Assignment 10

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Download all latex-tikz codes from

https://github.com/surbhi0912/EE5609/

1 Problem

Given that there are real constants a, b, c, d such that the identity

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2 \quad (1.0.1)$$

holds for all $x, y \in \mathbb{R}$. This implies

1) $\lambda = -5$
2) $\lambda \geq 1$
3) $0 < \lambda < 1$
4) there is no such $\lambda \in \mathbb{R}$

2 Solution

Given that

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2 \quad (2.0.1)$$

Arranging this in form of a matrix,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (2.0.2)$$

From this, we get

$$\lambda = a^2 + c^2 \quad (2.0.3)$$

$$ab + cd = 1 \quad (2.0.4)$$

$$b^2 + d^2 = 1 \quad (2.0.5)$$

Let

$$b = \cos\theta, d = \sin\theta \quad (2.0.6)$$

$$\therefore \cos^2\theta + \sin^2\theta = 1, \quad \forall\theta \in \mathbb{R} \quad (2.0.7)$$

Substituting (2.0.6) in (2.0.4)

$$a\cos\theta + c\sin\theta = 1 \quad (2.0.8)$$

$$\text{let } a = k\cos\beta, c = k\sin\beta \quad (2.0.9)$$

Substituting this in (2.0.8)

$$k\cos\beta\sin\theta + k\sin\beta\cos\theta = 1 \quad (2.0.10)$$

$$\implies k\sin(\theta + \beta) = 1 \quad (2.0.11)$$

Substituting (2.0.9) in (2.0.3),

$$\lambda = k^2\cos^2\beta + k^2\sin^2\beta \quad (2.0.12)$$

$$= k^2(\cos^2\beta + \sin^2\beta) \quad (2.0.13)$$

$$= k^2 \quad (2.0.14)$$

Now, since $|\sin(\theta + \beta)| \leq 1$, then from (2.0.11), we get $|k| \geq 1$, hence $k^2 \geq 1$. Using this in (2.0.14),

$$\lambda \geq 1 \quad (2.0.15)$$

So from the given options, option 2) $\lambda \geq 1$ is correct.