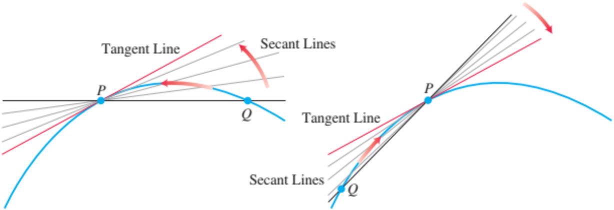
Calculus and its Applications

(Limits and Continuity - Limits)



LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

TEXT BOOKS:

1. Thomas G B Jr., Maurice D Wier, Joel Hass, Frank R. Giordano, Thomas' Calculus, Pearson Education, 2018.
2. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley, 2014.

Limit

$\lim_{x \to \infty} f(x) = a$

Small change in x produce only small change in $f(x)$.

$f(x) = \frac{1}{x} + 1$

$\lim_{x \to \infty} f(x) = 1$ Think!

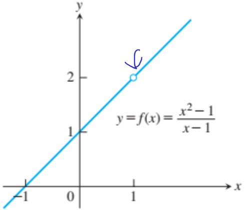
Range

Limit of a function

When studying a function $y = f(x)$, we find ourselves interested in the function's behavior near a point c but not at c itself.

Example 1

How does the function $f(x) = \frac{x^2-1}{x-1}$ behave near $x = 1$?

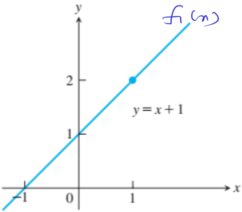


$= \frac{(x+1)(x-1)}{x-1}, x \neq 1$

$f_1(x) = x+1$

$D(f) = R - \{1\}$

$D(f_1) = R = (-\infty, \infty)$



Definition of limit of a function

may not be defined

Let $f(x)$ be a function defined on an interval about c, except possibly at c itself. If $f(x)$ is arbitrarily close to the number L for all x sufficiently close to c other than c itself, then we say that $f(x)$ approaches the limit L as x approaches c, that is,

$$\lim_{x \to c} f(x) = L$$

For $f(x) = \frac{x^2-1}{x-1}$,

$$\lim_{x \to 1} f(x) = 2.$$

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

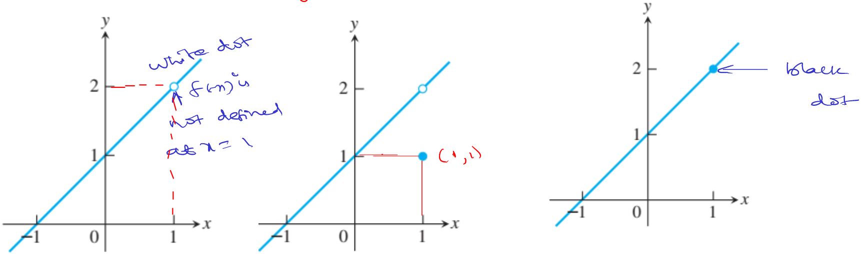
$$= \lim_{x \to 1} x+1$$

$$= 1+1 = 2$$

Example 2 Find the limits of the following functions as x approaches 1

$f(x) = \frac{x^2-1}{x-1}$ $g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1; \\ 1, & x = 1. \end{cases}$ $h(x) = x + 1$ $x \in (-\infty, \infty)$

$g(1) = 1$



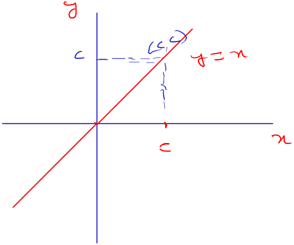
(a) $f(x) = \dfrac{x^2-1}{x-1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

Example 3 Find the limits of the identity function and of a constant function as x approaches c.

