Riemann solver for non-linear Green-Naghdi II Thermal Equations

Suresh Chaudhary

Supervisor: **Prof. Thomas Huezé**

Institut de Recherche en Génie Civil et Mécanique (GEM)
Computational Mechanics
Centrale Nantes

This dissertation is submitted for the degree of
Master of Science

August 2019

Abstract

This thesis presents the Riemann solution for non linear Green Naghdi dissipationless energy thermal equation which is also know as type II. The classical theory of heat conduction based on the Fourier law allows infinite diffusion speed which is not well accepted from a physical point of view. The Green and Naghdi [3] employ a procedure which differs from the usual one.They introduce the notion of thermal displacement so that thermal waves propagate at finite speed and constitutive equation for entropy flux is determined by potential function.
This work is focused on the solution of a Riemann problem for Green-Naghdi II equations, which are hyperbolic and these equations are formulated according to some internal energy or free energy potential describing the constitutive response. The solution to the Riemann problem is derived according to a linearized thermal [6] response as well as a nonlinear one. It is shown that the solution of such a Riemann problem can be used in a finite volume solver.

Contents

List of figures

List of tables

Symbols

K^i eigen vector

S_0 scale of entropy

Φ conserved state variable ϕ, averaged in a cell

α thermal displacement

α_0 thermal displacement at t_0

β thermal displacement gradient

$\dot{\alpha}$ thermal displacement rate

γ temperature gradient

λ eigen values

∇ gradient

$\overline{\phi}$ cell average function

ϕ in vector equation notation: conserved state variable, first order tensor or scalar

ρ density

τ relaxation factor

θ absolute temperature

$\tilde{\phi}$ integral curve

ξ ratio x/t

$\widehat{K}$ thermal conductivity

c	specific heat constant
$f(\phi)$	flux function vector
h	entropy flux
K	fourier thermal conductivity
q	heat flux

Other Symbols

X	position in reference configuration
x	position in current configuration
r	external source supply
S	entropy
s	shock speed
T	temperature
t	time
T_0	initial temperature
U	iternal energy
W	free energy

Chapter 1

Heat propagation theories

1.1 Introduction

First sound is the classical mechanism which allows a disturbance of pressure (or density) propagating through a continuous medium such as air or water. **Second Sound** involves the propagation of heat as a thermal wave.The classical theory of heat propagation is via diffusion where heat diffuses through a continuous body.However, experiments in the late 1960's and early 1970's on some particular material showed that a thermal disturbance could travel as a wave and this has acted as an impulse to much subsequent theoretical work in this area. In addition to a thermal wave, the experiments also showed that second sound was also important in Thermoelasticity [4] this Experiments revealed the existence of three distinct waves that are longitudinal elastic wave which travels fastest, transverse elastic wave, thermal wave. The Maxwell-Cattaneo theory [1] which try to solves the problem of infinite speed by introducing relaxation factor.The models for producing thermal wave which travels at finite speed all been based on a time delay between the heat flux q and the gradient of temperature $\nabla\theta$, or have involved Taylor expansion which leads to the introduction of a thermal relaxation time.In 1991,the paper of Green and Naghdi[3] brought a new way of thinking to the area of heat wave propagation and their article has influenced many subsequent developments.There have been several theories proposed to solve heat wave propagation.

This work is concerned with the dissipationless theory (the so-called type II) of the Green-Naghdi equations, which are hyperbolic and describe the propagation of heat as a thermal wave without dissipation. These equations are usually a second order PDE in time on the thermal displacement, and they have been solved numerically with the classical finite element method [9]. However, their hyperbolic nature also allows a writing as a first order hyperbolic system, hence allowing to employ analysis and solution procedures known for such type of equations [11]. This work particularly focuses on the solution of a Riemann problem with such equations. This solution is interesting first because it allows to identify the nature of characteristic fields and the states propagated from the simplest set of prescribed initial conditions, and second because such a solution is used in upwind finite volume schemes which are famous and efficient numerically to solve such equations[5]. In this work, the dissipationless G-N thermal equation is first written as a first order system of balance laws. These equations are formulated according to some internal energy or free energy potential describing the constitutive response which are explained in next section.The solution of the Riemann problem is then derived according to a linearized thermal response as well as a nonlinear one. The solution for the linearized response is quite standard and been studied previously[6]. The Riemann solution associated with that nonlinear response yields nonlinear (rarefaction or shock) characteristic fields so that the solution to the Riemann problem is slightly more complex to derive, and is computed numerically by finding the root of a nonlinear equation.

1.2 Maxwell-Cattaneo Theory

The Maxwell-Cattaneo Theory is one of major influence on thermal waves and it begins with the classical diffusion equation for heat reads in one dimension (1D) infinite medium

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}, \qquad x \in \mathbb{R}, t > 0 \tag{1.1}$$

with the initial data

$$T(x,0) = f(x), \qquad x \in \mathbb{R}$$

the solution for the above equation is given by

$$T(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} f(\xi) \exp\left[\frac{-(x-\xi)^2}{4Dt}\right] d\xi$$

We see that as soon as t > 0,T$\neq$ 0 everywhere. Thus, we can see that T is having an infinite speed of propagation which is an undesirable effect and, therefore, we seek to find a method where by T will propagate with a finite speed. The Maxwell Cattaneo model has been studied to overcome the problem of infinite speed of propagation.

If T denotes the temperature of a rigid solid, ρ its density, c its specific heat, and k its thermal conductivity, then equation (1.1) arises from the energy balance law and with the

Fourier law of heat conduction

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} \ ; \qquad q = -k \frac{\partial T}{\partial x}$$

In order to obtain a finite speed of propagation,Cattaneo [10] employs a time relaxed relation with the heat flux density and the temperature gradient.

$$\tau \frac{\partial q}{\partial t} + q = -k \frac{\partial T}{\partial x} \tag{1.2}$$

eliminating q leads to second order equation on T as the damped wave equation

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{k}{\rho C} \frac{\partial^2 T}{\partial x^2} \tag{1.3}$$

Maxwell-Cattaneo model the linkage between the heat conduction equation and the energy conservation introduces the hyperbolic equation which describes a heat propagation with finite speed. The finiteness of heat propagation speed provided by the generalized thermoelasticity theories based on Maxwell- Cattaneo model and are supposed to be more realistic than the conventional theory to deal with practical problems with very large heat flux.

1.3 Green Naghdi model

The propagation of thermal waves in rigid and elastic solids has received considerable attention in recent years.The Green and Naghdi [3] model employ a procedure which differs from the usual one. They make use of a general entropy balance rather than an entropy inequality. The basic development is quite general and The characterization of material response for thermal phenomena is based on three types of constitutive functions, which they are divided into the type I, II and III. The type I theory is the classical heat conduction theory (based on Fourier's law) while the type II theory permits propagation of thermal waves at finite speed without energy dissipation and the type III is the combination of type I and II (with Energy dissipation). In the following, we restrict our examination to type II theory which is Without Energy Dissipation.The classical theory type I and type III is well documented in the paper Green-Naghdi [3].

The Green–Naghdi theory introduces a scalar variable α by analogy to the electromagnetism, Green and Naghdi call it thermal displacement. It is the time integral of the temperature T:

$$\alpha(\mathbf{X},t) := \int_{t_0}^{t} T(\mathbf{X},\tau)\mathrm{d}\tau + \alpha_0 \tag{1.4}$$

Here,$\mathbf{X} \in \mathbb{R}^3$ denotes the position in the reference configuration $t \in \mathbb{R}$ the time and α_0 is the initial value at the reference time t_0. Differentiating with respect to time $\dot{\alpha} = T$ The absolute temperature $\theta = \theta(T)$ is defined as a monotonically increasing function of T. Without loss of generality we sub-sequently choose $T \equiv \theta$

One of the drawbacks of the classical heat conduction theory is that thermal waves may propagate at infinite speed. In order to overcome this unnatural behavior, the constitutive equations postulated for W, T, ξ, h are assumed to be functions of the temperature T, the thermal displacement α, the termal displacement gradient $\beta = \nabla\alpha$. The main difference between type I,II and III [4]are the Constitutive functions. The type I - constitutive equations postulated for W, T, ξ, q are assumed to be functions of temperature T, the temperature gradient ∇T which gives The classical Fourier theory and with thermal Disturbance which is quiet not useful.Similarly, for type III -the constitutive equations postulated for W, T, ξ, q are assumed to be functions of temperature T, the thermal displacement α, the thermal displacement gradient $\nabla\alpha$ and temperature gradient ∇ T

