Calculus and its Applications
(Limits and Continuity - Combining Functions)

KRISHNASAMY R

email: rky.amcs@psgtech.ac.in
Mobile No.: 9843245352

September 15, 2021

LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

TEXT BOOKS:

1. Thomas G B Jr., Maurice D Wier, Joel Hass, Frank R. Giordano, Thomas' Calculus, Pearson Education, 2018.

Sums, Differences, Products, and Quotients

If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f+g$, $f-g$ and fg by the formulas

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (fg)(x) &= f(x)g(x)\end{aligned}$$

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{where } g(x) \neq 0$$

EXAMPLE 1

$1-x \geq 0$, $1 \geq x$, $(-\infty, 1]$ $D(f) = [0, \infty)$, $D(g) = (-\infty, 1]$, $D(f) \cap D(g) = [0, 1]$

Let the functions be defined by the formulas $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$. Find the domains of $f(x)$ and $g(x)$. Also find the formulas and domains for the algebraic combinations defined below.

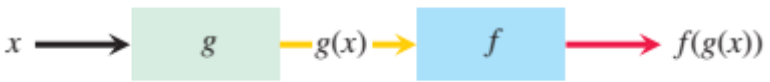
Function	Formula (x)	domain (y)
$f+g$	$\sqrt{x} + \sqrt{1-x}$	$D(f) \cap D(g) = [0,1]$
$f-g$	$\sqrt{x} - \sqrt{1-x}$	$[0,1]$
$g-f$	$\sqrt{1-x} - \sqrt{x}$	$[0,1]$
$f \cdot g$	$\sqrt{x}\sqrt{1-x}$	$[0,1]$
f/g	$\frac{\sqrt{x}}{\sqrt{1-x}}$	$[0,1)$, $x=1$ is excluded because $g(1)=0$
g/f	$\frac{\sqrt{1-x}}{\sqrt{x}}$	$(0,1]$, $x=0$ is excluded $f(0)=0$

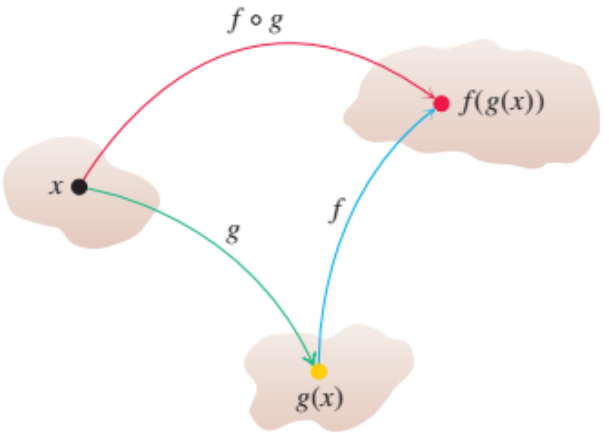
Composite functions

If f and g are functions, the composite function $f \circ g$ (f composed with g) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.





EXAMPLE 2

$f(y) = \sqrt{y}$ $\quad g(y) = y+1$

If $f(x) = \sqrt{x}$ and $g(x) = x + 1$ find the formulas and domains for the following: $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$ and $(g \circ g)(x)$.

		Domain
$f \circ g = f[x+1] = \sqrt{x+1}$	$x+1 \geq 0$, $x \geq -1$	$[-1, \infty)$
$g \circ f = g[\sqrt{x}] = \sqrt{x} + 1$		$[0, \infty)$
$f \circ f = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4} = \sqrt[4]{x}$		$[0, \infty)$
$g \circ g = g(x+1) = (x+1)+1 = x+2$		$(-\infty, \infty)$

$(f \circ g)(x) = f[g(x)] = f[x+1] = f[y] = \sqrt{y} = \sqrt{x+1}$

$((g \circ f)(x) = g[f(x)] = g[\sqrt{x}] = g[y] = y + 1 = \sqrt{x} + 1$

Shifting a graph of a function

$f(x)$

$y = x^2$

$y = x^2 + 1$

$y = x^2 - 1$

- **Vertical Shifts**

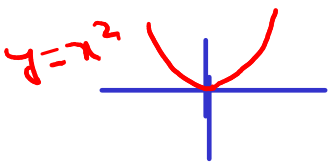
  $y = f(x) + k$ shifts the graph of f up k units if $k > 0$ or shifts it down $|k|$ units if $k < 0$

