and combining Equations 2.0.7 and 2.0.8,

$$\begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \end{pmatrix} \left((\mathbf{A}_2 - \mathbf{A}_1) + \begin{pmatrix} \mathbf{m}_2 & \mathbf{m}_1 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} \right) = 0 \tag{2.0.9}$$

$$\implies \begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \end{pmatrix} (\mathbf{A}_2 - \mathbf{A}_1) + \begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \end{pmatrix} \begin{pmatrix} \mathbf{m}_2 & \mathbf{m}_1 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = 0 \tag{2.0.10}$$

Now substituting the values from Equation (1.0.1) in (2.0.10)

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -5 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = 0 \tag{2.0.11}$$

$$\implies \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 13 & 6 \\ 38 & 13 \end{pmatrix}\begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = 0 \tag{2.0.12}$$

$$\implies \begin{pmatrix} 13 & 6 \\ 38 & 13 \end{pmatrix}\begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.13}$$

$$\implies \begin{pmatrix} 13 & 6 \\ 38 & 13 \end{pmatrix}\begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{2.0.14}$$

The augmented matrix will be

$$\begin{pmatrix} 13 & 6 & -1 \\ 38 & 13 & -1 \end{pmatrix} \tag{2.0.15}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 13 & 6 & -1 \\ 38 & 13 & -1 \end{pmatrix} \xleftrightarrow{R_2=38R_1-13R_2} \begin{pmatrix} 13 & 6 & -1 \\ 0 & 59 & -25 \end{pmatrix} \tag{2.0.16}$$

This gives us,

$$\begin{pmatrix} 13 & 6 \\ 0 & 59 \end{pmatrix}\begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -25 \end{pmatrix} \tag{2.0.17}$$

Solving Equation 2.0.17,

$$-59\lambda_1 = -25 \tag{2.0.18}$$

$$\implies \lambda_1 = \frac{25}{59} \tag{2.0.19}$$

$$13\lambda_2 + 6\lambda_1 = -1 \tag{2.0.20}$$

$$\implies \lambda_2 = \frac{7}{59} \tag{2.0.21}$$

Putting the values of λ_1, $\mathbf{A}_1$, $\mathbf{m}_1$ in $\mathbf{E}$ and λ_2, $\mathbf{A}_2$, $\mathbf{m}_2$ in $\mathbf{F}$, we get

$$\mathbf{E} = \begin{pmatrix} \frac{109}{59} \\ \frac{34}{59} \\ \frac{25}{59} \end{pmatrix} \tag{2.0.22}$$

$$= \begin{pmatrix} 1.85 \\ 0.58 \\ 0.42 \end{pmatrix} \tag{2.0.23}$$

$$\mathbf{F} = \begin{pmatrix} \frac{139}{59} \\ \frac{24}{59} \\ \frac{-45}{59} \end{pmatrix} \tag{2.0.24}$$

$$= \begin{pmatrix} 2.36 \\ 0.41 \\ -0.76 \end{pmatrix} \tag{2.0.25}$$

The figure obtained is shown in Fig 1

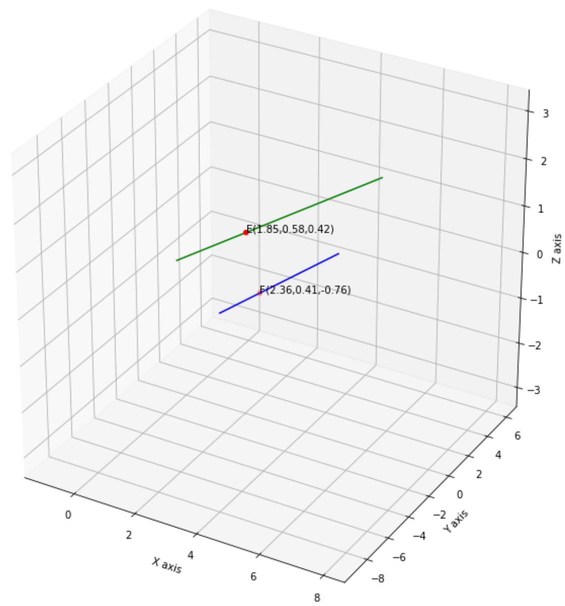

Fig. 1: Closest points between skew lines L_1 and L_2