The difference in all 3 types of Green Naghdi model based on constitutive functions
type I - T , ∇T - Classical fourier theory and thermal disturbance of wave
type II - T , α , β -thermal waves with finite speed without energy dissipation
type III - T, α , $\nabla\alpha$, γ -thermal waves with finite speed with energy dissipation.
There have been several research done heat conduction models describing heat propagation by means of thermal waves in addition to diffusive propagation the one paper attracts towards is [2] which concluded that the type II model predicts a finite speed of heat propagation without any damping or smoothing, the type III generates thermal waves which travel at finite speed and smoothen during the propagation, containing the type II as a dissipationless special case and Approximated Taylor shock waves with finite-speed wavefront can also be obtained in the genuine type III model.The Green-Naghdi are neither the only nor the firstly appeared non-classical heat conduction models able to predict heat-wave propagation. We may quote for instance the Maxwell-Cattaneo law, introduced in order to generate a hyperbolic heat equation, so to remove the infinite velocity paradox of the Fourier heat diffusion. Unlike Fourier and Green-Naghdi theories, the Maxwell-Cattaneo theory is based on a rate-type constitutive equation for the heat flux.The below figure shows a typical profile of heat propagation for all 3 types.The type I with dotted line shows a profile similar to that of Fourier, the type II model predicts a finite speed of heat propagation without any damping or smoothing and the type III the propagation of the thermal front is depicted from which it is apparent that type III sharply approaches the form of a Taylor shock wave, but with an infinite tail, due to the parabolic character of the evolution equation and the solution consists of a wave front propagating forever[8].

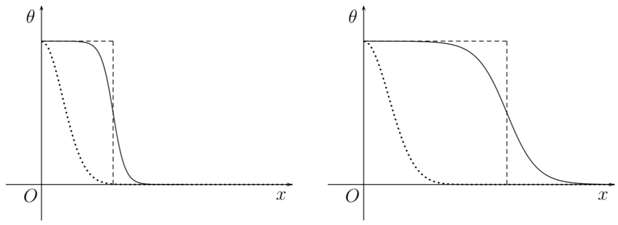

Fig. 1.1 *Temperature profiles* $\theta(x, t_0)$ *and* $\theta(x, t_1)$ *for type I (dotted), type II (dashed) and type III (solid) with* $\theta = T - T_0$

from type II non linear dissipation-less theory, the free energy equation W and Internal Energy U as follows [4],

$$W(T,\beta) = \rho C\left((T - T_0) - T\ln\left(\frac{T}{T_0}\right)\right) + \frac{1}{2}\frac{\widehat{K}}{T_0}\beta^2 \tag{1.5}$$

$$U(S,\beta) = \rho C T_0\left(\mathrm{e}^{\left(\frac{S}{\rho C}\right)} - 1\right) + \frac{1}{2}\frac{\widehat{K}}{T_0}\beta^2 \tag{1.6}$$

The main difference between linear and non linear is assumption we made.so,Similarly for type II linear dissipation-less theory,

$$W(T,\beta) = -\frac{\rho C}{2T_0}(T - T_0)^2 + \frac{1}{2}\frac{\widehat{K}}{T_0}\beta^2 \tag{1.7}$$

$$U(S,\beta) = -\frac{T_0}{2\rho C}S^2 + S_0 + \frac{1}{2}\frac{\widehat{K}}{T_0}\beta^2 \tag{1.8}$$

previous study on type II linear dissipation-less theory[6] and gave some sort of results here we are focus non linear theory.
The relation for free energy equation W internal energy U and entropy S is given by,

$$U(S,\beta) \;=\; W(T,\beta) + \;TS$$

The conservation of energy equation

$$\frac{\partial S}{\partial t} + \frac{\partial h}{\partial x} = \frac{\rho r}{T} \qquad ; \tag{1.9}$$

our classical theory of conservation of energy gives in the form,

$$\dot{U} + \operatorname{div} q = \rho r \tag{1.10}$$

$$\dot{S} + \operatorname{div} h = \frac{\rho r}{T} \tag{1.11}$$

The relation between U internal energy, entropy S, entropy flux h, heat flux q, T and W free energy

$$h = \frac{\partial U}{\partial \beta} = \frac{\widehat{K}\beta}{T_0} \qquad S = -\frac{\partial W}{\partial T} = \rho C \ln\left(\frac{T}{T_0}\right) \qquad \dot{S} = \frac{\rho C \dot{T}}{T}$$

and q = T h (valid in 1D assumption otherwise)
The Equation 1.9 becomes

$$\frac{\rho C \dot{T}}{T} + \frac{\widehat{K}}{T_0}\left(\frac{\partial \beta}{\partial x}\right) = \frac{\rho r}{T} \tag{1.12}$$

with T = $\dot{\alpha}$

$$\frac{\rho C \ddot{\alpha}}{\dot{\alpha}} - \frac{\widehat{K}}{T_0}\left(\frac{\partial^2 \alpha}{\partial x^2}\right) = \frac{\rho r}{T} \tag{1.13}$$

$$\frac{\rho C \ddot{\alpha}}{\dot{\alpha}} = \frac{\widehat{K}}{T_0}\nabla^2 \alpha + \frac{\rho r}{T} \tag{1.14}$$

This is clearly a hyperbolic wave equation and the thermal displacement then travels as a wave with finite speed with no dissipation. The first order system of balance laws given by,

$$\boxed{\frac{\partial S}{\partial t} + \frac{\partial h}{\partial x} = \frac{\rho r}{T}}$$

$$\boxed{\frac{\partial \beta}{\partial t} + \frac{\partial T}{\partial x} = 0}$$

The above equation can be written in the form of scalar conservation laws with flux function f(q).

$$q_t + f(q)_x = 0$$

The equation becomes more complicated if $f(q)$ is a nonlinear function of q. The solution no longer simply translates uniformly. Instead, it deforms as it evolves, and in particular shock waves and Rarefaction can form, across which the solution is discontinuous.

Chapter 2

Riemann Problem

The dissipationless Green Naghdi equations are set of nonlinear hyperbolic equations.The nonlinear character of the equations means that the use of analytical techniques to solve them can only be successful in very special circumstances.Hyperbolic equations may admit discontinuous solutions, in addition to smooth or classical solutions. Even for the case in which the initial data is smooth everywhere, the nonlinear character combined with the hyperbolic type of the equations can lead to discontinuous solutions in a finite time. Here, the discontinuities are associated with shock.

The exact solution of the Riemann problem is useful in a number of ways. First, it represents the solution to a system of hyperbolic conservation laws subject to the simplest, non–trivial, initial conditions; in spite of the simplicity of the initial data the solution of the Riemann problem contains the fundamental physical and mathematical character of the relevant set of conservation laws. The solution of a Riemann problem, exact or approximate, can also be used locally in the method of Godunov and high–order extensions of it; this is the main role we assign to the Riemann problem here.The solution for non-linear type II GN with detail explanation in next section.Previous study [6]the solution for linear type have been solved and here more focus on non-linear and the solution is no longer simply translates uniformly. Instead, it deforms as it evolves, and in particular shock waves and rarefaction waves can form, across which the solution is discontinuous.

Strategy for solving the Riemann problem- In order to apply Riemann-solver-based finite volume methods, we must be able to solve the riemann problem with provided left and right states U_L and U_R. To compute the exact solution, we must do the following:

1. characterstic analysis of system.

2. determine whether each of the two waves is a shock or a rarefaction wave(perhaps using an appropriate entropy condition).

3. determine the intermediate state * between the two waves.

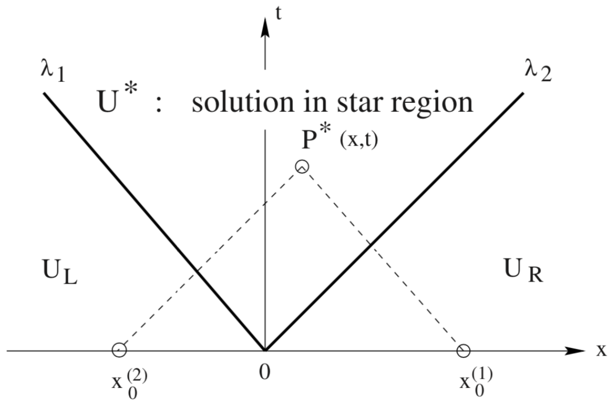

Fig. 2.1 *The Riemann problem solution in x-t plane Left data, Right data and Star Region*

The Riemann problem for the one–dimensional non linear Green Naghdi type II model is the Initial Value Problem (IVP) for the conservation laws,

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}, \tag{2.1}$$

$$\mathbf{U} = \begin{bmatrix} S \\ \beta \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} h \\ T \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial \beta} \\ \frac{\partial U}{\partial S} \end{bmatrix}$$

with initial condition

$$\mathbf{U}(x,0) = \mathbf{U}^{(0)}(x) = \begin{cases} \mathbf{U}_\mathrm{L} \\ \mathbf{U}_\mathrm{R} \end{cases}$$

The provided initial condition that consist of left data $\mathbf{U}_\mathrm{L}$ and right data $\mathbf{U}_\mathrm{R}$ the problem amounts to compute the intermediate State * In solving the Riemann problem we shall frequently have a U as a vector of conserved variables and $\mathbf{F}(\mathbf{U})_x$ is the vector of fluxes and each of its components F_i of F is a function of the components U_j of U.

we can see from above figure 2.2 which tells us possible wave pattern of Riemann solution for Non-Linear type II theory. It also tells that the solution can have 4 different cases. Firstly,case (a)- Left Rarefaction and Right Shock; case (b)- Left Shock and Right Rarefaction; case (c)- Both Rarefaction; case (d)- both Shock.

