Calculus and its Applications
(Limits and Continuity - Differentiability)

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LIMITS AND CONTINUITY: Standard functions – Graphs - Limit - continuity - piecewise continuity - periodic - differentiable functions - Riemann sum - integrable functions - fundamental theorem of calculus

TEXT BOOKS:

1. Thomas G B Jr., Joel Hass, Christopher Heil, Maurice D Wier, Thomas' Calculus, Pearson Education, 2018.

Tangent Lines and the Derivative at a Point

To find a tangent line to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we calculate the slope of the secant line through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$. If the limit exists, we call it the slope of the curve at P and define the tangent line at P to be the line through P having this slope.

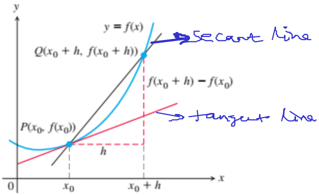


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Definition The slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The tangent line to the curve at P is the line through P with this slope.

Example 1

1. Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$?.What is the slope at the point $x = -1$?
2. Where does the slope equal $-1/4$?
3. What happens to the tangent line to the curve at the point $(a, 1/a)$ as a changes?

1) $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \frac{-1}{a^2}$

Slope of $y = \frac{1}{x}$ at $x = -1$ is $\frac{-1}{(-1)^2} = -1$

② Slope of $y = \frac{1}{x}$ is $-\frac{1}{4}$

$\frac{-1}{a^2} = \frac{-1}{4} \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$

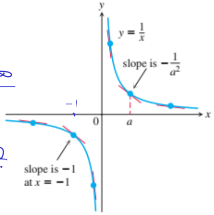
Slope $= \frac{-1}{a^2}$

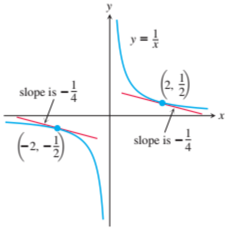
$a \to 0^+$

Slope $\to -\infty$

$a \to 0^-$

Slope $\to$?





The tangent line slopes,
steep near the origin, become more gradual as the point of tangency moves away

The two tangent lines to
y = 1/x having slope -1/4

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0+h)-f(x_0)}{h}, \quad h \neq 0$$

is called the difference quotient of f at x_0 with increment h.

Definition The derivative of a function f at a point x_0, denoted $f'(x_0)$, is

$$\underline{f'(x_0)} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

provided this limit exists.

H.W

Example 2 The rock fall freely from rest near the surface of the earth and its corresponding mathematical expression is given by $y = 16t^2$ feet during the first t sec, and used as a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's exact speed at this time?

Remark

The following are all interpretations for the limit of the difference quotient

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at the $x = x_0$
4. The derivative $f'(x_0)$ at $x = x_0$

The Derivative as a Function

The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

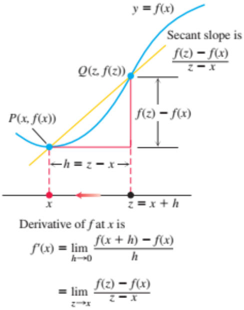
provided the limit exists.

The domain of f' is the set of points in the domain of f for which the limit exists, which means that the domain may be the same as or smaller than the domain of f. If f' exists at a particular x, we say that f is differentiable (has a derivative) at x. If f' exists at every point in the domain of f, we call f differentiable.

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, an equivalent definition of the derivative is as follows (see Figure 3.4). This formula is sometimes more convenient to use when finding a derivative function, and focuses on the point z that approaches x.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$



Example 1 Differentiate $f(x) = \frac{x}{x-1}$. $\Rightarrow f'(x) = \frac{-1}{(x-1)^2}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{-1}{(x-1)^2}$$

Example 2

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.
(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

(a) $f'(x) = \frac{1}{2\sqrt{x}}$ $f'(4) = \frac{1}{4}$

(b) $y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{4}(x - 4)$

$y = \frac{x}{4} + 1$

Differentiable on an Interval

[a,b] [a, ∞)

A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{Right-hand derivative at } a$$

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \quad \text{Left-hand derivative at } b$$

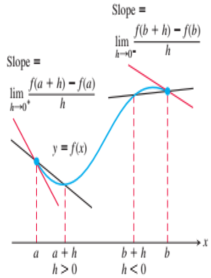
exist at the endpoints.

