Calculus and its Applications
(Limits and Continuity - Combining Functions)

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LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

TEXT BOOKS:

1. Thomas G B Jr., Maurice D Wier, Joel Hass, Frank R. Giordano, Thomas' Calculus, Pearson Education, 2018.

Sums, Differences, Products, and Quotients

If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f+g$, $f-g$ and fg by the formulas

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \\ (f-g)(x) &= f(x)-g(x) \\ (fg)(x) &= f(x)g(x) \end{aligned}$$

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{where } g(x) \neq 0$$

EXAMPLE 1

$1-x \geq 0$, $1 \geq x$, $(-\infty, 1]$

$D(f) = [0,\infty)$, $D(g) = (-\infty, 1]$, $D(f) \cap D(g) = [0, 1]$

Let the functions be defined by the formulas $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$. Find the domains of $f(x)$ and $g(x)$. Also find the formulas and domains for the algebraic combinations defined below.

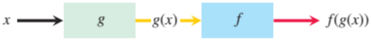
Function	Formula (x)	domain (y)
$f+g$	$\sqrt{x} + \sqrt{1-x}$	$D(f) \cap D(g) = [0,1]$
$f-g$	$\sqrt{x} - \sqrt{1-x}$	$[0,1]$
$g-f$	$\sqrt{1-x} - \sqrt{x}$	$[0,1]$
$f \cdot g$	$\sqrt{x}\sqrt{1-x}$	$[0,1]$
f/g	$\dfrac{\sqrt{x}}{\sqrt{1-x}}$	$[0,1)$, $x=1$ is excluded because $g(1)=0$
g/f	$\dfrac{\sqrt{1-x}}{\sqrt{x}}$	$(0,1]$, $x=0$ is excluded $f(0)=0$

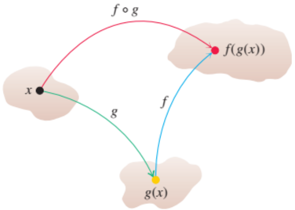
Composite functions

If f and g are functions, the composite function $f \circ g$ (f composed with g) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.





EXAMPLE 2

$f(y) = \sqrt{y}$ $\qquad g(y) = y+1$

If $f(x) = \sqrt{x}$ and $g(x) = x + 1$ find the formulas and domains for the following: $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$ and $(g \circ g)(x)$.

Domain

$f \circ g = f[x+1] = \sqrt{x+1}$ $\qquad x+1 \geq 0$, $x \geq -1$ $\qquad [-1, \infty)$

$g \circ f = g[\sqrt{x}] = \sqrt{x} + 1$ $\qquad [0, \infty)$

$f \circ f = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4} = \sqrt[4]{x}$ $\qquad [0, \infty)$

$g \circ g = g(x+1) = (x+1)+1 = x+2$ $\qquad (-\infty, \infty)$

$(f \circ g)(x) = f[g(x)] = f[\underbrace{x+1}_{y}] = f[y] = \sqrt{y} = \sqrt{x+1}$

$(g \circ f)(x) = g[f(x)] = g[\underbrace{\sqrt{x}}_{y}] = g[y] = y+1 = \sqrt{x}+1$

Shifting a graph of a function

$f(x)$

$y = x^2$

$y = x^2 + 1$

$y = x^2 - 1$

- **Vertical Shifts**

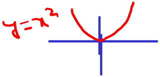
  $y = f(x) + k$ shifts the graph of f up k units if $k > 0$ or shifts it down $|k|$ units if $k < 0$

- **Horizontal Shifts**

  $y = f(x + h)$ shifts the graph of f left h units if $h > 0$ or shifts it right $|h|$ units if $h < 0$

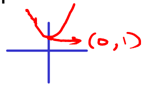
$y = (x+1)^2$

$y = (x-1)^2$

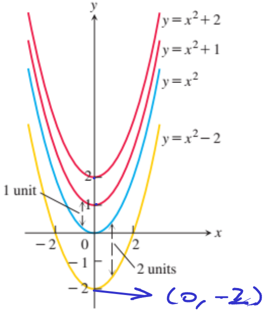


Example 3a

If $y = x^2$ then mention the type of shifts for the following operations and hence sketch the graph in each cases:



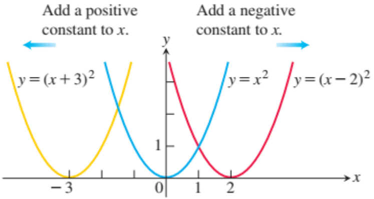
- adding 1 to the right hand side of y, $\Rightarrow y = x^2 + 1$ (VS)
- adding 2 to the right hand side of y, $\Rightarrow y = x^2 + 2$ (VS)
- adding -2 to the right hand side of y, $\Rightarrow y = x^2 - 2$ (VS)