In order to solve hyperbolic system it requires some condition in order to get the unique solution and physical admissibility. Following gas dynamics, these are called entropy conditions which ensure in the original problem that the second law of thermodynamics is

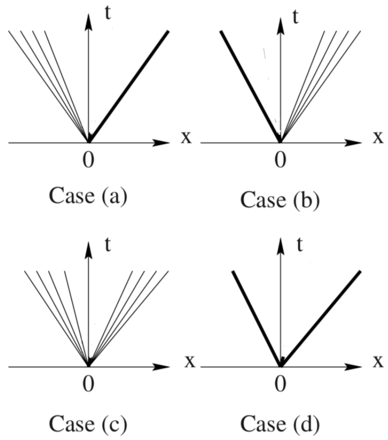

Fig. 2.2 *Possible wave patterns in the solution of the Riemann problem*

hold mathematically. With reference to the characteristics solution of the Riemann problem, this signifies that the entropy increasing and thus entropy violating solution depicted in fig2.3a across expansion shocks has to be prevented. In this figure characteristics come out of the shock. They build an expansion shock with increasing entropy which is physically not admissible. The realistic solution is a rarefaction fan as shown in 2.3b, it is in contrast to the former stable to perturbations and results if the initial profile is slightly smeared out or if viscosity is taken into account.In adding entropy conditions, it is ensured that physically admissible, "right" solutions to the weak form, are computed. The Lax-entropy condition for

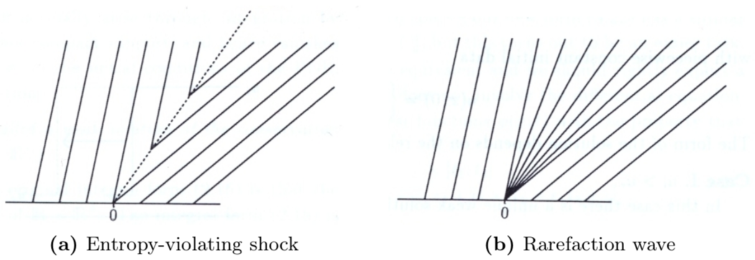

Fig. 2.3 *Two possible weak solutions*

a convex scalar conservation law for example reads

$$f'(\phi_l) > s > f'(\phi_r)$$

where the discontinuity is propagating with speed s and $f'(\phi_l)$ and $f'(\phi_r)$ are the characteristic speed of the state to the left or right, respectively, of the shock. As in case of convex or concave flux functions a the Rankine-Hugoniot condition satisfying speed s lies between

$f'(\phi_l)$ and $f'(\phi_r)$ the entropy condition reduces to

$$f'(\phi_l) > f'(\phi_r)$$

2.1 Characteristic analysis of the non linear dissipationless GN equation

$$\phi_{,t} + (f(\phi))_{,x} = 0 \tag{2.2}$$

Here, ϕ denotes a vector of the considered state variables, f(ϕ) are flux function vector. The subscript, x denotes a partial differentiation with respect to x

$$\phi = \begin{bmatrix} S \\ \beta \end{bmatrix} \qquad f(\phi) = \begin{bmatrix} h \\ T \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial \beta} \\ \frac{\partial U}{\partial S} \end{bmatrix}$$

Under the assumption that ϕ is smooth, a quasi-linear system can be considered.The one-dimensional system thus becomes, with the flux Jacobian matrix A $= \frac{\partial \mathbf{f}}{\partial \phi}$

$$\phi_{,t} + A\phi_{,x} = 0$$

In this expression the chain rule was applied, yielding

$$(f(\phi))_{,x} = \frac{\partial f}{\partial x} = \underbrace{\frac{\partial f}{\partial \phi}}_{A} \frac{\partial \phi}{\partial x} = A\phi_{,x}$$

A one-dimensional system

$$\phi_{,t} + A(\phi, x, t)\phi_{,x} = 0$$

is hyperbolic at a point (ϕ, x, t) if the matrix A(ϕ, x, t) satisfies at this point the hyperbolicity conditions at each physical relevant value of ϕ i.e. if it is diagonalizable with real eigenvalues and distinct eigenvectors. Our Conservation Form of Equation is given by,

$$\frac{\partial S}{\partial t} + \frac{\partial h}{\partial x} = \frac{\rho r}{T} \tag{2.3}$$

$$\frac{\partial \beta}{\partial t} + \frac{\partial T}{\partial x} = 0 \tag{2.4}$$

The above equation can be written in vector notation

$$\phi = \begin{bmatrix} S \\ \beta \end{bmatrix} \quad ; \quad f(\phi) = \begin{bmatrix} h \\ T \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial \beta} \\ \frac{\partial U}{\partial S} \end{bmatrix}$$

$$\phi_{,t} + A\phi_{,x} = 0 \tag{2.5}$$

Jacobian Matrix A reads,

$$f' = A = \begin{bmatrix} \frac{\partial h}{\partial S} & \frac{\partial h}{\partial \beta} \\ \frac{\partial T}{\partial S} & \frac{\partial T}{\partial \beta} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 U}{\partial S \partial \beta} & \frac{\partial^2 U}{\partial \beta^2} \\ \frac{\partial^2 U}{\partial S^2} & \frac{\partial^2 U}{\partial \beta \partial S} \end{bmatrix}$$

This yields the Two real and distinct eigenvalues

$$\lambda_2^1 = \frac{\partial^2 U}{\partial S \partial \beta} \pm \sqrt{\frac{\partial^2 U}{\partial S^2} \frac{\partial^2 U}{\partial \beta^2}}$$

$$\lambda_2^1 = \pm \sqrt{\frac{\widehat{K}}{\rho C} \mathrm{e}^{\left(\frac{S}{\rho C}\right)}} \tag{2.6}$$

The eigenvectors are given by,

$$K^1 = \begin{bmatrix} \xi \\ \alpha \end{bmatrix} \qquad K^2 = \begin{bmatrix} -\xi \\ \alpha \end{bmatrix}$$

with $\xi = \frac{\sqrt{\widehat{K}\rho C}}{T_0 \mathrm{e}^{\left(\frac{S}{2\rho C}\right)}}$

Characterstic Field- A Hyperbolic system of conservation laws in the form (2.3) with real eigenvalues λ_2^1 and corresponding right eigenvectors $\mathbf{K}^{(1)}$,$\mathbf{K}^{(2)}$. A characteristic field is defined by both the characteristic speed and the eigenvector associated with it.

