Assignment 11

Surbhi Agarwal

***Abstract*—This document illustrates linear transformation matrices with respect to a set of linearly independent eigenvectors.**

1 Problem

Let $\mathbf{S} : \mathbb{R}^n \to \mathbb{R}^n$ be given by $\mathbf{S}(\mathbf{v}) = \alpha\mathbf{v}$, for a fixed $\alpha \in \mathbb{R}, \alpha \neq 0$. Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $\mathbf{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of linearly independent eigenvectors of $\mathbf{T}$. Then

1) The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is diagonal
2) The matrix of $(\mathbf{T} - \mathbf{S})$ with respect to $\mathbf{B}$ is diagonal
3) The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is not necessarily diagonal, but is upper triangular
4) The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is diagonal but the matrix of $(\mathbf{T} - \mathbf{S})$ with respect to $\mathbf{B}$ is not diagonal.

2 Solution

Given that $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and B represents a set of linearly independent eigenvectors of $\mathbf{T}$ given as follows

$$\mathbf{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \tag{2.0.1}$$

So,

$$\mathbf{T}(\mathbf{v}_i) = \mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{2.0.2}$$

where λ_i represents the eigenvalue corresponding to $\mathbf{v}_i$. Hence, the matrix $\mathbf{T}$ with respect to $\mathbf{B}$ can be represented as

$$[\mathbf{T}]_B = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \tag{2.0.3}$$

And,

$$\begin{aligned} (\mathbf{T} - \mathbf{S})\mathbf{v}_i &= \mathbf{T}(\mathbf{v}_i) - \mathbf{S}(\mathbf{v}_i) && (2.0.4) \\ &= \lambda_i \mathbf{v}_i - \alpha \mathbf{v}_i && (2.0.5) \\ &= (\lambda_i - \alpha)\mathbf{v}_i && (2.0.6) \end{aligned}$$

Hence, matrix of $\mathbf{T} - \mathbf{S}$ with respect to $\mathbf{B}$ can be represented as

$$[\mathbf{T} - \mathbf{S}]_B = \begin{pmatrix} \lambda_1 - \alpha & 0 & \dots & 0 \\ 0 & \lambda_2 - \alpha & \dots & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \lambda_n - \alpha \end{pmatrix} \tag{2.0.7}$$

1. The matrix of $\mathbf{T}$ w.r.t to $\mathbf{B}$ is diagonal	True, as seen from (2.0.3)
2. The matrix of $(\mathbf{T} - \mathbf{S})$ w.r.t $\mathbf{B}$ is diagonal	True, as seen from (2.0.7)
3. The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is not necessarily diagonal but is upper triangular	False, as already proved $[\mathbf{T}]_B$ is diagonal
4. The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is diagonal but the matrix of $(\mathbf{T} - \mathbf{S})$ with respect to $\mathbf{B}$ is not diagonal	False, as already proved $[\mathbf{T} - \mathbf{S}]_B$ is diagonal

TABLE 1: Verifying the given options

3 Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$\mathbf{T}(x) = \mathbf{A}\mathbf{x} = \begin{pmatrix} 4 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.0.1}$$

Here, the eigenvalues of the above trasformation matrix are $\lambda_1 = 3, \lambda_2 = -2$. And the corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Thus,

$$\mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2\} \tag{3.0.2}$$

Now,

$$\begin{aligned}\mathbf{T}(\mathbf{v}_1) &= \mathbf{A}\mathbf{v}_1 && (3.0.3)\\ &= \begin{pmatrix}4 & -2\\ 3 & -3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} && (3.0.4)\\ &= \begin{pmatrix}6\\3\end{pmatrix} && (3.0.5)\\ &= 3\begin{pmatrix}2\\1\end{pmatrix} && (3.0.6)\\ &= \lambda_1\mathbf{v}_1 && (3.0.7)\end{aligned}$$

And,

$$\begin{aligned}\mathbf{T}(\mathbf{v}_2) &= \mathbf{A}\mathbf{v}_2 && (3.0.8)\\ &= \begin{pmatrix}4 & -2\\ 3 & -3\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} && (3.0.9)\\ &= \begin{pmatrix}-2\\-6\end{pmatrix} && (3.0.10)\\ &= -2\begin{pmatrix}1\\3\end{pmatrix} && (3.0.11)\\ &= \lambda_2\mathbf{v}_2 && (3.0.12)\\ & && (3.0.13)\end{aligned}$$

For any vector $\mathbf{v} \in \mathbb{R}^2, \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$

$$\begin{aligned}[\mathbf{v}]_B &= \begin{pmatrix}c_1\\c_2\end{pmatrix} && (3.0.14)\\ \mathbf{T}(\mathbf{v}) &= \mathbf{T}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) && (3.0.15)\\ &= c_1\mathbf{T}(\mathbf{v}_1) + c_2\mathbf{T}(\mathbf{v}_2) && (3.0.16)\\ &= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 && (3.0.17)\\ [\mathbf{T}(\mathbf{v})]_B &= \begin{pmatrix}\lambda_1 c_1\\ \lambda_2 c_2\end{pmatrix} && (3.0.18)\\ &= \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} && (3.0.19)\\ &= \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}[\mathbf{v}]_B && (3.0.20)\\ &= \begin{pmatrix}3 & 0\\ 0 & -2\end{pmatrix}[\mathbf{v}]_B\mathbf{S}(\mathbf{v}) \quad = \alpha\mathbf{v}, \alpha \neq 0 && (3.0.21)\\ &= \alpha(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) && (3.0.22)\\ &= \alpha c_1\mathbf{v}_1 + \alpha c_2\mathbf{v}_2 && (3.0.23)\\ & && (3.0.24)\end{aligned}$$

$$\begin{aligned}[\mathbf{S}(\mathbf{v})]_B &= \begin{pmatrix}\alpha c_1\\ \alpha c_2\end{pmatrix} && (3.0.25)\\ [(\mathbf{T}-\mathbf{S})(\mathbf{v})]_B &= \begin{pmatrix}\lambda_1 c_1 - \alpha c_1\\ \lambda_2 c_2 - \alpha c_2\end{pmatrix} && (3.0.26)\\ &= \begin{pmatrix}\lambda_1 - \alpha & 0\\ 0 & \lambda_2 - \alpha\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} && (3.0.27)\\ &= \begin{pmatrix}\lambda_1 - \alpha & 0\\ 0 & \lambda_2 - \alpha\end{pmatrix}[\mathbf{v}]_B && (3.0.28)\\ &= \begin{pmatrix}3 - \alpha & 0\\ 0 & -2 - \alpha\end{pmatrix}[\mathbf{v}]_B && (3.0.29)\end{aligned}$$

Hence, shown from (3.0.21) and (3.0.29) that the matrix of $\mathbf{T}$ and of $\mathbf{T} - \mathbf{S}$ w.r.t to $\mathbf{B}$ is diagonal.