Generalized linear models

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Exponential Family: ~~Any~~ group/class of distributions ~~which~~ ~~can~~ for which pmf / pdf can be written in the form of

$$p(y;\eta) = b(y)\exp(\eta^T \cdot T(y) - a(\eta))$$

$\eta \rightarrow$ natural parameter

$T(y) \rightarrow$ Sufficient static (usually $T(y) = y$)

Fixing $b(y)$, $T(y)$, $a(\eta)$, then $P(y;\eta)$ defines a set of distributions as we vary η.

Exponential Family = {Bernouli, Normal, poisson, Geometric, .. multinomial}

Prove bernoulli distribution is Generalized model ?

logistic regression: $y|x;\theta \sim$ Bernoulli(ϕ) Prob. of success

$$P(y=1|x;\theta) = \phi, \quad P(y=0|x;\theta) = 1-\phi$$

we define hypothesis ; $h_\theta(x) = \phi$

$$P(y|x;\theta) = h_\theta(x)^y (1-h_\theta(x))^{1-y} = \phi^y(1-\phi)^{1-y}$$

$$P(y|x;\theta) = \exp(\log \phi^y (1-\phi)^{1-y}) = \exp(y\log\phi + (1-y)\log(1-\phi))$$

$$= \exp\left(y \log\frac{\phi}{1-\phi} + \log(1-\phi)\right)$$

Comparing : $P(y;\eta) = b(y)\exp(\eta^T T(y) - a(\eta))$

$$b(y) = 1, \quad \eta = \log\frac{\phi}{1-\phi}, \quad a(\eta) = -\log(1-\phi), \quad T(y) = y$$

in terms of η: $\phi = \dfrac{1}{1+e^{-\eta}}$ $\quad a(\eta) = \log(1+e^\eta)$

Hence, Bernouli distribution is in Exponential Family

Generalized linear model:

Assumption-1: $y|x;\theta \sim \text{ExpF}(\phi)$; $\text{Bern}(\phi)$

Assumption-2: $h_\theta(x) = E[T(y)|x] = E[y|x] = P(y=1|x;\theta)$
$= \phi = \frac{1}{1+e^{-\eta}}$

Assumption-3: $\eta = \theta^T x$

$$\boxed{h_\theta(x) = \phi = \frac{1}{1+e^{-\theta^T x}}}$$

Prove Gaussian/Normal distribution is an Exponential family?

random variable: $y|x;\theta \sim \text{Gaussian}(\mu, \sigma^2)$

$\sim \mathcal{N}(\mu, 0)$ $\quad$ σ does not matter in linear regress

Probability density function:

$$P(y|x;\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y-\mu)^2\right)$$

$$P(y|x;\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y^2+\mu^2-2y\mu)\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp^{-\frac{1}{2}y^2} \exp\left(y\mu - \frac{\mu^2}{2}\right)$$

~~b(y)~~

Comparing: $P(y;\eta) = b(y)\exp(\eta^T T(y) - a(\eta))$

$b(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right)$ $\quad$ $T(y) = y$, $\eta = \mu$

$a(\eta) = \frac{\mu^2}{2} = \frac{\eta^2}{2}$

Generalized linear model:

Assumption-1: $y|x;\theta \sim \mathcal{N}(\mu, \sigma)$

Assumption-2: $h_\theta(x) = E[T(y)|x] = E[y|x] = \mu = \eta$

Assumption-3: $\eta = \theta^T x$

$$\boxed{h_\theta(x) = \eta = \theta^T x}$$

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