A characteristic field is said to be linearly degenerate fields if -

$$\nabla \lambda_i(\phi) \cdot \mathbf{K}^{(i)}(\phi) = 0, \quad \forall \phi \in \Re^m$$

where $\Re^m$ is the set of real-valued vectors of m components

similarly, a field is said to be genuinely nonlinear fields if -

$$\nabla \lambda_i(\phi) \cdot \mathbf{K}^{(i)}(\phi) \neq 0, \quad \forall \phi \in \Re^m$$

where $\Re^m$ is the set of real-valued vectors of m components

$$\nabla\lambda_1(\phi)\cdot\mathbf{K}^{(1)}(\phi)=\nabla\lambda_2(\phi)\cdot\mathbf{K}^{(2)}(\phi)=\frac{\widehat{K}}{2\rho C T_0}\neq 0$$

Therefore both the characteristic field are **genuinely nonlinear**

2.2 Rarefaction

A rarefaction wave is a genuinely non linear field that connects the two data states ϕ_L and ϕ_R through a smooth transition. The states spanned during this smooth transition can be computed using the Riemann invariants,which should be constant along any ray of the rarefaction. These are defined by:

$$\frac{dw_1}{k_1^i}=\frac{dw_2}{k_2^i}=\ldots=\frac{dw_m}{k_m^i}$$

these gives us the set of ODEs which are integrated between two states connected by the rarefaction to determine the states spanned with in the rarefaction. If $\mathbf{K}$ are the right eigenvectors of Matrix $\mathbf{A}$ corresponding to the eigenvalues λ then the inverse matrix $\mathbf{K}^{-1}$ makes it possible to define a new set of dependent variables $\mathbf{W}=(w_1,w_2,\ldots,w_m)^T$ via the transformation,

$$\mathbf{W}=\mathbf{K}^{-1}\phi$$

The dependent variables read for the dissipationless Green Naghdi equation:

$$K=\begin{bmatrix}\xi & -\xi\\ 1 & 1\end{bmatrix}\qquad K^{-1}=\begin{bmatrix}\frac{1}{2\xi} & \frac{1}{2}\\ \frac{-1}{2\xi} & \frac{1}{2}\end{bmatrix}\qquad \phi=\begin{bmatrix}S\\ \beta\end{bmatrix}$$

$$W=\begin{bmatrix}\frac{1}{2}\left(\frac{S}{\xi}+\beta\right)\\ \frac{1}{2}\left(\frac{-S}{\xi}+\beta\right)\end{bmatrix}=\begin{bmatrix}w_1\\ w_2\end{bmatrix}$$

2.2.1 1-Riemann Invariant

The 1-Riemann invariant associated with the 1-characteristic. therefore,Solving for Right Rarefaction between Right region and Star Region.

$$\frac{dw_1}{k_1^1} = \frac{dw_2}{k_2^1}$$

$$K^1 = \begin{bmatrix} \xi \\ \alpha \end{bmatrix} \text{ and } \quad W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{S}{\xi}+\beta\right) \\ \frac{1}{2}\left(\frac{-S}{\xi}+\beta\right) \end{bmatrix} \quad \text{with } \alpha = 1$$

so we get

$$\frac{d\left(\frac{S}{\xi}+\beta\right)}{\xi} = \frac{d\left(-\frac{S}{\xi}+\beta\right)}{\alpha} \tag{2.7}$$

Integrating the above equation between right region and star region we get,

$$\int_{\left(\frac{S}{\xi}\right)_R}^{\left(\frac{S}{\xi}\right)_*} \left(\frac{\alpha+\xi}{\alpha-\xi}\right) \mathrm{d}\left(\frac{S}{\xi}\right) + \int_{\left(\frac{S}{\xi}\right)_R}^{\left(\frac{S}{\xi}\right)_*} \mathrm{d}\beta = 0$$

The solution of the right rarefaction reads:

$$\boxed{\beta^* = \ \beta_R - \int_{\left(\frac{S}{\xi}\right)_R}^{\left(\frac{S}{\xi}\right)_*} \left(\frac{\alpha+\xi}{\alpha-\xi}\right) \mathrm{d}\left(\frac{S}{\xi}\right)} \tag{2.8}$$

The integral in equation (2.8) can be computed analytically,whose details in appendix **Right Rarefaction** - The solution within the right rarefaction hence satisfies:

$$\begin{aligned}\beta^* = \beta_R - \frac{a}{b}&\left[\frac{-2b}{\alpha}\ln\left|\frac{\alpha - b\exp\left(\frac{S^*}{a}\right)}{\alpha - b\exp\left(\frac{S_R}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_R}{a}\right)} + \frac{2b}{\alpha a}(S^* - S_R)\right] \\ &- \frac{2a}{\alpha}\left[\ln\left|\alpha - b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a} - \ln\left|\alpha - b\exp\left(\frac{S_R}{a}\right)\right|\frac{S_R}{a} + \mathrm{Li}_2\left(b\exp\left(\frac{S^*}{a}\right)\right)\right. \\ &\left. - \mathrm{Li}_2\left(b\exp\left(\frac{S_R}{a}\right)\right)\right) + \frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a-S^*) - \exp\left(\frac{-S_R}{a}\right)(a-S_R)\right) + \left(\frac{(S^*)^2 - (S_R)^2}{a\alpha}\right)\end{aligned}$$

2.2.2 2-Riemann Invariant

The 2-Riemann invariant associated with the 2-Characteristic field. therefore Solving for left rarefaction between star region and left region.

$$\frac{dw_1}{k_1^2} = \frac{dw_2}{k_2^2}$$

$$K^2 = \begin{bmatrix} -\xi \\ \alpha \end{bmatrix} \text{ and } \quad W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{S}{\xi}+\beta\right) \\ \frac{1}{2}\left(\frac{-S}{\xi}+\beta\right) \end{bmatrix} \text{ with } \alpha = 1$$

we get

$$\frac{d\left(\frac{S}{\xi}+\beta\right)}{-\xi} = \frac{d\left(-\frac{S}{\xi}+\beta\right)}{\alpha} \tag{2.9}$$

Integrating the above equation between star region and left region we get,

$$\int_{\left(\frac{S}{\xi}\right)_*}^{\left(\frac{S}{\xi}\right)_L} \left(\frac{\alpha-\xi}{\alpha+\xi}\right) \mathrm{d}\left(\frac{S}{\xi}\right) + \int_{\left(\frac{S}{\xi}\right)_*}^{\left(\frac{S}{\xi}\right)_L} \mathrm{d}\beta = 0 \tag{2.10}$$

The solution of the left rarefaction reads:

$$\boxed{\beta^* = \beta_L - \int_{\left(\frac{S}{\xi}\right)_*}^{\left(\frac{S}{\xi}\right)_L} \left(\frac{\alpha-\xi}{\alpha+\xi}\right) \mathrm{d}\left(\frac{S}{\xi}\right)} \tag{2.11}$$

The integral in equation (2.11) can be computed analytically,as detailed in appendix.

Left Rarefaction-The solution within the left rarefaction hence satisfies:

$$\begin{aligned}
\beta^* = \beta_L - \frac{a}{b}&\left[\frac{2b}{\alpha}\ln\left|\frac{\alpha+b\exp\left(\frac{S^*}{a}\right)}{\alpha+b\exp\left(\frac{S_L}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_L}{a}\right)} + \frac{2b}{\alpha}(S^*-S_L)\right] \\
&+ \frac{2a}{\alpha}\left[\ln\left|\alpha+b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a} - \ln\left|\alpha+b\exp\left(\frac{S_L}{a}\right)\right|\frac{S_L}{a} + \mathrm{Li}_2\left(-b\exp\left(\frac{S^*}{a}\right)\right)\right. \\
&\left. - \mathrm{Li}_2\left(-b\exp\left(\frac{S_L}{a}\right)\right) + \frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a-S^*) - \exp\left(\frac{-S_L}{a}\right)(a-S_L)\right) - \left(\frac{(S^*)^2-(S_R)^2}{a\alpha}\right)\right.
\end{aligned}$$

2.3 Shock

For a shock wave the two constant states ϕ_L and ϕ_R are connected through a single jump discontinuity in a genuinely non–linear field i and the following conditions apply.
1. The Rankine–Hugoniot conditions

$$s\left(\phi_i-\phi_*\right)=f\left(\phi_i\right)-f\left(\phi_*\right)$$

2. The entropy condition arising from Lax theorem

$$\lambda_i\left(\phi_{\mathrm{L}}\right)>s_i>\lambda_i\left(\phi_{\mathrm{R}}\right)$$

A shock wave is a discontinuous wave satisfying the Rankine-Hugoniot condition recalling our equation (2.5)

$$\phi=\begin{bmatrix} S \\ \beta \end{bmatrix} \qquad f(\phi)=\begin{bmatrix} h \\ T \end{bmatrix}$$

$$s\left(\phi_i-\phi_*\right)=f\left(\phi_i\right)-f\left(\phi_*\right)$$

where s is shock speed and the subscript i $\in\{L,R\}$ represents a fixed left or right state ϕ_l or ϕ_r to which the searched state ϕ_* may be connected through a shock

For Non-linear Green Naghdi type II equations the Rankine-Hugoniot relation reads

$$s\left(S_i-S_*\right)=\left(h_i-h_*\right) \tag{2.12}$$

$$s\left(\beta_i-\beta_*\right)=\left(T_i-T_*\right) \tag{2.13}$$

$$s=\frac{\left(T_i-T_*\right)}{\left(\beta_i-\beta_*\right)}$$

In addition to the Rankine-Hugoniot condition, one has to consider the Lax entropy conditions, stating that characteristic curves collide in a shock wave.The detail solution explained in Appendix.