[a, b]

right hand derivative at x = a (RHD)

left hand derivative at x = b (LHD)

at all other interior points LHD = RHD



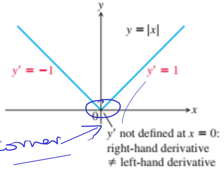
Remark

- Right-hand and left-hand derivatives may or may not be defined at any point of a function's domain.
- A function has a derivative at an interior point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

Problem Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ (a) and (b) on $(0, \infty)$ but (c) has no derivative at $x = 0$.

$y = |x|$

$= \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$



corner

The function y = |x| is not differentiable at the origin where the graph has a "corner"

a) $(-\infty, 0)$

$-\infty \quad 0 \quad \infty$

$y(x) = -x \qquad y(x) = x$

$y(x) = 0$

$y = -x$

$y' = -1 \Rightarrow y$ is diff. $(-\infty, 0)$

b) $(0, \infty)$

$y(x) = x$

$y' = 1$

$\Rightarrow y$ is diff in $(0, \infty)$

$\boxed{\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}}$

$a = 0$

c) LHD: $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} =$

$$= \lim_{h \to 0^-} \frac{f(h) - 0}{h}$$

$$= \lim_{h \to 0^-} \frac{-\cancel{h}}{\cancel{h}}$$

$$= -1$$

RHD

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{h}{h}$$

$$= 1$$

$\Rightarrow$ LHD $\neq$ RHD

$\Rightarrow$ $y = |x|$ is not differentiable at $x = 0$

Domain of $y = (-\infty, \infty)$

Domain of $y' = (-\infty, 0) \cup (0, \infty)$

Problem Verify whether the function, $f(x) = \sqrt{x}$ has a derivative at $x = 0$.

Domain of $f(x) = [0, \infty)$

right hand derivative at $x = 0$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h) - 0}{h}$$

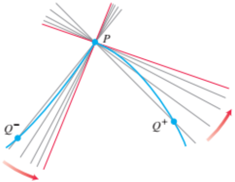
$$= \lim_{h \to 0^+} \frac{\sqrt{h}}{h}$$

$$= \lim_{h \to 0^+} \frac{1}{\sqrt{h}}$$

$= \infty$ (not a finite numbers)

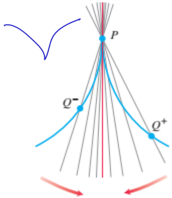
$\Rightarrow$ $f(x)$ has no derivative at $x = 0$

When does a function fails to have a derivative at a point

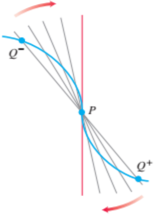


1. a *corner*, where the one-sided derivatives differ

$f(x) = |x|$



2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other

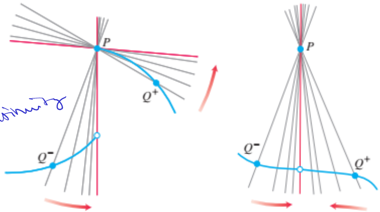


3. a *vertical tangent line*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)

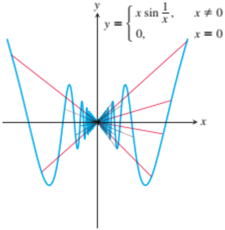
When does a function fails to have a derivative at a point

removable discontinuity

jump discontinuity



4. a *discontinuity* (two examples shown)



5. wild oscillation

Infinite discontinuity

$f(x) = \frac{1}{x^2}$

$\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to 0^-} f(x) = \infty$

Oscillating discontinuity

Differentiable $\Rightarrow$ Continuous $\Rightarrow$ limit exists at all points

Differentiable Functions are Continuous

A function is continuous at every point where it has a derivative.

Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

Remark

The converse of Theorem 1 is false.

A function need not have a derivative at a point where it is continuous.

Differentiation rules

$f(x) = c \Rightarrow f'(x) = 0$

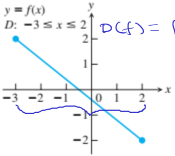
- Derivative of a constant function is zero.
- If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$
- If u is a differentiable function of x, and c is a constant, then $\frac{d}{dx}cu = c\frac{du}{dx}$
- $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{uv' - vu'}{v^2}$

Practice problems

Each figure given below shows the graph of a function over a closed interval D. At what domain points does the function appear to be a. differentiable? b. continuous but not differentiable? c. neither continuous nor differentiable? Give reasons for your answers.

Hw

1



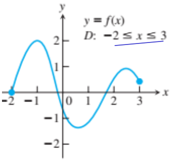
$D(f) = [-3,2]$

a) $[-3,2]$

b) –

c) –

2

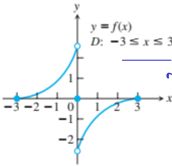


a) $[-2,3]$

b) —

c) –

3



not continuous at $x=0$

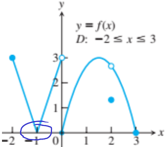
a) $[-3,0) \cup (0,3]$

b) —

c) 0

discontinuous at $x=0, x=2$

4



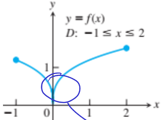
$x=-1$ corner point

a) $[-2,-1) \cup (-1,0) \cup (0,2) \cup (2,3]$ differentiable

b) $x=-1$ continuous but not differentiable

c) $x=0, \ x=2$ neither continuous nor differentiable

5



cusp

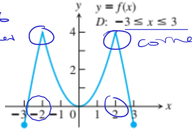
a) $[-1,0) \cup (0,2]$

b) $x=0$

c) —

6

corner



corner

a) $[-3,-2) \cup (-2,2) \cup (2,3]$

b) $x=-2, 2$

c) —

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangent lines? If so, where?

$$\frac{dy}{dx} = 4x^3 - 4x$$

horizontal tangent line $\Rightarrow$ slope $= 0$

$$4x^3 - 4x = 0$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$\Rightarrow \underline{x = 0, 1, -1}$$

at $x = 0, 1, -1$, $y = x^4 - 2x^2 + 2$ has horizontal tangent lines

Find the derivative of $y = \frac{t^2-1}{t^3+1}$

$$d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\frac{dy}{dt} = \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}$$

H.W

The area A of a circle is related to its diameter by the equation $A = \frac{\pi}{4}D^2$. How fast does the area change with respect to the diameter when the diameter is 10 m?

$$\frac{dA}{dD} = \frac{\pi}{4}(2D)$$

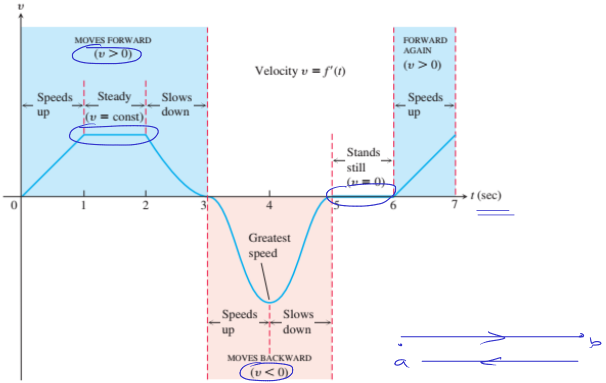
$$= \frac{\pi}{4}2(10)$$

$$\left.\frac{dA}{dD}\right|_{D=10} = 5\pi$$

Definitions

$\frac{ds}{dt} = v$, $\frac{dv}{dt} = \frac{d^2s}{dt^2}$,

1. Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is
$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$
2. Jerk is the derivative of acceleration with respect to time
$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}$$



