Design & Analysis of Algorithms

[Asymptotic Notations]

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Asymptotic Notations

The O-Notation

Definition :

Let $f(n)$ and $g(n)$ be two functions from the set of natural numbers to the set of nonnegative real numbers. $f(n)$ is said to be $O(g(n))$ (or $f(n) = O(g(n))$) if there exists a natural number n_0 and a constant $c > 0$ such that $\forall\ n \geq n_0,\ f(n) \leq cg(n)$.

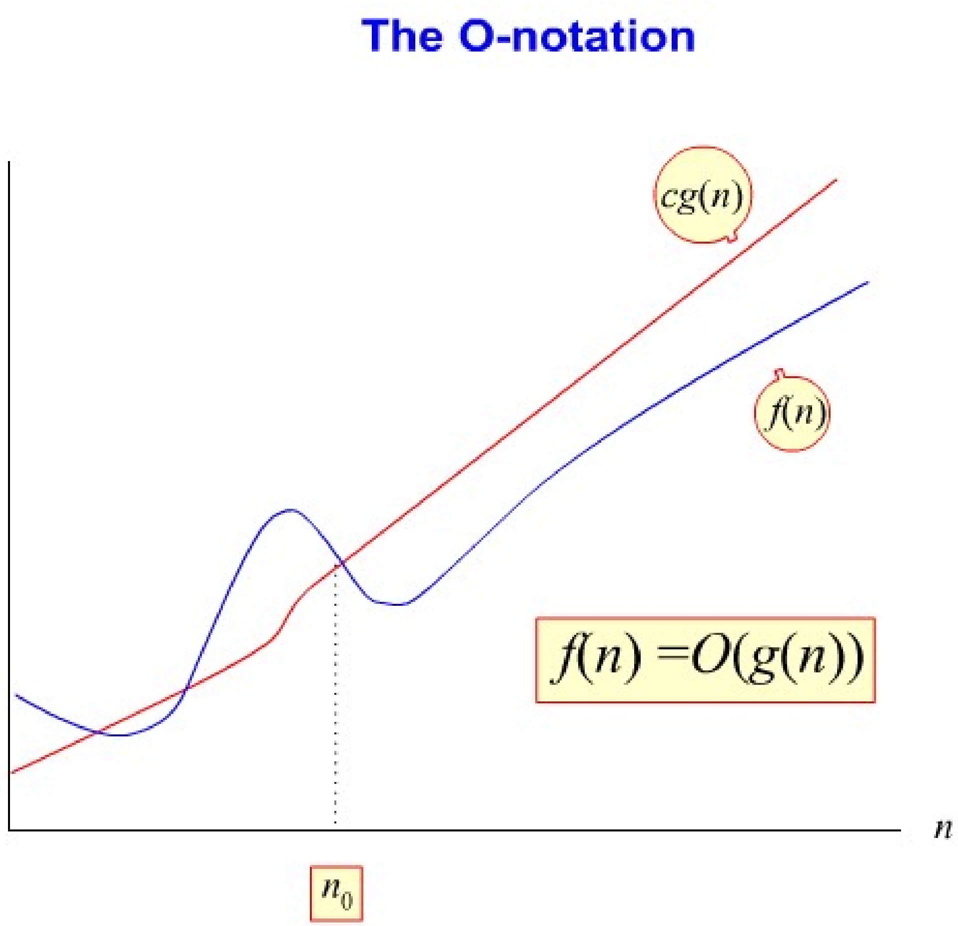
Example :

Let $f(n) = 10n^2 + 20n$. Since $\forall\ n \geq 1$

$$\begin{aligned} f(n) &= 10n^2 + 20n \\ &\leq 30n^2 \end{aligned}$$

Therefore, $f(n) = O(n^2)$ as there exists a natural number $n_0 = 1$ and a constant $c = 30 > 0$ such that $\forall\ n \geq n_0,\ f(n) \leq cg(n)$.

Growth of function



Alternative definition of O-notation

The O-notation (continued)

The definition for O-notation is also equivalent to the following:

if $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq \infty$ implies $f(n) = O(g(n))$.

Example

$f(n) = 10n^2 + 20n$ and $g(n) = n^2$.

$$\text{Since } \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{10n^2 + 20n}{n^2}$$

$$= \lim_{n\to\infty} \left(\frac{10n^2}{n^2} + \frac{20n}{n^2} \right) = \lim_{n\to\infty} \left(10 + \frac{20}{n} \right)$$

$$= 10 + \lim_{n\to\infty} \left(\frac{20}{n} \right)$$

$= 10 \neq \infty$. Hence, $f(n)$ is $O(n)$.

Note that f grows no faster than some constant times g.

Asymptotic Notations

The Ω-Notation

Definition

Let $f(n)$ and $g(n)$ be two functions from the set of natural numbers to the set of nonnegative real numbers. $f(n)$ is said to be $\Omega(g(n))$ if there exists a natural number n_0 and a constant $c > 0$ such that $\forall\ n \geq n_0$, $f(n) \geq\ c\, g(n)$.