- **Horizontal Shifts**

  $y = f(x + h)$ shifts the graph of f left h units if $h > 0$ or shifts it right $|h|$ units if $h < 0$

$y = (x+1)^2$

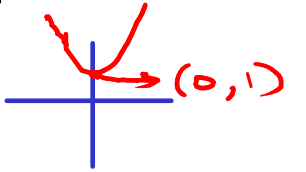
$y = (x-1)^2$

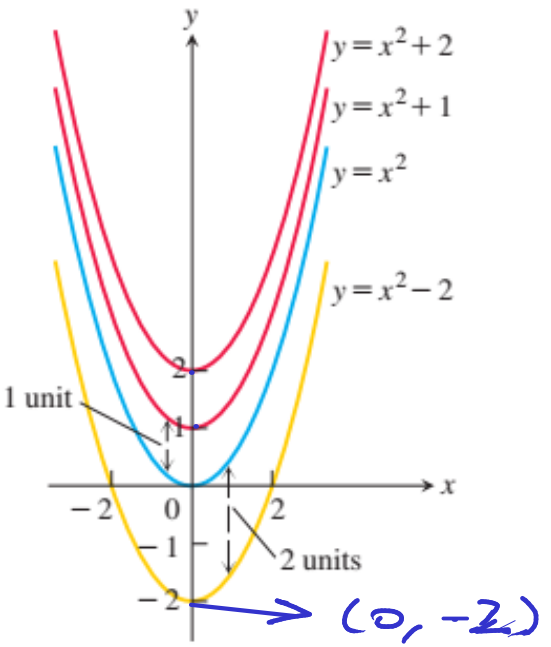


Example 3a

If $y = x^2$ then mention the type of shifts for the following operations and hence sketch the graph in each cases:

- adding 1 to the right hand side of y, $\Rightarrow y = x^2 + 1$ (VS)
- adding 2 to the right hand side of y, $\Rightarrow y = x^2 + 2$ (VS)
- adding -2 to the right hand side of y, $\Rightarrow y = x^2 - 2$ (VS)

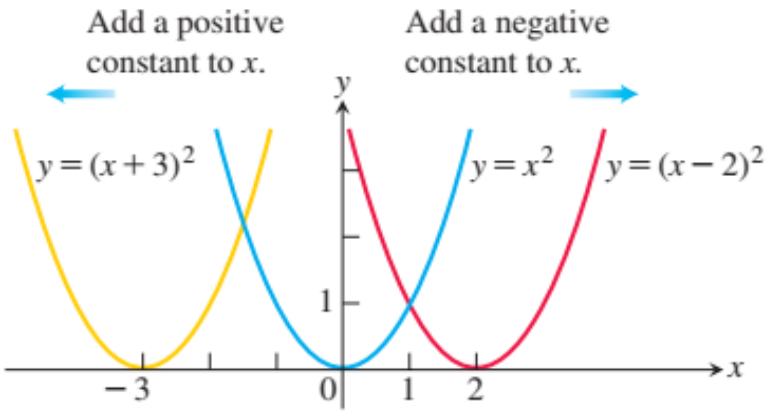




Example 3b

If $y = x^2$ then mention the type of shifts for the following operations and hence sketch the graph in each cases:

