DISCRETE STRUCTURE

METHODS OF PROOF

DIFFERENT METHODS OF PROOF :-

* DIRECT PROOF
* INDIRECT PROOF
* — i] PROOF BY CONTRAPOSITION
  — ii] PROOF BY CONTRADICTION
* VACOUS PROOF
* TRIVIAL PROOF
* PROOF BY COUNTER EXAMPLES
* PROOF BY EXHAUSTION
* PROOF BY CASES
* MATHEMATICAL INDUCTION (WEAK)
* STRONG MATHEMATICAL INDUCTION

THEOREM : we can show that the set is true

LEMMA : are kind of theorem used to prove the theorem.

COROLLARY : consequence of theorems, proofing a statement using the theorem

CONJECTURE : it is a statement which cannot be proven false, let assume it as true.

AXIOM : with the help of axioms we can prove the theorem.

Even $n = 2k$ (some integer k)

odd $n = 2k+1$

perfect square $n = a^2$

rational no $n = \frac{p}{q}$ where $q \neq 0$, $p, q \in Z$

$\therefore \gcd(p, q) = 1$

DIRECT PROOF:-

A direct proof shows that a conditional statement $P \to Q$ is true by showing that if P is true then Q must also be true.

STEPS TO PROVE:-

* Assume P is true
* use P to show that Q must also be true.

PROBLEM:-

If $0 < a < b$ then $a^2 < b^2$ where a and b are integers.

PROOF:-

Let a and b are integer and $0 < a < b$ then $a < b$.

Multiply both sides by a

$a^2 < ab \to ①$

Multiply both sides by 'b'

$ab < b^2 \to ②$

By ① and ②,

$a^2 < ab < b^2$

$\Rightarrow a^2 < b^2$

Hence proved

Problem :-

The sum of two even numbers x and y is even.

PROOF :-

P : x and y are even numbers

Q : $x + y$ is even

Claim,

$P \rightarrow Q$ is true

Assume : P is true

i.e x and y are even numbers

x is even $\Rightarrow x = 2k$ (k and m are integers)

y is even $\Rightarrow y = 2m$

Consider, $x + y = 2k + 2m$

$= 2(k+m)$ $(k+m = n)$ where n is a integer

$= 2n$

Exercise:-

1. If n is odd integer then n^2 is also an odd integer.

2. If a divides b and * a divides c then" a divides b+c "

3. If a is an integer, divisible by 4, a is the difference of two perfect square.

INDIRECT PROOF :-

Proofs of theorem which do not start with the hypothesis and end with conclusion are called indirect proof.

Two types of indirect proof :-

* Proof by contraposition
* Proof by contradiction

PROOF BY CONTRAPOSITION :-

Use $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$

$\downarrow$

Start with negation of conclusion and end with negation of premises.

PROBLEM :-

Prove that if $x^3 < 0$ then $x < 0$

Proof :-

Here $P : x^3 < 0$ and $Q = x < 0$

proof by contraposition,

Assume $\neg Q$ i.e Assume $x \geq 0$

Multiply x^2 on both sides,

$$x \geq 0$$

$$x^3 \geq 0 \times x^2 = 0$$

Hence $x^3 \geq 0$

QUESTION :- Let $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even then x is odd.

Proof :-

Here $P : x^2 - 6x + 5$ is even

$Q : x$ is odd

proof by contraposition,

Assume $\neg Q$. i.e Assume x is even

$\Rightarrow$ $x = 2k$ for some integer k, $(k \in Z)$

to proove $\neg P : x^2 - 6x + 5$ is odd

$\therefore$ Since x is even, product of two even number is always even

So x^2 is even and

$6x$ is also even

$\therefore$ $x^2 - 6x$ is also even

Since we add odd no 5 to even no it becomes odd.

$\therefore$ Hence $\neg P : x^2 - 6x + 5$ is odd

(or)

when $x = 2k$

$$\neg P = x^2 - 6x + 5$$
$$= (2k)^2 - 6(2k) + 5$$
$$= 4k^2 - 12k + 4 + 1$$
$$= 4(k^2 - 3k + 1) + 1$$
$$= 2(2k^2 - 6k + 2) + 1$$

Since $2k^2 - 6k + 2$ is an integer,

$2(2k^2 - 6k + 2) + 1$ it is odd.

Hence $x^2 - 6x + 5$ is odd.

Proof by Contradiction:-

The idea of proof by contradiction is to assume our statement is false, and then show that

This leads to a contradiction or something that is obviously not true.

STEPS TO PROVE :-

1) Assume that P is true

2) Assume that $\neg Q$ is true

3) use P and $\neg Q$ to derive a contradiction.

NOTATION :- $\Rightarrow \Leftarrow$

Prove that $\sqrt{2}$ is an irrational number

Proof :-

Assume that $\sqrt{2}$ is an rational

$\Rightarrow \sqrt{2} = \frac{p}{q}$ where $q \neq 0$, $\gcd(p,q) = 1$

Squaring on both sides,

$$2 = \frac{p^2}{q^2}$$

$$p^2 = 2q^2$$

$\Rightarrow$ p is an even number

$$p = 2k, \ k \in \mathbb{Z}$$

$$\Rightarrow p^2 = 4k^2$$

$$2q^2 = \cancel{2}p^2$$

$$= 4k^2$$

$$\boxed{q^2 = 2k^2}$$

$\Rightarrow$ q is also even

Say $q = 2m$

$\gcd(p,q) = \gcd(2k, 2m) \geq 2$

This is contradiction $\Rightarrow\Leftarrow$, since $\gcd(n,m) \neq 1$, it should be 1.

Ques: proof that there is an infinite even number

Ques: prove that for all integer n, if n^2 is odd then n is ~~even~~ odd

Proof :-

Here n^2 is odd the n is odd

Assume that n^2 is odd the n is even

Since n is even, $n = 2K, K \in \mathbb{Z}$

$$n^2 = (2K)^2 = 4K^2$$

$$n^2 = 2(2K^2)$$

$$\Rightarrow n^2 \text{ is even}$$

$\Rightarrow\Leftarrow$ where n^2 should be odd

$\therefore$ Hence n^2 is odd the n is odd.

Exercise:- (2)

PROOF BY CONTRAPOSITION

1. Suppose $x \in \mathbb{Z}$, If $7x+9$ is even then x is odd.
2. If a^2 is not divisible by 4 then a is odd.

PROVE BY CONTRADICTION

1. If x is a real number such that $x^2 = 2$, then x is an irrational number.
2. For all real number x and y if $x \neq y$, $x>0$ and $y>0$ then $\frac{x}{y} + \frac{y}{x} > 2$

Exercise :1:-

1. SOLUTION:-

p: n is an odd Integer

q: n^2 is also an odd Integer

To prove:-

$P \to Q$ is true

Assume P is true

$n = 2K - 1$

$n^2 = (2K-1)^2$

$n^2 = 4K^2 \pm 4K + 1$

Since both $4K^2$ and $4K$ are even

$4K^2 - 4K$ is also even so adding 1 makes it an odd value

Hence n is an odd integer then n^2 is odd.

2. SOLUTION:-

p: a divides b and a divides c

q: a divides b+c

To prove:

$p \to q$ is true

Assume p is true

a divides b	a divides c
$\Rightarrow \frac{b}{a} = k$ $\to$ ①	$\frac{c}{a} = 1$ $\to$ ②

Add ① and ②

$\frac{b+c}{a} = 1+k \to$ ③

From ③ we proved that

a divides b and a divides c then a divides b+c.

3. SOLUTION:-

GIVEN:-

p : a is an integer divisible by 4

q : a is the difference of two perfect squares.

To Prove:

$p \to q$ is true

$\begin{bmatrix} 4 = 2^2 - 0^2 \\ 8 = 3^2 - 1^2 \\ 12 = 4^2 - 2^2 \end{bmatrix}$

Assume that p is true

$a \in Z$

$\boxed{a = 4K}$ where $K \in Z$

$= (K+1)^2 - (K-1)^2$

From theorem it is proved that difference between the squares of two odd consecutive number is divisible by 4.

Hence proved.

$\frac{4}{x} = K$

$4 =$

$\frac{x}{4} = K \Rightarrow x = 4K$

$x = 4K = (K+1)^2 - (K-1)^2$

$= [(K+1) - (K-1)][(K+1) + (K-1)]$

$= 2 \cdot (2K)$

$= 4K.$

EXERCISE : 2 :-

1. SOLUTION:-

P : $7x+9$ is even

Q : x is odd

To prove

∴ By using contraposition

Assume that $\neg Q$ is true

∴ x is even

to prove: $7x+9$ is odd

since x is even, $7x$ is also even

but $7x+9$ becomes odd

Hence $\neg p$: $7x+9$ is odd.

2. SOLUTION:-

p : a^2 is not divisible by 4

q : a is odd

To prove using contraposition,

Assume $\neg Q$ is true

$\neg Q$: a is even

To prove that $\neg p$: a^2 is divisible by 4

Since $a = 2k$ $\quad k \in Z$

$a^2 = 4k^2$ $\quad k \in Z$

Since a^2 is multiple of 4 it is divisible by 4.

Hence $\neg p$: a^2 is divisible by 4.

1. SOLUTION:-

p : x is a real number such that $x^2 = 2$ then x is a rational number.

Proof :-

x is a real number

$x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$

which implies that x is cannot be a rational number

$\Rightarrow \Leftarrow$ where x should be an irrational number

$\therefore$ Hence x is a real number such that $x^2 = 2$ then x is a rational number.

2. SOLUTION:-

For all $\mathbb{R}$ $x \neq y$ where $x > 0$ and $y > 0$ then

$\frac{x}{y} + \frac{y}{x} < 2$

Since for all $x>0$ and $y>0$

$x \neq y$

Let us take an example

$x=3,\ y=2$ where $x \neq y$

$\frac{x}{y} = \frac{3}{2} = 1.5$

$\frac{y}{x} = \frac{2}{3} = 0.67$

$\frac{x}{y} + \frac{y}{x} = 1.5 + 0.67 = 2.17 > 2$

$\Rightarrow\Leftarrow \quad \frac{x}{y} + \frac{y}{x} < 2$

$\therefore$ Hence when $x>0$ and $y>0$, $x \neq y$ then

$\frac{x}{y} + \frac{y}{x} \geq 2.$

MISTAKES IN PROOF :-

PROOF :- A proof is a valid argument that establish the truth of theorem.

1. Let $a, b, c \in Z$, If $a \mid bc$ then either $a \mid b$ or $a \mid c$.

PROOF :- Let $a=5,\ b=3$ and $c=10$. Then $5 \mid 30$ Also, since $5 \mid 10$. It is true that $a \mid c$, so the claim is true.

- This proof is wrong.

2. Let $a, b, c \in Z$ where $a \equiv 1 \pmod 3$, $b = 2 \pmod 3$ then $(a+b) = 0 \pmod 3$

Since $a \equiv 1 \pmod 3$ there is an integer k such that $a = 3k+1$

Since $b = 2 \pmod 3$ there is an integer k such that $b = 3k+2$

$$a + b = 3K+2 + 3K+1$$
$$= 6K+3$$
$$= 3(2K+1)$$

So $a+b \equiv 0 \pmod 3$

$\therefore$ The proof is correct.

3. Show that $1 = 2$:

PROOF :-

Let $a \in \mathbb{Z}$ and $b = a$

$$a = b$$
$$a^2 = ab$$
$$a^2 - b^2 = ab - b^2$$
$$(a+b)(a-b) = b(a-b) \implies$$ Since $a-b=0$, we cannot cancel 0 on both sides.
$$a+b = b$$
$$2b = b$$
$$2 = 1$$

– This proof is wrong.

4. Show that if x is real number then x^2 is positive

PROOF :-

There are two cases

CASE : i]

x is positive

Product of two positive number's positive.

So x^2 is positive.

case : ii]

x is negative

Product of two negative numbers is negative

So x^2 is negative.

Thus we obtain the same conclusion in all cases, so that the original statement is true.

5. Prove that n is even iff n^2 is even.

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

p : n is even

q : n^2 is even

case : i : $p \to q$:

If n is even,

$$n = 2K$$

$$n^2 = (2K)^2 = 4K^2$$

$$n^2 = 2(2K^2) \text{ is even}$$

$$p \to q \equiv T \longrightarrow ①$$

case : ii) $q \to p$

By contrapositive:

$$q \to p \equiv q \wedge \neg p$$

when q and $\neg p$ are true,

when n^2 is even,

$$n^2 = 2K \qquad K \in Z$$

when n is odd, $n = 2l+1 \qquad l \in Z$

$$q \wedge \neg p \Rightarrow \quad n^2 = (2l+1)(2l+1) \neq 2K$$

$\Rightarrow \Leftarrow$ So our assumption is wrong

Hence n is even

$$q \Rightarrow p \equiv T \longrightarrow ②$$

From ① and ② we get,

$$(p \to q) \wedge (q \to p) \equiv T \wedge T \equiv T \equiv q[p \to q \equiv T]$$

Hence proved.

MATHEMATICAL INDUCTION:-

Mathematical Induction is a proof technique which is a proof technique which is used to prove a statement or a formula or a theorem is true for every natural number.

Steps to prove:-

* Base step:-

Prove that the statement is true for initial value (i.e), $P(1) \equiv T$, 1 is the initial value.

* Inductive step:-

Assume the statement is true for any value of $n=k$. Then prove that the stmt is true for $n=k+1$

i.e, It proves that the conditional statement $p(k) \rightarrow p(k+1)$ is true for positive integer k.

Ques: Prove that $3^n - 1$ is the multiple of 2 for $n \geq 1$; $n \in N$

PROOF:-

Here $p(n) = 3^n - 1$ is a multiple of 2

To prove: $p(1)$ is true

$p(1)$: $3^1 - 1 = 2$ is a multiple of 2

$\therefore$ $p(1)$ is true

Assume that $p(k)$ is true

$p(k)$: 3^{k-1} is a multiple of 2

(i.e) $3^k - 1 = 2t$ $\boxed{t \in N}$

To prove that,

$p(k+1) = 3^{k+1} - 1$ is a multiple of 2

$\therefore p(k+1) = 3 \cdot 3^k - 1$

$= 3(2t+1) - 1$

$= 6t + 3 - 1$

$= 6t + 2$

$= 2(3t+1)$ which is a multiple of 2.

