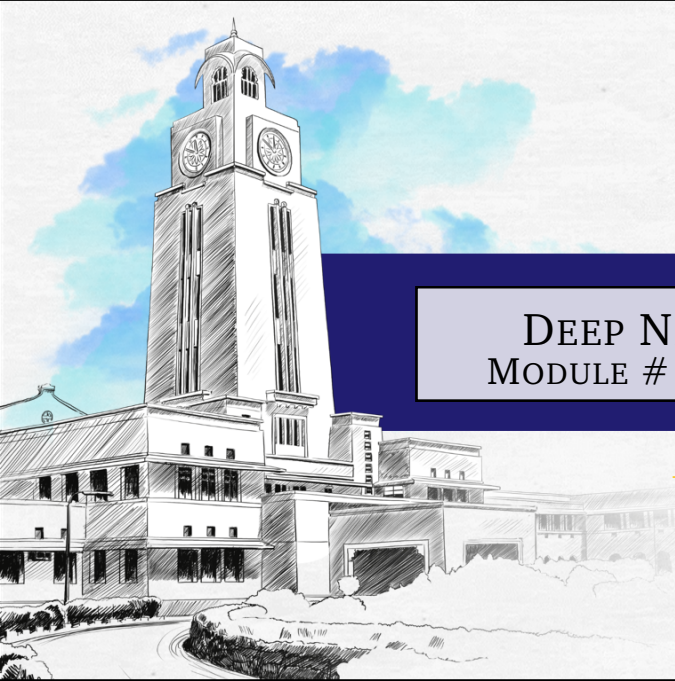

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Deep Neural Network
Module # 5: DFNN Numerical

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The instructor is gratefully acknowledging
the authors who made their course
materials freely available online.
This deck is prepared by Seetha Parameswaran.

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Numerical Problem : Forward Propagation

Problem: Consider a simple 2-layer neural network with the following parameters. Calculate the forward propagation step by step.

Network Architecture:

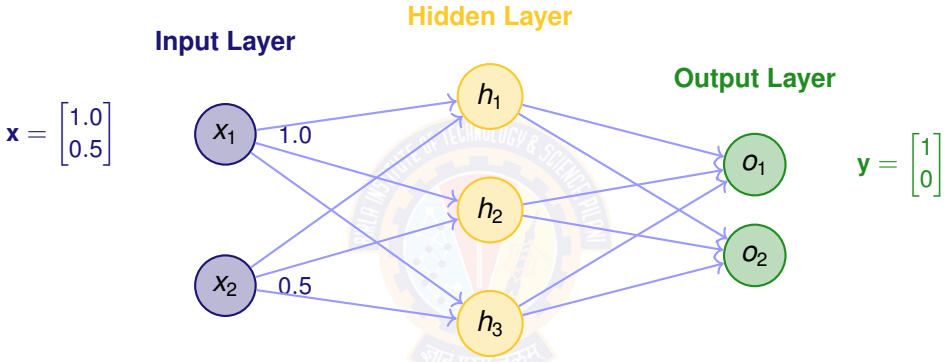
- Input layer: 2 neurons
- Hidden layer: 3 neurons (ReLU activation)
- Output layer: 2 neurons (Sigmoid activation)

Given Parameters:

$$x = \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix} \quad W^{[1]} = \begin{bmatrix} 0.5 & 0.2 \\ -0.3 & 0.8 \\ 0.1 & -0.4 \end{bmatrix}, \quad b^{[1]} = \begin{bmatrix} 0.1 \\ -0.2 \\ 0.3 \end{bmatrix}$$

$$W^{[2]} = \begin{bmatrix} 0.4 & -0.1 & 0.6 \\ 0.2 & 0.7 & -0.3 \end{bmatrix}, \quad b^{[2]} = \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix}$$

Forward Prop: Step 0 - Initial Network



Ready to compute forward propagation!

Solution: Forward Prop - Layer 1

Step 1: Calculate $z^{[1]}$

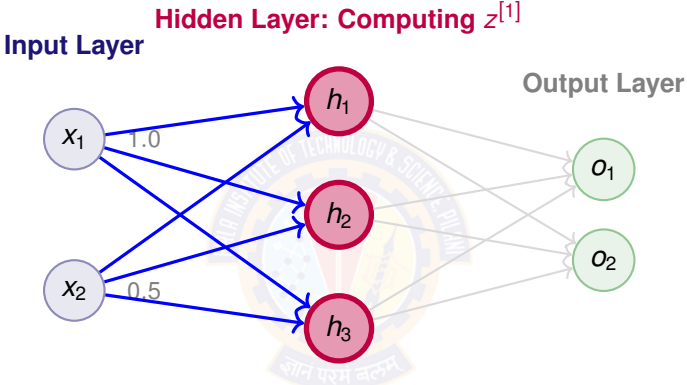
$$z^{[1]} = W^{[1]}x + b^{[1]}$$

$$z^{[1]} = \begin{bmatrix} 0.5 & 0.2 \\ -0.3 & 0.8 \\ 0.1 & -0.4 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.2 \\ 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5(1.0) + 0.2(0.5) \\ -0.3(1.0) + 0.8(0.5) \\ 0.1(1.0) + (-0.4)(0.5) \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.2 \\ 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 \\ 0.1 \\ -0.1 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.2 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ -0.1 \\ 0.2 \end{bmatrix}$$

Forward Prop: Step 1 - Computing $z^{[1]}$



$$z^{[1]} = W^{[1]}x + b^{[1]} = \begin{bmatrix} 0.5 & 0.2 \\ -0.3 & 0.8 \\ 0.1 & -0.4 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.2 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ -0.1 \\ 0.2 \end{bmatrix}$$

Solution: Forward Prop - Layer 1 Activati

Step 2: Apply ReLU activation $a^{[1]} = \text{ReLU}(z^{[1]})$

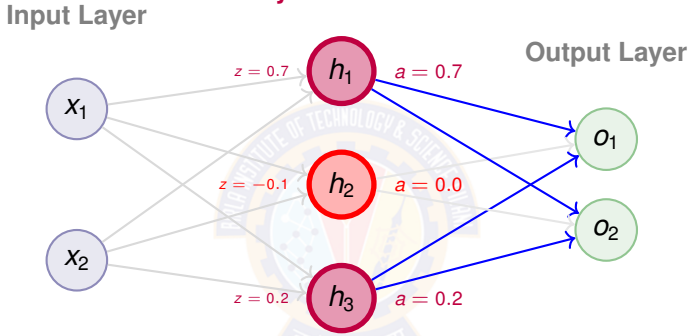
ReLU function: $\text{ReLU}(x) = \max(0, x)$

$$a^{[1]} = \text{ReLU}\begin{bmatrix} 0.7 \\ -0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} \max(0, 0.7) \\ \max(0, -0.1) \\ \max(0, 0.2) \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.0 \\ 0.2 \end{bmatrix}$$

Hidden layer activation: $a^{[1]} = \begin{bmatrix} 0.7 \\ 0.0 \\ 0.2 \end{bmatrix}$

Forward Prop: Step 2 - Applying ReLU

Hidden Layer: ReLU Activation



$$a^{[1]} = \text{ReLU}\begin{bmatrix} 0.7 \\ -0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.0 \\ 0.2 \end{bmatrix}$$

Neuron 2 is **dead** ($z < 0$)

