Assignment 6

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***Abstract*—This document deals with QR decomposition and Singlular Value Decomposition**

Download all python codes from

https://github.com/surbhi0912/EE5609/

and latex-tikz codes from

https://github.com/surbhi0912/EE5609/

1 Problem

1. Find the QR decomposition of $\mathbf{V} = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix}$
2. Find the vertex of a parabola

$$(4x + 3y + 15)^2 = 5(3x - 4y)$$

using SVD and verify solution using least squares.

2 Solution

2.1 QR decomposition of V

Let the column vectors of $\mathbf{V}$ be α and β:

$$\alpha = \begin{pmatrix} 16 \\ 12 \end{pmatrix} \tag{2.1.1}$$

$$\beta = \begin{pmatrix} 12 \\ 9 \end{pmatrix} \tag{2.1.2}$$

We can express

$$\alpha = k_1 \mathbf{u}_1 \tag{2.1.3}$$

$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \tag{2.1.4}$$

where

$$k_1 = \|\alpha\| = \sqrt{16^2 + 12^2} = 20 \tag{2.1.5}$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} = \frac{1}{20}\begin{pmatrix} 16 \\ 12 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \tag{2.1.6}$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} = 15 \tag{2.1.7}$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.8}$$

$$k_2 = \mathbf{u}_2^T \beta = 0 \tag{2.1.9}$$

From (2.1.3) and (2.1.4),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.1.10}$$

where,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \tag{2.1.11}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.1.12}$$

$\mathbf{R}$ should be an upper triangular matrix and $\mathbf{Q}$ an orthogonal matrix such that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

Here, we see that the second column vector of $\mathbf{Q}$ is zero since the column vectors of $\mathbf{V}$ are dependent. Therefore, we can effecively write $\mathbf{Q}$ as:

$$\mathbf{Q} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \tag{2.1.13}$$

Verifying $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} = 1 \tag{2.1.14}$$

Therefore, for the given matrix $\mathbf{V}$, we can write $\mathbf{QR}$ decomposition as the product of respective row and column vectors as,

$$\mathbf{V} = \mathbf{QR} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \begin{pmatrix} 20 & 15 \end{pmatrix} \tag{2.1.15}$$

2.2 Singular Value Decomposition for finding Vertex

The given equation can be rewritten as

$$16x^2 + 24xy + 9y^2 + 105x + 110y + 225 = 0 \tag{2.2.1}$$

Comparing this to the standard equation,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \frac{105}{2} \\ 55 \end{pmatrix}, \quad f = 225 \tag{2.2.2}$$

The characteristic equation of $\mathbf{V}$ is given as

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{2.2.3}$$

$$\implies \begin{vmatrix} \lambda - 16 & -12 \\ -12 & \lambda - 9 \end{vmatrix} = 0 \tag{2.2.4}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{2.2.5}$$

The eigenvalues are the roots of the equation (2.2.5), which are as follows :

$$\lambda_1 = 0, \quad \lambda_2 = 25 \tag{2.2.6}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.2.7}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{2.2.8}$$

For $\lambda_1 = 0$

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 3R_1]{R_1 \leftarrow \frac{1}{4}R_1} \begin{pmatrix} -4 & -3 \\ 0 & 0 \end{pmatrix} \tag{2.2.9}$$

$$\implies \mathbf{p_1} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{2.2.10}$$

For $\lambda_2 = 25$

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 4R_1]{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \tag{2.2.11}$$

$$\implies \mathbf{p_2} = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.2.12}$$

So, using Eigenvalue decomposition, $\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D}$, where

$$\mathbf{P} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \tag{2.2.13}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{2.2.14}$$

Then, for the parabola

$$\text{focal length} = \left|\frac{2\eta}{\lambda_2}\right| \tag{2.2.15}$$

$$\eta = \mathbf{p}_1^T\mathbf{u} = \frac{25}{2} \tag{2.2.16}$$

Substituting values from (2.2.16) and (2.2.6) in (2.2.15), we get

$$\text{focal length} = 1 \tag{2.2.17}$$

The standard equation of the parabola is given by

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.2.18}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u}^T + 2\eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ 2\eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{2.2.19}$$

Substituting values from (2.2.2),(2.2.16),(2.2.10) in (2.2.19),

$$\begin{pmatrix} \frac{75}{2} & 75 \\ 16 & 12 \\ 12 & 9 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -225 \\ \frac{-135}{2} \\ -35 \end{pmatrix} \tag{2.2.20}$$