$$\beta^* \quad = \quad \beta_i \quad \pm \quad \frac{2T_0}{\sqrt{K}} \; \sqrt{S_i-S^*} \; \sqrt{\exp\left(\frac{S_i}{\rho C}\right)-\exp\left(\frac{S^*}{\rho C}\right)}$$

left shock-The solution within the left shock hence satisfies:

$$\beta^* = \beta_L - \frac{2T_0}{\sqrt{K}} \sqrt{S_L - S^*} \sqrt{\exp\left(\frac{S_L}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)}$$

Right Shock-The solution within the right shock hence satisfies:

$$\beta^* = \beta_R + \frac{2T_0}{\sqrt{K}} \sqrt{S_R - S^*} \sqrt{\exp\left(\frac{S_R}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)}$$

2.4 Riemann solution for the non linear GN II equations

given the set of conservation laws

$$\phi_t + \mathbf{f}(\phi)_x = \mathbf{0}, \tag{2.14}$$

with initial condition

$$\begin{cases} \phi_L = \begin{bmatrix} S_L \\ \beta_L \end{bmatrix} \\ \\ \phi_R = \begin{bmatrix} S_R \\ \beta_R \end{bmatrix} \end{cases}$$

note: from equation (2.6),the characterstic speed are monotone functions of the entropy variable S. If initial conditions are given such that $S_L < S_R$, left-going characteristics will collide while right-going ones will move away from one another as shown in figure (2.4) below.In that case, the first and second characteristic fields are respectively referred to as a **1-Rarefaction** and a **2-Shock** . Conversely, if $S_L > S_R$, the solution corresponds to a **1-Shock** and a **2-Rarefaction**.

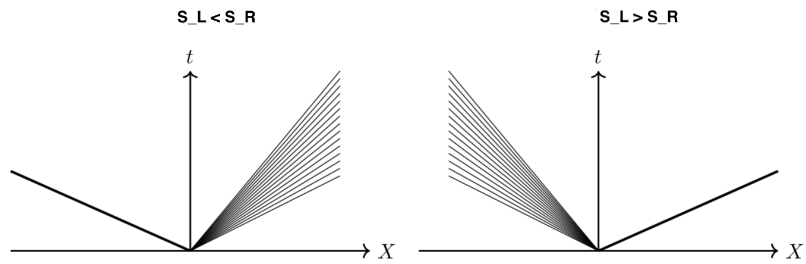

Fig. 2.4 *Solution of the Riemann problem in the x–t plane*

For the **1-rarefaction 2-shock** solution one then seeks a state U^* that is connected to U_L and U_R through a shock wave and a rarefaction wave respectively. Hence, U must satisfy equations,

$$\beta^* = \beta_L - \frac{2T_0}{\sqrt{K}}\sqrt{S_L - S^*}\sqrt{\exp\left(\frac{S_L}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)}$$

$$\beta^* = \beta_R - \frac{a}{b}\left[\frac{-2b}{\alpha}\ln\left|\frac{\alpha - b\exp\left(\frac{S^*}{a}\right)}{\alpha - b\exp\left(\frac{S_R}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_R}{a}\right)} + \frac{2b}{\alpha a}(S^* - S_R)\right]$$
$$-\frac{2a}{\alpha}\left[\ln\left|\alpha - b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a} - \ln\left|\alpha - b\exp\left(\frac{S_R}{a}\right)\right|\frac{S_R}{a} + \mathrm{Li}_2\left(b\exp\left(\frac{S^*}{a}\right)\right)\right.$$
$$\left. - \mathrm{Li}_2\left(b\exp\left(\frac{S_R}{a}\right)\right)\right) + \frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a - S^*) - \exp\left(\frac{-S_R}{a}\right)(a - S_R)\right) + \left(\frac{(S^*)^2 - (S_R)^2}{a\alpha}\right)$$

similarly for the **1-shock 2-rarefaction** the solution is given by,

$$\beta^* = \beta_R + \frac{2T_0}{\sqrt{K}}\sqrt{S_R - S^*}\sqrt{\exp\left(\frac{S_R}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)}$$

$$\beta^* = \beta_L - \frac{a}{b}\left[\frac{2b}{\alpha}\ln\left|\frac{\alpha + b\exp\left(\frac{S^*}{a}\right)}{\alpha + b\exp\left(\frac{S_L}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_L}{a}\right)} - \frac{2b}{\alpha}(S^* - S_L)\right]$$
$$+\frac{2a}{\alpha}\left[\ln\left|\alpha + b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a} - \ln\left|\alpha + b\exp\left(\frac{S_L}{a}\right)\right|\frac{S_L}{a} + \mathrm{Li}_2\left(-b\exp\left(\frac{S^*}{a}\right)\right)\right.$$
$$\left. - \mathrm{Li}_2\left(-b\exp\left(\frac{S_L}{a}\right)\right)\right) + \frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a - S^*) - \exp\left(\frac{-S_L}{a}\right)(a - S_L)\right) - \left(\frac{(S^*)^2 - (S_R)^2}{a\alpha}\right)$$

The above equations represent the solution for non linear Green Naghdi type II dissipationless energy for two different cases and these two equations to show the nonlinear scalar equation whose root is sought.The solution for both shocks and rarefaction can not be solved and The numerical method to solve these equations is discussed in next section.

Chapter 3

Numerical Method

The previous Riemann solver can be combined with upwind finite volume scheme. This idea was initially proposed by Godunov, and has then been extended to high-resolution and less diffusive methods.

3.1 The Godunov Method

The Godunov method proceeds with an explicit time stepping and relies on the solution of Rieman of problems,used at each cell interface, which allows to use the characteristic information of the problem and to introduce the sufficient amount of numerical diffusion to make the explicit scheme stable. The solution of the Riemann problem is utilised locally, the solution can be exact or approximate. It is a generalization of the first-order upwind method to conservation law systems; the basic scheme is first-order accurate in space and time, higher-order extensions are possible. In this section the wave propagation form of the Godunov Method is presented. The average of the state variable $\phi\left(x,t_n\right)$ at time t_n in the cell i is considered, it is denoted by the quantity Φ_i^n

$$\Phi_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \phi\left(x,t_n\right)\mathrm{d}x \tag{3.1}$$

From this cell average a function $\overline{\phi}\left(x,t_n\right)$ for the shape of the variable in the cell is defined; in the simplest case it is piecewise constant, thus

$$\overline{\phi}\left(x,t_n\right) = \Phi_i^n$$

The basic idea is that the cell average for a new time step is affected by a set of waves. Each wave $\mathscr{W}$ consists of a jump in ϕ which is computable as a scalar multiple of an right eigenvector K hence across the p -th wave the jump can be expressed as

$$\mathscr{W}^p = \alpha^p K^p$$

The jump propagates at the velocity λ^p, the p-th characteristic speed; after an increment Δt of time is has moved a distance $\lambda^p\ \Delta t$ and enters a cell of width Δx up to the ratio $\lambda^p\ \Delta t\ /\ \Delta x$ At this fraction of the cell the average value is modified by the propagating wave. Therefore an updated average cell value, influenced by several waves defined at the neighbouring cell interfaces i-1 / 2 and i+1 / 2 and travelling with positive velocities λ^+ as right going wave from i-1 / 2 or with negative velocities λ^- as left-going wave from i+1 / 2, is obtained by

$$\Phi_i^{n+1} = \Phi_i^n - \frac{\Delta t}{\Delta x}\left[\underbrace{\sum^p (\lambda^p)^+ \mathscr{W}^p_{i-1/2}}_{\text{right-going waves from } x_{i-1/2}} + \underbrace{\sum^p (\lambda^p)^- \mathscr{W}^p_{i+1/2}}_{\text{left-going waves from } x_{i+1/2}} \right] \tag{3.2}$$

This expression represents a generalization of a first order upwind method.
Introducing for the change in Φ_i^n through left- or right-going waves, respectively, the symbols

$$\mathscr{A}^+\Delta\Phi_{i+1/2} = \sum^p (\lambda^p)^+ \mathscr{W}^p_{i-1/2} = \sum^p (\lambda^p)^+ \alpha^p_{i-1/2} r^p$$

$$\mathscr{A}^-\Delta\Phi_{i-1/2} = \sum^p (\lambda^p) \mathscr{W}^p_{i+1/2} = \sum^p (\lambda^p)^- \alpha^p_{i+1/2} r^p$$

The symbol $\mathscr{A}^+\Delta\Phi_{i+1/2}$ can be interpreted as a the net effect of all right-going waves from the cell interface $x_{i-1/2}$ measuring single entity; analogeously $\mathscr{A}^-\Delta\Phi_{i-1/2}$ measures the effect of all left-going waves from the cell interface $x_{i+1/2}$.

