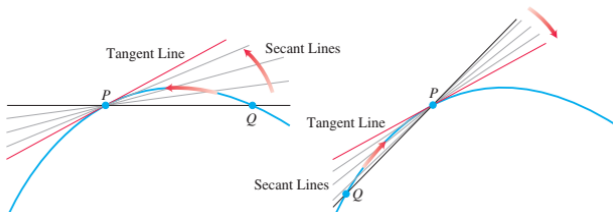


Calculus and its Applications

(Limits and Continuity - Limits)



LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

TEXT BOOKS:

- 1 Thomas G B Jr., Maurice D Wier, Joel Hass, Frank R. Giordano, Thomas' Calculus, Pearson Education, 2018.
- 2 Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley, 2014.

Limit

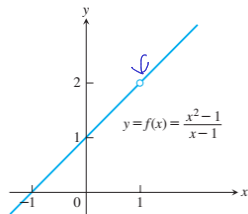
Small change in x produce only small change in $f(x)$.

Limit of a function

When studying a function $y = f(x)$, we find ourselves interested in the function's behavior near a point c but not at c itself.

Example 1

How does the function $f(x) = \frac{x^2-1}{x-1}$ behave near $x = 1$?

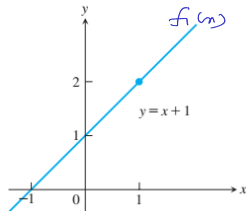


$$= \frac{(x+1)(x-1)}{x-1}, x \neq 1$$

$$f_1(x) = x+1$$

$$D(f) = \mathbb{R} - \{1\}$$

$$D(f_1) = \mathbb{R} = (-\infty, \infty)$$



$$\lim_{x \rightarrow a} f(x) = a$$

$$f(x) = \frac{1}{x} + 1$$

$$\lim_{x \rightarrow 0} f(x) = \text{There!}$$

Range

Definition of limit of a function

Let $f(x)$ be a function defined on an interval about c , *may not be defined* except possibly at c itself. If $f(x)$ is arbitrarily close to the number L for all x sufficiently close to c other than c itself, then we say that $f(x)$ approaches the limit L as x approaches c , that is,

$$\lim_{x \rightarrow c} f(x) = L$$

For $f(x) = \frac{x^2-1}{x-1}$,

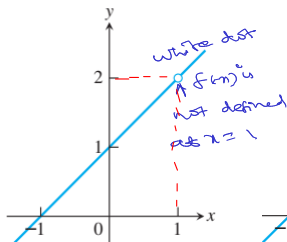
$$\lim_{x \rightarrow 1} f(x) = \underline{\underline{2.}}$$

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2-1}{x-1} &= \lim_{x \rightarrow 1} \frac{(x+1)(x-1)}{\cancel{(x-1)}} \\ &= \lim_{x \rightarrow 1} x+1 \\ &= 1+1 = 2 \end{aligned}$$

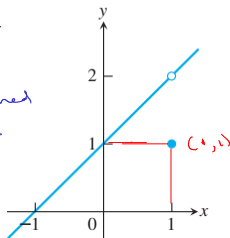
Example 2 Find the limits of the following functions as x approaches 1

$$f(x) = \frac{x^2-1}{x-1} \quad g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1; \\ 1, & x = 1. \end{cases} \quad h(x) = x + 1, \quad x \in (-\infty, \infty)$$

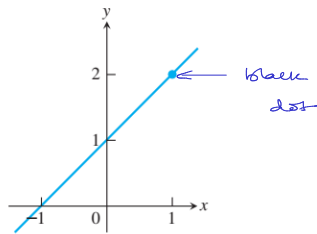
$\rightarrow g(1) = 1$



(a) $f(x) = \frac{x^2-1}{x-1}$



(b) $g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$



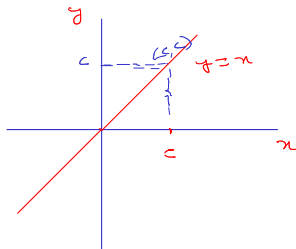
(c) $h(x) = x + 1$

Example 3 Find the limits of the identity function and of a constant function as x approaches c .

