Riemann solver for non-linear Green-Naghdi II Thermal Equations

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State of the art

The previous work, Finite element approach to Green Naghdi model type II been solved but numerical oscillation observed due to hyperbolic equations[4].
We propose the **Riemann solver** to non linear thermal response Green Naghdi-II equations which are hyperbolic and formulated by potential function also describe the constitutive relation and can be solved numerically for FVM (Gudnov method or even higher order).

Contents

Objective

- To provide a Riemann solution to non linear thermal response Green Naghdi model type-II dissipationless.

Introduction

- Existence of second sound - heat propgates as thermal wave
- Possibility of second sound in solids
- classical Fourier law of conduction is based on assumption
- the heat flux is an immediate response to a temperature change
- due to parabolic nature the infinite wave speed
- the Fourier law gives a macroscopic description of the microscopic phenomena associated with heat diffusion
- Cattaneo resolve the problem of infinite speed

Maxwell-Cattaneo theory

- Cattaneo employs a time relaxed relation with the heat flux density and the temperature gradient.
- finite speed of propagation

$$\tau \frac{\partial q}{\partial t} + q = -k \frac{\partial T}{\partial x} \tag{1}$$

eliminating q leads to second order equation on T as the damped wave equation

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{k}{\rho C} \frac{\partial^2 T}{\partial x^2} \tag{2}$$

- leads to hyperbolic heat equation

Green Naghdi model

- Thermal displacement - time integral of the temperature T

$$\alpha(\mathbf{X}, t) := \int_{t_0}^{t} T(\mathbf{X}, \tau)\mathrm{d}\tau + \alpha_0 \tag{3}$$

- it holds $\dot{\alpha} = T$
- heat propagates as a thermal wave with finite speed
- 3 different model known as
    - type I - classical Fourier based
    - type II - dissipationless
    - type III - with dissipation
- this work focus on non-linear thermal response for type II

Green Naghdi - type II

Constitutive equations - non linear thermal response

- free energy W and internal energy U

$$W(T,\beta) = \rho C\left((T - T_0) - T\ln\left(\frac{T}{T_0}\right)\right) + \frac{1}{2}\frac{\widehat{K}}{T_0}\beta^2$$

$$\mathrm{U}(\mathrm{S},\beta) = \rho C T_0\left(\mathrm{e}^{\left(\frac{S}{\rho C}\right)} - 1\right) + \tfrac{1}{2}\tfrac{\widehat{K}}{T_0}\beta^2$$

- free energy W and internal energy U an potential for entropy S and entropy flux h

$$S = -\frac{\partial W}{\partial T} = \rho C\ln\left(\frac{T}{T_0}\right)$$

$$h = \frac{\partial U}{\partial \beta} = \frac{\widehat{K}\beta}{T_0}$$

with β thermal displacement gradient and T_0 reference temperature

Green Naghdi - II

non linear thermal response

- The thermodynamic relation between heat flux q and entropy flux h

$$q = T \;\; h$$

- The conservation equation

$$\dot{S} + \text{div}\, h = \frac{\rho r}{T}$$

with no external source r = 0

$$\boxed{\dot{S} + \text{div}\, h = 0} \tag{4}$$

$$\boxed{\dot{\beta} + \text{div}\, T = 0} \tag{5}$$

Green Naghdi - II

linear thermal response

- free energy W and internal energy U for linear thermal response

$$W(T,\beta) = -\frac{\rho C}{2T_0}(T - T_0)^2 + \frac{1}{2}\frac{\widehat{K}}{T_0}\beta^2 \tag{6}$$

$$U(S,\beta) = -\frac{T_0}{2\rho C}S^2 + S_0 + \frac{1}{2}\frac{\widehat{K}}{T_0}\beta^2 \tag{7}$$

the difference between linear and non linear thermal response depend on assumption[4].

