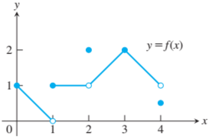
Calculus and its Applications
(Limits and Continuity - Problems)

KRISHNASAMY R

email: rky.amcs@psgtech.ac.in
Mobile No.: 9843245352



LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

TEXT BOOKS:

1. Thomas G B Jr., Joel Hass, Christopher Heil, Maurice D Wier, Thomas' Calculus, Pearson Education, 2018.

Closed Interval

$f(x)$ defined $[a, b]$

for all $c \in (a, b)$

LHL = RHL = limit

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x)$$

at $x = a$ (left end point)

$$\lim_{x \to a^+} f(x) = \lim_{x \to a} f(x)$$

at $x = b$ (right end point)

$$\lim_{x \to b^-} f(x) = \lim_{x \to b} f(x)$$

Remark:

For the existence of limit at $x = a$ (x approaches a (right or left)) the function may or may not be defined at $x = a$)

Open Interval

$f(x)$ defined over (a,b)

limits do not exists at end points $(a$ & $b)$

because a & b are not included in the domain

for all $c \in (a,b)$

LHL = RHL = limit

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x)$$

Problem

H.W

greatest integer function

Evaluate (a) $\lim_{x\to 3^+} \frac{\lfloor x \rfloor}{x}$ (b) $\lim_{x\to 3^-} \frac{\lfloor x \rfloor}{x}$ $= \lim_{x\to 3^-} \frac{x-1}{x}$

$= \lim_{x\to 3^+} \frac{x}{x}$

$= 1$

$= \frac{2}{3}$

$\lfloor 3.1 \rfloor = 3$

$\lfloor 3.2 \rfloor = 3$

$\lfloor 2.9 \rfloor = 2$

$\lfloor 2.8 \rfloor = 2$

Continuity

Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f. The function f is continuous at c if

LHL RHL

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x) = f(c).$$

→ value of the function $f(x)$ at $x = c$

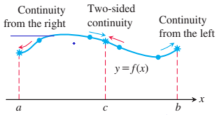
The function f is right-continuous at c (or continuous from the right) if

$$\lim_{x \to c^+} f(x) = f(c).$$

RHL =

The function f is left-continuous at c (or continuous from the left) if

$$\lim_{x \to c^-} f(x) = f(c).$$



$[a, b]$

Example Let $f(x) = \sqrt{4-x^2}$. Find out the points at which $f(x)$ is left-continuous, right-continuous and continuous.

$f(2) = \sqrt{4-2^2} = \sqrt{4-4} = 0$

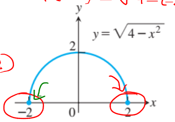
$f(-2) = \sqrt{4-(-2)^2} = 0$

$f(x)$ is left continuous at 2 $\left(\lim_{x\to 2^-} f(x) = f(2) \right.$, $0 = 0$

$f(x)$ is right continuous at -2 $\left(\lim_{x\to -2^+} f(x) = f(-2) \right.$, $0 = 0$

$f(x)$ is continuous

$[-2, 2]$



Domain = $[-2, 2]$

end points are -2 & 2

left end

right end

Example Unit step function

$U(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$\lim_{x\to 0^-} U(x) = 0$, $U(0) = 1$

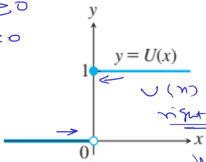
$\lim_{x\to 0^-} U(x) \neq U(0) \Rightarrow U(x)$ is not left continuous

$\lim_{x\to 0^-} U(x) \neq \lim_{x\to 0^+} U(x) \Rightarrow$ limit does not exists at $x = 0$

$0 \neq 1$

$\Rightarrow U(x)$ is not continuous at $x = 0$

function $U(x)$ is continuous in $R - \{0\}$



$U(x)$ is right continuous at $x = 0$

$\lim_{x\to 0^+} U(x) = U(0)$

$1 = 1$

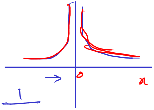
Domain $(-\infty, \infty)$

Range $\{0, 1\}$

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

- $f(c)$ exists (c lies in the domain of f)
- $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$) $\Rightarrow$ $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$
- $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)



$f(x) = \frac{1}{x^2}$

Domain $f = \mathbb{R} - \{0\}$

$\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to 0^-} f(x) = \infty$

limit does not exists

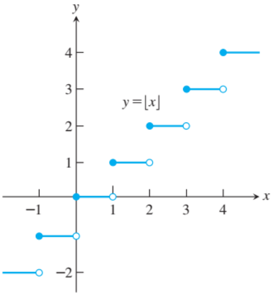
∞ is not a finite number

Greatest integer function (GIF)