Solution: Forward Prop - Layer 2

Step 3: Calculate $z^{[2]}$

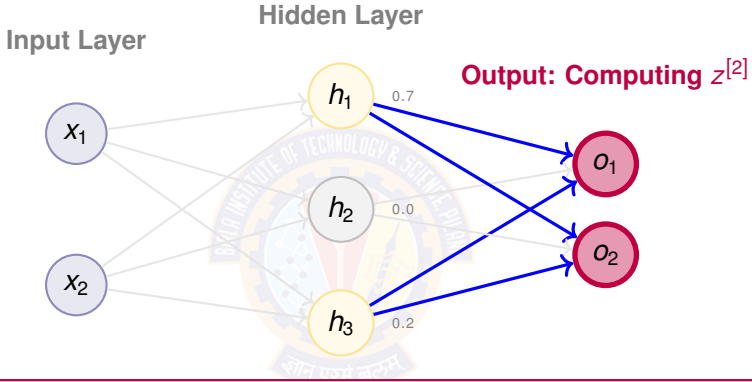
$$z^{[2]} = W^{[2]} a^{[1]} + b^{[2]}$$

$$z^{[2]} = \begin{bmatrix} 0.4 & -0.1 & 0.6 \\ 0.2 & 0.7 & -0.3 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.0 \\ 0.2 \end{bmatrix} + \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4(0.7) + (-0.1)(0.0) + 0.6(0.2) \\ 0.2(0.7) + 0.7(0.0) + (-0.3)(0.2) \end{bmatrix} + \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.28 + 0 + 0.12 \\ 0.14 + 0 - 0.06 \end{bmatrix} + \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.4 + 0.2 \\ 0.08 - 0.1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ -0.02 \end{bmatrix}$$

Forward Prop: Step 3 - Computing $z^{[2]}$



$$z^{[2]} = W^{[2]} a^{[1]} + b^{[2]} = \begin{bmatrix} 0.4 & -0.1 & 0.6 \\ 0.2 & 0.7 & -0.3 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.0 \\ 0.2 \end{bmatrix} + \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ -0.02 \end{bmatrix}$$

Solution: Forward Prop - Output Layer

Step 4: Apply Sigmoid activation $a^{[2]} = \sigma(z^{[2]})$

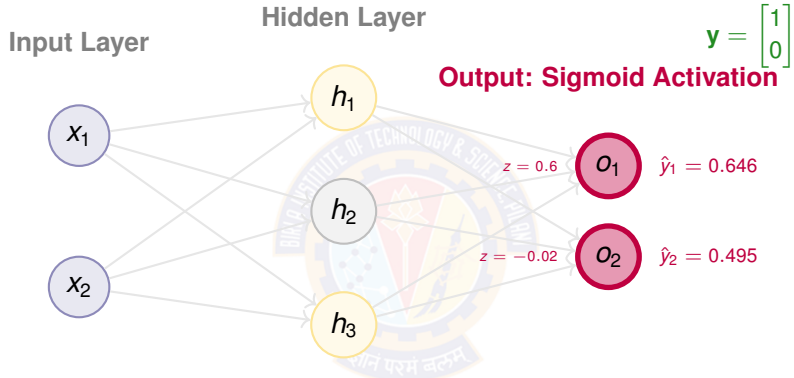
Sigmoid function: $\sigma(x) = \frac{1}{1 + e^{-x}}$

$$a^{[2]} = \sigma \begin{bmatrix} 0.6 \\ -0.02 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1+e^{-0.6}} \\ \frac{1}{1+e^{0.02}} \end{bmatrix}$$

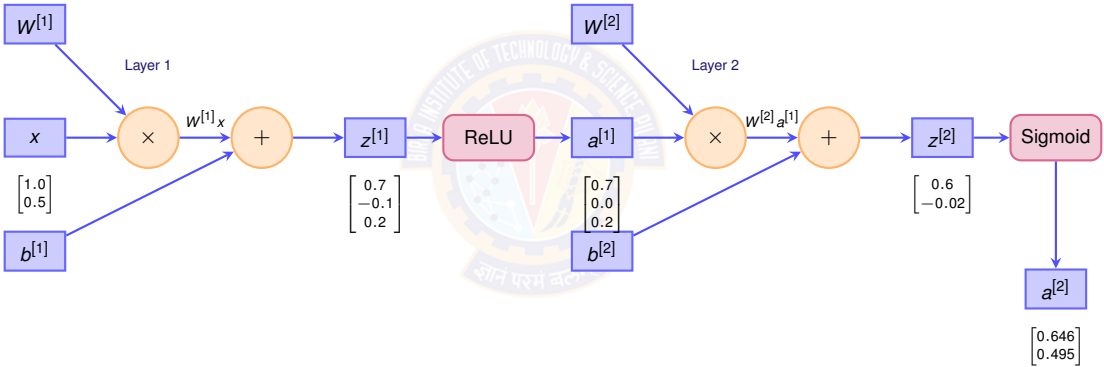
Output activation: $a^{[2]} = \begin{bmatrix} 0.646 \\ 0.495 \end{bmatrix}$

Forward Prop: Step 4 - Applying Sigmoid



Forward propagation complete! Prediction: $\hat{\mathbf{y}} = [0.646, 0.495]^T$

Computational Graph: Forward Propagatio



Numerical Problem: Backward Propagation

Problem: Using the previous network, calculate the backward propagation.

Given:

- Forward propagation results from previous problem
- Target output: $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Loss function: Binary Cross-Entropy (BCE)

Calculate the gradients $\frac{\partial L}{\partial W^{[2]}}$, $\frac{\partial L}{\partial b^{[2]}}$, $\frac{\partial L}{\partial W^{[1]}}$, and $\frac{\partial L}{\partial b^{[1]}}$.

Solution: Backward Propagation - Loss

Step 1: Calculate the Binary Cross-Entropy loss

For our outputs: $a^{[2]} = \begin{bmatrix} 0.646 \\ 0.495 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

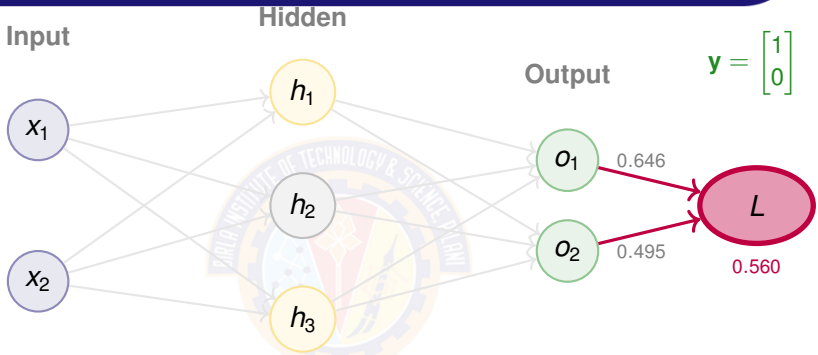
BCE Loss Formula: $L = \frac{-1}{n} \sum_{i=1}^{n} [y_i \log(a_i^{[2]}) + (1 - y_i) \log(1 - a_i^{[2]})]$

$$L = \frac{-1}{2} [1 \cdot \log(0.646) + (1 - 1) \cdot \log(1 - 0.646) + 0 \cdot \log(0.495) + (1 - 0) \cdot \log(1 - 0.495)]$$

$$= \frac{-1}{2} [\log(0.646) + \log(0.505)]$$

$$= \frac{-1}{2} [-0.437 + (-0.683)] = 0.560$$

Computing Loss



$$L = -\frac{1}{2}[y_1 \log(\hat{y}_1) + (1 - y_1) \log(1 - \hat{y}_1) + y_2 \log(\hat{y}_2) + (1 - y_2) \log(1 - \hat{y}_2)]$$
$$L = -\frac{1}{2}[\log(0.646) + \log(0.505)] = -\frac{1}{2}[-0.437 - 0.683] = 0.560$$

Now we backpropagate to update weights!

