Calculus and its Applications
(Limits and Continuity - Integrable Functions)

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LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

TEXT BOOKS:

1. Thomas G B Jr., Joel Hass, Christopher Heil, Maurice D Wier, Thomas' Calculus, Pearson Education, 2018.

The definite integral

Consider the limit of general Riemann sums as the norm of the partitions of a closed interval $[a, b]$ approaches zero. This leads to the concept of the definite integral.

Definition Let $f(x)$ be defined on $[a, b]$. J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if: as $\|P\| \to 0$, $\sum_{k=1}^{n} f(c_k)\Delta x_k \to J$

Given any $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ with $\|P\|$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \varepsilon$$

- When the limit exists,

$$J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k$$

and we say that the definite integral exists.

- Leibniz introduced a notation for the definite integral as a limit of Riemann sums.
- He visualized the finite sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ as an infinite sum of function values $f(x)$ multiplied by infinitesimal subinterval widths dx. 'infinitely small'
- The sum symbol $\sum$ is replaced in the limit by the integral symbol $\int$, whose origin is in the letter "S" (for sum). elongated 's'
- The subinterval widths Δx_k become the differential dx.

- If the definite integral exists, then we write J as

$$\int_a^b f(x)dx.$$

- When the definite integral exists, we say that the Riemann sums of f on $[a, b]$ converge to $J = \int_a^b f(x)dx$ and f is integrable over $[a, b]$.

A Formula for the Riemann Sum with Equal-Width Subintervals

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$$

This is an explicit formula to compute definite integrals.

Integrable and Nonintegrable Functions

$|f(x)| \leq M$

- Not every function defined over a closed interval $[a, b]$ is integrable even if the function is bounded.
- Riemann sums for some functions might not converge to the same limiting value, or to any value at all.
- Every continuous function over $[a, b]$ is integrable over this interval, and so the functions with finite number of jump discontinuities.

Theorem: Integrability of Continuous Functions

If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

Remark

- When f is continuous we can choose each c_k so that $f(c_k)$ gives the maximum value of f on $[x_{k-1}, x_k]$, results in an upper sum.
- Likewise, we can choose c_k to give the minimum value of f on $[x_{k-1}, x_k]$ to obtain a lower sum.
- The upper and lower sums can be shown to converge to the same limiting value as the norm of the partition P tends to zero. as $\|P\| \to 0$
  lower sum $\to J$, upper sum $\to J$,
- Every Riemann sum is trapped between the values of the upper and lower sums, so every Riemann sum converges to the same limit as well.
- Therefore, J in the definition of the definite integral exists, and the continuous function f is integrable over $[a, b]$.
- For non-integrable, a function needs to be sufficiently discontinuous that the region between its graph and the x-axis cannot be approximated well by increasingly thin rectangles.

Problem

Show that the function

H.W

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

has no Riemann integral over $[0, 1]$.

f(x) is not continuous

f(x) has

Jump discontinuity

at $x = c$

either

$\lim_{x \to c^-} f(x) = f(c)$ or

left continuous

$\lim_{x \to c^+} f(x) = f(c)$

right continuous

Properties of Definite integrals

When f and g are integrable over the interval $[a, b]$ then

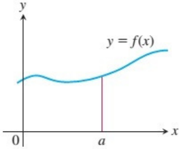
- Order of integration : $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- Zero width interval :$\int_a^a f(x)dx = 0$
- constant multiple :$\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$
- sum and difference : $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- Additivity : $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

- Max-min inequality : If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then

$$(\min\ f)\cdot(b-a) \leq \int_a^b f(x)dx \leq (\max\ f)\cdot(b-a)$$

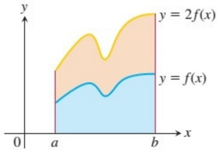
- Domination : If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

  If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x)dx \geq 0$.



