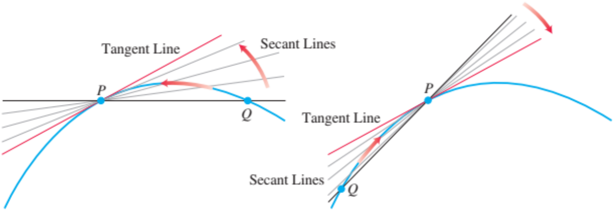
Calculus and its Applications

(Limits and Continuity - Limits)



LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

TEXT BOOKS:

1. Thomas G B Jr., Maurice D Wier, Joel Hass, Frank R. Giordano, Thomas' Calculus, Pearson Education, 2018.
2. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley, 2014.

Limit

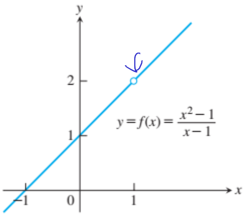
Small change in x produce only small change in $f(x)$.

Limit of a function

When studying a function $y = f(x)$, we find ourselves interested in the function's behavior near a point c but not at c itself.

Example 1

How does the function $f(x) = \frac{x^2-1}{x-1}$ behave near $x = 1$?

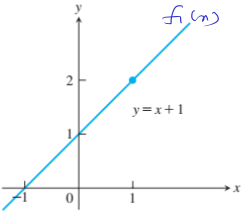


$= \frac{(x+1)(x-1)}{x-1}, x \neq 1$

$f_1(x) = x+1$

$D(f) = R - \{1\}$

$D(f_1) = R = (-\infty, \infty)$



Definition of limit of a function

may not be defined

Let $f(x)$ be a function defined on an interval about c, except possibly at c itself. If $f(x)$ is arbitrarily close to the number L for all x sufficiently close to c other than c itself, then we say that $f(x)$ approaches the limit L as x approaches c, that is,

$$\lim_{x \to c} f(x) = L$$

For $f(x) = \frac{x^2-1}{x-1}$,

$$\lim_{x \to 1} f(x) = 2.$$

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} x+1$$

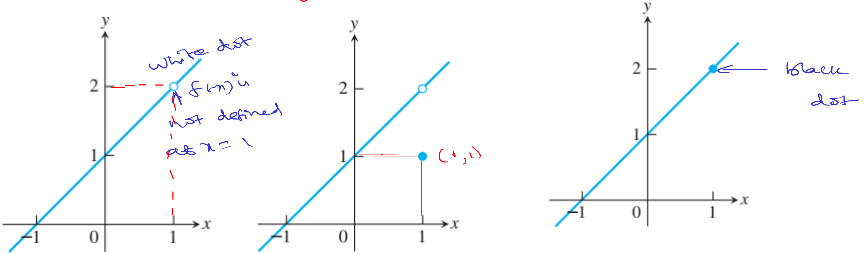
$$= 1+1 = 2$$

Example 2 Find the limits of the following functions as x approaches 1

$$f(x) = \frac{x^2-1}{x-1} \qquad g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1; \\ 1, & x = 1. \end{cases} \qquad h(x) = x + 1$$

$x \in (-\infty, \infty)$

$g(1) = 1$



(a) $f(x) = \dfrac{x^2-1}{x-1}$

(b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

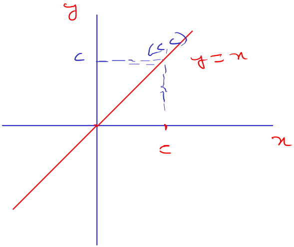
(c) $h(x) = x + 1$

Example 3 Find the limits of the identity function and of a constant function as x approaches c.

Identity fun $f(x) = x$

$$\lim_{x \to c} f(x) = \lim_{x \to c} x$$

$$= c$$



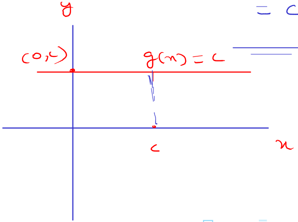
Constant function

$g(x) = c$

$$\lim_{x \to c} g(x) = \lim_{x \to c} c$$

$$= c \lim_{x \to c} 1$$

$$= c$$



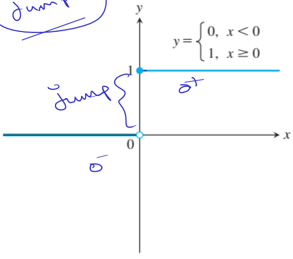
Example 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$

$$U(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases} \qquad g(x) = \begin{cases} \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

$$f(x) = \begin{cases} 0, & x \leq 0; \\ \sin\left(\frac{1}{x}\right), & x > 0. \end{cases}$$

