Clamped Splines

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A clamped spline is a type of spline interpolation where the first derivatives at the endpoints of the interpolation interval are specified. That is, if we are interpolating a function $f(x)$ over $[a, b]$, then we require that

$$S'(a) = f'(a) \quad \text{and} \quad S'(b) = f'(b).$$

This leads to a better approximation near the boundaries of the interpolation interval.

Clamped Cubic Spline

Given nodes $x_0 < x_1 < \cdots < x_n$ and function values $f(x_0), f(x_1), \ldots, f(x_n)$, the clamped cubic spline $S(x)$ is the unique piecewise cubic polynomial such that:

- $S(x_j) = f(x_j), \quad j = 0, 1, \ldots, n$
- $S'(x_0) = f'(x_0), \quad S'(x_n) = f'(x_n)$
- $S(x), S'(x), S''(x)$ are continuous on $[x_0, x_n]$

Construction

Let $h_i = x_{i+1} - x_i$. Define the cubic spline on each subinterval $[x_i, x_{i+1}]$ as:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \quad j = 0, 1, \ldots, n - 1.$$

The algorithm to compute the coefficients a_j, b_j, c_j, d_j is as follows:

Step 1: For $i = 1, \ldots, n - 1$, set $h_i = x_{i+1} - x_i$.

Step 2: Set

$$\alpha_0 = 3\frac{a_1 - a_0}{h_0} - 3FPO, \quad \alpha_n = 3FPN - 3\frac{a_n - a_{n-1}}{h_{n-1}}.$$

Step 3: For $i = 1, \ldots, n - 1$, set

$$\alpha_i = 3\left(\frac{a_{i+1} - a_i}{h_i} - \frac{a_i - a_{i-1}}{h_{i-1}}\right).$$

Step 4: Set $l_0 = 2h_0$, $\mu_0 = 0.5$, $z_0 = \alpha_0/l_0$.
Step 5: For $i = 1, \ldots, n-1$, set:

$$l_i = 2(x_{i+1} - x_{i-1}) - h_{i-1}\mu_{i-1}, \quad \mu_i = \frac{h_i}{l_i}, \quad z_i = \frac{\alpha_i - h_{i-1}z_{i-1}}{l_i}.$$

Step 6: Set $l_n = h_{n-1}(2 - \mu_{n-1})$,

$$z_n = \frac{\alpha_n - h_{n-1}z_{n-1}}{l_n}, \quad c_n = z_n.$$

Step 7: For $j = n-1, \ldots, 0$, set

$$c_j = z_j - \mu_j c_{j+1}, \quad b_j = \frac{a_{j+1} - a_j}{h_j} - \frac{h_j}{3}(c_{j+1} + 2c_j), \quad d_j = \frac{c_{j+1} - c_j}{3h_j}.$$

Step 8: Output (a_j, b_j, c_j, d_j) for $j = 0, \ldots, n-1$.

Theorem

Theorem 3.12: If f is defined at $x_0 < x_1 < \cdots < x_n$ and differentiable at x_0 and x_n, then f has a unique clamped spline S on the nodes satisfying the clamped conditions $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$.

Proof: Since $f'(a) = S'(x_0) = b_0$, from the spline formula:

$$f'(a) = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1),$$

and similarly:

$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a).$$

Analogously for $f'(b)$,

$$f'(b) = \frac{a_n - a_{n-1}}{h_{n-1}} - \frac{h_{n-1}}{3}(c_{n-1} + 2c_n).$$

Multiplying and rearranging gives:

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}).$$

Thus, the full system can be solved using tridiagonal matrix methods.

Example

We use the points $(0,1), (1,e), (2,e^2), (3,e^3)$, and derivatives $f'(0) = 1, f'(3) = e^3$.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3(e-2) \\ 3(e^2 - 2e + 1) \\ 3(e^3 - 2e^2 + e) \\ 3e^2 \end{bmatrix}$$

Solving this gives (rounded to 5 decimals):

$$c_0 = 0.44468, \qquad c_1 = 1.26548,$$
$$c_2 = 3.35087, \qquad c_3 = 9.40815$$

Using back-substitution:

$$b_0 = 1.00000, \qquad d_0 = 0.27360$$
$$b_1 = 2.71016, \qquad d_1 = 0.69513$$
$$b_2 = 7.32652, \qquad d_2 = 2.01909$$

The spline is:

$$S(x) = \begin{cases} 1 + x + 0.44468x^2 + 0.27360x^3, & 0 \le x < 1 \\ 2.71828 + 2.71016(x-1) + 1.26548(x-1)^2 + 0.69513(x-1)^3, & 1 \le x < 2 \\ 7.38906 + 7.32652(x-2) + 3.35087(x-2)^2 + 2.01909(x-2)^3, & 2 \le x \le 3 \end{cases}$$

Computing integral approximation:

$$\int_0^3 s(x)dx \approx (a_0 + a_1 + a_2) + \frac{1}{2}(b_0 + b_1 + b_2) + \frac{1}{3}(c_0 + c_1 + c_2) + \frac{1}{4}(d_0 + d_1 + d_2)$$

$$= (1{+}2.71828{+}7.38906){+}\frac{1}{2}(1{+}2.71016{+}7.32652){+}\frac{1}{3}(0.44468{+}1.26548{+}3.35087){+}\frac{1}{4}(0.27360{+}0.69513{+}2.0$$

$$= 19.05965 \quad \text{(approx.)}$$

Exact value:

$$\int_0^3 e^x dx = e^3 - 1 \approx 20.08554 - 1 = 19.08554$$

Error using clamped spline:

$$|19.08554 - 19.05965| = 0.02589$$

Clamped splines are more accurate than natural splines when the derivatives at endpoints are known.