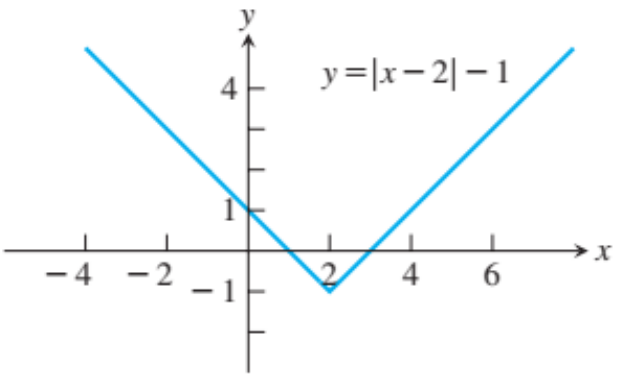
- adding 3 to x in $y = x^2$, $\Rightarrow$ $y = (x+3)^2$
- adding -2 to x in $y = x^2$. $\Rightarrow$ $y = (x-2)^2$



Example 3c

hw

If $y = |x|$ then mention the type of shifts when -2 is added to x in y and then -1 is added to the result. Also sketch the graph for the above.

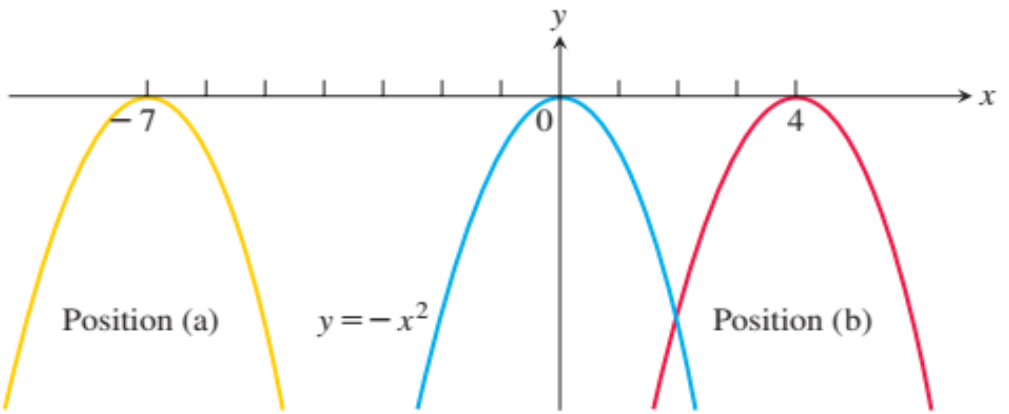


Practice problems

1. Find the domain and ranges of f, g $f+g$, $f \cdot g$, for $f(x) = x$ and $g(x) = \sqrt{x-1}$
2. Find the domain and ranges of f, g, f/g and g/f for $f(x) = \sqrt{x}$ and $g(x) = |x-3|$