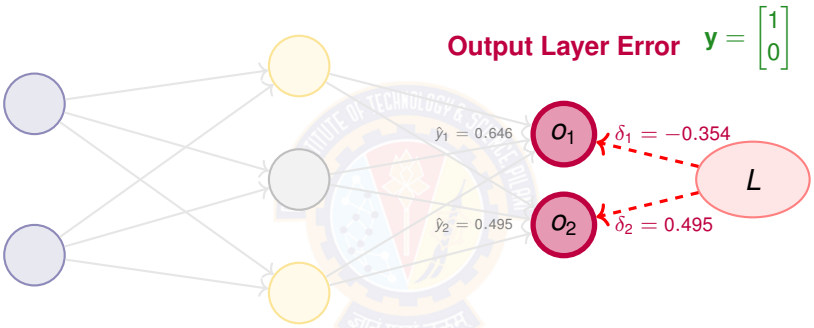
Solution: BackProp - Output Layer Error

Step 2: Calculate output layer error $\delta^{[2]}$

$$\frac{\partial L}{\partial z^{[2]}} = \delta^{[2]} = a^{[2]} - y$$

$$\delta^{[2]} = \begin{bmatrix} 0.646 \\ 0.495 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix}$$

Back-Prop Step 1 - Output Layer Error



$$\delta^{[2]} = \begin{bmatrix} 0.646 \\ 0.495 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix}$$

Solution: Backward Prop - Output Layer

Step 3: Calculate gradients for output layer

Recall: $a^{[1]} = \begin{bmatrix} 0.7 & 0.0 & 0.2 \end{bmatrix}^\top$

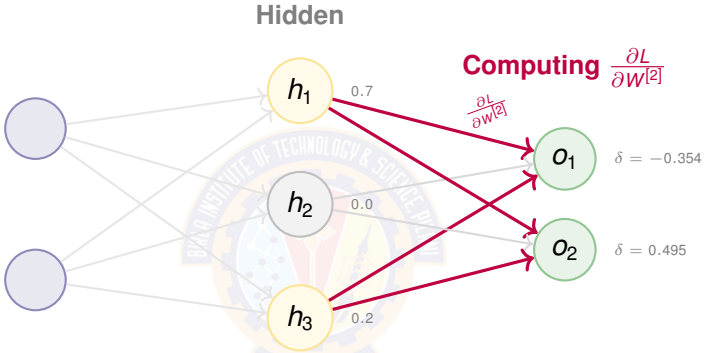
$$\frac{\partial L}{\partial W^{[2]}} = \delta^{[2]} (a^{[1]})^T = \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix} \begin{bmatrix} 0.7 & 0.0 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.354 \times 0.7 & -0.354 \times 0.0 & -0.354 \times 0.2 \\ 0.495 \times 0.7 & 0.495 \times 0.0 & 0.495 \times 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.248 & 0.000 & -0.071 \\ 0.347 & 0.000 & 0.099 \end{bmatrix}$$

$$\frac{\partial L}{\partial b^{[2]}} = \delta^{[2]} = \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix}$$

Back-Prop: Step 2 - Output Layer Gradient



$$\frac{\partial L}{\partial W^{[2]}} = \delta^{[2]}(a^{[1]})^T = \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix} \begin{bmatrix} 0.7 & 0.0 & 0.2 \end{bmatrix} = \begin{bmatrix} -0.248 & 0.000 & -0.071 \\ 0.347 & 0.000 & 0.099 \end{bmatrix}$$

$$\frac{\partial L}{\partial b^{[2]}} = \delta^{[2]} = \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix}$$

Solution: BackProp - Hidden Layer Error

Step 4: Calculate hidden layer error $\delta^{[1]}$

Backpropagate error to hidden layer: $\delta^{[1]} = (W^{[2]})^T \delta^{[2]} \odot g'^{[1]}(z^{[1]})$

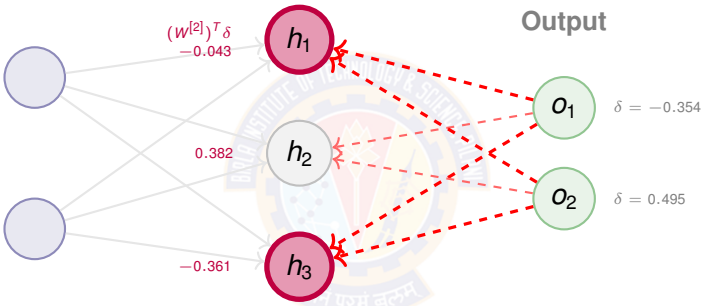
$$(W^{[2]})^T \delta^{[2]} = \begin{bmatrix} 0.4 & 0.2 \\ -0.1 & 0.7 \\ 0.6 & -0.3 \end{bmatrix} \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4(-0.354) + 0.2(0.495) \\ -0.1(-0.354) + 0.7(0.495) \\ 0.6(-0.354) + (-0.3)(0.495) \end{bmatrix}$$

$$= \begin{bmatrix} -0.142 + 0.099 \\ 0.035 + 0.347 \\ -0.212 - 0.149 \end{bmatrix} = \begin{bmatrix} -0.043 \\ 0.382 \\ -0.361 \end{bmatrix}$$

Back-Prop: Step 3 - Backprop to Hidden La

Backprop: $(W^{[2]})^T \delta^{[2]}$



$$(W^{[2]})^T \delta^{[2]} = \begin{bmatrix} 0.4 & 0.2 \\ -0.1 & 0.7 \\ 0.6 & -0.3 \end{bmatrix} \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix} = \begin{bmatrix} -0.043 \\ 0.382 \\ -0.361 \end{bmatrix}$$

Solution: BackProp - Hidden Layer Error

Step 4 (continued):

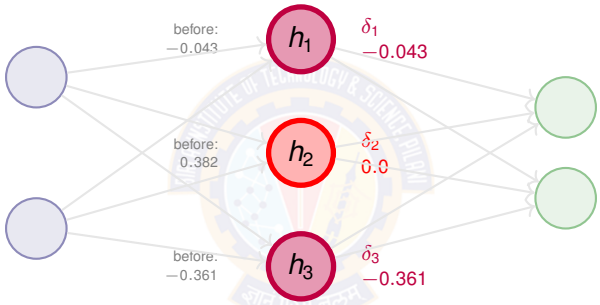
$$g'^{[1]}(z^{[1]}) = \begin{bmatrix} 1 & (\text{since } z_1^{[1]} = 0.7 > 0) \\ 0 & (\text{since } z_2^{[1]} = -0.1 \leq 0) \\ 1 & (\text{since } z_3^{[1]} = 0.2 > 0) \end{bmatrix}$$

Error to Layer 1: $\delta^{[1]} = (W^{[2]})^T \delta^{[2]} \odot g'^{[1]}(z^{[1]})$

$$\delta^{[1]} = \begin{bmatrix} -0.043 \\ 0.382 \\ -0.361 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.043 \\ 0.000 \\ -0.361 \end{bmatrix}$$

Back-Prop: Step 4 - Apply ReLU Derivative

Apply ReLU': $\delta^{[1]} = (W^{[2]})^T \delta^{[2]} \odot g'(z^{[1]})$



$$\delta^{[1]} = \begin{bmatrix} -0.043 \\ 0.382 \\ -0.361 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.043 \\ 0.0 \\ -0.361 \end{bmatrix}$$

Dead neuron gradient = 0 (no update!)

Step 5: Calculate gradients for hidden layer

$$\frac{\partial L}{\partial W^{[1]}} = \delta^{[1]} x^T = \begin{bmatrix} -0.043 \\ 0.000 \\ -0.361 \end{bmatrix} \begin{bmatrix} 1.0 & 0.5 \end{bmatrix}$$

$$\frac{\partial L}{\partial W^{[1]}} = \begin{bmatrix} -0.043 \times 1.0 & -0.043 \times 0.5 \\ 0.000 \times 1.0 & 0.000 \times 0.5 \\ -0.361 \times 1.0 & -0.361 \times 0.5 \end{bmatrix}$$

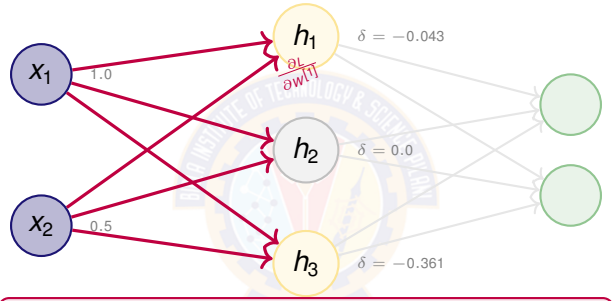
$$= \begin{bmatrix} -0.043 & -0.022 \\ 0.000 & 0.000 \\ -0.361 & -0.181 \end{bmatrix}$$

$$\frac{\partial L}{\partial b^{[1]}} = \delta^{[1]} = \begin{bmatrix} -0.043 \\ 0.000 \\ -0.361 \end{bmatrix}$$