3.2 Approximate Riemann Solvers

Approximate Riemann solvers do not determine the entire structure of the Riemann problem but an approximating state $\hat{\Phi}_{i-1/2}(x,t)$ based on given data $\Phi_{i-1/2}$ and Φ_i. This is considered to be sufficient with respect to the fact that the application of Godunov's method does not require the entire solution of the Riemann problem, e.g. the exact solution is averaged over each grid cell.The approximate Riemann solution can be obtained through replacing the nonlinear problem by a linearized one which is defined locally at each cell interface,

$$\hat{\phi}_{,t} + \hat{A}_{i-1/2}\hat{\phi}_{,x} = 0 \tag{3.3}$$

The matrix $\hat{A}_{i-1/2}$ represents as a local linearization of the flux Jacobian $f'(\phi)$ an approximation to the latter. It is valid in a neighbourhood of the data $\Phi_{i-1/2}$ and Φ_i and replaces the Jacobian; the resulting linear Riemann problem is solved with respect to this approximation. As in the case of a linear Riemann problem, the solution will consist of a set of waves propagating at speeds which are calculated through the eigenvalues of the constant matrix of the flux derivatives. A distinction to the circumstances of linear Riemann problem lies in the number of waves which can now differ from the number of equations. The wave speeds are not necessarily identical to the eigenvalues as well.
A conservative, i.e. conservation law obeying, approximate solution will satisfy

$$f(\phi_r) - f(\phi_l) = \sum_p s^p \mathscr{W}^p$$

which means that flux evaluated at the far sides of the changing region equals the by the jump disconinuity caused time rate of change of the solution

Chapter 4

Numerical Result

This chapter presents the numerical solution of the exact Riemann solver derived in chapter 2, for a set of material parameters extracted from the paper [7].

4.1 Material Parameters

The second sound phenomenon have been observed in Bismuth below 3.5 K see paper [7] Bismuth is a brittle metal and one of the few non-toxic heavy metals and possesses a rhombohedral crystal structure. Bismuth has many outstanding properties. For example, it is the metal with the highest natural diamagnetism, the highest Hall effect and the second lowest thermal conductivity.

A paper Narayanamurti and Dynes [7] which proved the existence of second sound in Bi for the first time and numerical simulation by Bargmann, P. Steinmann [9] using Galerkin finite element method are capable of modeling the second sound phenomenon for Non linear Green Naghdi type II.

Iterative schemes- Initally the GN II equations solved by newton raphson method but mathematical constraint did not satisfied such as square root,logarithm,Li_2 dilogarithm involved.The problem is solved by optimization using SLSQP method in the Scipy library that solves constrained minimization problems.

4.2 Case I

When $S_L > S_R$ we have **1-shock 2-rarefaction** the combine two equation(2.4) show the nonlinear scalar equation whose root is to be solved by given left and Right data and obtain star state.The Below table show the different test case for right shock and left rarefaction with given left and right data,

Test Case	S_L	S_R	β_L	β_R	S_*	β_*
1.	0.009	0.006	1.0	1.0	0.00101038	0.99954
2.	0.0104	0.0100	0.0	0.0	$7.441401e^{-06}$	$1.5416999e^{-07}$
3.	0.0118	0.0114	0.0	-1.0	0.00107168	-0.00163836
4.	0.0121	0.0115	-1.0	-1.0	-0.00074492	-0.99973143
5.	0.0121	0.01	1.0	1.0	0.0001	1.0002685

Table 4.1 Test cases for right shock and left rarefaction with given left and right state

The Below figure (4.1) shows the hugoniot locus in which integral curves associated with the two waves are plotted from U_L and U_R. Riemann Solution for GN II dissipationless Energy for Right shock and Left Rarefaction.Test case No 5,the point of intersection between the two curves is the star state S_* and β_*,

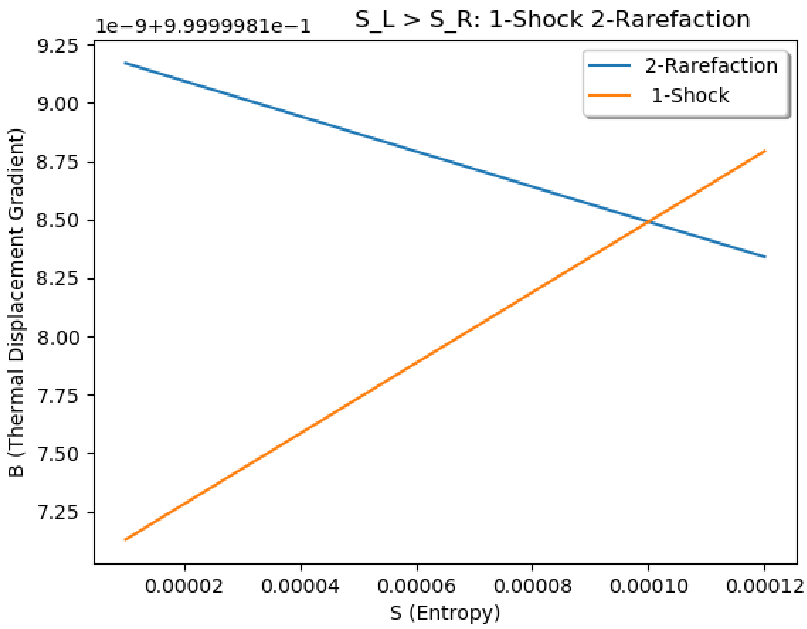

Fig. 4.1 *Riemann solution for GN II with left rarefaction and right Shock*

4.3 Case II

When $S_L < S_R$ - **1-Rarefaction 2-Shock** the combine two equation (2.4) show the nonlinear scalar equation whose root is to be solved by given left and Right data and obtain star state. The below table show the different test case for left shock and right rarefaction with given left and right data,

Test Case	S_L	S_R	β_L	β_R	S_*	β_*
1.	0.0182	0.0184	2.0	1.0	0.018197	5.0303712
2.	0.0192	0.0198	1.0	2.0	0.019196	45.4961
3.	0.0201	0.0206	0.0	0.0	0.020096	-5.844 e^{-5}
4.	0.0311	0.0318	-1.0	1.0	0.03109047	-1.44158e^{-5}
5.	0.0142	0.0144	1.0	1.0	0.0141999	1.0017792

Table 4.2 Test cases for left shock and right rarefaction with given left and right state

The Below figure (4.2) shows the hugoniot locus in which integral curves associated with the two waves are plotted from U_L and U_R. Riemann Solution for GN II dissipationless Energy for Left shock and Right Rarefaction.The test Case No 5 the point of intersection between the two curves is the star state S_* and β_*,

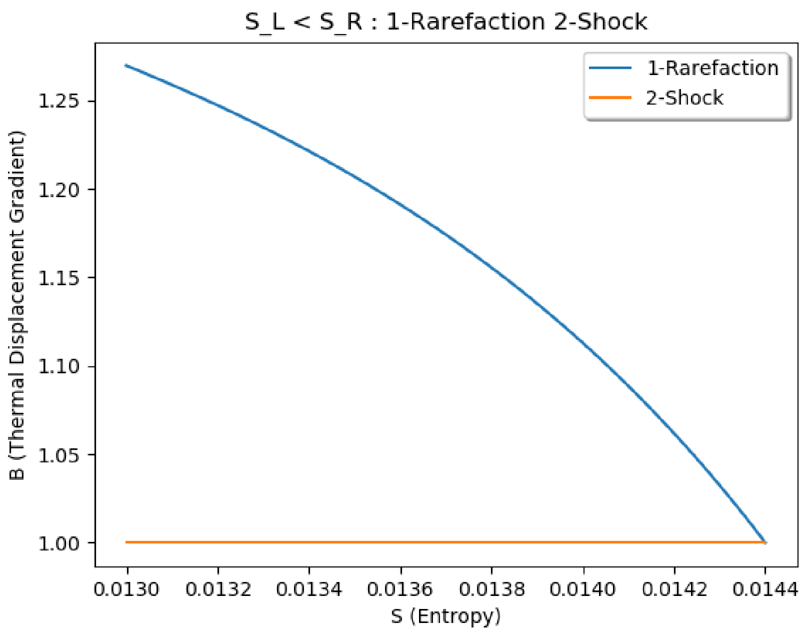

Fig. 4.2 *Riemann Solution for GN II with right rarefaction and left shock*

Chapter 5

Conclusion

In this work, a Riemann solution has been proposed to non linear Green Naghdi type II thermal equations. The nonlinear thermal response extracted from the classical generalized thermoelasticity [4], written in the form of free energy potential, the dissipationless Green-Naghdi equations consist of a set of nonlinear hyperbolic equations. In a 1D medium, these equations consist of two genuinely nonlinear characteristic fields. It has been shown that the characteristic structure exhibited in the solution of a Riemann problem necessarily consists of a combination of a shock and a rarefaction(i.e. 1-shock 2-rarefaction or 1-rarefaction 2-shock). It has also been shown that equations linked to a shock and a rarefaction can be derived analytically, though the solution of the star state requires to find numerically the root of a nonlinear scalar equation. Finally, different test cases have been shown and integral curves plotted for a given set of material parameters associated with Bismuth.