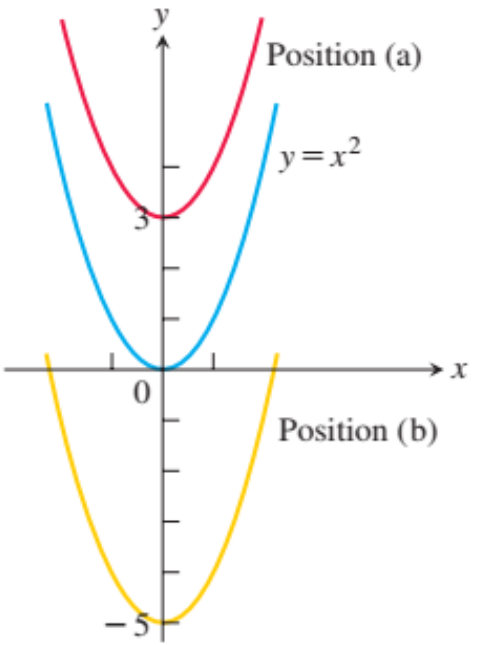
Shifting graphs

Graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs



Shifting graphs

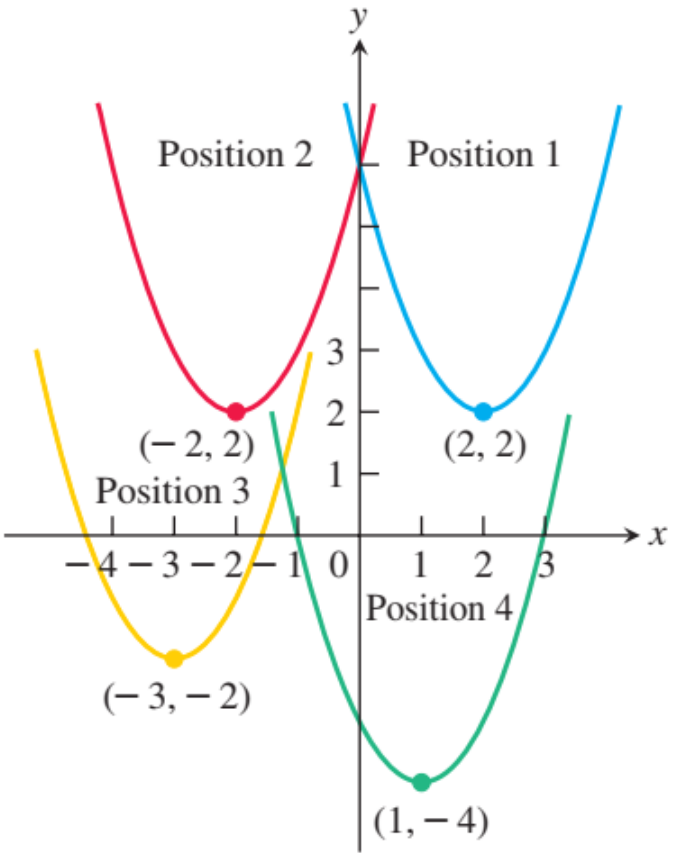
Graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs



Shifting graphs

Match the equations listed in part (1)-(4) to the graphs in the following figure.

1. $y = (x-1)^2 - 4$
2. $(x-2)^2 + 2$
3. $y = (x+2)^2 + 2$
4. $y = (x+3)^2 - 2$



Vertical and Horizontal Scaling

For $c > 1$, the graph is scaled

- $y = cf(x)$ Stretches the graph of f vertically by a factor of c.
- $y = \frac{1}{c}f(x)$ Compresses the graph of f vertically by a factor of c.
- $y = f(cx)$ Compresses the graph of f horizontally by a factor of c.
- $y = f(x/c)$ Stretches the graph of f horizontally by a factor of c.

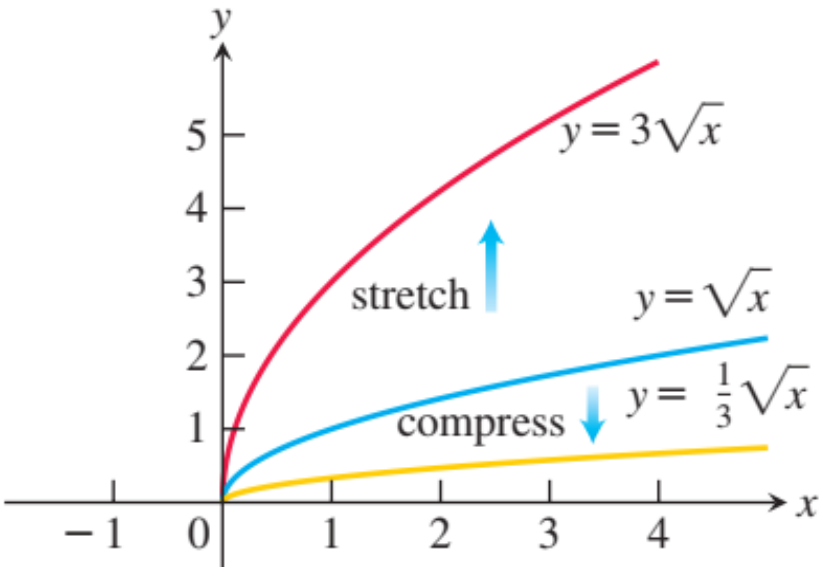
Reflection

For $c = -1$, the graph is reflected

- $y = -f(x)$ Reflects the graph of y across the x-axis.
- $y = f(-x)$ Reflects the graph of y across the y-axis

