Introduction to Topological Manifolds, Lee - Homework 1

suremark

August 23, 2023

1 Problem 1

(a). f is continuous.
(b). f is an open and closed mapping.
(c). Let us first show that continuity on X implies continuity at every point in X.
Let $x_0 \in X$ and $U \in N(f(x_0))$ be arbitrary, where $N(x)$ denotes the collection of open neighborhoods of x. Then there exists $V \in N(x_0)$ such that $f(V) \subseteq U$; simply take $V = f^{-1}(U)$ which is open (by continuity on X) and contains x_0.
To show the converse, suppose f is continuous at every point in X, and let $A \subseteq Z$ be an arbitrary open set. Take

$$V = \bigcup \{ \ V_x \ : \ x \in f^{-1}(A) \ \}$$

where each V_x is a neighborhood of x such that $f(V_x) \subseteq A$. The existence of each V_x is guaranteed by continuity at every point in X (simply take $x_0 = x$ and $U = A$ in the definition of continuity at a point). By construction, $f(V) \subseteq A$ since $f(V_x) \subseteq A$, which imples $V \subseteq f^{-1}(A)$. But V contains every point in $f^{-1}(A)$ since $x \in V_x$, which implies that $f^{-1}(A) \subseteq V$. Hence $V = f^{-1}(A)$. Since V is a union of open sets V_x, V is clearly open, so $f^{-1}(A)$ is open.
Since A was arbitrary, continuity on all of X is proved. $\blacksquare$

2 Problem 2

A manifold is a second-countable Hausdorff space locally homeomorphic to the Euclidean d-ball. For each $x \in M$ let $U_x \in N(x)$ such that there exists a homeomorphism $\varphi_x : U_x \to B^d$. Let $\{B_{x,i}\}_{i \in I}$ be a basis for U_x induced by the homeomorphism to B^d. Our basis will simply be the union of every $\{B_{x,i}\}_{i \in I}$.
Now let $A \subseteq M$ be an arbitrary open set. One has that

$$A = A \cap M = A \cap \bigcup_{x \in M} U_x = \bigcup_{x \in M} A \cap U_x$$

which is a decomposition of A into a union of open sets. Now, one has for each $x \in M$,

$$A \cap U_x = \bigcup_{i \in I_x \subseteq I} B_{x,i}$$

essentially saying $A \cap U_x$ can be represented as a union of elements from $\{B_{x,i}\}_{i \in I}$, since $B_{x,i}$ is a basis for U_x. It follows that

$$A = \bigcup_{x \in M} \bigcup_{i \in I_x \subseteq I} B_{x,i}$$

which gives A as a union of open sets from our basis.

3 Problem 3

(a). $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x \mod 1$.
(b). $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \lfloor x \rfloor$.
(c). $f : \mathbb{R} - \{0\} \to [-1, 1]$ given by $f(x) = \sin x^{-1}$.
(d). $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{x}{x^2+1}$.
(e). $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \exp((x(1-x))^{-1})$ if $x \notin [0, 1]$, and 0 otherwise
(f). $f : S^1 \to [0, 1)$ where $f(z)$ is the unique $t \in [0, 1)$ such that $\exp(2i\pi t) = z$.

4 Problem 4

(a). The empty set is represented by the zeroes of $p(x) = 1$. $\mathbb{C}$ is represented by the zeroes of $p(x) = 0$.

(b). Let $\{p_i(x)\}_{i\in I}$ be a family of polynomials (representing closed sets). Let R_i be the set of roots for each p_i. Since each R_i is finite, there is a unique monic polynomial whose roots are $\bigcap_{i\in I} R_i$, provided that it is nonempty.
If R_i is empty then just choose $p(x) = 1$.

(c). $\{\ x\ :\ p(x) = 0\ \} \cup \{\ x\ :\ q(x) = 0\ \} = \{\ x\ :\ p(x)q(x) = 0\ \}$

5 Problem 5

Let $\mathcal{U}$ be an open cover of X, i.e. a collection of open sets s.t. $\bigcup_{u\in\mathcal{U}} u = X$.

(a). Let U be open in X. Let B_u be a basis for each $u \in \mathcal{U}$. Then

$$U = U \cap X = U \cap (\bigcup_{u\in\mathcal{U}} u) = \bigcup_{u\in\mathcal{U}} U \cap u$$

Now, let $\mathcal{F}_u \subseteq B_u$ such that $\bigcup_{f\in\mathcal{F}_u} f = U \cap u$ (since B_u is a basis). Then

$$U = \bigcup_{u\in\mathcal{U}} \bigcup_{f\in\mathcal{F}_u} f$$

which is a union of open sets from $\bigcup_{u\in\mathcal{U}} B_u$ which forms a basis for X.

