# Challenging Problem 1

Surbhi Agarwal

*Abstract*—**This document shows the method to find the closest points on two skew lines in 3-Dimension.**

Download all python codes from

https://github.com/surbhi0912/EE5609/tree/master/challenging_problem/challenging1/codes

and latex-tikz codes from

https://github.com/surbhi0912/EE5609/tree/master/challenging_problem/challenging1

## 1 Problem

In 3-Dimensional Space, find the points on the two skew lines

$$L_1 : \mathbf{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{1.0.2}$$

such that the points are closest to each other

## 2 Solution

In the given problem,

$$\mathbf{A}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix}2\\1\\-1\end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{2.0.1}$$

where $L_1$ is passing through the point $A_1(1, 1, 0)$ and direction vector $\mathbf{m}_1$,
And $L_2$ is passing through the point $A_2(2, 1, -1)$ and direction vector $\mathbf{m}_2$

Let us take a point $\mathbf{E}$ on Line $L_1$ and $\mathbf{F}$ on Line $L_2$ such that they are closest to each other.

Then $\mathbf{E}$ and $\mathbf{F}$ can be expressed using Equation 1.0.1 and 1.0.2 respectively as follows :

$$\mathbf{E} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{F} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\5\\-2\end{pmatrix} \tag{2.0.3}$$

Now, the position vector from $\mathbf{E}$ to $\mathbf{F}$, ie $\mathbf{F}-\mathbf{E}$ is given as,

$$\mathbf{F}-\mathbf{E} = (\mathbf{A}_2 + \lambda_2\mathbf{m}_2) - (\mathbf{A}_1 + \lambda_1\mathbf{m}_1) \tag{2.0.4}$$

$$= (\mathbf{A}_2 - \mathbf{A}_1) + (\lambda_2\mathbf{m}_2 - \lambda_1\mathbf{m}_1) \tag{2.0.5}$$

$$= (\mathbf{A}_2 - \mathbf{A}_1) + \begin{pmatrix}\mathbf{m}_2 & -\mathbf{m}_1\end{pmatrix}\begin{pmatrix}\lambda_2\\\lambda_1\end{pmatrix} \tag{2.0.6}$$

$$\implies \mathbf{F}-\mathbf{E} = \begin{pmatrix}2\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}3 & -2\\-5 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}\lambda_2\\\lambda_1\end{pmatrix} \tag{2.0.7}$$

$$= \begin{pmatrix}1\\0\\-1\end{pmatrix} + \begin{pmatrix}3 & -2\\-5 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}\lambda_2\\\lambda_1\end{pmatrix} \tag{2.0.8}$$

Since the points $\mathbf{E}$ and $\mathbf{F}$ are closest to each other, position vector $\mathbf{F}-\mathbf{E}$ is perpendicular to the skew lines $L_1$ and $L_2$, thus we can say that $\mathbf{F}-\mathbf{E}$ is perpendicular to the direction vectors of these lines, ie $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively. Therefore,

$$\mathbf{m}_1^T(\mathbf{F}-\mathbf{E}) = 0 \tag{2.0.9}$$

$$\mathbf{m}_2^T(\mathbf{F}-\mathbf{E}) = 0 \tag{2.0.10}$$

Using the values of $\mathbf{F}-\mathbf{E}$ from Equation 2.0.6 and combining Equations 2.0.9 and 2.0.10,

$$\begin{pmatrix}\mathbf{m}_1^T\\\mathbf{m}_2^T\end{pmatrix}\left((\mathbf{A}_2 - \mathbf{A}_1) + \begin{pmatrix}\mathbf{m}_2 & -\mathbf{m}_1\end{pmatrix}\begin{pmatrix}\lambda_2\\\lambda_1\end{pmatrix}\right) = 0 \tag{2.0.11}$$

$$\implies \begin{pmatrix}\mathbf{m}_1^T\\\mathbf{m}_2^T\end{pmatrix}(\mathbf{A}_2 - \mathbf{A}_1) + \begin{pmatrix}\mathbf{m}_1^T\\\mathbf{m}_2^T\end{pmatrix}\begin{pmatrix}\mathbf{m}_2 & -\mathbf{m}_1\end{pmatrix}\begin{pmatrix}\lambda_2\\\lambda_1\end{pmatrix} = 0 \tag{2.0.12}$$

Now substituting the values from Equation (1.0.1) in (2.0.12)

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} = 0 \tag{2.0.13}$$

$$\implies \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 13 & -6 \\ 38 & -13 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} = 0 \tag{2.0.14}$$

$$\implies \begin{pmatrix} 13 & -6 \\ 38 & -13 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.15}$$

$$\implies \begin{pmatrix} 13 & -6 \\ 38 & -13 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{2.0.16}$$

This gives us,

$$13\lambda_2 - 6\lambda_1 = -1 \tag{2.0.17}$$

$$38\lambda_2 - 13\lambda_1 = -1 \tag{2.0.18}$$

Solving Equation 2.0.17 and 2.0.18, $\lambda_1 = \frac{25}{59}$ and $\lambda_2 = \frac{7}{59}$

$$\mathbf{E} = \begin{pmatrix} \frac{109}{59} \\ \frac{34}{59} \\ \frac{25}{59} \end{pmatrix} \tag{2.0.19}$$

$$= \begin{pmatrix} 1.85 \\ 0.58 \\ 0.42 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{F} = \begin{pmatrix} \frac{139}{59} \\ \frac{24}{59} \\ \frac{-45}{59} \end{pmatrix} \tag{2.0.21}$$

$$= \begin{pmatrix} 2.36 \\ 0.41 \\ -0.76 \end{pmatrix} \tag{2.0.22}$$

The figure obtained is shown in Fig 1

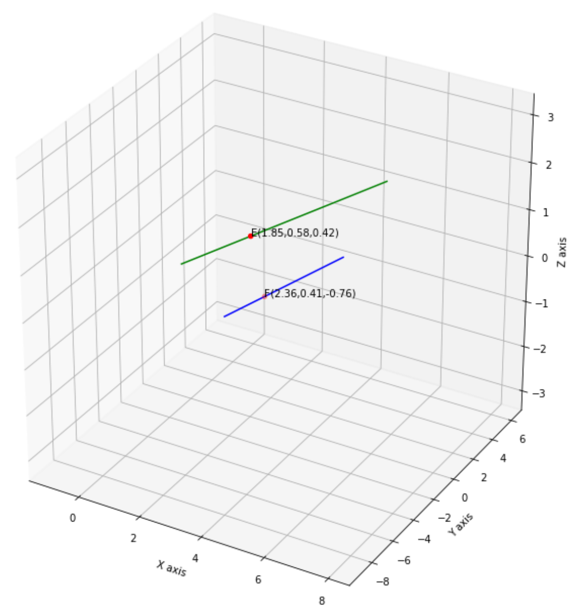


Fig. 1: Closest points between skew lines $L_1$ and $L_2$