Example 3b

If $y = x^2$ then mention the type of shifts for the following operations and hence sketch the graph in each cases: HSL

- adding 3 to x in $y = x^2$, $\Rightarrow$ $y = (x+3)^2$
- adding -2 to x in $y = x^2$. $\Rightarrow$ $y = (x-2)^2$



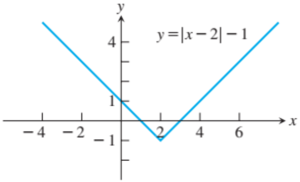
Example 3c

hw

If $y = |x|$ then mention the type of shifts when (i) -2 is added to x in y and then -1 (ii) is added to the result. Also sketch the graph for the above.

(i) $y = |x-2|$ shifted horizontally to the right by 2 units

(ii) $y = |x-2|-1$ shifted vertically downward by 1 unit



Practice problems

new

1. Find the domain and ranges of f, g $f+g$, $f \cdot g$, for $f(x) = x$ and $g(x) = \sqrt{x-1}$
2. Find the domain and ranges of f, g, f/g and g/f for $f(x) = \sqrt{x}$ and $g(x) = |x-3|$

1

function	domain	range
$f = x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$g = \sqrt{x-1}$	$[1, \infty)$	$[0, \infty)$
$f+g = x+\sqrt{x-1}$	$[1, \infty)$	$[1, \infty)$
$f \cdot g = x\sqrt{x-1}$	$[1, \infty)$	$[0, \infty)$

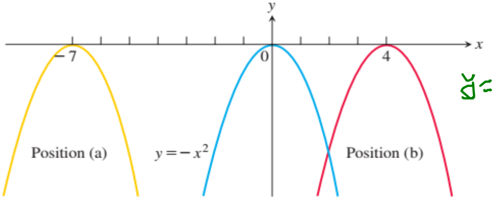
2

function	domain	range
$f = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$g = \lvert x-3 \rvert$	$(-\infty, \infty)$	$[0, \infty)$
$\frac{f}{g} = \frac{\sqrt{x}}{\lvert x-3 \rvert}$	$[0, \infty) - \{3\}$ or $[0,3) \cup (3, \infty)$	$[0, \infty)$
$\frac{g}{f} = \frac{\lvert x-3 \rvert}{\sqrt{x}}$	$(0, \infty)$	$[0, \infty)$

Shifting graphs

Graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs

$y = -(x+7)^2$

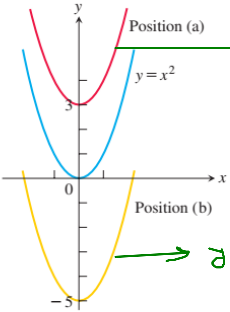


$y = -(x-4)^2$

Shifting graphs

hw

Graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs



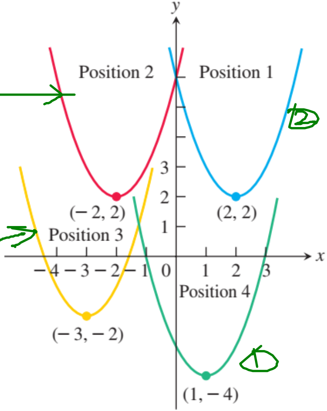
$y = x^2 + 3$

$y = x^2 - 5$

Shifting graphs

Match the equations listed in part (1)-(4) to the graphs in the following figure.

1. $y = (x-1)^2 - 4$
2. $(x-2)^2 + 2$
3. $y = (x+2)^2 + 2$
4. $y = (x+3)^2 - 2$



Vertical and Horizontal Scaling

For $c > 1$, the graph is scaled

- $y = cf(x)$ Stretches the graph of f vertically by a factor of c.
- $y = \frac{1}{c}f(x)$ Compresses the graph of f vertically by a factor of c.
- $y = f(cx)$ Compresses the graph of f horizontally by a factor of c.
- $y = f(x/c)$ Stretches the graph of f horizontally by a factor of c.

