Sequence → list of ordered elements.
• every finite sequence is convergent.

Ch: Advanced Counting Techniques

Recurrence relation:

A relation in which n^{th} term of the sequence is defined using one or more of its preceding terms.

Why we study R.R?

→ Many counting problems in science and engineering are modelled as recurrence relations.

→ Recurrence relations can be solved systematically.

eg. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots c_k a_{n-k} + f(n)$

(Linear)

$c_1 a_{n-1} + c_2 a_{n-2} a_{n-3} + c_3 a_{n-3} \dots$

(Non-linear)

of degree k, $c_k \neq 0$.

Homogeneous for $f(n) = 0$

Non-homogeneous for $f(n) \neq 0$.

Solution of recurrence:

A sequence $\{a_n\}$ is said to be solution of the recurrence relation if its terms satisfy the equation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} \dots c_k a_{n-k}$.

Initial condition:

The initial condition for a sequence specifies the terms that precedes the first term where the recurrence relation takes effect.

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Eg: 1 Determine the terms of the sequence $\{a_n\}$, if $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4,$ etc. and suppose that $a_0 = 3$ and $a_1 = 5$.

$\{a_n\} = ?$ $\qquad a_0 = 3, a_1 = 5$

$a_n = a_{n-1} - a_{n-2}$ $\qquad n = 2, 3, 4, \ldots$

$n = 2, \quad a_2 = a_1 - a_0$

$= 5 - 3$

$a_2 = 2$

$a_3 = a_2 - a_1$

$a_3 = 2 - 5 = -3$

$a_4 = -5$

$\{a_n\} = \{3, 5, 2, -3, -5, \ldots\}$

Eg: 2 Determine whether the sequence $\{a_n\}$, $a_n = 3n$ is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

$a_{n-1} = 3(n-1) = 3n - 3$

$a_{n-2} = 3(n-2) = 3n - 6$

RHS: $2(3n-3) - (3n-6)$

$= 6n - \cancel{6} - 3n + \cancel{6}$

$= 3n = \text{LHS } (a_n)$

Thus, sequence is a solution.

Eg: Same as eg: 2, use $a_n = 2^n$ and $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

$a_{n-1} = 2^{n-1}$

$a_{n-2} = 2^{n-2}$

$$a_n = 2(2^{n-1}) - (2^{n-2})$$
$$= 2^n - \frac{2^n}{4}$$
$$= \frac{4 \cdot 2^n - 2^n}{4}$$
$$= \frac{3}{4} \cdot 2^n \neq a_n$$

$\therefore$ Not a solution.

Eg: 4 $a_n = 5$, $a_n = 2a_{n-1} - a_{n-2}$, $n = 2, 3, 4, \ldots$

$a_{n-1} = 5$
$a_{n-2} = 5$

RHS:

$$a_n = 2(5) - 5$$
$$= 10 - 5$$
$$= 5 \quad \therefore a_n = LHS$$

$\therefore$ It is a solution.

NOTE: A recurrence relation can have many solutions. But, a recurrence relation with initial condition always has unique solution.

Solving recurrence relation:

1) Homogeneous:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \cdots c_k a_{n-k}$$

$\{a_n\}$, $a_n = 0$ (trivial solution)

Assume $a_n = r^n$ be a trial solution, where r is a real number.

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots c_k r^{n-k}$$

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$\div$ by r^{n-k}

$$r^k = C_1 r^{k-1} + C_2 r^{k-2} + \dots C_k$$

$$r^k - C_1 r^{k-1} - C_2 r^{k-2} - \dots - C_k = 0$$

which is an algebraic equation in 'r' of degree 'k' and it has 'k' no. of solutions, namely $r_1, r_2, r_3, \dots r_k$.

Therefore, $r_1^n, r_2^n, \dots, r_k^n$ are solutions of given equation. In addition, $\alpha_1 r_1^n, \alpha_2 r_2^n, \dots \alpha_k r_k^n$ are also solutions of the equation.

The general solution is,

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots \alpha_k r_k^n.$$

Eg:1 What is the soln of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$. $a_0 = 2$, $a_1 = 7$.

