* Convex functions *

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1. what are convex functions ?

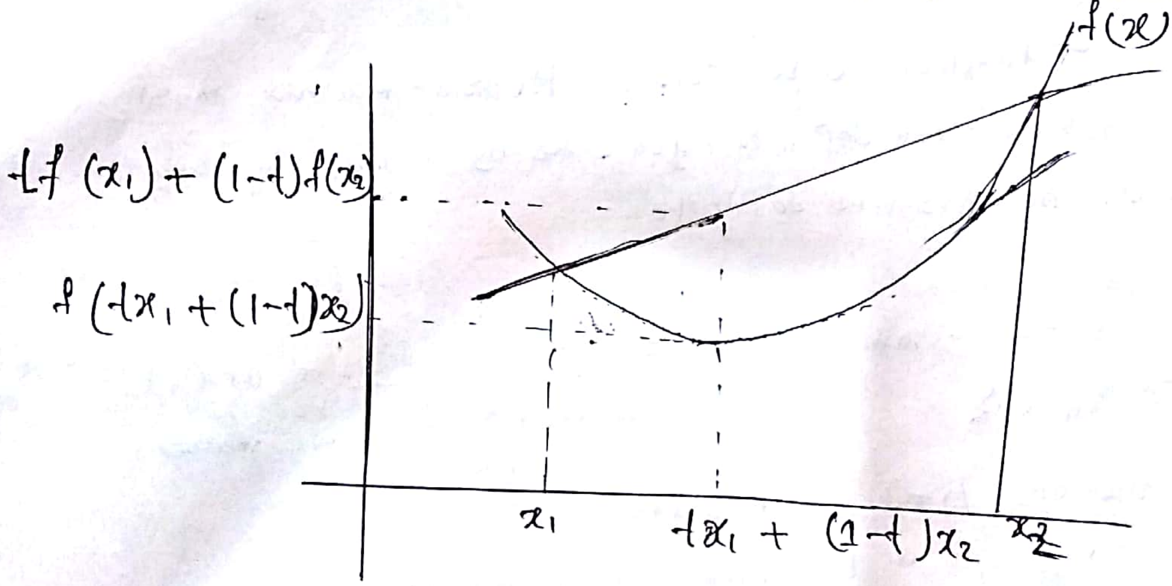
It is a real-valued function defined on an n-dimensional interval if the line segment between any two points on the graph of the function lies above or on the graph.

Well known convex functions:

quadratic functions x^2 and the exponential function e^x

Convex function has "no more one minimum." This property used in optimization of convex function using gradient descent and newtons optimization.

A function f is said to be convex if $-f$ is concave and vice versa



* f is called convex if:

$$\forall x_1, x_2 \in X, \forall t \in (0,1) ; \ f(tx_1+(1-t)x_2) \le tf(x_1)+(1-t)f(x_2)$$

* f is called strictly convex if:

$$\forall x_1 \neq x_2 \in X \ \forall t \in (0,1) : \ f(tx_1+(1-t) < tf(x_1) + (1-t) f(x_2)$$

2. How to identify if a given function is convex ?

Let $f(x,y) = x^2 + y^2$, function in R^2

gradient $\nabla f(x,y) = \begin{bmatrix} \frac{\partial}{\partial x} f(x,y) \\ \frac{\partial}{\partial y} f(x,y) \end{bmatrix}$

Since $f(x,y)$ has two independent variab[les] ∇f has shape 2×1 i.e for n independent variables $n \times 1$

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Hessian matrix : second derivative matrix

$$Hf(x,y) = \begin{array}{c|cc} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \hline \frac{\partial}{\partial x} & \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial}{\partial y} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{array} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

For a ~~no~~ function to be convex Hessian matrix must be positive semidefinite. for strictly convex, Hessian must be positive definite.

3. How to tell given matrix is positive definite and positive semidefinite ?

(positive definite / semi-definite) must be symmetric matrix

1 - Solution: Find eigenvalues

a. All eigenvalues +ve → positive definite (strictly convex)

b. All eigenvalues -ve → negative definite (strictly concave)

c. Some are ~~positive~~ 0's and rest +ve → positive semidefinite (convex)

d. Some are 0's and rest -ve → negative semidefinite (concave)

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Solution: Reduce the given matrix using elimination.

a. If All Pivot Variables are +ve → Positive definite (strictly convex)

b. If All Pivot Variables are -ve → negative definite (strictly convex)

c. If Some Pivot Variables are +ve and Some 0's → Positive semi definite (convex)

d. If Some pivot Variables -ve and Some 0's → negative semi definite (concave)

Tutorial on Positive definite/Semi definite:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

← Symmetric ✓

For a matrix to be definite it must be Symmetric.

Reduce A:

$$\begin{bmatrix} \boxed{2} & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad R_2 - \frac{R_1}{2}$$

$$\begin{bmatrix} \boxed{2} & -1 & 0 \\ 0 & \boxed{3/2} & -1 \\ 0 & 0 & \boxed{\frac{4}{3}} \end{bmatrix}$$

Pivots are $2, \frac{3}{2}, \frac{4}{3}$

all Positive ✓

Hence A is Positive definite.

Tutorial on Convexity of a function :

$f(x,y) = x^2 + y^2$

check for convexicity :

gradient ~~of~~ vector: ~~matrix~~ $\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. x, y are independent Variable hence Δf Vector has 2×1

Hessian matrix : $Hf = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2 \times 2}$

Hf has two pivot variables 2, 2 both positive, hence ~~is~~ its eigenvalues are both positive.

Hence ~~Δf~~ Hf is positive definite, hence it is convex function, has only one minimum (global minimum)