Assignment 3

Surbhi Agarwal

Abstract—This document proves that a given equation represents two straight lines and finds the point of intersection and angle between them

1 PROBLEM

Prove that the following equations represent two straight lines; and also find their point of intersection and the angle between them

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

2 SOLUTION

2.1 Proving that given equation represents two straight lines

The given equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0 \quad (2.1.1)$$

Comparing this to the standard equation,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \quad (2.1.2)$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (2.1.3)$$

$$f = -2 \quad (2.1.4)$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (2.1.5)$$

Equation (2.1.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.1.6)$$

$$\delta = \begin{vmatrix} 1 & \frac{-5}{2} & \frac{1}{2} \\ \frac{-5}{2} & 4 & 1 \\ \frac{1}{2} & 1 & -2 \end{vmatrix} \quad (2.1.7)$$

$$= 0 \quad (2.1.8)$$

Hence, proved that given equation represents two straight lines.

2.2 Finding point of intersection between the straight lines

$$\det V = \begin{vmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{vmatrix} \quad (2.2.1)$$

$$= \frac{-9}{4} < 0 \quad (2.2.2)$$

Thus, the two straight lines intersect. Let the equation of the straight lines be given as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (2.2.3)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (2.2.4)$$

with their slopes as $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively.

Then the equation of the pair of straight lines is

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (2.2.5)$$

Using (2.1.5) and (2.2.5),

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 \quad (2.2.6)$$

Comparing both sides,

$$c_2\mathbf{n}_1 + c_1\mathbf{n}_2 = -2\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (2.2.7)$$

$$c_1 c_2 = -2 \quad (2.2.8)$$

Slopes of the lines are roots of the equation

$$cm^2 + 2bm + a = 0 \qquad (2.2.9)$$

$$\implies m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \qquad (2.2.10)$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \qquad (2.2.11)$$

Substituting (2.1.1) in (2.2.9),

$$4m^2 - 5m + 1 = 0 \qquad (2.2.12)$$

$$\implies m_i = \frac{\frac{5}{2} \pm \frac{3}{2}}{4} \qquad (2.2.13)$$

$$\implies m_1 = 1, m_2 = \frac{1}{4} \qquad (2.2.14)$$

Therefore,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad (2.2.15)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} \qquad (2.2.16)$$

We know that

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \qquad (2.2.17)$$

$$k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \qquad (2.2.18)$$

$$\implies k_1 k_2 = 4 \qquad (2.2.19)$$

Taking $k_1 = 1$, $k_2 = 4$, we get

$$\mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{n}_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \qquad (2.2.20)$$

For verifying values of $\mathbf{n}_1$ and $\mathbf{n}_2$, we compute the convolution by representing $\mathbf{n}_1$ as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \qquad (2.2.21)$$

Now, obtaining c_1 and c_2 using (2.2.20) and (2.2.7)

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \qquad (2.2.22)$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \qquad (2.2.23)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 4 & -2 \end{pmatrix} \xrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & -2 \end{pmatrix} \qquad (2.2.24)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \qquad (2.2.25)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \qquad (2.2.26)$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$c_1 = -1 \qquad (2.2.27)$$

$$c_2 = 2 \qquad (2.2.28)$$

Thus, equation of lines can be written as

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = -1 \qquad (2.2.29)$$

$$\begin{pmatrix} -1 & 4 \end{pmatrix} \mathbf{x} = 2 \qquad (2.2.30)$$

Augmented matrix for these set of equations is

$$\begin{pmatrix} -1 & 1 & -1 \\ -1 & 4 & 2 \end{pmatrix} \xrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 4 & 2 \end{pmatrix} \qquad (2.2.31)$$

$$\xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad (2.2.32)$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \qquad (2.2.33)$$