$\to$ Degree = 2 (k value)

$$a_2 = a_1 + 2a_0$$
$$= 7 + 4 = 11$$

$$a_3 = a_2 + 2a_1$$
$$= 11 + 14 = 35$$

$$a_4 = a_3 + 2a_2$$
$$= 35 + 22 = 57.$$

$$a_n = 2, 7, 11, 35, 57, \dots$$

(or)

$$a_n = a_{n-1} + 2a_{n-2}$$

Put $a_n = r^n$

$$r^n = r^{n-1} + 2r^{n-2}$$

$\div$ by r^n,

$$1 = r^{-1} + 2r^{-2}$$

$$1 = \frac{1}{r} + \frac{2}{r^2} \qquad 1 = \frac{2+r}{r^2}$$

$$r^2 - r - 2 = 0$$

$(r+1)(r-2)=0$

$$\boxed{r_1=-1, \quad r_2=2}$$

General solution,

$$a_n = \alpha_1(-1)^n + \alpha_2(2)^n \quad -①$$

For $n=0$,

$$a_0 = \alpha_1(-1)^0 + \alpha_2(2)^0$$

$$2 = \alpha_1 + \alpha_2 \quad -②$$

For $n=1$,

$$a_1 = \alpha_1(-1)^1 + \alpha_2(2)^1$$

$$7 = -\alpha_1 + 2\alpha_2 \quad -③$$

From ② and ③,

$$\begin{aligned} -\cancel{\alpha_1} + 2\alpha_2 &= 7 \\ (+)\ \cancel{\alpha_1} + \alpha_2 &= 2 \\ \hline 3\alpha_2 &= 9 \end{aligned}$$

$\therefore \alpha_2 = 3$

$\Rightarrow \alpha_1 = -1$

Subs in ①,

$$a_n = (-1)(-1)^n + 3(2^n)$$

$$\boxed{a_n = 3(2^n) - (-1)^n}$$

Eg: 2 Find explicit formula for the following sequence 0, 1, 1, 2, 3, 5, 8, etc.

$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2, \ldots, a_6 = 8.$

$a_{100} = ? \quad a_{101} = ? \quad a_n = ?$

$a_n = a_{n-1} + a_{n-2} \Rightarrow$ Implicit formula

degree $= 2$

Put $a_n = r^n$

$$r^n = r^{n-1} + r^{n-2}$$

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$\div$ by r^n

$$1 = r^{-1} + r^{-2}$$

$$1 = \frac{1}{r} + \frac{1}{r^2}$$

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

$$r = \frac{-(-1) \pm \sqrt{1+4}}{2}$$

$$\boxed{r = \frac{1 \pm \sqrt{5}}{2}}$$

General solution,

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

For $n=0$,

$$a_0 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\alpha_1 + \alpha_2 = 0$$

$$\alpha_1 = -\alpha_2$$

For $n=1$,

$$a_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 - \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)$$

$$1 - \frac{\alpha_2}{2} + \frac{\alpha_2\sqrt{5}}{2} = -\frac{\alpha_2}{2}(1+\sqrt{5})$$

$$1 - \cancel{\frac{\alpha_2}{2}} + \frac{\alpha_2\sqrt{5}}{2} = -\cancel{\frac{\alpha_2}{2}} - \frac{\alpha_2\sqrt{5}}{2}$$

$$\alpha_2\sqrt{5} = -1$$

$$\alpha_2 = \frac{-1}{\sqrt{5}}$$

$$\Rightarrow \alpha_1 = \frac{1}{\sqrt{5}}$$

$\therefore$ General solution is,

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Note :

→ The algebraic equation we get "$r^k - c_1 r^{k-1} - c_2 r^{k-2} - c_k = 0$" is called characteristic equation and its roots $r_1, r_2, \ldots, r_k$ is called characteristic roots.

→ If 2 characteristic roots are equal, say $r_1 = r_2$, then the general solution is $(\alpha_1 + \alpha_2 n) r^n + \alpha_3 r_3^n + \alpha_4 r_4^n + \ldots \alpha_k r_k^n$

Say $r_1 = r_2 = r_3 = r, r_4, r_5, \ldots r_k$

$GS = (\alpha_1 + \alpha_2 n + \alpha_3 n^2) r^n + \alpha_4 r_4^n + \ldots \alpha_k r_k^n$

and so on.

