Assignment 5

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***Abstract*—This document traces a parabola when it's general second degree equation is given.**

Download all python codes from

https://github.com/surbhi0912/EE5609/

and latex-tikz codes from

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1 Problem

Trace the parabola

$$(4x + 3y + 15)^2 = 5(3x - 4y)$$

2 Solution

The given equation can be rewritten as

$$16x^2 + 24xy + 9y^2 + 105x + 110y + 225 = 0 \tag{2.0.1}$$

Comparing this to the standard equation,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \frac{105}{2} \\ 55 \end{pmatrix}, \quad f = 225 \tag{2.0.2}$$

The characteristic equation of $\mathbf{V}$ is given as

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{2.0.3}$$

$$\implies \begin{vmatrix} \lambda - 16 & -12 \\ -12 & \lambda - 9 \end{vmatrix} = 0 \tag{2.0.4}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{2.0.5}$$

The eigenvalues are the roots of the equation (2.0.5), which are as follows :

$$\lambda_1 = 0, \quad \lambda_2 = 25 \tag{2.0.6}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.7}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{2.0.8}$$

For $\lambda_1 = 0$

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 3R_1]{R_1 \leftarrow \frac{1}{4}R_1} \begin{pmatrix} -4 & -3 \\ 0 & 0 \end{pmatrix} \tag{2.0.9}$$

$$\implies \mathbf{p_1} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{2.0.10}$$

For $\lambda_2 = 25$

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 4R_1]{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \tag{2.0.11}$$

$$\implies \mathbf{p_2} = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.0.12}$$

So, using Eigenvalue decomposition, $\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D}$, where

$$\mathbf{P} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \tag{2.0.13}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{2.0.14}$$

Then, for the parabola

$$\text{focal length} = \left|\frac{2\eta}{\lambda_2}\right| \tag{2.0.15}$$

$$\eta = \mathbf{p}_1^T\mathbf{u} = \frac{25}{2} \tag{2.0.16}$$

Substituting values from (2.0.16) and (2.0.6) in (2.0.15), we get

$$\text{focal length} = 1 \tag{2.0.17}$$

The standard equation of the parabola is given by

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.18}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{2.0.19}$$

Substituting values from (2.0.2),(2.0.16),(2.0.10) in (2.0.19),

$$\begin{pmatrix} 45 & 65 \\ 16 & 12 \\ 12 & 9 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -225 \\ -60 \\ -45 \end{pmatrix} \tag{2.0.20}$$

To find $\mathbf{c}$, performing row reduction on the augmented matrix as follows:

$$\begin{pmatrix} 45 & 65 & -225 \\ 16 & 12 & -60 \\ 12 & 9 & -45 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{1}{45} R_1]{R_3 \leftarrow R_3 - \frac{3}{4} R_2} \begin{pmatrix} 1 & \frac{13}{9} & -5 \\ 16 & 12 & -60 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.21}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 16 R_1} \begin{pmatrix} 1 & \frac{13}{9} & -5 \\ 0 & \frac{-100}{9} & 20 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.22}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-9}{100} R_2} \begin{pmatrix} 1 & \frac{13}{9} & -5 \\ 0 & 1 & \frac{-9}{5} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.23}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{13}{9} R_2} \begin{pmatrix} 1 & 0 & \frac{-12}{5} \\ 0 & 1 & \frac{-9}{5} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.24}$$

Thus,

$$\mathbf{c} = \begin{pmatrix} \frac{-12}{5} \\ \frac{-9}{5} \end{pmatrix} = \begin{pmatrix} -2.4 \\ -1.8 \end{pmatrix} \tag{2.0.25}$$

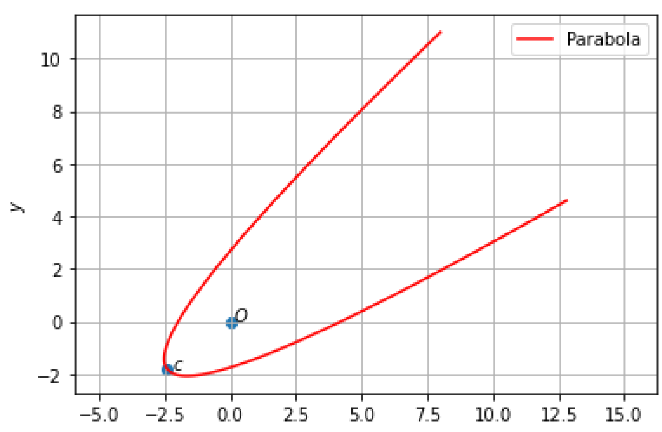

Fig. 1: Parabola with vertex c