Back-Prop: Step 5 - Hidden Layer Gradient

Computing $\frac{\partial L}{\partial W^{[1]}}$

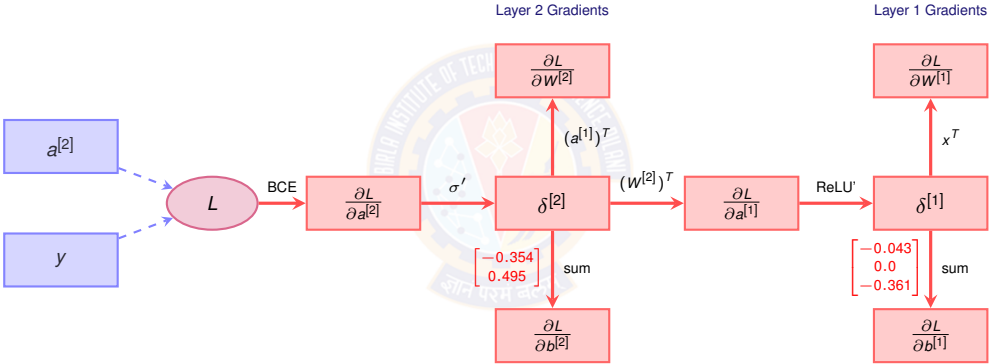
Input



$$\frac{\partial L}{\partial W^{[1]}} = \delta^{[1]}\mathbf{x}^T = \begin{bmatrix} -0.043 \\ 0.0 \\ -0.361 \end{bmatrix} \begin{bmatrix} 1.0 & 0.5 \end{bmatrix} = \begin{bmatrix} -0.043 & -0.022 \\ 0.0 & 0.0 \\ -0.361 & -0.181 \end{bmatrix}$$

$$\frac{\partial L}{\partial b^{[1]}} = \delta^{[1]} = \begin{bmatrix} -0.043 \\ 0.0 \\ -0.361 \end{bmatrix}$$

Computational Graph: Backward Propagat



Numerical Problem: Weight Updation

Problem: Using the gradients computed in previous problem, update the network parameters with learning rate $\alpha = 0.1$.

Given gradients:

$$\frac{\partial L}{\partial W^{[2]}} = \begin{bmatrix} -0.248 & 0.000 & -0.071 \\ 0.347 & 0.000 & 0.099 \end{bmatrix} \qquad \frac{\partial L}{\partial b^{[2]}} = \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix}$$

$$\frac{\partial L}{\partial W^{[1]}} = \begin{bmatrix} -0.043 & -0.022 \\ 0.000 & 0.000 \\ -0.361 & -0.181 \end{bmatrix} \qquad \frac{\partial L}{\partial b^{[1]}} = \begin{bmatrix} -0.043 \\ 0.000 \\ -0.361 \end{bmatrix}$$

Calculate the updated parameters.

Solution: Parameter Updates

Update rule: $\theta_{\text{new}} = \theta_{\text{old}} - \alpha \frac{\partial L}{\partial \theta}$

$$W^{[2]}_{\text{new}} = W^{[2]} - 0.1 \times \frac{\partial L}{\partial W^{[2]}}$$

$$W^{[2]}_{\text{new}} = \begin{bmatrix} 0.4 & -0.1 & 0.6 \\ 0.2 & 0.7 & -0.3 \end{bmatrix} - 0.1 \times \begin{bmatrix} -0.248 & 0.000 & -0.071 \\ 0.347 & 0.000 & 0.099 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 + 0.025 & -0.1 + 0.000 & 0.6 + 0.007 \\ 0.2 - 0.035 & 0.7 + 0.000 & -0.3 - 0.010 \end{bmatrix}$$

$$= \begin{bmatrix} 0.425 & -0.100 & 0.607 \\ 0.165 & 0.700 & -0.310 \end{bmatrix}$$

$$b^{[2]}_{\text{new}} = \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix} - 0.1 \times \begin{bmatrix} -0.354 \\ 0.495 \end{bmatrix} = \begin{bmatrix} 0.235 \\ -0.150 \end{bmatrix}$$

Solution: Parameter Updates (Continued)

$$W_{\text{new}}^{[1]} = W^{[1]} - 0.1 \times \frac{\partial L}{\partial W^{[1]}}$$

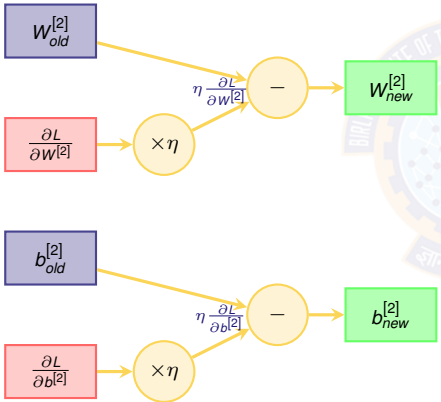
$$W_{\text{new}}^{[1]} = \begin{bmatrix} 0.5 & 0.2 \\ -0.3 & 0.8 \\ 0.1 & -0.4 \end{bmatrix} - 0.1 \times \begin{bmatrix} -0.043 & -0.022 \\ 0.000 & 0.000 \\ -0.361 & -0.181 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 + 0.004 & 0.2 + 0.002 \\ -0.3 + 0.000 & 0.8 + 0.000 \\ 0.1 + 0.036 & -0.4 + 0.018 \end{bmatrix} = \begin{bmatrix} 0.504 & 0.202 \\ -0.300 & 0.800 \\ 0.136 & -0.382 \end{bmatrix}$$

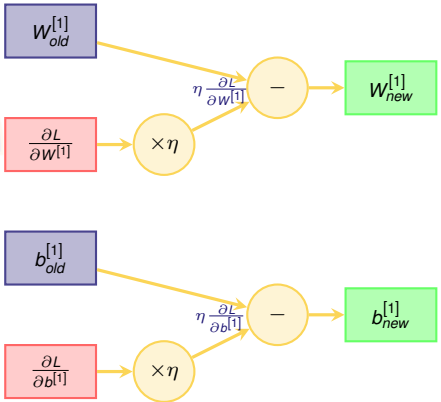
$$b_{\text{new}}^{[1]} = \begin{bmatrix} 0.1 \\ -0.2 \\ 0.3 \end{bmatrix} - 0.1 \times \begin{bmatrix} -0.043 \\ 0.000 \\ -0.361 \end{bmatrix} = \begin{bmatrix} 0.104 \\ -0.200 \\ 0.336 \end{bmatrix}$$

Computational Graph: Weight Update Step

Layer 2 Updates



Layer 1 Updates



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Sigmoid Function: Definition and Properties

The sigmoid function maps any real number to the interval $(0, 1)$.

Definition:

$$\sigma(z) = \frac{1}{1 + e^{-z}} \tag{1}$$

Properties:

- Output range: $(0, 1)$
- Smooth and differentiable everywhere
- $\sigma(0) = 0.5$
- $\sigma(-z) = 1 - \sigma(z)$ (symmetry)
- As $z \to \infty$: $\sigma(z) \to 1$
- As $z \to -\infty$: $\sigma(z) \to 0$

Use cases:

- Binary classification (output layer)
- Gate mechanisms in LSTMs

Key values:

- $\sigma(-2) \approx 0.12$
- $\sigma(0) = 0.5$
- $\sigma(2) \approx 0.88$

Sigmoid Function: Derivative Derivation

Goal: Find $\frac{d\sigma}{dz}$ where $\sigma(z) = \frac{1}{1+e^{-z}}$

Step 1: Rewrite and apply chain rule

$$\sigma(z) = (1 + e^{-z})^{-1} \quad \Rightarrow \quad \frac{d\sigma}{dz} = -(1 + e^{-z})^{-2} \cdot (-e^{-z}) = \frac{e^{-z}}{(1 + e^{-z})^2} \tag{2}$$

Step 2: Factor to express in terms of $\sigma(z)$

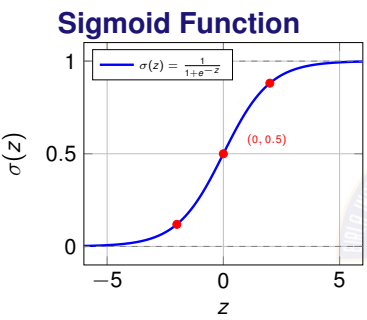
$$\frac{d\sigma}{dz} = \frac{e^{-z}}{(1 + e^{-z})^2} = \frac{1}{1 + e^{-z}} \cdot \frac{e^{-z}}{1 + e^{-z}} \tag{3}$$

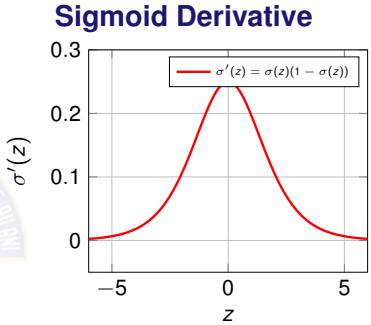
Step 3: Recognize that $\sigma(z) = \frac{1}{1+e^{-z}}$ and $1 - \sigma(z) = \frac{e^{-z}}{1+e^{-z}}$

$$\frac{d\sigma}{dz} = \sigma(z) \cdot (1 - \sigma(z)) \tag{4}$$

Beautiful property: Derivative computed using only the function value itself!

