9.6.7

EE24BTECH11066 - YERRA AKHILESH

Question:
Find the general solution for the given differential equation.

$$x \log x \cdot \frac{dy}{dx} + y = \frac{2}{x} \log x$$

Solution: Divide the given Differential equation by $x \log x$,

$$\frac{dy}{dx} + \left(\frac{1}{x \log x}\right) \cdot y = \frac{2}{x^2} \tag{0.1}$$

Above Differential equation is of form **first-order linear differential equation** and its general form is expressed as:

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \tag{0.2}$$

and the general solution is,

$$\mu(x) \cdot y = \int \mu(x) \, Q(x) \, dt \tag{0.3}$$

where $\mu(x)$ is an integrating factor, expressed as

$$\mu(x) = e^{\int P(x) \, dx} \tag{0.4}$$

for the given question,

$$\mu(x) = e^{\int \frac{1}{x \log x} \, dx} \tag{0.5}$$

$$\log x = t \tag{0.6}$$

Differentiate with x on both sides,

$$\frac{1}{x} \cdot dx = dt \tag{0.7}$$

Therefore $\mu(x)$:

$$\mu(x) = e^{\int \frac{1}{t} \, dt} \tag{0.8}$$

$$\mu(x) = e^{\log t} \tag{0.9}$$

$$\mu(x) = t = \log x \tag{0.10}$$

Substitute this integrating factor in general solution,

$$y \cdot \log x = \int \log x \cdot \frac{2}{x^2} \, dx \tag{0.11}$$

Integrate the right-side part using by-parts,

$$y \cdot \log x = 2\left(\left(\log x \cdot \int \frac{1}{x^2}\,dx\right) + \left(\int \frac{1}{x^2}\,dx\right)\right) \tag{0.12}$$

$$y \cdot \log x = 2\left(\log x \cdot \frac{x^{-1}}{-1} - \frac{1}{x}\right) + c \tag{0.13}$$

$$y \cdot \log x = -2 \cdot \frac{\log x}{x} - \frac{2}{x} + c \tag{0.14}$$

Divide with $\log x$ on the both sides and assume $c = 0$,

$$y(x) = \frac{-2}{x}\left(1 + \frac{1}{\log x}\right) \tag{0.15}$$

Now let us verify this computationally From definition of $\frac{dy}{dx}$,

$$y_{n+1} = y_n + \frac{dy}{dx} \cdot h \tag{0.16}$$

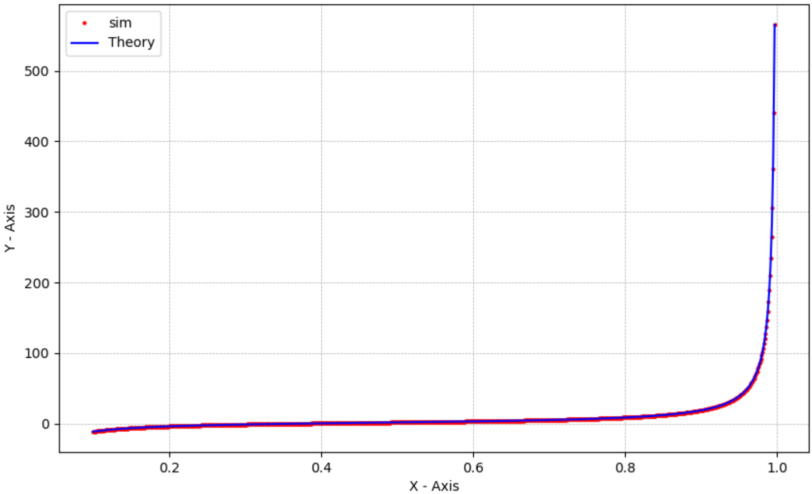
(where h is small number tending to zero) From the differential equation given,

$$\frac{dy}{dx} = \frac{2}{x^2} - \frac{y}{x \log x} \tag{0.17}$$

By substituting,

$$y_{n+1} = y_n + \left(\frac{2}{x_n^2} - \frac{y_n}{x_n \log x_n}\right) \cdot h \tag{0.18}$$

By taking $x_1 = 0.1$ and $y_1 = -11.32$ and $h = 0.001$ going till $x = 1$ by iterating through the loop and finding $y_2, y_3, \cdots, y_{500}$ and plotting the graph, we can verify if the function we got by solving the differential equation mathematically.


sim
Theory
Y - Axis
X - Axis
0
100
200
300
400
500
0.2
0.4
0.6
0.8
1.0