References

[1] Cattaneo, C. (1948). *Sulla conduzione del calore*. Atti Sem. Mat. Fis. Modena, 3:83–101.

[2] claudio G, d. G. and vittorino P (2005). *on the green naghdi type III heat conduction model*.

[3] Green and Naghdi (1991a). *A re-examination of the basic postulates of themomechanics*.

[4] Green and Naghdi (1991b). *Thermoelasticity without energy dissipation*.

[5] LeVeque, R. (2004). *finite volume methods for hyperbolic problems*. Cambridge University.

[6] Loïc, D. Z. and Julien, C. (2015). *Formulation conservative et modélisation d'ondes thermiques*. ECN.

[7] Narayanamurti V, Dynes, R. (1972). *Observation of second sound in bismuth*.

[8] Quintanilla, R. (2002). *existence in thermoelasticity without energy dissipation*.

[9] S. Bargmann, P. S. (2005). *Modeling and simulation of first and second sound in solids*. International Journal of Solids and Structure.

[10] Straughan, B. (2011). *Heat Waves*. Springer.

[11] Toro, E. F. (2009). *Riemann Solvers and Numerical Methods for Fluid Dynamics*. Springer.

Chapter 6

Appendix

In these section, the detail explanation of Rarefaction and shock equations

6.1 Rarefaction

Solving for rarefaction we begin with equation (2.8) which is solution for right rarefaction (between star region and right region) we get 1-Riemann invariant ,

$$\beta^{*} - \beta_R + \int_{\left(\frac{S}{\xi}\right)_R}^{\left(\frac{S}{\xi}\right)_*} \left(\frac{\alpha+\xi}{\alpha-\xi}\right) \mathrm{d}\left(\frac{S}{\xi}\right) = 0 \tag{6.1}$$

similarly, 2-Riemann invariant (i.e. solution for left rarefa tion)

$$\beta^{*} - \beta_L - \int_{\left(\frac{S}{\xi}\right)_L}^{\left(\frac{S}{\xi}\right)_*} \left(\frac{\alpha-\xi}{\alpha+\xi}\right) \mathrm{d}\left(\frac{S}{\xi}\right) = 0 \tag{6.2}$$

Solve first for Right Rarefaction the integral equation (6.1)

$$\int_{\left(\frac{S}{\xi}\right)_R}^{\left(\frac{S}{\xi}\right)_*} \left(\frac{\alpha+\xi}{\alpha-\xi}\right) \mathrm{d}\left(\frac{S}{\xi}\right) \tag{6.3}$$

we know that ξ from equation (2.6)

$$\xi = \frac{\sqrt{\rho c \hat{K}}}{T_o} \mathrm{e}^{\left(\frac{-S}{2\rho c}\right)} \tag{6.4}$$

For simplicity Let,

$$a=\left(\frac{-1}{2\rho c}\right) \qquad ; \qquad b=\frac{\sqrt{\rho c\hat{K}}}{T_o}$$

Therefore,

$$\xi = b\,\mathrm{e}^{\left(\frac{S}{a}\right)}$$

by using change of variable The above equation (6.3) become

$$=\int_{S_R}^{S_*}\left(\frac{\alpha+b\,\mathrm{e}^{\left(\frac{S}{a}\right)}}{\alpha-b\,\mathrm{e}\left(\frac{S}{a}\right)}\right)\left(\frac{a-S}{ab}\right)\exp\left(\frac{-S}{a}\right)\mathrm{d}S$$

$$\frac{a}{ab}\int_{S_R}^{S_*}\left(\frac{\alpha+b\exp\left(\frac{S}{a}\right)}{\alpha-b\exp\left(\frac{S}{a}\right)}\right)\exp\left(\frac{-S}{a}\right)\mathrm{d}S-\frac{S}{ab}\int_{S_R}^{S_*}\left(\frac{\alpha+b\exp\left(\frac{S}{a}\right)}{\alpha-b\exp\left(\frac{S}{a}\right)}\right)\exp\left(\frac{-S}{a}\right)\mathrm{d}S \quad (6.5)$$

let solving I part of above of equation (6.5) we get,

$$\int_{S_R}^{S_*}\left(\frac{\alpha+b\exp\left(\frac{S}{a}\right)}{\alpha-b\exp\left(\frac{S}{a}\right)}\right)\exp\left(\frac{-S}{a}\right)\mathrm{d}S=a\left[2b\ln\left|\alpha-b\exp\left(\frac{S}{a}\right)\right|-\frac{1}{\exp\left(\frac{S}{a}\right)}-2b\ln\left|\exp\left(\frac{S}{a}\right)\right|\right]_{S_R}^{S^*}$$

$$=a\left[2b\ln\left|\frac{\alpha-b\exp\left(\frac{S^*}{a}\right)}{\alpha-b\exp\left(\frac{S_R}{a}\right)}\right|-\frac{1}{\exp\left(\frac{S^*}{a}\right)}+\frac{1}{\exp\left(\frac{S_R}{a}\right)}+\frac{2b}{\alpha\,a}(S^*-S_R)\right] \quad (6.6)$$

similarly solving II part of equation(6.5) we get,

$$=-\frac{2a}{\alpha}\left[\ln\left|\alpha-b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a}-\ln\left|\alpha-b\exp\left(\frac{S_R}{a}\right)\right|\frac{S_R}{a}+\mathrm{Li}_2\left(b\exp\left(\frac{S^*}{a}\right)\right)\right.$$
$$\left.-\mathrm{Li}_2\left(b\exp\left(\frac{S_R}{a}\right)\right)\right)+\frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a-S^*)-\exp\left(\frac{-S_R}{a}\right)(a-S_R)\right)+\left(\frac{(S^*)^2-(S_R)^2}{a\alpha}\right)$$

adding I and II the solution for right rarefaction,

$$\beta^* = \beta_R - \frac{a}{b}\left[\frac{-2b}{\alpha}\ln\left|\frac{\alpha - b\exp\left(\frac{S^*}{a}\right)}{\alpha - b\exp\left(\frac{S_R}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_R}{a}\right)} + \frac{2b}{\alpha a}(S^* - S_R)\right]$$
$$-\frac{2a}{\alpha}\left[\ln\left|\alpha - b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a} - \ln\left|\alpha - b\exp\left(\frac{S_R}{a}\right)\right|\frac{S_R}{a} + \mathrm{Li}_2\left(b\exp\left(\frac{S^*}{a}\right)\right)\right.$$
$$\left. - \mathrm{Li}_2\left(b\exp\left(\frac{S_R}{a}\right)\right)\right) + \frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a - S^*) - \exp\left(\frac{-S_R}{a}\right)(a - S_R)\right) + \left(\frac{(S^*)^2 - (S_R)^2}{a\alpha}\right)$$

similarly for 2-Riemann invariant we have equation (6.2),

$$\beta^* - \beta_L - \int_{\left(\frac{S}{\xi}\right)_L}^{\left(\frac{S}{\xi}\right)_*}\left(\frac{\alpha - \xi}{\alpha + \xi}\right)\mathrm{d}\left(\frac{S}{\xi}\right) = 0$$

solving first integral,

$$\int_{\left(\frac{S}{\xi}\right)_L}^{\left(\frac{S}{\xi}\right)_*}\left(\frac{\alpha - \xi}{\alpha + \xi}\right)\mathrm{d}\left(\frac{S}{\xi}\right)$$

by using change of variable The above equation become

$$\int_{S_L}^{S_*}\left(\frac{\alpha - b\exp\left(\frac{S}{a}\right)}{\alpha + b\exp\left(\frac{S}{a}\right)}\right)\left(\frac{a - S}{ab}\right)\exp\left(\frac{-S}{a}\right)\mathrm{d}S$$