Sigmoid Function: Visualization





Key observations:

- Sigmoid squashes inputs to $(0, 1)$ - ideal for probabilities
- Derivative peaks at $z = 0$ with value 0.25
- Derivative vanishes for large $|z|$ (vanishing gradient problem!)

Softmax Function: Definition

The softmax function converts a vector of real numbers into a probability distribution.

For input vector $\mathbf{z} = [z_1, z_2, \ldots, z_C]^T$:

$$\text{softmax}(\mathbf{z})_j = \frac{e^{z_j}}{\sum_{k=1}^{C} e^{z_k}} \quad \text{for } j = 1, 2, \ldots, C \tag{5}$$

Properties:

- All outputs in $(0, 1)$: $0 < \text{softmax}(\mathbf{z})_j < 1$
- Outputs sum to 1: $\sum_{j=1}^{C} \text{softmax}(\mathbf{z})_j = 1$
- Monotonic: larger $z_j \rightarrow$ larger output
- Translation invariant: $\text{softmax}(\mathbf{z} + c) = \text{softmax}(\mathbf{z})$

Use case: Multi-class classification output layer

Softmax Function: Notation

Let's denote the softmax outputs as:

$$\hat{y}_j = \frac{e^{z_j}}{\sum_{k=1}^{C} e^{z_k}} \quad \text{where} \quad \hat{\mathbf{y}} = [\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_C]^T \tag{6}$$

Example: $\mathbf{z} = [1.0, 2.0, 3.0]^T$

$$\hat{y}_1 = \frac{e^{1.0}}{e^{1.0} + e^{2.0} + e^{3.0}} = \frac{2.718}{2.718 + 7.389 + 20.086} \approx 0.090$$

$$\hat{y}_2 = \frac{e^{2.0}}{e^{1.0} + e^{2.0} + e^{3.0}} = \frac{7.389}{30.193} \approx 0.245$$

$$\hat{y}_3 = \frac{e^{3.0}}{e^{1.0} + e^{2.0} + e^{3.0}} = \frac{20.086}{30.193} \approx 0.665$$

Verify: $0.090 + 0.245 + 0.665 = 1.0$ ✓

Softmax + Cross-Entropy: Beautiful

Cross-entropy Loss: $\ell = -\sum_c y_c \log(\hat{y}_c)$ where true class is k (7)

Chain rule:
$$\frac{\partial \ell}{\partial z_j} = \sum_{i=1}^{C} \frac{\partial \ell}{\partial \hat{y}_i} \cdot \frac{\partial \hat{y}_i}{\partial z_j} \tag{8}$$

$$\frac{\partial \ell}{\partial z_j} = -\frac{1}{\hat{y}_k} \cdot \frac{\partial \hat{y}_k}{\partial z_j} \quad \text{where} \quad \frac{\partial \ell}{\partial \hat{y}_i} = -\frac{y_i}{\hat{y}_i} \quad y_k = 1 \tag{9}$$

$$= -\frac{1}{\hat{y}_k}[\hat{y}_k(\delta_{kj} - \hat{y}_j)] \tag{10}$$

$$= -(\delta_{kj} - \hat{y}_j) \tag{11}$$

$$= \hat{y}_j - \delta_{kj} = \hat{y}_j - y_j \tag{12}$$

Tanh Function: Definition and Properties

The hyperbolic tangent (tanh) maps real numbers to the interval $(-1, 1)$.

Definition:

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}} \tag{13}$$

Properties:

- Output range: $(-1, 1)$
- Zero-centered: $\tanh(0) = 0$
- Odd function: $\tanh(-z) = -\tanh(z)$
- As $z \to \pm\infty$: $\tanh(z) \to \pm 1$

Advantages over sigmoid:

- Zero-centered
- Stronger gradients (max = 1)
- Faster convergence

Key values:

- $\tanh(-1) \approx -0.76$
- $\tanh(0) = 0$
- $\tanh(1) \approx 0.76$

Tanh Function: Derivative Derivation

Quotient Rule Method (where $\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$)

Let $u = e^z - e^{-z}$, $v = e^z + e^{-z}$, so $u' = e^z + e^{-z}$, $v' = e^z - e^{-z}$

$$\frac{d\tanh}{dz} = \frac{u'v - uv'}{v^2} = \frac{(e^z + e^{-z})^2 - (e^z - e^{-z})^2}{(e^z + e^{-z})^2} = \frac{4}{(e^z + e^{-z})^2} \tag{14}$$

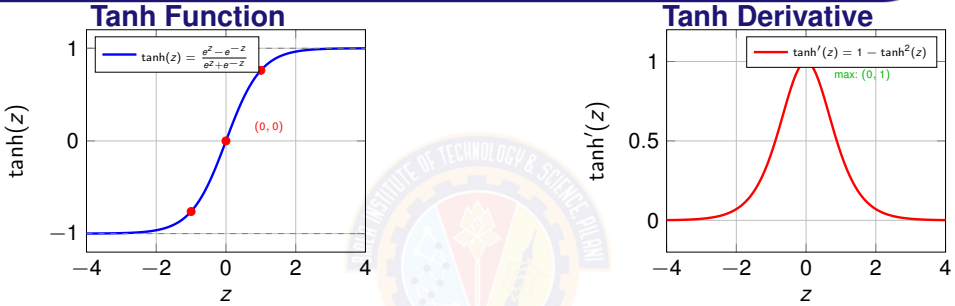
Note: $(e^z + e^{-z})^2 - (e^z - e^{-z})^2 = (e^{2z} + 2 + e^{-2z}) - (e^{2z} - 2 + e^{-2z}) = 4$

Tanh Derivative

$$\boxed{\tanh'(z) = 1 - \tanh^2(z) = \frac{4}{(e^z + e^{-z})^2}} \tag{15}$$

Beautiful property: Derivative computed using only the function value! Maximum at $z = 0$: $\tanh'(0) = 1$

Tanh Function: Visualization



Key observations:

- Tanh is zero-centered (unlike sigmoid) - better for hidden layers
- Derivative peaks at $z = 0$ with value 1 (4× larger than sigmoid!)
- Derivative vanishes for large $|z|$ but slower than sigmoid
- Symmetric derivative (even function): $\tanh'(-z) = \tanh'(z)$

Numerical Problem: Activation Functions

Problem: Given the input vector $z = \begin{bmatrix} 2.0 \\ -1.5 \\ 0.8 \\ -0.3 \end{bmatrix}$, calculate the output for each activation function. Also compute the derivatives for each function at the given input.