Eg:3 What is soln of $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$, $a_1 = 6$.

$$a_n = 6a_{n-1} - 9a_{n-2}$$

degree 'k' = 2

Put $a_n = r^n$

$$r^n = 6r^{n-1} - 9r^{n-2}$$

÷ by r^n

$$1 = 6r^{-1} - 9r^{-2}$$

$$1 = \frac{6}{r} - \frac{9}{r^2}$$

$$r^2 = 6r - 9$$

$r^2 - 6r + 9 = 0$ → characteristic eqn

$$(r-3)(r-3) = 0$$

$\boxed{r = 3}$ → characteristic root

$r_1 = 3$, $r_2 = 3$.

3 is the characteristic root with multiplicity 2

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Note:

Suppose a recurrence relation has characteristic roots 5 and 7 with multiplicity 4 and 2, general soln is:

$$a_n = (\alpha_1 + \alpha_2 n + \alpha_3 n^2 + \alpha_4 n^3)5^n + (\alpha_5 + \alpha_6 n)7^n$$

Eg: 4 What is general form of soln of linear homogeneous recurrence relation if its characteristic eqn has roots 1, 1, 1, 1, -2, -2, -2, 3, 3, -4 ?

$$a_n = (\alpha_1 + \alpha_2 n + \alpha_3 n^2 + \alpha_4 n^3)1^n + (\alpha_5 + \alpha_6 n + \alpha_7 n^2)(-2^n) + (\alpha_8 + \alpha_9 n)3^n + (\alpha_{10})(-4^n).$$

Eg: 5 Find soln of recurrence relation

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

degree = 3

Put $a_n = r^n$

$$r^n = 6r^{n-1} - 11r^{n-2} + 6r^{n-3}$$

÷ by r^n

$$1 = 6r^{-1} - 11r^{-2} + 6r^{-3}$$

$$1 = \frac{6}{r} - \frac{11}{r^2} + \frac{6}{r^3}$$

$$r^3 - 6r^2 + 11r - 6 = 0$$

1
$x - 1$
-1
$x + 1$

$$\begin{array}{r|rrrr} -1 & 1 & -6 & 11 & -6 \\ & 0 & -1 & -4 & \\ \hline & 1 & & 7 & \end{array}$$

$$\begin{array}{l} r^2 - 5r + 6 \\ r - 1 \overline{\big)\, r^3 - 6r^2 + 11r - 6} \\ \underline{r^3 - r^2} \\ -5r^2 + 11r \\ \underline{-5r^2 + 5r} \\ 6r - 6 \\ \underline{6r - 6} \\ 0 \end{array}$$

$$r^3 - 6r^2 + 11r - 6 = (r-1)(r^2 - 5r + 6)$$

$$= (r-1)(r-2)(r-3)$$

$r_1 = 1$, $r_2 = 2$, $r_3 = 3$.

General soln.

$$a_n = \alpha_1 (1)^n + \alpha_2 (2)^n + \alpha_3 (3)^n$$

For $n = 0$.

$$\alpha_1 + \alpha_2 + \alpha_3 = 2$$

For $n = 1$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$

For $n = 2$,

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15.$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 2$$
$$-\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$
$$-\alpha_2 - 2\alpha_3 = -3$$

$$\alpha_2 + 2\alpha_3 = 3 \quad — 1$$

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$$
$$-\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$
$$2\alpha_2 + 6\alpha_3 = 10$$
$$\alpha_2 + 3\alpha_3 = 5 \quad — 2$$

From 1 & 2,

$$\cancel{\alpha_2} + 2\alpha_3 = 3$$
$$-\ \cancel{\alpha_2} + 3\alpha_3 = 5$$
$$-\alpha_3 = -2$$

$$\boxed{\alpha_3 = 2}$$

$$\boxed{\alpha_2 = -1}$$

$$\boxed{\alpha_1 = 1}$$

Subs in GS,

$$a_n = 1^n - 2^n + 2(3^n)$$

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Solving linear non-homogeneous equations:

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots c_k a_{n-k} + F(n)$

$c_k \neq 0$ & $F(n) \neq 0$.

