Challenging Problem 2

Surbhi Agarwal

Abstract—**This document looks at the cases when the matrix multiplication of two matrices is commutative.**

Download latex-tikz codes from

https://github.com/surbhi0912/EE5609/tree/master/challenging_problem/challenging2

1 Problem

We know that in general, $\mathbf{AB} \neq \mathbf{BA}$, i.e. matrix multiplication is not commutative in general. What are the conditions on $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$?

2 Explanation

Two matrices $\mathbf{A}$ and $\mathbf{B}$ commute on matrix multiplication if they are Simultaneously Diagonalizable.

An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if it has n linearly independent eigen vectors. Then, it is similar to a diagonal matrix and can be expressed as

$$\mathbf{A} = \mathbf{P}\mathbf{D_1}\mathbf{P}^{-1} \tag{2.0.1}$$

for some invertible matrix $\mathbf{P}$ and diagonal matrix $\mathbf{D_1}$. Here, the columns of $\mathbf{P}$ are n linearly independent eigen vectors of $\mathbf{A}$, and the diagonal entries of $\mathbf{D_1}$ are eigenvalues of $\mathbf{A}$ that correspond to respective eigen vectors in $\mathbf{P}$

A matrix $\mathbf{B}$ is said to be Simultaneously diagonalizable with $\mathbf{A}$ if the same $\mathbf{P}$ diagonalizes both matrices, so $\mathbf{B}$ can be expressed as

$$\mathbf{B} = \mathbf{P}\mathbf{D_2}\mathbf{P}^{-1} \tag{2.0.2}$$

3 Proof

Using above concept, we see that

$$\begin{aligned} \mathbf{AB} &= (\mathbf{P}\mathbf{D_1}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D_2}\mathbf{P}^{-1}) && (3.0.1)\\ &= \mathbf{P}\mathbf{D_1}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D_2}\mathbf{P}^{-1} && (3.0.2)\\ &= \mathbf{P}\mathbf{D_1}\mathbf{D_2}\mathbf{P}^{-1} && (3.0.3) \end{aligned}$$

$$\begin{aligned} \mathbf{BA} &= (\mathbf{P}\mathbf{D_2}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D_1}\mathbf{P}^{-1}) && (3.0.4)\\ &= \mathbf{P}\mathbf{D_2}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D_1}\mathbf{P}^{-1} && (3.0.5)\\ &= \mathbf{P}\mathbf{D_2}\mathbf{D_1}\mathbf{P}^{-1} && (3.0.6) \end{aligned}$$

Now, note that since $\mathbf{D_1}$ and $\mathbf{D_2}$ are diagonal matrices,

$$\mathbf{D_1}\mathbf{D_2} = \mathbf{D_2}\mathbf{D_1} \tag{3.0.7}$$

because the product of two diagonal matrices is the product of their corresponding diagonal elements, and multiplication for the diagonal elements is commutative since they are scalar. From Equations (3.0.3),(3.0.6) and (3.0.7), we conclude

$$\mathbf{AB} = \mathbf{BA} \tag{3.0.8}$$

Hence, proved the case where matrix multiplication is commutative.