1. Sigmoid: $\sigma(z)$
2. Tanh: $\tanh(z)$
3. ReLU: $\text{ReLU}(z)$

Solution: Sigmoid Activation

Sigmoid Function: $\sigma(z) = \dfrac{1}{1 + e^{-z}}$

$$\sigma(z) = \begin{bmatrix} \frac{1}{1+e^{-2.0}} \\ \frac{1}{1+e^{1.5}} \\ \frac{1}{1+e^{-0.8}} \\ \frac{1}{1+e^{0.3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{1+0.135} = 0.881 \\ \frac{1}{1+4.482} = 0.182 \\ \frac{1}{1+0.449} = 0.690 \\ = \frac{1}{1+1.350} = 0.426 \end{bmatrix} = \begin{bmatrix} 0.881 \\ 0.182 \\ 0.690 \\ 0.426 \end{bmatrix}$$

Sigmoid Derivative: $\sigma'(z) = \begin{bmatrix} 0.881(1 - 0.881) \\ 0.182(1 - 0.182) \\ 0.690(1 - 0.690) \\ 0.426(1 - 0.426) \end{bmatrix} = \begin{bmatrix} 0.105 \\ 0.149 \\ 0.214 \\ 0.245 \end{bmatrix}$

Solution: Tanh Activation

Tanh Function: $\tanh(z) = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$

$$\tanh(z) = \begin{bmatrix} \tanh(2.0) \\ \tanh(-1.5) \\ \tanh(0.8) \\ \tanh(-0.3) \end{bmatrix} = \begin{bmatrix} \frac{7.389-0.135}{7.389+0.135} = 0.964 \\ \frac{0.223-4.482}{0.223+4.482} = -0.905 \\ \frac{2.226-0.449}{2.226+0.449} = 0.664 \\ \frac{0.741-1.350}{0.741+1.350} = -0.291 \end{bmatrix} = \begin{bmatrix} 0.964 \\ -0.905 \\ 0.664 \\ -0.291 \end{bmatrix}$$

Tanh Derivative: $\tanh'(z) = \begin{bmatrix} 1 - 0.964^2 \\ 1 - (-0.905)^2 \\ 1 - 0.664^2 \\ 1 - (-0.291)^2 \end{bmatrix} = \begin{bmatrix} 0.070 \\ 0.181 \\ 0.559 \\ 0.915 \end{bmatrix}$

Solution: ReLU

ReLU Function: $\text{ReLU}(z) = \max(0, z)$

$$= \begin{bmatrix} \max(0, 2.0) \\ \max(0, -1.5) \\ \max(0, 0.8) \\ \max(0, -0.3) \end{bmatrix} = \begin{bmatrix} 2.0 \\ 0.0 \\ 0.8 \\ 0.0 \end{bmatrix}$$

ReLU Derivative: $\text{ReLU}'(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases}$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Numerical Problem: Softmax Function

Problem: Given the logit vector z, compute:

1. The softmax output
2. The cross-entropy loss with true label y
3. The gradient of the loss with respect to the logits

$$z = \begin{bmatrix} 2.0 \\ 1.0 \\ 0.1 \\ -1.0 \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Cross-entropy Loss:

$$L = -\sum_{i=1}^{n} y_i \log(\text{softmax}(z_i))$$

Solution: Softmax Computation

Step 1: Compute softmax values First, for numerical stability, subtract the maximum:

$$z_{\text{stable}} = z - \max(z) = \begin{bmatrix} 2.0 \\ 1.0 \\ 0.1 \\ -1.0 \end{bmatrix} - 2.0 = \begin{bmatrix} 0.0 \\ -1.0 \\ -1.9 \\ -3.0 \end{bmatrix}$$

$$e^{z_{\text{stable}}} = \begin{bmatrix} e^{0.0} \\ e^{-1.0} \\ e^{-1.9} \\ e^{-3.0} \end{bmatrix} = \begin{bmatrix} 1.000 \\ 0.368 \\ 0.150 \\ 0.050 \end{bmatrix}$$

$$\sum_{j=1}^{4} e^{z_j - \max(z)} = 1.000 + 0.368 + 0.150 + 0.050 = 1.568$$

Solution: Softmax Computation

$$\text{softmax}(z) = \frac{1}{1.568}\begin{bmatrix}1.000\\0.368\\0.150\\0.050\end{bmatrix} = \begin{bmatrix}0.638\\0.235\\0.096\\0.032\end{bmatrix}$$

$$\text{Verification: } = 0.638 + 0.235 + 0.096 + 0.032 = 1.001 \approx 1$$

Solution: Cross-entropy Loss and Gradient

Step 2: Compute cross-entropy loss

$$L = -\sum_{i=1}^{4} y_i \log(\text{softmax}(z_i))$$

$$y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ only the second term contributes

$$L = -y_2 \log(\text{softmax}(z_2)) = -1 \times \log(0.235) = -\log(0.235) = 1.447$$

Solution: Cross-entropy Loss and Gradient

Step 3: Compute gradient For cross-entropy loss with softmax, the gradient is:

$$\frac{\partial L}{\partial z_i} = \text{softmax}(z_i) - y_i$$

$$\frac{\partial L}{\partial z} = \begin{bmatrix} 0.638 \\ 0.235 \\ 0.096 \\ 0.032 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.638 \\ -0.765 \\ 0.096 \\ 0.032 \end{bmatrix}$$

This result shows that the gradient is simply the difference between predicted and true probabilities!

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Numerical Problem: Gradient Flow Analys

Problem: Consider a 4-layer network with sigmoid activations. Analyze gradient flow through the network.

Given:

- All weight matrices have spectral norm $\|W^{[l]}\| = 0.8$
- Sigmoid derivatives at each layer: $g'^{[l]} = 0.25$ (maximum sigmoid derivative)
- Initial gradient magnitude at output: $|\delta^{[4]}| = 1.0$

Calculate the gradient magnitude at each layer and discuss the implications.

Gradient Propagation Formula:

$$|\delta^{[l-1]}| = |\delta^{[l]}| \cdot \|W^{[l]}\| \cdot |g'^{[l-1]}|$$

Solution: Gradient Flow Analysis

Layer-by-layer gradient magnitude:

$$\textbf{Layer 4 (Output):} |\delta^{[4]}| = 1.0 \quad \text{(given)}$$

$$\textbf{Layer 3:} |\delta^{[3]}| = |\delta^{[4]}| \cdot \|W^{[4]}\| \cdot |g'^{[3]}| = 1.0 \times 0.8 \times 0.25 = 0.2$$

$$\textbf{Layer 2:} |\delta^{[2]}| = |\delta^{[3]}| \cdot \|W^{[3]}\| \cdot |g'^{[2]}| = 0.2 \times 0.8 \times 0.25 = 0.04$$

$$\textbf{Layer 1:} |\delta^{[1]}| = |\delta^{[2]}| \cdot \|W^{[2]}\| \cdot |g'^{[1]}| = 0.04 \times 0.8 \times 0.25 = 0.008$$

Analysis:

- Gradient magnitude decreases by factor of $0.8 \times 0.25 = 0.2$ per layer
- After 3 backward steps: gradient is reduced to 0.8% of original
- First layer receives extremely weak learning signal
- This demonstrates the vanishing gradient problem in deep sigmoid networks

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Example 1: Small Tabular Dataset

Problem Setup: Compute the number of layers and number of neurons in each layer

- $d = 20$ features (customer data)
- $M = 800$ training samples
- $c = 3$ classes (customer segments)

Example 1: Small Tabular Dataset

- Step 1: Apply constraints
  $L \leq \log_2(800/(20 \times 3)) + 1 = \log_2(13.33) + 1 \approx 4.7 \Rightarrow L \leq 4$
- Step 2: Choose starting architecture Since $M < 1000$, start with $L = 2$:
  $[20, 40, 3]$
- Step 3: Parameter count check Parameters
  $= 20 \times 40 + 40 + 40 \times 3 + 3 = 800 + 40 + 120 + 3 = 963$
  Constraint: $963 \leq 800/5 = 160$ ✗ Too many parameters!
- Step 4: Reduce complexity
  - Try $[20, 20, 3]$: Parameters = $20 \times 20 + 20 + 20 \times 3 + 3 = 483$ ✗ Still too many!
  - Try $[20, 10, 3]$: Parameters = $20 \times 10 + 10 + 10 \times 3 + 3 = 243$ ✗
  - Try $[20, 6, 3]$: Parameters = $20 \times 6 + 6 + 6 \times 3 + 3 = 147$ ✔
- Final recommendation: $[20, 6, 3]$ with regularization