This is of the form

$$\mathbf{A}\mathbf{c} = \mathbf{b} \tag{2.2.21}$$

To solve this, we perform Singular Value Decomposition of $\mathbf{A}$ as follows :

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.2.22}$$

where columns of $\mathbf{V}$ are eigen vectors of $\mathbf{A}^T\mathbf{A}$, columns of $\mathbf{U}$ are eigen vectors of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{S}$ is the diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T\mathbf{A}$. Now, using (2.2.22) in (2.2.21), we get

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{c} = \mathbf{b} \tag{2.2.23}$$

$$\implies \mathbf{c} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \tag{2.2.24}$$

where $\mathbf{S}_+$ is the Moore-Penrose Pseduo-Inverse of $\mathbf{S}$. Now, we see

$$\mathbf{A}\mathbf{A}^T = \begin{pmatrix} \frac{28125}{4} & 1500 & 1125 \\ 1500 & 400 & 300 \\ 1125 & 300 & 225 \end{pmatrix} \tag{2.2.25}$$

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} \frac{7225}{4} & \frac{6225}{2} \\ \frac{6225}{2} & 5850 \end{pmatrix} \tag{2.2.26}$$

Eigen values and vectors for $\mathbf{A}\mathbf{A}^T$

$$\left|\mathbf{A}\mathbf{A}^T - \lambda\mathbf{I}\right| = 0 \tag{2.2.27}$$

$$\implies \begin{vmatrix} \frac{28125}{4} - \lambda & 1500 & 1125 \\ 1500 & 400 - \lambda & 300 \\ 1125 & 300 & 225 - \lambda \end{vmatrix} = 0 \tag{2.2.28}$$

$$\implies -\lambda^3 + \frac{30625}{4}\lambda^2 - \frac{3515625}{4}\lambda = 0 \tag{2.2.29}$$

Solving (2.2.29), we get

$$\lambda_1 = \frac{-625\sqrt{2257}+30625}{8} \tag{2.2.30}$$

$$\lambda_2 = \frac{625\sqrt{2257}+30625}{8} \tag{2.2.31}$$

$$\lambda_3 = 0 \tag{2.2.32}$$

The normalized eigen vector corresponding to these eigen values is:

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-205+5\sqrt{2257}}{\sqrt{(205-5\sqrt{2257})^2+14400}} \\ \frac{-96}{\sqrt{(205-5\sqrt{2257})^2+14400}} \\ \frac{-72}{\sqrt{(205-5\sqrt{2257})^2+14400}} \end{pmatrix} \tag{2.2.33}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{205+5\sqrt{2257}}{\sqrt{(205+5\sqrt{2257})^2+14400}} \\ \frac{96}{\sqrt{(205+5\sqrt{2257})^2+14400}} \\ \frac{72}{\sqrt{(205+5\sqrt{2257})^2+14400}} \end{pmatrix} \tag{2.2.34}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix} \tag{2.2.35}$$

Thus,

$$\mathbf{U} = \begin{pmatrix} \frac{-205+5\sqrt{2257}}{\sqrt{(205-5\sqrt{2257})^2+14400}} & \frac{205+5\sqrt{2257}}{\sqrt{(205+5\sqrt{2257})^2+14400}} & 0 \\ \frac{-96}{\sqrt{(205-5\sqrt{2257})^2+14400}} & \frac{96}{\sqrt{(205+5\sqrt{2257})^2+14400}} & \frac{-3}{5} \\ \frac{-72}{\sqrt{(205-5\sqrt{2257})^2+14400}} & \frac{72}{\sqrt{(205+5\sqrt{2257})^2+14400}} & \frac{4}{5} \end{pmatrix} \tag{2.2.36}$$

$\mathbf{S}$ corresponding to eigen-values $\lambda_1, \lambda_2, \lambda_3$ is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{-625\sqrt{2257}+30625}}{2\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{625\sqrt{2257}+30625}}{2\sqrt{2}} \\ 0 & 0 \end{pmatrix} \tag{2.2.37}$$

Eigen values and vectors for $\mathbf{A}^T\mathbf{A}$

$$\left|\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{2.2.38}$$

$$\begin{vmatrix} \frac{7225}{4} - \lambda & \frac{6225}{2} \\ \frac{6225}{2} & 5850 - \lambda \end{vmatrix} = 0 \tag{2.2.39}$$

$$\implies \lambda^2 - \frac{30625}{4}\lambda + \frac{3515625}{4} = 0 \tag{2.2.40}$$

