9.5.14

EE24BTECH11066 - YERRA AKHILESH

Question:
For the following differential equation, find the particular solution satisfying the given condition:

$\frac{dy}{dx} - \frac{y}{x} + \csc\left(\frac{y}{x}\right) = 0; y = 0$ when $x = 1$

Solution: Given Differential equation:

$$\frac{dy}{dx} - \frac{y}{x} + \csc\left(\frac{y}{x}\right) = 0 \tag{0.1}$$

let t be a variable,

$$\frac{y}{x} = t \tag{0.2}$$

$$y = xt \tag{0.3}$$

Differentiate on both sides with x,

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \tag{0.4}$$

Substitute the above $\frac{dy}{dx}$ in given differential equation,

$$t + x\frac{dt}{dx} - t + \csc(t) = 0 \tag{0.5}$$

$$\frac{dt}{dx} = -\frac{\csc(t)}{x} \tag{0.6}$$

$$-\sin t\, dt = \frac{dx}{x} \tag{0.7}$$

Integrating on both sides,

$$\int -\sin t\, dt = \int \frac{1}{x}\, dx \tag{0.8}$$

$$\cos t = \log|cx| \tag{0.9}$$

where c be integrating constant and it can be obtained by substituting intial values($y = 0, x = 1$)

On substituting,

$$c = e \tag{0.10}$$

Therefore final solution for $y(x)$ by substituting $t = \frac{y}{x}$,

$$\cos\frac{y}{x} = (1 + \log x) \tag{0.11}$$

$$y(x) = x \cdot \cos^{-1}(1 + \log x) \tag{0.12}$$

Now let us verify this computationally From definition of $\frac{dy}{dx}$,

$$y_{n+1} = y_n + \frac{dy}{dx} \cdot h \tag{0.13}$$

(where h is small number tending to zero) From the differential equation given,

$$\frac{dy}{dx} = \frac{y}{x} - \csc\frac{y}{x} \tag{0.14}$$

By substituting 0.14 in 0.13,

$$y_{n+1} = y_n + \left(\frac{y_n}{x_n} - \csc\frac{y_n}{x_n}\right) \cdot h \tag{0.15}$$

By taking $x_1 = 0.2$ and $y_1 = 0.4361$ and $h = 0.0032$ going till $x = 1$ by iterating through the loop and finding $y_2, y_3, \cdots, y_{500}$ and plotting the graph, we can verify if the function we got by solving the differential equation mathematically.