Example

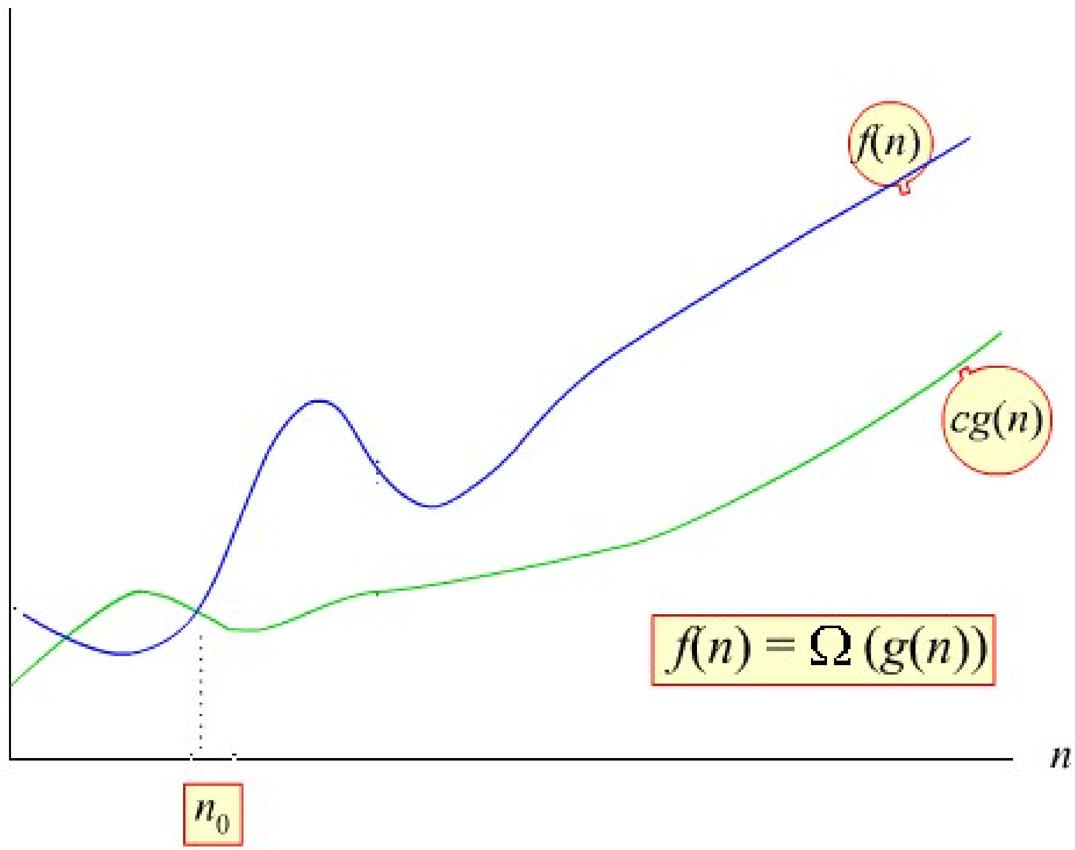
Let $f(n) = 10\,n^2 + 20n$. Since $\forall\ n\ \geq 1$,

$$\begin{aligned} f(n) &= 10\,n^2 + 20n \\ &\geq 10\,n^2 \end{aligned}$$

Therefore, $f(n) =\ \Omega(n^2)$ as there exists a natural number $n_0 = 1$ and a constant $c\ = 10 > 0$ such that $\forall\ n \geq n_0$, $f(n) \geq\ c\, g(n)$.

Growth of function

The Ω-notation



Alternative definition of Ω-notation

The Ω-Notation (Cont.)

The definition for Ω-notation is also equivalent to the following:

if $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$ implies $f(n) = \Omega(g(n))$.

Example

Consider the same problem as $f(n) = 10n^2 + 20n$ and $g(n) = n^2$.

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 10 \neq \infty$.

Hence, $f(n)$ is $\Omega(n^2)$.

It can be noted that f grows at least as fast as some constant times g. It is clear from the definition that $f(n) = \Omega(g(n))$ if and only if $g(n) = O(f(n))$

Asymptotic Notations

The Θ-Notation

Definition

Let $f(n)$ and $g(n)$ be two functions from the set of natural numbers to the set of nonnegative real numbers. $f(n)$ is said to be $\Theta(g(n))$ if there exists a natural number n_0 and two positive constants c_1 and c_2 such that $\forall\ n \geq n_0,\ \ c_1 g(n) \leq f(n) \leq c_2 g(n)$.

Example

Let $f(n) = 10n^2 + 20n$.

Then, $f(n) = O(n^2)$ since $\forall\ n \geq 1,\ \ f(n) \leq 30n^2$.

