Calculus and its Applications
(Limits and Continuity - Riemann Sums)

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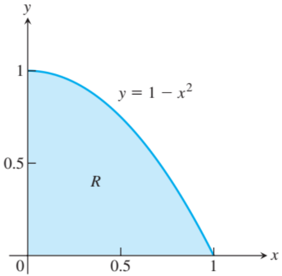
LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

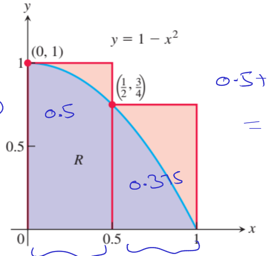
TEXT BOOK~~S~~:

1. Thomas G B Jr., Joel Hass, Christopher Heil, Maurice D Wier, Thomas' Calculus, Pearson Education, 2018.



$$\int_0^1 (1-x^2)\,dx = \frac{2}{3} = 0.667.$$

①



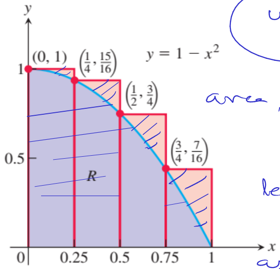
(a)

0.5 + 0.375

= 0.875

0.875 >

②



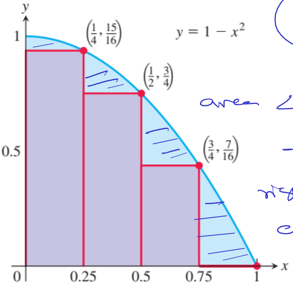
(b)

upper sum

area > 0.667

left end point as height of the rectangle

③



[0, 0.25]

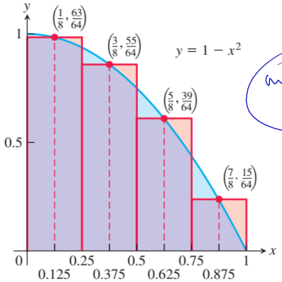
(a)

lower sum

area < 0.667

right end point

④



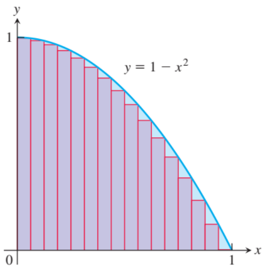
(b)

mid point sum

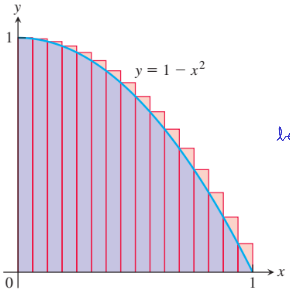
[0, 0.25]

area(3) < area(4) < area(2)

left end point



(a)



(b)

left end point

actual area $\simeq$ 0.667

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

Finite sums

index ends at $k=n$

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

sigma

formula for the k^{th} term.

index starts at $k=1$

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$1+3+\cdots+9 = \sum_{n=1}^{5}(2n-1)$$

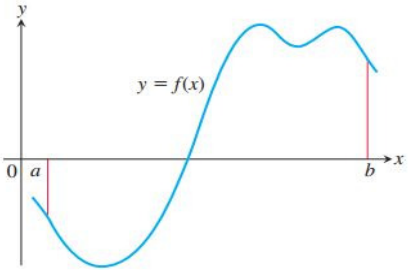
$$\sum_{n=0}^{4}(2n+1)$$

Algebra rules for finite sums

- sum rule : $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
- difference rule : $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
- constant multiple rule : $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$
- constant value rule : $\sum_{k=1}^{n} c = n \cdot c$

$\sum_{k=1}^{10} 1 = 10$

The theory of limits of finite approximations was made precise by the German Mathematician Bernhard Riemann. The notion of a Riemann sum underlies the theory of the definite integral. ?



- Begin with an arbitrary bounded function f defined on a closed interval $[a, b]$.
- f may have negative as well as positive values.
- Subdivide the interval $[a, b]$ into subintervals, not necessarily of equal widths (or lengths), and form sums in the same way as for the finite approximations.

- Choose $n-1$ points $x_1, x_2, \cdots, x_{n-1}$ between a and b that are in increasing order, so that $a < x_1 < x_2 < \cdots < x_{n-1} < b$.
- Set $x_0 = a$ and $x_n = b$, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

$[0,1]$

- The set of all of these points,

$[0, 0.5]$ $[0.5, 1]$

$$P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\},$$

$P = \{0, 0.5, 1\}$

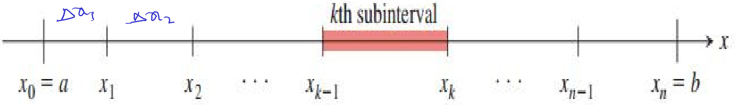
is called a partition of $[a, b]$.