GIF is continuous

at every non integer values

$[0,1)$

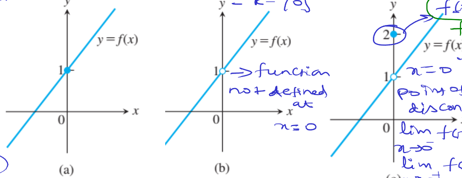


Domain(f) $= (-\infty, \infty)$

Types of discontinuity

- Jump discontinuity ✓
- Infinite discontinuity ✓
- Oscillating discontinuity
- Removable discontinuity (f g c)

Domain = R   Domain = R − {0}   Domain = R



(a) (b) (c)

function not defined at $x = 0$

$f(0) = 2$

$f(0) = 1$

$x = 0$ is the point of discontinuity

$\lim_{x \to 0^-} f(x) \neq f(0)$

$\lim_{x \to 0^+} f(x) \neq f(0)$

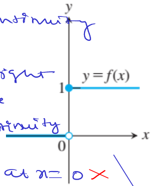
$\lim_{x \to 0^+} f(x) = \infty$ $\lim_{x \to 0^-} f(x) = \infty$ not a finite number

wild oscillation around $x = 0$

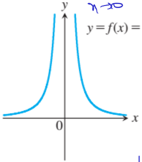
$\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$

$\lim_{x \to 0^-} \sin\left(\frac{1}{x}\right)$

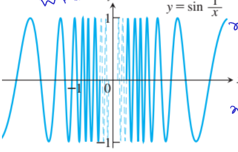
Jump discontinuity either left continuous or right continuous at the point of discontinuity



(d)



(e)



(f)

left continuous at $x = 0$ ✗

$\lim_{x \to 0^-} f(x) = f(0)$ ✗

$0 = 1$ ✗

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\Rightarrow f(x)$ is not left continuous at $x = 0$

right continuous at $x = 0$ ✓

$\lim_{x \to 0^+} f(x) = f(0)$ ✓

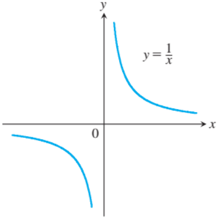
$1 = 1$ ✓

Continuous function Continuous at every point of its domain

Discontinuous function Discontinuous at one or more points of its domain.

Example

- $f(x) = \frac{1}{x}$.
- Identity function $f(x) = x$
- constant function $f(x) = c$



Domain $\mathbb{R} - \{0\}$

Properties of continuous functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$

- Sums: $f + g$
- Differences: $f - g$
- Constant multiples $k \cdot f$, for any number k
- Products: $f \cdot g$
- Quotients: f/g, provided $g(c) \neq 0$
- Powers: f^n, where n is the positive integer
- Roots: $\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer.

Problems

Polynomial functions, rational functions, $|x|$, $\sin x$, $\cos x$, $\tan x$.

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$\cdots \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \cdots$

$r(x) = \frac{p(x)}{q(x)}$, $q(c) \neq 0$

Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$ (b) $y = \frac{x^{2/3}}{1+x^4}$ (c) $y = \left|\frac{x-2}{x^2-2}\right|$

H.W

$(-\infty, -2) \cup [4, \infty)$ ×

$x^2 - 2x - 5 \geq 0$

Domain $= R$

Domain $= R - \{\sqrt{2}, -\sqrt{2}\}$

Theorem Limits of Continuous Functions

If $\lim_{x \to c} f(x) = b$ and g is continuous at the point b, then

$$\lim_{x \to c} g(f(x)) = g(b).$$

$\hookrightarrow \quad g\left(\lim_{x \to c} f(x)\right) = g(b)$

Problem Evaluate

$$\lim_{x \to \frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = -1$$

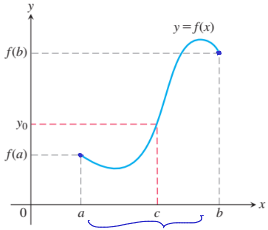
$= \cos\left(\cancel{2}\frac{\pi}{\cancel{2}} + \sin\left(\frac{3\pi}{2} + \frac{\pi}{2}\right)\right)$

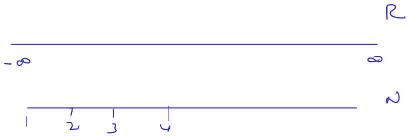
$= 0$

$= -1$

The Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c \in [a, b]$.

