# Assignment 3

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***Abstract*—This document proves that a given equation represents two straight lines and finds the point of intersection and angle between them**

Download all python codes from

https://github.com/surbhi0912/EE5609/

and latex-tikz codes from

https://github.com/surbhi0912/EE5609/

## 1 Problem

Prove that the following equations represent two straight lines; and also find their point of intersection and the angle between them

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

## 2 Solution

### *2.1 Proving that given equation represents two straight lines*

The given equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0 \tag{2.1.1}$$

Comparing this to the standard equation,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \tag{2.1.2}$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.1.3}$$

$$f = -2 \tag{2.1.4}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{2.1.5}$$

Equation (2.1.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.1.6}$$

$$\delta = \begin{vmatrix} 1 & \frac{-5}{2} & \frac{1}{2} \\ \frac{-5}{2} & 4 & 1 \\ \frac{1}{2} & 1 & -2 \end{vmatrix} \tag{2.1.7}$$

$$= 0 \tag{2.1.8}$$

Hence, proved that given equation represents two straight lines.

### *2.2 Finding point of intersection between the straight lines*

$$\det V = \begin{vmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{vmatrix} \tag{2.2.1}$$

$$= \frac{-9}{4} < 0 \tag{2.2.2}$$

Thus, the two straight lines intersect. Let the equation of the straight lines be given as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \tag{2.2.3}$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \tag{2.2.4}$$

with their slopes as $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively.

Then the equation of the pair of straight lines is

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \tag{2.2.5}$$

Using (2.1.5) and (2.2.5),

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 \tag{2.2.6}$$

Comparing both sides,

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.2.7}$$

$$c_1 c_2 = -2 \tag{2.2.8}$$

Slopes of the lines are roots of the equation

$$cm^2 + 2bm + a = 0 \tag{2.2.9}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.2.10}$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.2.11}$$

Substituting (2.1.1) in (2.2.9),

$$4m^2 - 5m + 1 = 0 \tag{2.2.12}$$

$$\implies m_i = \frac{\frac{5}{2} \pm \frac{3}{2}}{4} \tag{2.2.13}$$

$$\implies m_1 = 1, m_2 = \frac{1}{4} \tag{2.2.14}$$

Therefore,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.2.15}$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} \tag{2.2.16}$$

We know that

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.2.17}$$

$$k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \tag{2.2.18}$$

$$\implies k_1 k_2 = 4 \tag{2.2.19}$$

Taking $k_1 = 1$, $k_2 = 4$, we get

$$\mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{n}_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{2.2.20}$$

For verifying values of $\mathbf{n}_1$ and $\mathbf{n}_2$, we compute the convolution by representing $\mathbf{n}_1$ as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \tag{2.2.21}$$

Now, obtaining $c_1$ and $c_2$ using (2.2.20) and (2.2.7)

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.2.22}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \tag{2.2.23}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 4 & -2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & -2 \end{pmatrix} \tag{2.2.24}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \tag{2.2.25}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \tag{2.2.26}$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$c_1 = -1 \tag{2.2.27}$$

$$c_2 = 2 \tag{2.2.28}$$

Thus, equation of lines can be written as

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{2.2.29}$$

$$\begin{pmatrix} -1 & 4 \end{pmatrix} \mathbf{x} = 2 \tag{2.2.30}$$

Augmented matrix for these set of equations is

$$\begin{pmatrix} -1 & 1 & -1 \\ -1 & 4 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 4 & 2 \end{pmatrix} \tag{2.2.31}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.2.32}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.2.33}$$

Thus, the point of intersection is $\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

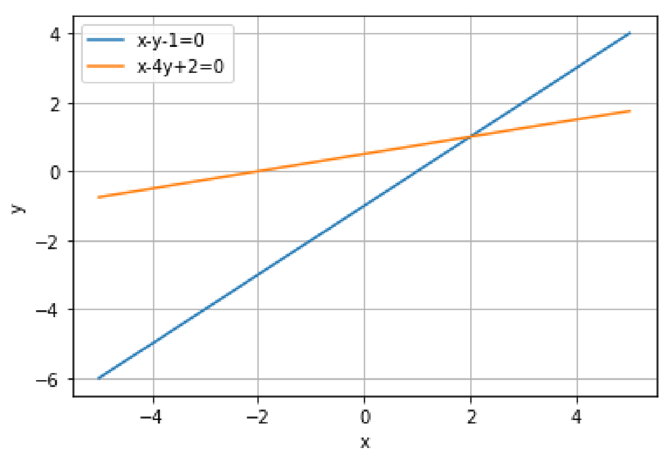


Fig. 1: Intersection of pair of straight lines

Using (2.2.20) and (2.2.28) in (2.2.5), equation of the pair of straight lines is

$$(x - y - 1)(x - 4y + 2) = 0 \tag{2.2.34}$$

### 2.3 Angle between lines

Angle between pair of lines is,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) \tag{2.3.1}$$

