Problem Set 1.2

ILA_CH01_02_01

1) $\vec{u} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\vec{\omega} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

↳ $\vec{u} \cdot \vec{v} = (-0.6) \times 4 + (0.8) \times 3 = 0$

$\Rightarrow \vec{u} \perp^r \vec{v}$

↳ $\vec{u} \cdot (\vec{v} + \vec{\omega}) = \begin{bmatrix} -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 1$

↳ $\vec{u} \cdot \vec{\omega} = \begin{bmatrix} -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1.0$

↳ $\vec{\omega} \cdot \vec{v} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 10$

2) $\|\vec{u}\| = \sqrt{(0.6)^2 + (0.8)^2} = 1$

$\|\vec{v}\| = \sqrt{4^2 + 3^2} = 5$

$\|\vec{\omega}\| = \sqrt{1 + 4} = \sqrt{5}$

$|\vec{u} \cdot \vec{v}| = 0 < 1 \times 5 \ (= 5)$

$|\vec{v} \cdot \vec{\omega}| = 10 < 5\sqrt{5} \ (= 11.18)$

3) ↳ unit vector in the direction of $\vec{v}$ $= \vec{v}_u = \dfrac{\vec{v}}{\|\vec{v}\|} = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$

↳ unit vector in the direction of $\vec{\omega}$ $= \vec{\omega}_u = \dfrac{\vec{\omega}}{\|\vec{\omega}\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

↳ $\cos\theta = \vec{v}_u \cdot \vec{w}_u = \frac{2}{\sqrt{5}}$

↳ vector $\vec{a} = m \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, $m > 0$

↳ vector $\vec{b} = k \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $k \in \mathbb{R}$

↳ vector $\vec{c} = m \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, $m < 0$.

4) a) $\vec{v} \cdot (-\vec{v}) = -\|\vec{v}\|^2 = -1$.

b) $(\vec{v}+\vec{w}) \cdot (\vec{v}-\vec{w}) = (\vec{v}\cdot\vec{v}) - (\vec{v}\cdot\vec{w}) + (\vec{w}\cdot\vec{v}) - (\vec{w}\cdot\vec{w})$

$= 0$

c) $(\vec{v}-2\vec{w}) \cdot (\vec{v}+2\vec{w}) = (\vec{v}\cdot\vec{v}) + 2(\vec{v}\cdot\vec{w}) - 2(\vec{w}\cdot\vec{v}) - 4(\vec{w}\cdot\vec{w})$

$= -3$.

5) ↳ $\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$; $\vec{u}_2 = \frac{\vec{w}}{\|\vec{w}\|} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

↳ $\vec{U}_1 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$ or $\begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$

↳ $\vec{U}_2 \Longrightarrow$ The plane defined by $2\vec{x} + \vec{y} + 2\vec{z} = 0$ is perpendicular to $\vec{u}_2$. The set of unit vectors in this plane define a circle. Two unit vectors $\perp^r$ to $\vec{u}_2$ are $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

6 (a) $\vec{\omega} \perp^r \vec{v}$

$\Rightarrow 2\omega_1 - \omega_2 = 0 \quad \Rightarrow \omega_2 = 2\omega_1$.

$\Rightarrow$ All vectors of the form $\begin{pmatrix} c \\ 2c \end{pmatrix}$ are $\perp^r$ to $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(b) All vectors $\perp^r$ to $\vec{V} = (1, 1, 1)$ lie on a plane in 3 dimensions. The plane is defined by $\vec{\omega_1} + \vec{\omega_2} + \vec{\omega_3} = 0$.

(c) The vectors $\perp^r$ to both $(1, 1, 1)$ and $(1, 2, 3)$ lie on a line defined by $\begin{bmatrix} c \\ -2c \\ c \end{bmatrix}$

7 (a) $\vec{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$; $\vec{\omega} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \cos\theta = \dfrac{1}{2 \times 1} = \dfrac{1}{2}$

$\Rightarrow \theta = 60°$

(b) $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$; $\vec{\omega} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90°$

(c) $\vec{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$; $\vec{\omega} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \Rightarrow \cos\theta = \dfrac{2}{2 \times 2} = \dfrac{1}{2}$

$\Rightarrow \theta = 60°$.

(d) $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$; $\vec{\omega} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \Rightarrow \cos\theta = \dfrac{-5}{\sqrt{10} \times \sqrt{5}} = \dfrac{-1}{\sqrt{2}}$

$\Rightarrow \theta = 135°$

8 a) All vectors $\perp^r$ to $\vec{u} = (1, 1, 1)$ lie in a plane defined by $x + y + z = 0 \Rightarrow$ two vectors $\vec{v}$ & $\vec{\omega}$ on this plane need not be parallel to each other.

b) ↳ $\vec{u}$ is $\perp^r$ to $\vec{v}$ & $\vec{\omega} \Rightarrow \vec{u} \cdot \vec{v} = 0$ &
$\vec{u} \cdot \vec{\omega} = 0$.

↳ $\vec{u} \cdot (\vec{v} + 2\vec{\omega}) = \vec{u} \cdot \vec{v} + 2\vec{u} \cdot \vec{\omega} = 0$

$\Rightarrow \vec{u}$ is $\perp^r$ to $(\vec{v} + 2\vec{\omega})$.

c) $\vec{u}$ is $\perp^r$ to $\vec{v} \Rightarrow \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = 0$

$\|u - v\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2\vec{u} \cdot \vec{v}$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2 = 2$.