Similarly, we have $f(n) = \Omega(n^2)$ since $\forall\ n \geq 1, f(n) \geq n^2$

Thus, one can find,

$$n_0 = 1,\ c_1 = 1, \text{ and } c_2 = 30,$$

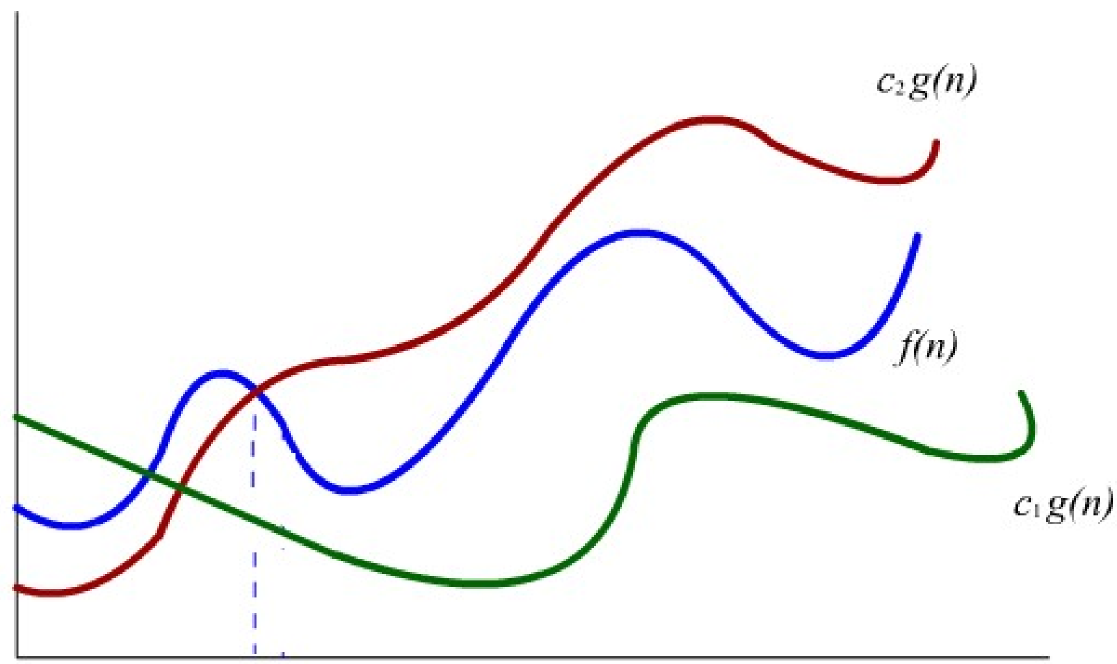
such that

$$\forall\ n \geq 1,\ \ n^2 \leq f(n) \leq 30n^2.$$

Growth of function

The Θ-Notation (Cont.)

Demonstration: A graphic representation when $f(n) = \Theta(g(n))$.



$f(n) = \Theta(g(n))$

Alternative definition of Θ-notation

The Θ-Notation (Cont.)

The definition for Θ-notation is also equivalent to the following:

if $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ implies $f(n) = \Theta(g(n))$, where c is a constant strictly greater than 0.

Example Consider the same problem as $f(n) = 10n^2 + 20n$ and $g(n) = n^2$.

$$
\begin{aligned}
\text{Since } \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{10n^2 + 20n}{n^2} \\
&= \lim_{n\to\infty} \left(\frac{10n^2}{n^2} + \frac{20n}{n^2} \right) = \lim_{n\to\infty} \left(10 + \frac{20}{n} \right) \\
&= 10 + \lim_{n\to\infty} \left(\frac{20}{n} \right) = 10
\end{aligned}
$$

Hence, $f(n)$ is $\Theta(n^2)$.

Asymptotic Notations

The o-Notation

Definition

Let $f(n)$ and $g(n)$ be two functions from the set of natural numbers to the set of nonnegative real numbers. $f(n)$ is said to be $o(g(n))$ if, for every constant $c > 0$, there exists a positive integer n_0 such that $f(n) < cg(n)\ \forall\ n \geq n_0$.

Example

Let $g(n) = n^3 + 10n^2 + 20n$ and $f(n) = n^2 + 20n$. Then $f(n) = o(g(n))$ since $\forall\ n \geq 1,\ f(n) < g(n)$.

Alternative definition of o-notation

The o-Notation (cont.)

The definition for o-notation is also equivalent to the following:

if $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ implies $f(n) = o(g(n))$.

Example Let $f(n) = 10n^2 + 20n$ and $g(n) = n^3$. Since

$$\lim_{n\to\infty} \frac{10n^2 + 20n}{n^3} = \lim_{n\to\infty}\left(\frac{10}{n} + \frac{20}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{10}{n} + \lim_{n\to\infty} \frac{20}{n^2} = 0,$$

we see that $f(n)$ is $o(g(n))$.

Example Consider the same problem as $f(n) = 10n^2 + 20n$ and $g(n) = n^2$. Since

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{10n^2 + 20n}{n^2} = 10.$$

Hence, $f(n)$ is $\Theta(n^2)$ but not $o(n^2)$.

End of Lecture