$F(n) = (b_0 + b_1 n + b_2 n^2 \dots b_i n^i) s^n$

$b_0, b_1, \dots b_i, s$ are real

~~G.S~~ GS: $a_n = a_n^{(h)} + a_n^{(P)}$

where $a_n^{(h)}$ is the solution of associated homogeneous equation and $a_n^{(P)}$ is the particular solution of the non-homogeneous equation.

$a_n^{(P)}$ is of the form,

case: (i) $a_n^{(P)} = (P_0 + P_1 n + P_2 n^2 + \dots P_i n^i) s^n$

provided, s is not a characteristic root.

Suppose, s is a characteristic root of multiplicity m, then the particular solution is of the form,

case: (ii)

$a_n^{(P)} = n^m (P_0 + P_1 n + P_2 n^2 + \dots P_i n^i) s^n$

Eg: 1 Solve the following recurrence relation,

$a_n = 3a_{n-1} + 2n$; $a_1 = 3$

↳ non-homogeneous [$F(n) = 2n$]

GS: $a_n = a_n^{(h)} + a_n^{(P)}$ — ①

To find $a_n^{(h)}$

Consider $a_n = 3a_{n-1}$

Put $a_n = r^n$

$r^n = 3r^{n-1}$

$\div$ by r^n

$1 = \frac{3}{r}$ => $\boxed{r = 3}$

$a_n^{(h)} = \alpha_1 3^n$

To find $a_n^{(P)}$

$F(n) = 2n$

$= (b_0 + b_1 n) s^n$

where $b_0 = 0$, $b_1 = 2$, $s = 1$

$2n = (0 + 2n) 1^n$

As $s \neq 3$

$a_n^{(P)} = (P_0 + P_1 n) 1^n$ [from case : i]

$a_n = P_0 + P_1 n$

$a_{n-1} = P_0 + P_1 (n-1)$

Given,

$a_n = 3a_{n-1} + 2n$

$3[P_0 + P_1(n-1)] + 2n = P_0 + P_1 n$

$P_0 + P_1 n = 3P_0 + 3P_1 n - 3P_1 + 2n$

$= (3P_0 - 3P_1) + (3P_1 + 2)n$

$P_0 = 3P_0 - 3P_1$; $P_1 = 3P_1 + 2$

$P_0 = -3/2$ $\qquad$ $P_1 = -1$

$\Rightarrow a_n^{(P)} = (-3/2 - n) 1^n$

GS: $a_n = \alpha_1 3^n - n - 3/2$

as $a_1 = 3$,

$n = 1$ in GS,

$a_1 = \alpha_1 3 - 1 - 3/2$

$3 + 1 + 3/2 = 3\alpha_1$

$3\alpha_1 = \cancel{\frac{7}{2}}\ \frac{11}{2}$

$\alpha_1 = \frac{11}{6}$

$\therefore a_n = \frac{11}{6} 3^n - n - \frac{3}{2}$

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Eg: 2 $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$F(n) = 7^n$

GS: $a_n = a_n^{(h)} + a_n^{(p)}$

To find $a_n^{(h)}$,

consider $a_n = 5a_{n-1} - 6a_{n-2}$

Put $a_n = r^n$

$r^n = 5r^{n-1} - 6r^{n-2}$

$\div$ by r^n

$1 = \frac{5}{r} - \frac{6}{r^2}$

$r^2 = 5r - 6$

$r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0$

$r_1 = 3,\ r_2 = 2$

G.S: $a_n = \alpha_1 (3)^n + \alpha_2 (2)^n$

To find $a_n^{(P)}$,

$F(n) = 7^n$

$= (b_0) s^n$

$b_0 = 1,\quad s = 7$

As $s \neq 2$ or $s \neq 3$,

$a_n^{(P)} = (P_0)7^n$

$a_{n-1} = (P_0)7^{n-1}$

$a_{n-2} = (P_0)7^{n-2}$

Given,

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$(P_0)7^n = 5P_0 7^{n-1} - 6P_0 7^{n-2} + 7^n$