$\boxed{t \in N}$ $\boxed{k \in N}$

Here $p(k+1)$ is true

Thus by mathematical induction,

$p(n) : 3^n - 1$ is a multiple of 2. for $n \geq 1, n \in N$

Ques : Prove that for any $n \in N$ $1 + 3 + 5 + \cdots + (2n-1) = n^2$

Proof :-

Here $p(n) : 1 + 3 + 5 + \cdots + (2n-1) = n^2$

To prove: $p(1)$ is true [Base step]

LHS: $p(1) = 1$ $\rightarrow$ (1)

RHS: $p(1) = 1^2 = 1$ $\rightarrow$ (2)

From (1) and (2) we get,

$p(1)$ is true

Lets assume that, [Inductive step]

$p(k) : 1 + 3 + 5 + \cdots + (2k-1) = k^2$ is true

to prove : $p(k+1)$ is true

$p(k+1) = 1 + 3 + 5 + \cdots + (2k-1) + 2(k+1) - 1 = (k+1)^2$

$= \cancel{k^2} + \cancel{k}$

LHS :-

$p(K) = 1+3+5+ \dots + 2K-1 + 2(K+1)-1$

$= K^2 + 2K + 2 - 1$

$= K^2 + 2K + 1$

$p(K) = (K+1)^2$

LHS = RHS

$\therefore$ $p(K+1)$ is true

By mathematical induction it is proved that

$p(n) : 1+3+5+ \dots + n = n^2$ for $n \in N$

Ques : Prove that the sum of first n naturals numbers is

$1+2+3+ \dots + n = \frac{n(n+1)}{2}$

PROOF :-

Here $p(n) : 1+2+3+ \dots + n = \frac{n(n+1)}{2}$

Base step :-

To claim $p(1)$ is true

LHS : $p(1) = 1$

RHS : $p(1) = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$

LHS = RHS

Hence $p(1)$ is true

Induction step :-

Assume that, $p(K)$ is true

(i.e) $p(K) = 1+2+3+ \dots + K = \frac{K(K+1)}{2}$

To claim that

$p(K+1)$ is true

$p(K+1) = 1+2+3+\cdots\cdots+K+K+1 = \frac{(K+1)(K+2)}{2}$

Proof:-

$$p(K+1) = 1+2+3+\cdots\cdots+K+K+1$$

$$= \frac{K(K+1)}{2} + (K+1)$$

$$= \frac{K(K+1)+2(K+1)}{2}$$

$$= \frac{(K+1)(K+2)}{2}$$

$$p(K+1) = \frac{(K+1)(K+2)}{2}$$

LHS = RHS

$\therefore$ $p(K+1)$ is true

Hence by mathematical induction,

$$p(n) = 1+2+3+\cdots\cdots+n = \frac{n(n+1)}{2}, \; n \in N$$

Prove that $n < 2^n$ for all positive integer n,

Proof:-

Here $p(n) = n < 2^n$ for all n

Base step:- $p(1)$ is true,

LHS : $p(1)$: $1 < 2^1$

$1 < 2$

$\therefore$ $p(1)$ is true

Inductive step:

Assume that $p(K)$ is true

$p(K) = K < 2^K$ for all +ve K,

To claim: $p(K+1) = K+1 < 2^{K+1}$ for all +ve k, LHS : $p(K+1) = K+1$

From

$$K < 2^K$$

Add ① on both sides,

$$K+1 < 2^K + 1$$

Since $1 \le 2^K$ we can write,

$$K+1 < 2^K + 1 \le 2^K + 2^K$$

$$K+1 < 2 \times 2^K$$

$$K+1 < 2^{K+1}$$

$\therefore$ $p(K+1)$ is true

By mathematical induction,

$p(n) : n < 2^n$ for all positive integer n.

$K < 2^K$

$K+1 < 2^{K+1}$

2^{K+1}

$\underline{\underline{2 \cdot 2^K}}$

Prove that $2^n < n!$ for positive integer $n \ge 4$

Proof : $p(n) : 2^n < n!$ for $n \ge 4$

Base step: To prove $p(4)$ is true

$p(4) : 2^4 < 4!$

$16 < 24$

$\therefore$ $p(4)$ is true

Inductive Step:-

Assume that $p(K)$ is true,

$p(K) = 2^K < K!$ for $K \ge 4$

To prove that $p(K+1)$ is true

$p(K+1) = 2^{K+1} < (K+1)!$

$(K+1)! > 2^K \cdot 2$

$2^K < K!$

$2^{K+1} < 2K! < (K+1)K!$

because $K \ge 4$

$2^{K+1} < (K+1)!$

$(K+1)! > 2^K + 2^K$

Proof:-

Let us take the LHS:-

$(K+1)! = \cancel{K+K}\ (K+1)\ K!$

$= K K! + K!$

$> K 2^K + 2^K$

Since $K \geq 4$ we can include here

$\geq 4 2^K + 2^K$

Since $1 < 4$ we can add 1 here

$> 1 2^K + 2^K$

$(K+1)! > 2 2^K$

$(K+1)! > 2^{K+1}$

$\therefore$ $p(K+1)$ is true

By mathematical Induction,

$p(n) : 2^n < n!$ for all $n \geq 4$

Hence proved.

Ques:- Let a, b, c are integers

If $a | bc$ then either $a | b$ or $a | c$

$a | b$: a divides b

Counterexample,

eg:- $6 | 3 \times 2$

but $6 \nmid 3$ & $6 \nmid 3$

eg:- $8 | 6 \times 4$

but $8 \nmid 6$ & $8 \nmid 4$

$\therefore$ It is mistake.

Ques: Prove that $2n^3 + 3n^2 + n$ is divisiblye by 6 for every integer $n \geq 1$

Proof:- $p(n) : 2n^3 + 3n^2 + n$ divisble by 6

BS:

To prove : $p(1)$ is true

$p(1) : 2(1) + 3(1) + 1$

$: 6$ (6 is divisible by 6)

$\therefore$ $p(1)$ is true

Inductive step:-

Assume that $p(K) : 2K^3 + 3K^2 + K$ is divisible by 6

$$2K^3 + 3K^2 + K = 6t \qquad t \in Z \geq 1$$

To prove $p(K+1)$ is true,

$p(K+1) : 2(K+1)^3 + 3(K+1)^2 + (K+1)$ is divisible by 6.

L.H.S : $p(K+1)$

$$= 2(K+1)^3 + 3(K+1)^2 + (K+1)$$

$$= 2(K^3 + 3K^2 + 3K + 1) + 3(K^2 + 2K + 1) + K + 1$$

$$= 2K^3 + 6K^2 + 6K + 2 + 3K^2 + 6K + 3 + K + 1$$

$$= 2K^3 + 9K^2 + 13K + 6$$

$$= (6t - 3K^2 - K) + 9K^2 + 13K + 6$$

$$= 6t + 6K^2 + 12K + 6$$

$$= 6(t + K^2 + 2K + 1)$$

$\therefore$ $p(K+1)$ is divisible by 6

$\therefore$ $p(K+1)$ is true.

By mathematical induction,

$p(n)$: $2n^3 + 3n^2 + n$ is divisible by 6.

STRONG MATHEMATICAL INDUCTION :-

To prove $p(n)$ is true for all positive integer n, where $p(n)$ is a propositional function, we complete two steps.

Base step:-

Verify that $P(i)$ is true, where i is the initial value.

Inductive step :-

Show that the conditional,

Statement $[P(1) \wedge P(2) \wedge \cdots P(K)] \longrightarrow P(K+1)$ is true for all positive integer k.

PROBLEM:- Prove that every integer $n \geq 2$ either is a prime or can be written as a product of primes.

PROOF:-

Here $p(n)$, n is either prime or it can be written as product of primes, $n \geq 2$.

Base step:-

To prove: $p(2)$ is a prime (or) product of primes

$\therefore$ $p(2)$: 2 is a prime

$\therefore$ $P(2)$ is true.

Inductive step:-

Assume $p(2) \wedge p(3) \wedge \cdots \wedge p(K)$ is True.

is for any t, $2 \leq t \leq k$ where t either is prime or can be written as product of prime.

clear $p(k+1)$ is true to claim.

If $p(k+1)$ is prime, then we are done.

If $k+1$ is not a prime then

$k+1 = x \times y$, where x, y are some +ve Integer

where $2 \leq x, y \leq k$.

By induction hypothesis,

$p(x)$ and $p(y)$ are true. $p(x)$ and $p(y)$ are either prime or they can be written as product of primes.

since $k+1 = x \times y$, $k+1$ is also a product of two or more primes.

Hence $p(n)$ is true for all $n \geq 2$.

Ques :- prove them every positive integer n can be written as a sum of distinct non-negative integer powers of 2 using S.M.I.

Proof :-

Here $p(n)$, n is a sum of distinct non-negative integer powers of 2.

Base step :- To prove $p(1)$ is true

$\therefore$ $p(1) = 2^0 = 1$

$\therefore$ $p(1)$ is true

Inductive Step :-

Assume $p(1) \wedge p(2) \wedge \ldots \wedge p(k)$ is true i.e for any m, $1 \le m < k$ is a sum of distinct non-negative integers of 2

claim $p(k+1)$ is true

Let l be the largest integer such that,

$2^l < k+1$

Let $m = (k+1) - 2^l$

Since $1 \le m \le k$, $p(m)$ is true by induction hypothesis.

i.e) $m = 2^{r_1} + 2^{r_2} + \dots + 2^{r_p}$

so $k+1 = m + 2^l$

$= 2^{r_1} + 2^{r_2} + \dots + 2^{r_p} + 2^l$

Here $p(n)$ is true.

Prove or Disprove, for every positive integer n, $n! \le n^2$

$\therefore$ It is false because $n = 4$, $4! > 4^2$.

Prooving condition start $P \to Q$

VACOUS PROOF :- If P is false, then the implication

$\therefore$ $P \to Q$ is always true

TRIVIAL PROOF :- If Q is true, then the implication

$\therefore$ $P \to Q$ is always true.

Example: Vacous proof

1. For all n, if n is both odd and even then $n^2 = n + n$

2. If $x^2 + 1 < 0$ then $x^2 \ge 4$

Example (Trivial proof)

① Let $x \in R$, If $x > 0$ then $x^2 + 5 > 0$

② Let $x \in R$, If $0 < x < 1$ then $x^2 - 2x + 2 \neq 0$.

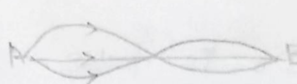
COUNTING PRINCIPLE:-

Sum rule:

If a task can be done in one of n_1, ways or in one of n_2 ways where none of the set of n_1 ways is the same as any of the n_2 ways then there are $n_1 + n_2$ ways to do the task.

Eg: A student can choose a computer project from 1 of 3 list, the three list contain 23, 15 and 19 possible projects. No project is on more than one list. How many possible projects to choose from?

ANS: $23 + 15 + 19 = 57$



Eg:

```
K := 0
for i1 = 1 to n1
    K = K+1
for i2 = 1 to n2
    K = K+1
  :        :
for im = 1 to nm
    K = K+1
```

ANS: Finally $K := n_1 + n_2 + \dots\dots + n_m$.

Product rule :-

Suppose that a ~~product~~cedure can be broken down into a sequence of two tasks. If there are n ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \times n_2$ ways to do the procedure.

Eg:

No of ways $= 3 \times 4 = 12$ ways

Ques :- A new company with two employees A and B, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

ANS : $12 \times 11 = 132$

$^{12}P_2 = \frac{12!}{10!}$

$^{12}C_2 = \frac{12 \times 11}{2}$

Ques :- How many bit strings of length seven are there?

$$\underset{0,1}{2} \quad \underset{0,1}{2} \quad \underset{0,1}{2} \quad \underset{0,1}{2} \quad \underset{0,1}{2} \quad \underset{0,1}{2} \quad \underset{0,1}{2} = 2^7$$

Ques :-

$A \rightarrow \{2, 3\}$ (m)

$B \rightarrow \{a, b, c\}$ (n)

$f : A \rightarrow B$

No of functions : $n^m = 3^2 = 9$

No of one-one function : $n \times (n-1) \times \dots \times (n-(m-1))$

$= 3 \times 2 = 6$

PROBLEM SHEET : 4

1. $7 \overset{*}{\cancel{\times}} 5 \overset{+}{\cancel{\times}} 9 = \cancel{35}\ \cancel{315}\ 44$

2. i) $5 \times 3 \times 2 = 30$ – different deserts

$30 - [s\ c\ wc + s\ c\ sc]$
$= 30 - 2 = 28$

ii) ~~$4 \times 2 \times 2 = 16 - 2 = 14$~~
-2

3. $4 \times 8 \times 6 = 192$

4. $10 \times 7 \times 3 \times 2 \times 2 \times 2 \times 1 \times 1 = 1680$

5. $\underline{1}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underline{1} = 2^8$

6. $P\{4\} + P\{3\} + P\{2\} + P\{1\} + P\{0\}$ 0

$(26)^4 + (26)^3 + (26)^2 + (26)^1 + (26)^0$ 0

7. $n(S) = 100$ elements.

Subset of total having n elements $= 2^{100}$

Subset having one element $= 100$

Subset having more than one element $= 2^{100} - 101$

9. numbers = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

a) $\underline{10}\ \underline{10}\ \underline{10}\ \underline{10} = 10^4$

b) $\underline{10}\ \underline{9}\ \underline{8}\ \underline{7} = 5040$

c) $\underline{2}\ \underline{10}\ \underline{10}\ \underline{10} = \cancel{1000}\ 2000$

4/5

8. $10 \times 5 \times 2 = 100$

10. $\underline{6}$ $\underline{6}$ $\underline{6}$ $\underline{6}$ $\underline{6}$ $\underline{6}$ $\underline{6}$ = 7^{th} time.

1, 2, 3, 4, 5, 6

11. $P(A \cup B) = ?$

$P(A) = 13$, $P(B) = 10$, $P(A \cap B) = 4$

$P(A \cup B) = 13 + 10 - 4 = 19$

12. To find :-

$n(\bar{A} \cap \bar{B}) = n(\overline{A \cup B})$

$= 137 - (n(A \cup B))$

$= 137 - [56 + 38 - 17]$

$= 137 - [94 - 17]$

$= 137 - 77$

$= 60$

13. SOLN:-

$n(A \cup B \cup C) = ?$

Given that if it is equal or < 450 then it is confidence

$n(A \cup B \cup C) = 425 + 397 + 340 - 284 - 315 - 219 + 147$

$= 491 \neq 450$

∴ So no confidence.

14. SOLN:-

a) 50

b) 27

c) 5

(15) SOLUTION:

Leap-year : 367

Non-leap year : 366

(16) SOLUTION:-

a) $(68)^8 + (68)^9 + (68)^{10} + (68)^{11} + (68)^{12}$

b) $6 \times ((68)^7 + (68)^8 + (68)^9 + (68)^{10} + (68)^{11})$

(17) SOLN:- length = 10

a) $2^8 + 2^7 - 2^5$

b) $6 \times 2^5 + 6 \times 2^5 - 1$

c) $2^7 + 2^8 - 2^5$

d) $8 \times 2^{7} + 7 \times 2^6 - 10 \times 2^3$

(18) SOLN:-

(19) SOLN:-

$\left\lfloor \frac{677}{38} \right\rfloor = 17$

PRINCIPLE OF INCLUSION AND EXCLUSION

Suppose that a task can be done in n ways, or in n_2 ways but that some of the set of n_1 ways to do the task are same as the same of n_2 other ways to do the task,

Say,
$$|A_1| = n_1$$
$$|A_2| = n_2$$

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|.$$

$$|A_1 \cup A_2 \cup A_3 \overset{\cdots}{\cup} A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i,j=1}^{n} |A_i \cap A_j| + \sum_{\substack{i \neq j \neq k \\ =1}}^{n} |A_i \cap A_j \cap A_k| \cdots\cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots\cdots A_n|$$

NOTE:-

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots\cdots \cap \bar{A}_n| = |U| - \sum_{i=1}^{n} |A_i| + \sum_{i \neq j} |A_i \cap A_j| - \sum_{\substack{i \neq j \neq k \\ 1}}^{n} |A_i \cap A_j \cap A_k| + \cdots\cdots + (-1)^{n} |A_1 \cap A_2 \cap \cdots\cdots \cap A_n|$$

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |U| - |\bar{A}_1 \cap \bar{A}_2 \cdots \cap \bar{A}_n|$$

problem:-

A computer company receives 350 applications from computer graduates for a job planning a line of new web series. Suppose that 220 of these people majored in computer science, 147 majored in business and 51 majored in CS and business. How many of these applications majored in neither and nor in business?

