Assignment 3

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***Abstract*—This document proves that a given equation represents two straight lines and finds the point of intersection and angle between them**

Download all python codes from

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https://github.com/surbhi0912/EE5609/
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and latex-tikz codes from

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https://github.com/surbhi0912/EE5609/
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1 Problem

Prove that the following equations represent two straight lines; and also find their point of intersection and the angle between them

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

2 Solution

2.1 Proving that given equation represents two straight lines

The given equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0 \tag{2.1.1}$$

Comparing this to the standard equation,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \tag{2.1.2}$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.1.3}$$

$$f = -2 \tag{2.1.4}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{2.1.5}$$

Equation (2.1.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.1.6}$$

$$\delta = \begin{vmatrix} 1 & \frac{-5}{2} & \frac{1}{2} \\ \frac{-5}{2} & 4 & 1 \\ \frac{1}{2} & 1 & -2 \end{vmatrix} \tag{2.1.7}$$

$$= 0 \tag{2.1.8}$$

Hence, proved that given equation represents two straight lines.

2.2 Finding point of intersection between the straight lines

$$\det V = \begin{vmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{vmatrix} \tag{2.2.1}$$

$$= \frac{-9}{4} < 0 \tag{2.2.2}$$

Thus, the two straight lines intersect. Let the equation of the straight lines be given as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \tag{2.2.3}$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \tag{2.2.4}$$

with their slopes as $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively.

Then the equation of the pair of straight lines is

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \tag{2.2.5}$$

Using (2.1.5) and (2.2.5),

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 \tag{2.2.6}$$

Comparing both sides,

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.2.7}$$

$$c_1 c_2 = -2 \tag{2.2.8}$$

Slopes of the lines are roots of the equation

$$cm^2 + 2bm + a = 0 \tag{2.2.9}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.2.10}$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.2.11}$$

Substituting (2.1.1) in (2.2.9),

$$4m^2 - 5m + 1 = 0 \tag{2.2.12}$$

$$\implies m_i = \frac{\frac{5}{2} \pm \frac{3}{2}}{4} \tag{2.2.13}$$

$$\implies m_1 = 1, m_2 = \frac{1}{4} \tag{2.2.14}$$

Therefore,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.2.15}$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} \tag{2.2.16}$$

We know that

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.2.17}$$

$$k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \tag{2.2.18}$$

$$\implies k_1 k_2 = 4 \tag{2.2.19}$$

Taking $k_1 = 1$, $k_2 = 4$, we get

$$\mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{n}_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{2.2.20}$$

For verifying values of $\mathbf{n}_1$ and $\mathbf{n}_2$, we compute the convolution by representing $\mathbf{n}_1$ as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \tag{2.2.21}$$

Now, obtaining c_1 and c_2 using (2.2.20) and (2.2.7)

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.2.22}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \tag{2.2.23}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 4 & -2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & -2 \end{pmatrix} \tag{2.2.24}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \tag{2.2.25}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \tag{2.2.26}$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$c_1 = -1 \tag{2.2.27}$$

$$c_2 = 2 \tag{2.2.28}$$

Thus, equation of lines can be written as

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{2.2.29}$$

$$\begin{pmatrix} -1 & 4 \end{pmatrix} \mathbf{x} = 2 \tag{2.2.30}$$

Augmented matrix for these set of equations is

$$\begin{pmatrix} -1 & 1 & -1 \\ -1 & 4 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 4 & 2 \end{pmatrix} \tag{2.2.31}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.2.32}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.2.33}$$

Thus, the point of intersection is $\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

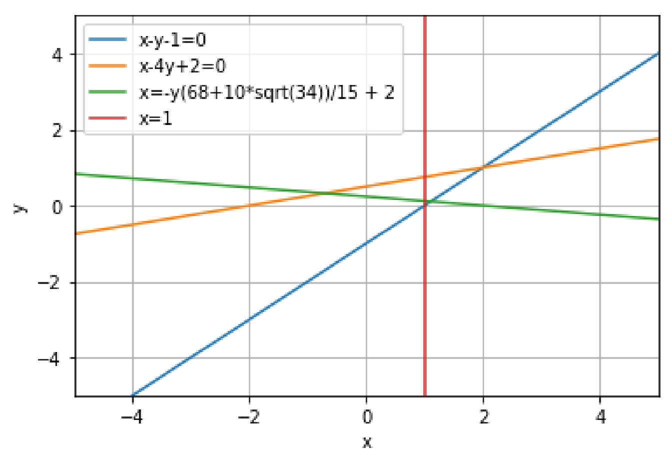

Fig. 1: Intersection of pair of original pair of straight lines and the pair of straight lines after affine transform