$P_0 7^n = 7^n(\frac{5}{7}P_0 - \frac{6}{49}P_0 + 1)$

$P_0 \cancel{7^n} = \cancel{7^n}(\frac{29}{49}P_0 + 1)$

$\frac{20P_0}{49} = 1$

$P_0 = \frac{49}{20}$ $\Rightarrow a_n^{(P)} = (\frac{49}{20})7^n$

$\therefore a_n = \alpha_1 (3)^n + \alpha_2 (2)^n + \left(\frac{49}{20}\right) 7^n$

Eg: 3 solve $a_{n+2} - 3a_{n+1} + 2a_n = 3^n$

degree $= 2$

Put $a_n = r^n$

$$a_n = \frac{3a_{n+1} - a_{n+2} + 3^n}{2}$$

$$r^n = \frac{3r^{n+1} - r^{n+2}}{2}$$

$\div$ by r^n

$$2 = 3r - r^2$$

$$r^2 - 3r + 2 = 0$$

$$\cancel{r = \frac{3 \pm \sqrt{9-4}}{2}}$$

$$\cancel{r = \frac{3 \pm \sqrt{5}}{2}}$$

$$(r-2)(r-1) = 0$$

$$r_1 = 1, \quad r_2 = 2$$

$$a_n^{(h)} = \alpha_1 (1)^n + \alpha_2 (2)^n$$

To find $a_n^{(P)}$,

$$F(n) = 3^n$$
$$= (b_0) s^n$$
$$b_0 = 1$$
$$s = 3$$

As $s \neq 1$ or $s \neq 2$,

$$a_n^{(P)} = (P_0) 3^n$$
$$a_{n+1} = (P_0) 3^{n+1}$$
$$a_{n+2} = (P_0)^{3n+2}$$

$$a_n = \frac{1}{2}(3a_{n+1} - a_{n+2} + 3^n)$$

$$(P_0) 3^n = \frac{1}{2}\left(3P_0 \cdot 3^{n+1} - (P_0)^{3n+2} + 3^n\right)$$

$$\cancel{3^n} P_0 = \frac{1}{2} \cancel{3^n} (9P_0 - 9P_0 + 1)$$

$$\boxed{P_0 = 1/2}$$

$a_n^{(P)} = \left(\frac{1}{2}\right)3^n$

$\therefore \ a_n = \alpha_1(1)^n + \alpha_2(2)^n + \left(\frac{1}{2}\right)3^n$

Eg: 4 $\quad a_{n+2} - 5a_{n+1} + 6a_n = 5^n$

$6a_n = 5a_{n+1} - a_{n+2} + 5^n$

Put $a_n = r^n$

$6r^n = 5r^{n+1} - r^{n+2}$

$\div$ by r^n,

$6 = 5r - r^2$

$r^2 - 5r + 6 = 0$

$(r-2)(r-3)$

$r_1 = 2, \quad r_2 = 3$

$a_n^{(h)} = \alpha_1(2)^n + \alpha_2(3)^n$

$F(n) = 5^n$

$= (b_0 + b_1 n + b_2 n^2 \dots b_i n^i)\,5^n$

$b_0 = 1$, rest all $= 0$, $s = 5$.

As $s \neq 2$ or $s \neq 3$,

$a_n^{(P)} = (P_0)5^n$

$a_{n+1} = (P_0)5^{n+1}$

$a_{n+2} = (P_0)5^{n+2}$

Subs in a_n,

$\therefore \ 6(P_0)5^n = 5P_0(5^{n+1}) - (P_0)5^{n+2} + 5^n$

$6(P_0)\cancel{5^n} = \cancel{5^n}(25P_0 - 25P_0 + 1)$

$\boxed{P_0 = \frac{1}{6}}$

$a_n^{(P)} = \left(\frac{1}{6}\right)5^n$

$\therefore \ a_n = \alpha_1(2)^n + \alpha_2(3^n) + \left(\frac{1}{6}\right)5^n$

Eg:5 $a_{n+2} - 2a_{n+1} + a_n = 3n+5$

$a_n = 2a_{n+1} - a_{n+2} + 3n+5$

For $a_n^{(h)}$

$a_n = 2a_{n+1} - a_{n+2}$

Put $a_n = r^n$

$r^n = 2 r^{n+1} - r^{n+2}$

$\div$ by r^n

$1 = 2r - r^2$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r_1 = 1,\ r_2 = 1$

