

Calculus and its Applications

(Limits and Continuity - Combining Functions)

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LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

TEXT BOOKS:

- 1 Thomas G B Jr., Maurice D Wier, Joel Hass, Frank R. Giordano, Thomas' Calculus, Pearson Education, 2018.

Sums, Differences, Products, and Quotients

If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$ and fg by the formulas

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{where } \underline{g(x) \neq 0}$$

EXAMPLE 1

$$1-x \geq 0 \Rightarrow x \leq 1 \Rightarrow x \in (-\infty, 1] \quad D(f) = [0, \infty), \quad D(g) = (-\infty, 1], \quad D(f) \cap D(g) = [0, 1]$$

Let the functions be defined by the formulas $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$. Find the domains of $f(x)$ and $g(x)$. Also find the formulas and domains for the algebraic combinations defined below.

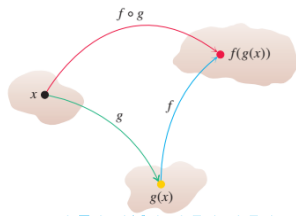
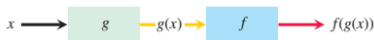
Function	Formula (x)	domain (y)
$f + g$	$\sqrt{x} + \sqrt{1-x}$	$D(f) \cap D(g) = [0, 1]$
$f - g$	$\sqrt{x} - \sqrt{1-x}$	$[0, 1]$
$g - f$	$\sqrt{1-x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$\sqrt{x} \sqrt{1-x}$	$[0, 1]$
f/g	$\frac{\sqrt{x}}{\sqrt{1-x}}$	$[0, 1)$, $x=1$ is excluded because $g(1)=0$
g/f	$\frac{\sqrt{1-x}}{\sqrt{x}}$	$(0, 1]$, $x=0$ is excluded because $f(0)=0$

Composite functions

If f and g are functions, the composite function $f \circ g$ (f composed with g) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f .



EXAMPLE 2

$$f(y) = \sqrt{y}$$

$$g(y) = y + 1$$



If $f(x) = \sqrt{x}$ and $g(x) = x + 1$ find the formulas and domains for the following: $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$ and $(g \circ g)(x)$.

$$f \circ g = f[x+1] = \sqrt{x+1}$$

$$\begin{aligned} x+1 &\geq 0 \\ x &\geq -1 \end{aligned}$$

Domain

$$[-1, \infty)$$

$$g \circ f = g[\sqrt{x}] = \sqrt{x} + 1$$

$$[0, \infty)$$

$$f \circ f = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4} = \sqrt[4]{x}$$

$$[0, \infty)$$

$$g \circ g = g(x+1) = (x+1)+1 = x+2$$

$$(-\infty, \infty)$$

$$(f \circ g)(x) = f[g(x)] = f[x+1] = f(y) = \sqrt{y} = \sqrt{x+1}$$

$$(g \circ f)(x) = g[f(x)] = g[\sqrt{x}] = g(y) = y+1 = \sqrt{x}+1$$

Shifting a graph of a function

$$f(x)$$

$$y = x^2$$

$$y = x^2 + 1$$

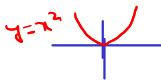
$$y = x^2 - 1$$

Vertical Shifts

$y = f(x) + k$ shifts the graph of f up k units if $k > 0$ or shifts it down $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ shifts the graph of f left h units if $h > 0$ or shifts it right $|h|$ units if $h < 0$



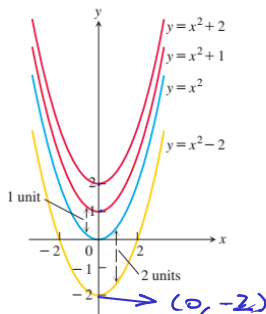
$$y = (x+1)^2$$

$$y = (x-1)^2$$

Example 3a

If $y = x^2$ then mention the type of shifts for the following operations and hence sketch the graph in each cases:

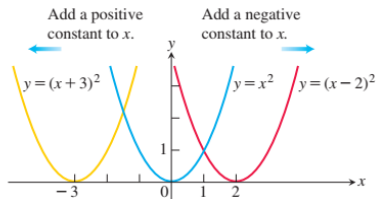
- adding 1 to the right hand side of y , $\Rightarrow y = x^2 + 1$ (V.S)
- adding 2 to the right hand side of y , $\Rightarrow y = x^2 + 2$ (V.S)
- adding -2 to the right hand side of y , $\Rightarrow y = x^2 - 2$ (V.S)



Example 3b

If $y = x^2$ then mention the type of shifts for the following operations and hence sketch the graph in each cases:

- adding 3 to x in $y = x^2$, $\Rightarrow y = (x+3)^2$
- adding -2 to x in $y = x^2$. $\Rightarrow y = (x-2)^2$



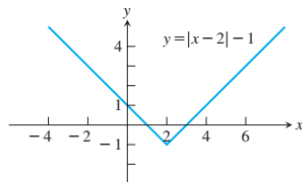
Example 3c

hw

If $y = |x|$ then mention the type of shifts when -2 is added to x in y and then -1 is added to the result. Also sketch the graph for the above.