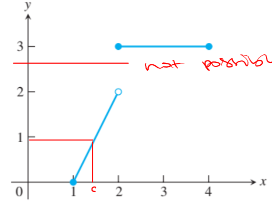




- Continuous functions over finite closed intervals have this property.
- Geometrically, the IVT says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.
- The proof of IVT depends on the completeness property.
- The completeness property implies that $\mathbb{R}$ have no holes or gaps while $\mathbb{Q}$ do not satisfy the completeness property.

A Consequence for Graphing: Connectedness

- IMVT implies that the graph of a function that is continuous on an interval cannot have any breaks over the interval.
- It will be connected - a single, unbroken curve.
- It will not have jumps such as the ones found in the graph of the greatest integer function, or separate branches as found in the graph of $1/x$.



not possible to find $c \ni y_0 = f(c)$

$f(x)$ is not continuous

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

A Consequence for Root Finding

- We call a solution of the equation $f(x) = 0$ a root of the equation or zero of the function f.
- The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function.
- Somewhere between a point where a continuous function is positive and a second point where it is negative, the function must be equal to zero   If $f(c_1) < 0$ & $f(c_2) > 0$
  then root lies between c_1 & c_2

Problem Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

$f(x) = x^3 - x - 1$

$$\left.\begin{array}{l} f(1) = 1 - 1 - 1 = -1 < 0 \\ f(2) = 8 - 2 - 1 = 5 > 0 \end{array}\right\} \Rightarrow f(x) = 0 \text{ has a root between } 1 \text{ \& } 2$$

Continuous extension to a point

Example $f(x) = \dfrac{\sin x}{x}$.

Domain $= \mathbb{R} - \{0\}$

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ \_\_, & x = 0 \end{cases}$$

'0' is an interior point

Suppose if $f(x)$ is continuous at $x=0$

then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

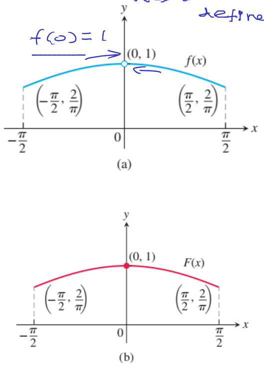
To be find out

$$\lim_{x\to 0^-} \frac{\sin x}{x} = 1 \quad \Rightarrow \quad f(0) = 1$$

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

$f(0)$ is not defined

$f(0) = 1$



$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Problem Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2.$$

$x = -2$

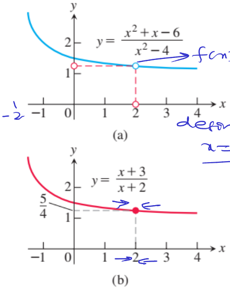
$$f(x) = \frac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

$$f(x) = \frac{x+3}{x+2}$$

$$\lim_{x \to 2^-} f(x) = \frac{5}{4}, \quad \lim_{x \to 2^+} f(x) = \frac{5}{4}$$

$$f(2) = \frac{5}{4}$$

$$f(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-4}, & x \neq 2 \\ \dfrac{5}{4}, & x = 2 \end{cases}$$

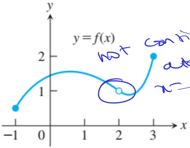


f(x) is not defined at x = 2

H.W

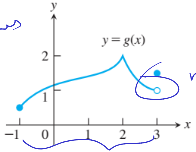
Check whether the functions graphed below are continuous on [-1,3]?
If not, give reasons.

1.



not continuous at $x=2$

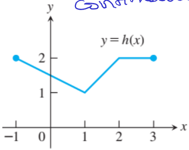
2.



not left continuous at $x=3$

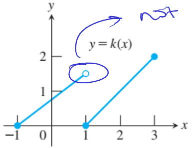
3.

continuous



4.

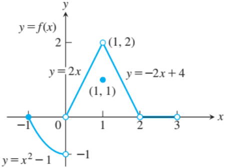
not continuous at $x=1$ (not right continuous)



H.W

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



The graph for Exercises 5–10.

5. **a.** Does $f(-1)$ exist?

b. Does $\lim_{x\to -1^+} f(x)$ exist?

c. Does $\lim_{x\to -1^+} f(x) = f(-1)$?

d. Is f continuous at $x = -1$?

6. **a.** Does $f(1)$ exist?

b. Does $\lim_{x\to 1} f(x)$ exist?

c. Does $\lim_{x\to 1} f(x) = f(1)$?

d. Is f continuous at $x = 1$?

7. **a.** Is f defined at $x = 2$? (Look at the definition of f.)

b. Is f continuous at $x = 2$?

