Assignment 3

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***Abstract*—This document proves that a given equation represents two straight lines and finds the point of intersection and angle between them**

Download all python codes from

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https://github.com/surbhi0912/EE5609/
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and latex-tikz codes from

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https://github.com/surbhi0912/EE5609/
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1 Problem

Prove that the following equations represent two straight lines; and also find their point of intersection and the angle between them

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

2 Solution

2.1 Proving that given equation represents two straight lines

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.1.1}$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.2}$$

The given equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0 \tag{2.1.3}$$

Comparing this to the standard equation (2.1.2), we find

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \tag{2.1.4}$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.1.5}$$

$$f = -2 \tag{2.1.6}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{2.1.7}$$

Equation (2.1.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.1.8}$$

$$\delta = \begin{vmatrix} 1 & \frac{-5}{2} & \frac{1}{2} \\ \frac{-5}{2} & 4 & 1 \\ \frac{1}{2} & 1 & -2 \end{vmatrix} \tag{2.1.9}$$

$$= 0 \tag{2.1.10}$$

Hence, proved that given equation represents two straight lines.

2.2 Finding point of intersection between the straight lines

$$\det V = \begin{vmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{vmatrix}$$

$$= \frac{-9}{4} < 0 \tag{2.2.1}$$

Thus, the two straight lines intersect. Let the equation of the straight lines be given as

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{2.2.2}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{2.2.3}$$

with their slopes as $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively.

Then the equation of the pair of straight lines is

$$(\mathbf{n}_1^T\mathbf{x} - c_1)(\mathbf{n}_2^T\mathbf{x} - c_2) = 0 \tag{2.2.4}$$

Using (2.1.7) and (2.2.4),

$$(\mathbf{n}_1^T\mathbf{x} - c_1)(\mathbf{n}_2^T\mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 \tag{2.2.5}$$

Comparing both sides,

$$c_2\mathbf{n}_1 + c_1\mathbf{n}_2 = -2\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.2.6}$$

$$c_1c_2 = -2 \tag{2.2.7}$$

Slopes of the lines are roots of the equation

$$cm^2 + 2bm + a = 0 \tag{2.2.8}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.2.9}$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.2.10}$$

Substituting (2.1.1) in (2.2.8),

$$4m^2 - 5m + 1 = 0 \tag{2.2.11}$$

$$\implies m_i = \frac{\frac{5}{2} \pm \frac{3}{2}}{4} \tag{2.2.12}$$

$$\implies m_1 = 1, m_2 = \frac{1}{4} \tag{2.2.13}$$

Therefore,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.2.14}$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} \tag{2.2.15}$$

We know that

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.2.16}$$

$$k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \tag{2.2.17}$$

$$\implies k_1 k_2 = 4 \tag{2.2.18}$$

Taking $k_1 = 1$, $k_2 = 4$, we get

$$\mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{n}_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{2.2.19}$$

For verifying values of $\mathbf{n}_1$ and $\mathbf{n}_2$, we compute the convolution by representing $\mathbf{n}_1$ as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \tag{2.2.20}$$

Now, obtaining c_1 and c_2 using (2.2.19) and (2.2.6)

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.2.21}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \tag{2.2.22}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 4 & -2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & -2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \tag{2.2.23}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \tag{2.2.24}$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$c_1 = -1 \tag{2.2.25}$$

$$c_2 = 2 \tag{2.2.26}$$

Thus, equation of lines can be written as

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{2.2.27}$$

$$\begin{pmatrix} -1 & 4 \end{pmatrix} \mathbf{x} = 2 \tag{2.2.28}$$

Augmented matrix for these set of equations is

$$\begin{pmatrix} -1 & 1 & -1 \\ -1 & 4 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 4 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \tag{2.2.29}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.2.30}$$

Thus, the point of intersection is $\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Using (2.2.19) and (2.2.26) in (2.2.4), equation of the pair of straight lines is

$$(x - y - 1)(x - 4y + 2) = 0 \tag{2.2.31}$$

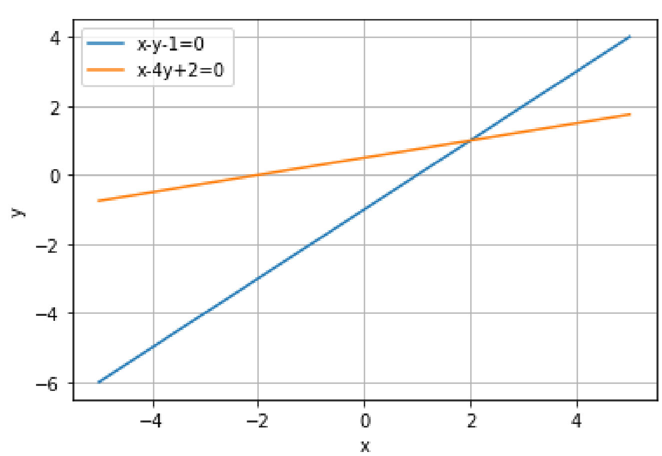

Fig. 1: Intersection of lines

2.3 Angle between lines

Angle between pair of lines is,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) \tag{2.3.1}$$

$$\mathbf{n}_1^T\mathbf{n}_2 = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}-1\\4\end{pmatrix} = 5 \tag{2.3.2}$$

$$\|\mathbf{n}_1\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \tag{2.3.3}$$

$$\|\mathbf{n}_2\| = \sqrt{(-1)^2 + 4^2} = \sqrt{17} \tag{2.3.4}$$

Substituting these values (2.3.1)

$$\theta = 30.9^\circ \tag{2.3.5}$$

Hence, angle between the given pair of straight lines is 30.9°