- The partition P divides $[a, b]$ into the n closed subintervals

$$[x_0, x_1], [x_1, x_2], \cdots [x_{n-1}, x_n].$$

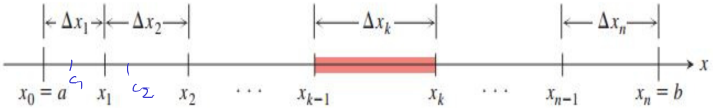
- The first of these subintervals is $[x_0, x_1]$,the second is $[x_1, x_2]$, and the k^{th} subinterval is $[x_{k-1}, x_k]$ (where k is an integer between 1 and n).

$\Delta x_k = x_k - x_{k-1}$



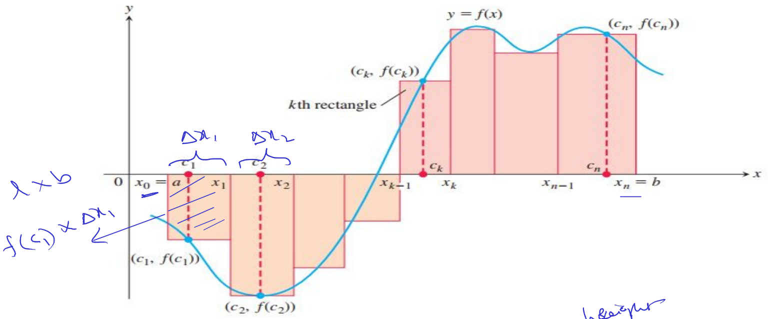
$[a, b]$

The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the k^{th} subinterval is $\Delta x_k = x_k - x_{k-1}$.



$\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \Delta x$

- If all n subintervals have equal width, then $\Delta x = (b - a)/n$.
- In each subinterval we select some point.
- The point chosen in the k^{th} subinterval $[x_{k-1}, x_k]$ is called c_k.
- Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$.
- These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on the x-axis if $f(c_k) = 0$.



→ height → width

- On each subinterval, form the product $f(c_k) \cdot \Delta x_k$ which is positive, negative, or zero, depending on the sign of $f(c_k)$.
- When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k.
- When $f(c_k) < 0$,the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

- Next, sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

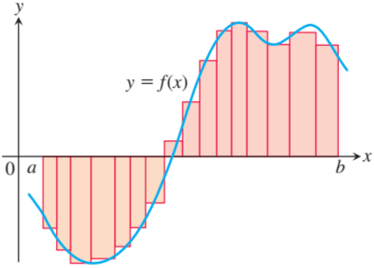
- S_P is called a Riemann sum for f on the interval $[a, b]$.
- There are many such sums, depending on the partition P we choose, and the choices of the points c_k in the subintervals.
- For instance, we could choose n subintervals all having equal width $\Delta x = (b-a)/n$ to partition $[a, b]$, and then choose the point c_k to be the right-hand endpoint of each subinterval when forming the Riemann sum.
- This choice leads to the Riemann sum formula

$\Delta x_k = \Delta x$

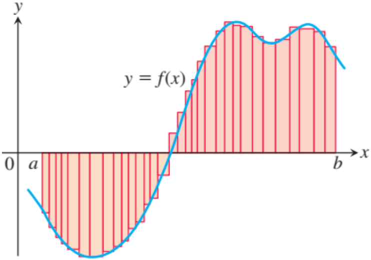
$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right) \cdot \left(\frac{b-a}{n}\right).$$

Check!

- Similar formulas can be obtained if instead we choose c_k to be the left-hand endpoint, or the midpoint, of each subinterval.
- If all subintervals have equal width $\Delta x = \frac{(b-a)}{n}$, we can make them thinner by increasing their number n.
- When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval.
- We define the norm of a partition P, written $\|P\|$, to be the largest of all the subinterval widths.
- If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width.



(a)

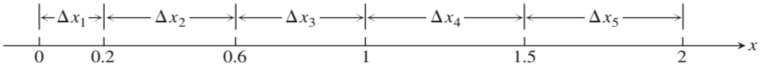


(b)

Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

Problem

- The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$.
- There are five subintervals of P : $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$
- The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$.
- The longest subinterval length is 0.5, so the norm of the partition is $\|P\| =0.5$.



Remark

- Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function f and the x-axis.
- Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy.
- If the function f is continuous over the closed interval $[a, b]$, then no matter how we choose the partition P and the points c_k in its subintervals, the Riemann sums corresponding to these choices will approach a single limiting value as the subinterval widths (which are controlled by the norm of the partition) approach zero.

THANK YOU