Identity fun $f(x) = x$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x$$

$$= c$$



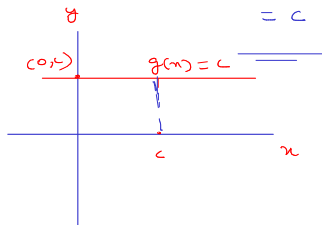
Constant function

$$g(x) = c$$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} c$$

$$= c \lim_{x \rightarrow c} 1$$

$$= c$$

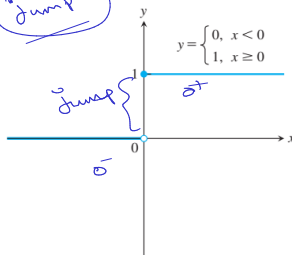


Example 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \rightarrow 0$

$$U(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases} \quad g(x) = \begin{cases} \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

$$f(x) = \begin{cases} 0, & x \leq 0; \\ \sin\left(\frac{1}{x}\right), & x > 0. \end{cases}$$

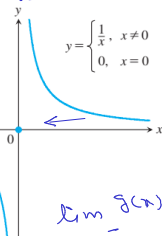
jump



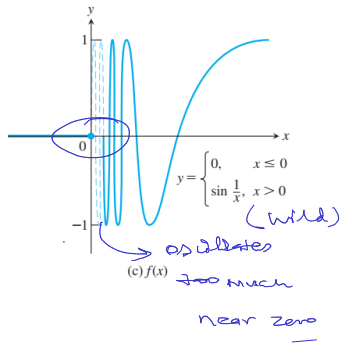
(a) Unit step function $U(x)$

g(x) = 0

$\lim_{x \rightarrow 0^+} g(x) = \infty$



(b) $g(x)$



(c) $f(x)$

Limit laws

Theorem If L , M , c and k are real numbers and $\lim_{x \rightarrow c} f(x) = L$, $\lim_{x \rightarrow c} g(x) = M$ then

- $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm M$

$$\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = L + M$$
- $\lim_{x \rightarrow c} cf(x) = cL$
- $\lim_{x \rightarrow c} f(x) \cdot g(x) = L \cdot M$
- $\lim_{x \rightarrow c} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{M}, M \neq 0$

$$\lim_{x \rightarrow c} [f(x)]^n = \left[\lim_{x \rightarrow c} f(x) \right]^n = L^n$$
- $\lim_{x \rightarrow c} [f(x)]^n = L^n$, where n is the positive integer

$$= \left[\lim_{x \rightarrow c} f(x) \right]^n = L^n = \sqrt[n]{L^n}$$
- $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L}$, where n is the positive integer
 (If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c)

Example 5 Find the limits of the following functions if $\lim_{x \rightarrow c} k = k$ and

$$\lim_{x \rightarrow c} x = c$$

$$\textcircled{1} \lim_{x \rightarrow c} [x^3 + 4x^2 - 3] = \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 + \lim_{x \rightarrow c} (-3) = c^3 + 4c^2 - 3$$

$$\textcircled{2} \lim_{x \rightarrow c} \left[\frac{x^4 + x^2 - 1}{x^2 + 5} \right] = \frac{c^4 + c^2 - 1}{c^2 + 5}$$

$$\textcircled{3} \lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)} = \sqrt{16 - 3} = \sqrt{13}$$

Limits of polynomials If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$, then
 $\lim_{x \rightarrow c} p(x) = p(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_0$.

Limits of rational functions If $p(x)$ and $q(x)$ are polynomials and $q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \left[\frac{p(x)}{q(x)} \right] = \frac{p(c)}{q(c)} \quad \begin{array}{l} \xrightarrow{\text{Handwritten}} \frac{\lim_{x \rightarrow c} p(x)}{\lim_{x \rightarrow c} q(x)} \end{array}$$

Ex 6

Example 6 Evaluate $\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$.