SOLN:-

where $A_1 \to C$

$A_2 \to B$

ANSWER:-

$n(C) = 220 \Rightarrow |A_1| = 220$

$n(B) = 147 \Rightarrow |A_2| = 147$

$n(B \cap C) = 51 \Rightarrow |A_1 \cap A_2| = 51$

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$= 220 + 147 - 51$$
$$= 316$$

$$|\bar{A}_1 \cap \bar{A}_2| = |U| - |A_1 \cup A_2|$$
$$= 350 - 316$$
$$= 34$$

problem:

How many bit strings of length eight start with a 1 bit or end with the two bits 00?

ANSWER:-

A_1 = No of bit strings of length start with 1

A_2 = No of bit strings of length end with 00

$A_1 \cap A_2$ = No of bit strings start with 1 and end with 00.

$|A_1| = 2^7$

$|A_2| = 2^6$

$|A_1 \cap A_2| = 2^5$

$|A_1 \cup A_2| = 2^7 + 2^6 - 2^5$

QUES:-

How many integers from 1 to 100 is divisible by 2 or 3?

SOLUTION:-

A_1 = set of numbers divisible by 2

$|A_1| = 50 \Leftarrow \lfloor \frac{100}{2} \rfloor$

A_2 = set of numbers divisible by 3

$|A_2| = 33 \Leftarrow \lfloor \frac{100}{3} \rfloor$

$A_1 \cap A_2$ = set of numbers divisible by 2 and 3

$|A_1 \cap A_2| = 16 \Leftarrow \lfloor \frac{100}{6} \rfloor$

NOW,

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$= 50 + 33 - 16$

$= 50 + 17$

$= 67$

Ques:- How many numbers from 1 to 1000 not divisible by 2, 3 and 4

SOLN:-

A_1 = set of numbers divisible by 2

$|A_1| = \lfloor \frac{1000}{2} \rfloor = 500$

A_2 = set of numbers divisibly by 3

$|A_2| = \lfloor \frac{1000}{3} \rfloor = 333$

A_3 = set of numbers divisible by 4

$|A_3| = \lfloor \frac{1000}{4} \rfloor = 250$

$A_1 \cap A_2$ = set of number divisible by 2 and 3
lcm(2,3)

$|A_1 \cap A_2| = 166$

$A_2 \cap A_3$ = set of numbers divisible by 3 and 4

$|A_2 \cap A_3| = 83$

$A_1 \cap A_3$ = set of numbers divisible by 2 and 4
lcm(2,4)

$|A_2 \cap A_3| = 250$

$A_1 \cap A_2 \cap A_3$ = set of no divisible by 2, 3, 4

$|A_1 \cap A_2 \cap A_3| = 83$

Now,

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = 1000 - [|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|]$$

$$= 1000 - [500 + 333 + 250 - 166 - 83 - 250 + 83]$$

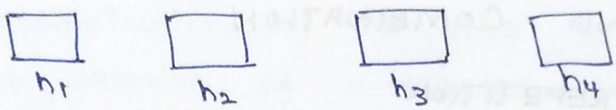
$$= 1000 - 667$$

= 333

PigeonHole Principle:

If K is a positive Integer and $K+1$ or more objects are placed into K boxes, then there is atleast one box containing one or more of the objects.

$\{P_1, P_2, P_3, P_4, P_5\}$



Generalized pigeon-hole principle:

If N objects are placed into K boxes. then there is atleast one box containing $\lceil \frac{N}{K} \rceil$ objects

Note:-

The minimum number of objects required such that atleast t objects must be in one of K boxes is,

$$N = K(t-1)+1$$

Ques:

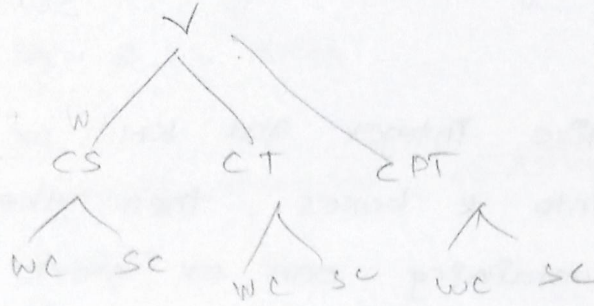
How many cards must be selected from a standard deck of 52 cards to gurantee that atleast 3 cards of the same suite are chosen.

ANS:-

$K = 4$ boxes $= 4$

$t =$ atleast 3 cards $= 3$

$$\begin{aligned} N &= K(t-1)+1 \\ &= 4(3-1)+1 \\ &= 4(2)+1 \\ &= 9 \end{aligned}$$



strowberry

PERMUTATION AND COMBINATION

WITHOUT REPETITION

A <u>permutation</u> of a set of distinct objects is an ordered arrangements of these objects.

An object ordered arrangement of r-elements of a set is called r-permutation.

$S = \{a, b, c\}$

abc	ab
acb	ac
bca	ba
bac	bc
cab	ca
cba	cb
3-permutation	2-permutation.

Theorem:

If n is a positive integer and r is a integer such that $1 \leq r \leq n$ then

$$P(n,r) = n(n-1)\ldots\ldots(n-(r-1))$$

$$= n(n-1)\ldots\ldots(n-r+1)$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

$P(n,1) = n$

$P(n,n) = n!$

$P(n,0) = 1$

A combination of a set of distinct objects is an unordered arrangement / selection of these objects

An r-combination of elements of a set is an unordered selection of r-elements from the set. Thus r-combination is simply a subset of the set with r-elements.

Theorem :-

The number of r-combination of a set with n-elements, where n is a non-negative integer and r is a integer $0 \le r \le n$ equals,

$$C(r,n) = \frac{n!}{r!\,(n-r)!}$$

Note :

$C(n,0) = 1$, $C(n,1) = n$, $C(n,n) = 1$

Eg: $S = \{a,b,c,d\}$

3 - permutation :

abc	abd	bcd	acd
acb	adb	bdc	adc
bac	bad	cbd	cad
bca	bda	cdb	cda
cab	dab	dbc	dac
cba	dba	dcb	dca

Ques:

In how many ways are there to select a first prize winner, a second prize winner and a third prize winner from hundred different people to have enter a contest:

$$= 100 \times 99 \times 98 = {}^{100}P_3$$

Ques:

How many different committee of three students can be formed from 100 people

$$= {}^{100}C_3$$

PROBLEM SHEET:-

1. The permutation of all 4 letters $= 4! = 24$ ways
2. The permutation of 9 man is $= 9!$ ways
3. No of listing possible is $= 14!$ ways
4. a) Different arrangement for 8 letters $= 7!$ ways.
   b) Start with t end with c $= 6!$ ways
5. a) No restrictions $= 7!$ ways
   b) language alternate $=$ JCJCJCJ + $\underbrace{\text{CJCJCJC}}_{\text{not possible}}$

   $= \underbrace{4\,3\,3\,2\,2\,1\,1}$

   $= 4! \times 3!$

   c) all C++ together $= 5! \times 3!$

   $\boxed{c|c|c}$

   d) all C++ together and all Java together $= 4! \times 3! \times 2!$

   3

   $\boxed{c|c|c}\ \boxed{J|J|J|J}$

   $3! \times 4! \times 2!$

6. a) No of ways $= 2! \times 2! \times 2! \cdots 2!$

   $= 10 \times 2!$

   $= 2^{10}$

b) 2 3 3 ... 3 3 3 3 3

$= 2^{10}$

7] Different batch chips = $^{300}C_{25}$

8] Different no of novels = $^{21}C_4 * {}^{11}C_3$

9] Include Siman or yuvan =

= Include Simon + Include yuvan − both Simon & yuvan

$= 1 \cdot {}^{24}C_4 + 1 \cdot {}^{24}C_4 - 1 \cdot 1 \; {}^{23}C_3$

10] Different teams possible = $^{18}C_{11}$

11] Different no of juries = $^{14}C_5 \times {}^{23}C_4$

12] _ _ _ _ _ _ _ _ _ _ _ _

a) exactly three 1s : $^{12}C_3$

b) atmost three 1s :

1 = 12	0 = 12
3	9
2	10
1	11
0	12

$= {}^{12}C_3$

$= {}^{12}C_2 \quad = {}^{12}C_3 + {}^{12}C_2 + {}^{12}C_1 + {}^{12}C_0$

$= {}^{12}C_1$

$= {}^{12}C_0$

c) atleast three 1s :

1 = 12	0 = 12
3	9
4	8
5	7
6	6
7	5
8	4
9	3
10	2
11	1
12	0

$= {}^{12}C_3 + {}^{12}C_4 + {}^{12}C_5 + {}^{12}C_6 +$

${}^{12}C_7 + {}^{12}C_8 + {}^{12}C_9 + {}^{12}C_{10}$ $^{12}C_2$

$+ {}^{12}C_{11} + {}^{12}C_{11}$

13) a) 2^8 times

b) ${}^8C_3$

c) ${}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$

d) ${}^8C_4$

14) a) $7!$

b) $6!$

c) $5!$

d) $4!$

e) $4!$

f) 0.

ED B B C F G H

BA FGH C D E

AB DE GH CF

CABED _ _ _

$\underbrace{\_\ \_}_{{}^5C_2}\ \underbrace{\_\ \_\ \_\ \_}_{{}^{21}C_4}$

15) a) ${}^5C_2 \times {}^{21}C_4 \times 6!$

b) $26^5 - [{}^{21}C_6 \times 6!]$

c) $[{}^5C_3\,{}^{21}C_3 + {}^5C_2\,{}^{21}C_4 + {}^5C_1\,{}^{21}C_5 + {}^5C_0\,{}^{21}C_6] \times 6!$

d) $[{}^5C_1\,{}^{21}C_5 + {}^5C_0\,{}^{21}C_6] \times 6!$

16) a) ${}^{25}C_4$

b) $25 \times 24 \times 23 \times 22 = {}^{25}P_4$

${}^{48}C$

17) a) ${}^4C_4 \times {}^{48}C_1$

1 = _ _ _ _ _

b) ${}^{13}C_5$

$\frac{13}{C_1}, \frac{13}{C_1}, \frac{13}{C_1}, \frac{13}{C_1}, {}^{48}C_1$

c) ${}^{13}C_3 \times {}^{13}C_2$

d) ${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{48}C_1$

$\begin{array}{r} 48 \\ -12 \\ \hline 36 \end{array}$

e) ${}^{12}C_5$

f) ${}^{13}C_2 \times {}^{13}C_2 \times {}^{26}C_1$

g) ${}^4C_2 \times [{}^{12}C_1 \times {}^{36}C_2 + {}^{12}C_2 \times {}^{36}C_1 + {}^{12}C_3 \times {}^{36}C_0]$

h) ${}^4C_2 \times [{}^{12}C_2\,{}^{36}C_0 + {}^{12}C_1\,{}^{36}C_1 + \cancel{{}^{12}C_0} \times {}^{36}C_2]$

18) SOLUTION:-

a) $^{12}C_4$

b) $^5C_2 \times {}^7C_2$

c) $^5C_4 + {}^7C_4 - [0] = {}^5C_4 + {}^7C_4$

d)

quaters (7)	dimes (5)	
3	1	$= {}^7C_3 \times {}^5C_1$
4	0	$= {}^7C_4 \times {}^5C_0$

$= {}^7C_3 \times 5 + {}^7C_4 \times 1$

19) SOLUTION:-

a) $^{12}C_3$

b)

	Independents (4)	Republicans (3)	Demo (5)
No Inde:-	0	3	0
		2	1
		1	2
		0	3

$= {}^{12}C_3 - [{}^3C_3 \times {}^5C_0 + {}^3C_2 \times {}^5C_1 + {}^3C_1 \times {}^5C_2 + {}^3C_0 \times {}^5C_3]$

c) $^4C_3$

d) $^{12}C_3 -$ [No democrat and no republican are possible]

only republican

$= {}^{12}C_3 - [{}^4C_3 + {}^5C_3 + 1]$

only democrat

Problem: What is the number of ways to order the 26 letters of the alphabet so that no two vowels {a, e, i, o, u} occur consecutively?

ANS : $\underbrace{{}^{26}C_1 \;\; {}^{21}C_1}_{1\ \text{pair}}$ . . . . 13 pairs

(or)

$= 21! \times {}^{22}C_5 \times 5!$

Problem:- How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Ans $= {}^{52}C_5$

Problem:- How many bit string of length 10 contain exactly five 1's?

1 1 1 1 1 _ _ _ _ _

$= {}^{10}C_5$

Ques:

a) exactly four 1's? $= {}^{10}C_4$ exactly 4 ones + . . + exactly 10 1's

b) atleast four 1's? $= {}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$

c) at ~~least~~ most four 1's? $= {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4$

d) an equal no of 0's and 1's? $= {}^{10}C_5$

Problem: How many permutation of the letters A B C D E F G contain

a) the string with BCD _ _ _ _ _

$P(5,5) = 5!$

b) the string CFGA ?

$P(4,4) = 4!$

c) the string BA and GF ?

$P(5,5) = 5!$

d) the strings ABC and DE ?

$P(4,4) = 4!$

e) the string ABC and CDE ?

$P(3,3) = 3!$

f) the string CBA and BED?

$= 0$ (not possible)

Permutation & Combination with repetition allowed

1. SOLN:-

$\{a, b, c, d, e, e, e, e, e\}$

$\underset{1}{^9C_1}\ \underset{2}{^4C_1}\ \underset{3}{^9C_1}\ \underset{4}{^4C_1}\ \underset{5}{^9C_1}\ \underset{6}{^4C_1}\ \underset{7}{^9C_1}\ \underset{8}{^4C_1}\ \underset{9}{^9C_1}$ (and also alternate ways)

$= 4! \times {}^5C_5$

Generalized permutations and Combinations

Permutation with repetition:

The number of r-permutation of a set of n objects with repetition allowed is n^r.