The velocity graph of a particle moving along a horizontal line

Derivatives of trigonometric functions

Find derivatives of (a) $y = 5x + \cos x$ **(b)** $y = \sin x \cos x$. $= \frac{1}{2} 2 \sin x \cos x$

$= \frac{1}{2} \sin 2x$

$\frac{dy}{dx} = 5 - \sin x$

$\frac{dy}{dx} = \frac{1}{2} \cos 2x \,(2)$

$= \cos 2x$

$\frac{dy}{dx} = \sin x(-\sin x) + \cos x \cos x$

$= \cos^2 x - \sin^2 x$

$\frac{dy}{dx} = \cos 2x$

A weight hanging from a spring is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is $s = 5\cos t$. What are its velocity and acceleration at time t?

$s = 5\cos t$

$\frac{ds}{dt} = v = -5\sin t$

$\frac{d^2s}{dt^2} = a = -5\cos t$

Chain rule

Find the derivative of $y = (3x^2 + 1)^2$**.**

$\frac{dy}{dx} = 2(3x^2+1)(6x+0)$

$\frac{dy}{dx} = 12x(3x^2+1)$

Theorem - The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function

$$(f \circ g)(x) = f(g(x))$$

$\frac{d}{dx}[f(g(x))]$

$= [f'(g(x))]g'(x)$

is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

$$\frac{dx}{dt} = -\sin(t^2+1)\ (2t)$$

$$\frac{dx}{dt} = -2t\sin(t^2+1)$$

Differentiate $\sin(x^2+x)$ **with respect to** x **and** $g(t) = \tan(5 - \sin 2t)$ **with respect to** t**.**

$f(x) = \sin(x^2+x)$

$\frac{df}{dx} = \cos(x^2+x)\,(2x+1)$

$\frac{df}{dx} = (2x+1)\cos(x^2+x)$

$g(t) = \tan(5 - \sin 2t)$

$\frac{dg}{dt} = \sec^2(5 - \sin 2t) \cdot (-2\cos 2t)$

$\frac{dg}{dt} = -2\cos 2t \sec^2(5 - \sin 2t)$

H.W

Find the derivative of (a) $(5x^3 - x^4)^7$**, (b)** $\frac{1}{3x-2}$ **and (c)** $\sin^5 x$**.**

$y = \sin^5 x$

$\frac{dy}{dx} = 5\sin^4 x \cos x$

Find the derivative of $y = |x|$ for non zero x.

$\sqrt{x^2}$

$x \neq 0$

$y = \sqrt{x^2} = (x^2)^{1/2}$

$\frac{dy}{dx} = \frac{1}{\cancel{2}\sqrt{x^2}}(\cancel{2}x)$

$= \frac{x}{\sqrt{x^2}}$

$\frac{dy}{dx} = \frac{x}{|x|}$

$\frac{dy}{dx} = \begin{cases} \frac{x}{x}, & x > 0 \\ \frac{x}{-x} & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

$y = |x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$

$x \neq 0$

$\frac{dy}{dx} = \begin{cases} 1, & \text{for } x > 0 \\ -1, & \text{for } x < 0 \end{cases}$

HW

Show that the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

Implicit differentiation

Find $\frac{dy}{dx}$ if $y^2 = x$.

$$2y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

$$x^2 + y^2 = 25$$

diff. w.r.to 'x'

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = \frac{3}{4}$$

H.W

Find dy/dx **if** $y^2 = x^2 + \sin xy$.

Find d^2y/dx^2 **if** $2x^3 - 3y^2 = 8$**.**

term

Show that the point $(2, 4)$ lies on the curve $x^3 + y^3 - 9xy = 0$ .Then find the tangent and normal to the curve there.

tangent line $y - y_1 = m \; (x - x_1)$

normal line: $y - y_1 = \frac{-1}{m}(x - x_1)$

THANK YOU