Spectral Decomposition

Surbhi Agarwal

***Abstract*—This document looks into when Spectral (eigenvalue) Decomposition exists for a matrix.**

Download latex-tikz codes from

https://github.com/surbhi0912/EE5609/

1 Problem

Does the Spectral (Eigenvalue) decomposition always exist for any matrix?

2 Solution

Eigenvalue decompisition is possible for a diagonalizable matrix only. An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if it has n linearly independent eigen vectors. Then, it is similar to a diagonal matrix and can be expressed as

$$\mathbf{A} = \mathbf{PDP}^{-1} \tag{2.0.1}$$

for some invertible matrix $\mathbf{P}$ and diagonal matrix $\mathbf{D}$. Here, the columns of $\mathbf{P}$ are n linearly independent eigen vectors of $\mathbf{A}$, and the diagonal entries of $\mathbf{D}$ are eigenvalues of $\mathbf{A}$ that correspond to respective eigen vectors in $\mathbf{P}$

3 Proof

If $\mathbf{P}$ is any $n \times n$ matrix with columns $\mathbf{v}_1, \cdots, \mathbf{v}_n$, and if $\mathbf{D}$ is any diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, then

$$\mathbf{AP} = \mathbf{A}\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \tag{3.0.1}$$

$$= \begin{pmatrix} \mathbf{Av}_1 & \mathbf{Av}_2 & \cdots & \mathbf{Av}_n \end{pmatrix} \tag{3.0.2}$$

And

$$\mathbf{PD} = \mathbf{P}\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{3.0.3}$$

$$= \begin{pmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{pmatrix} \tag{3.0.4}$$

Now, suppose

$$\mathbf{A} = \mathbf{PDP}^{-1} \tag{3.0.5}$$

Right multiplying by $\mathbf{P}$

$$\mathbf{AP} = (\mathbf{PDP}^{-1})\mathbf{P} \tag{3.0.6}$$

$$\implies \mathbf{AP} = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P}) \tag{3.0.7}$$

$$\implies \mathbf{AP} = \mathbf{PD} \tag{3.0.8}$$

From (3.0.2) and (3.0.4)

$$\implies \begin{pmatrix} \mathbf{Av}_1 & \mathbf{Av}_2 & \cdots & \mathbf{Av}_n \end{pmatrix} = \begin{pmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{pmatrix} \tag{3.0.9}$$

Equating columns

$$\mathbf{Av}_1 = \lambda_1\mathbf{v}_1, \quad \cdots \quad, \mathbf{Av}_n = \lambda_n\mathbf{v}_n \tag{3.0.10}$$

Since $\mathbf{P}$ is invertible, its columns $\mathbf{v}_1, \cdots, \mathbf{v}_n$ must be linearly independent. And since these columns are nonzero, from (3.0.10), we get that $\lambda_1, \cdots, \lambda_n$ are eigen values and $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are their corresponding eigenvector. Hence, proved.