# Calculus and its Applications
## (Limits and Continuity - Combining Functions)

**KRISHNASAMY R**

email: rky.amcs@psgtech.ac.in
Mobile No.: 9843245352

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**LIMITS AND CONTINUITY:** Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

**TEXT BOOKS:**

1. Thomas G B Jr., Maurice D Wier, Joel Hass, Frank R. Giordano, Thomas' Calculus, Pearson Education, 2018.

# Sums, Differences, Products, and Quotients

If $f$ and $g$ are functions, then for every $x$ that belongs to the domains of both $f$ and $g$ (that is, for $x \in D(f) \cap D(g)$), we define functions $f+g$, $f-g$ and $fg$ by the formulas

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\(f-g)(x) &= f(x)-g(x)\\(fg)(x) &= f(x)g(x)\end{aligned}$$

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function $f/g$ by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{where } \underline{g(x) \neq 0}$$

# EXAMPLE 1

$1-x \geq 0$
$1 \geq x$ $(-\infty, 1]$
$D(f) = [0,\infty)$, $D(g) = (-\infty, 1]$ $D(f) \cap D(g) = [0, 1]$

Let the functions be defined by the formulas $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$. Find the domains of $f(x)$ and $g(x)$. Also find the formulas and domains for the algebraic combinations defined below.

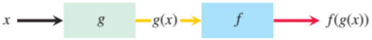
| Function | Formula $(x)$ | domain $(y)$ |
|---|---|---|
| $f+g$ | $\sqrt{x} + \sqrt{1-x}$ | $D(f) \cap D(g) = [0, 1]$ |
| $f-g$ | $\sqrt{x} - \sqrt{1-x}$ | $[0, 1]$ |
| $g-f$ | $\sqrt{1-x} - \sqrt{x}$ | $[0, 1]$ |
| $f \cdot g$ | $\sqrt{x}\sqrt{1-x}$ | $[0, 1]$ |
| $f/g$ | $\dfrac{\sqrt{x}}{\sqrt{1-x}}$ | $[0, 1)$   $x=1$ is excluded because $g(1)=0$ |
| $g/f$ | $\dfrac{\sqrt{1-x}}{\sqrt{x}}$ | $(0, 1]$   $x=0$ is excluded $f(0)=0$ |

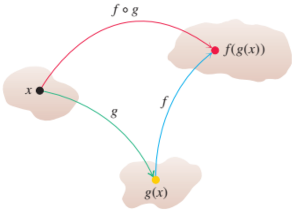
# Composite functions

If $f$ and $g$ are functions, the composite function $f \circ g$ ($f$ composed with $g$) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers $x$ in the domain of $g$ for which $g(x)$ lies in the domain of $f$.



# EXAMPLE 2

$f(y) = \sqrt{y}$ $\quad$ $g(y) = y+1$


If $f(x) = \sqrt{x}$ and $g(x) = x + 1$ find the formulas and domains for the following: $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$ and $(g \circ g)(x)$.

| | Domain |
|---|---|
| $f \circ g = f[x+1] = \sqrt{x+1}$ $\quad$ $x+1 \geq 0$, $x \geq -1$ | $[-1, \infty)$ |
| $g \circ f = g[\sqrt{x}] = \sqrt{x} + 1$ | $[0, \infty)$ |
| $f \circ f = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4} = \sqrt[4]{x}$ | $[0, \infty)$ |
| $g \circ g = g(x+1) = (x+1)+1 = x+2$ | $(-\infty, \infty)$ |

$(f \circ g)(x) = f[\underbrace{g(x)}_{y}] = f[x+1] = f[y] = \sqrt{y} = \sqrt{x+1}$

$(g \circ f)(x) = g[f(x)] = g[\underbrace{\sqrt{x}}_{y}] = g[y] = y+1 = \sqrt{x}+1$