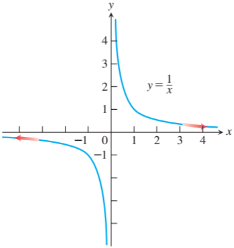
8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

10. To what new value should $f(1)$ be changed to remove the discontinuity?

Limits Involving Infinity; Asymptotes of Graphs

Behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \to \pm\infty$. We further extend the concept of limit to infinite limits.



Finite Limits as $x \to \pm\infty$

Example $f(x) = \frac{1}{x}$.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

Evaluate $\lim_{x\to\infty} \left(5 + \frac{1}{x}\right)$. $= 5$

$f(x) = x$

$$\lim_{x\to\infty} f(x) = \infty$$

$$\lim_{x\to-\infty} f(x) = -\infty$$

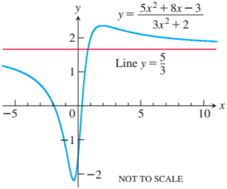
Limits at Infinity of Rational Functions

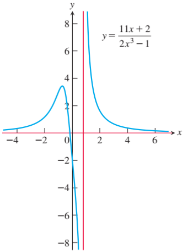
To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

Examples

(a) $\lim\limits_{x\to\infty} \dfrac{5x^2+8x-3}{3x^2+2}$ $= \lim\limits_{x\to\infty} \dfrac{\cancel{x^2}\left(5+\frac{8}{x}-\frac{3}{x^2}\right)}{\cancel{x^2}\left(3+\frac{2}{x^2}\right)}$

$= \dfrac{5}{3}$





(b) $\lim\limits_{x\to-\infty} \dfrac{11x+2}{2x^3-1}$ $= 0$

Horizontal Asymptotes

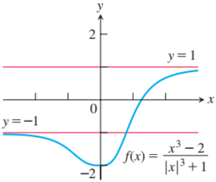
Definition A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Problem Find the asymptotes of the graph of

$$f(x) = \frac{x^3 - 3}{|x|^3 + 1}.$$



$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3-3}{x^3+1} = \lim_{x\to\infty} \frac{\cancel{x^3}(1-3/x^3)}{\cancel{x^3}(1+1/x^3)}$$

$$= \frac{1}{1} = 1$$

$y = 1$ is the horizontal asymptote of $f(x)$ as $x \to \infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3-3}{-x^3+1} = \lim_{x\to-\infty} \frac{\cancel{x^3}(1-3/x^3)}{\cancel{x^3}(-1+1/x^3)} = \frac{1}{-1} = -1$$

$\Rightarrow$ $y = -1$ is the horizontal asymptote of $f(x)$ as $x \to -\infty$

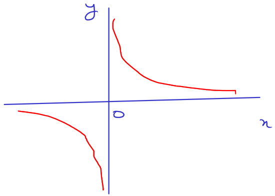
Horizontal Asymptotes

Find the horizontal asymptote for the function f(x)=1/x

Soln:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x} = 0$$



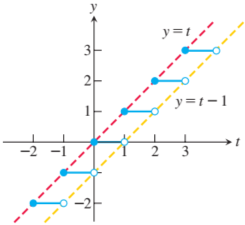
$\Rightarrow \underbrace{y=0}_{x\text{-axis}}$ is the horizontal asymptote for $f(x) = \frac{1}{x}$

H.W

Problem Find $\lim\limits_{x\to 0^+} x\left\lfloor \frac{1}{x} \right\rfloor$

$\frac{1}{x} = t \Rightarrow x = \frac{1}{t}$

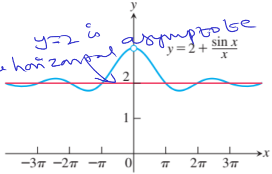
$\lim\limits_{\frac{1}{t}\to 0^+} \frac{1}{t}\lfloor t \rfloor = \lim\limits_{t\to\infty} \frac{\lfloor t \rfloor}{t} = 1$



H.W

Problem Using the Sandwich Theorem, find the horizontal asymptote of the curve $y = 2 + \frac{\sin x}{x}$.

$\lim\limits_{x\to\infty} f(x) = b$



y = 2 is the horizontal asymptote

$\lim\limits_{x\to\infty} 2 + \frac{\sin x}{x}$

$= \lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \frac{\sin x}{x}$

$2 + 0 = 2$

$-1 \le \sin x \le +1$

$0 \le |\sin x| \le 1$

$0 \le \left|\frac{\sin x}{x}\right| \le \frac{1}{x}$

$\lim\limits_{x\to\infty} \frac{1}{x} = 0$

$\lim\limits_{x\to\infty} 0 = 0$

$\lim\limits_{x\to\infty} \left|\frac{\sin x}{x}\right| = 0$

Oblique Asymptotes

- If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an oblique or slant line asymptote.
- We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