Thus, the point of intersection is $\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
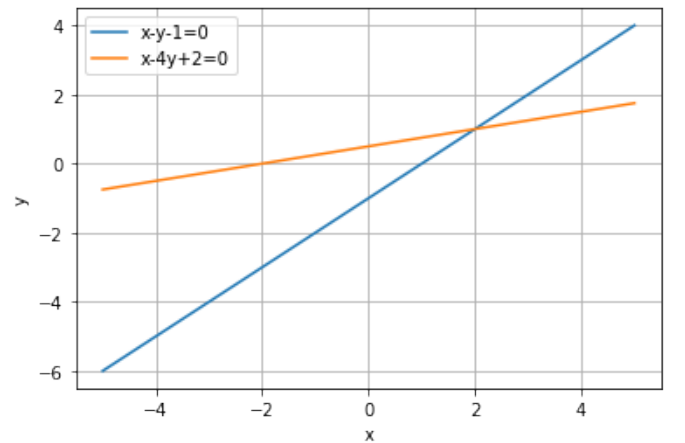

Fig. 1: Intersection of pair of straight lines

Using (2.2.20) and (2.2.28) in (2.2.5), equation of the pair of straight lines is

$$(x - y - 1)(x - 4y + 2) = 0 \qquad (2.2.34)$$

2.3 Angle between lines

Angle between pair of lines is,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) \qquad (2.3.1)$$

$$\mathbf{n}_1^T \mathbf{n}_2 = \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = 5 \qquad (2.3.2)$$

$$\|\mathbf{n}_1\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \qquad (2.3.3)$$
$$\|\mathbf{n}_2\| = \sqrt{(-1)^2 + 4^2} = \sqrt{17} \qquad (2.3.4)$$

Substituting these values (2.3.1)

$$\theta = 30.9° \qquad (2.3.5)$$

Hence, angle between the given pair of straight lines is 30.9°

2.4 Affine Transformation and Eigen Value decomposition

First, verifying if $\mathbf{u}^T \mathbf{V}^{-1}\mathbf{u} - f = 0$. To do this, finding V^{-1} by augmenting with identity matrix and row reducing as follows :

$$\begin{pmatrix} 1 & \frac{-5}{2} & 1 & 0 \\ \frac{-5}{2} & 4 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{5}{2}R_1} \begin{pmatrix} 1 & \frac{-5}{2} & 1 & 0 \\ 0 & \frac{-9}{4} & \frac{5}{2} & 1 \end{pmatrix} \qquad (2.4.1)$$

$$\xrightarrow{R_2 \leftarrow \frac{-4}{9}R_2} \begin{pmatrix} 1 & \frac{-5}{2} & 1 & 0 \\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \qquad (2.4.2)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{5}{2}R_2} \begin{pmatrix} 1 & 0 & \frac{-16}{9} & \frac{-10}{9} \\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \qquad (2.4.3)$$

$$\implies \mathbf{V}^{-1} = \begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \qquad (2.4.4)$$

$$u^T V^{-1} u - f = \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} - (-2) \qquad (2.4.5)$$

$$= 0 \qquad (2.4.6)$$

The characteristic equation of $\mathbf{V}$ is given as :

$$\left|\lambda \mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 1 & \frac{5}{2} \\ \frac{5}{2} & \lambda - 4 \end{vmatrix} = 0 \qquad (2.4.7)$$

$$\implies (\lambda - 1)(\lambda - 4) - \frac{25}{4} = 0 \qquad (2.4.8)$$

$$\implies 4\lambda^2 - 20\lambda - 9 = 0 \qquad (2.4.9)$$

The roots of (2.4.9), i.e. the eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = \frac{5 + \sqrt{34}}{2}, \lambda_2 = \frac{5 - \sqrt{34}}{2} \qquad (2.4.10)$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{Vp} = \lambda\mathbf{p} \qquad (2.4.11)$$
$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \qquad (2.4.12)$$

For $\lambda_1 = \dfrac{5 + \sqrt{34}}{2}$

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{3+\sqrt{34}}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} \end{pmatrix} \qquad (2.4.13)$$

