10.4.1.2.4

EE24BTECH11066 - YERRA AKHILESH

Question:

On comparing the ratio's $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the following pair of linear equations are consistent, or inconsistent.

$$2x - 3y = 8$$
$$4x - 6y = 9$$

Solution :

For the first equation:

$$a_1 = 2, \tag{0.1}$$
$$b_1 = -3, \tag{0.2}$$
$$c_1 = 8 \tag{0.3}$$

and For the second equation:

$$a_2 = 4, \tag{0.4}$$
$$b_2 = -6, \tag{0.5}$$
$$c_2 = 9 \tag{0.6}$$

Compare the ratio's,

$$\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2} \tag{0.7}$$
$$\frac{b_1}{b_2} = \frac{-3}{-6} = \frac{1}{2} \tag{0.8}$$
$$\frac{c_1}{c_2} = \frac{8}{9} \tag{0.9}$$

from the above results

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \tag{0.10}$$

We know that if:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \tag{0.11}$$

then the equations are **inconsistent**, meaning they represent parallel lines with **no solution**

We aim to solve the system of equations using LU decomposition:

$$2x - 3y = 8, \tag{0.12}$$

$$4x - 6y = 9. \tag{0.13}$$

Step 1: Represent the system in matrix form

$$A = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}. \tag{0.14}$$

Thus, the system becomes:

$$A\mathbf{x} = \mathbf{b}. \tag{0.15}$$

Step 2: Decompose A into L and U

To find L (lower triangular matrix) and U (upper triangular matrix), we perform Gaussian elimination.

Row Reduction

Start with:

$$A = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}. \tag{0.16}$$

Perform the row operation $R_2 \to R_2 - 2R_1$:

$$\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}. \tag{0.17}$$

The second row becomes zero, indicating that the matrix A is singular ($\det A = 0$).

Define L and U

From the row reduction, we can write:

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}. \tag{0.18}$$

Thus:

$$A = L \cdot U. \tag{0.19}$$

Step 3: Solve the system using forward and back substitution

Forward Substitution: Solve $L\mathbf{y} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}. \tag{0.20}$$

From the first row:

$$y_1 = 8. \tag{0.21}$$

From the second row:

$$2y_1 + y_2 = 9, \quad \text{which gives } y_2 = -7. \tag{0.22}$$

Thus:

$$\mathbf{y} = \begin{bmatrix} 8 \\ -7 \end{bmatrix}. \tag{0.23}$$

Back Substitution: Solve $U\mathbf{x} = \mathbf{y}$

$$\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \end{bmatrix}. \tag{0.24}$$

From the second row:

$$0x + 0y = -7 \quad \text{(contradiction!)}. \tag{0.25}$$

Conclusion

The LU decomposition process reveals that the system is **inconsistent**. The matrix A is singular (its determinant is zero), and the system does not have a solution. The inconsistency arises because the equations contradict each other.

LU Decomposition computation

The LU decomposition can be efficiently computed using Doolittle's algorithm. This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:

Elements of the U Matrix:

For each column j:

$$U_{ij} = A_{ij} \quad \text{if } i = 0, \tag{0.26}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \tag{0.27}$$

Elements of the L Matrix:

For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \tag{0.28}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \tag{0.29}$$

This systematic approach ensures that the matrix A is decomposed into L and U without requiring row swaps, provided A is nonsingular.

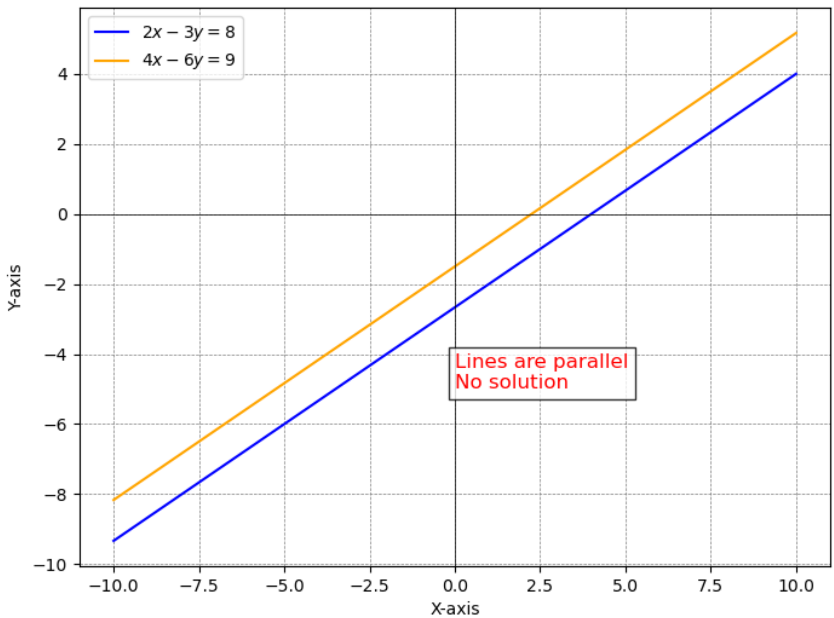

Fig. 0.1: Solution of the system of linear equations