Assignment 9

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***Abstract*—This document deals with properties of subspaces of a finite dimensional vector space.**

1 Problem

Let W_1 and W_2 be subspaces of a finite-dimensional vector space $\mathbb{V}$. Prove that

1) $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$
2) $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$

2 Solution

Given	W_1 and W_2 are subspaces of a finite dimensional vector space $\mathbb{V}$
1. To prove	$(W_1 + W_2)^0 = W_1^0 \cap W_2^0$
Proof of $(W_1 + W_2)^0 \subseteq (W_1^0 \cap W_2^0)$	Let $f \in (W_1 + W_2)^0$ $\forall \mathbf{v} \in (W_1 + W_2)$ $f(\mathbf{v}) = 0$ $\implies \forall \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2$ $\mathbf{w_1} + \mathbf{w_2} \in (W_1 + W_2)$ $\therefore f(\mathbf{w_1} + \mathbf{w_2}) = 0$ $\implies$ When $\mathbf{w}_2 = 0$, then $\forall \mathbf{w}_1 \in W_1$ $f(\mathbf{w_1}) = 0$ $\therefore f \in W_1^0$ And when $\mathbf{w_1} = 0, \forall \mathbf{w_2} \in W_2$ $f(\mathbf{w_2}) = 0$ $\therefore f \in W_2^0$ $\because f \in W_1^0, f \in W_2^0$ $\implies f \in (W_1^0 \cap W_2^0)$ $\therefore (W_1 + W_2)^0 \subseteq (W_1^0 \cap W_2^0)$
Proof of $(W_1^0 \cap W_2^0) \subseteq (W_1 + W_2)^0$	Let $f \in (W_1^0 \cap W_2^0)$ $\implies f \in W_1^0, f \in W_2^0$ $\implies \forall \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2$ $f(\mathbf{w}_1) = 0$ and $f(\mathbf{w}_2) = 0$ $\forall \mathbf{v} \in (W_1 + W_2)$, $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ $\implies f(\mathbf{v}) = f(\mathbf{w}_1 + \mathbf{w}_2)$ $\implies f(\mathbf{v}) = f(\mathbf{w}_1) + f(\mathbf{w}_2)$ $\implies f(\mathbf{v}) = 0$ $\implies f \in (W_1 + W_2)^0$ $\therefore (W_1^0 \cap W_2^0) \subseteq (W_1 + W_2)^0$
	Hence, $(W_1 + W_2)^0 = (W_1^0 \cap W_2^0)$
2. To prove	$(W_1 \cap W_2)^0 = W_1^0 + W_2^0$
Proof of $(W_1^0 + W_2^0) \subseteq (W_1 \cap W_2)^0$	Let $f \in (W_1^0 + W_2^0)$ for some $f_1 \in W_1^0, f_2 \in W_2^0$, $f = f_1 + f_2$ Now, for $\mathbf{v} \in (W_1 \cap W_2)$ $f(\mathbf{v}) = (f_1 + f_2)(\mathbf{v})$ $\implies f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v})$ $\because \mathbf{v} \in (W_1 \cap W_2)$ $\implies \mathbf{v} \in W_1$, and $\mathbf{v} \in W_2$ So, $f_1(\mathbf{v}) = 0$, and $f_2(\mathbf{v}) = 0$ $\implies f(\mathbf{v}) = 0 + 0 = 0$ $\implies f \in (W_1 \cap W_2)^0$ $\therefore (W_1^0 + W_2^0) \subseteq (W_1 \cap W_2)^0$

Proof of $(W_1 \cap W_2)^0 \subseteq (W_1^0 + W_2^0)$	Let $f \in (W_1 \cap W_2)^0$ Assuming Basis of W_1 as $\{\alpha_1, \ldots, \alpha_{\mathbf{k}}, \beta_1, \ldots, \beta_l\}$ Basis of W_2 as $\{\alpha_1, \ldots, \alpha_{\mathbf{k}}, \gamma_1, \ldots, \gamma_m\}$ $\therefore$ Basis of $(W_1 \cap W_2)$ is $\{\alpha_1, \ldots, \alpha_{\mathbf{k}}\}$ and Basis of $W_1 + W_2$ is $\{\alpha_1, \ldots, \alpha_{\mathbf{k}}, \beta_1, \ldots, \beta_l, \gamma_1, \ldots, \gamma_m\}$ Now, for $\mathbf{v} \in (W_1 + W_2)$ $\mathbf{v} = \sum_{i=1}^{k} x_i\alpha_i + \sum_{i=1}^{l} y_i\beta_i + \sum_{i=1}^{m} z_i\gamma_i$ $f(\mathbf{v}) = \sum_{i=1}^{k} a_i\alpha_i + \sum_{i=1}^{l} b_i\beta_i + \sum_{i=1}^{m} c_i\gamma_i$ $\forall \mathbf{v} \in (W_1 \cap W_2)$ $\mathbf{v} = \sum_{i=1}^{k} x_i\alpha_i$ $f(\mathbf{v}) = \sum_{i=1}^{k} a_i\alpha_i$ But since $f \in (W_1 \cap W_2)^0$, thus $f(\mathbf{v}) = 0$ $\therefore a_1 = a_2 = \ldots = a_k = 0$ So, we can now write $f(\mathbf{v}) = \sum_{i=1}^{l} b_i\beta_i + \sum_{i=1}^{m} c_i\gamma_i$ Now, $\forall \mathbf{v} \in W_1$, $\mathbf{v} = \sum_{i=1}^{k} x_i\alpha_i + \sum_{i=1}^{l} y_i\beta_i$ $\implies f_1(\mathbf{v}) = \sum_{i=1}^{k} a_i\alpha_i + \sum_{i=1}^{l} b_i\beta_i$ Comparing this to the original equation, we can say $c_i = 0$ And $\forall \mathbf{v} \in W_2$, $\mathbf{v} = \sum_{i=1}^{k} x_i\alpha_i + \sum_{i=1}^{m} z_i\gamma_i$ $\implies f_2(\mathbf{v}) = \sum_{i=1}^{k} a_i\alpha_i + \sum_{i=1}^{m} c_i\gamma_i$ Comparing this to the original equation, we can say $b_i = 0$ $f = f_1 + f_2$ $\because a_i = b_i = c_i = 0$, $f_1(\mathbf{v}) = 0$ $\implies f_1 \in W_1^0$ Also, $f_2(\mathbf{v}) = 0$ $\implies f_2 \in W_2^0$ So, $f_1 + f_2 \in (W_1^0 + W_2^0)$ $\implies f \in (W_1^0 + W_2^0)$ $\therefore (W_1 \cap W_2)^0 \subseteq (W_1^0 + W_2^0)$
	Hence, $(W_1 \cap W_2)^0 = (W_1^0 + W_2^0)$