$g(0)=0$

jump



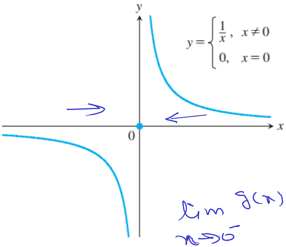
jump

0^+

0^-

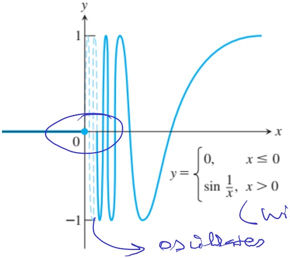
(a) Unit step function $U(x)$

$\lim_{x \to 0^+} g(x) = \infty$



$\lim_{x \to 0^-} g(x) = -\infty$

(b) $g(x)$



(wild)

oscillates

(c) $f(x)$

too much

near zero

Limit laws

Theorem If L, M, c and k are real numbers and $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$ then

- $\lim_{x\to c} [f(x) \pm g(x)] = L \pm M$

$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

$= L + M$

- $\lim_{x\to c} cf(x) = cL$
- $\lim_{x\to c} f(x) \cdot g(x) = L \cdot M$
- $\lim_{x\to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$, $M \neq 0$

$\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n = L^n$

- $\lim_{x\to c} [f(x)]^n = L^n$, where n is the positive integer

$= [\lim_{x\to c} f(x)]^{1/n} = L^{1/n} = \sqrt{L}$

- $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$, where n is the positive integer (If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c)

Example 5 Find the limits of the following functions if $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$

1. $\lim_{x\to c}[x^3 + 4x^2 - 3]$ $= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 + \lim_{x\to c} (-3) = c^3 + 4c^2 - 3$

2. $\lim_{x\to c}\left[\frac{x^4+x^2-1}{x^2+5}\right]$ $= \frac{c^4+c^2-1}{c^2+5}$

3. $\lim_{x\to -2}\sqrt{4x^2-3}$ $= \sqrt{\lim_{x\to -2}(4x^2 - 3)} = \sqrt{16-3} = \sqrt{13}$

Limits of polynomials If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\lim_{x\to c} p(x) = p(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0$.

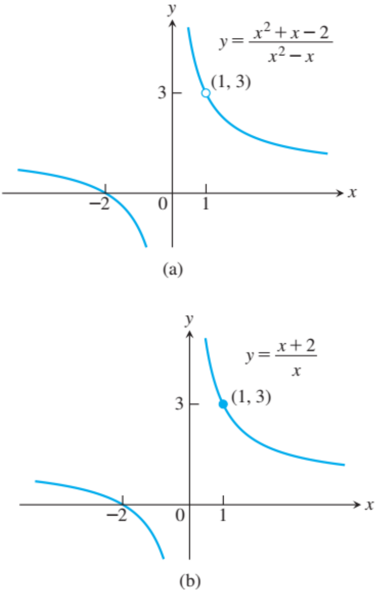
Limits of rational functions If $p(x)$ and $q(x)$ are polynomials and $q(c) \neq 0$, then

$$\lim_{x\to c}\left[\frac{p(x)}{q(x)}\right] = \frac{p(c)}{q(c)} \qquad \frac{\lim_{x\to c} p(x)}{\lim_{x\to c} q(x)}$$

HW

Example 6 Evaluate $\lim_{x\to -1} \frac{x^3+4x^2-3}{x^2+5}$. $= \frac{-1+4-3}{1+5} = \frac{0}{6} = 0$

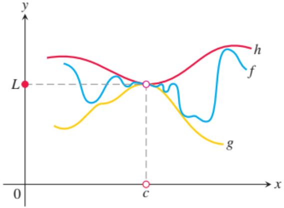
Example 7 Evaluate $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x}$



Sandwich theorem or Squeeze theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for every x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} h(x) = L$ then

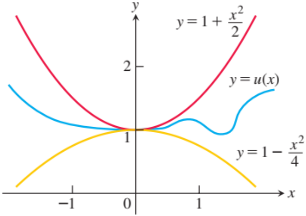
$$\lim_{x\to c} f(x) = L.$$



Example 8 Given a function u that satisfies

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{4}, \quad \forall x \neq 0$$

find $\lim\limits_{x\to 0} u(x)$.



