Assignment 13

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***Abstract*—This document illustrates properties of subspaces of a vectorspace.**

1 Problem

For arbitrary subspaces, U, V and W of a finite dimensional vectorspace, which of the following hold :

1) $U \cap (V + W) \subset (U \cap V) + (U \cap W)$
2) $U \cap (V + W) \supset (U \cap V) + (U \cap W)$
3) $(U \cap V) + W \subset (U + W) \cap (V + W)$
4) $(U \cap V) + W \supset (U + W) \cap (V + W)$

2 Solution

1. $U \cap (V + W) \subset (U \cap V) + (U \cap W)$	False. Counter Example: Let $\mathbf{u}_1 = (\mathbf{v}_1 + \mathbf{w}_1) \in U \cap (V + W)$ such that $(\mathbf{v}_1 + \mathbf{w}_1) \in U, \mathbf{v}_1 \in V, \mathbf{w}_1 \in W$ But since $\mathbf{w}_1 \notin V$, hence $\mathbf{v}_1 + \mathbf{w}_1 \notin V$ $\implies (\mathbf{v}_1 + \mathbf{w}_1) \notin (U \cap V)$ Also, $\mathbf{v}_1, \mathbf{w}_1 \notin U \implies \mathbf{v}_1, \mathbf{w}_1 \notin (U \cap V)$ And since $\mathbf{v}_1 \notin W$, hence $\mathbf{v}_1 + \mathbf{w}_1 \notin W$ $\implies (\mathbf{v}_1 + \mathbf{w}_1) \notin (U \cap W)$ Also, $\mathbf{v}_1, \mathbf{w}_1 \notin U \implies \mathbf{v}_1, \mathbf{w}_1 \notin (U \cap W)$ Therefore, $(\mathbf{v}_1 + \mathbf{w}_1) \notin (U \cap V) + (U \cap W)$ There exists an element in LHS that does not belong to RHS. $\therefore U \cap (V + W) \not\subset (U \cap V) + (U \cap W)$
2. $U \cap (V + W) \supset (U \cap V) + (U \cap W)$	Let $(\mathbf{u}_1 + \mathbf{u}_2) \in (U \cap V) + (U \cap W)$ such that $\mathbf{u}_1 \in U \cap V$ and $\mathbf{u}_2 \in U \cap W$ $\implies \mathbf{u}_1 \in U, V$ and $\mathbf{u}_2 \in U, W$ Since $\mathbf{u}_1 \in V, \mathbf{u}_2 \in W$ $\implies (\mathbf{u}_1 + \mathbf{u}_2) \in (V + W)$ And since $\mathbf{u}_1, \mathbf{u}_2 \in U$ $\implies (\mathbf{u}_1 + \mathbf{u}_2) \in U$ $\therefore (\mathbf{u}_1 + \mathbf{u}_2) \in U \cap (V + W)$

	So, $(\mathbf{u}_1 + \mathbf{u}_2) \in (U \cap V) + (U \cap W) \implies (\mathbf{u}_1 + \mathbf{u}_2) \in U \cap (V + W)$ Hence, $U \cap (V + W) \supset (U \cap V) + (U \cap W)$ The given option is true.
3. $(U \cap V) + W \subset (U + W) \cap (V + W)$	Let $(\mathbf{u}_1 + \mathbf{w}_1) \in (U \cap V) + W$, such that $\mathbf{u}_1 \in (U \cap V)$ and $\mathbf{w}_1 \in W$ Since, $\mathbf{u}_1 \in (U \cap V), \implies \mathbf{u}_1 \in U, V$ Now, since $\mathbf{u}_1 \in U, \mathbf{w}_1 \in W$ $(\mathbf{u}_1 + \mathbf{w}_1) \in (U + W)$ And since, $\mathbf{u}_1 \in V, \mathbf{w}_1 \in W$ $(\mathbf{u}_1 + \mathbf{w}_1) \in (V + W)$ $\therefore (\mathbf{u}_1 + \mathbf{w}_1) \in (U + W) \cap (V + W)$ Hence, $(\mathbf{u}_1 + \mathbf{w}_1) \in (U \cap V) + W \implies (\mathbf{u}_1 + \mathbf{w}_1) \in (U + W) \cap (V + W)$ $(U \cap V) + W \subset (U + W) \cap (V + W)$ The given option is true.
4. $(U \cap V) + W \supset (U + W) \cap (V + W)$	False. Counter Example: Let $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{w}_1 \in U$ $\mathbf{v}_1 \in V, \mathbf{w}_1 \in W$ Then, since $\mathbf{v}_1 + \mathbf{w}_1 \in U \implies \mathbf{v}_1 + \mathbf{w}_1 \in U + W$ And since, $\mathbf{v}_1 \in V, \mathbf{w}_1 \in W \implies \mathbf{v}_1 + \mathbf{w}_1 \in V + W$ $\therefore \mathbf{v}_1 + \mathbf{w}_1 \in (U + W) \cap (V + W)$ Now, since $\mathbf{w}_1 \notin V \implies \mathbf{v}_1 + \mathbf{w}_1 \notin V$ $\implies \mathbf{v}_1 + \mathbf{w}_1 \notin U \cap V$ Also, $\mathbf{v}_1 \notin U \implies \mathbf{v}_1 \notin (U \cap V)$ And since, $\mathbf{v}_1 \notin W \implies \mathbf{v}_1 + \mathbf{w}_1 \notin W$ $\implies \mathbf{v}_1 + \mathbf{w}_1 \notin (U \cap V) + W$ There exists an element in RHS that does not exist in LHS $\therefore (U \cap V) + W \not\supset (U + W) \cap (V + W)$

TABLE 1: Proving properties of subspaces of a vectorspace