$\Rightarrow y = |x-2|$ shifted horizontally to the right by 2 units
 $\vee y = |x-2|-1$

shifted vertically downward by 1 unit



Practice problems

new

- Find the domain and ranges of f , g , $f + g$, $f \cdot g$, for $f(x) = x$ and $g(x) = \sqrt{x-1}$
- Find the domain and ranges of f , g , f/g and g/f for $f(x) = \sqrt{x}$ and $g(x) = |x-3|$

①

function	domain	range
$f = x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$g = \sqrt{x-1}$	$[1, \infty)$	$[0, \infty)$
$f+g = x+\sqrt{x-1}$	$[1, \infty)$	$[1, \infty)$
$f \cdot g = x\sqrt{x-1}$	$[1, \infty)$	$[0, \infty)$

②

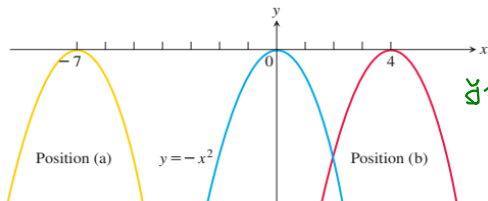
function	domain	range
$f = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$g = x-3 $	$(-\infty, \infty)$	$[0, \infty)$
$\frac{f}{g} = \frac{\sqrt{x}}{ x-3 }$	$[0, \infty) - \{3\}$ $[0, 3) \cup (3, \infty)$	$[0, \infty)$
$\frac{g}{f} = \frac{ x-3 }{\sqrt{x}}$	$(0, \infty)$	$[0, \infty)$

Shifting graphs

vvv

Graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs

$$y = -(x+7)^2$$

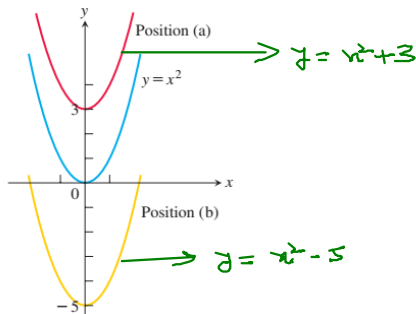


$$y = -(x-4)^2$$

Shifting graphs

hw

Graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs



Shifting graphs

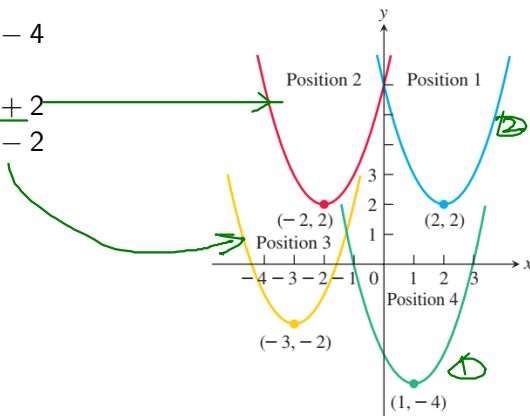
Match the equations listed in part (1)-(4) to the graphs in the following figure.

① $y = (x - 1)^2 - 4$

② $(x - 2)^2 + 2$

③ $y = (x + 2)^2 + 2$

④ $y = (x + 3)^2 - 2$



Vertical and Horizontal Scaling

For $c > 1$, the graph is scaled

- $y = cf(x)$ Stretches the graph of f vertically by a factor of c .
- $y = \frac{1}{c}f(x)$ Compresses the graph of f vertically by a factor of c .
- $y = f(cx)$ Compresses the graph of f horizontally by a factor of c .
- $y = f(x/c)$ Stretches the graph of f horizontally by a factor of c .

