Challenging Problem 1

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***Abstract*—This document shows the method to find the closest points on two skew lines in 3-Dimension.**

Download all python codes from

https://github.com/surbhi0912/EE5609/tree/master/challenging_problem/challenging1/codes

and latex-tikz codes from

https://github.com/surbhi0912/EE5609/tree/master/challenging_problem/challenging1

1 Problem

In 3-Dimensional Space, find the points on the two skew lines

$$L_1 : \mathbf{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{1.0.2}$$

such that the points are closest to each other

2 Solution

In the given problem,

$$\mathbf{A_1} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{m_1} = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{A_2} = \begin{pmatrix}2\\1\\-1\end{pmatrix}, \mathbf{m_2} = \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{2.0.1}$$

where L_1 is passing through the point $A_1(1, 1, 0)$ and direction vector $\mathbf{m_1}$,
And L_2 is passing through the point $A_2(2, 1, -1)$ and direction vector $\mathbf{m_2}$

Let us take a point $\mathbf{E}$ on Line L_1 and $\mathbf{F}$ on Line L_2 such that they are closest to each other.

Then $\mathbf{E}$ and $\mathbf{F}$ can be expressed using Equation 1.0.1 and 1.0.2 respectively as follows :

$$\mathbf{E} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{2.0.2}$$

$$\implies \mathbf{E} = \begin{pmatrix}1 + 2\lambda_1\\1 - \lambda_1\\\lambda_1\end{pmatrix} \tag{2.0.3}$$

$$\mathbf{F} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\5\\-2\end{pmatrix} \tag{2.0.4}$$

$$\implies \mathbf{F} = \begin{pmatrix}2 + 3\lambda_2\\1 - 5\lambda_2\\-1 + 2\lambda_2\end{pmatrix} \tag{2.0.5}$$

Now, the position vector from $\mathbf{E}$ to $\mathbf{F}$, ie $\mathbf{EF}$ is given as,

$$\mathbf{EF} = \begin{pmatrix}(2 + 3\lambda_2) - (1 + 2\lambda_1)\\(1 - 5\lambda_2) - (1 - \lambda_1)\\(-1 + 2\lambda_2) - (\lambda_1)\end{pmatrix} \tag{2.0.6}$$

$$\implies \mathbf{EF} = \begin{pmatrix}1 + 3\lambda_2 - 2\lambda_1\\-5\lambda_2 + \lambda_1\\-1 + 2\lambda_2 - \lambda_1\end{pmatrix} \tag{2.0.7}$$

Since the points $\mathbf{E}$ and $\mathbf{F}$ are closest to each other, position vector $\mathbf{EF}$ is perpendicular to the skew lines L_1 and L_2

Thus we can say that $\mathbf{EF}$ is perpendicular to the direction vectors of these lines, ie $\mathbf{m_1}$ and $\mathbf{m_2}$ respectively. Therefore,

$$(\mathbf{EF})^T \mathbf{m_1} = 0 \tag{2.0.8}$$

$$(\mathbf{EF})^T \mathbf{m_2} = 0 \tag{2.0.9}$$

Using the values of $\mathbf{EF}$, $\mathbf{m_1}$, $\mathbf{m_2}$ from Equations 2.0.7 and 2.0.1 in Equations 2.0.8 and 2.0.9, we get

$$\begin{pmatrix}1 + 3\lambda_2 - 2\lambda_1\\-5\lambda_2 + \lambda_1\\-1 + 2\lambda_2 - \lambda_1\end{pmatrix}^T \begin{pmatrix}2\\-1\\1\end{pmatrix} = 0 \tag{2.0.10}$$

$$\begin{pmatrix}1 + 3\lambda_2 - 2\lambda_1\\-5\lambda_2 + \lambda_1\\-1 + 2\lambda_2 - \lambda_1\end{pmatrix}^T \begin{pmatrix}3\\-5\\2\end{pmatrix} = 0 \tag{2.0.11}$$

This gives us,

$$6\lambda_1 - 13\lambda_2 = 1 \tag{2.0.12}$$

$$13\lambda_1 - 38\lambda_2 = 1 \tag{2.0.13}$$

Solving Equation 2.0.12 and 2.0.13, $\lambda_1 = \frac{25}{59}$ and $\lambda_2 = \frac{7}{59}$

$$\mathbf{E} = \begin{pmatrix} 109/59 \\ 34/59 \\ 25/59 \end{pmatrix} \tag{2.0.14}$$

$$= \begin{pmatrix} 1.84 \\ 0.57 \\ 0.42 \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{F} = \begin{pmatrix} 139/59 \\ 24/59 \\ -45/59 \end{pmatrix} \tag{2.0.16}$$

$$= \begin{pmatrix} 2.35 \\ 0.40 \\ -0.76 \end{pmatrix} \tag{2.0.17}$$

The figure obtained is shown below

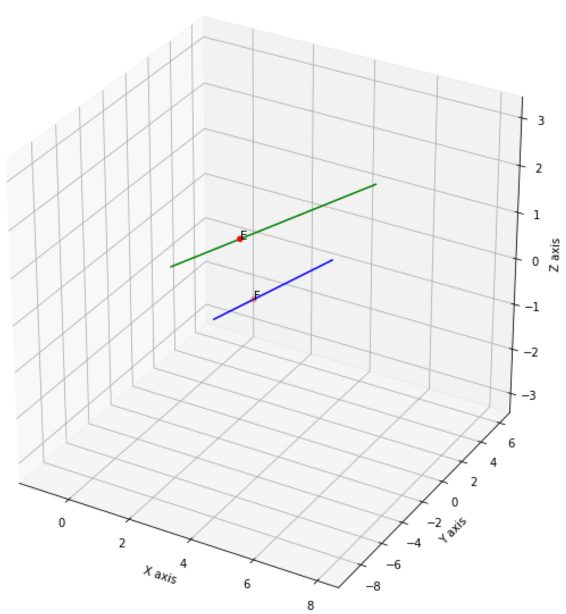

Fig. 0: Closest points between skew lines L_1 and L_2