$$\stackrel{\text{Handwritten}}{=} \frac{-1 + 4 - 3}{1 + 5} = \frac{0}{6} = 0$$

Example 7 Evaluate $\lim_{x \rightarrow 1} \frac{x^2+x-2}{x^2-x}$

$$f(x) = \frac{x^2+x-2}{x^2-x} \quad \text{at } x=1 \text{ not defined}$$

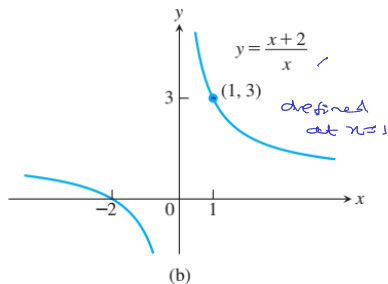
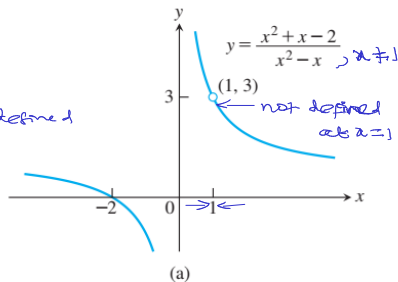
$$f(x) = \frac{x^2+x-2}{x^2-x} \quad (x \neq 1)$$

$$\lim_{x \rightarrow 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \rightarrow 1} \frac{(x+2)(x-1)}{x(x-1)}$$

$$\underline{\underline{= 3}}$$

$$\text{as } x \rightarrow 1$$

$$f(x) \rightarrow 3$$



Sandwich theorem or Squeeze theorem

Set of all $y \in \mathbb{R}$
 $\rightarrow a < y < b$
 \downarrow
 (a, b) $a < y < b$

Suppose that $g(x) \leq f(x) \leq h(x)$ for every x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$ then

$$\lim_{x \rightarrow c} f(x) = L.$$

$\Rightarrow \lim_{n \rightarrow \infty} g(n) = \lim_{n \rightarrow \infty} h(n) = \lim_{n \rightarrow \infty} f(n)$

Example 8 Given a function u that satisfies

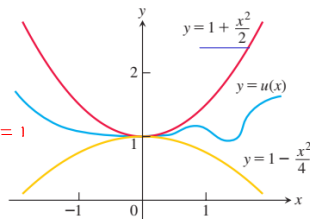
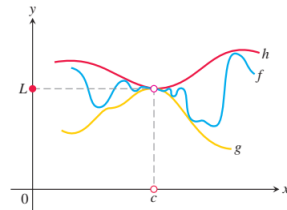
$$1 - \frac{x^2}{4} \leq \underline{u(x)} \leq 1 + \frac{x^2}{4}, \quad \forall x \neq 0$$

find $\lim_{x \rightarrow 0} u(x)$.

$$\lim_{x \rightarrow 0} 1 - \frac{x^2}{4} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} 1 + \frac{x^2}{4} = 1$$

$\Rightarrow L = 1$

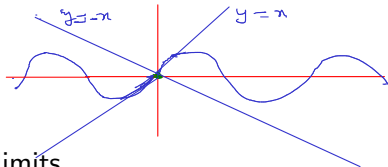
$$\lim_{x \rightarrow 0} u(x) = L = 1$$



think

$$g(x) \leq \sin x \leq h(x)$$

$$-x \leq \sin x \leq x$$



Example 9 Evaluate the following limits

• $\lim_{x \rightarrow 0} \sin \theta = 0$

think • $\lim_{x \rightarrow 0} \cos \theta = 1$ find $g(x)$ & $h(x) \Rightarrow g(x) \leq \cos x \leq h(x)$

• $\lim_{x \rightarrow c} |f(x)| = 0$ implies $\lim_{x \rightarrow c} f(x) = 0$?

$$-|f(x)| \leq f(x) \leq |f(x)|$$

Problems

1. Check whether the limit exist for the following cases or not. In either case give reasons.

(a) $\lim_{x \rightarrow 0} \frac{x}{|x|}$ and (b) $\lim_{x \rightarrow 1} \frac{1}{x-1}$.

$$(a) \quad |x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

When we approach $x=0$ from the left $\lim_{x \rightarrow 0^-} f(x) = -1$,

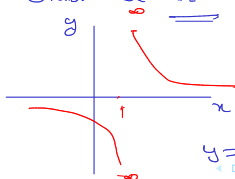
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right $\lim_{x \rightarrow 0^+} f(x) = 1$

\Rightarrow limit does not exist at $x=0$

(b) $\lim_{x \rightarrow 1} \frac{1}{x-1}$

H Shifts right



limit does not exist

$$y = \frac{1}{x-1}$$

2. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x \rightarrow c} f(x)$? Give reasons for your answer.