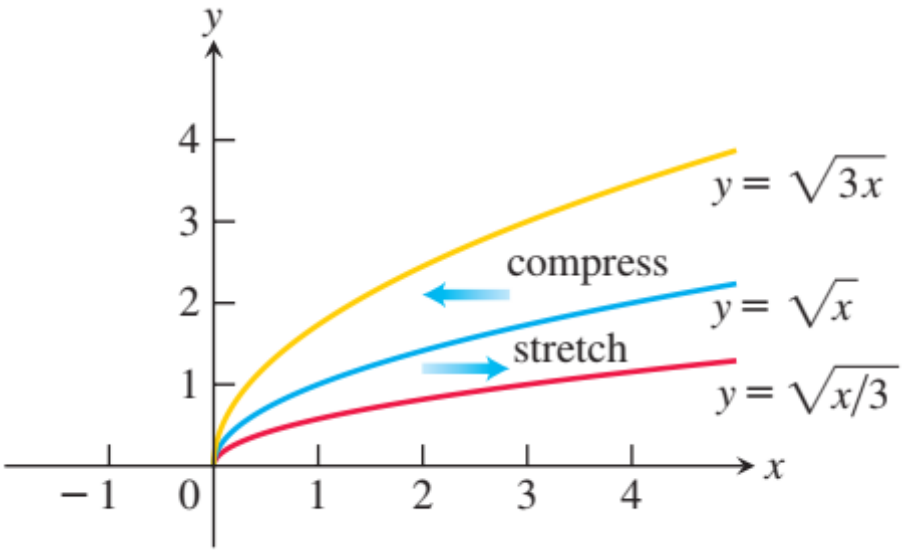
Example 4a Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases cases:

- multiplying the right hand side of y by 3,
- multiplying the right hand side of y by $1/3$



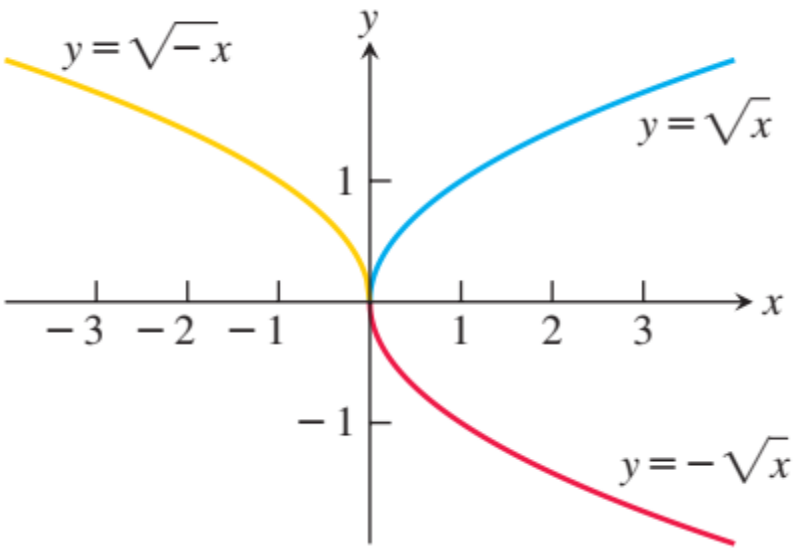
Example 4b Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases cases:

- multiplying x by 3 in the right hand side of y
- multiplying x by $1/3$ in the right hand side of y



Example 4c Scale and sketch the graph of $y = \sqrt{x}$ in each of the following cases cases:

- Reflects the graph of f across the x-axis.
- Reflects the graph of f across the y-axis



Example 5 Given the function $f(x) = x^4 - 4x^3 + 10$, find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis.