Permutation with indistinguishable object:

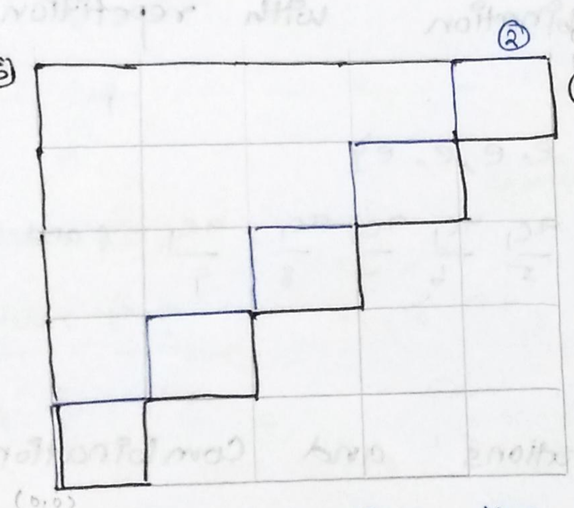
The number of different permutation of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ...

..... and n_k indistinguishable object of type K is $\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$

How many different strings can be made from the letters in SUCCESS (using all letters) =

$$ANS = \frac{7!}{3!\ 2!}$$

Count the number of paths in xy-plane between the origin (0,0) and point (5,5) such that each path is made up of a series of steps, where each step is a move one unit to the right (R) or a move one unit upward (U). [No moves to the left or down are allowed].



It is nothing but arranging the the 5 UP's and 5 Right's. = {U U U U U R R R R R}

$$= \frac{10!}{5!\ 5!}$$

Permutation & Combination - with repetition:

2. SOLUTION:-

a) {3, 4, 4, 5, 5, 6, 7}

$$= \frac{7!}{2! \times 2!}$$

b)

— (— — — — — —)

5, 6, 7

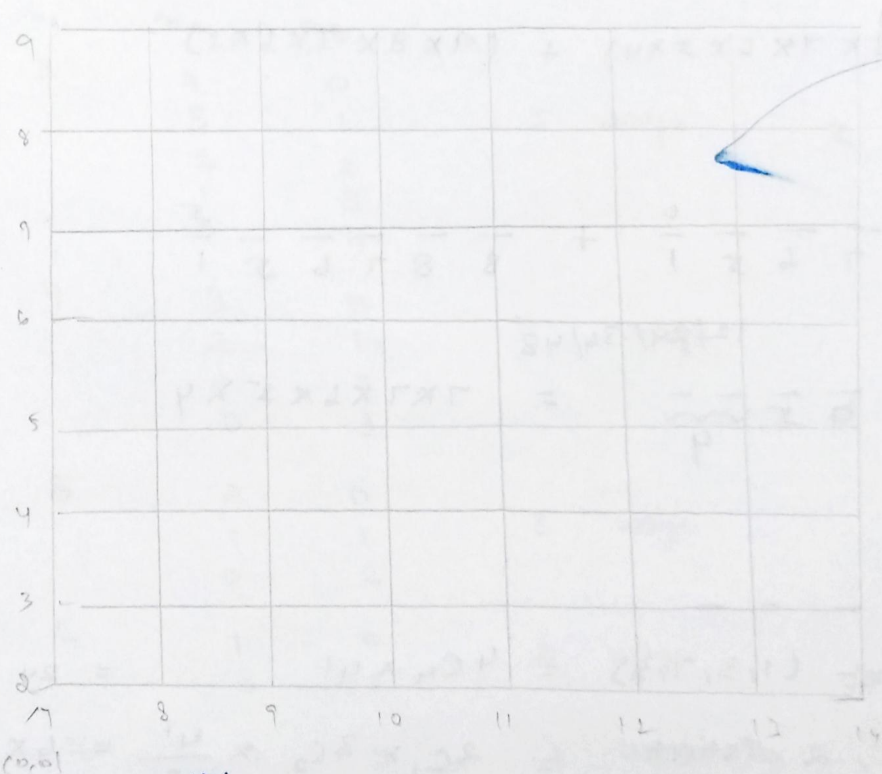
$$= 2 \times \frac{6!}{2!\,2!} + 1 \times \frac{6!}{2!}$$

3. SOLUTION:-

a) {UUUUUUU RRRRRRR}

$$= \frac{14!}{7!\,7!}$$

b)



$$= \frac{14!}{7!\,7!}$$

4. SOLUTION:-

$\{0,1,2,3,4,5,6,7,8,9\}$

a) $\underline{9}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5} = 9\times9\times8\times7\times6\times5 =$

b) $\underline{9}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10} = 9\times(10^5$

c) Total digit - no digit repeated

$= 9\times10^5 - (9\times9\times8\times7\times6\times5$

$=$

d) not 0: $\underline{8}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{\cancel{5}4}$

i]

if last is 0:

$\underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{1}$

$= (8\times8\times7\times6\times5\times4) + (9\times8\times7\times6\times5)$

ii] Divisibly 5

$\overline{9}\ \overline{8}\ \overline{7}\ \overline{6}\ \overline{5}\ \overset{0}{\overline{1}} + \overline{8}\ \overline{8}\ \overline{7}\ \overline{6}\ \overline{5}\ \overset{5}{\overline{1}}$

iii] 12/24/36/48

$\overline{\cancel{8}}\ \overline{\cancel{8}}\ \overline{6}\ \overline{5}\ \underbrace{\overline{\ }\ \overline{\ }}_{4} = 7\times7\times6\times5\times4$

5 SOLUTION:-

$\_\ \_\ \_\ \_$

4 distinct (1,3,7,8) $= {}^4C_4 \times 4! = 24$

2 alike, 2 distinct $= {}^2C_1 \times {}^3C_2 \times \frac{4!}{2!} = 6\times12 = 72$

(33, 77) (1,8)

2 set of like $= {}^2C_2 \times \frac{4!}{2!\,2!} = 6$

(33, 77)

$\overline{102}$

b) $\frac{6!}{2!\,2!}$

6] $\frac{12!}{4!\,3!\,3!}$

7] a) $\frac{8!}{2!\,2!}$ AAA II WN

b) H _ _ _ _ _ _ _ = $\frac{7!}{3!\,2!}$

8] a) $\frac{12!}{4!\,2!\,2!}$

b) $\frac{11!}{4!\,2!}$

9] $x_1 = 3, 4, 5, 6, 7$

Explanation:

x_1	x_2	x_3	
3	4	0	
	3	1	5 ways
	2	2	
	1	3	
	0	4	
4	3	0	
	2	1	4 ways
	1	2	
	0	3	
5	2	0	
	1	1	3 ways
	0	2	
6	1	0	2 ways
	0	1	
7	0	0	1 way
			5! ways

COMBINATION WITH REPETITION

1. How many ways are three to select four pieces of fruit from a bowl containing apple, oranges and pears. if the order in which the pieces are selected does not matter, only the type of fruit and not the individual ~~fruit~~ pieces matters and there are atleast four pieces of each type of fruit in the bowl.

SOLN:-

Apple	Oorange	Pears
4	0	0
3	1	0
3	0	1
2	2	0
2	1	1
2	0	2
1	3	0
1	2	1
1	1	2
1	0	3
0	4	0
0	0	4
0	3	1
0	2	2
0	1	3

15 ways

_ _ _ _ _ _

X X X X Y Y

$= \frac{6!}{4!\,2!} = 15$ ways

X - 4 objects

Y - 2 partition

A | O | P

1 2

$r+n-1 \; C_2$ $\quad r = 4$ (objects)

$\downarrow \quad n = 3$ (distinct boxes)

$n-1$

$\therefore \; C(r+n-1, n-1)$

↓ Distributing identical objects into distinct boxes.

$\therefore \; C(n, r) \Rightarrow n \geq r$

$\therefore \; (n+r-1, n-1) \Rightarrow n$ may be greater than equal to or less than r.

Ques :-

How many non-negative integer solutions are there, $x_1 + x_2 + x_3 = 4$, $x_i \geq 0$

x_1	x_2	x_3
4	0	0
3	1	0
3	0	1
⋮	⋮	⋮
0	0	4

$\therefore$ Total ways $= 15$

where : $n = 3$, $r = 4$ $\quad | \quad {}^{n+r-1}C_{n-1} = {}^{6}C_{2} = 15$ ways

If $x_i > 0$ then

$${}^{n-1}C_{n-1} = {}^{3}C_{2} = 3 \text{ ways} \quad [{}^{r-1}C_{n-1}]$$

Ques :

How many different ways you can distribute 7 chocolate to 3 children?

$r = 7, \; n = 3$

$${}^{n+r-1}C_{n-1} = {}^{10-1}C_{3-1} = {}^{9}C_{2} = 36 \text{ ways}$$

* each student should have atleast 1 chocolate

$${}^{r-1}C_{n-1} = {}^{6}C_{2} = 15 \text{ ways}$$

Ques :

How many different ways you can distribute ~~chocolate~~ $x_1 + x_2 + x_3 = 7 \quad , x_i \geq 1$

How many positive integer soln?

Soln:-

$r = 7, \quad n = 3$

$^{r-1}C_{n-1} = {}^{7-1}C_{3-1} = {}^{6}C_{2} = 15$ ways

Let

$y_i = x_i - 1 \quad , \quad i = 1, 2, 3$

$\Rightarrow x_i = y_i + 1$

$x_1 + x_2 + x_3 = 7$

$y_1 + 1 + y_2 + 1 + y_3 + 1 = 7$

$y_1 + y_2 + y_3 = 4$

$\therefore \quad {}^{n+r-1}C_{r-1} = {}^{6}C_{2} = 15$ ways

Ques :

How many different ways are there to choose a ~~dessert~~ dozen donot from 21 varieties of a dessert donout shop?

ANS :- $r = 12$
$n = 21$

$^{n+r-1}C_{n-1} = {}^{33-1}C_{20} = {}^{32}C_{20}$

Ques :-

How many integer soln are there to the equation

$x_1 + x_2 + x_3 + x_4 = 21$

where $x_i \geq 0 \quad , \quad 1 \leq i \leq 4$ such that

$n = 4 \quad , \quad r = 21$

i] $x_2 \geq 1$

$$x_1 + x_2 + x_3 + x_4 = 21 \rightarrow (1)$$

Now

$$x_2 - 1 \geq 0$$

Let $y_2 = x_2 - 1$

and $y_i = x_i$ for $i = 1, 3, 4$

$(1) \Rightarrow$ $x_1 + x_2 + x_3 + x_4 = 21$

$$y_1 + y_2 + 1 + y_3 + y_4 = 21$$

$$y_1 + y_2 + y_3 + y_4 = 20$$

where $n = 4$, $r = 20$

ANS $= {}^{n+r-1}C_{n-1} = {}^{23}C_3$

ii] $x_i \geq i$ for $i = 1, 3, 4$

$x_1 \geq 1$	$x_3 \geq 3$	$x_4 \geq 4$
$x_1 - 1 \geq 0$	$x_3 - 3 \geq 0$	$x_4 - 4 \geq 0$
$y_1 = x_1 - 1$	$y_3 = x_3 - 3$	$y_4 = x_4 - 4$

$$x_1 + x_2 + x_3 + x_4 = 21$$

$$y_1 + 1 + y_2 + y_3 + 3 + y_4 + 4 = 21$$

$$y_1 + y_2 + y_3 + y_4 = 13$$

where $n = 4$, $r = 13$

$${}^{n+r-1}C_{n-1} = {}^{16}C_3$$

iii] $x_3 \le 5$

$\begin{bmatrix} \text{\# of all possible solution} \\ \text{(without any restriction on } x_i) \end{bmatrix} - \begin{bmatrix} \text{\# of soln} \\ x_3 > 5 \end{bmatrix}$

$x_3 \ge 6$

~~Let~~

$\cancel{x_3} - \cancel{5}$

$= \cancel{x_3}$

$= n = 4, \ r = 21$

${}^{n+r-1}C_{n-1} = {}^{24}C_3$

$x_3 - \cancel{5} \ge 0$

where $y_3 = x_3 - \cancel{5}$

$y_1 + y_2 + y_3 + \cancel{5} + y_4 = 21$

$y_1 + y_2 + y_3 + y_4 = 15$

$n = 4, \ r = 15$

${}^{n+r-1}C_{n-1} = {}^{18}C_3$

$\therefore$ ANS $= {}^{24}C_3 - {}^{18}C_3$

iv) $x_2 < 6 \quad \equiv x_2 \le 5$

$= n = 4, \ r = 21$

${}^{n+r-1}C_{n-1} = {}^{24}C_3$

$x_2 \ge 5$

$\equiv x_2 \ge 6$

${}^{18}C_3$

$\therefore {}^{24}C_3 - {}^{18}C_3$

v) $3 \le x_3 \le 7$

$= \begin{bmatrix} \text{\# of solution with} \\ x_3 \ge 3 \end{bmatrix} - \begin{bmatrix} \text{\# of solutinon with} \\ x_3 > 7 \\ x_3 \ge 8 \end{bmatrix}$

$= y_1 + y_2 + y_3 + 3 + y_4 = 21 \quad - \quad y_1 + y_2 + y_3 + 8 + y_4 = 21$

$y_1 + y_2 + y_3 + y_4 = 18 \qquad y_1 + y_2 + y_3 + y_4 = \cancel{23}\ 13$

$\Rightarrow {}^{21}C_3 \qquad \Rightarrow {}^{16}C_3$

$= {}^{21}C_3 - {}^{16}C_3$

$x_1 + x_2 + x_3 + x_4 \leq 21$, $x_i \geq 0$

Introduce a dummy variable $x_5 \geq 0$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

if $x_5 = 0$

$$x_1 + x_2 + x_3 + x_4 = 21 \Rightarrow {}^{24}C_3$$

if $x_5 = 1$

$$x_1 + x_2 + x_3 + x_4 = 20 \Rightarrow {}^{23}C_3$$

if $x_5 = 2$

$$x_1 + x_2 + x_3 + x_4 = 19 \Rightarrow {}^{22}C_3$$

$\vdots \qquad \vdots$

RECURRENCE RELATION

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \ldots a_{n-1}$ for all integer n.