(b). Second-countability of each element in the cover implies the existence of a family of countable bases $\{B_u\}_{u\in\mathcal{U}}$. Then it follows that, since $\mathcal{U}$ is a countable open cover, $\bigcup_{u\in\mathcal{U}} B_u$ is a countable open cover of all of X, since the union of a countable family of countable sets is also countable.

6 Problem 6

(a).

$$d(f, g) = |f - g| = \left(\int_{\mathbb{R}} |f(t) - g(t)|^2 dt \right)^{\frac{1}{2}}$$

$$d(f, g) = d(g, f) \quad \text{by symmetry of } |\cdot|$$

$$d(f, g) \geq 0 \quad \text{by non-negativity of } |\cdot|\text{, and } d(f, g) = 0 \iff |f(t) - g(t)|^2 \sim 0$$

$$
\begin{aligned}
d(f,h) &= \left(\int_{\mathbb{R}} |f(t)-g(t)|^2 dt\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}} |g(t)-g(t)+g(t)-h(t)|^2 dt\right)^{\frac{1}{2}} \\
&\le \left(\int_{\mathbb{R}} |f(t)-g(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}} |g(t)-h(t)|^2 dt\right)^{\frac{1}{2}} \quad \text{by Minkowski's inequality} \\
&\le d(f,g)+d(g,h). \quad \blacksquare
\end{aligned}
$$

(b).

$$B_r(f) = \{\ g\ :\ \int_{\mathbb{R}} |f(t)-g(t)|^2 dt \le r^2\ \} \cong B_r(0)+f$$

U is first-countable and Hausdorff.
$f_n \to f \iff (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \ge N)|f_n - f| < \varepsilon$, which is not equivalent to uniform continuity for functions on $\mathbb{R}$.

(c). It suffices to show the primage of an open ball is open.

$$+^{-1}(B_r(h)) = \{\ (f,g)\ :\ |f+g-h| < r\ \}$$

Now suppose $(f,g) \in +^{-1}(B_r(h))$. Let $\delta > 0$, and let (f',g') be such that $(|f'-f|^2+|g'-g|^2)^{\frac{1}{2}} < \delta$.
Then of course $|f'-f| < \delta$ and $|g'-g| < \delta$ by the properties of the Euclidean metric. Now,

$$|f'-f-(-(g'-g))| \le |f'-f| + |-(g'-g)| = |f'-f| + |g'-g| < 2\delta \quad \text{by the triangle inequality.}$$

Note that $|f'+g'-(f+g)| < 2\delta$ and $|(f+g)-h| < r$. Then if we choose δ such that $2\delta < r - |f+g-h|$,

$$|f'+g'-h| = |f'+g'-(f+g)+(f+g)-h| \le |f'+g'-(f+g)| + |(f+g)-h| < 2\delta + |(f+g)-h| < r$$

From this it follows that every $(f,g) \in +^{-1}(B_r(h))$ has an open neighborhood of radius $\delta < \frac{1}{2}(r-|f+g-h|)$ inside the preimage, which proves continuity in the metric sense.

Now let us prove continuity of $\times : \mathbb{R} \times U \to U$:

$$\times^{-1}(B_r(g)) = \{\ \lambda f\ :\ |\lambda f - g| < r\ \}.$$

Let (λ', f') such that $((\lambda'-\lambda)^2 + |f'-f|^2)^{\frac{1}{2}} < \delta$.
Then of course $|\lambda'-\lambda| < \delta$ and $|f'-f| < \delta$ by the properties of the Euclidean metric. Now,

$$
\begin{aligned}
|\lambda' f' - \lambda f| &= |(\lambda'-\lambda)f - \lambda(f'-f)| \\
&\le |\lambda'-\lambda||f| + |\lambda||f'-f| \\
&\le \delta(|f|+|\lambda|).
\end{aligned}
$$

Now, $|\lambda f - g| < r$, so $\delta(|f|+|\lambda|) < r - |\lambda f - g| \implies |\lambda' f' - g| < r$ in a similar fashion to the previous argument with the $+$ function. If $|f| = |\lambda| = 0$ then $\delta > 0$ can be arbitrary. Otherwise, just choose $\delta < \frac{r-|\lambda f - g|}{|f|+|\lambda|}$.
Then $B_\delta((\lambda, f)) \subseteq \times^{-1}(g)$. $\blacksquare$