Identity fun $f(x) = x$

$$\lim_{x \to c} f(x) = \lim_{x \to c} x$$

$$= c$$



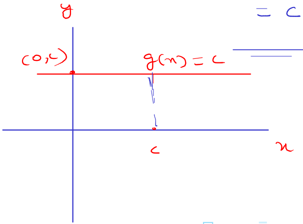
Constant function

$$g(x) = c$$

$$\lim_{x \to c} g(x) = \lim_{x \to c} c$$

$$= c \lim_{x \to c} 1$$

$$= c$$



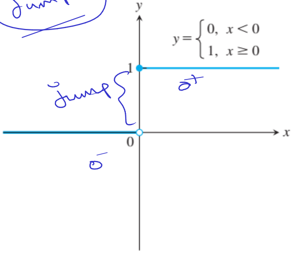
Example 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$

$$U(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases} \quad g(x) = \begin{cases} \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

$$f(x) = \begin{cases} 0, & x \leq 0; \\ \sin\left(\frac{1}{x}\right), & x > 0. \end{cases}$$

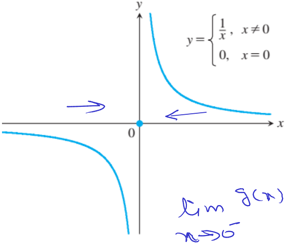
$g(0)=0$

jump



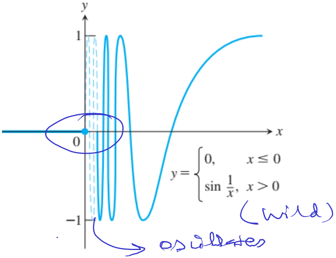
(a) Unit step function $U(x)$

$\lim_{x \to 0^+} g(x) = \infty$



(b) $g(x)$

$\lim_{x \to 0^-} g(x) = -\infty$



(c) $f(x)$

(wild)

oscillates too much near zero

Limit laws

Theorem If L, M, c and k are real numbers and $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$ then

- $\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$ $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$
- $\lim_{x \to c} cf(x) = cL$
- $\lim_{x \to c} f(x) \cdot g(x) = L \cdot M$
- $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$, $M \neq 0$ $\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n = L^n$
- $\lim_{x \to c} [f(x)]^n = L^n$, where n is the positive integer
  $= [\lim_{x\to c} f(x)]^{1/n} = L^{1/n} = \sqrt[n]{L}$
- $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$, where n is the positive integer
  (If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c)

Example 5 Find the limits of the following functions if $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$

1. $\lim_{x \to c} [x^3 + 4x^2 - 3]$ $= \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 + \lim_{x \to c} (-3) = c^3 + 4c^2 - 3$
2. $\lim_{x \to c} \left[\frac{x^4+x^2-1}{x^2+5}\right]$ $= \frac{c^4+c^2-1}{c^2+5}$
3. $\lim_{x \to -2} \sqrt{4x^2 - 3}$ $= \sqrt{\lim_{x \to -2} (4x^2 - 3)} = \sqrt{16-3} = \sqrt{13}$

Limits of polynomials If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\lim_{x\to c} p(x) = p(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0$.

Limits of rational functions If $p(x)$ and $q(x)$ are polynomials and $q(c) \neq 0$, then

$$\lim_{x\to c}\left[\frac{p(x)}{q(x)}\right] = \frac{p(c)}{q(c)}$$

$$\frac{\lim_{x\to c} p(x)}{\lim_{x\to c} q(x)}$$

HW

Example 6 Evaluate $\lim_{x\to -1} \frac{x^3+4x^2-3}{x^2+5}$. $= \frac{-1+4-3}{1+5} = \frac{0}{6} = 0$

Example 7 Evaluate $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$

$f(x) = \frac{x^2+x-2}{x^2-x}$ at $x=1$ not defined

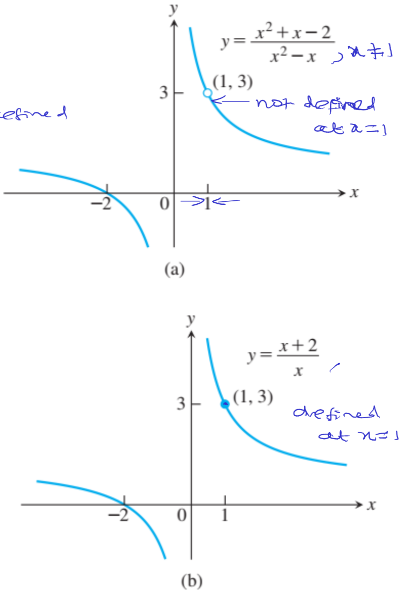
$f(x) = \frac{x^2+x-2}{x^2-x}$ $x \neq 1$