Solution :-

A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

9- SOLUTION :-

$$x_1 + x_2 + x_3 = 7 \quad \text{for } x_1 \geq 3$$

Since $x_1 \geq 3$

$x_1 - 3 \geq 0$

$y_1 = x_1 - 3$ | $x_i = x_i$ for $i = 1, 2$

$$x_1 + x_2 + x_3 = 7$$

$$y_1 + 3 + x_2 + y_3 = 7$$

$$x_1 + y_2 + y_3 = 4$$

Now, $n = 3$, $r = 4$

$${}^{n+r-1}C_{n-1} = {}^{6}C_{2} = 15 \text{ ways}$$

10. SOLUTION:-

$$x_1 + x_2 + x_3 + x_4 = 10 \quad \text{in} \quad x_i \geq 0$$

ANS :- $n = 4$, $r = 10$

$${}^{n+r-1}C_{n-1} = {}^{13}C_{3}$$

11. SOLUTION:-

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$$

where,

i] $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$

$$x_i \geq 2$$

$$x_i - 2 \geq 0 \Rightarrow y_i = x_i - 2$$

$$x_i = y_i + 2$$

$$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$$

$$y_1 + 2 + y_2 + 2 + y_3 + 2 + \dots + y_6 + 2 = 29$$

$$y_1 + x_2 + y_3 + x_4 + y_5 + y_6 = 29 - 12$$

$$= 17$$

$n = 6$, $r = 17$

ANS :- ${}^{n+r-1}C_{n-1} = {}^{22}C_{5}$

ii] $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 \geq 5$,

$x_6 \geq 6$

$x_1 - 1 \geq 0$ | $x_2 - 2 \geq 0$ | $x_3 - 3 \geq 0$ | $x_4 - 4 \geq 0$ | $x_5 - 5 \geq 0$

$y_1 = x_1 - 1$ | $y_2 = x_2 - 2$ | $y_3 = x_3 - 3$ | $y_4 = x_4 - 4$ | $y_5 = x_5 - 5$

$$\boxed{\begin{array}{l} x_6 - 6 \geq 0 \\ y_6 = x_6 - 6 \end{array}}$$

$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$

$y_1 + 1 + y_2 + 2 + y_3 + 3 + y_4 + 4 + y_5 + 5 + y_6 + 6 = 29$

$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 29 - 21 = 8$

$n = 6, \; r = 8$

$\Longrightarrow {}^{n+r-1}C_{n-1} = {}^{13}C_5$

c) $x_1 \leq 5$

Total $-$ $[x_1 > 5 \equiv x_1 \geq 6]$

Total $=$ $n = 6, \; r = 29$

$\Rightarrow {}^{34}C_5$

$x_1 \geq 6 \Rightarrow x_1 - 6 \geq 0$

$y_1 \geq 0$

$\Rightarrow y_1 + 6 + y_2 + y_3 + y_4 + y_5 + y_6 = 29$

$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 23$

$n = 6, \; r = 23$

$\Rightarrow {}^{28}C_5$

$\therefore {}^{34}C_5 - {}^{28}C_5$

d) $x_1 < 8$ and $x_2 \geq 9$

$x_1 < 8$

$\Rightarrow$ Total $-$ $[x_1 > 8 \equiv x_1 \geq 9$ and $x_2 \geq 9]$

Total $= {}^{34}C_5$

$x_1 - 9 \geq 0$ | $x_2 - 9 \geq 0$

$y_1 \geq 0$ | $y_2 \geq 0$

$$y_1 + 9 + y_2 + 9 + y_3 + y_4 + y_5 + y_6 = 29$$

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 11$$

$n = 6, \quad r = 11$

$\Rightarrow \quad {}^{n+r-1}C_{n-1} = {}^{16}C_5$

12. SOLUTION:-

$$x_1 + x_2 + \cdots + x_{34} = 48$$

where $x_i \geq 0$, $n = 34$, $r = 48$

Total way $= {}^{n+r-1}C_{n-1} = {}^{34+48-1}C_{33} = {}^{81}C_{33}$

13. SOLUTION:-

$$\left.\begin{array}{l} \text{Small} = 2 \\ \text{Medium} = 3 \\ \text{large} = 6 \\ \text{extra large} = 2 \end{array}\right\} \text{in stock}$$

$$= \frac{13!}{2!\,3!\,6!\,2!}$$

14. SOLUTION:-

$$\frac{12!}{5!\,3!\,4!}$$

15. SOLUTION:-

$r = 3000$

$n = 3$

${}^{n+r-1}C_{n-1} = {}^{3002}C_2$

16. SOLUTION:-

i) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 6$

$n = 8, \quad r = 6$

${}^{13}C_7$

b) $x_1 + x_2 + \cdots + x_8 = 12$

$n = 8, \ r = 12$

$^{n+r-1}C_{n-1} = {}^{19}C_7$

c) $x_1 + x_2 + \cdots + x_8 = 24$

$n = 8, \ r = 24$

$^{n+r-1}C_{n-1} = {}^{31}C_7$

d) $x_i \geq 1$

$x_i - 1 \geq 0$

$y_i \geq 0$

$x_1 + x_2 + x_3 + \cdots + x_8 = 12$

$y_1 + y_2 + y_3 + \cdots + y_8 = 12 - 8 = 4$

$n = 8, \ r = 4$

$^{n+r-1}C_{n-1} = {}^{11}C_7$

e) $x_3 \geq 3$

$x_3 - 3 \geq 0$

$y_3 \geq 0$

$x_4 \leq 2$

12

Total $- [x_4 > 2 \equiv x_4 \geq 3]$

$x_4 - 3 \geq 0$

$^{19}C_7 - \left[x_1 + x_2 + x_3 + x_4 + \cdots + x_8 = 9 \right]$

$= {}^{19}C_7 - {}^{16}C_7$

17. a) $52!$

b) $\dfrac{104!}{(2!)^{52}} \rightarrow [(2!) \text{ 52 times}]$

18. $\dfrac{8!}{4!\,2!\,2!}$

19. $= 40 + 120 + 180 + 180$

20.

15

222, 888, 77, 9

1. 3 same 2 same

$2C_1 \times 7C_1 * \dfrac{5!}{3!\,2!} = 40$

2. 3 same, 2 distinct

$2C_1 \times 3C_2 = \dfrac{5!}{3!} = 120$

3. 2 same 2 same 1 distinct

2 set of 2 same

$^3C_2 \times {}^2C_1 \times \dfrac{5!}{2!\,2!} = 180$

4. 2 same 3 distinct

$3C_1 \times {}^3C_3 \times \dfrac{5!}{2!} = 180$

(50) 11, 777777, 88, 999

— — — — — —

(1) 5 digit same + 1 digit (distinct)

$^{1}C_1 \times {}^{3}C_1 \times \frac{6!}{5!} = 18$

(2) 4 digit same + 2 distinct

$^{1}C_1 \times {}^{3}C_2 \times \frac{6!}{4!} = 90.$

(3) 4 digit same + 2 same

$^{1}C_1 \times {}^{2}C_1 \times \frac{6!}{4!\,2!} = 30$

(4) 3 digit same (2 sets)

$^{2}C_2 \times \frac{6!}{3!\,3!} = 20$

(5) 3 digit same + 2 same + 1 dis

$^{2}C_1 \times {}^{3}C_1 \times {}^{2}C_1 \times \frac{6!}{3!\,2!} = \cancel{1260}\ 720$

(6) 3 digit same + 3 distinct

$^{2}C_1 \times {}^{3}C_3 \times \frac{6!}{3!} = 240$

(7) 2 digit same (3 sets)

$^{4}C_3 \times \frac{6!}{2!\,2!\,2!} = 90 \times 4 = 360$

(8) 2 digit same (2 sets) + 2 distinct

$^{4}C_2 \times {}^{2}C_2 \times \frac{6!}{2!\,2!} = \cancel{2160}\ 1080$

(9) ~~1 set of 2 same~~ +

Total = 18 + 90 + 30 + 20 + 720 +
240 + 360 + 1080

RECURRENCE RELATION

$\{a_n\} = a_1, a_2, \ldots\ldots a_{n-1}, a_n, \ldots\ldots$

$a_n = f(a_1 a_2, \ldots\ldots a_{n-1})$

eg:-1) $1, 2, 3, \ldots\ldots n-1, n$

$a_n = a_{n-1} + 1$

2) $1, 3, 5, \ldots\ldots 2n, 2n+1$

$a_n = a_{n-1} + 2$

$a_n = 2n - 1 \quad , n \geq 1$

3) $0, 1, 1, 2, 3, 5, 8, \ldots\ldots$

$a_n = a_{n-1} + a_{n-2}$ → recurrence relation for n^{th} term

$\therefore$ we can't directly find n^{th} term of the above sequence.

~~Let~~ Suppose that the person deposit Rs 10,000 in a saving account at a bank yielding ~~Rs~~ 11,~~000~~ % year with interest compounded anually. How much will be in the account after 30 years.

SOLN :

a_n = Total amount in the account after n years.

$a_0 = 10,000$

$a_1 = (10,000) + (11\%$ of $10,000)$

$= 11,100$

$a_2 = 11,100 + (11\%$ of $10,000)$

$= 12200 \quad (a_1 + 11\%$ of $10,000)$

$a_3 = 12,200 + (11\%$ of $10,000)$

$= (a_2 + 11\%$ of $10,000)$

$$a_n = a_{n-1} + 11\% \text{ of } a_{n-1}$$

$$= a_{n-1} + 0.011 \ a_{n-1}$$

$$= a_{n-1} (1 + 0.11)$$

$$a_n = 1.11 \ a_{n-1} \qquad n \geq 1$$

where a_0 = initial amount

$$a_1 = 1.11 \ a_0$$

$$a_2 = 1.11 (1.11 \ a_0)$$

$$= (1.11)^2 a_0$$

$$a_3 = (1.11) \ a_2$$

$$= (1.11)^3 a_0$$

$$\therefore \ a_n = (1.11)^n a_0$$

Find a_{30}, $a_{30} = (1.11)^{30} a_0$

A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbit does not breed until they are 2 months old. After they are 2 months old each pair of rabbits produce another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months assuming that no rabbits ever die.

Reproducing pairs	Young pairs	Month	No of reproducing pairs	No of Young pairs	Total
–	XX	1	0	1	1
–	XX	2	0	1	1
XX	XX	3	1	1	2
XX	XX XX	4	1	2	3
XX XX	XX XX XX	5	2	3	5
XX XX XX	XX XX XX XX	6	3	4	7

$$a_n = a_{n-1} + a_{n-2}$$

Ques:-

Suppose that no of bacteria in a colony triples every hour, set up a recurrerce relation for no of bacteria after n hours have elapsed. If 100 bacteria are used to begin a new colony, how many bacteria will be in colony in 10 hours.

Let a_n = no of bacteria after n hours elapsed.

$a_1 = 3a_0$

$a_2 = 3a_1$

$\vdots \quad \vdots$

$\boxed{a_n = 3a_{n-1}}$ → recurrence relation

where,

$a_1 = 3a_0$

$a_2 = 3^2 a_0$

$\vdots \quad \vdots$

$a_n = 3^n a_0$

when $n=10$, since $a_0 = 100$

$$a_n = 3^{10} a_0$$

$$a_n = 3^{10}(100)$$

$$= 3^{10} \cdot 10^2$$

Ques

A factory makes custom cars at an increasing rate. In the first month only one can is made, in the second month two cars are made and so on, with n cars ~~one~~ are made in the n^{th} month. Set up a recurrence relation for the number of kars produced in the first n month by this factory.

Soln:-

C_n = No of cars produced after n months

$$C_n = C_{n-1} + n$$

$$\Rightarrow C_1 = 1$$

$$\Rightarrow C_2 = 1 + 2$$

$$\Rightarrow C_3 = 1 + 2 + 3$$

$\vdots \quad \vdots$

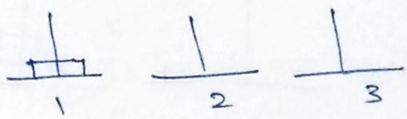
$$C_n = 1 + 2 + 3 + \cdots + n = \frac{n(n-1)}{2}$$

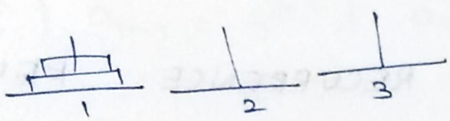
Ques :

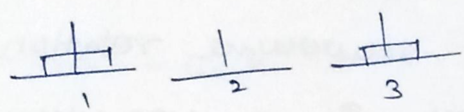
The tower of Hanoi puzzle consists of 3 pegs around mounted on a brand together with disks placed of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the buttons. The rules of the puzzle allow disks to be moved one at a time from one's peg to another pegs as long as a disk is never placed on top of a smaller disk. The goal of the puzzle, is to have all the disks on the second peg in order of size with the largest on the button.

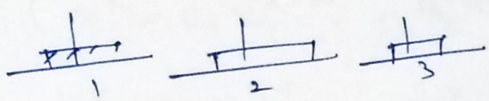
SOLUTION:-

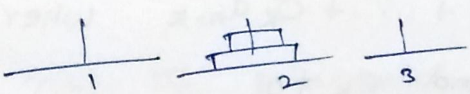
Let H_n denotes the number of moves required to solve the problem. Set up a recurrence relation to the sequence $\{H_n\}$.

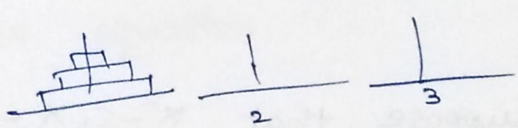
Eg:- 1)  ⇒ 1 move

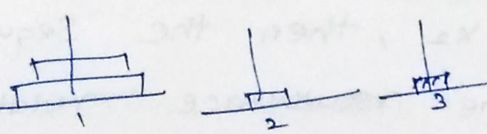
2) 

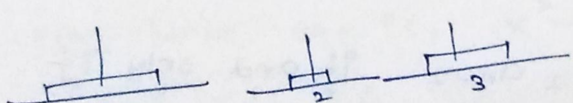
⇒  ⇒ 3 move

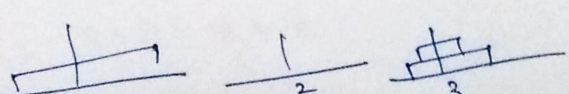
⇒ 

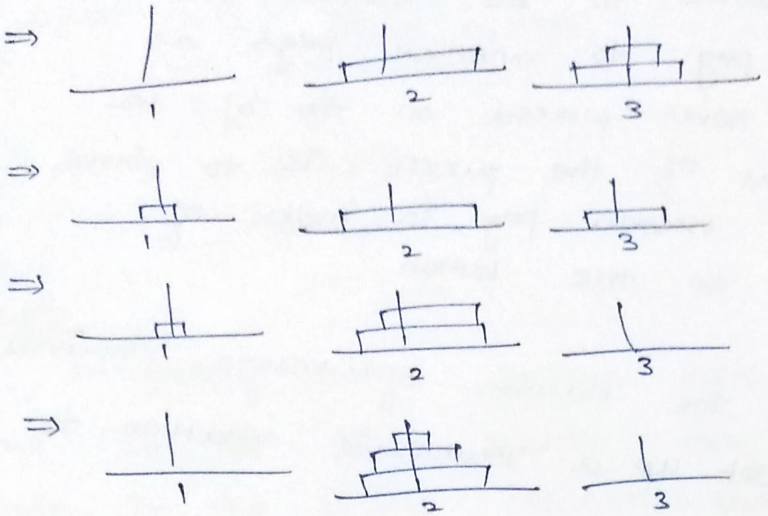
⇒ 

3) 

⇒ 

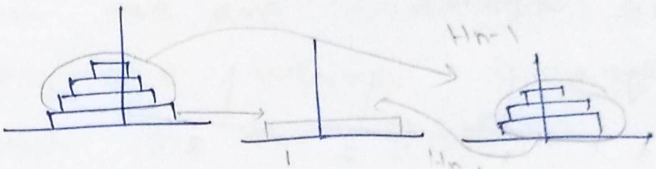
⇒ 

⇒ 



$\Rightarrow 7^{th}$ move

$= 1, 3, 7, \ldots = 2^n - 1$



$H_n = H_{n-1} + 1 + H_{n-1}$

$H_n = 2H_{n-1} + 1$

SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATION :

DEFN :-

A linear homogeneous recurrence relation of degree k with constant co-eff is a recurrence relation of the form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ where

$c_1, c_2, \ldots c_k$ are constants and $c_k \neq 0$

THEOREM :1 :

Let $c_1, c_2 \in \mathbb{R}$, suppose that $x^2 - c_1 x - c_2 = 0$ has two roots x_1 and x_2, then the sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 x_1^n + \alpha_2 x_2^n$$ where α_1 and α_2 are constants and $n = 0, 1, 2, \ldots$

THEOREM:2:

Let $c_1, c_2 \in \mathbb{R}$ and $c_2 \neq 0$, suppose that $x^2 - c_1 x - c_2 = 0$ has only one root x_0. then the seq $\{a_n\}$ is a solution of the rec rel,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$ if and only if

$$a_n = \alpha_1 (x_1)^n + \alpha_2 n (x_1)^n$$ where α_1 and α_2 are constants and $n = 0, 1, 2, \ldots$

$$\Rightarrow \quad a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

$$a_n = x^n \mid a_{n-1} = x^{n-1} \mid a_{n-2} = x^{n-2} \mid \ldots \mid a_{n-k} = x^{n-k}$$

Now

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} \ldots + c_k a_{n-k} = 0$$

$$x^n - c_1 x^{n-1} - c_2 x^{n-2} \ldots - c_k x^{n-k} = 0$$

$$x^{n-k}(x^k - c_1 x^{k-1} - c_2 x^{k-2} \ldots - c_k) = 0$$

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} \ldots - c_k = 0$$

Every root of polynomial is the solution of the recurrence equation.

