Question-12.6.5.2.2

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Problem Statement

The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is 105, and for a journey of 15 km, the charge paid is 155. Find the fixed charge and the charge per km.

Theoretical Solution

Let the fixed charge be x and the charge per kilometer be y. From the question, we can frame the following equations:

$$x + 10y = 105 \tag{1}$$

$$x + 15y = 155 \tag{2}$$

$$\begin{bmatrix} 1 & 10 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105 \\ 155 \end{bmatrix} \tag{3}$$

Any non-singular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U:

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b} \tag{4}$$

The upper triangular matrix U is found by row reducing A:

$$\begin{bmatrix} 1 & 10 \\ 1 & 15 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix} \tag{5}$$

Let

$$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \tag{6}$$

l_{21} is the multiplier used to eliminate the element a_{21} in the matrix A.
Therefore $l_{21} = 1$
Now,

$$A = \begin{bmatrix} 1 & 10 \\ 1 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix} \tag{7}$$

Now we can get the solution to our problem by the two-step process:

$$L\mathbf{y} = \mathbf{b} \tag{8}$$

$$U\mathbf{x} = \mathbf{y} \tag{9}$$

Using forward substitution to solve the first equation:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 105 \\ 155 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 105 \\ 50 \end{bmatrix} \tag{11}$$

Using this we solve for x in $y = \mathbf{U}x$ using back-substitution knowing that $\mathbf{U}$ is upper triangular. LU Factorizing $\mathbf{A}$, we get:

$$\begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 105 \\ 50 \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \tag{13}$$

Therefore, the fixed charge is 5 and the charge per kilometer is 10.

LU Decomposition Computation

The LU decomposition is computed using Doolittle's algorithm. This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:

Elements of the U Matrix:

For each column j:

$$U_{ij} = A_{ij} \quad \text{if } i = 0, \tag{14}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \tag{15}$$

Elements of the L Matrix:
For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \tag{16}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \tag{17}$$

Using this systematic approach, the matrix A is decomposed into L and U:

$$A = \begin{bmatrix} 1 & 10 \\ 1 & 15 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix}. \tag{18}$$

Conclusion

This decomposition confirms that the **fixed charge is** 5 and the **charge per kilometer is** 10.
The LU decomposition process reveals that the system is **consistent**. The matrix A is non-singular (its determinant is non-zero), and the system has a unique solution. This indicates that the fixed charge and the charge per kilometer can be accurately determined.

Plot

Figure_1.png