First, we recognize that our prior distribution on $\boldsymbol{\beta}$ actually results from a mixture of multivariate normals.

$\boldsymbol{\beta} \mid \lambda^{\beta} \sim \sum_{m=1}^{M} \lambda^{\beta} \mathcal{N}(\mathbf{0},),$ (1)

where for each W$_m$in$M, wehavethatW_m =\phi_m + \omega_m$ on the diagonal and ω on the off-diagonal.

In computing a mixture distribution, we recognize that

p$(\theta \mid D) \propto f(x \mid \theta) \sum w_i p_i(theta)) \propto \sum w_i p_i(theta) f(x \mid \theta)) \propto$

0.1 BF and posterior prob approximation

MLE for π_{sl}^{α} and posterior on α_{sl} can be computed as described in the previous sections. This section describes derivations for the approximate Bayes Factor (ABF) and posterior on $\boldsymbol{\beta}_{sl}$ when $\boldsymbol{\beta}_{sl} \sim \mathcal{N}(\mathbf{0}, \sigma_{sl,m}^2 \mathbf{I}_2)$ $(Z_{sl} = m)$ in the prior above. Then, MLE for π_{sl}^{β} and posterior on $\boldsymbol{\beta}_{sl}$ can be computed as described in the previous sections.

I drop the subscript s and l for convenience. Let $\Sigma = I(\hat{\boldsymbol{\beta}})^{-1}$ and $A = \phi_m^2 + \omega_m^2 \mathbf{I}_2$+omega$_m^2 ontheoff-diagonalto$ P(D $\mid Z = m) = \int \frac{C'}{2\pi|\Sigma|^{\frac{1}{2}}} \exp[-\frac{(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\Sigma^{-1}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}{2}] \frac{1}{2\pi|A|^{\frac{1}{2}}} \exp[-\frac{\boldsymbol{\beta}' A^{-1}\boldsymbol{\beta}}{2}] \mathrm{d}\boldsymbol{\beta}$
$= \frac{C'}{(2\pi)^2|\Sigma|^{\frac{1}{2}}|A|^{\frac{1}{2}}} \int \exp[-\frac{(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\Sigma^{-1}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})+\boldsymbol{\beta}' A^{-1}\boldsymbol{\beta}}{2}] \mathrm{d}\boldsymbol{\beta}$(2)Let $\mathbf{b} = (\Sigma^{-1}+A^{-1})^{-1}\Sigma^{-1}\hat{\boldsymbol{\beta}}$, then $\mathsf{P}(D \mid Z = m)$

$$
\begin{aligned}
&= \frac{C'}{(2\pi)^2|\Sigma|^{\frac{1}{2}}|A|^{\frac{1}{2}}} \int \exp[-\frac{(\boldsymbol{\beta}-\mathbf{b})'(\Sigma^{-1}+A^{-1})(\boldsymbol{\beta}-\mathbf{b}) - \hat{\boldsymbol{\beta}}'\Sigma^{-1}(\Sigma^{-1}+A^{-1})^{-1}\Sigma^{-1}\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{\beta}}'\Sigma^{-1}\hat{\boldsymbol{\beta}}}{2}] \\
&= \frac{C' 2\pi |(\Sigma^{-1}+A^{-1})^{-1}|^{\frac{1}{2}}}{(2\pi)^2|\Sigma|^{\frac{1}{2}}|A|^{\frac{1}{2}}} \exp[-\frac{-\hat{\boldsymbol{\beta}}'\Sigma^{-1}(\Sigma^{-1}+A^{-1})^{-1}\Sigma^{-1}\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{\beta}}'\Sigma^{-1}\hat{\boldsymbol{\beta}}}{2}].
\end{aligned}
$$

And

$$
\mathsf{P}(D \mid \boldsymbol{\beta} = \mathbf{0}) = \frac{C'}{2\pi|\Sigma|^{\frac{1}{2}}} \exp[-\frac{\hat{\boldsymbol{\beta}}'\Sigma^{-1}\hat{\boldsymbol{\beta}}}{2}]. \qquad (5)
$$

Then, ABF can be written as

$$
\begin{aligned}
\mathrm{BF_i}(\sigma_{\mathrm{m}}^2) &= \frac{\mathsf{P}(D \mid Z=m)}{\mathsf{P}(D \mid \boldsymbol{\beta}=\mathbf{0})} && (6)\\
&= \frac{|(\Sigma^{-1}+A^{-1})^{-1}|^{\frac{1}{2}}}{|A|^{\frac{1}{2}}} \exp\left[-\frac{-\hat{\boldsymbol{\beta}}'\Sigma^{-1}(\Sigma^{-1}+A^{-1})^{-1}\Sigma^{-1}\hat{\boldsymbol{\beta}}}{2}\right]. && (7)\\
&= \frac{1}{|I(\hat{\boldsymbol{\beta}})+\frac{1}{\sigma_m^2}\mathbf{I}_2|^{\frac{1}{2}}\sigma_m^2} \exp\left[-\frac{-\hat{\boldsymbol{\beta}}'I(\hat{\boldsymbol{\beta}})(I(\hat{\boldsymbol{\beta}})+\frac{1}{\sigma_m^2}\mathbf{I}_2)^{-1}I(\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}}}{2}\right]. && (8)
\end{aligned}
$$

And a posterior on $\boldsymbol{\beta}$ is

$$
\begin{aligned}
\mathsf{P}(\boldsymbol{\beta} \mid D, Z=m) &\propto \mathcal{L}(\boldsymbol{\beta})\mathsf{P}(\boldsymbol{\beta} \mid Z=m) && (9)\\
&\propto \exp\left[-\frac{(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\Sigma^{-1}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}{2}\right]\exp\left[-\frac{\boldsymbol{\beta}'A^{-1}\boldsymbol{\beta}}{2}\right] && (10)\\
&\propto \exp\left[-\frac{(\boldsymbol{\beta}-\mathbf{b})'(\Sigma^{-1}+A^{-1})(\boldsymbol{\beta}-\mathbf{b})}{2}\right], && (11)
\end{aligned}
$$

leading to

$$
\begin{aligned}
\boldsymbol{\beta} \mid D, Z=m &\sim \mathcal{N}(\mathbf{b}, (\Sigma^{-1}+A^{-1})^{-1}) && (12)\\
&\sim \mathcal{N}((\Sigma^{-1}+A^{-1})^{-1}\Sigma^{-1}\hat{\boldsymbol{\beta}}, (\Sigma^{-1}+A^{-1})^{-1}) && (13)\\
&\sim \mathcal{N}((I(\hat{\boldsymbol{\beta}})+\frac{1}{\sigma_m^2}\mathbf{I}_2)^{-1}I(\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}}, (I(\hat{\boldsymbol{\beta}})+\frac{1}{\sigma_m^2}\mathbf{I}_2)^{-1}). && (14)
\end{aligned}
$$

Let $\mathbf{b}=(b_1,b_2)$ and $(\Sigma^{-1}+A^{-1})^{-1}=\Omega$, then

$$
\begin{aligned}
\mathsf{P}(\beta_1 \mid D, Z=m) &\sim \mathcal{N}(b_1, \omega_{1,1}) && (15)\\
\mathsf{P}(\beta_1+\beta_2 \mid D, Z=m) &\sim \mathcal{N}(b_1+b_2, \omega_{1,1}+\omega_{1,2}+\omega_{2,1}+\omega_{2,2}). && (16)
\end{aligned}
$$