Ques :-

$$a_n = 5a_{n-1} - 6a_{n-2}$$

Now,

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

The characteric eqn is, $x^2 - 5x + 6 = 0$

$$(x-3)(x-2) = 0$$

$$x_1 = 3 \mid x_2 = 2$$

$\therefore \quad a_n = \alpha_1(2)^n + \alpha_2(3)^n$

when initial condition $a_0 = 1, a_1 = 4$

$a_0 = 1, n = 0$ $\qquad a_1 = 4, n = 1$

$a_0 = \alpha_1 + \alpha_2$ $\qquad a_1 = 2\alpha_1 + 3\alpha_2$

$1 = \alpha_1 + \alpha_2$ $\qquad 4 = 2\alpha_1 + 3\alpha_2$

$\Rightarrow \quad 2\alpha_1 + 3\alpha_2 = 4$

$\underline{2\alpha_1 + 2\alpha_2 = 2}$

$\boxed{\alpha_2 = 2}$

when $\alpha_2 = 2$

$\Rightarrow \alpha_1 = -\alpha_2 + 1$

$\boxed{\alpha_1 = -1}$

$\therefore$

$$a_n = -(2)^n + 2(3)^n, \quad n \geq 0$$

Ques:

$$a_n = 6a_{n-1} - 9a_{n-2} \mid a_0 = 4 \mid a_1 = 6$$

Soln:-

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

The characteristic eqn is,

$$x^2 - 6x + 9 = 0$$

$$(x-3)(x-3) = 0$$

$$x_0 = 3 \mid x_0 = 3$$

$$\therefore \quad a_n = \alpha_1(3)^n + \alpha_2 n(3)^n$$

when $a_0 = 4, n = 0$

$a_0 = \alpha_1 + 0$

$\boxed{\alpha_1 = 4}$

when $a_1 = 6, n = 1$

$a_1 = 3\alpha_1 + 3\alpha_2$

$6 = 12 + 3\alpha_2$

$$3\alpha_2 = -6$$

$$\boxed{\alpha_2 = -2}$$

$$\therefore \quad a_n = 4\,3^n - 2n\,3^n \quad , n \geq 0$$

Ques:

$a_n = \alpha_1$ degree $= 5$

roots $= -1, -1, 2, 2, 3$

Soln:-

$$a_n = \alpha_1 (-1)^n + \alpha_2 n (-1)^n + \alpha_3 (2)^n + \alpha_4 n (2)^n + \alpha_5 (3)^n$$

Ques:

degree $= 10$

roots $= -2, -2, -2, \cancel{-3}\ 3, 3, 3, 4, 4, 5$

Soln :-

$$a_n = \alpha_1 (-2)^n + \alpha_2 n (-2)^n + \alpha_3 (n)^2 (-2)^n + \alpha_4 n^3 (-2)^n +$$

$$\alpha_5 (3)^n + \alpha_6 n (3)^n + \alpha_7 n^2 (3)^n + \alpha_8 (4)^n + \alpha_9 n (4)^n + \alpha_{10} (5)^n$$

State the recurrence relation:-

$$x_n = 6x_{n-1} - 11 x_{n-2} + 6 x_{n-3}$$

$$x_0 = 2, \quad x_1 = 5, \quad x_2 = 15$$

Solution:

$$x_n - 6x_{n-1} + 11 x_{n-2} - 6x_{n-3} = 0$$

The characteristic eqn is

$$x^3 - 6x^2 + 11x - 6 = 0$$

Let $x = 1$,

$$1 - 6 + 11 - 6 = 0$$

$\therefore$ $(x-1)$ is the root of the equation.

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\therefore \quad x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$\therefore \quad x_2 = 3, \quad x_3 = 2$

$\therefore$ The roots are $x_1 = 1, \quad x_2 = 2, \quad x_3 = 3$

The soln is,

$x_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n \quad , n \geq 3 \rightarrow (*)$

when $x_0 = 2$

$x_0 = \alpha_1 + \alpha_2 + \alpha_3 \Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 2 \rightarrow (1)$

When $x_1 = 5$

$x_1 = \alpha_1 (1) + \alpha_2 (2) + \alpha_3 (3)$

$\Rightarrow \quad \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \quad \rightarrow (2)$

when $x_2 = 15$

$x_2 = \alpha_1 (1) + \alpha_2 (4) + \alpha_3 (9)$

$\Rightarrow \quad \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \quad \rightarrow (3)$

$(1) \Rightarrow \quad \boxed{\alpha_1 = 2 - \alpha_2 - \alpha_3} \quad \Rightarrow \alpha_1 = 2 - 3 + 2\alpha_3 - \alpha_3$

$(2) \Rightarrow \quad \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \qquad \boxed{\alpha_1 = -1 + \alpha_3}$

$2 - \alpha_2 - \alpha_3 + 2\alpha_2 + 3\alpha_3 = 5$

$2 + \alpha_2 + 2\alpha_3 = 5$

$\boxed{\alpha_2 = 3 - 2\alpha_3}$

③ $\Rightarrow$ $\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$

$-1 + \alpha_3 + 4(3 - 2\alpha_3) + 9\alpha_3 = 15$

$-1 + \alpha_3 + 12 - 8\alpha_3 + 9\alpha_3 = 15$

$2\alpha_3 = 4$

$\boxed{\alpha_3 = 2}$

$\alpha_1 = -1 + \alpha_3 \Rightarrow \boxed{\alpha_1 = 1}$

$\alpha_2 = 3 - 2\alpha_3 \Rightarrow \boxed{\alpha_2 = -1}$

$\therefore \alpha_1 = 1, \quad \alpha_2 = -1, \quad \alpha_3 = 2$

(*) $\Rightarrow$ $x_n = 1^n - 2^n + 2\,3^n$

Solve the recurrence relation,

$f_n = f_{n-1} + f_{n-2}, \quad f_0 = 0, \quad f_1 = 1$

Soln:-

$f_n - f_{n-1} - f_{n-2} = 0$

The characteristic eqn,

$x^2 - x - 1 = 0$

$x = \dfrac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$

$\therefore$ The roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$

The soln is,

$f_n = x_n = \alpha_1 \left[\frac{1+\sqrt{5}}{2}\right]^n + \alpha_2 \left[\frac{1-\sqrt{5}}{2}\right]^n \rightarrow$ (*)

when $f_0 = 0$

$f_0 = \alpha_1 + \alpha_2$

$\boxed{\alpha_1 + \alpha_2 = 0} \rightarrow$ ①

$\alpha_1 = -\alpha_2$

when, $f_1 = 1$

$$f_1 = \alpha_1\left[\frac{1+\sqrt{5}}{2}\right] + \alpha_2\left[\frac{1-\sqrt{5}}{2}\right]$$

$$1 = \alpha_1\frac{(1+\sqrt{5})}{2} + \alpha_2\frac{(1-\sqrt{5})}{2} \longrightarrow (2)$$

when $\alpha_1 = -\alpha_2$

$$1 = -\alpha_2\frac{(1+\sqrt{5})}{2} + \alpha_2\frac{(1-\sqrt{5})}{2}$$

$$1 = \frac{-\cancel{\alpha_2} - \alpha_2\sqrt{5} + \cancel{\alpha_2} - \alpha_2\sqrt{5}}{2}$$

$$1 = -\frac{\cancel{2}\alpha_2\sqrt{5}}{\cancel{2}}$$

$$\alpha_2 = -1/\sqrt{5}$$

$$\alpha_1 = 1/\sqrt{5}$$

$$(*) \Rightarrow f_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right]^n$$

$$= \frac{1}{\sqrt{5}}\left[\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^n\right]$$

Solve the equation:-

$$a_n = 8a_{n-2} - 16a_{n-4}$$

Soln:-

$$a_n - 8a_{n-2} + 16a_{n-4} = 0$$

The characteristic eqn is,

$$x^4 - 8x^2 + 16 = 0$$

Let $y = x^2$

$$y^2 - 8y + 16 = 0$$

$$(y-4)(y-4) = 0$$

$$\boxed{y=4} \mid \boxed{y=4}$$

16
-4 -4
-8

$\therefore$ Then the value of x is $+2, -2, +2, -2$

The gen eqn;

$$x_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 (-2)^n + \alpha_4 n (-2)^n$$

A recursive relation of the form,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

for $F(n) \neq 0$ is said to be non-homogeneous

Its associated homogeneous rec rel is,

$$a_n^{(h)} = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \rightarrow (2)$$

Theorem,

The solution of the non-homogeneous rec rel (1) is of the form,

$$a_n = (a_n)^{(h)} + a_n^{(p)}$$

where $a_n^{(h)}$ = Soln of eqn (2)

$a_n^{(p)}$ = particular soln of eqn (1)

Note, Here we consider only,

$F(n)$ = Exponential $\times$ Polynomial in 'n'.

Eg:

1. $F(n) = 3^n (n^2 + 2n - 3)$
2. $F(n) = (-1)^n (1)$
3. $F(n) = 3n + 1 \; (1^n)$

Theorem:-

Consider the non-homogeneous rec rel

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$

where $F(n) = t^n$ (Polynomial in n)

<u>case i</u>:- If t is not a root of $x^k - c_1 x^{k-1} - c_2 x^{k-2} \cdots c_k = 0$ then there is particular solution of the form,

$$a_n^{(p)} = t^n (p_0 + p_1 n + p_2 n^2 + \cdots + p_q n^q)$$

<u>case ii</u>:- If t is a root of $x^k - c_1 x^{k-1} \cdots - c_k = 0$ of multiplicity m, then there is a particular soln of the form.

$$a_n^{(p)} = t^n \times n^m (p_0 + p_1 n + \cdots + p_q n^q)$$

<u>Solve</u>:

i] $a_n = 5a_{n-1} - 6a_{n-2} + 5^n (3n^2+1)$

ii] $a_n = 5a_{n-1} - 6a_{n-2} + 2^n (n-7)$

i] <u>Soln</u> The associated rec rel is

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \quad \rightarrow 1$$

$\therefore$ The soln of 1 is,

$$a_n^{(h)} = \alpha_1 2^n + \alpha_2 3^n$$

Particular soln is $F(n) = 5^n (3n^2+1)$

$\therefore$ 5 is not a root of $\kappa a n$

$$a_n^{(p)} = 5^n (p_0 + p_1 n + p_2 n^2)$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = \alpha_1 2^n + \alpha_2 3^n + 5^n (P_0 + P_1 n + P_2 n^2)$$

ii) SOLN:-

$$a_n - 5a_{n-1} + 6a_{n-2} = 2^n (n-7)$$

where the chara eqn,

$$x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0$$

$$x = 2, x = 3$$

$\therefore$ The Homogeneous eqn,

$$a_n^{(h)} = \alpha_1 (2)^n + \alpha_2 (3)^n$$

Here $F(n) = 2^n (n-7)$

$$a_n^{(p)} = 2^n \times n^1 \times (P_0 + P_1 n)$$

$$\therefore a_n = a_n^{(h)} + a_n^{(p)}$$

$$= \alpha_1 (2)^n + \alpha_2 (3)^n + n 2^n (P_0 + P_1 n)$$

Ques :

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

i) $F(n) = 3^n$

ii) $F(n) = (-3)^n (5n+1)$

iii) $F(n) = 3^n n^3$

iv) $F(n) = 2^n (n-1)$

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

The charc eqn,

$$x^2 - 6x + 9 = 0$$

$$(x-3)(x-3) = 0$$

$$\therefore x = 3 \mid x = 3$$

The homogeneous eqn is

$$a_n^{(h)} = \alpha_1 (3)^n + \alpha_2 n (3)^n$$

i] The particular soln,

$$F(n) = 3^n$$

$$a_n^{(p)} = 3^n n^2 \times P_0$$

ii] The particular soln,

$$F(n) = (-3)^n (5n+1)$$

$$a_n^{(p)} = (-3)^n (P_0 + P_1 n)$$

iii) The particular soln,

$$F(n) = 3^n n^3$$

$$a_n^{(p)} = 3^n n^2 \times (P_0 + P_1 n + P_2 n^2 + P_3 n^3)$$

iv) The particular soln,

$$F(n) = 2^n (n-1)$$

$$a_n^{(p)} = 2^n [P_0 + P_1 n]$$

v] The particular soln,

$$F(n) = n 3^n$$

$$a_n^{(p)} = 3^n n^2 (P_0 + nP_1)$$

vi] The particular soln,

$$F(n) = n^2 2^n$$

$$a_n^{(p)} = 2^n (P_0 + nP_1 + n^2 P_2)$$

vii] The particular soln

$$F(n) = 3^n (n^2+1)$$

$$a_n^{(p)} = 3^n n^2 (P_0 + P_1 n + n^2 P_2)$$

Solve the rec rel:

$$a_n = 4a_{n-1} - 3a_{n-2} - 200$$

where $a_0 = 3000$, $a_1 = 3500$

Solution,

$$a_n - 4a_{n-1} + 3a_{n-2} = -200$$

Now, The associated homogeneous rec rel is,

$$x^2 - 4x + 3 = 0$$

$$(x-1)(x-3) = 0$$

$$x = 1 \mid x = 3$$

Now,

$$a_n^{(h)} = \alpha_1 (1)^n + \alpha_2 (3)^n$$

$$a_n^{(h)} = \alpha_1 + (\alpha_2)(3)^n$$

where $a_0 = 3000$,

$$3000 = \alpha_1 + \alpha_2 \longrightarrow (1)$$

when $a_1 = 3300$

$$3300 = \alpha_1 + 3\alpha_2 \longrightarrow (2)$$

Eqn (2) - (1),

$$2\alpha_2 = 300$$

$$\boxed{\alpha_2 = 150}$$

$$\alpha_1 = 2850 \quad] \longrightarrow \text{no need.}$$

Here

$$F(n) = -200 \times (1)^n$$

$$a_n^{(P)} = (1)^n \times P_0 \times n$$

$$a_n^{(P)} = nP_0 \longrightarrow (3)$$

Rewrite the given equation in P,

$$a_n^{(P)} - 4a_{n-1}^{(P)} + 3a_{n-2}^{(P)} = -200$$

where $a_n^{(P)} = nP_0$

$$a_{n-1}^{(P)} = (n-1)P_0$$

$$a_{n-2}^{(P)} = (n-2)P_0$$

Now

$$nP_0 - 4(n-1)P_0 + 3(n-2)P_0 = -200$$

$$nP_0 - 4nP_0 + 4P_0 + 3nP_0 - 6P_0 = -200$$

$$-2P_0 = -200$$

$$P_0 = 100$$

$$\boxed{a_n^{(P)} = 100n}$$

The soln is,

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = \alpha_1 + \alpha_2 (3)^n + 100n$$