Example 2: Medium Image Dataset

Problem Setup: Compute the number of layers and number of neurons in each layer

- $d = 784$ features (28×28 grayscale images)
- $M = 5000$ training samples
- $c = 10$ classes (digit classification)

Example 2: Medium Image Dataset

- **Step 1: Relaxed approach**
  For medium datasets, start with $L = 3$: $[784, 2352, 784, 10]$
- **Step 2: Parameter count check**
  Parameters = $784 \times 2352 + 2352 + 2352 \times 784 + 784 + 784 \times 10 + 10 \approx 3.7M$
  Constraint $5000/5 = 1000$ ✗ Way too many!
- **Step 3: Practical architecture**
  - Try $[784, 128, 64, 10]$:
    Parameters = $784 \times 128 + 128 + 128 \times 64 + 64 + 64 \times 10 + 10 = 109,000$ ✗
  - Try $[784, 64, 32, 10]$:
    Parameters = $784 \times 64 + 64 + 64 \times 32 + 32 + 32 \times 10 + 10 = 52,458$ ✗
  - Try $[784, 32, 10]$: Parameters = $784 \times 32 + 32 + 32 \times 10 + 10 = 25,450$ ✗
  - Try $[784, 16, 10]$: Parameters = $784 \times 16 + 16 + 16 \times 10 + 10 = 12,714$ ✗
- **Reality check:** Need to use regularization techniques (dropout, L2) with larger architectures.

Example 3: Large Text Dataset

Problem Setup: Compute the number of layers and number of neurons in each layer

- $d = 1000$ features (TF-IDF vectors)
- $M = 50000$ training samples
- $c = 5$ classes (sentiment categories)

Example 3: Large Text Dataset

- Step 1: Apply constraints
  $L \leq \log_2(50000/(1000 \times 5)) + 1 = \log_2(10) + 1 \approx 4.3 \Rightarrow L \leq 4$
- Step 2: Choose architecture
  Since $M \geq 10000$, can use $L = 4$: $[1000, 5000, 2000, 1000, 5]$
- Step 3: Parameter count check Parameters =
  $1000 \times 5000 + 5000 + 5000 \times 2000 + 2000 + 2000 \times 1000 + 1000 + 1000 \times 5 + 5 =$
  $5M + 10M + 2M + 5K = 17M$
  Constraint: $17M \leq 50000/5 = 10000$ ✗ Way too many!
- Step 4: Reasonable architecture
  Try $[1000, 512, 256, 128, 5]$:
  Parameters = $1000 \times 512 + 512 + 512 \times 256 + 256 + 256 \times 128 + 128 + 128 \times 5 + 5$
  $= 512K + 131K + 32K + 640 = 675K$ parameters ✔
  This is $675000/50000 = 13.5$ parameters per sample, which is reasonable.
- Final recommendation: $[1000, 512, 256, 128, 5]$ with batch normalization and dropout.

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Overview: Loss Functions

Loss functions measure the discrepancy between predicted and actual values.

We will include:

1. Mean Squared Error (MSE) Loss
2. Mean Absolute Error (MAE) Loss
3. Binary Cross-Entropy (BCE) Loss
4. Cross-Entropy Loss

Key Goal: Understand how to compute gradients for backpropagation

MSE Loss: Definition

Mean Squared Error (MSE) is commonly used for regression tasks.

$$\ell(\hat{y}, y) = \frac{1}{2}(\hat{y} - y)^2 \tag{16}$$

Properties:

- Always non-negative: $\ell \geq 0$
- Convex function (for linear models)
- Differentiable everywhere
- Penalizes large errors quadratically
- Factor of $\frac{1}{2}$ simplifies derivative

MSE Loss: Derivative Derivation

Goal: Find $\frac{\partial \ell}{\partial \hat{y}}$ where $\ell = \frac{1}{2}(\hat{y} - y)^2$

Let $u = \hat{y} - y$, so $\ell = \frac{1}{2}u^2$

$$\frac{\partial \ell}{\partial \hat{y}} = \frac{\partial \ell}{\partial u} \cdot \frac{\partial u}{\partial \hat{y}} \quad \text{(chain rule)} \tag{17}$$

$$= \left(\frac{1}{2} \cdot 2u\right) \cdot (1) \quad \text{(compute derivatives)} \tag{18}$$

$$= u = \hat{y} - y \quad \text{(substitute back)} \tag{19}$$

MSE Loss Derivative:

$$\boxed{\frac{\partial \ell}{\partial \hat{y}} = \hat{y} - y} \tag{20}$$

MSE Loss: Numerical Example

Example 1: $y = 3$, $\hat{y} = 5$

$$\ell = \frac{1}{2}(5-3)^2 = \frac{1}{2}(2)^2 = 2$$

$$\frac{\partial \ell}{\partial \hat{y}} = 5 - 3 = 2$$

Example 2: $y = 8$, $\hat{y} = 5$

$$\ell = \frac{1}{2}(5-8)^2 = \frac{1}{2}(-3)^2 = 4.5$$

$$\frac{\partial \ell}{\partial \hat{y}} = 5 - 8 = -3$$

Note: Gradient is negative, telling us to *increase* $\hat{y}$

MAE Loss: Definition

Mean Absolute Error (MAE) is more robust to outliers than MSE.

$$\ell(\hat{y}, y) = |\hat{y} - y| \tag{21}$$

Properties:

- Always non-negative: $\ell \geq 0$
- Less sensitive to outliers than MSE
- Not differentiable at $\hat{y} = y$
- Linear penalty for errors

MAE Loss: Derivative Derivation

Goal: Find $\frac{\partial \ell}{\partial \hat{y}}$ where $\ell = |\hat{y} - y|$

Recall: Derivative of absolute value function

$$\frac{d|x|}{dx} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \end{cases} \tag{22}$$

Apply to $\ell = |\hat{y} - y|$ using chain rule:

$$\frac{\partial \ell}{\partial \hat{y}} = \frac{\partial}{\partial \hat{y}}|\hat{y} - y| = \begin{cases} 1 & \text{if } \hat{y} > y \\ -1 & \text{if } \hat{y} < y \\ 0 & \text{if } \hat{y} = y \text{ (convention)} \end{cases} = \text{sign}(\hat{y} - y) \tag{23}$$

MAE Loss Derivative:

$$\boxed{\frac{\partial \ell}{\partial \hat{y}} = \text{sign}(\hat{y} - y)} \tag{24}$$

MAE Loss: Numerical Examples

Example 1: $y = 3$, $\hat{y} = 5$

$$\ell = |5 - 3| = 2$$
$$\frac{\partial \ell}{\partial \hat{y}} = \text{sign}(5 - 3) = \text{sign}(2) = +1$$

Example 2: $y = 8$, $\hat{y} = 5$

$$\ell = |5 - 8| = 3$$
$$\frac{\partial \ell}{\partial \hat{y}} = \text{sign}(5 - 8) = \text{sign}(-3) = -1$$

Comparison with MSE:

- MAE gradient is constant: ± 1
- MSE gradient varies with error magnitude: $\hat{y} - y$

BCE Loss: Definition

Binary Cross-Entropy (BCE) is used for binary classification where $y \in \{0, 1\}$ and $\hat{y} \in (0, 1)$.

For a single example:

$$\ell(\hat{y}, y) = -[y \log(\hat{y}) + (1 - y) \log(1 - \hat{y})] \tag{25}$$

Properties:

- Measures divergence between predicted and true probability distributions
- Always non-negative: $\ell \geq 0$
- Heavily penalizes confident wrong predictions

BCE Loss: Understanding the Formula

Why this form?

