Assignment 10

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Download all latex-tikz codes from

https://github.com/surbhi0912/EE5609/

1 Problem

Given that there are real constants a, b, c, d such that the identity

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2 \quad (1.0.1)$$

holds for all $x, y \in \mathbb{R}$. This implies

1) $\lambda = -5$
2) $\lambda \geq 1$
3) $0 < \lambda < 1$
4) there is no such $\lambda \in \mathbb{R}$

2 Solution

Given that

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2 \quad (2.0.1)$$

Arranging this in form of a matrix,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (2.0.2)$$

From this, we get

$$\lambda = a^2 + c^2 \quad (2.0.3)$$

$$ab + cd = 1 \quad (2.0.4)$$

$$b^2 + d^2 = 1 \quad (2.0.5)$$

Let

$$\mathbf{u} = \begin{pmatrix} a \\ c \end{pmatrix} \quad (2.0.6)$$

$$\mathbf{v} = \begin{pmatrix} b \\ d \end{pmatrix} \quad (2.0.7)$$

$$||\mathbf{u}||^2 = a^2 + c^2 = \lambda \quad (2.0.8)$$

$$||\mathbf{v}||^2 = b^2 + d^2 = 1 \quad (2.0.9)$$

Then,

$$\mathbf{u}^T \mathbf{v} = \begin{pmatrix} a & c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = ab + cd = 1 \quad (2.0.10)$$

Using the Cauchy-Schwartz Inequality, we get

$$|\mathbf{u}^T \mathbf{v}|^2 \leq ||\mathbf{u}||^2 \, ||\mathbf{v}||^2 \quad (2.0.11)$$

Now, substituing values from (2.0.8), (2.0.9), (2.0.10) above,

$$\implies 1 \leq \lambda \quad (2.0.12)$$

So from the given options, option 2) $\lambda \geq 1$ is correct.