When $a_0 = 3000$, put $n=0$

$$3000 = \alpha_1 + \alpha_2 + 0$$

$$\alpha_1 + \alpha_2 = 3000 \longrightarrow (4)$$

when $a_1 = 3300$, put $n=1$

$$3300 = \alpha_1 + 3\alpha_2 + 100$$

$$\alpha_1 + 3\alpha_2 = 3200 \longrightarrow (5)$$

Solving (4) and (5), (5) − (4) ⇒

$$2\alpha_2 = 200$$

$$\boxed{\alpha_2 = 100}$$

$$\alpha_1 = 2900$$

∴ The soln is,

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = 2900 + 100(3)^n + 100n$$

Solve :

$$a_n = 3a_{n-1} + 2n$$

where $a_1 = 3$

Soln,

Let, $a_n - 3a_{n-1} = 2n$

$\therefore$ The associated homogeneous equation is

$$x - 3 = 0$$

$$\boxed{x = 3}$$

$$\therefore \quad a_n^{(h)} = \alpha_1 (3)^n$$

The function, $F(n) = 2n$

$$a_n^{(p)} = P_0 + nP_1 \quad \rightarrow (1)$$

Now

Substitute in the given eqn,

$$a_n - 3a_{n-1} = 2n \quad \rightarrow (2)$$

where

$$a_n = P_0 + nP_1$$

$$a_{n-1} = P_0 + (n-1)P_1$$

$$\Rightarrow \quad P_0 + nP_1 - 3[P_0 + (n-1)P_1] = 2n$$

$$P_0 + nP_1 - 3P_0 - 3nP_1 + 3P_1 = 2n$$

$$-2P_0 - 2nP_1 + 3P_1 = 2n$$

Equating the co-efficients of like powers of n

$$-2P_1 = 2 \qquad\qquad -2P_0 + 3P_1 = 0$$

$$P_1 = -1 \qquad\qquad -2P_0 = 0 + 3$$

$$P_0 = -3/2$$

$a_n^{(p)} = -n - 3/2$

Now,

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = \alpha_1 (3)^n - n - 3/2 \longrightarrow$ (*)

when $a_1 = 3$ where $n = 1$

$a_1 = \alpha_1 (3) - 1 - 3/2$

$3 = 3\alpha_1 - 1 - 3/2$

$3\alpha_1 = 4 + 3/2$

$3\alpha_1 = 11/2$

$\alpha_1 = 11/6$

The solution is,

(*) $\Rightarrow$ $a_n = \frac{11}{6}(3)^n - n - \frac{3}{2}$

Ques :

$a_n - 8a_{n-1} + 16a_{n-2} = 8(5)^n + 6(4)^n$

$\therefore$ Soln $\Rightarrow$ $a_n = a_n^{(h)} + a_n^{(P_1)} + a_n^{(P_2)}$

Solving linear Homogeneous Recurrence relations with constant co-efficients :-

1) QUES :-

$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$

$a_n - a_{n-1} - 2a_{n-2} = 0$

The charac eqn is,

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, \ x = -1$

<u>Now</u>

$$a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n$$

when $a_0 = 2, n = 0$

$$a_0 = \alpha_1 + \alpha_2$$

$$2 = \alpha_1 + \alpha_2 \longrightarrow ①$$

when $a_1 = 7, \ n = 1$

$$a_1 = \alpha_1 (2) + \alpha_2 (-1)$$

$$7 = 2\alpha_1 - \alpha_2 \longrightarrow ②$$

Add ① and ②,

$$3\alpha_1 = 9$$

$$\boxed{\alpha_1 = 3}$$

① $\Rightarrow$ $\alpha_1 + \alpha_2 = 2$

$$3 + \alpha_2 = 2$$

$$\boxed{\alpha_2 = -1}$$

$$\therefore a_n = -1 (2)^n + 3(-1)^n$$

2) QUES:-

$$a_n = 6a_{n-1} - 9a_{n-2}$$

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

The charac eqn is,

$$x^2 - 6x + 9 = 0$$

$$(x-3)(x-3) = 0$$

$$x_0 = 3, \ x_0 = 3$$

<u>Now</u>,

$$a_n = \alpha_1 (3)^n + \alpha_2 n (3)^n$$

when $a_0 = 1$

$$a_0 = \alpha_1 + 0$$

$$\boxed{\alpha_1 = 1}$$

When $a_1 = 6$

$$a_1 = 3\alpha_1 + 3\alpha_2$$

$$6 = 3 + 3\alpha_2$$

$$3\alpha_2 = 3$$

$$\boxed{\alpha_2 = 1}$$

$\therefore$ The soln is,

$$a_n = 1(3)^n + n(3)^n$$

3) <u>QUES:-</u>

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

$$a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$$

The charac eqn is

$$x^3 - 6x + 11x - 6 = 0$$

Since $x = 1$, is a root, $(x-1)$ is a root

1	1	−6	11	−6
	0	1	−5	6
	1	−5	6	0

$$\therefore x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0$$

$$\therefore x = 2 \mid x = 3$$

$\therefore$ $x_0 = 1, x_1 = 2, x_2 = 3$ is the roots.

$$a_n = \alpha_1(1)^n + \alpha_2(2)^n + \alpha_3(3)^n$$

When $a_0 = 2$, $n = 0$

$a_0 = \alpha_1 + \alpha_2 + \alpha_3$

$\alpha_1 + \alpha_2 + \alpha_3 = 2 \longrightarrow (1)$

When $a_1 = 5$, $n = 1$

$a_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3$

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \longrightarrow (2)$

When $a_2 = 15$, $n = 2$

$a_2 = \alpha_1 + 4\alpha_2 + 9\alpha_3$

$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \longrightarrow (3)$

(1) $\Rightarrow$ $\alpha_3 = 2 - \alpha_1 - \alpha_2 \Rightarrow \alpha_3 = 2 - \alpha_1 \mp 1 + 2\alpha_1$

$= 1\cancel{3} + \cancel{3}\alpha_1$

$= 1 + \alpha_1$

(2) $\Rightarrow$ $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$

$\alpha_1 + 2\alpha_2 + 3(2 - \alpha_1 - \alpha_2) = 5$

$\alpha_1 + 2\alpha_2 + 6 - 3\alpha_1 - 3\alpha_2 = 5$

$-2\alpha_1 - \alpha_2 = -1$

$\alpha_2 = -2\alpha_1 + 1$

$\alpha_2 = 1 - 2\alpha_1$

(3) $\Rightarrow$

$\alpha_1 + 4(1 - 2\alpha_1) + 9(1 + \alpha_1) = 15$

$\alpha_1 + 4 - 8\alpha_1 + 9 + 9\alpha_1 = 15$

$\overset{2}{\cancel{-16}}\alpha_1 + 13 = 15$

$\overset{2}{\cancel{-16}}\alpha_1 = 2$

$2\alpha_1 = \cancel{-1/8}\ 2$

$\boxed{\alpha_1 = 1}$

$\alpha_2 = -1$ | $\alpha_3 = 2$

$\therefore \quad a_n = (1)^n - (2)^n + 2(3)^n$

4. **QUES**:-

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

where

$$a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$$

$$x^3 + 3x^2 + 3x + 1 = 0$$

wher $x = -1$

$$-1 + 3 - 3 + 1 = 0$$

$\therefore \quad (x+1)$ is a root,

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$$\therefore \quad x^2 + 2x + 1 = 0$$

$$(x+1)(x+1) = 0$$

$$x = -1 \mid x = -1$$

$\therefore \quad x_0 = -1, \; x_0 = -1, \; x_0 = -1$ is the roots,

$$a_n = \alpha_1(-1)^n + \alpha_2 n(-1)^n + \alpha_3 n^2 (-1)^n$$

when $a_0 = 1$

$$\boxed{1 = \alpha_1}$$

when $a_1 = -2$

$$-2 = \alpha_1(-1) + \alpha_2(-1)^1 + \alpha_3(-1)$$

$$-2 = -\alpha_1 - \alpha_2 - \alpha_3$$

$$-2 = -1 - \alpha_2 - \alpha_3$$

$$\alpha_2 + \alpha_3 = 1 \longrightarrow (1)$$

when,

$a_2 = -1$

$a_2 = \alpha_1(1) + 2\alpha_2 + 4\alpha_3$

$-1 = 1 + 2\alpha_2 + 4\alpha_3$

$2\alpha_2 + 4\alpha_3 = -2$

$\alpha_2 + 2\alpha_3 = -1 \longrightarrow (3)$

Sub (2) $-$ (1),

$\boxed{\alpha_3 = -2}$

(4) $\Rightarrow$ $\alpha_2 - 2 = 1$

$\boxed{\alpha_2 = 3}$

$\therefore \quad a_n = (-1)^n + 3n(-1)^n - 2(-1)^n n^2$

(5) **QUES:-**

$a_n = 5a_{n-1} - 6a_{n-2}$

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

The charac eqn,

$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$\therefore$ The $x_1 = 2, \ x_2 = 3$

$a_n = \alpha_1 (2)^n + \alpha_2 (3)^n$

when $a_0 = 1$

$a_0 = \alpha_1 + \alpha_2$

$\alpha_1 + \alpha_2 = 1 \longrightarrow (1)$

when $a_1 = 0$

$a_1 = 2\alpha_1 + 3\alpha_2$

$2\alpha_1 + 3\alpha_2 = 0 \longrightarrow (2)$

$$2\alpha_1 + 3\alpha_2 = 0$$
$$2\alpha_1 + 2\alpha_2 = 2$$

$$\boxed{\alpha_2 = -2}$$

$\Rightarrow$ $\alpha_1 + \alpha_2 = 1$

$$\boxed{\alpha_1 = 3}$$

$\therefore$ $a_n = 3(2)^n - 2(3)^n$

6. <u>QUES</u> :

$$a_n = 4a_{n-1} - 4a_{n-2}$$

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

The charac eqn,

$$x^2 - 4x + 4 = 0$$

$$(x-2)(x-2) = 0$$

$$x_0 = 2 \mid x_0 = 2$$

$\therefore$ The eqn is

$$a_n = \alpha_1(2)^n + \alpha_2 n(2)^n$$

when $a_0 = 6$

$$a_0 = \alpha_1 + 0$$

$$\boxed{\alpha_1 = 6}$$

when $a_1 = 8$

$$a_1 = \alpha_1(2) + \alpha_2(2)$$

$$8 = 12 + 2\alpha_2$$

$$2\alpha_2 = -4$$

$$\boxed{\alpha_2 = -2}$$

$\therefore$ $a_n = 6(2)^n - 2n(2)^n$

7] QUES :-

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$$

where

$$x^3 - 2x^2 - x + 2 = 0$$

$$\Rightarrow x = 1 \Rightarrow 1 - 2 - 1 + 2 = 0$$

$\therefore$ $(x-1)$ is a root.

$$\begin{array}{c|cccc} 1 & 1 & -2 & -1 & 2 \\ & 0 & +1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$$\therefore x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \mid x = -1$$

$\therefore$ The roots are $x_0 = 1$, $x_1 = 2$, $x_2 = -1$

~~when $a_0 = 3$~~

$$\Rightarrow a_n = \alpha_1 (1)^n + \alpha_2 (2)^n + \alpha_3 (-1)^n$$

when $a_0 = 3$

$$a_0 = \alpha_1 + \alpha_2 + \alpha_3$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 3 \longrightarrow (1)$$

when $a_1 = 6$

$$a_1 = \alpha_1 + 2\alpha_2 + (-1)\alpha_3$$

$$\alpha_1 + 2\alpha_2 - \alpha_3 = 6 \longrightarrow (2)$$

when $a_2 = 0$

$$a_2 = \alpha_1 + 4\alpha_2 + \alpha_3$$

$$\alpha_1 + 4\alpha_2 + \alpha_3 = 0 \longrightarrow (3)$$

Add ① and ②,

$$2\alpha_1 + 3\alpha_2 = 9 \longrightarrow ③$$

Add ② and ③,

$$2\alpha_1 + 6\alpha_2 = 6 \longrightarrow ④$$

Sub ③ - ④,

$$-3\alpha_2 = 3$$

$$\boxed{\alpha_2 = -1}$$

$$\Rightarrow \quad 2\alpha_1 - 3 = 9$$

$$2\alpha_1 = 12$$

$$\boxed{\alpha_1 = 6}$$

$$① \Rightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 = 3$$

$$6 + (-1) + \alpha_3 = 3$$

$$5 + \alpha_3 = 3$$

$$\boxed{\alpha_3 = -2}$$

$\therefore$

$$a_n = 6(1)^n - 1(2)^n - 2(-1)^n$$

8] <u>Ques</u> :

$$a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$$

$$a_n - 2a_{n-1} - 5a_{n-2} + 6a_{n-3} = 0$$

The charc eqn,

$$x^3 - 2x - 5x + 6 = 0$$

when $x = 1$, $1 - 2 - 5 + 6 = 0$

$(x-1)$ is a root,

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & 0 & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$$\begin{array}{ccc} & -6 & \\ -3 & \times & 2 \\ & -1 & \end{array}$$

$\therefore \; x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$\therefore \; x = 3 \;|\; x = -2$

$\therefore$ The soln is $x = 1, 3, -2$

$\therefore \; a_n = \alpha_1 (1)^n + \alpha_2 (3)^n + \alpha_3 (-2)^n$

when $a_0 = 7$

$\alpha_1 + \alpha_2 + \alpha_3 = 7 \longrightarrow ①$

when $a_1 = -4$

$\alpha_1 + 3\alpha_2 - 2\alpha_3 = -4 \longrightarrow ②$

when, $a_2 = 8$

$\alpha_1 + 9\alpha_2 + 4\alpha_3 = 8 \longrightarrow ③$

② − ① :

$2\alpha_2 - 3\alpha_3 = -11 \longrightarrow ④$

③ − ① :