Since $y \in \{0, 1\}$, the loss simplifies based on the true label:

Case 1: When $y = 1$

$$\ell = -[1 \cdot \log(\hat{y}) + (1 - 1)\log(1 - \hat{y})] \tag{26}$$

$$= -\log(\hat{y}) \tag{27}$$

Case 2: When $y = 0$

$$\ell = -[0 \cdot \log(\hat{y}) + (1 - 0)\log(1 - \hat{y})] \tag{28}$$

$$= -\log(1 - \hat{y}) \tag{29}$$

Intuition:

- If $y = 1$, want $\hat{y} \to 1$ to minimize $-\log(\hat{y})$
- If $y = 0$, want $\hat{y} \to 0$ to minimize $-\log(1 - \hat{y})$

BCE Loss: Derivative Derivation

Goal: Find $\frac{\partial \ell}{\partial \hat{y}}$ where $\ell = -[y \log(\hat{y}) + (1-y)\log(1-\hat{y})]$

Differentiate both terms (using $\frac{d}{dx}\log(x) = \frac{1}{x}$ and chain rule):

$$\frac{\partial \ell}{\partial \hat{y}} = -\frac{\partial}{\partial \hat{y}}[y \log(\hat{y})] - \frac{\partial}{\partial \hat{y}}[(1-y)\log(1-\hat{y})] \tag{30}$$

$$= -\frac{y}{\hat{y}} - (1-y) \cdot \frac{1}{1-\hat{y}} \cdot (-1) = -\frac{y}{\hat{y}} + \frac{1-y}{1-\hat{y}} \tag{31}$$

Simplify with common denominator:

$$\frac{\partial \ell}{\partial \hat{y}} = \frac{-y(1-\hat{y}) + (1-y)\hat{y}}{\hat{y}(1-\hat{y})} = \frac{-y + y\hat{y} + \hat{y} - y\hat{y}}{\hat{y}(1-\hat{y})} = \frac{\hat{y} - y}{\hat{y}(1-\hat{y})} \tag{32}$$

BCE Loss Derivative:

$$\boxed{\frac{\partial \ell}{\partial \hat{y}} = \frac{\hat{y} - y}{\hat{y}(1-\hat{y})}} \tag{33}$$

BCE Loss: Derivative Examples

Example 1: $y = 1, \hat{y} = 0.9$

$$\ell = -\log(0.9) \approx 0.105$$

$$\frac{\partial \ell}{\partial \hat{y}} = \frac{0.9 - 1}{0.9(1 - 0.9)} = \frac{-0.1}{0.09} \approx -1.11$$

Example 2: $y = 1, \hat{y} = 0.1$

$$\ell = -\log(0.1) \approx 2.303$$

$$\frac{\partial \ell}{\partial \hat{y}} = \frac{0.1 - 1}{0.1(1 - 0.1)} = \frac{-0.9}{0.09} = -10$$

Example 3: $y = 0, \hat{y} = 0.9$

$$\ell = -\log(0.1) \approx 2.303$$

$$\frac{\partial \ell}{\partial \hat{y}} = \frac{0.9 - 0}{0.9(1 - 0.9)} = \frac{0.9}{0.09} = 10$$

Cross-Entropy Loss: Definition

Cross-Entropy Loss generalizes BCE to multi-class classification where y is one-hot encoded and $\hat{\mathbf{y}}$ is a probability distribution.

For C classes and single example:

$$\ell(\hat{\mathbf{y}}, \mathbf{y}) = -\sum_{c=1}^{C} y_c \log(\hat{y}_c) \tag{34}$$

where:

- $\mathbf{y} = [y_1, y_2, \ldots, y_C]^T$ is one-hot encoded (one $y_c = 1$, rest are 0)
- $\hat{\mathbf{y}} = [\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_C]^T$ are predicted probabilities
- $\sum_{c=1}^{C} \hat{y}_c = 1$ (output of softmax)

Cross-Entropy Loss: Derivative Derivation

Goal: Find $\frac{\partial \ell}{\partial \hat{y}_j}$ for each class j, where $\ell = -\sum_{c=1}^{C} y_c \log(\hat{y}_c)$

Step 1: Take derivative with respect to $\hat{y}_j$

$$\frac{\partial \ell}{\partial \hat{y}_j} = -\frac{\partial}{\partial \hat{y}_j}\left[\sum_{c=1}^{C} y_c \log(\hat{y}_c)\right] \tag{35}$$

Step 2: Only the $c = j$ term depends on $\hat{y}_j$

$$\frac{\partial \ell}{\partial \hat{y}_j} = -\frac{\partial}{\partial \hat{y}_j}\left[y_j \log(\hat{y}_j)\right] = -y_j \cdot \frac{1}{\hat{y}_j} \tag{36}$$

Cross-Entropy Loss

$$\boxed{\frac{\partial \ell}{\partial \hat{y}_j} = -\frac{y_j}{\hat{y}_j}} \tag{37}$$

Cross-Entropy Loss: Gradient Vector

The gradient with respect to all outputs forms a vector:

$$\nabla_{\hat{\mathbf{y}}} \ell = \begin{bmatrix} \frac{\partial \ell}{\partial \hat{y}_1} \\ \frac{\partial \ell}{\partial \hat{y}_2} \\ \vdots \\ \frac{\partial \ell}{\partial \hat{y}_C} \end{bmatrix} = \begin{bmatrix} -\frac{y_1}{\hat{y}_1} \\ -\frac{y_2}{\hat{y}_2} \\ \vdots \\ -\frac{y_C}{\hat{y}_C} \end{bmatrix} \tag{38}$$

Since $\mathbf{y}$ is one-hot (only $y_k = 1$):

- For the true class k: $\frac{\partial \ell}{\partial \hat{y}_k} = -\frac{1}{\hat{y}_k}$
- For all other classes $j \neq k$: $\frac{\partial \ell}{\partial \hat{y}_j} = 0$

Important: This is the gradient w.r.t. softmax outputs. When combined with softmax derivative, we get a simpler form!

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Activation Functions: Comparison

Property	Sigmoid	Tanh	Softmax
Formula	$\frac{1}{1+e^{-z}}$	$\frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}$	$\frac{e^{z_j}}{\sum_k e^{z_k}}$
Range	$(0, 1)$	$(-1, 1)$	$(0, 1)$
Zero-centered?	No	Yes	No
Derivative	$\sigma(1-\sigma)$	$1 - \tanh^2$	Complex
Max Derivative	0.25	1.0	Varies
Output Sum	N/A	N/A	1.0
Use Case	Binary class	Hidden layers	Multi-class

Key Advantages:

- **Sigmoid:** Output is probability for binary classification
- **Tanh:** Zero-centered, stronger gradients
- **Softmax:** Outputs form probability distribution

Loss Functions: Comparison

Loss	Formula	Derivative
MSE	$\frac{1}{2}(\hat{y} - y)^2$	$\hat{y} - y$
MAE	$\lvert\hat{y} - y\rvert$	$\text{sign}(\hat{y} - y)$
BCE	$-[y \log \hat{y} + (1 - y) \log(1 - \hat{y})]$	$\frac{\hat{y} - y}{\hat{y}(1 - \hat{y})}$
Cross-Entropy	$-\sum_c y_c \log \hat{y}_c$	$-\frac{y_j}{\hat{y}_j}$

Use Cases:

- **MSE:** Regression tasks
- **MAE:** Regression with outliers
- **BCE:** Binary classification ($y \in \{0, 1\}$)
- **Cross-Entropy:** Multi-class classification

Special case: Softmax + Cross-Entropy gradient: $\hat{y}_j - y_j$ (very clean!)

Summary: Key Takeaways

Loss Functions:

- MSE penalizes errors quadratically, derivative is linear
- MAE is robust to outliers, derivative is constant ± 1
- BCE for binary classification, heavily penalizes confident errors
- Cross-Entropy for multi-class, only cares about correct class probability

Activation Functions:

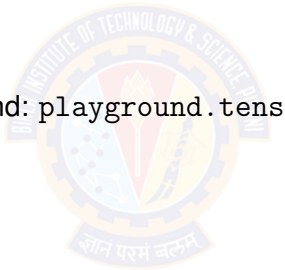
- Sigmoid squashes to $(0, 1)$, useful for probabilities
- Tanh squashes to $(-1, 1)$, zero-centered is better for hidden layers
- Softmax converts logits to probability distribution

Practical Tips:

- Regression: Use MSE or MAE loss
- Binary classification: Use sigmoid + BCE
- Multi-class: Use softmax + cross-entropy

Try

- Neural Network Playground: `playground.tensorflow.org`



Thank You!