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}}$$

$$= 3$$

as $x \to 1$

$f(x) \to 3$



Sandwich theorem or Squeeze theorem

Set of all $y \in \mathbb{R} \ni a < y < b$

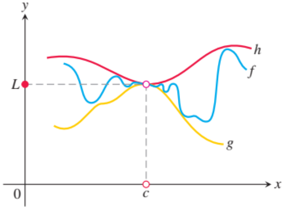
(a, b) $a<y<b$

$-\infty \quad a \quad b \quad \infty$

Suppose that $g(x) \leq f(x) \leq h(x)$ for every x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$ then

$$\lim_{x \to c} f(x) = L.$$

$\Rightarrow \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = \lim_{x \to c} f(x)$



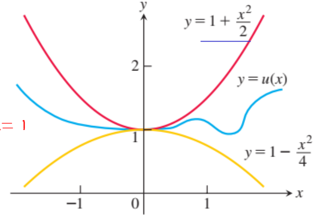
Example 8 Given a function u that satisfies

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{4}, \quad \forall x \neq 0$$

find $\lim_{x \to 0} u(x)$.

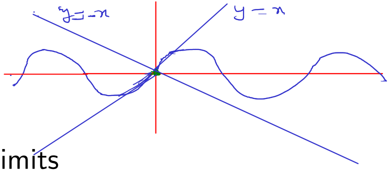
$\lim_{x \to 0} 1 - \frac{x^2}{4} = 1$ (L) $\quad \lim_{x \to 0} 1 + \frac{x^2}{4} = 1$ (L) $\Rightarrow L = 1$

$\lim_{x \to 0} u(x) = L = 1$



thank $g(x) \le \sin x \le h(x)$

$-x \le \sin x \le x$



Example 9 Evaluate the following limits

- $\lim\limits_{x\to 0} \sin\theta$ $= 0$
- thm $\lim\limits_{x\to 0} \cos\theta$ $= 1$ find $g(x)$ & $h(x)$ $\ni$ $g(x) \le \cos x \le h(x)$
- $\lim\limits_{x\to c} |f(x)| = 0$ implies $\lim\limits_{x\to c} f(x)$ $= 0$?

$-|f(x)| \le f(x) \le |f(x)|$

Problems

1. Check whether the limit exist for the following cases or not. In either case give reasons.
(a) $\lim\limits_{x\to 0} \frac{x}{|x|}$ and (b) $\lim\limits_{x\to 1} \frac{1}{x-1}$.

(a) $|x| = \begin{cases} x, & x>0 \\ 0, & x=0 \\ -x, & x<0 \end{cases}$

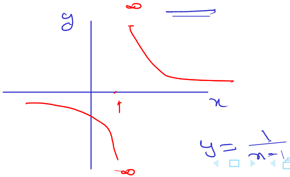
When we approach $x=0$ from the left $\lim\limits_{x\to 0} f(x) = -1$.

" right $\lim\limits_{x\to 0} f(x) = 1$

$\Rightarrow$ limit does not exist at $x=0$

(b) $\lim\limits_{x\to 1} \frac{1}{x-1}$

H shift right



limit does not exist

2. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x \to c} f(x)$? Give reasons for your answer.

$$f(x) = \begin{cases} \text{defined} & x \neq c \\ \text{not defined} & x = c \end{cases}$$

$\lim_{x \to c} f(x) = \,?$

HW

(i) limit exists

(ii) limit doesnot exists

(iii) no conclusion

✓ $f(x) = \dfrac{1}{x-1}$, limit doesnot exists at $x = 1$

✓ $f(x) = \dfrac{x^2-1}{x-1}$, $x \neq 1$ limit exists at $x = 1$

3. If $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$ for $-1 \leq x \leq 1$ find $\lim_{x\to 0} f(x)$.

$\lim_{x\to 0} f(x)$

$\lim_{x\to 0} \sqrt{5-2x^2} = \sqrt{5}$

11(equal) $\Rightarrow$ $\lim_{x\to 0} f(x) = \sqrt{5}$

$\lim_{x\to 0} \sqrt{5-x^2} = \sqrt{5}$

H.W

4. It can be shown that the inequalities $1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$ hold for all values of x close to zero. What, if anything does this tell you about $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$.