$8\alpha_2 + 3\alpha_3 = 1 \longrightarrow ⑤$

Add ④ and ⑤,

$10\alpha_2 = -10$

$\boxed{\alpha_2 = -1}$

④ $\Rightarrow \; -2 - 3\alpha_3 = -11$

$-3\alpha_3 = -9$

$\boxed{\alpha_3 = 3}$

$\boxed{\alpha_1 = 5}$

$\therefore \; a_n = 5(1)^n - 1(3)^n + 3(-2)^n$

9] <u>QUES</u> :

$$a_n = 5a_{n-2} - 4a_{n-4}$$

$$a_n - 5a_{n-2} + 4a_{n-3} = 0$$

The charc eqn is

$$x^4 - 5x^2 + 4 = 0$$

where $y = x^2$

$$y^2 - 5y + 4 = 0$$

$$(y-1)(y-4) = 0$$

$y = 1 \mid y = 4$

$x^2 = 1 \mid y = 4$

$x = 1, -1 \mid x^2 = 4$

$x = 1, -1 \mid x = 2, -2$

$$\therefore \quad a_n = \alpha_1 (1)^n + \alpha_2 (-1)^n + \alpha_3 (2)^n + \alpha_4 (-2)^n$$

when $a_0 = 3$,

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 3 \rightarrow (1)$

when $a_1 = 2$

$\alpha_1 - \alpha_2 + \overset{2}{4}\alpha_3 + \overset{2}{4}\alpha_4 = 2 \rightarrow (2)$

when $a_2 = 6$

$\alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4 = 6 \rightarrow (3)$

when $a_3 = 8$

$\alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4 = 8 \rightarrow (4)$

$\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = 0$

$$a_n = (1)^n + (-1)^n + (2)^n$$

$2\alpha_1 + 3\alpha_3 + 3\alpha_4 = 5$

$2\alpha_1 + 12\alpha_3 + 4\alpha_4 = 14$

$9\alpha_3 - 7\alpha_4 = 9 \rightarrow (1)$

$2\alpha_1 + 9\alpha_3 - 7\alpha_4 = 11$

$2\alpha_1 + 6\alpha_3 + 2\alpha_4 = 8$

$3\alpha_3 - 9\alpha_4 = 3 \rightarrow (2)$

$9\alpha_3 - 27\alpha_4 = 9 \rightarrow (3)$

$20\alpha_4 = 0$

$\boxed{\alpha_4 = 0}$

$\boxed{\alpha_3 = 1}$

$\alpha_1 + \alpha_2 = 2$

$\alpha_1 - \alpha_2 = 0$

$2\alpha_1 = 2$

$\boxed{\alpha_1 = 1}$

$\boxed{\alpha_2 = 1}$

10) QUES :-

$$f_n = 3f_{n-2} + 2f_{n-3}$$

$$f_n - 3f_{n-2} - 2f_{n-3} = 0$$

Now, the charac eqn,

$$x^3 + 0x^2 - 3x - 2 = 0$$

$$x^3 - 3x - 2 = 0$$

$\boxed{8-8=0}$

$\therefore$ $(x-2)$ is the root,

2	1	0	-3	-2
	0	2	4	2
	1	2	+1	0

$\therefore$ $x^2 + 2x + 1 = 0$

$(x+1)(x+1) = 0$

$x = -1,\ x = -1$

$\therefore$ $x_0 = 2 \mid x_1 = -1 \mid x_2 = -1$

$\therefore f_n = a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n + \alpha_3 n (-1)^n$

when $f_0 = 2$

$f_0 = \alpha_1 + \alpha_2$

$\alpha_1 + \alpha_2 = 2 \longrightarrow (1)$

$\boxed{\alpha_3 = 1 \quad \alpha_1 = 1 \quad \alpha_2 = 1}$

when $f_2 = 7$

$f_2 = 4\alpha_1 + \alpha_2 + 2\alpha_3$

$4\alpha_1 + \alpha_2 + 2\alpha_3 = 7 \longrightarrow (2)$

$\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 1$

$a_n = (2)^n + (-1)^n + n(-1)^n$

Problems :-

1) Ques :-

Refer front

2) Ques :-

$$a_n = 6a_{n-1} \mp 12a_{n-2} + 8a_{n-3} + F(n)$$

Now,

$$a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$$

Now

$$x^3 - 6x^2 + 12x - 8 = 0$$

when $x = 2$ | $(x-2)$ is a root

$\therefore$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & 0 & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\therefore$ $x^2 - 4x + 4 = 0$

$(x-2)(x-2) = 0$

$x = 2$ / $x = 2$

$\therefore$ The roots are $x_0 = 2,\ x_1 = 2,\ x_2 = 2$

$$a_n^{(h)} = \alpha_1 (2)^n + \alpha_2 n (2)^n + \alpha_3 n^2 (2)^n$$

a) $F(n) = n^2$

$a_n^{(P)} = (P_0 + P_1 n)$

b) $F(n) = 2^n$

$a_n^{(P)} = 2^n n^3 P_0$

c) $F(n) = n2^n$

$a_n^{(P)} = 2^n n^3 (P_0 + P_1 n)$

d) $F(n) = (-2)^n$

$a_n^{(P)} = (-2)^n P_0$

e) $F(n) = n^2 2^n$

$a_n^{(P)} = 2^n n^3 (P_0 + P_1 n + P_2 n^2)$

g) $F(n) = 3(1)^n$

$a_n^{(P)} = P_0 (1)^n$

f) $F(n) = n^3 (-2)^n$

$a_n^{(P)} = (-2)^n [P_0 + P_1 n + P_2 n^2 + P_3 n^3]$

g h) $F(n) = (-2)^n (n^3 + 3n - 1)$

$a_n^{(P)} = (-2)^n \{P_0 + P_1 n + P_2 n^2 + P_3 n^3]$

3 Ques :-

$a_n = 8a_{n-2} - 16a_{n-4} + F(n)$

Let, $a_n - 8a_{n-2} + 16a_{n-4} = 0$

The charac eqn,

$x^4 - 8x^2 + 16 = 0$

Let $y = x^2$

$y^2 - 8y + 16 = 0$

$(y-4)(y-4) = 0$

$y = 4 \quad | \quad y = 4$

$x^2 = 4 \quad x^2 = 4$

$x = 2, -2 \quad | \quad x = 2, -2$

16

-4×-4

-8

$\therefore$ The roots are $x_0 = 2,\ x_1 = 2,\ x_2 = -2,\ x_3 = -2$

$a_n^{(h)} = \alpha_1 (2)^n + \alpha_2 n (2)^n + \alpha_3 (-2)^n + \alpha_4 n (-2)^n$

a) $F(n) = n^3$

$a_n^{(P)} = P_0 + P_1 n + P_2 n^2 + P_3 n^3$

b) $F(n) = (-2)^n$

$a_n^{(P)} = (-2)^n n^2 P_0$

c) $F(n) = n 2^n$

$a_n^{(P)} = 2^n n^2 (P_0 + P_1 n)$

d) $F(n) = n^2 4^n$

$a_n^{(P)} = 4^n (P_0 + P_1 n + P_2 n^2)$

e) $F(n) = (n^2 - 2)(-2)^n$

$a_n^{(P)} = (-2)^n n^2 (P_0 + P_1 n + P_2 n^2)$

f) $F(n) = 2$

$a_n^{(P)} = (1)^n P_0$

g) $F(n) = n^4 2^n$

$a_n^{(P)} = 2^n n^2 (P_0 + P_1 n + P_2 n^2 + P_3 n^3 + P_4 n^4)$

h) $F(n) = n^4 4^n$

$a_n^{(P)} = 4^n (P_0 + P_1 n + P_2 n^2 + P_3 n^3 + P_4 n^4)$

4] QUES :-

$$a_n = 5a_{n-1} - 6a_{n-2} + F(n)$$

where,

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

Now

$$x^2 - 5x + 6 = 0$$

$$(x-3)(x-2) = 0$$

$$x = 3, \; x = 2$$

$$\therefore a_n^{(h)} = \alpha_1 (3)^n + \alpha_2 (2)^n$$

Now

a) $F(n) = 2n^2$

$$a_n^{(P)} = (P_0 + P_1 n + P_2 n^2)$$

b) $F(n) = 5^n$

$$a_n^{(P)} = 5^n P_0$$

c) $F(n) = 3^n$

$$a_n^{(P)} = 3^n n P_0$$

d) $F(n) = n(-2)^n$

$$a_n^{(P)} = (-2)^n (P_0 + P_1 n)$$

e) $F(n) = 5^n (3n^2 + 2n + 1)$

$$a_n^{(P)} = 5^n (P_0 + P_1 n + P_2 n^2)$$

f) $F(n) = 2^n (3n + 1)$

$$a_n^{(P)} = 2^n n (P_0 + P_1 n)$$

g) $F(n) = n(-2)^n$

$$a_n^{(P)} = (-2)^n (P_0 + P_1 n)$$

h) $F(n) = n^3 2^n + n^3 5^n$

$$a_n^{(P)} = 2^n n (P_0 + P_1 n + P_2 n^2 + P_3 n^3) + 5^n (P_0 + P_1 n + P_2 n^2 + P_3 n^3)$$

i) $F(n) = n^4 2^n + n^3 3^n$

$$a_n^{(P)} = 2^n n (P_0 + P_1 n + P_2 n^2 + P_3 n^3 + P_4 n^4) + 3^n n (P_0 + P_1 n + P_2 n^2 + P_3 n^3)$$

j) $F(n) = n^4 2^n + n^3 3^n$

$$a_n^{(P)} = 2^n n (P_0 + P_1 n + P_2 n^2 + P_3 n^3 + P_4 n^4) + 3^n n (P_0 + P_1 n + P_2 n^2 + P_3 n^3)$$

5] <u>QUES</u> :-

$$a_n = 2a_{n-1} + 2n^2 \longrightarrow (1)$$

<u>Now</u>

$$a_n = 2a_{n-1} \quad | \quad F(n) = 2n^2$$

$$a_n - 2a_{n-1} = 0$$

$$x - 2 = 0$$

$$\boxed{x = 2}$$

$\therefore$ The root are $x = 2$

$$a_n^{(h)} = \alpha_1 (2)^n$$

The particular soln,

$$F(n) = 2n^2$$

$$a_n^{(P)} = P_0 + nP_1 + n^2 P_2$$

Now,

$$a_n = P_0 + nP_1$$

$$a_{n-1} = P_0 + (n-1)P_1 + (n-1)$$

$$(1) \Rightarrow \quad a_n = 2a_{n-1} + 2n^2$$

$$P_0 + nP_1 - 2P_0 - 2(n-1)P_1 = 2n^2$$

$$P_0 + nP_1 - 2P_0 - 2nP_1 + 2P_1 = 2n^2$$

$$-P_0 - nP_1 + 2P_1 = 2n^2$$

$$-P_0 + 2P_1 - nP_1 = 2n^2$$

Compare co-eff on both sides,

$-P_1 = 0$	$-P_0 + 2P_1 = 0$
$P_1 = 0$	$-P_0 + 0 = 0$
	$P_0 = 0$

$$a_n = P_0 + nP_1 + n^2 P_2$$

$$a_n = P_0 + nP_1 + n^2 P_2$$
$$a_{n-1} = P_0 + (n-1)P_1 + (n-1)^2 P_2$$

$$a_n = 2a_{n-1} + 2n^2$$

$$P_0 + nP_1 + n^2P_2 - 2(P_0 + (n-1)P_1 + (n^2 - 2n + 1)P_2) = 2n^2$$

$$P_0 + nP_1 + n^2P_2 - 2P_0 - 2nP_1 + 2P_1 - 2n^2P_2 + 4nP_2 - 2P_2 = 2n^2$$

$$-P_0 + 2P_1 - 2P_2 - nP_1 + 4nP_2 - n^2P_2 = 2n^2$$

Compare on both sides,

$$-P_2 = 2$$
$$\boxed{P_2 = -2}$$

$$-P_1 + 4P_2 = 0$$
$$-P_1 - 8 = 0$$
$$-P_1 = 8$$
$$\boxed{P_1 = -8}$$

$$-P_0 + 2P_1 - 2P_2 = 0$$

$$-P_0 - 16 + 4 = 0$$

$$-P_0 = +12$$

$$P_0 = -12$$

$$\therefore \quad a_n^{(P)} = -12 - 8n - 2n^2$$

$$\therefore \quad a_n = \alpha_1 (2)^n - 12 - 8n - 2n^2 \qquad [a_n^{(h)} + a_n^{(P)}]$$

$\boxed{a_1 = 4}$

$$a_1 = \alpha_1(2) - 12 - 8 - 2$$

$$4 = 2\alpha_1 - 22$$

$$2\alpha_1 = 26$$

$$\boxed{\alpha_1 = 13}$$

$$\therefore \quad a_n = 13(2)^n - 12 - 8n - 2n^2$$

6. Ques :-

$$a_n = 2a_{n-1} + 3^n \qquad a_1 = 5 \longrightarrow (1)$$

where $a_n = 2a_{n-1}$

$$a_n - 2a_{n-1} = 0$$

$$x - 2 = 0 \mid x = 2$$

$\therefore$ The root is $x_0 = 2$

$$\therefore a_n^{(h)} = \alpha_1 (2)^n$$

The particular soln,

$$F(n) = 3^n$$

$$a_n^{(p)} = 3^n p_0$$

$$a_{n-1}^{(p)} = 3^{n-1} p_0$$

$$\therefore a_n - 2a_{n-1} = 3^n$$

$$3^n p_0 - 2(3^{n-1} p_0) = 3^n$$

$$3^n p_0 - 2(\frac{3^n}{3} p_0) = 3^n$$

$$3^n p_0 - \frac{2}{3} 3^n p_0 = 3^n$$

$$3^n (p_0 - \frac{2}{3} p_0) = 3^n$$

Compare on both sides,

$$p_0 - \frac{2}{3} p_0 = 1$$

$$p_0 (1 - 2/3) = 1$$

$$p_0 (1/3) = 1$$

$$\boxed{p_0 = 3}$$

$$a_n^{(p)} = 3^n \cdot 3 = 3^{n+1}$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = \alpha_1 (2)^n + 3^{n+1}$$

$$a_1 = 5$$

$$a_1 = \alpha_1 (2) + 3^2$$

$$5 = 2\alpha_1 + 9$$

$$2\alpha_1 = -4$$
$$\alpha_1 = -2$$
$$a_n = (-2)(2)^n + 3^{n+1}$$

FORMAL LANGUAGE

1. Alphabet / Vocabulary :-

$V = \{a, b, c\}$

eg:- a, b, c → symbols [either be alphabet, letters, symbols]

2. String / words :-

eg: λ, a, aaa, abbcd....

3. Language :-

- Normal Language : speaking languages
- Formal Language : Language genrated using some terms.

$V^* = \{\lambda, a, b, c, aa, \ldots\}$

Vocabulary → null set V^* → null set

4. Grammar :- (Invented by Noam Chomsky)

Four types of Grammar

* Type : 0 - (unrestricted / force structed grammar)
* Type : 1 - Contex Sensitive → Linear
* Type : 2 - Contex Free → Push down
* Type : 3 - Regular → Finite state automata