Solving (2.2.40), we get

$$\lambda_4 = \frac{-625\sqrt{2257}+30625}{8} \tag{2.2.41}$$

$$\lambda_5 = \frac{625\sqrt{2257}+30625}{8} \tag{2.2.42}$$

The normalized eigen vector corresponding to these eigen values is :

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-25\sqrt{2257}-647}{\sqrt{(25\sqrt{2257}+647)^2+992016}} \\ \frac{996}{\sqrt{(25\sqrt{2257}+647)^2+992016}} \end{pmatrix} \tag{2.2.43}$$

$$\mathbf{v}_2 = \begin{pmatrix} \frac{25\sqrt{2257}-647}{\sqrt{(25\sqrt{2257}-647)^2+992016}} \\ \frac{996}{\sqrt{(25\sqrt{2257}-647)^2+992016}} \end{pmatrix} \tag{2.2.44}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} \frac{-25\sqrt{2257}-647}{\sqrt{(25\sqrt{2257}+647)^2+992016}} & \frac{25\sqrt{2257}-647}{\sqrt{(25\sqrt{2257}-647)^2+992016}} \\ \frac{996}{\sqrt{(25\sqrt{2257}+647)^2+992016}} & \frac{996}{\sqrt{(25\sqrt{2257}-647)^2+992016}} \end{pmatrix} \tag{2.2.45}$$

Using (2.2.36), (2.2.37), (2.2.45) in (2.2.22), we can write the Singular Value Decomposition of $\mathbf{A}$. Now, the Moore-Penrose Pseudo Inverse of $\mathbf{S}$ is given by:

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2\sqrt{2}}{\sqrt{-625\sqrt{2257}+30625}} & 0 \\ 0 & \frac{2\sqrt{2}}{\sqrt{625\sqrt{2257}+30625}} \\ 0 & 0 \end{pmatrix}^T \tag{2.2.46}$$

$$= \begin{pmatrix} \frac{2\sqrt{2}}{\sqrt{-625\sqrt{2257}+30625}} & 0 & 0 \\ 0 & \frac{2\sqrt{2}}{\sqrt{625\sqrt{2257}+30625}} & 0 \end{pmatrix} \tag{2.2.47}$$

Now, using values from (2.2.45), (2.2), (2.2.36) in (2.2.24), we get:

$$\mathbf{c} = \begin{pmatrix} -2.4 \\ -1.7999 \end{pmatrix} \tag{2.2.48}$$

Verifying this solution using least squares,

$$\mathbf{A}^T\mathbf{A}\mathbf{c} = \mathbf{A}^T\mathbf{b} \tag{2.2.49}$$

Substituting values here, we get

$$\begin{pmatrix} \frac{7225}{4} & \frac{6225}{2} \\ \frac{6225}{2} & 5850 \end{pmatrix} \mathbf{c} = \begin{pmatrix} \frac{-19875}{2} \\ -18000 \end{pmatrix} \tag{2.2.50}$$

Solving the augmented matrix

$$\begin{pmatrix} \frac{7225}{4} & \frac{6225}{2} & \frac{-19875}{2} \\ \frac{6225}{2} & 5850 & -18000 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{4}{7225} R_1} \begin{pmatrix} 1 & \frac{498}{289} & \frac{-1590}{289} \\ \frac{6225}{2} & 5850 & -18000 \end{pmatrix} \tag{2.2.51}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{6225}{2} R_1} \begin{pmatrix} 1 & \frac{498}{289} & \frac{-1590}{289} \\ 0 & \frac{140625}{289} & \frac{-253125}{289} \end{pmatrix} \tag{2.2.52}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{289}{140625} R_2} \begin{pmatrix} 1 & \frac{498}{289} & \frac{-1590}{289} \\ 0 & 1 & \frac{-9}{5} \end{pmatrix} \tag{2.2.53}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{498}{289} R_2} \begin{pmatrix} 1 & 0 & \frac{-12}{5} \\ 0 & 1 & \frac{-9}{5} \end{pmatrix} \tag{2.2.54}$$

Therefore,

$$\mathbf{c} = \begin{pmatrix} \frac{-12}{5} \\ \frac{-9}{5} \end{pmatrix} = \begin{pmatrix} -2.4 \\ -1.8 \end{pmatrix} \tag{2.2.55}$$

Hence, verified.