# Assignment 8

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*Abstract*—**This document deals with linear operators and basis of a finite dimensional vector space over a field.**

## 1 Problem

Let $\mathbb{V}$ be finite dimensional vector space over the field $\mathbb{F}$, and let $\mathbf{S}$ and $\mathbf{T}$ be linear operators on $\mathbb{V}$. When do there exist ordered bases $\mathcal{B}$ and $\mathcal{B}'$ for $\mathbb{V}$ such that $[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$? Prove that such bases exist if and only if there is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$

## 2 Solution

Assume $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$, where $\mathbf{S}$ and $\mathbf{T}$ are linear operators on $\mathbb{V}$, and $\mathbb{V}$ has bases as follows

$$\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$$
$$\mathcal{B}' = \{\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_n\}$$

Now, let $\mathbf{v} \in \mathbb{V}$. Then we can write $\mathbf{v}$ as a linear combination of the vectors of $\mathcal{B}$

$$\mathbf{v} = a_1\mathbf{x_1} + a_2\mathbf{x}_2 + \dots + a_n\mathbf{x}_n \tag{2.0.1}$$

Let $\mathbf{U}$ be the operator which carries $\mathcal{B}$ to $\mathcal{B}'$. Then,

$$\mathbf{w} = \mathbf{U}(\mathbf{v}) = a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \dots + a_n\mathbf{x}'_n \tag{2.0.2}$$

Given

$$[S]_{\mathcal{B}} = [T]'_{\mathcal{B}} \tag{2.0.3}$$
$$\implies \mathbf{S}(\mathbf{v}) = c_1\mathbf{x_1} + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n \quad \text{and} \tag{2.0.4}$$
$$\mathbf{T}(\mathbf{w}) = c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \dots + c_n\mathbf{x}'_n \tag{2.0.5}$$

Now using (2.0.2),(2.0.5) and (2.0.4) in the following expresssion, we get

$$\mathbf{U}^{-1}\mathbf{T}\mathbf{U}(\mathbf{v}) = \mathbf{U}^{-1}\mathbf{T}(\mathbf{w}) \tag{2.0.6}$$
$$= \mathbf{U}^{-1}(c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \dots + c_n\mathbf{x}'_n) \tag{2.0.7}$$
$$= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n \tag{2.0.8}$$
$$= \mathbf{S}(\mathbf{v}) \tag{2.0.9}$$
$$\implies \mathbf{U}^{-1}\mathbf{T}\mathbf{U} = \mathbf{S} \tag{2.0.10}$$
$$\implies \mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1} \tag{2.0.11}$$

Hence, proved that there exist ordered bases $\mathcal{B}$ and $\mathcal{B}'$ for $\mathbb{V}$ such that if $[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$, then there is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$

Conversely, assume there exists an invertible operator $\mathbf{U}$ such that $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$. Let

$$\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$$
$$\mathcal{B}' = \{\mathbf{U}(\mathbf{x}_1), \mathbf{U}(\mathbf{x}_2), \dots, \mathbf{U}(\mathbf{x}_n)\}$$

For $\mathbf{v} \in \mathbb{V}$,

$$\mathbf{v} = a_1\mathbf{x_1} + a_2\mathbf{x}_2 + \dots + a_n\mathbf{x}_n \tag{2.0.12}$$

By definition,

$$\mathbf{T}(\mathbf{v}) = \mathbf{T}(a_1\mathbf{x_1} + a_2\mathbf{x}_2 + \dots + a_n\mathbf{x}_n) \tag{2.0.13}$$
$$= c_1\mathbf{x_1} + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n \tag{2.0.14}$$
$$= \mathbf{U}\mathbf{S}\mathbf{U}^{-1}(\mathbf{v}) \tag{2.0.15}$$

We know that $\mathbf{U}^{-1}(\mathbf{v}) = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \dots + a_n\mathbf{x}_n$. So $\mathbf{S}$ in basis $\mathcal{B}$ has the same entries as $\mathbf{T}$ in basis $\mathcal{B}'$. Henced, proved that that there exist ordered bases $\mathcal{B}$ and $\mathcal{B}'$ for $\mathbb{V}$ such that if there is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$, then $[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$