The constitutive assumptions and relations leads to

$$\dot{S} + \operatorname{div} h = \frac{\rho r}{T}$$

we have $\dot{S} = \frac{\rho C \dot{T}}{T}$

$$\frac{\rho C \dot{T}}{T} + \frac{\widehat{K}}{T_0}\left(\frac{\partial \beta}{\partial x}\right) = \frac{\rho r}{T} \tag{8}$$

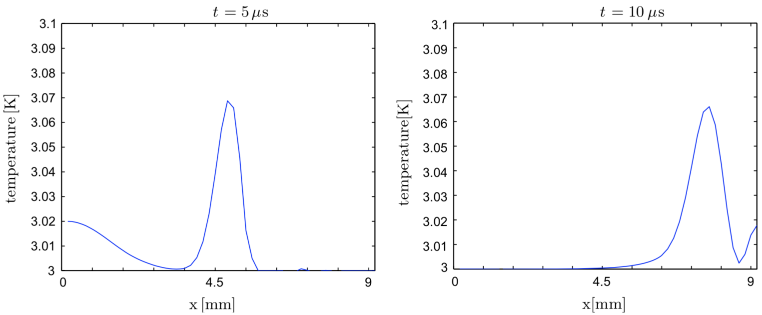
with $\mathrm{T} = \dot{\alpha}$

$$\frac{\rho C \ddot{\alpha}}{\dot{\alpha}} - \frac{\widehat{K}}{T_0}\left(\frac{\partial^2 \alpha}{\partial x^2}\right) = \frac{\rho r}{T} \tag{9}$$

$$\frac{\rho C \ddot{\alpha}}{\dot{\alpha}} = \frac{\widehat{K}}{T_0}\nabla^2 \alpha + \frac{\rho r}{T} \tag{10}$$

- a hyperbolic wave equation
- thermal displacement travel as a wave with finite speed with no dissipation (undamped wave).

The non linear thermal response Green Naghdi model type II been solved by FEM[10].



- plot shows temperature vs time
- thermal waves at finite speed
- numerical oscillations

Riemann problem

The Initial Value Problem (IVP) for the conservation laws,

$$q_{,t} + (f(q))_{,x} = 0$$

q as a vector of conserved variables and f(q) is the vector of fluxes with piecewise constant initial data,

$$\mathbf{q}(x,0) = \mathbf{q}^{(0)}(x) = \begin{cases} \mathbf{q}_{\mathrm{L}} \\ \mathbf{q}_{\mathrm{R}} \end{cases}$$

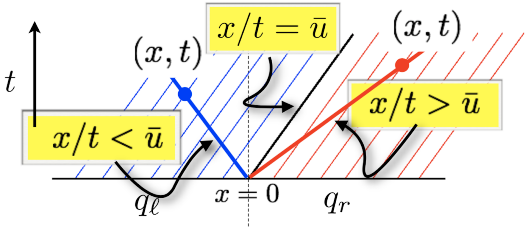
- solution based on characterstic field
- represent the one solution of hyperbolic system
- used in Godunov method based on FVM
- linear riemann problem easy to solve but non linear complicated

Linear riemann problem

$$q_t + \bar{u} q_x = 0$$

subject to initial conditions

$$q(x,0) = \begin{cases} q_\ell & x < 0 \\ q_r & x > 0 \end{cases}$$



Solution :
$$q(x,t) = \begin{cases} q_\ell & x/t < \bar{u} \\ q_r & x/t > \bar{u} \end{cases}$$

non linear green naghdi II equations

The non linear Green naghdi II equations, given by

$$\frac{\partial S}{\partial t} + \frac{\partial h}{\partial x} = 0 \tag{11}$$

$$\frac{\partial \beta}{\partial t} + \frac{\partial T}{\partial x} = 0 \tag{12}$$

is an example of a system of equations written in conservative form. More generally, we can write PDEs in conservative form as

$$q_{,t} + (f(q))_{,x} = 0$$

q denotes a vector of the considered state variables, f(q) are flux function vector. The subscript, x denotes a partial differentiation with respect to x

$$q_t + f\left(q\right)_x = 0$$

where

$$q = \left[\begin{array}{c} S \\ \beta \end{array}\right] \qquad f(q) = \left[\begin{array}{c} h \\ T \end{array}\right] = \left[\begin{array}{c} \frac{\partial U}{\partial \beta} \\ \frac{\partial U}{\partial S} \end{array}\right]$$

for smooth solutions, this can also be written as

$$q_t + f'\left(q\right) q_x = 0$$

where

$$f'(q) = \left[\begin{array}{cc} \frac{\partial h}{\partial S} & \frac{\partial h}{\partial \beta} \\ \frac{\partial T}{\partial S} & \frac{\partial T}{\partial \beta} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial^2 U}{\partial S \partial \beta} & \frac{\partial^2 U}{\partial \beta^2} \\ \frac{\partial^2 U}{\partial S^2} & \frac{\partial^2 U}{\partial \beta \partial S} \end{array}\right]$$

is the flux Jacobian matrix.