Reflection

For $c = -1$, the graph is reflected

- $y = -f(x)$ Reflects the graph of y across the x -axis.
- $y = f(-x)$ Reflects the graph of y across the y -axis

vertical scaling

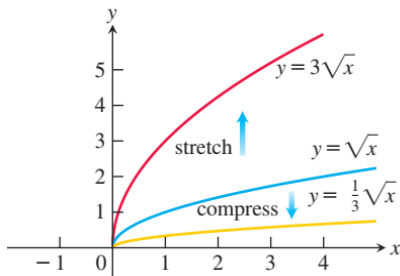
Example 4a Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases

- multiplying the right hand side of y by 3, $y = 3\sqrt{x}$
- multiplying the right hand side of y by $\frac{1}{3}$, $y = \frac{1}{3}\sqrt{x}$

$$y = \sqrt{x}$$

$$x=1 \Rightarrow y=1$$

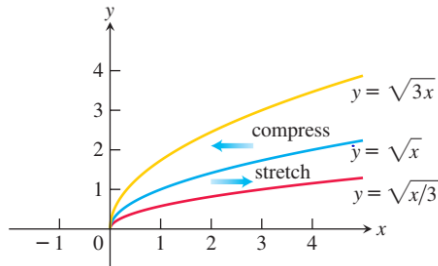
$$y = 3\sqrt{x} \Rightarrow y=3$$



Horizontal scaling

Example 4b Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases

- multiplying x by 3 in the right hand side of y $y = \sqrt{3x}$
- multiplying x by $1/3$ in the right hand side of y $y = \sqrt{\frac{1}{3}x}$



Reflection

Example 4c Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases

- Reflects the graph of f across the x -axis. $y = -\sqrt{x}$
- Reflects the graph of f across the y -axis. $y = \sqrt{-x}$

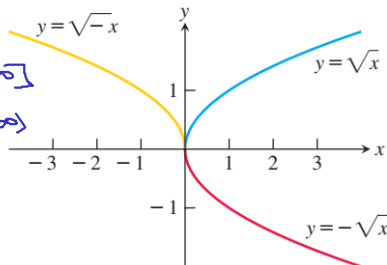
$$y = \sqrt{-x}$$

$$-x \geq 0$$

$$x \leq 0$$

$$\text{Domain} = (-\infty, 0]$$

$$\text{Range} = [0, \infty)$$



$$\text{domain} = [0, \infty)$$

$$\text{range} = [0, \infty)$$

$$\text{domain} = [0, \infty)$$

$$\text{range} = (-\infty, 0]$$

Example 5 Given the function $f(x) = x^4 - 4x^3 + 10$, find formulas to

- (a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis. $f(2x)$
 $f(-2x) =$
- (b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis. $\frac{1}{2}f(x)$
 $-\frac{1}{2}f(x)$

$$1a) f(2x) = 16x^4 - 32x^3 + 10$$

$$f(-2x) = 16x^4 + 32x^3 + 10$$

$$1b) \frac{1}{2}f(x) = \frac{1}{2}x^4 - 2x^3 + 5$$

$$-\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5$$

Periodic function

A function $f(x)$ is periodic if there is a positive number p such that

$$f(x + p) = f(x)$$

for every value of x . The smallest such value of p is the period of f .

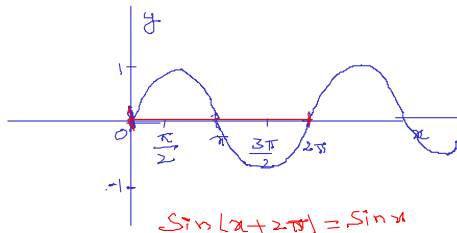
eg:

$$\sin x = 2\pi$$

$$\cos x = 2\pi$$

$$\tan x = \pi$$

		all the	
sin	90° + θ	90° - θ	cos
	180° - θ	360° + θ	
cos	180° + θ	270° + θ	tan
	270° - θ	360° - θ	
		270°	sec



$$\sin(x + 2\pi) = \sin x$$

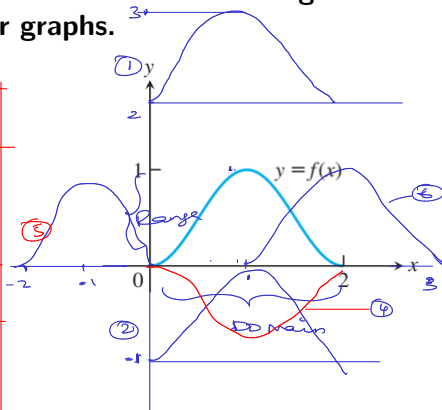
$$\sin(90^\circ + \theta) = +\cos \theta$$

180° are even 90° odd
360° 270°

Practice problems

The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.