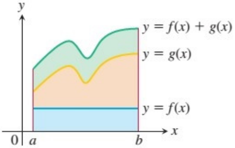
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



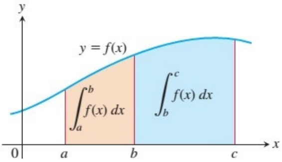
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



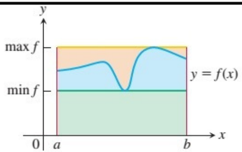
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



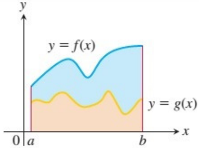
(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx$$
$$\le (\max f)\cdot(b-a)$$



(f) *Domination:*

If $f(x) \ge g(x)$ on $[a, b]$ then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

H.W

Problem Show that the value of $\int_0^1 \sqrt{1+\cos x}\,dx$ is less than or equal to $\sqrt{2}$.

$[0, 1]$

Soln:

$\int_0^1 \sqrt{1+\cos x}\,dx \leq \sqrt{2}$

$\min f\,(b-a) \leq \int_a^b f(x)\,dx \leq \max f\,(b-a)$

$a=0,\ b=1,\quad \max f = \sqrt{1+1} = \sqrt{2}$

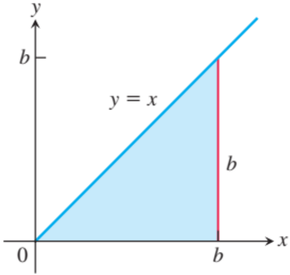
$\Rightarrow \int_0^1 \sqrt{1+\cos x}\,dx \leq \sqrt{2}$

Definition If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$A = \int_a^b f(x)dx.$$

te.w

Problem Compute $\int_0^b x dx$ and find the area A under $y = x$ over the interval $[0, b],\ b > 0$.



Definition If f is integrable on $[a, b]$, then its average value on $[a, b]$, which is also called its mean, is

$$av(f) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Problem Find the average value of $f(x) = \sqrt{24 - x^2}$ on $[-2, 2]$.

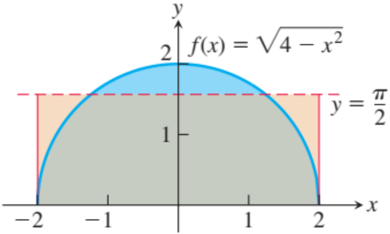


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

1 $\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^{\,2}\,\Delta x_k$, where P is a partition of $[0, 2]$

2 $\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k^{\,3}\,\Delta x_k$, where P is a partition of $[-1, 0]$

3 $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k^{\,2} - 3c_k)\,\Delta x_k$, where P is a partition of $[-7, 5]$

4 $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right)\Delta x_k$, where P is a partition of $[1, 4]$

5 $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k}\,\Delta x_k$, where P is a partition of $[2, 3]$

6 Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

find

a. $\int_2^2 g(x)\,dx$ **b.** $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$ **d.** $\int_2^5 f(x)\,dx$

e. $\int_1^5 [f(x) - g(x)]\,dx$ **f.** $\int_1^5 [4f(x) - g(x)]\,dx$

7 Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\int_1^2 f(u)\,du$ **b.** $\int_1^2 \sqrt{3}f(z)\,dz$

c. $\int_2^1 f(t)\,dt$ **d.** $\int_1^2 [-f(x)]\,dx$

8 Suppose that f is integrable and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a. $\int_3^4 f(z)\,dz$ **b.** $\int_4^3 f(t)\,dt$

9 Use known area formulas to evaluate the integrals

a. $\int_0^b \frac{x}{2}\,dx, \quad b > 0$ b. $\int_a^b 2s\,ds, \quad 0 < a < b$

10. What values of a and b maximize the value of

$$\int_a^b (x - x^2)\, dx?$$

(*Hint:* Where is the integrand positive?)

11. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2} dx.$$

THANK YOU