# Shifting a graph of a function

$f(x)$

$y = x^2$

$y = x^2 + 1$

$y = x^2 - 1$

- **Vertical Shifts**

  $y = f(x) + k$ shifts the graph of $f$ up $k$ units if $k > 0$ or

  shifts it down $|k|$ units if $k < 0$

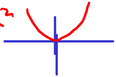
- **Horizontal Shifts**

  $y = f(x + h)$ shifts the graph of $f$ left $h$ units if $h > 0$ or

  shifts it right $|h|$ units if $h < 0$

$y = (x+1)^2$

$y = x^2$

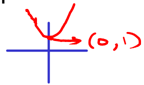


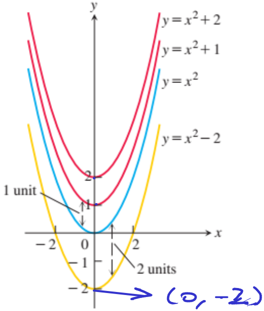
$y = (x-1)^2$

# Example 3a

If $y = x^2$ then mention the type of shifts for the following operations and hence sketch the graph in each cases:

- adding 1 to the right hand side of $y$, $\Rightarrow y = x^2 + 1$ (VS)
- adding 2 to the right hand side of $y$, $\Rightarrow y = x^2 + 2$ (VS)
- adding -2 to the right hand side of $y$, $\Rightarrow y = x^2 - 2$ (VS)

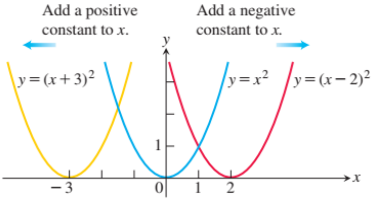


# Example 3b

If $y = x^2$ then mention the type of shifts for the following operations and hence sketch the graph in each cases: HSL

- adding 3 to $x$ in $y = x^2$, $\Rightarrow$ $y = (x+3)^2$
- adding -2 to $x$ in $y = x^2$. $\Rightarrow$ $y = (x-2)^2$



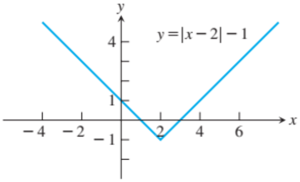
# Example 3c

hw

If $y = |x|$ then mention the type of shifts when (i) -2 is added to $x$ in $y$ and then $-1$ is added (ii) to the result. Also sketch the graph for the above.

(i) $y = |x-2|$ shifted horizontally to the right by 2 units

(ii) $y = |x-2|-1$ shifted vertically downward by 1 unit



# Practice problems

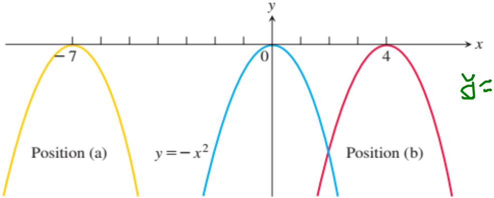
H.W

1. Find the domain and ranges of $f$, $g$ $f+g$, $f \cdot g$, for $f(x)=x$ and $g(x)=\sqrt{x-1}$
2. Find the domain and ranges of $f$, $g$, $f/g$ and $g/f$ for $f(x)=\sqrt{x}$ and $g(x)=|x-3|$

# Shifting graphs

**Graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs**

$y = -(x+7)^2$

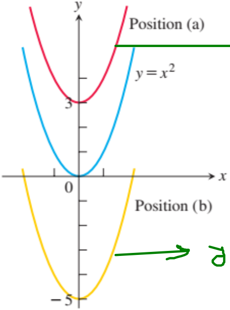



$y = -(x-4)^2$

# Shifting graphs

hw

**Graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs**



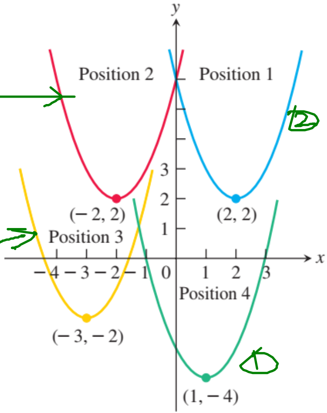

$y = x^2 + 3$

$y = x^2 - 5$

# Shifting graphs

**Match the equations listed in part (1)-(4) to the graphs in the following figure.**

1. $y = (x-1)^2 - 4$
2. $(x-2)^2 + 2$
3. $y = (x+2)^2 + 2$
4. $y = (x+3)^2 - 2$



# Vertical and Horizontal Scaling

**For $c > 1$, the graph is scaled**

- $y = cf(x)$ Stretches the graph of $f$ vertically by a factor of $c$.
- $y = \frac{1}{c}f(x)$ Compresses the graph of $f$ vertically by a factor of $c$.
- $y = f(cx)$ Compresses the graph of $f$ horizontally by a factor of $c$.
- $y = f(x/c)$ Stretches the graph of $f$ horizontally by a factor of $c$.