Definition of limit

(READ)

ε, δ are positive real numbers

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write $\lim_{x\to c} f(x) = L$, if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

range

domain

(a) $0 < |x-c| < \delta$

centre

radius

open

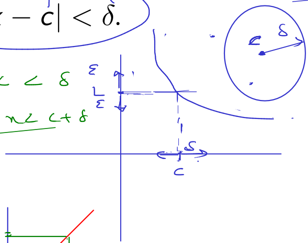
(b)

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad |x-c| < \delta.$$

$\lim_{x\to c} f(x) = L$

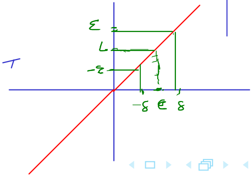
$-\delta < x-c < \delta$

$c-\delta < x < c+\delta$



H.W

Example Show that $\lim_{x\to 1}(5x-3) = 2.$



Approaching a limit from one side

$-\ c\ -$

Suppose a function f is defined on an interval that extends to both sides of a number c. In order for f to have a limit L as x approaches c, the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is two-sided.

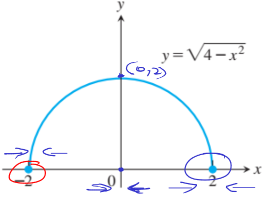
$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a right-hand limit or limit from the right. From the left, it is a left-hand limit or limit from the left.

Left-hand is denoted by $\lim\limits_{x \to c^-} f(x)$.
Right-hand is denoted by $\lim\limits_{x \to c^+} f(x)$.

$f(x) = x \quad (-\infty, \infty) \quad [-1, 1]$

Example 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$. Find the left-hand and right-hand limits of $f(x)$ as x approaches 2.



(a) $\lim_{x \to 2^-} f(x) = 0$

(b) $\lim_{x \to 2^+} f(x)$ does not exists (function is defined only on $[-2,2]$)

$\lim_{x \to -2^-} f(x)$ does not exists

$\lim_{x \to -2^+} f(x) = 0$

LHL = RHL (a) (b)

Theorem Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

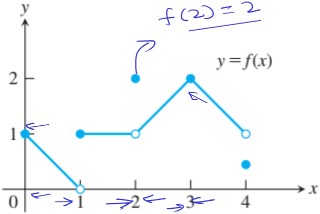
Note:

(i) above theorem is applicable when c is an interior point

(ii) At boundary points of its domain, function has a limit when its has an appropriate one sided limit

Example 2 Find left-hand limits, right-hand limits and limits at $x = 0, 1, 2, 3, 4$ for the function graphed below

$f(2) = 2$



Domain $= [0, 4]$

left end point $= 0$

right end point $= 4$

at $x = 0$ (left end point)

$$\lim_{x \to 0} f(x) = \lim_{x \to 0^+} f(x) = 1$$

at $x = 1$ (interior point)

$$\lim_{x \to 1^-} f(x) = 0, \quad \lim_{x \to 1^+} f(x) = 1 \Rightarrow LHL \neq RHL$$

$$0 \neq 1$$

$\Rightarrow$ limit not exists at $x = 1$

at $x = 2$ (interior point)

$$\lim_{x \to 2^-} f(x) = 1, \quad \lim_{x \to 2^+} f(x) = 1 \Rightarrow LHL = RHL$$

limit exists at $x = 2$

at $x = 3$ (interior point)

$LHL = 2$, $RHL = 2$ $\Rightarrow$ limit exists at $x = 3$

at $x = 4$ (end point)

$LHL = 1$, RHL (no need to find) $\Rightarrow$ limit exists at $x = 4$

If c is an interior point

$$\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

If c is a left end point

$$\lim_{x \to c} f(x) = \lim_{x \to c^+} f(x)$$

If c is a right end point

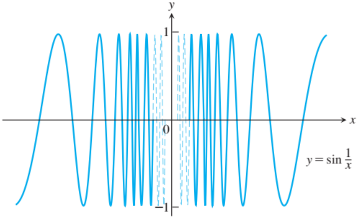
$$\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x)$$

H.W

Example 3 Find $\lim_{x \to 0^+} \sqrt{x}$.

H.W

Example 4 Show that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ has no limit as x approaches zero from either side.



Result $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$

H.W

Example 5 Show that (a) $\lim_{y \to 0} \frac{\cos y - 1}{y} = 0$ and (b) $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Example 6 Find $\lim_{t\to 0} \frac{\tan t \sec 2t}{3t}$.

