# Assignment 8

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***Abstract*—This document deals with linear operators and basis of a finite dimensional vector space over a field.**

## 1 Problem

Let $\mathbb{V}$ be finite dimensional vector space over the field $\mathbb{F}$, and let $\mathbf{S}$ and $\mathbf{T}$ be linear operators on $\mathbb{V}$. When do there exist ordered bases $\mathcal{B}$ and $\mathcal{B}'$ for $\mathbb{V}$ such that $[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$? Prove that such bases exist if and only if there is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$

## 2 Solution

| Assume $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$ | Assume $\mathbf{T} = \mathbf{USU}^{-1}$ |
|---|---|
| **Given**<br><br>$\mathbb{V}$ is a finite dimensional vector space over field $\mathbb{F}$<br>$\mathbf{S}$ and $\mathbf{T}$ are linear operators on $\mathbb{V}$<br>$\mathcal{B}$ and $\mathcal{B}'$ are ordered bases for $\mathbb{V}$<br>$[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$ | **Given**<br><br>$\mathbb{V}$ is a finite dimensional vector space over field $\mathbb{F}$<br>$\mathbf{S}$ and $\mathbf{T}$ are linear operators on $\mathbb{V}$<br>$\mathcal{B}$ and $\mathcal{B}'$ are ordered bases for $\mathbb{V}$<br>There is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$<br>such that $\mathbf{T} = \mathbf{USU}^{-1}$ |
| **To prove**<br><br>There is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$<br>such that $\mathbf{T} = \mathbf{USU}^{-1}$ | **To prove**<br><br>$[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$ |
| **Assumptions**<br><br>Let $\mathbf{U}$ be the operator which carries $\mathcal{B}$ to $\mathcal{B}'$<br>$\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$<br>$\mathcal{B}' = \{\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n\}$ | **Assumptions**<br><br>Let $\mathbf{U}$ be the operator which carries $\mathcal{B}$ to $\mathcal{B}'$<br>$\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$<br>$\mathcal{B}' = \{\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n\}$ |
| **Proof**<br><br>For $\mathbf{v} \in \mathbb{V}$, expressed as a linear combination<br>of the vectors of $\mathcal{B}$<br>$\mathbf{v} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_n\mathbf{x}_n$<br><br>$\mathbf{w} = \mathbf{U}(\mathbf{v}) = a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n$<br><br>$\because [S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$<br>$\mathbf{S}(\mathbf{v}) = c_1\mathbf{x_1} + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$ and<br>$\mathbf{T}(\mathbf{w}) = c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_n\mathbf{x}'_n$<br><br>$\mathbf{U}^{-1}\mathbf{TU}(\mathbf{v})$<br>$= \mathbf{U}^{-1}\mathbf{T}(\mathbf{w})$<br>$= \mathbf{U}^{-1}(c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_n\mathbf{x}'_n)$<br>$= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$<br>$= \mathbf{S}(\mathbf{v})$<br><br>$\implies \mathbf{U}^{-1}\mathbf{TU} = \mathbf{S}$<br>$\implies \mathbf{T} = \mathbf{USU}^{-1}$ | **Proof**<br><br>For $\mathbf{v} \in \mathbb{V}$, expressed as a linear combination<br>of the vectors of $\mathcal{B}'$<br>$\mathbf{v} = a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n$<br><br>$\mathbf{USU}^{-1}(\mathbf{v}) = \mathbf{T}(\mathbf{v})$<br>$= \mathbf{T}(a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n)$<br>$= c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_n\mathbf{x}'_n$<br><br>But we know that<br>$\mathbf{USU}^{-1}(\mathbf{v}) = \mathbf{USU}^{-1}(a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n)$<br>$= \mathbf{US}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_n\mathbf{x}_n)$<br><br>So, $\mathbf{S}$ in basis $\mathcal{B}$ has the same entries as $\mathbf{T}$ in basis $\mathcal{B}'$<br>$\therefore [S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$ |

TABLE 0