

Challenging Problem 1

Surbhi Agarwal

Abstract—This document shows the method to find the closest points on two skew lines in 3-Dimension.

Download all python codes from

https://github.com/surbhi0912/EE5609/tree/master/challenging_problem/challenging1/codes

and latex-tikz codes from

https://github.com/surbhi0912/EE5609/tree/master/challenging_problem/challenging1

1 PROBLEM

In 3-Dimensional Space, find the points on the two skew lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (1.0.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (1.0.2)$$

such that the points are closest to each other

2 SOLUTION

In the given problem,

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (2.0.1)$$

where L_1 is passing through the point $A_1(1, 1, 0)$ and direction vector \mathbf{m}_1 ,

And L_2 is passing through the point $A_2(2, 1, -1)$ and direction vector \mathbf{m}_2

Let us take a point \mathbf{E} on Line L_1 and \mathbf{F} on Line L_2 such that they are closest to each other.

Then \mathbf{E} and \mathbf{F} can be expressed using Equation 1.0.1 and 1.0.2 respectively as follows :

$$\mathbf{E} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{F} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} \quad (2.0.3)$$

Now, the position vector from \mathbf{E} to \mathbf{F} , ie $\mathbf{F} - \mathbf{E}$ is given as,

$$\mathbf{F} - \mathbf{E} = (\mathbf{A}_2 + \lambda_2 \mathbf{m}_2) - (\mathbf{A}_1 + \lambda_1 \mathbf{m}_1) \quad (2.0.4)$$

$$= (\mathbf{A}_2 - \mathbf{A}_1) + (\lambda_2 \mathbf{m}_2 - \lambda_1 \mathbf{m}_1) \quad (2.0.5)$$

$$\Rightarrow \mathbf{F} - \mathbf{E} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} - \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (2.0.6)$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \quad (2.0.7)$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ -5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \quad (2.0.8)$$

Since the points \mathbf{E} and \mathbf{F} are closest to each other, position vector $\mathbf{F} - \mathbf{E}$ is perpendicular to the skew lines L_1 and L_2

Thus we can say that $\mathbf{F} - \mathbf{E}$ is perpendicular to the direction vectors of these lines, ie \mathbf{m}_1 and \mathbf{m}_2 respectively. Therefore,

$$(\mathbf{F} - \mathbf{E})^T \mathbf{m}_1 = 0 \quad (2.0.9)$$

$$(\mathbf{F} - \mathbf{E})^T \mathbf{m}_2 = 0 \quad (2.0.10)$$

Using the values of $\mathbf{F} - \mathbf{E}$, \mathbf{m}_1 , \mathbf{m}_2 from Equations 2.0.8 and 2.0.1 in Equations 2.0.9 and 2.0.10, we get

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \left(\begin{pmatrix} 3 & -2 \\ -5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \right)^T \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0 \quad (2.0.11)$$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + \left(\begin{pmatrix} 3 & -2 \\ -5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \right)^T \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 0 \quad (2.0.12)$$

This gives us,

$$6\lambda_1 - 13\lambda_2 = 1 \quad (2.0.13)$$

$$13\lambda_1 - 38\lambda_2 = 1 \quad (2.0.14)$$

Solving Equation 2.0.13 and 2.0.14, $\lambda_1 = \frac{25}{59}$ and $\lambda_2 = \frac{7}{59}$

$$\mathbf{E} = \begin{pmatrix} \frac{109}{59} \\ \frac{34}{59} \\ \frac{25}{59} \end{pmatrix} \quad (2.0.15)$$

$$= \begin{pmatrix} 1.85 \\ 0.58 \\ 0.42 \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{F} = \begin{pmatrix} \frac{139}{59} \\ \frac{24}{59} \\ \frac{-45}{59} \end{pmatrix} \quad (2.0.17)$$

$$= \begin{pmatrix} 2.36 \\ 0.41 \\ -0.76 \end{pmatrix} \quad (2.0.18)$$

The figure obtained is shown below

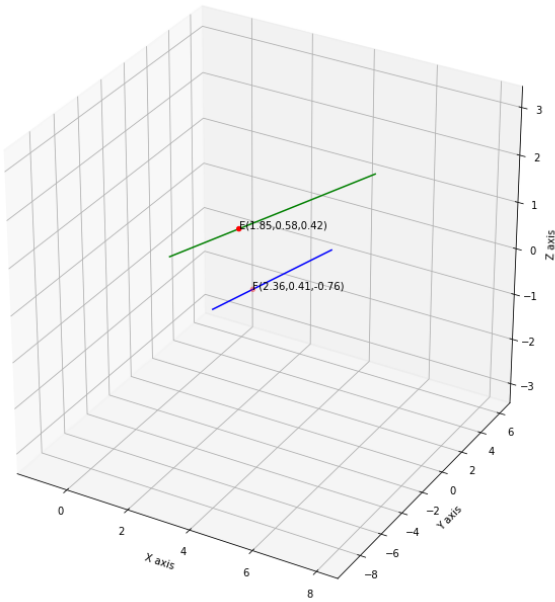


Fig. 1: Closest points between skew lines L_1 and L_2