Reflection

For $c = -1$, the graph is reflected

- $y = -f(x)$ Reflects the graph of y across the x-axis.
- $y = f(-x)$ Reflects the graph of y across the y-axis

vertical scaling

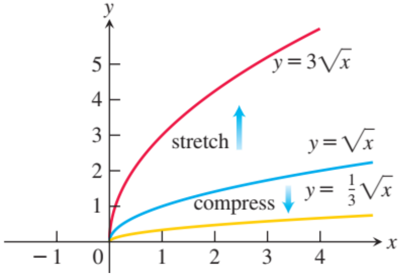
Example 4a Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases

- multiplying the right hand side of y by 3, $y = 3\sqrt{x}$
- multiplying the right hand side of y by $1/3$ $y = \frac{1}{3}\sqrt{x}$

$y = \sqrt{x}$

$x = 1 \Rightarrow y = 1$

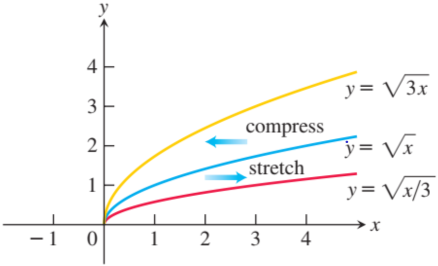
$y = 3\sqrt{x} \Rightarrow y = 3$



Horizontal scaling

Example 4b Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases

- multiplying x by 3 in the right hand side of y $y = \sqrt{3x}$
- multiplying x by $1/3$ in the right hand side of y $y = \sqrt{\frac{1}{3}x}$



Reflection

Example 4c Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases

- Reflects the graph of f across the x-axis. $y = -\sqrt{x}$
- Reflects the graph of f across the y-axis $y = \sqrt{-x}$

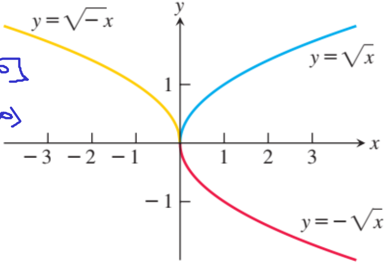
$y = \sqrt{-x}$

$-x \geq 0$

$x \leq 0$

Domain $= [-\infty, 0]$

Range $= [0, \infty)$



domain $=$ $[0, \infty)$

range $=$ $[0, \infty)$

domain$=$ $[0, \infty)$

range$=$ $(-\infty, 0]$

Example 5 Given the function $f(x) = x^4 - 4x^3 + 10$, find formulas to

$f(2x)$

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis. $f(-2x) =$

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis $-\frac{1}{2} f(x)$ $\frac{1}{2} f(x)$

(a) $f(2x) = 16x^4 - 32x^3 + 10$

$f(-2x) = 16x^4 + 32x^3 + 10$

(b) $\frac{1}{2} f(x) = \frac{1}{2} x^4 - 2x^3 + 5$

$-\frac{1}{2} f(x) = -\frac{1}{2} x^4 + 2x^3 - 5$

Periodic function

Period of $\sin nx = \frac{2\pi}{n}$

A function $f(x)$ is periodic if there is a positive number p such that

$$f(x+p) = f(x)$$

$\sin(x+4\pi) = \sin x$

$\sin(x+6\pi) = \sin x$

for every value of x. The smallest such value of p is the period of f.

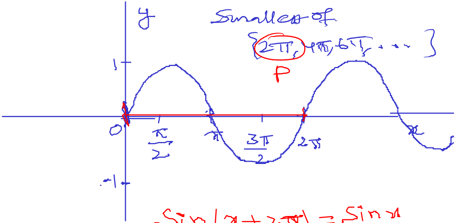
Eg:

Period

$\sin x = 2\pi$

$\cos x = 2\pi$

$\tan x = \pi$



$\sin(x+2\pi) = \sin x$

$\sin(90^\circ+\theta) = +\cos\theta$

sin +ve $90^\circ+\theta$ | 90° all +ve

cosec $180^\circ-\theta$ | $90^\circ-\theta$

$360+\theta$

180°

tan +ve $180^\circ+\theta$ | $270^\circ+\theta$ $0 = 360^\circ$

cot $270-\theta$ | $360^\circ-\theta$ cos +ve

270° sec

180° 360° same even

90° 270° odd

Practice problems

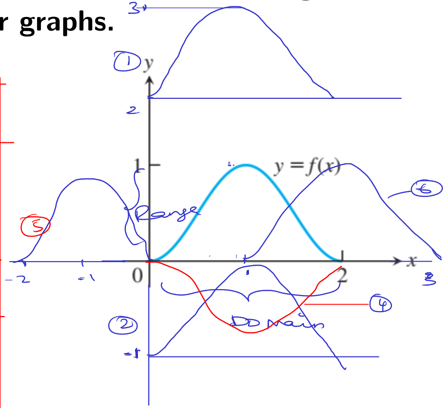
H.W

The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.