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis

Periodic function

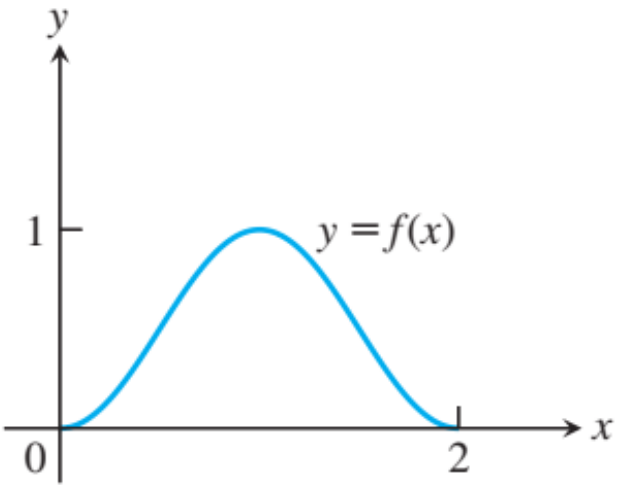
A function $f(x)$ is periodic if there is a positive number p such that

$$f(x+p) = f(x)$$

for every value of x. The smallest such value of p is the period of f.

Practice problems

The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



- $f(x) + 2$
- $f(x) - 1$
- $2f(x)$
- $-f(x)$
- $f(x + 2)$
- $f(x - 1)$
- $-f(x + 1) + 1$

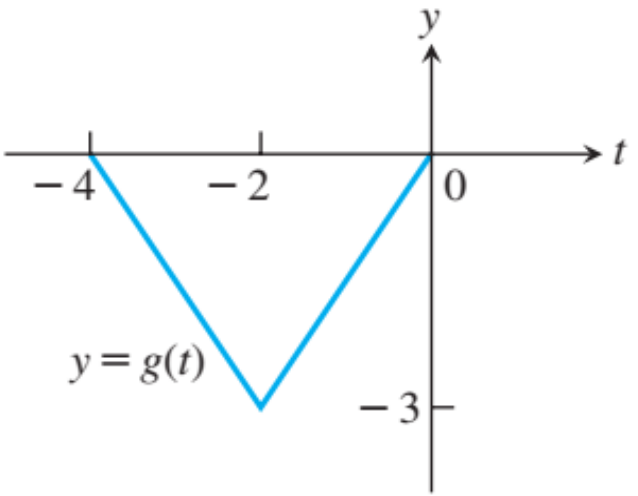
Practice problems

Find by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph

- $y = x^2 - 1$ stretched vertically by a factor of 3
- $y = x^2 - 1$ compressed horizontally by a factor of 2
- $y = 1 + \frac{1}{x^2}$ compressed vertically by a factor of 2
- $y = 1 + \frac{1}{x^2}$ stretched horizontally by a factor of 3
- $y = \sqrt{x+1}$ stretched vertically by a factor of 3
- $y = \sqrt{4 - x^2}$ stretched horizontally by a factor of 2

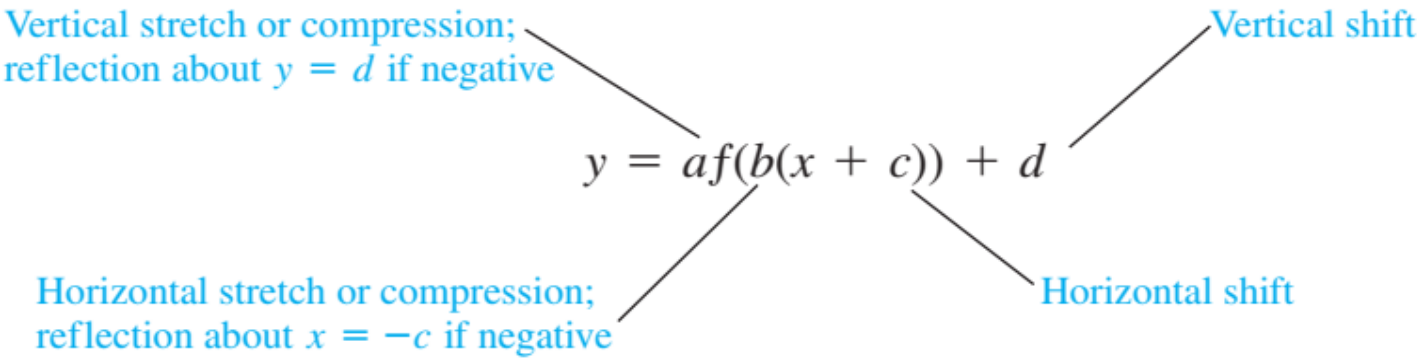
Practice problems

The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions and sketch their graphs