Using (2.2.20) and (2.2.28) in (2.2.5), equation of the pair of straight lines is

$$(x - y - 1)(x - 4y + 2) = 0 \tag{2.2.34}$$

2.3 Angle between lines

Angle between pair of lines is,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) \tag{2.3.1}$$

$$\mathbf{n}_1^T\mathbf{n}_2 = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}-1\\4\end{pmatrix} = 5 \tag{2.3.2}$$

$$\|\mathbf{n}_1\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \tag{2.3.3}$$

$$\|\mathbf{n}_2\| = \sqrt{(-1)^2 + 4^2} = \sqrt{17} \tag{2.3.4}$$

Substituting these values (2.3.1)

$$\theta = 30.9^\circ \tag{2.3.5}$$

Hence, angle between the given pair of straight lines is 30.9°

2.4 Affine Transformation and Eigen Value decomposition

First, verifying if $\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 0$. To do this, finding V^{-1} by augmenting with identity matrix and row reducing as follows :

$$\begin{pmatrix}1 & \frac{-5}{2} & 1 & 0\\ \frac{-5}{2} & 4 & 0 & 1\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2+\frac{5}{2}R_1} \begin{pmatrix}1 & \frac{-5}{2} & 1 & 0\\ 0 & \frac{-9}{4} & \frac{5}{2} & 1\end{pmatrix} \tag{2.4.1}$$

$$\xleftrightarrow{R_2\leftarrow \frac{-4}{9}R_2} \begin{pmatrix}1 & \frac{-5}{2} & 1 & 0\\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9}\end{pmatrix} \tag{2.4.2}$$

$$\xleftrightarrow{R_1\leftarrow R_1+\frac{5}{2}R_2} \begin{pmatrix}1 & 0 & \frac{-16}{9} & \frac{-10}{9}\\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9}\end{pmatrix} \tag{2.4.3}$$

$$\implies \mathbf{V}^{-1} = \begin{pmatrix}\frac{-16}{9} & \frac{-10}{9}\\ \frac{-10}{9} & \frac{-4}{9}\end{pmatrix} \tag{2.4.4}$$

$$u^TV^{-1}u - f = \begin{pmatrix}\frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}\frac{-16}{9} & \frac{-10}{9}\\ \frac{-10}{9} & \frac{-4}{9}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix} - (-2) \tag{2.4.5}$$

$$= 0 \tag{2.4.6}$$

The characteristic equation of $\mathbf{V}$ is given as :

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda - 1 & \frac{5}{2}\\ \frac{5}{2} & \lambda - 4\end{vmatrix} = 0 \tag{2.4.7}$$

$$\implies (\lambda - 1)(\lambda - 4) - \frac{25}{4} = 0 \tag{2.4.8}$$

$$\implies 4\lambda^2 - 20\lambda - 9 = 0 \tag{2.4.9}$$

The roots of (2.4.9), i.e. the eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = \frac{5 + \sqrt{34}}{2}, \lambda_2 = \frac{5 - \sqrt{34}}{2} \tag{2.4.10}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.4.11}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{2.4.12}$$

For $\lambda_1 = \frac{5 + \sqrt{34}}{2}$

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix}\frac{3+\sqrt{34}}{2} & \frac{5}{2}\\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2}\end{pmatrix} \tag{2.4.13}$$

To find $\mathbf{p}_1$, let's look at Augmented form of $(\lambda_1\mathbf{I}-\mathbf{V})$

$$\begin{pmatrix}\frac{3+\sqrt{34}}{2} & \frac{5}{2} & 0\\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0\end{pmatrix} \tag{2.4.14}$$

$$\xleftrightarrow{R_1\leftarrow \frac{2}{3+\sqrt{34}}R_1} \begin{pmatrix}1 & \frac{-3+\sqrt{34}}{5} & 0\\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0\end{pmatrix} \tag{2.4.15}$$

$$\xleftrightarrow{R_2\leftarrow \frac{2}{5}R_2-R_1} \begin{pmatrix}1 & \frac{-3+\sqrt{34}}{5} & 0\\ 0 & 0 & 0\end{pmatrix} \tag{2.4.16}$$