# Reflection

**For $c = -1$, the graph is reflected**

- $y = -f(x)$ Reflects the graph of $y$ across the $x$-axis.
- $y = f(-x)$ Reflects the graph of $y$ across the $y$-axis

vertical scaling

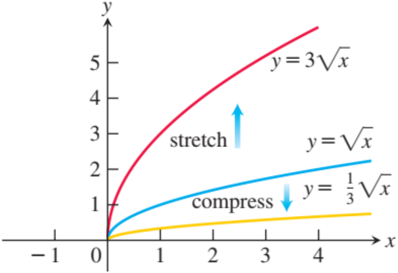
**Example 4a** Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases

- multiplying the right hand side of $y$ by 3, $y = 3\sqrt{x}$
- multiplying the right hand side of $y$ by $1/3$ $y = \frac{1}{3}\sqrt{x}$

$y = \sqrt{x}$

$x = 1 \Rightarrow y = 1$

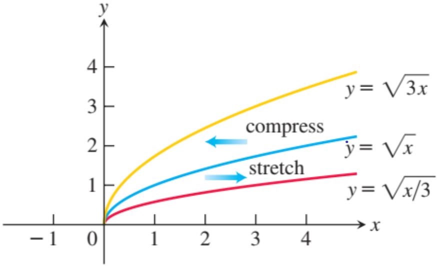
$y = 3\sqrt{x} \Rightarrow y = 3$



Horizontal scaling

**Example 4b** Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases

- multiplying $x$ by 3 in the right hand side of $y$ $y = \sqrt{3x}$
- multiplying $x$ by $1/3$ in the right hand side of $y$ $y = \sqrt{\frac{1}{3}x}$



## Reflection

**Example 4c** Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases

- Reflects the graph of $f$ across the $x$-axis. $y = -\sqrt{x}$
- Reflects the graph of $f$ across the $y$-axis $y = \sqrt{-x}$

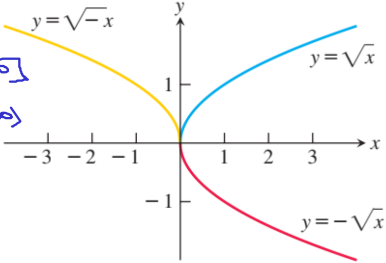
$y = \sqrt{-x}$

$-x \geq 0$

$x \leq 0$

Domain $= [-\infty, 0]$

Range $= [0, \infty)$




domain $= [0, \infty)$

range $= [0, \infty)$

domain$= [0, \infty)$

range$= (-\infty, 0]$

**Example 5** Given the function $f(x) = x^4 - 4x^3 + 10$, find formulas to

$f(2x)$

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the $y$-axis. $f(-2x) =$

(b) compress the graph vertically by a factor of 2 followed by a reflection across the $x$-axis $-\frac{1}{2}f(x)$ $\frac{1}{2}f(x)$

(a) $f(2x) = 16x^4 - 32x^3 + 10$

$f(-2x) = 16x^4 + 32x^3 + 10$

(b) $\frac{1}{2}f(x) = \frac{1}{2}x^4 - 2x^3 + 5$

$-\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5$

# Periodic function

A function $f(x)$ is periodic if there is a positive number $p$ such that

$$f(x+p) = f(x)$$

for every value of $x$. The smallest such value of $p$ is the period of $f$.