Eigenvalues and eigenvectors of the flux Jacobian $f'(q)$

$$\lambda_2^1 = \pm\sqrt{\frac{\widehat{K}}{\rho C}\mathrm{e}^{\left(\frac{S}{\rho C}\right)}}$$

The eigenvectors are given by,

$$K^1 = \begin{bmatrix} \xi \\ \alpha \end{bmatrix} \qquad K^2 = \begin{bmatrix} -\xi \\ \alpha \end{bmatrix}$$

with $\xi = \dfrac{\sqrt{\widehat{K}\rho C}}{T_0\mathrm{e}^{\left(\frac{S}{2\rho C}\right)}}$

Eigenvalues and eigenvectors depend on q (i.e. S)

Characterstic Field

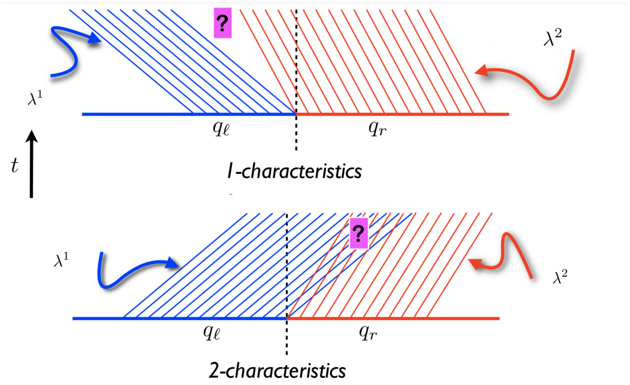
Characterstic Field- A characteristic field is defined by both the characteristic speed $\lambda$and the eigenvector K associated with it.

linearly degenerate fields - $\nabla\lambda_i(q)\cdot\mathbf{K}^{(i)}(q)=0, \quad \forall q\in\Re^m$
genuinely nonlinear fields - $\nabla\lambda_i(q)\cdot\mathbf{K}^{(i)}(q)\neq 0, \quad \forall q\in\Re^m$
where $\Re^m$ is the set of real-valued vectors of m components

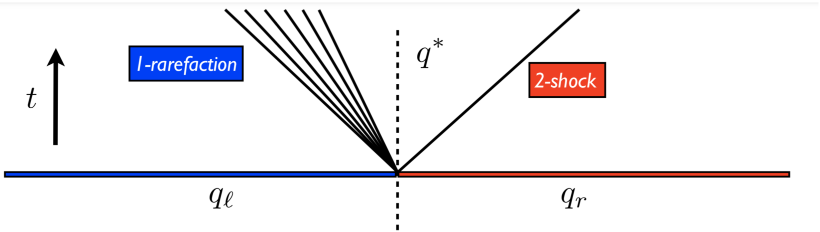
$$\nabla\lambda_1(q)\cdot\mathbf{K}^{(1)}(q)=\nabla\lambda_2(q)\cdot\mathbf{K}^{(2)}(q)=\frac{\widehat{K}}{2\rho C T_0}\neq 0$$

Therefore both the characteristic field are **genuinely nonlinear**

Problem with non linear



Solving the non linear riemann problem



find a state q^* such that q_l is connected to q^* by physically correct 1-shock wave or 1-rarefaction, and q_r is connected to q^* by a physically correct 2-shock or 2-rarefaction.

find a state q^* that simultaneously satisfies both conditions. use a nonlinear root-finder to find the middle state q^* .

