Lecture 3 Note Thursday, 02/10/2022

Announcement
- For students who haven't attend the first three classes, make sure you attend at least one of the first five in-person classes to avoid the WN grade and being withdrawn by the registrar.
- The Lecture Recordings will be available on the following YouTube Playlists Link: https://youtube.com/playlist?list=PLZaTmV9UMKliRBuEs0-dL968iQqE0qwpc

Representations of Integers
- Let b be an integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form $n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$, where k is a nonnegative integer, $a_0, a_1, \ldots, a_k$ are nonnegative integers less than b, and $a_k \neq 0$.

Example of Expended Forms
- Find the expended form of 12034 in base 10.

  $12034 = 1 \cdot 10^4 + 2 \cdot 10^3 + 0 \cdot 10^2 + 3 \cdot 10 + 4$

  $(12034)_{10} = (12034)_{10}$

- Find the expended form of 46 in base 2.

  $2^6 = 64,\ 2^5 = 32,\ 2^4 = 16,\ 2^3 = 8,\ 2^2 = 4,\ 2^1 = 2,\ 2^0 = 1.$

  $46 = 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 0$

  $(46)_{10} = (101110)_2$

Representations of Integers
- The representation of n above is called **the base b expansion of n.** The base b expansion of n is denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$.
  - o For instance, $(245)_8$ represents $2 \cdot 82 + 4 \cdot 8 + 5 = 165$. Typically, the subscript 10 is omitted for base 10 expansions of integers because base 10, or decimal expansions, are commonly used to represent integers.

Examples of Integer Representations
- What is the decimal expansion of the number with binary expansion $(1\,0101\,1111)_2$?

  $$\begin{aligned}(1\,0101\,1111)_2 &= 1 \cdot 2^8 + 0 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 1 \\ &= 256 + 0 + 64 + 0 + 16 + 8 + 4 + 2 + 1 = (351)_{10}\end{aligned}$$

- What is the decimal expansion of the number with octal expansion $(7016)_8$?

  $(7016)_8 = 7 \cdot 8^3 + 0 \cdot 8^2 + 1 \cdot 8 + 6 = 7 \cdot 512 + 0 + 8 + 6 = (3598)_{10}$

- What is the decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

  In hexadecimal, $10 = A,\ 11 = B,\ 12 = C,\ 13 = D,\ 14 = E,\ 15 = F$.

  $(2AE0B)_{16} = 2 \cdot 16^4 + 10 \cdot 16^3 + 14 \cdot 16^2 + 0 \cdot 16 + 11 = (175627)_{10}$

- Find the decimal expansion of $(12034)_{10}$.

  $12034 = 10(1203) + \mathbf{4}$

  $1203 = 10\,(120) + \mathbf{3}$

  $120 = 10\,(12) + \mathbf{0}$

  $12 = 10\,(1) + \mathbf{2}$

  $1 = 10\,(0) + \mathbf{1}$ Stop when quotient is zero.

  Take the remainders from bottom up, you got $(12034)_{10}$.

- Find the binary expansion of $(46)_{10}$.
  $46 = 2(23) + \mathbf{0}$
  $23 = 2(11) + \mathbf{1}$
  $11 = 2(5) \ + \mathbf{1}$
  $5 = 2(2) \ + \mathbf{1}$
  $2 = 2(1) \ + \mathbf{0}$
  $1 = 2(0) \ + \mathbf{1}$ Stop when quotient is zero.
  Take the remainders from bottom up, you got $(101110)_2$.
  From the previous exercise, we know that $(46)_{10} = (101110)_2$.

Base Conversion

- We will now describe an algorithm for constructing the base b expansion of an integer n. First, divide n by b to obtain a quotient and remainder, that is,
  $n = bq_0 + a_0,\ 0 \le a_0 < b$.
  The remainder, a_0, is the rightmost digit in the base b expansion of n. Next, divide q_0 by b to obtain
  $q_0 = bq_1 + a_1,\ 0 \le a_1 < b$.
  We see that a_1 is the second digit from the right in the base b expansion of n. Continue this process, successively dividing the quotients by b, obtaining additional base b digits as the remainders. This process terminates when we obtain a quotient equal to zero. It produces the base b digits of n from the right to the left.

Examples of Base Conversion

- Find the octal expansion of $(12345)_{10}$.
  $12345 = 8(1543) + \mathbf{1}$
  $1543 = 8(192) \ + \mathbf{7}$
  $192 = 8(24) \ + \mathbf{0}$
  $24 = 8(3) \ + \mathbf{0}$
  $3 = 8(0) \ + \mathbf{3}$ Stop when quotient is zero.
  Take the remainders from bottom up, you got $(17003)_8$.