$a_n^{(h)} = \alpha_1 + \alpha_2.$

For $a_n^{(p)}$

$F(n) = 3n+5$

$= (b_0 + b_1 n) s^n$

$s = 1,\ b_0 = 5,\ b_1 = 3$

$\because s = 1,$

$a_n^{(p)} = n^2 (P_0 + P_1 n) \cancel{1^n}$

$a_n^{(p)} = n^2 (P_0 + P_1 n)$

$a_{n+1} = (n+1)^2 (P_0 + P_1 (n+1))$

$a_{n+2} = (n+2)^2 (P_0 + P_1 (n+2))$

$\therefore n^2(P_0 + P_1 n) = 2(n+1)^2 (P_0 + P_1 n + P_1) - (n+2)^2 (P_0 + P_1 (n+2)) + 3n+5$

$n^2(P_0 + P_1 n) = 2(n^2 + 2n + 1)(P_0 + P_1 n + P_1) - (n^2 + 4n + 4)(P_0 + P_1 n + 2P_1) + 3n+5$

$\cancel{n^2}(P_0 + P_1 n) = \cancel{n^2}(2 + \frac{4}{n} + 2)(P_0 + P_1 n + P_1) - \cancel{n^2}(1 + \frac{4}{n} + \frac{4}{n^2})(P_0 + P_1 n + 2P_1) + \cancel{n^2}(\frac{3}{n} + \frac{5}{n^2})$

$$P_0 + P_1 n = \left(4+\frac{4}{n}\right)(P_0+P_1 n+P_1) - \left(1+\frac{4}{n}+\frac{4}{n^2}\right)(P_0+P_1 n+2P_1) + \frac{3}{n} + \frac{5}{n^2}$$

$$P_0+P_1 n = 4P_0 + 4P_1 n + 4\cancel{P_1} + \frac{4}{n}\cancel{P_0} + \frac{4}{\cancel{n}}P_1 \cancel{n} + \frac{4}{\cancel{n}}\cancel{P_1} + \frac{3}{n}+\frac{5}{n^2}$$

$$- \cancel{P_0} - P_1 n - 2P_1 - \frac{4}{n}\cancel{P_0} - 4\cancel{P_1} - \frac{8}{n}P_1 - \frac{4}{n^2}P_0$$

$$- \frac{4}{\cancel{n}}\cancel{P_1} - \frac{8}{n^2}P_1$$

$$P_0+P_1 n = 3P_0 + 4P_1 n + 4P_1 + \frac{3}{n}+\frac{5}{n^2} - P_1 n - 2P_1$$

$$- \frac{8}{n}P_1 - \frac{4}{n^2}P_0 - \frac{8}{n^2}P_1$$

$$P_0+P_1 n = 3P_0 + 3P_1 n + 2P_1 + \frac{3}{n} + \frac{5}{n^2} - \frac{8}{n}P_1$$

$$- \frac{4}{n^2}P_0 - \frac{8}{n^2}P_1$$

$$P_0 n^2 + P_1 n^3 = 3P_0 n^2 + 3P_1 n^3 + 2P_1 n^2 + 3n + 5 - 8P_1 n$$

$$- 4P_0 - 8P_1$$

$$-3n-5 = 2P_0 n^2 + 2P_1 n^3 + 2P_1 n^2 - 8P_1 n - 4P_0 - 8P_1$$

$$-3n-5 = \cancel{n(2P_1 n^2 + 2P_0 n} - 8P_1 n - 4P_0 - 8P_1$$

$$(8P_1 n) + (4P_0 + 8P_1) = 3n+5$$

$$8P_1 = 3$$

$$\boxed{P_1 = 3/8} \qquad P_1 = 1/2$$

$$P_0 = 1$$

$$4P_0 + 3 = 5$$

$$\boxed{P_0 = 1/2} \quad \times$$

* Equate coeff.