Shock wave

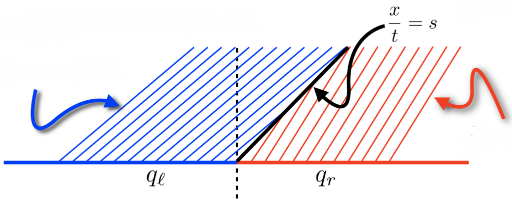
Consider the case where $\lambda^p(q_l) > \lambda^p(q_r)$:

- the two characteristic field will collide each other (i.e. shock)

This is the required jump condition across shocks. More generally we can write this condition as **Rankie Hugoniot jump condition**

$$f\left(q_r\right) - f\left(q_l\right) = s\left(q_r - q_l\right) \tag{13}$$

where s is the shock speed



shock

The solution within the **left shock** hence satisfies :

$$\beta^* \;=\; \beta_L \;-\; \frac{2T_0}{\sqrt{K}}\;\sqrt{S_L - S^*}\;\sqrt{\exp\left(\frac{S_L}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)} \tag{14}$$

The solution within the **right shock** hence satisfies :

$$\beta^* \;=\; \beta_R \;+\; \frac{2T_0}{\sqrt{K}}\;\sqrt{S_R - S^*}\;\sqrt{\exp\left(\frac{S_R}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)} \tag{15}$$

Rarefaction

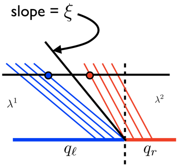
Consider the case where $\lambda^p(q_l) < \lambda^p(q_r)$:

- connects the left and right state through a smooth transition.
- this smooth transition can be computed using the Riemann invariants
- this invariants constant along any ray of the rarefaction.

$$\frac{dw_1}{k_1^i} = \frac{dw_2}{k_2^i} = \ldots = \frac{dw_m}{k_m^i}$$

- gives the set of ODEs which are integrated between two states connected by the rarefaction to determine the states spanned with in the rarefaction

slope $\xi = \frac{x}{t}$ of the characteristic.
we need to find $q(\xi)$



Rarefaction

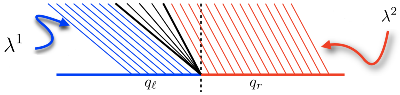
1-Riemann Invariant

The 1-Riemann invariant associated with the characteristic feild.
the ODE given,

$$\frac{d\left(\frac{S}{\xi}+\beta\right)}{-\xi}=\frac{d\left(-\frac{S}{\xi}+\beta\right)}{\alpha} \tag{16}$$

Integrating the above equation between star region and left region we get,

$$\beta^* = \quad \beta_L - \int_{\left(\frac{S}{\xi}\right)_*}^{\left(\frac{S}{\xi}\right)_L} \left(\frac{\alpha-\xi}{\alpha+\xi}\right) \mathrm{d}\left(\frac{S}{\xi}\right) \tag{17}$$



Rarefaction

2-Riemann Invariants

similarly, 2-Riemann invariant ODE,

$$\frac{d\left(\frac{S}{\xi}+\beta\right)}{\xi}=\frac{d\left(-\frac{S}{\xi}+\beta\right)}{\alpha} \tag{18}$$

Integrating the above equation between right region and star region we get,

$$\int_{\left(\frac{S}{\xi}\right)_R}^{\left(\frac{S}{\xi}\right)_*}\left(\frac{\alpha+\xi}{\alpha-\xi}\right)\mathrm{d}\left(\frac{S}{\xi}\right)+\int_{\left(\frac{S}{\xi}\right)_R}^{\left(\frac{S}{\xi}\right)_*}\mathrm{d}\beta=0 \tag{19}$$

$$\beta^* = \ \beta_R-\int_{\left(\frac{S}{\xi}\right)_R}^{\left(\frac{S}{\xi}\right)_*}\left(\frac{\alpha+\xi}{\alpha-\xi}\right)\mathrm{d}\left(\frac{S}{\xi}\right) \tag{20}$$

it leads to,

Rarefaction

The solution within the **left rarefaction** hence satisfies :