- Find the hexadecimal expansion of $(177130)_{10}$.
  $177130 = 16(11070) + \mathbf{10} \rightarrow \boldsymbol{A}$
  $11070 = 16(691) \ + \mathbf{14} \rightarrow \boldsymbol{E}$
  $691 = 16(43) \ + \ \mathbf{3}$
  $43 = 16(2) \ + \mathbf{11} \rightarrow \boldsymbol{B}$
  $2 = 16(0) \ + \ \mathbf{2}$ Stop when quotient is zero.
  Take the remainders from bottom up, you got $(2B3EA)_{16}$.

Divisibility Rules

- Divisibility Rule for 2: Check the last digit.
  - An integer N is divisible by 2 if and only if the last digit of N is divisible by 2.
  - A number that's divisible by 2 will congruent to $0 (\text{mod } 2)$.
  - If we can show that $N \equiv \textit{last digit of } N \ (\text{mod } 2)$, then if one of them divisible by 2, so does the other one.
  - To show $N \equiv \textit{last digit of } N \ (\text{mod } 2)$, we need to rewrite N such that its last digit appears. We can put N in the division algorithm form with $d = 10$, then we have $N = 10q + r,\ 0 \le r < 10$, and r will be the last digit of N.
  - Compute $N (\text{mod } 2)$: $N \equiv 10q + r \equiv 0q + r \equiv r \ (\text{mod } 2)$.

- Since $N \equiv \textit{last digit of } N \pmod 2$, you can check whether the number is divisible by 2 by checking its last digits.
- Similar Divisibility Rules that check the last (few) digit(s):
  - Divisibility Rule for 5: Check the last digit.
  - Divisibility Rule for 10: Check the last digit.
  - Divisibility Rule for 4: Check the last two digits.
  - Divisibility Rule for 8: Check the last three digits.

- Divisibility Rule for 3: Check the sum of the digits.
  - An integer N is divisible by 3 if and only if the sum of all digits of N is divisible by 3.
  - Similar to divisibility rule for 2, if we can show that $N \equiv \textit{sum of all digits of } N \pmod 3$, then we know this divisibility rule is true.
  - To have all digits of N appears, we can write N in the expended form of base 10, i.e., $N = a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 \cdot 10 + a_0$, where $a_0, a_1, \ldots, a_k$ are digits of N.
  - Compute $N \pmod 3$: $N \equiv a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 \cdot 10 + a_0 \pmod 3$.
    - $10^x \equiv 10 \cdot 10 \cdot \ldots \cdot 10 \equiv 1 \cdot 1 \cdot \ldots \cdot 1 \equiv 1^k \equiv 1 \pmod 3$

$$\begin{aligned} N &\equiv a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 \cdot 10 + a_0 \pmod 3 \\ &\equiv a_k \cdot 1 + a_{k-1} \cdot 1 + \cdots + a_1 \cdot 1 + a_0 \pmod 3 \\ &\equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \pmod 3 \end{aligned}$$

  - Since $N \equiv \textit{sum of all digits of } N \pmod 3$, you can check whether the number is divisible by 3 by checking the sum of its digits.
- Similar Divisibility Rule that checks the sum of its digits.
  - Divisibility Rule for 9: Check the sum of its digits.
  - Divisibility Rule for 11: Check the alternate sum of its digits.
- Divisibility Rule for 11:
  - Let's observe the following:
    Let $N = a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 \cdot 10 + a_0$, where $a_0, a_1, \ldots, a_k$ are digits of N.
  - Compute $N \pmod{11}$: $N \equiv a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 \cdot 10 + a_0 \pmod{11}$.
    - $10^x \equiv 10 \cdot 10 \cdot \ldots \cdot 10 \equiv (-1) \cdot (-1) \cdot \ldots \cdot (-1) \equiv (-1)^k \pmod{11}$
      $10^x \equiv 1 \pmod{11}$ when x is even, $10^x \equiv -1 \pmod{11}$ when x is odd.

    Then, $N \equiv a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 \cdot 10 + a_0 \pmod{11}$

$$\begin{aligned} &\equiv a_k(-1)^k + a_{k-1}(-1)^{k-1} + \cdots + a_1 \cdot (-1) + a_0 \pmod{11} \\ &\equiv a_k(-1)^k + a_{k-1}(-1)^{k-1} + \cdots + a_4 - a_3 + a_2 - a_1 + a_0 \pmod{11} \end{aligned}$$

  - Since $N \equiv \textit{alternate digits sum of } N \pmod{11}$, you can check whether the number is divisible by 11 by checking the alternate sum of its digits.
- Divisibility Rules Reference Link:
  https://brilliant.org/wiki/proof-of-divisibility-rules/

What to expect or prepare for the next class:
- Prime and composite
- Greatest Common Divisor

Suggested Problems (You don't need to hand in.)
- Discrete Mathematics and its Application 4.2 # 1,3,5,7
- zyBook Additional Exercises #1.3.1, 1.3.2