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Soln

$$\lim_{t\to 0} \frac{\tan t \sec 2t}{3t} = \lim_{t\to 0} \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \cdot \frac{1}{3t}$$

$$= \frac{1}{3} \lim_{t\to 0} \underbrace{\frac{\sin t}{t}}_{\to 1} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t}$$

$$= \frac{1}{3}\left(\lim_{t\to 0} \frac{\sin t}{t}\right)\left(\lim_{t\to 0} \frac{1}{\cos t}\right)\left(\lim_{t\to 0} \frac{1}{\cos 2t}\right)$$

$$= \frac{1}{3}(1)(1)(1)$$

$$= \frac{1}{3}$$

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

a. $\lim_{x\to -1^+} f(x) = 1$ T

b. $\lim_{x\to 0^-} f(x) = 0$ T

c. $\lim_{x\to 0^-} f(x) = 1$ F

d. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ T

e. $\lim_{x\to 0} f(x)$ exists. T ⇒ interior point

f. $\lim_{x\to 0} f(x) = 0$ T

g. $\lim_{x\to 0} f(x) = 1$ F

h. $\lim_{x\to 1} f(x) = 1$ F

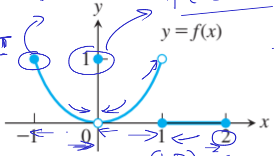
i. $\lim_{x\to 1} f(x) = 0$ F

j. $\lim_{x\to 2^-} f(x) = 2$ F

k. $\lim_{x\to -1^-} f(x)$ does not exist. T

l. $\lim_{x\to 2^+} f(x) = 0$ F

$f(-1)=1$ $f(0)=1$



(1, 0)

Domain = [−1, 2]

$f(1.5)=0$
$f(1.7)=0$
$f(1.9)=0$
$f(1.99)=0$
$\downarrow$ 2 $\downarrow$ 0

(e) $\lim_{x\to0} f(x)$ exists means $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = 0$

$0 = 0$ $\Rightarrow \lim_{x\to0} f(x) = 0$

(h) $\lim_{x\to1^-} f(x) = 1$ $\lim_{x\to1^+} f(x) = 0$ $\Rightarrow$ LHL $\neq$ RHL

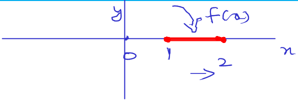
$\Rightarrow$ limit of f(x) doesnot exists at $x=1$

(j) $\lim_{x\to2^-} f(x) = 0$

$$\lim_{x \to 2^-} f(x)$$

x	$f(x)$
1.5	$f(1.5) = 0$
1.7	$f(1.7) = 0$
1.85	$f(1.85) = 0$
1.9	$f(1.9) = 0$
1.999	$f(1.999) = 0$
⋮	
1.99999	$f(1.99999) = 0$

approaches 2   approaches 0



Domain $[1, 2]$

Range $= \{0\}$

$f(x) = 0, \quad 1 \le x \le 2, \quad \forall x \in [1, 2]$

Note: Suppose If domain of $f(x)$ is $[a, b]$ then $\lim_{x \to a^-} f(x)$ does not exist & $\lim_{x \to b^+} f(x)$ does not exist

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

LHL=1, RHL =1 → limit exists

a. $\lim_{x\to -1^+} f(x) = 1$ T

b. $\lim_{x\to 2} f(x)$ does not exist. F

c. $\lim_{x\to 2} f(x) = 2$ F

d. $\lim_{x\to 1^-} f(x) = 2$ T

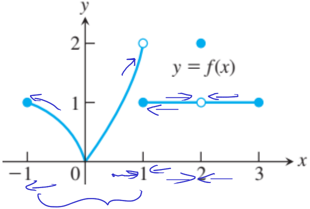
e. $\lim_{x\to 1^+} f(x) = 1$ T

f. $\lim_{x\to 1} f(x)$ does not exist. T

g. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$ T

h. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$. T

i. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$. T



j. $\lim_{x\to -1^-} f(x) = 0$

false

doesnot exist

k. $\lim_{x\to 3^+} f(x)$ does not exist.

True

as $x \to a$

$f(x) \to f(a)$

$x \to 1$

$f(x) \to 2$

THANK YOU