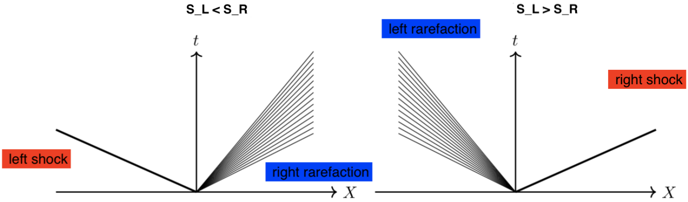
$$\beta^* = \beta_L - \frac{a}{b}\left[\frac{2b}{\alpha}\ln\left|\frac{\alpha + b\exp\left(\frac{S^*}{a}\right)}{\alpha + b\exp\left(\frac{S_L}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_L}{a}\right)} + \frac{2b}{\alpha}(S^* - S_L)\right]$$
$$+ \frac{2a}{\alpha}\left[\ln\left|\alpha + b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a} - \ln\left|\alpha + b\exp\left(\frac{S_L}{a}\right)\right|\frac{S_L}{a} + \mathrm{Li}_2\left(-b\exp\left(\frac{S^*}{a}\right)\right)\right.$$
$$\left.- \mathrm{Li}_2\left(-b\exp\left(\frac{S_L}{a}\right)\right)\right) + \frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a - S^*) - \exp\left(\frac{-S_L}{a}\right)(a - S_L)\right) - \left(\frac{(S^*)^2 - (S_R)^2}{a\alpha}\right)$$

The solution within the **right rarefaction** hence satisfies :

$$\beta^* = \beta_R - \frac{a}{b}\left[\frac{-2b}{\alpha}\ln\left|\frac{\alpha - b\exp\left(\frac{S^*}{a}\right)}{\alpha - b\exp\left(\frac{S_R}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_R}{a}\right)} + \frac{2b}{\alpha a}(S^* - S_R)\right]$$
$$- \frac{2a}{\alpha}\left[\ln\left|\alpha - b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a} - \ln\left|\alpha - b\exp\left(\frac{S_R}{a}\right)\right|\frac{S_R}{a} + \mathrm{Li}_2\left(b\exp\left(\frac{S^*}{a}\right)\right)\right.$$
$$\left.- \mathrm{Li}_2\left(b\exp\left(\frac{S_R}{a}\right)\right)\right) + \frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a - S^*) - \exp\left(\frac{-S_R}{a}\right)(a - S_R)\right) + \left(\frac{(S^*)^2 - (S_R)^2}{a\alpha}\right)$$

Riemann solution for GN-II

- the characterstic speed are monotone functions of the entropy S
- case I - left shock and right rarefaction (1-shock 2-rarefaction)
- case II - left rarefaction and right shock (1-rarefaction 2-shock)



Case I : $S_L > S_R$

left rarefaction and right shock

The 1-rarefaction 2-shock solution one then seeks a state q^* that is connected to q_L and q_R through a shock wave and a rarefaction wave respectively. Hence, q must satisfy equations,

$$\beta^* = \beta_R + \frac{2T_0}{\sqrt{K}}\sqrt{S_R - S^*}\sqrt{\exp\left(\frac{S_R}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)}$$

$$\beta^* = \beta_L - \frac{a}{b}\left[\frac{2b}{\alpha}\ln\left|\frac{\alpha + b\exp\left(\frac{S^*}{a}\right)}{\alpha + b\exp\left(\frac{S_L}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_L}{a}\right)} - \frac{2b}{\alpha}(S^* - S_L)\right]$$
$$+ \frac{2a}{\alpha}\left[\ln\left|\alpha + b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a} - \ln\left|\alpha + b\exp\left(\frac{S_L}{a}\right)\right|\frac{S_L}{a} + \mathrm{Li}_2\left(-b\exp\left(\frac{S^*}{a}\right)\right)\right.$$
$$\left. - \mathrm{Li}_2\left(-b\exp\left(\frac{S_L}{a}\right)\right)\right) + \frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a - S^*) - \exp\left(\frac{-S_L}{a}\right)(a - S_L)\right) - \left(\frac{(S^*)^2 - (S_R)^2}{a\alpha}\right)$$

Case II : $S_L < S_R$

left shock and right rarefaction

similarly, for case II q must satisfy equation

$$\beta^* = \beta_L - \frac{2T_0}{\sqrt{K}}\sqrt{S_L - S^*}\sqrt{\exp\left(\frac{S_L}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)}$$

$$\beta^* = \beta_R - \frac{a}{b}\left[\frac{-2b}{\alpha}\ln\left|\frac{\alpha - b\exp\left(\frac{S^*}{a}\right)}{\alpha - b\exp\left(\frac{S_R}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_R}{a}\right)} + \frac{2b}{\alpha a}(S^* - S_R)\right]$$
$$- \frac{2a}{\alpha}\left[\ln\left|\alpha - b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a} - \ln\left|\alpha - b\exp\left(\frac{S_R}{a}\right)\right|\frac{S_R}{a} + \mathrm{Li}_2\left(b\exp\left(\frac{S^*}{a}\right)\right)\right.$$
$$\left. - \mathrm{Li}_2\left(b\exp\left(\frac{S_R}{a}\right)\right) + \frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a - S^*) - \exp\left(\frac{-S_R}{a}\right)(a - S_R)\right) + \left(\frac{(S^*)^2 - (S_R)^2}{a\alpha}\right)\right]$$