$\Rightarrow \|\vec{u} - \vec{v}\| = \sqrt{2}$

9) ↳ $\vec{v} \cdot \vec{\omega} = v_1 \omega_1 + v_2 \omega_2 = 0$

$\Rightarrow \vec{v}$ is $\perp^r$ to $\vec{\omega}$

↳ Consider $\vec{v} = (1, 4)$. This lies on the line $y = 4x$ w/ slope $= 4$ (S_1)

↳ Consider $\vec{\omega} = (1, -\frac{1}{4})$. This lies on the line $y = -\frac{1}{4}x$ w/ slope $= -\frac{1}{4}$ (S_2)

$S_1 \times S_2 = -1 \Rightarrow \vec{v} \perp^r \vec{w}$

$\Rightarrow (1, 4)$ is $\perp^r$ to $(1, -\frac{1}{4})$

In general $\vec{v} = (x_1, x_2)$ is $\perp^r$ to $\vec{w} = (x_1, -\frac{x_1^2}{x_2})$

10) $\vec{v} \cdot \vec{w} = -2+2=0 \Rightarrow \vec{v}$ & $\vec{w}$ are $\perp^r$.

11) ↳ $\vec{v} \cdot \vec{w}$ is $-Ve \Rightarrow$ angle between $\vec{v}$ & $\vec{w}$ lies between $90°$ & $270°$.

↳ Consider a 3-D vector $\vec{v} = (1, 0, 0)$ which is a unit vector along x-axis. All vectors $\vec{w}$ in the 3D plane for which x is $-Ve$ will have $\vec{v} \cdot \vec{w} < 0$

12) ↳ $\vec{w} - c\vec{v} = (1-c, 5-c)^T$

$(\vec{w} - c\vec{v}) \perp^r \vec{v} \Rightarrow (1-c) + (5-c) = 0$

$\Rightarrow c = 3$.

↳ For general $\vec{v}, \vec{w}$,

$(\vec{w} - c\vec{v}) \cdot \vec{v} = 0$

$\Rightarrow c = \dfrac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}$

13) For vectors $\vec{v}$ & $\vec{w}$ to be perpendicular to $(1, 0, 1)$, we need | ILA_CH01_02_06

$$\left.\begin{aligned} v_1 + v_3 &= 0 \\ w_1 + w_3 &= 0 \end{aligned}\right\} \text{ where } \vec{v} = (v_1, v_2, v_3)^T \text{ \& } \vec{w} = (w_1, w_2, w_3)^T$$

↳ choose $\vec{v} = (-1, 0, 1)$

↳ Since both $(1, 0, 1)$ & $\vec{v} = (-1, 0, 1)$ lie in the x-z plane, the entire y-axis is orthogonal to both these vectors.

So, choose $\vec{w} = (0, 1, 0)$.

14) The vectors $\vec{u}, \vec{v}, \vec{w}$ $\perp^r$ to $(1, 1, 1, 1)$ will lie in the 3D-hyperplane defined by

$$v_1 + v_2 + v_3 + v_4 = 0$$

One Possible option for $\vec{u}, \vec{v}, \vec{w}$ is

$$\vec{u} = (1, -1, -1, 1)$$
$$\vec{v} = (-1, 0, 0, 1)$$
$$\vec{w} = (0, 1, -1, 0)$$

$\vec{u}, \vec{v}, \vec{w}$ can be rotated in their 3D-hyperplane and they will stay $\perp^r$.

15) Arithmetic mean $= \frac{1}{2}(x+y) = 5$

& $\sqrt{xy} < \frac{1}{2}(x+y)$.

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{8}{10}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{5}\right).$$

16) ↳ $\|\vec{v}\| = \underset{(9 \text{ times})}{\sqrt{1^2+1^2+\cdots+1^2}} = 3$

unit vector $\vec{u}$ in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{1}{3}, \frac{1}{3}, \cdots, \frac{1}{3}\right).$$

↳ Any vector $\vec{w}$ $\perp^r$ to $\vec{u}$ satisfies

$$w_1 + w_2 + \cdots + w_9 = 0$$

choose $\vec{w} = (1, -1, 0, 0, \cdots 0)$

unit vector $\vec{x}$ in the direction of $\vec{w}$ is

$$\vec{x} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0, 0, \cdots 0\right).$$

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17) $\cos\alpha = \dfrac{(1,0,-1)\cdot(1,0,0)}{1\times\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \alpha = 45^\circ$

$\cos\beta = \dfrac{(1,0,-1)\cdot(0,1,0)}{1\times\sqrt{2}} = 0 \Rightarrow \beta = 90^\circ$

$\cos\theta = \dfrac{(1,0,-1)\cdot(0,0,1)}{1\times\sqrt{2}} = \dfrac{-1}{\sqrt{2}} \Rightarrow \beta = 135^\circ$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\theta = \frac{1}{2} + \frac{1}{2} = 1.$

For any vector $\vec{v} = (v_1, v_2, v_3)$

$\cos\alpha = \dfrac{v_1}{\|\vec{v}\|}$; $\cos\beta = \dfrac{v_2}{\|\vec{v}\|}$; $\cos\theta = \dfrac{v_3}{\|\vec{v}\|}$

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\theta$

$= \dfrac{v_1^2 + v_2^2 + v_3^2}{\|\vec{v}\|^2} = 1.$