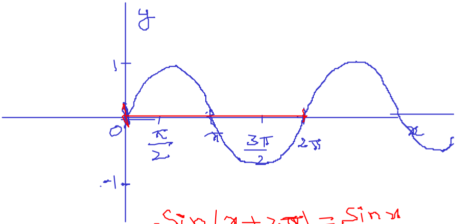
Period

eg:

$\sin x = 2\pi$

$\cos x = 2\pi$

$\tan x = \pi$




$\sin(x + 2\pi) = \sin x$

$\sin(90^\circ + \theta) = +\cos\theta$

Sin, cosec +ve: $90^\circ + \theta$, $180^\circ - \theta$

$90^\circ$ — all +ve: $90^\circ - \theta$, $360^\circ + \theta$

$180^\circ$

tan, cot +ve: $180^\circ + \theta$, $270^\circ - \theta$

cos, sec +ve: $270^\circ + \theta$, $360^\circ - \theta$

$270^\circ$

$0 = 360^\circ$

$180^\circ$, $360^\circ$ +ve even

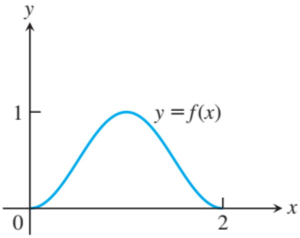
$90^\circ$, $270^\circ$ odd

# Practice problems

H.W

**The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.**

- $f(x) + 2$
- $f(x) - 1$
- $2f(x)$
- $-f(x)$
- $f(x + 2)$
- $f(x - 1)$
- $-f(x + 1) + 1$



# Practice problems

H.W

**Find by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph**

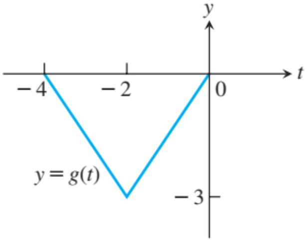
- $y = x^2 - 1$ stretched vertically by a factor of 3
- $y = x^2 - 1$ compressed horizontally by a factor of 2
- $y = 1 + \frac{1}{x^2}$ compressed vertically by a factor of 2
- $y = 1 + \frac{1}{x^2}$ stretched horizontally by a factor of 3
- $y = \sqrt{x+1}$ stretched vertically by a factor of 3
- $y = \sqrt{4 - x^2}$ stretched horizontally by a factor of 2

# Practice problems

H.W

The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions and sketch their graphs

1. $g(-t)$
2. $-g(t)$
3. $g(t) + 3$
4. $1 - g(t)$
5. $g(-t + 2)$
6. $g(t - 2)$
7. $g(1 - t)$
8. $-g(t - 4)$



# Transformation of trigonometric graphs

Vertical stretch or compression; reflection about $y = d$ if negative

Vertical shift

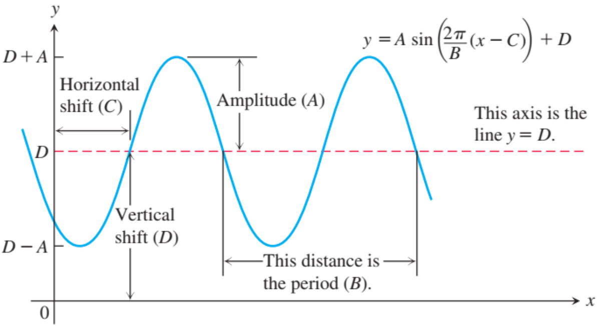
$$y = af(b(x + c)) + d$$

Horizontal stretch or compression; reflection about $x = -c$ if negative

Horizontal shift

# General sine function or sinusoid formula

$$f(x) = A\sin\left(\frac{2\pi}{B}(x-c)\right) + D$$



# Problem

Find the period of each function

- $\sin 2x$
- $\sin(x/2)$
- $\cos \pi x$
- $\cos\left(\frac{\pi x}{2}\right)$
- $\sin\left(x + \frac{\pi}{6}\right)$

## Practice problems

1. Graph the functions $y = 2\cos(x - \pi/3)$ and $y = 1 + \sin(x + \pi/4)$
2. Describe how each graph is obtained from the graph of $y = f(x)$
   - $y = f(x - 5)$
   - $y = f(4x)$
   - $y = f(-3x)$
   - $y = f(2x + 1)$
   - $y = f(x/3) - 4$
   - $y = -3f(x) + 1/4$

# THANK YOU