$$\mathbf{n}_1^T\mathbf{n}_2 = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}-1\\4\end{pmatrix} = 5 \tag{2.3.2}$$

$$\|\mathbf{n}_1\| = \sqrt{(-1)^2+1^2} = \sqrt{2} \tag{2.3.3}$$

$$\|\mathbf{n}_2\| = \sqrt{(-1)^2+4^2} = \sqrt{17} \tag{2.3.4}$$

Substituting these values (2.3.1)

$$\theta = 30.9^\circ \tag{2.3.5}$$

Hence, angle between the given pair of straight lines is 30.9°

### 2.4 Affine Transformation and Eigen Value decomposition

First, verifying if $\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 0$. To do this, finding $V^{-1}$ by augmenting with identity matrix and row reducing as follows :

$$\begin{pmatrix}1 & \frac{-5}{2} & 1 & 0\\ \frac{-5}{2} & 4 & 0 & 1\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2+\frac{5}{2}R_1} \begin{pmatrix}1 & \frac{-5}{2} & 1 & 0\\ 0 & \frac{-9}{4} & \frac{5}{2} & 1\end{pmatrix} \tag{2.4.1}$$

$$\xleftrightarrow{R_2\leftarrow\frac{-4}{9}R_2} \begin{pmatrix}1 & \frac{-5}{2} & 1 & 0\\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9}\end{pmatrix} \tag{2.4.2}$$

$$\xleftrightarrow{R_1\leftarrow R_1+\frac{5}{2}R_2} \begin{pmatrix}1 & 0 & \frac{-16}{9} & \frac{-10}{9}\\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9}\end{pmatrix} \tag{2.4.3}$$

$$\implies \mathbf{V}^{-1} = \begin{pmatrix}\frac{-16}{9} & \frac{-10}{9}\\ \frac{-10}{9} & \frac{-4}{9}\end{pmatrix} \tag{2.4.4}$$

$$u^TV^{-1}u - f = \begin{pmatrix}\frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}\frac{-16}{9} & \frac{-10}{9}\\ \frac{-10}{9} & \frac{-4}{9}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix} - (-2) \tag{2.4.5}$$

$$= 0 \tag{2.4.6}$$

The characteristic equation of $\mathbf{V}$ is given as :

$$\left|\lambda\mathbf{I}-\mathbf{V}\right| = \begin{vmatrix}\lambda-1 & \frac{5}{2}\\ \frac{5}{2} & \lambda-4\end{vmatrix} = 0 \tag{2.4.7}$$

$$\implies (\lambda-1)(\lambda-4) - \frac{25}{4} = 0 \tag{2.4.8}$$

$$\implies 4\lambda^2 - 20\lambda - 9 = 0 \tag{2.4.9}$$

The roots of (2.4.9), i.e. the eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = \frac{5+\sqrt{34}}{2}, \lambda_2 = \frac{5-\sqrt{34}}{2} \tag{2.4.10}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.4.11}$$

$$\implies (\lambda\mathbf{I}-\mathbf{V})\mathbf{p} = 0 \tag{2.4.12}$$

For $\lambda_1 = \frac{5+\sqrt{34}}{2}$

$$(\lambda_1\mathbf{I}-\mathbf{V}) = \begin{pmatrix}\frac{3+\sqrt{34}}{2} & \frac{5}{2}\\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2}\end{pmatrix} \tag{2.4.13}$$

To find $\mathbf{p}_1$, let's look at Augmented form of $(\lambda_1\mathbf{I}-\mathbf{V})$

$$\begin{pmatrix}\frac{3+\sqrt{34}}{2} & \frac{5}{2} & 0\\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0\end{pmatrix} \tag{2.4.14}$$

$$\xleftrightarrow{R_1\leftarrow\frac{2}{3+\sqrt{34}}R_1} \begin{pmatrix}1 & \frac{-3+\sqrt{34}}{5} & 0\\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0\end{pmatrix} \tag{2.4.15}$$

$$\xleftrightarrow{R_2\leftarrow\frac{2}{5}R_2-R_1} \begin{pmatrix}1 & \frac{-3+\sqrt{34}}{5} & 0\\ 0 & 0 & 0\end{pmatrix} \tag{2.4.16}$$

So we get

$$x_1 + \left(\frac{-3+\sqrt{34}}{5}\right)x_2 = 0 \tag{2.4.17}$$

Thus, our eigenvector corresponding to $\lambda_1$

$$\mathbf{p}_1 = \begin{pmatrix}\frac{3-\sqrt{34}}{5}\\ 1\end{pmatrix} \tag{2.4.18}$$

For $\lambda_2 = \frac{5-\sqrt{34}}{2}$

$$(\lambda_2\mathbf{I}-\mathbf{V}) = \begin{pmatrix}\frac{3-\sqrt{34}}{2} & \frac{5}{2}\\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2}\end{pmatrix} \tag{2.4.19}$$

