Assignment 8

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***Abstract*—This document deals with linear operators and basis of a finite dimensional vector space over a field.**

1 Problem

Let $\mathbb{V}$ be finite dimensional vector space over the field $\mathbb{F}$, and let $\mathbf{S}$ and $\mathbf{T}$ be linear operators on $\mathbb{V}$. When do there exist ordered bases $\mathcal{B}$ and $\mathcal{B}'$ for $\mathbb{V}$ such that $[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$? Prove that such bases exist if and only if there is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{USU}^{-1}$

2 Solution

Assume $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$	Assume $\mathbf{T} = \mathbf{USU}^{-1}$
Given $\mathbb{V}$ is a finite dimensional vector space over field $\mathbb{F}$ $\mathbf{S}$ and $\mathbf{T}$ are linear operators on $\mathbb{V}$ $\mathcal{B}$ and $\mathcal{B}'$ are ordered bases for $\mathbb{V}$ $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$	**Given** $\mathbb{V}$ is a finite dimensional vector space over field $\mathbb{F}$ $\mathbf{S}$ and $\mathbf{T}$ are linear operators on $\mathbb{V}$ $\mathcal{B}$ and $\mathcal{B}'$ are ordered bases for $\mathbb{V}$ There is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{USU}^{-1}$
To prove There is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{USU}^{-1}$	**To prove** $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$
Assumptions Let $\mathbf{U}$ be the operator which carries $\mathcal{B}$ to $\mathcal{B}'$ $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ $\mathcal{B}' = \{\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n\}$	**Assumptions** Let $\mathbf{U}$ be the operator which carries $\mathcal{B}$ to $\mathcal{B}'$ $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ $\mathcal{B}' = \{\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n\}$
Proof For $\mathbf{v} \in \mathbb{V}$, expressed as a linear combination of the vectors of $\mathcal{B}$ $\mathbf{v} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_n\mathbf{x}_n$ $\mathbf{w} = \mathbf{U}(\mathbf{v}) = a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n$ $\because [S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$ $\mathbf{S}(\mathbf{v}) = c_1\mathbf{x_1} + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$ and $\mathbf{T}(\mathbf{w}) = c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_n\mathbf{x}'_n$ $\mathbf{U}^{-1}\mathbf{TU}(\mathbf{v})$ $= \mathbf{U}^{-1}\mathbf{T}(\mathbf{w})$ $= \mathbf{U}^{-1}(c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_n\mathbf{x}'_n)$ $= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$ $= \mathbf{S}(\mathbf{v})$ $\implies \mathbf{U}^{-1}\mathbf{TU} = \mathbf{S}$ $\implies \mathbf{T} = \mathbf{USU}^{-1}$	**Proof** For $\mathbf{v} \in \mathbb{V}$, expressed as a linear combination of the vectors of $\mathcal{B}'$ $\mathbf{v} = a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n$ $\mathbf{USU}^{-1}(\mathbf{v}) = \mathbf{T}(\mathbf{v})$ $= \mathbf{T}(a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n)$ $= c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_n\mathbf{x}'_n$ But we know that $\mathbf{USU}^{-1}(\mathbf{v}) = \mathbf{USU}^{-1}(a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n)$ $= \mathbf{US}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_n\mathbf{x}_n)$ So, $\mathbf{S}$ in basis $\mathcal{B}$ has to have the same entries as $\mathbf{T}$ in basis $\mathcal{B}'$ $\therefore [S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$

TABLE 0