$$f(x) = \begin{cases} \text{defined} & x \neq c \\ \text{not defined} & x = c \end{cases}$$

$$\lim_{x \rightarrow c} f(x) = ?$$

How

(i) limit exists

(ii) limit does not exist

(iii) no conclusion

$$\checkmark f(x) = \frac{1}{x-1}, \text{ limit does not exist at } x=1$$

$$\checkmark f(x) = \frac{x^2-1}{x-1}, x \neq 1 \quad \text{limit exists at } x=1$$

3. If $\sqrt{5 - 2x^2} \leq f(x) \leq \sqrt{5 - x^2}$ for $-1 \leq x \leq 1$ find $\lim_{x \rightarrow 0} f(x)$.

$\lim_{x \rightarrow 0} f(x)$

$$\lim_{x \rightarrow 0} \sqrt{5 - 2x^2} = \sqrt{5}$$

1 (equal) \Rightarrow

$$\lim_{x \rightarrow 0} f(x) = \sqrt{5}$$

$$\lim_{x \rightarrow 0} \sqrt{5 - x^2} = \sqrt{5}$$

h.w

4. It can be shown that the inequalities $1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$ hold for all values of x close to zero. What, if anything does this tell you about

$$\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}.$$

Definition of limit

(Real)

ϵ, δ are positive real numbers

Let $f(x)$ be defined on an open interval about c , except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L , and write $\lim_{x \rightarrow c} f(x) = L$, if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad |x - c| < \delta$$

(a) $0 < |x - c| < \delta$

(b)

$$|x - c| < \delta$$

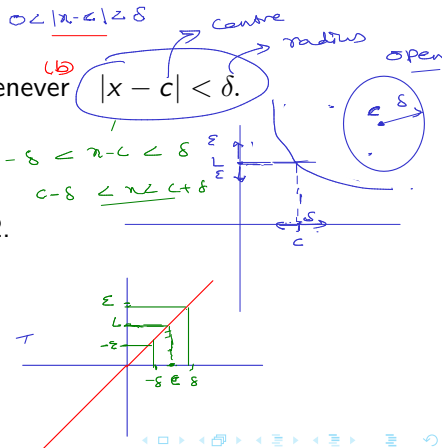
centre

radius

open

How
 $\lim_{x \rightarrow c} f(x) = L$

Example Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.



Approaching a limit from one side



Suppose a function f is defined on an interval that extends to both sides of a number c . In order for f to have a limit L as x approaches c , the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is two-sided.

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$$

If f fails to have a two-sided limit at c , it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a right-hand limit or limit from the right. From the left, it is a left-hand limit or limit from the left.

Left-hand is denoted by $\lim_{x \rightarrow c^-} f(x)$.

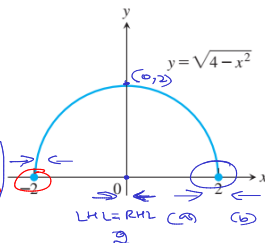
Right-hand is denoted by $\lim_{x \rightarrow c^+} f(x)$.

$$f(x) = x \quad (-\infty, \infty) \quad \text{---} \\ \text{---} \quad [-1, \square]$$

Example 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$. Find the left-hand and right-hand limits of $f(x)$ as x approaches 2.

$$(a) \lim_{x \rightarrow -2} f(x) = 0$$

(b) $\lim_{x \rightarrow 2} f(x)$ does not exist (function is defined only on $[-2, 2]$)
 $\lim_{x \rightarrow -2} f(x)$ does not exist
 $\lim_{x \rightarrow -2^+} f(x) = 0$



Theorem Suppose that a function f is defined on an open interval containing c , except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

Note:

- (i) above theorem is applicable when c is an interior point
 (ii) At boundary points of its domain, function has a limit when it has an appropriate one-sided limit

Example 2 Find left-hand limits, right-hand limits and limits at $x = 0, 1, 2, 3, 4$ for the function graphed below

at $x=0$ (left end point)

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0^+} f(x) = 1$$

at $x=1$ (interior point)

$$\lim_{x \rightarrow 1^-} f(x) = 0, \quad \lim_{x \rightarrow 1^+} f(x) = 1 \Rightarrow \text{LHL} \neq \text{RHL}$$

$$0 \neq 1$$

$$\Rightarrow \text{limit not exists at } x=1$$

at $x=2$ (interior point)

$$\lim_{x \rightarrow 2^-} f(x) = 1, \quad \lim_{x \rightarrow 2^+} f(x) = 1 \Rightarrow \text{LHL} = \text{RHL}$$

$$\text{limit exists at } x=2$$

at $x=3$ (interior point)

