Assignment 11

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***Abstract*—This document illustrates linear transformation matrices with respect to a set of linearly independent eigenvectors.**

1 Problem

Let $\mathbf{S} : \mathbb{R}^n \to \mathbb{R}^n$ be given by $\mathbf{S}(\mathbf{v}) = \alpha\mathbf{v}$, for a fixed $\alpha \in \mathbb{R}, \alpha \neq 0$. Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $\mathbf{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of linearly independent eigenvectors of $\mathbf{T}$. Then

1) The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is diagonal
2) The matrix of $(\mathbf{T} - \mathbf{S})$ with respect to $\mathbf{B}$ is diagonal
3) The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is not necessarily diagonal, but is upper triangular
4) The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is diagonal but the matrix of $(\mathbf{T} - \mathbf{S})$ with respect to $\mathbf{B}$ is not diagonal.

2 Solution

Given that $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and B represents a set of linearly independent eigenvectors of $\mathbf{T}$ given as follows

$$\mathbf{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \tag{2.0.1}$$

So,

$$\mathbf{T}\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{2.0.2}$$

where λ_i represents the eigenvalue λ_i corresponding to $\mathbf{v}_i$. Hence, the matrix $\mathbf{T}$ with respect to $\mathbf{B}$ can be represented as

$$\mathbf{T} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \tag{2.0.3}$$

And,

$$\begin{aligned} (\mathbf{T} - \mathbf{S})\mathbf{v}_i &= \mathbf{T}(\mathbf{v}_i) - \mathbf{S}(\mathbf{v}_i) && (2.0.4) \\ &= \lambda_i \mathbf{v}_i - \alpha \mathbf{v}_i && (2.0.5) \\ &= (\lambda_i - \alpha)\mathbf{v}_i && (2.0.6) \end{aligned}$$

Hence, matrix of $\mathbf{T} - \mathbf{S}$ with respect to $\mathbf{B}$ can be represented as

$$\mathbf{T} - \mathbf{S} = \begin{pmatrix} \lambda_1 - \alpha & 0 & \dots & 0 \\ 0 & \lambda_2 - \alpha & \dots & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \lambda_n - \alpha \end{pmatrix} \tag{2.0.7}$$

1. The matrix of $\mathbf{T}$ w.r.t to $\mathbf{B}$ is diagonal	True, as seen from (2.0.3)
2. The matrix of $(\mathbf{T} - \mathbf{S})$ w.r.t $\mathbf{B}$ is diagonal	True, as seen from (2.0.7)
3. The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is not necessarily diagonal but is upper triangular	False, as already proved $\mathbf{T}$ is diagonal
4. The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is diagonal but the matrix of $(\mathbf{T} - \mathbf{S})$ with respect to $\mathbf{B}$ is not diagonal	False, as already proved $\mathbf{T} - \mathbf{S}$ is diagonal

TABLE 1: Verifying the given options