Formulation of Recurrence relation:

Q. Suppose that a person deposits 10,000 Rs in a savings acct in a bank yielding 11% per year with interest compounded annually. How much will be there after 30 years?

Let a_n be the amount in the account at n^{th} year

$n=0$, $a_0 = 10,000$

$n=1$, $a_1 = 10,000 + \left(\frac{11}{100} \times 10,000\right)$

$$a_1 = a_0 + \frac{11}{100} a_0$$

$$a_1 = a_0 \left(1 + \frac{11}{100}\right)$$

$$a_2 = a_1 + \frac{11}{100} \times a_1$$

$$a_2 = a_1 \left(1 + \frac{11}{100}\right)$$

$$a_3 = a_2 \left(1 + \frac{11}{100}\right) \quad \text{--- (1)}$$

$\therefore$ $\boxed{A = P\left(1 + \frac{r}{100}\right)^n}$ $\rightarrow$ CI formula.

Deducing from (1),

$$a_n = a_{n-1}\left(1 + \frac{11}{100}\right)$$

$$a_n = (1.11) a_{n-1}$$

Put $a_n = r^n$ $\qquad\qquad$ $a_0 = 10,000$

$$r^n = (1.11)\, r^{n-1}$$

$\div$ by r^n

$$1 = \frac{1.11}{r}$$

$$\boxed{r = 1.11}$$

$$a_n = \alpha_1 (1.11)^n$$

$n = 0$,

$$a_0 = \alpha_1 (1.11)^0$$

$$\Rightarrow \alpha_1 = 10{,}000$$

$$a_n = 10{,}000 (1.11)^n$$

For $n = 30$,

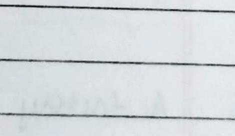
$$a_{30} = 10{,}000 (1.11)^{30}$$

—x—

3. Tower of Hanoi:

Let 'a_n' be the no. of moves required to transfer the discs.

Constraints: No larger disc is kept on smaller one. No two discs are allowed to shift at a time.



$n = 1$ $\quad a_n = 1$

$n = 2$ $\quad a_n = 1 + 1 + 1$

$n = 3$ $\quad a_n = 3 + 1 + 3$

$n = 4$ $\quad a_n = 7 + 1 + 7$

$$\therefore a_n = a_{n-1} + 1 + a_{n-1}$$

$$\boxed{a_n = 2a_{n-1} + 1}, \quad a_1 = 1$$

$$a_n = a_n(h) + a_n(p)$$

1) To find $a_n(h)$

$$a_n = 2a_{n-1}$$

Put $a_n = r^n$

$$r^n = 2r^{n-1}$$

$\div$ by r^n

$$1 = \frac{2}{r} \Rightarrow \boxed{r = 2}$$

$$a_n(h) = \alpha_1 2^n$$

ii) To find $a_n(P)$

$$F(n) = 1$$
$$= (b_0) s^n$$
$$b_0 = 1, \quad s = 1$$

$$\therefore a_n(P) = (P_0) 1^n$$
$$a_{n-1} = P_0$$

Subs,

$$P_0 = 2P_0 + 1$$
$$-P_0 = 1$$
$$\boxed{P_0 = -1}$$

$$\boxed{a_n = \alpha_1 2^n - 1}$$

$$a_1 = 2\alpha_1 - 1$$
$$2 = 2\alpha_1$$
$$\boxed{\alpha_1 = 1}$$

$$\therefore \boxed{a_n = 2^n - 1}$$

Q. A factory makes custom sports car at increasing rate. In the first month, only one car is made. In 2nd month, 2 cars are made and so on. n cars are made in n^{th} month. Model and find how many cars are produced in the first year?