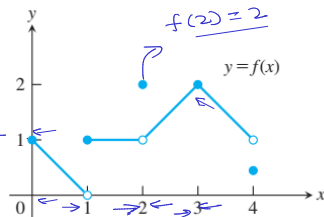
$$\text{LHL} = 2, \quad \text{RHL} = 2 \Rightarrow \text{limit exists at } x=3$$

at $x=4$ (end point)

$$\text{LHL} = 1, \quad \text{RHL (no need to find)} \Rightarrow \text{limit exists at } x=4$$

If c is an interior point

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$$



Domain: $[0, 4]$

left end point = 0

right end point = 4

If c is a left end point

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^+} f(x)$$

If c is a right end point

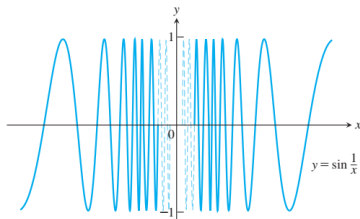
$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^-} f(x)$$

H.W

Example 3 Find $\lim_{x \rightarrow 0^+} \sqrt{x}$.

H.W

Example 4 Show that $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ has no limit as x approaches zero from either side.



Result $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$

tt.w
Example 5 Show that (a) $\lim_{y \rightarrow 0} \frac{\cos y - 1}{y} = 0$ and (b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}.$

Example 6 Find $\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t}$.

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Soln

$$\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t} = \lim_{t \rightarrow 0} \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \cdot \frac{1}{3t}$$

$$= \frac{1}{3} \lim_{t \rightarrow 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t}$$

$$= \frac{1}{3} \left(\lim_{t \rightarrow 0} \frac{\sin t}{t} \right) \left(\lim_{t \rightarrow 0} \frac{1}{\cos t} \right) \left(\lim_{t \rightarrow 0} \frac{1}{\cos 2t} \right)$$

$$= \frac{1}{3} (1) (1) (1)$$

$$= \frac{1}{3}$$

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

a. $\lim_{x \rightarrow -1^+} f(x) = 1$ F

c. $\lim_{x \rightarrow 0^-} f(x) = 1$ F

e. $\lim_{x \rightarrow 0} f(x)$ exists.

g. $\lim_{x \rightarrow 0} f(x) = 1$

i. $\lim_{x \rightarrow 1} f(x) = 0$

k. $\lim_{x \rightarrow -1^-} f(x)$ does not exist.

b. $\lim_{x \rightarrow 0^-} f(x) = 0$ F

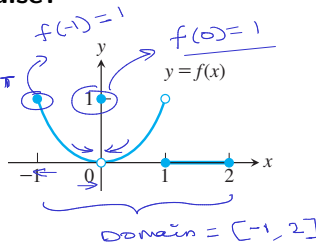
d. $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$ F

f. $\lim_{x \rightarrow 0} f(x) = 0$

h. $\lim_{x \rightarrow 1} f(x) = 1$

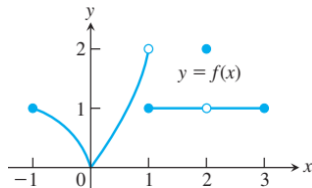
j. $\lim_{x \rightarrow 2^-} f(x) = 2$

l. $\lim_{x \rightarrow 2^+} f(x) = 0$



2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

- a. $\lim_{x \rightarrow -1^+} f(x) = 1$ b. $\lim_{x \rightarrow 2} f(x)$ does not exist.
- c. $\lim_{x \rightarrow 2} f(x) = 2$ d. $\lim_{x \rightarrow 1^-} f(x) = 2$
- e. $\lim_{x \rightarrow 1^+} f(x) = 1$ f. $\lim_{x \rightarrow 1} f(x)$ does not exist.
- g. $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$
- h. $\lim_{x \rightarrow c} f(x)$ exists at every c in the open interval $(-1, 1)$.
- i. $\lim_{x \rightarrow c} f(x)$ exists at every c in the open interval $(1, 3)$.
- j. $\lim_{x \rightarrow -1^-} f(x) = 0$ k. $\lim_{x \rightarrow 3^+} f(x)$ does not exist.



THANK YOU