10.3.3.3.4

EE24BTECH11066 - YERRA AKHILESH

Question:
The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is ₹105, and for a journey of 15 km, the charge paid is ₹155. Find the fixed charge and the charge per km.
Solution:
Let the fixed charge be x and the charge per kilometer be y. From the question, we can frame the following equations:

$$x + 10y = 105 \tag{0.1}$$
$$x + 15y = 155 \tag{0.2}$$
$$\begin{bmatrix} 1 & 10 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105 \\ 155 \end{bmatrix} \tag{0.3}$$

Any non-singular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U:

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b} \tag{0.4}$$

The upper triangular matrix U is found by row reducing A:

$$\begin{bmatrix} 1 & 10 \\ 1 & 15 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix} \tag{0.5}$$

Let

$$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \tag{0.6}$$

l_{21} is the multiplier used to zero a_{21}, so $l_{21} = 1$.
Now

$$A = \begin{bmatrix} 1 & 10 \\ 1 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix} \tag{0.7}$$

Now we can get the solution to our problem by the two-step process:

$$L\mathbf{y} = \mathbf{b} \tag{0.8}$$
$$U\mathbf{x} = \mathbf{y} \tag{0.9}$$

Using forward substitution to solve the first equation:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 105 \\ 155 \end{bmatrix} \tag{0.10}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 105 \\ 50 \end{bmatrix} \tag{0.11}$$

$$\begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 105 \\ 50 \end{bmatrix} \tag{0.12}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \tag{0.13}$$

Therefore, the fixed charge is ₹5 and the charge per kilometer is ₹10.

Conclusion

The LU decomposition process reveals that the system is **consistent**. The matrix A is non-singular (its determinant is non-zero), and the system has a unique solution. This indicates that the fixed charge and the charge per kilometer can be accurately determined.

LU Decomposition Computation

The LU decomposition is computed using Doolittle's algorithm. This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:

Elements of the U Matrix:

For each column j:

$$U_{ij} = A_{ij} \quad \text{if } i = 0, \tag{0.14}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \tag{0.15}$$

Elements of the L Matrix:

For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \tag{0.16}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \tag{0.17}$$

Using this systematic approach, the matrix A is decomposed into L and U:

$$A = \begin{bmatrix} 1 & 10 \\ 1 & 15 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix}. \tag{0.18}$$

This decomposition confirms that the fixed charge is ₹5 and the charge per kilometer is ₹10.

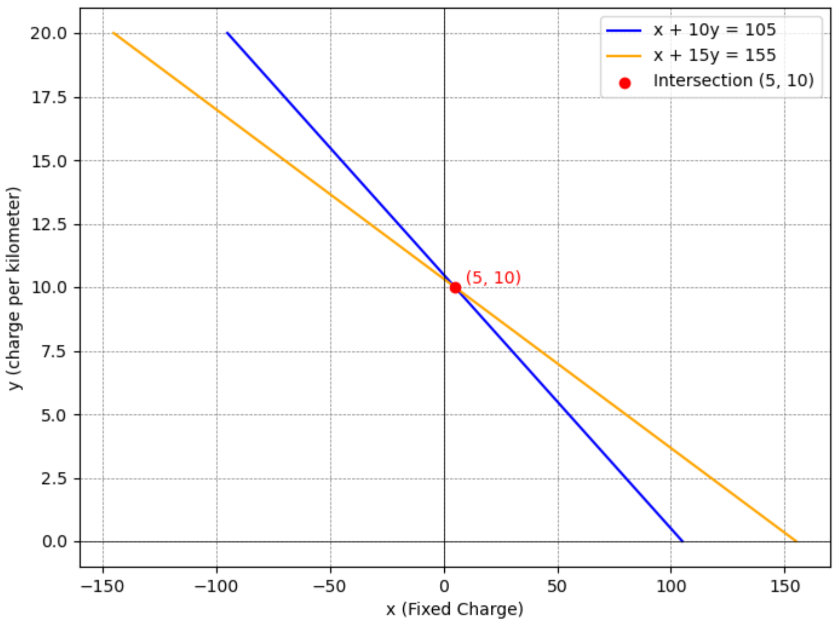

Fig. 0.1: Solution of the system of linear equations