Problem Find the oblique asymptote of $f(x) = \dfrac{x^2-3}{2x-4}$.

degree = 2

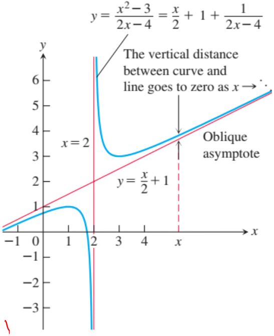
degree = 1

$$\begin{array}{r|l} & \frac{x}{2}+1 \\ \hline 2x-4 & x^2-3 \\ & x^2-2x \\ & \quad 2x-3 \\ & \quad 2x-4 \\ & \qquad 1 \end{array}$$

Remainder

$$\frac{x^2-3}{2x-4} = \frac{x}{2}+1 + \frac{1}{2x-4}$$

oblique asymptote



$$\lim_{x\to\infty} \frac{1}{2x-4} = 0$$

Vertical Asymptotes

A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

Problem Find the horizontal and vertical asymptotes of the curve $y = \frac{x+3}{x+2}$.

Horizontal asymptote

$$\lim_{x \to \infty} \frac{x+3}{x+2} = \lim_{x \to \infty} \frac{\cancel{x}(1+3/x)}{\cancel{x}(1+2/x)} = 1$$

$y = 1$ is the horizontal asymptote

Vertical asymptote

$$\lim_{x \to -2^+} y = \lim_{x \to -2^+} \frac{x+3}{x+2} \to \infty, \quad \lim_{x \to -2^-} \frac{x+3}{x+2} \to -\infty$$

$x = -2$ is the vertical asymptote

Problem

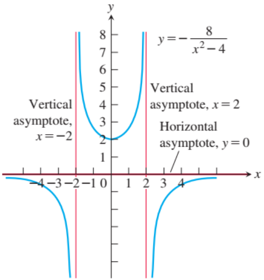
Find the horizontal and vertical asymptotes of the curve $y = -\frac{8}{x^2-4}$.

horizontal asymptote

$$\lim_{x\to\infty} \frac{-8}{x^2-4} = 0$$

$$\lim_{x\to-\infty} \frac{-8}{x^2-4} = 0$$

$\Rightarrow y = 0$ is the horizontal asymptote



Vertical asymptote (equalize the denominator to zero)

$\Rightarrow x^2-4=0 \Rightarrow x = \pm 2$ are vertical asymptotes

$\lim_{x\to 2^+} f(x) = -\infty$, $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to -2^+} f(x) = \infty$, $\lim_{x\to -2^-} f(x) = -\infty$

$\Rightarrow$

Finding Limits

1. For the function f whose graph is given, determine the following limits.

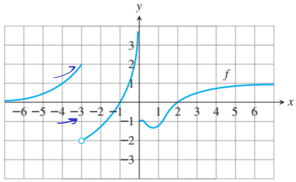
a. $\lim_{x \to 2} f(x)$ 0 **b.** $\lim_{x \to -3^+} f(x)$ -2

c. $\lim_{x \to -3^-} f(x)$ 2 **d.** $\lim_{x \to -3} f(x)$ LHL ≠ RHL

e. $\lim_{x \to 0^+} f(x)$ -1 **f.** $\lim_{x \to 0^-} f(x)$ Check! limit does not exist

g. $\lim_{x \to 0} f(x)$ **h.** $\lim_{x \to \infty} f(x) = 1$

i. $\lim_{x \to -\infty} f(x) = 0$



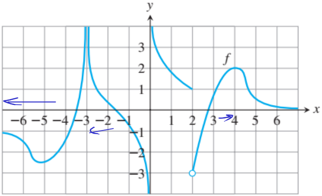
2. For the function f whose graph is given, determine the following limits.

a. $\lim_{x\to 4} f(x) = 2$ **b.** $\lim_{x\to 2^+} f(x) = -3$ **c.** $\lim_{x\to 2^-} f(x) = 1$

doesn't exists **d.** $\lim_{x\to 2} f(x)$ **e.** $\lim_{x\to -3^+} f(x) = \infty$ **f.** $\lim_{x\to -3^-} f(x) = \infty$

doesn't exists **g.** $\lim_{x\to -3} f(x)$ **h.** $\lim_{x\to 0^+} f(x)$ ∞ **i.** $\lim_{x\to 0^-} f(x) = -\infty$

n **j.** $\lim_{x\to 0} f(x)$ **k.** $\lim_{x\to \infty} f(x) = 0$ **l.** $\lim_{x\to -\infty} f(x) = -1$



as $x \to -\infty$

$f(x) \to -1$

THANK YOU