$$\frac{a}{ab}\int_{S_L}^{S_*}\left(\frac{\alpha - b\exp\left(\frac{S}{a}\right)}{\alpha + b\exp\left(\frac{S}{a}\right)}\right)\exp\left(\frac{-S}{a}\right)\mathrm{d}S - \frac{S}{ab}\int_{S_L}^{S_*}\left(\frac{\alpha - b\exp\left(\frac{S}{a}\right)}{\alpha + b\exp\left(\frac{S}{a}\right)}\right)\exp\left(\frac{-S}{a}\right)\mathrm{d}S \quad (6.7)$$

solving I part of above equation (6.7) we get,

$$\int_{S_L}^{S_*}\left(\frac{\alpha - b\exp\left(\frac{S}{a}\right)}{\alpha + b\exp\left(\frac{S}{a}\right)}\right)\exp\left(\frac{-S}{a}\right)\mathrm{d}S = a\left[\frac{2b}{\alpha}\ln\left|\alpha + b\exp\left(\frac{S}{a}\right)\right| - \frac{1}{\exp\left(\frac{S}{a}\right)} - \frac{2b}{\alpha a}\ln\left|\exp\left(\frac{S}{a}\right)\right|\right]_{S_L}^{S^*}$$

$$= a\left[\frac{2b}{\alpha}\ln\left|\frac{\alpha + b\exp\left(\frac{S^*}{a}\right)}{\alpha + b\exp\left(\frac{S_L}{a}\right)}\right| - \frac{1}{\exp\left(\frac{S^*}{a}\right)} + \frac{1}{\exp\left(\frac{S_L}{a}\right)} + \frac{2b}{\alpha a}(S^* - S_L)\right] \quad (6.8)$$

solving II part of equation (6.7),

$$
\begin{aligned}
&= +\frac{2a}{\alpha}\left[\ln\left|\alpha+b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a}-\ln\left|\alpha+b\exp\left(\frac{S_L}{a}\right)\right|\frac{S_L}{a}+\mathrm{Li}_2\left(-b\exp\left(\frac{S^*}{a}\right)\right)\right.\\
&-\mathrm{Li}_2\left(-b\exp\left(\frac{S_L}{a}\right)\right)+\frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a-S^*)-\exp\left(\frac{-S_L}{a}\right)(a-S_L)\right)-\left(\frac{(S^*)^2-(S_R)^2}{a\alpha}\right)
\end{aligned}
$$

adding I and II ,the solution for left rarefaction

$$
\begin{aligned}
\beta^* = \beta_L - \frac{a}{b}&\left[\frac{2b}{\alpha}\ln\left|\frac{\alpha+b\exp\left(\frac{S^*}{a}\right)}{\alpha+b\exp\left(\frac{S_L}{a}\right)}\right|-\frac{1}{\exp\left(\frac{S^*}{a}\right)}+\frac{1}{\exp\left(\frac{S_L}{a}\right)}-\frac{2b}{\alpha}(S^*-S_L)\right]\\
&+\frac{2a}{\alpha}\left[\ln\left|\alpha+b\exp\left(\frac{S^*}{a}\right)\right|\frac{S^*}{a}-\ln\left|\alpha+b\exp\left(\frac{S_L}{a}\right)\right|\frac{S_L}{a}+\mathrm{Li}_2\left(-b\exp\left(\frac{S^*}{a}\right)\right)\right.\\
&-\mathrm{Li}_2\left(-b\exp\left(\frac{S_L}{a}\right)\right)+\frac{1}{b}\left(\exp\left(\frac{-S^*}{a}\right)(a-S^*)-\exp\left(\frac{-S_L}{a}\right)(a-S_L)\right)-\left(\frac{(S^*)^2-(S_R)^2}{a\alpha}\right)
\end{aligned}
$$

6.2 Shock

we begin with The Rankine–Hugoniot condition,

$$s\,(q_* - q) = f\,(q_*) - f(q)$$

For Non-linear Green Naghdi type II equations the Rankine-Hugoniot relation reads

$$s\,(S_i - S_*) = (h_i - h_*) \tag{6.9}$$

$$s\,(\beta_i - \beta_*) = (T_i - T_*) \tag{6.10}$$

where s is shock speed and the subscript i $\in \{L,R\}$

$$s = \frac{(T_i - T_*)}{(\beta_i - \beta_*)}$$

eliminating s from equation (6.10)

$$\frac{(T_i - T_*)}{(\beta_i - \beta_*)}\,(S_i - S_*) = (h_i - h_*) \tag{6.11}$$

recalling relation from Green Naghdi model II we have,

$$T = \; T_0\,\mathrm{e}^{\left(\frac{S}{\rho C}\right)}$$

$$h = \frac{\widehat{K}\beta}{T_0}$$

inserting above relation into equation (6.11) finally we have,

$$\beta^* \quad = \quad \beta_i \quad \pm \; \frac{2T_0}{\sqrt{K}}\;\sqrt{S_i - S^*}\;\sqrt{\exp\left(\frac{S_i}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)}$$

left shock - The solution within the left shock hence satisfies:

$$\beta^* \quad = \quad \beta_L \quad - \; \frac{2T_0}{\sqrt{K}}\;\sqrt{S_L - S^*}\;\sqrt{\exp\left(\frac{S_L}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)} \tag{6.12}$$

right shock -The solution within the right shock hence satisfies:

$$\beta^* \quad = \quad \beta_R \quad + \quad \frac{2T_0}{\sqrt{K}} \; \sqrt{S_R - S^*} \; \sqrt{\exp\left(\frac{S_R}{\rho C}\right) - \exp\left(\frac{S^*}{\rho C}\right)} \tag{6.13}$$

6.3 Green Naghdi II equations

In these section the Green Naghdi II equations and some outcome relation is explained.

Conservation form of equation for type II dissipationless energy,

$$\frac{\partial S}{\partial t} + \frac{\partial h}{\partial x} = \frac{\rho r}{T} \tag{6.14}$$

$$\frac{\partial \beta}{\partial t} + \frac{\partial T}{\partial x} = 0 \tag{6.15}$$

The relation between U , W , S , β , T and q can be given,

$$h = \frac{\partial U}{\partial \beta} \qquad \beta = -\frac{\partial \alpha}{\partial x} \qquad T = \frac{\partial U}{\partial S} \qquad q = Th$$

$$U(S,\beta) \quad ; \quad W(T,\beta) \quad ; \quad S = -\frac{\partial W}{\partial T}$$

from type II non linear dissipationless theory we have free energy equation W and internal energy U as follows

$$\boxed{W(T,\beta) = \rho C \left((T - T_0) - T \ln\left(\frac{T}{T_0}\right) \right) + \frac{1}{2}\frac{\widehat{K}}{T_0}\beta^2} \tag{6.16}$$

$$S = -\frac{\partial W}{\partial T}$$

differentiation of equation (6.16) w.r.t T we get,

$$S = \rho C \ln\left(\frac{T}{T_0}\right)$$

$$T = \; T_0 \, \mathrm{e}^{\left(\frac{S}{\rho C}\right)}$$

The free energy potential is given,

$$U(S,\beta) \;\; = \;\; W(T,\beta) + TS$$

$$\boxed{U(S,\beta) = \rho C T_0 \left(\mathrm{e}^{\left(\frac{S}{\rho C}\right)} - 1 \right) + \frac{1}{2} \frac{\widehat{K}}{T_0} \beta^2} \tag{6.17}$$

$$h = \frac{\widehat{K}\beta}{T_0} \quad ; \quad S = \rho C \ln\left(\frac{T}{T_0}\right) \quad \frac{\partial S}{\partial t} = \frac{\rho C \dot{T}}{T} \quad ; \quad \dot{T} = \frac{\partial T}{\partial t}$$

our above equation

$$\frac{\partial S}{\partial t} + \frac{\partial h}{\partial x} = \frac{\rho r}{T}$$

become

$$\frac{\rho C \dot{T}}{T} + \frac{\widehat{K}}{T_0} \left(\frac{\partial \beta}{\partial x} \right) = \frac{\rho r}{T} \tag{6.18}$$

we know that

$$T = \frac{\partial \alpha}{\partial t} = \dot{\alpha} \quad ; \quad \beta = -\frac{\partial \alpha}{\partial x}$$

above equation become second order PDE

$$\boxed{\frac{\rho C \ddot{\alpha}}{\dot{\alpha}} - \frac{\widehat{K}}{T_0} \left(\frac{\partial^2 \alpha}{\partial x^2} \right) = \frac{\rho r}{T}} \tag{6.19}$$

The hyperbolic wave equation and the thermal displacement then travels as a wave with finite speed with no dissipation.