So we get

$$x_1 + \left(\frac{-3 + \sqrt{34}}{5}\right)x_2 = 0 \tag{2.4.17}$$

Thus, our eigenvector corresponding to λ_1

$$\mathbf{p}_1 = \begin{pmatrix}\frac{3-\sqrt{34}}{5}\\ 1\end{pmatrix} \tag{2.4.18}$$

For $\lambda_2 = \frac{5 - \sqrt{34}}{2}$

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix}\frac{3-\sqrt{34}}{2} & \frac{5}{2}\\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2}\end{pmatrix} \tag{2.4.19}$$

To find $\mathbf{p}_2$, let's look at Augmented form of $(\lambda_2\mathbf{I}-\mathbf{V})$

$$\begin{pmatrix}\frac{3-\sqrt{34}}{2} & \frac{5}{2} & 0\\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0\end{pmatrix} \tag{2.4.20}$$

$$\xleftrightarrow{R_1\leftarrow \frac{2}{3-\sqrt{34}}R_1} \begin{pmatrix}1 & \frac{-3-\sqrt{34}}{5} & 0\\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0\end{pmatrix} \tag{2.4.21}$$

$$\xleftrightarrow{R_2\leftarrow \frac{2}{5}R_2-R_1} \begin{pmatrix}1 & \frac{-3-\sqrt{34}}{5} & 0\\ 0 & 0 & 0\end{pmatrix} \tag{2.4.22}$$

So we get

$$x_1 + \left(\frac{-3-\sqrt{34}}{5}\right)x_2 = 0 \tag{2.4.23}$$

Thus, our eigenvector corresponding to λ_2

$$\mathbf{p}_2 = \begin{pmatrix} \frac{3+\sqrt{34}}{5} \\ 1 \end{pmatrix} \tag{2.4.24}$$

We know $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where $\mathbf{P}$ and the diagonal matrix $\mathbf{D}$ are given as:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.4.25}$$

$$= \begin{pmatrix} \frac{5+\sqrt{34}}{2} & 0 \\ 0 & \frac{5-\sqrt{34}}{2} \end{pmatrix} \tag{2.4.26}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \tag{2.4.27}$$

$$= \begin{pmatrix} \frac{3-\sqrt{34}}{5} & \frac{3+\sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \tag{2.4.28}$$

So, the equation of the pair of straight lines is given by :

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.4.29}$$

$$\mathbf{y}^T \begin{pmatrix} \dfrac{5+\sqrt{34}}{2} & 0 \\ 0 & \dfrac{5-\sqrt{34}}{2} \end{pmatrix} \mathbf{y} = 0 \tag{2.4.30}$$

$$\implies \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \dfrac{5+\sqrt{34}}{2} & 0 \\ 0 & \dfrac{5-\sqrt{34}}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \tag{2.4.31}$$

$$\implies (5+\sqrt{34})y_1^2 + (5-\sqrt{34})y_2^2 = 0 \tag{2.4.32}$$

So we get the equation of the pair of straight lines, as we can see this passes through the origin $(0, 0)$. The corresponding image is shown in Fig. 2

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.4.33}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.4.34}$$

And,

$$\mathbf{P}^T = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & 1 \\ \frac{3+\sqrt{34}}{5} & 1 \end{pmatrix} \tag{2.4.35}$$

Using affine transformation, we can express the equation as

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.4.36}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & \frac{3+\sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.4.37}$$

The corresponding image is shown in Fig. 1

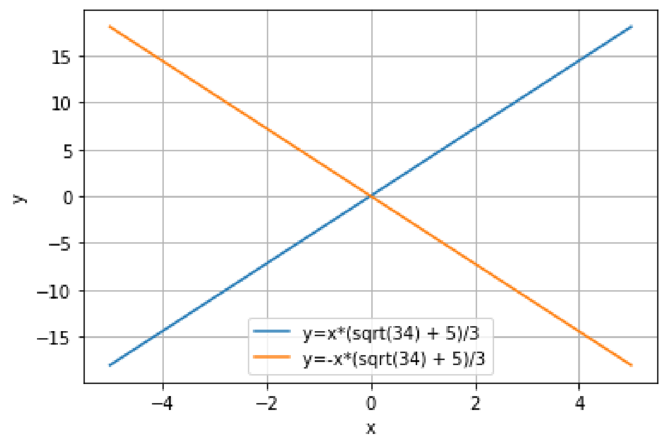

Fig. 2: Pair of straight lines passing through origin after eigenvalue decomposition