* Lagrange multipliers / optimization with constraints * 19.05.2019

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Problem descr.: $\min_{w} f(w)$

$\quad \text{s.t } h_i(w) = 0, \forall i \in \{1, 2, \dots, l\}$

Read as: find w which minimizes $f(w)$, such that

$h_i(w) = 0 \quad \forall i \in \{1, 2, \dots, l\}$

$$h(w) = \begin{bmatrix} h_1(w) \\ h_2(w) \\ h_l(w) \end{bmatrix} = \vec{0}$$

constraint: $h_i(w) = 0, \forall i \in \{1, 2, \dots, l\}$ ← multiple constraints

this is equality constraint.

solution:

Lagrangian function.

$$\mathcal{L}(w, \lambda) = f(w) + \sum_{i=1}^{l} \lambda_i h_i(w)$$

λ: langrangian multiplier.

solve. $\nabla_{w,\lambda} \mathcal{L}(w, \lambda) = 0$ $\iff$ $\nabla_w f(w) = \sum_{i=1}^{l} \lambda_i h_i(w)$

Constraint: $h_i(w) = 0$

$\forall i \in \{1, 2, \dots l\}$

$$\nabla_{w,\lambda} \mathcal{L}(w, \lambda) = \begin{bmatrix} \frac{\partial \mathcal{L}(w,\lambda)}{\partial w} \\ \frac{\partial \mathcal{L}(w,\lambda)}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

solve for ~~x~~ $w = w^*$, which gives

$f(w^*)$ minimum

tutorial Problems :

Problem-1: $\max_{x,y}$ / $\min_{x,y}$ $f(x,y) = 3x+4y$

s.t $x^2+y^2 = 100$ (equality constraint)

sol. lagrangian;

$$L(x,y,\lambda) = f(x,y) + \lambda(x^2+y^2-100)$$
$$= 3x+4y + \lambda(x^2+y^2-100)$$

$$\nabla_{x,y,\lambda} = \begin{bmatrix} 3 + 2\lambda x \\ 4 + 2\lambda y \\ x^2+y^2-100 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3 = 2\lambda x$ — ①

$4 = 2\lambda y$ — ②

$x^2+y^2 = 100$ — ③

from, ①, ② and ③ : $\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 100$

$9+16 = 100 \times 4\lambda^2$

$\lambda^2 = \frac{1}{16} \Rightarrow \lambda^2 - \frac{1}{16} = 0$

$$\left(\lambda - \frac{1}{4}\right)\left(\lambda + \frac{1}{4}\right) = 0$$

$$\boxed{\lambda = \pm\frac{1}{4}}$$

Substitute $\lambda = \pm\frac{1}{4}$ in equation ① and ② to get x, y that gives minimum and maximum

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$$\min_{\omega} f(\omega)$$

$$g_i(\omega) \le 0 \quad \forall i \in \{1, 2, \dots, m\}$$

$$h_i(\omega) = 0 \quad \forall i \in \{1, 2, \dots, m\}$$

Langrangian:

$$\mathcal{L}(\omega, b, \alpha, \beta) = f(\omega) + \sum_{i=1}^{m} \alpha_i g_i(\omega) + \sum_{i=1}^{m} \beta_i h_i(\omega)$$

Define primal problem,

$$\theta_p(\omega) = \max_{\substack{\alpha, \beta \\ \alpha_i \geq 0}} \mathcal{L}(\omega, b, \alpha, \beta)$$

$$= f(\omega)$$

$\sum_{i=1}^{m} \beta_i h_i(\omega) = 0$ because $h_i(\omega) = 0$

to $\max_{\substack{\alpha, \beta \\ \alpha_i \geq 0}} \mathcal{L}(\omega, b, \alpha, \beta)$ ~~all~~ α $\sum_{i=0}^{m} \alpha_i g_i(\omega) = 0$

$$p^* = \min_{\omega} \theta_p(\omega) = \min_{\omega} f(\omega)$$

← same as the original objective function

Define dual problem:

$$\theta_D(\alpha, \beta) = \min_{\omega} \mathcal{L}(\omega, b, \alpha, \beta)$$

→ will get low value of $f(\omega)$

$$d^* = \max_{\substack{\alpha, \beta \\ \alpha_i \geq 0}} \theta_D(\alpha, \beta) = \max_{\substack{\alpha, \beta \\ \alpha_i \geq 0}} \min_{\omega} \mathcal{L}(\omega, b, \alpha, \beta)$$

→ min $f(\omega)$

usually:

$$\max_{\cancel{\min}} \min_{\cancel{\max}} f(x) \leq \min \max f(x)$$

$\cancel{\theta_p(w)}$

$$\max_{\substack{\alpha, \beta \\ \alpha_i \geq 0}} \theta_D(\alpha, \beta) \leq \min_{w} \theta_p$$

$$d^* \leq p^*$$

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characteristics of Primal Problem:

$f(w)$ must be convex function (Hessian matrix must be positive-semi definite)

$\alpha_i \geq 0$ (langrange multiplier) $\forall i \in \{1, 2, \ldots, m\}$

$\alpha_i g_i(w) = 0$. (KKT conditions)