- The above equations represent the solution for non linear Green Naghdi type-II dissipationless
- These two equations to show the non linear scalar equation whose root is sought
- The problem is solved by optimization using SLSQP method
- Newton Raphson method failed due to constraint
- The Scipy library that solves constrained minimization problems
- material parameters extracted from paper[8]

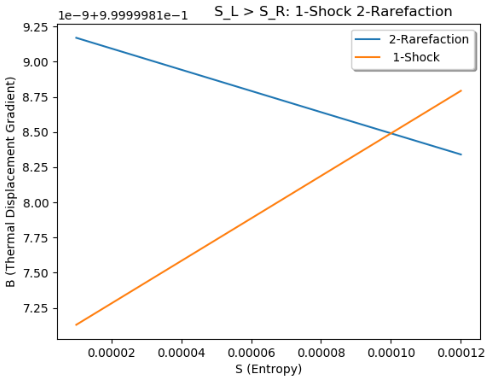
Case I : $S_L > S_R$

- The non linear scalar equation whose root is to be solved by given left and right data and obtain star state.
- The different test case for right shock and left rarefaction with given left and right data,

Test Case	S_L	S_R	β_L	β_R	S_*	β_*
1.	0.009	0.006	1.0	1.0	0.00101038	0.99954
2.	0.0104	0.0100	0.0	0.0	$7.441401e^{-06}$	$1.5416999e^{-07}$
3.	0.0118	0.0114	0.0	-1.0	0.00107168	-0.00163836
4.	0.0121	0.0115	-1.0	-1.0	-0.00074492	-0.99973143
5.	0.0121	0.01	1.0	1.0	0.0001	1.0002685

Case I : $S_L > S_R$

solution for left rarefaction and right shock



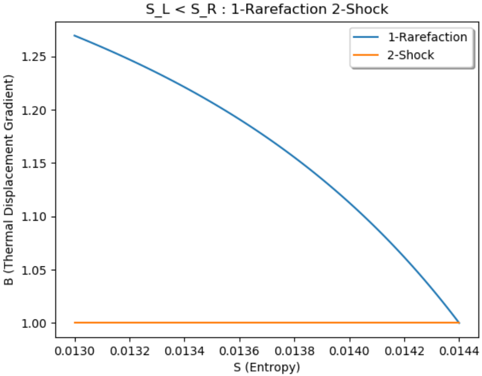
- The intersection between the two curves is the star state S_* and β_*

case II : $S_L < S_R$

- similarly ,the root is to be solved by given left and right data and obtain star state.
- The different test case for left shock and right rarefaction

Test Case	S_L	S_R	β_L	β_R	S_*	β_*
1.	0.0182	0.0184	2.0	1.0	0.018197	5.0303712
2.	0.0192	0.0198	1.0	2.0	0.019196	45.4961
3.	0.0201	0.0206	0.0	0.0	0.020096	-5.844 e^{-5}
4.	0.0311	0.0318	-1.0	1.0	0.03109047	-1.44158e^{-5}
5.	0.0142	0.0144	1.0	1.0	0.0141999	1.0017792

Case II : $S_L < S_R$



- The intersection between the two curves is the star state S_* and β_*

Conclusion

- The Riemann solution proposed to non linear thermal response Green Naghdi-II thermal equations.
- The shock and rarefaction equations derived analytically.
- The solution is combination of a shock and a rarefaction(i.e. 1-shock 2-rarefaction or 1-rarefaction 2-shock).
- The solution of the star state requires to find numerically the root of a non linear scalar equation.

Future scope

- The Riemann solution can be used for coupled problem (i.e. Thermoelasticity).
- Numerical simulation for FVM (Godunov method or even higher order).

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Thank You