Example 9 Evaluate the following limits

- $\lim\limits_{x \to 0} \sin\theta$
- $\lim\limits_{x \to 0} \cos\theta$
- $\lim\limits_{x \to c} |f(x)| = 0$ implies $\lim\limits_{x \to 0} f(x)$

Problems

1. Check whether the limit exist for the following cases or not. In either case give reasons.
(a) $\lim\limits_{x \to 0} \frac{x}{|x|}$ and (b) $\lim\limits_{x \to 1} \frac{1}{x-1}$.

2. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x \to c} f(x)$? Give reasons for your answer.

3. If $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$ for $-1 \leq x \leq 1$ find $\lim_{x \to 0} f(x)$.

4. It can be shown that the inequalities $1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$ hold for all values of x close to zero. What, if anything does this tell you about $\lim\limits_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$.

Definition of limit

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write $\lim_{x \to c} f(x) = L$, if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta.$$

Example Show that $\lim_{x \to 1} (5x - 3) = 2$.

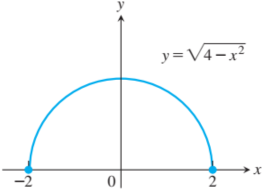
Approaching a limit from one side

Suppose a function f is defined on an interval that extends to both sides of a number c. In order for f to have a limit L as x approaches c, the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is two-sided.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a right-hand limit or limit from the right. From the left, it is a left-hand limit or limit from the left.

Left-hand is denoted by $\lim\limits_{x \to c^-} f(x)$.
Right-hand is denoted by $\lim\limits_{x \to c^+} f(x)$.

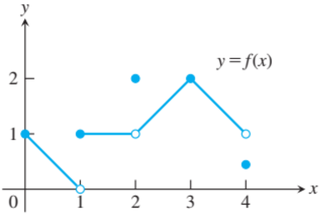
Example 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$. Find the left-hand and right-hand limits of $f(x)$ as x approaches 2.



Theorem Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

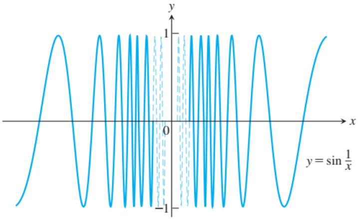
$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Example 2 Find left-hand limits, right-hand limits and limits at $x = 0, 1, 2, 3, 4$ for the function graphed below



Example 3 Find $\lim_{x \to 0^+} \sqrt{x}$.

Example 4 Show that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ has no limit as x approaches zero from either side.



Result $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$

Example 5 Show that (a) $\lim\limits_{y \to 0} \frac{\cos y - 1}{y} = 0$ and (b) $\lim\limits_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Example 6 Find $\lim_{t \to 0} \frac{\tan t \sec 2t}{3t}$.

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

a. $\lim_{x \to -1^+} f(x) = 1$

b. $\lim_{x \to 0^-} f(x) = 0$

c. $\lim_{x \to 0^-} f(x) = 1$

d. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

e. $\lim_{x \to 0} f(x)$ exists.

f. $\lim_{x \to 0} f(x) = 0$

g. $\lim_{x \to 0} f(x) = 1$

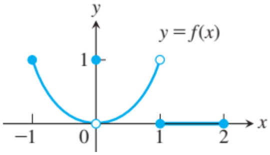
h. $\lim_{x \to 1} f(x) = 1$

i. $\lim_{x \to 1} f(x) = 0$

j. $\lim_{x \to 2^-} f(x) = 2$

k. $\lim_{x \to -1^-} f(x)$ does not exist.

l. $\lim_{x \to 2^+} f(x) = 0$



2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

a. $\lim_{x \to -1^+} f(x) = 1$

b. $\lim_{x \to 2} f(x)$ does not exist.

c. $\lim_{x \to 2} f(x) = 2$

d. $\lim_{x \to 1^-} f(x) = 2$

e. $\lim_{x \to 1^+} f(x) = 1$

f. $\lim_{x \to 1} f(x)$ does not exist.

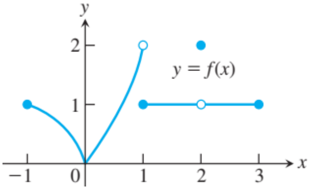
g. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

h. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(1, 3)$.

j. $\lim_{x \to -1^-} f(x) = 0$

k. $\lim_{x \to 3^+} f(x)$ does not exist.



THANK YOU