To find $\mathbf{p}_1$, let's look at Augmented form of $(\lambda_1\mathbf{I} - \mathbf{V})$

$$\begin{pmatrix} \frac{3+\sqrt{34}}{2} & \frac{5}{2} & 0 \\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0 \end{pmatrix} \qquad (2.4.14)$$

$$\xrightarrow{R_1 \leftarrow \frac{2}{3+\sqrt{34}}R_1} \begin{pmatrix} 1 & \frac{-3+\sqrt{34}}{5} & 0 \\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0 \end{pmatrix} \qquad (2.4.15)$$

$$\xrightarrow{R_2 \leftarrow \frac{2}{5}R_2 - R_1} \begin{pmatrix} 1 & \frac{-3+\sqrt{34}}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.4.16)$$

So we get

$$x_1 + \left(\frac{-3 + \sqrt{34}}{5}\right)x_2 = 0 \qquad (2.4.17)$$

Thus, our eigenvector corresponding to λ_1

$$\mathbf{p}_1 = \begin{pmatrix} \frac{3-\sqrt{34}}{5} \\ 1 \end{pmatrix} \qquad (2.4.18)$$

For $\lambda_2 = \dfrac{5 - \sqrt{34}}{2}$

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{3-\sqrt{34}}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} \end{pmatrix} \qquad (2.4.19)$$

To find $\mathbf{p}_2$, let's look at Augmented form of $(\lambda_2\mathbf{I} - \mathbf{V})$

$$\begin{pmatrix} \frac{3-\sqrt{34}}{2} & \frac{5}{2} & 0 \\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0 \end{pmatrix} \qquad (2.4.20)$$

$$\xrightarrow{R_1 \leftarrow \frac{2}{3-\sqrt{34}}R_1} \begin{pmatrix} 1 & \frac{-3-\sqrt{34}}{5} & 0 \\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0 \end{pmatrix} \qquad (2.4.21)$$

$$\xrightarrow{R_2 \leftarrow \frac{2}{5}R_2 - R_1} \begin{pmatrix} 1 & \frac{-3-\sqrt{34}}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.4.22)$$

So we get

$$x_1 + \left(\frac{-3 - \sqrt{34}}{5}\right)x_2 = 0 \qquad (2.4.23)$$

Thus, our eigenvector corresponding to λ_2

$$\mathbf{p}_2 = \begin{pmatrix} \frac{3+\sqrt{34}}{5} \\ 1 \end{pmatrix} \qquad (2.4.24)$$

We know $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where $\mathbf{P}$ and the diagonal matrix $\mathbf{D}$ are given as:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad (2.4.25)$$

$$= \begin{pmatrix} \frac{5+\sqrt{34}}{2} & 0 \\ 0 & \frac{5-\sqrt{34}}{2} \end{pmatrix} \qquad (2.4.26)$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \qquad (2.4.27)$$

$$= \begin{pmatrix} \frac{3-\sqrt{34}}{5} & \frac{3+\sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \qquad (2.4.28)$$

So, the equation of the pair of straight lines is given by :

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \qquad |\mathbf{V}| \neq 0 \qquad (2.4.29)$$

$$\mathbf{y}^T \begin{pmatrix} \dfrac{5+\sqrt{34}}{2} & 0 \\ 0 & \dfrac{5-\sqrt{34}}{2} \end{pmatrix} \mathbf{y} = 0 \qquad (2.4.30)$$

Now,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |\mathbf{V}| \neq 0 \qquad (2.4.31)$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad (2.4.32)$$

And,

$$\mathbf{P}^T = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & 1 \\ \frac{3+\sqrt{34}}{5} & 1 \end{pmatrix} \qquad (2.4.33)$$

Using affine transformation, we can express the equation as

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} + \mathbf{c} \qquad (2.4.34)$$

$$\implies \mathbf{y} = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & 1 \\ \frac{3+\sqrt{34}}{5} & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad (2.4.35)$$