Assignment 12

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***Abstract*—This document illustrates concepts of dimensions of image of a linear transformation and columnspace.**

1 Problem

Given a 4×4 matrix $\mathbf{A}$, let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by $\mathbf{Tv} = \mathbf{Av}$, where we think of $\mathbb{R}^4$ as the set of real 4×1 matrices. For which choices of $\mathbf{A}$ given below, do Image($\mathbf{T}$) and Image($\mathbf{T}^2$) have respective dimensions 2 and 1? ($*$ denotes a nonzero entry)

1) $\mathbf{A} = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$
2) $\mathbf{A} = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}$
3) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}$
4) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$

2 Solution

We can say,

$$\mathbf{T}(\mathbf{v}) = \mathbf{Av} = \text{Image}(\mathbf{T}) = C(\mathbf{A}) \tag{2.0.1}$$
$$\mathbf{T}^2(\mathbf{v}) = \mathbf{A}^2\mathbf{v} = \text{Image}(\mathbf{T}^2) = C(\mathbf{A}^2) \tag{2.0.2}$$

where $C(\mathbf{A})$ and $C(\mathbf{A}^2)$ denote the columnspace of $\mathbf{A}$ and $\mathbf{A}^2$ respectively. Therefore,

$$\text{dimension}(\text{Image}(\mathbf{T})) = \text{dimension}(C(\mathbf{A})) = \text{rank}(\mathbf{A}) \tag{2.0.3}$$
$$\text{dimension}(\text{Image}(\mathbf{T}^2)) = \text{dimension}(C(\mathbf{A}^2)) = \text{rank}(\mathbf{A}^2) \tag{2.0.4}$$

1. $\mathbf{A} = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$	The number of linearly independent columns in $\mathbf{A}$ is 2

<table>
<tr><td></td><td>hence, $dim(Image(\mathbf{T})) = dim(C(\mathbf{A})) = 2$

$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The number of linearly independent columns in $\mathbf{A}^2$ is 1 hence, $dim(Image(\mathbf{T}^2)) = dim(C(\mathbf{A}^2)) = 1$

$\therefore$ This option is true.</td></tr>
<tr><td>2. $\mathbf{A} = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}$</td><td>The number of linearly independent columns in $\mathbf{A}$ is 2

hence, $\mathrm{dim(Image(}\mathbf{T})) = \mathrm{dim(C(}\mathbf{A})) = 2$

$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}$

The number of linearly independent columns in $\mathbf{A}^2$ is 1 hence, $\mathrm{dim(Image(}\mathbf{T}^2)) = \mathrm{dim(C(}\mathbf{A}^2)) = 1$

$\therefore$ This option is true.</td></tr>
<tr><td>3. $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}$</td><td>The number of linearly independent columns in $\mathbf{A}$ is 2

hence, $dim(Image(\mathbf{T})) = dim(C(\mathbf{A})) = 2$

$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$

The number of linearly independent columns in $\mathbf{A}^2$ is 2 hence, $dim(Image(\mathbf{T}^2)) = dim(C(\mathbf{A}^2)) = 2 \neq 1$

$\therefore$ This option is false.</td></tr>
<tr><td>4. $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$</td><td>This option is false</td></tr>
</table>

	Counter example: For some non-zero $b, c \in \mathbb{R}$, let $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & b \\ 0 & 0 & c & c \end{pmatrix}$ The number of linearly independent columns in $\mathbf{A}$ is 1 hence, $dim(Image(\mathbf{T})) = dim(C(\mathbf{A})) = 1 \neq 2$

TABLE 1: Verifying with the options