	Domain	range
1 $f(x) + 2$ <i>VSU - 2 units</i>	$[0, 2]$	$[2, 3]$
2 $f(x) - 1$ <i>VSU</i>	$[0, 2]$	$[-1, 0]$
3 $2f(x)$ <i>VSU stretch</i>	$[0, 2]$	$[0, 2]$
4 $-f(x)$ <i>Reflection about x-axis</i>	$[0, 2]$	$[-1, 0]$
5 $f(x + 2)$ <i>Shift L</i>	$[-2, 0]$	$[0, 1]$
6 $f(x - 1)$ <i>Shift R</i>	$[1, 3]$	$[0, 1]$
7 $-f(x + 1) + 1$ <i>Ref about x-axis, Shift L, VSU</i>	$[-1, 1]$	$[0, 1]$



Practice problems

H.W

Find by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph

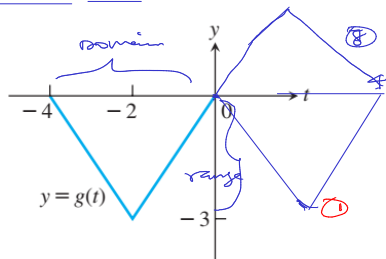
- $y = x^2 - 1$ stretched vertically by a factor of 3 $\Rightarrow y = 3(x^2 - 1)$
- $y = x^2 - 1$ compressed horizontally by a factor of 2 $\Rightarrow y = (2x)^2 - 1 = 4x^2 - 1$
- $y = 1 + \frac{1}{x^2}$ compressed vertically by a factor of 2 $\Rightarrow y = \frac{1}{2}(1 + \frac{1}{x^2})$
- $y = 1 + \frac{1}{x^2}$ stretched horizontally by a factor of 3 $\Rightarrow y = 1 + \frac{1}{(3x)^2} = 1 + \frac{1}{9x^2}$
- $y = \sqrt{x+1}$ stretched vertically by a factor of 3 $\Rightarrow y = 3(\sqrt{x+1})$
- $y = \sqrt{4-x^2}$ stretched horizontally by a factor of 2 $y = \sqrt{4 - (\frac{x}{2})^2}$
 $= \sqrt{4 - x^2/4}$

Practice problems

h.w

The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions and sketch their graphs

	Domain	Range
1 $g(-t)$ <small>reflect y</small>	$[0, 4]$	$[-3, 0]$
2 $-g(t)$ <small>reflect x axis</small>	$[-4, 0]$	$[0, 3]$
3 $g(t) + 3$	$[-4, 0]$	$[0, 3]$
4 $1 - g(t)$	$[-4, 0]$	$[1, 4]$
5 $g(-t + 2)$	$[-2, 2]$	$[-3, 0]$
6 $g(t - 2)$ <small>hsh on</small>	$[-2, 2]$	$[-3, 0]$
7 $g(1 - t)$	$[-1, 3]$	$[-3, 0]$
8 $-g(t - 4)$	$[0, 4]$	$[0, 3]$



Transformation of trigonometric graphs

Vertical stretch or compression;
reflection about $y = d$ if negative

Vertical shift

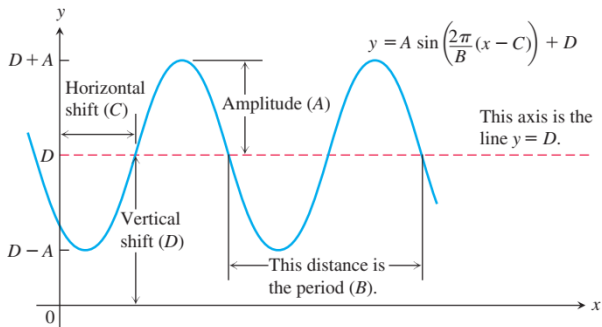
$$y = af(b(x + c)) + d$$

Horizontal stretch or compression;
reflection about $x = -c$ if negative

Horizontal shift

General sine function or sinusoid formula

$$f(x) = A \sin \left(\frac{2\pi}{B}(x - c) \right) + D$$



Problem

h.w

Find the period of each function

- $\sin 2x$
- $\sin(x/2)$
- $\cos \pi x$
- $\cos\left(\frac{\pi x}{2}\right)$
- $\sin\left(x + \frac{\pi}{6}\right)$

Practice problems

- ① Graph the functions $y = 2 \cos(x - \pi/3)$ and $y = 1 + \sin(x + \pi/4)$
- ② Describe how each graph is obtained from the graph of $y = f(x)$
 - $y = f(x - 5)$
 - $y = f(4x)$
 - $y = f(-3x)$
 - $y = f(2x + 1)$
 - $y = f(x/3) - 4$
 - $y = -3f(x) + 1/4$

THANK YOU