To find $\mathbf{p}_2$, let's look at Augmented form of $(\lambda_2\mathbf{I}-\mathbf{V})$

$$\begin{pmatrix}\frac{3-\sqrt{34}}{2} & \frac{5}{2} & 0\\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0\end{pmatrix} \tag{2.4.20}$$

$$\xleftrightarrow{R_1\leftarrow\frac{2}{3-\sqrt{34}}R_1} \begin{pmatrix}1 & \frac{-3-\sqrt{34}}{5} & 0\\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0\end{pmatrix} \tag{2.4.21}$$

$$\xleftrightarrow{R_2\leftarrow\frac{2}{5}R_2-R_1} \begin{pmatrix}1 & \frac{-3-\sqrt{34}}{5} & 0\\ 0 & 0 & 0\end{pmatrix} \tag{2.4.22}$$

So we get

$$x_1 + \left(\frac{-3-\sqrt{34}}{5}\right)x_2 = 0 \tag{2.4.23}$$

Thus, our eigenvector corresponding to $\lambda_2$

$$\mathbf{p}_2 = \begin{pmatrix} \frac{3+\sqrt{34}}{5} \\ 1 \end{pmatrix} \tag{2.4.24}$$

We know $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where $\mathbf{P}$ and the diagonal matrix $\mathbf{D}$ are given as:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.4.25}$$

$$= \begin{pmatrix} \frac{5+\sqrt{34}}{2} & 0 \\ 0 & \frac{5-\sqrt{34}}{2} \end{pmatrix} \tag{2.4.26}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \tag{2.4.27}$$

$$= \begin{pmatrix} \frac{3-\sqrt{34}}{5} & \frac{3+\sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \tag{2.4.28}$$

So, the equation of the pair of straight lines is given by :

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.4.29}$$

$$\mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{34}}{2} & 0 \\ 0 & \frac{5-\sqrt{34}}{2} \end{pmatrix} \mathbf{y} = 0 \tag{2.4.30}$$

$$\implies \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \frac{5+\sqrt{34}}{2} & 0 \\ 0 & \frac{5-\sqrt{34}}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \tag{2.4.31}$$

$$\implies (5+\sqrt{34})y_1^2 + (5-\sqrt{34})y_2^2 = 0 \tag{2.4.32}$$

So we get the equation of the pair of straight lines, as we can see this passes through the origin $(0, 0)$. The corresponding image is shown in Fig. 2

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.4.33}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.4.34}$$

And,

$$\mathbf{P}^T = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & 1 \\ \frac{3+\sqrt{34}}{5} & 1 \end{pmatrix} \tag{2.4.35}$$

Using affine transformation, we can express the equation as

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.4.36}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & \frac{3+\sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.4.37}$$

The corresponding image is shown in Fig. 3

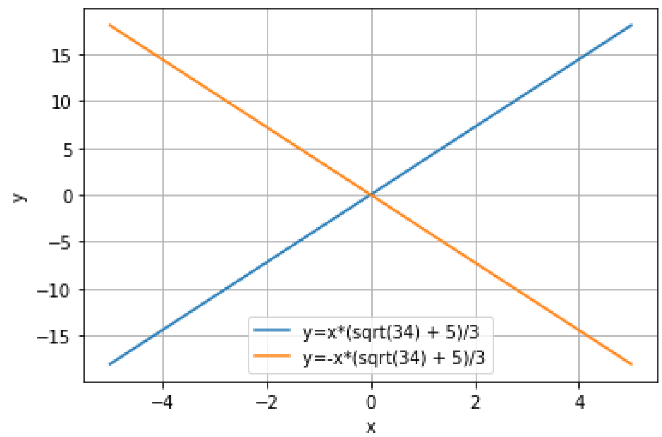


Fig. 2: Pair of straight lines passing through origin after eigenvalue decomposition

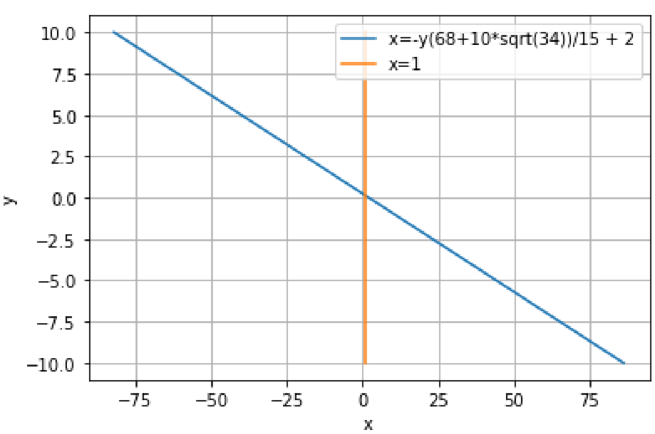


Fig. 3: Pair of straight lines after affine transform