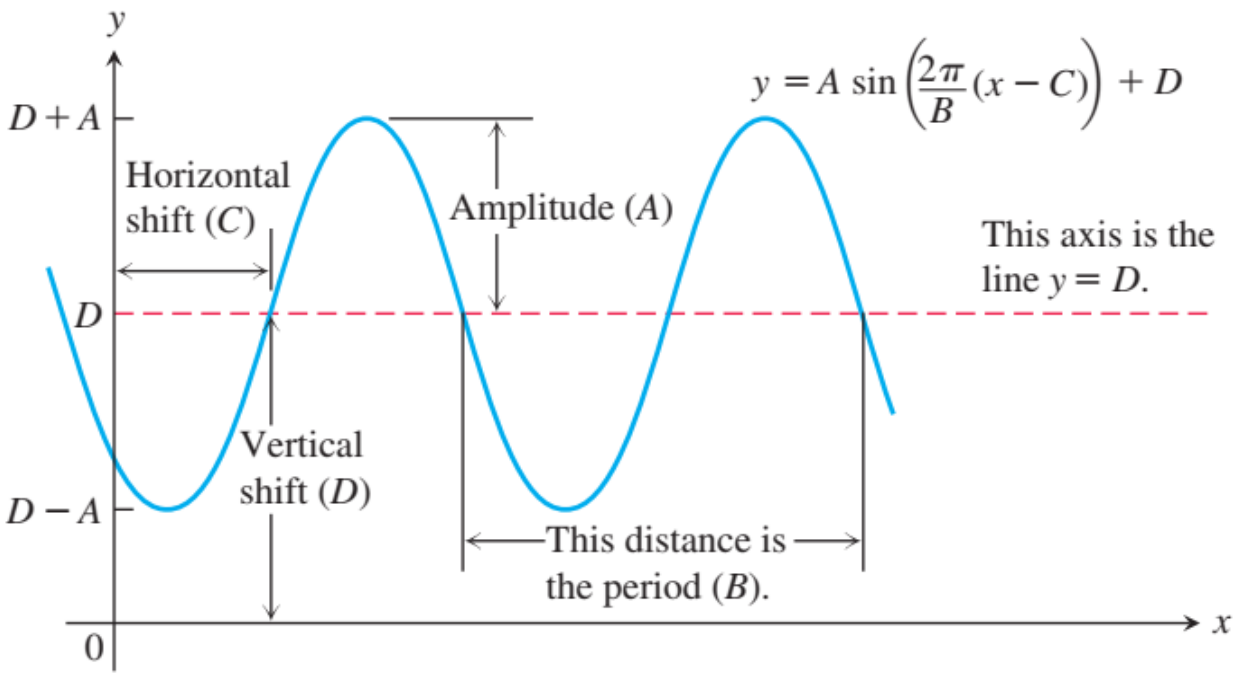
1. $g(-t)$
2. $-g(t)$
3. $g(t) + 3$
4. $1 - g(t)$
5. $g(-t + 2)$
6. $g(t - 2)$
7. $g(1 - t)$
8. $-g(t - 4)$

Transformation of trigonometric graphs



General sine function or sinusoid formula

$$f(x) = A\sin\left(\frac{2\pi}{B}(x-c)\right) + D$$



Problem

Find the period of each function

- $\sin 2x$
- $\sin(x/2)$
- $\cos \pi x$
- $\cos\left(\frac{\pi x}{2}\right)$
- $\sin\left(x + \frac{\pi}{6}\right)$

Practice problems

1. Graph the functions $y = 2\cos(x - \pi/3)$ and $y = 1 + \sin(x + \pi/4)$
2. Describe how each graph is obtained from the graph of $y = f(x)$
   - $y = f(x - 5)$
   - $y = f(4x)$
   - $y = f(-3x)$
   - $y = f(2x + 1)$
   - $y = f(x/3) - 4$
   - $y = -3f(x) + 1/4$

THANK YOU