$$a_n = a_{n-1} + n \quad , \quad a_1 = 1$$

i) $a_n(h)$,

$$a_n = a_{n-1}$$

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$r^n = r^{n-1}$

$\div$ by r^n

$1 = \frac{1}{r}$

$\boxed{r = 1}$

$\therefore a_n(h) = \alpha_1 (1)^n = \alpha_1$

ii) $a_n(P)$

$\cancel{a_n} = F(n) = n$

$= (\cancel{b_0} + b_1 n) s^n$

$b_0 = 0 \quad b_1 = 1 \quad s = 1$

$\because s = 1,$

$\therefore a_n(P) = n(P_0 + P_1 n) 1^n$

$a_{n-1} = (n-1)(P_0 + P_1 n - P_1)$

$n(P_0 + P_1 n) = (n-1)(P_0 + P_1 n - P_1) + n$

$n(P_0 + P_1 n) = nP_0 + P_1 n^2 - P_1 n - P_0 - P_1 n + P_1 + n$

$n\cancel{P_0} + n\cancel{P_1} n = n\cancel{P_0} + P_1 \cancel{n^2} - P_1 n - P_0 - P_1 n + P_1 + n$

$P_0 + 2P_1 n - P_1 = n$

$2P_1 = 1$

$\boxed{P_1 = 1/2} \quad \boxed{P_0 = +1/2}$

$a_n(P) = \left(\frac{1}{2} + \frac{n}{2}\right)$

$a_n = \alpha_1 + \left(\frac{1}{2} + \frac{n}{2}\right)$

$\because a_1 = 1,$

$1 = \alpha_1 + \frac{1}{2}$

Q. Find recurrence relation and give initial conditions for the no of bit strings of length n, that do not have 2 consecutive zeroes. How many such bit strings are there of length 5?

Let a_n be no. of bit strings of length n which has no consecutive 0's.

$\therefore$ a_{n-1} is the no. of valid bit strings of length n-1

Construction of valid bit string can be done in 2 cases

i) Append '1' with a valid bitstring of $l = n-1$.

ii) Append '0' with a valid bitstring of $l = n-1$ that does not end with 0.

$\boxed{a_5 = 13}$

Case (i) $a_n = a_{n-1}$

case (ii) $a_n = a_{n-1} + a_{n-2}$

$a_n = \cancel{2}a_{n-1} + a_{n-2}$

$r^n = \cancel{2}r^{n-1} + r^{n-2}$

$1 = \frac{1}{r} + \frac{1}{r^2}$

$r^2 - r - 1 = 0$

$r = \frac{2 \pm \sqrt{1 - 4(-1)(1)}}{2} = \frac{2 \pm \sqrt{5}}{2}$

$a_n(h) = \alpha_1\left(\frac{2+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{2-\sqrt{5}}{2}\right)^n$

For $n = 1$,

$a_1 = 2$.

$2 = \alpha_1\left(\frac{2+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{2-\sqrt{5}}{2}\right)$

$a_5 = a_4 + a_3$

$= a_3 + a_2 + a_3$

$= 2a_3 + a_2$

$= 2a_2 + 2a_1 + a_2$

$3a_2 + 2a_1$

$= 3a_1 + 3a_0 + 2a_1 = 5a_1 = 10$

For $n = 0$,

$a_0 = 0 \quad \Rightarrow \quad \alpha_1 + \alpha_2 = 0$

Solving,

$-\alpha_2\left(\frac{2+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{2-\sqrt{5}}{2}\right) = 2$

$-\frac{2\alpha_2}{2} - \frac{\sqrt{5}\alpha_2}{2} + \frac{2\alpha_2}{2} - \frac{\sqrt{5}\alpha_2}{2}$

$-\sqrt{5}\alpha_2 = 2 \qquad \boxed{\alpha_2 = -2/\sqrt{5}} \quad \boxed{\alpha_1 = 2/\sqrt{5}}$

$$a_n = \frac{2}{\sqrt{5}}\left(\frac{2+\sqrt{5}}{2}\right)^n - \frac{2}{\sqrt{5}}\left(\frac{2-\sqrt{5}}{2}\right)^n$$

a_5 = FML.

—x—

8. A comp system considers a string of decimal digits. A valid code word if it contains even no. of 0's. Let a_n be the no. of valid n digit code words. Find a recurrence relation for a_n. (For eg, 1230407869 is valid, 120987040406 is invalid).