VSU – 2 units

		Domain	range
1	$f(x) + 2$	$[0,2]$	$[2,3]$
2	$f(x) - 1$ VSD	$[0,2]$	$[-1,0]$
3	$2f(x)$ V Sca stretch	$[0,2]$	$[0,2]$
4	$-f(x)$ Reflection about x-axis	$[0,2]$	$[-1,0]$
5	$f(x+2)$ HShift L	$[-2,0]$	$[0,1]$
6	$f(x-1)$ HShi R	$[1,3]$	$[0,1]$
7	$-f(x+1)+1$	$[-1,1]$	$[0,1]$

Ref about x HShift VShift



Practice problems

H.W

Find by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph

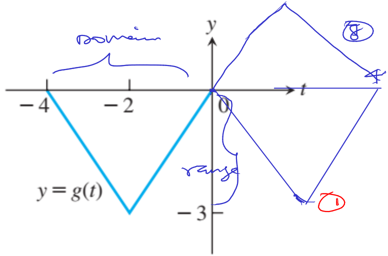
- $y = x^2 - 1$ stretched vertically by a factor of 3 $\Rightarrow$ $y = 3(x^2 - 1)$
- $y = x^2 - 1$ compressed horizontally by a factor of 2 $\Rightarrow$ $y = (2x)^2 - 1 = 4x^2 - 1$
- $y = 1 + \frac{1}{x^2}$ compressed vertically by a factor of 2 $\Rightarrow$ $y = \frac{1}{2}\left(1 + \frac{1}{x^2}\right)$
- $y = 1 + \frac{1}{x^2}$ stretched horizontally by a factor of 3 $\Rightarrow$ $y = 1 + \frac{1}{(x/3)^2} = 1 + \frac{9}{x^2}$
- $y = \sqrt{x+1}$ stretched vertically by a factor of 3 $\Rightarrow$ $y = 3(\sqrt{x+1})$
- $y = \sqrt{4 - x^2}$ stretched horizontally by a factor of 2 $y = \sqrt{4 - \left(\frac{x}{2}\right)^2}$
  $= \sqrt{4 - x^2/4}$

Practice problems

H.W

The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions and sketch their graphs

		Domain	range
1	$g(-t)$ refle -y'	[0, 4]	[-3, 0]
2	$-g(t)$ refl x-axis	[-4, 0]	[0, 3]
3	$g(t) + 3$	[-4, 0]	[0, 3]
4	$1 - g(t)$	[-4, 0]	[1, 4]
5	$g(-t + 2)$	[-2, 2]	[-3, 0]
6	$g(t - 2)$ H.shi ri	[-2, 2]	[-3, 0]
7	$g(1 - t)$	[-1, 3]	[-3, 0]
8	$-g(t - 4)$	[0, 4]	[0, 3]



Transformation of trigonometric graphs

$a > 1$ $a < 1$

Vertical stretch or compression; reflection about $y = d$ if negative

Vertical shift

$$y = af(b(x + c)) + d$$

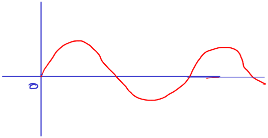
$b < 1$ $b > 1$

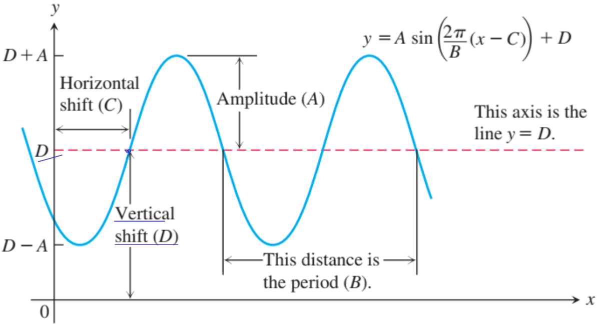
Horizontal stretch or compression; reflection about $x = -c$ if negative

Horizontal shift

General sine function or sinusoid formula

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - c)\right) + D$$





Problem

HW

Find the period of each function

1 - $\sin 2x$ π
2 - $\sin(x/2)$ 4π
3 - $\cos \pi x$ $\frac{2\pi}{\pi} = 2$
4 - $\cos\left(\frac{\pi x}{2}\right)$ $\frac{2\cancel{\pi}}{\cancel{\pi}/2} = 4$
5 - $\sin\left(x + \frac{\pi}{6}\right)$ 2π

(2) $\frac{2\pi}{n}$

$n = \frac{1}{2} \Rightarrow \frac{2\pi}{1/2} = 4\pi$

Practice problems

H.W

1. Graph the functions $y = 2\cos(x - \pi/3)$ and $y = 1 + \sin(x + \pi/4)$
2. Describe how each graph is obtained from the graph of $y = f(x)$

H.W

   - $y = f(x - 5)$
   - $y = f(4x)$
   - $y = f(-3x)$
   - $y = f(2x + 1)$
   - $y = f(x/3) - 4$
   - $y = -3f(x) + 1/4$

THANK YOU