CSCI 220 | Spring 2022

Discrete Structure

Induction

Discrete Mathematics and its Application

Section 5.1, 5.2

Mathematical Induction

Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for positive integer n.

Basis step: $n = 1$ $\quad 1 = 1^2$ ✓ $\quad | \; n = 2 \quad 1 + 3 = 2^2$, $4 = 4$ ✓ $\quad | \; n = 3 \quad 1 + 3 + 5 = 3^2$, $9 = 9$ ✓

Inductive step: IH: Assume $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ for $k \geq 1$. ← $P(k)$

Want to show $1 + 3 + 5 + \cdots + (2(k+1) - 1) = (k+1)^2$.

$(2k+2) - 1$

$$\underbrace{1 + 3 + 5 + \cdots (2k-1)}_{\text{by IH} \quad k^2} + (2k+1) = (k+1)^2$$

$$k^2 + 2k + 1 = (k+1)^2$$

$$(k+1)^2 = (k+1)^2 \checkmark$$

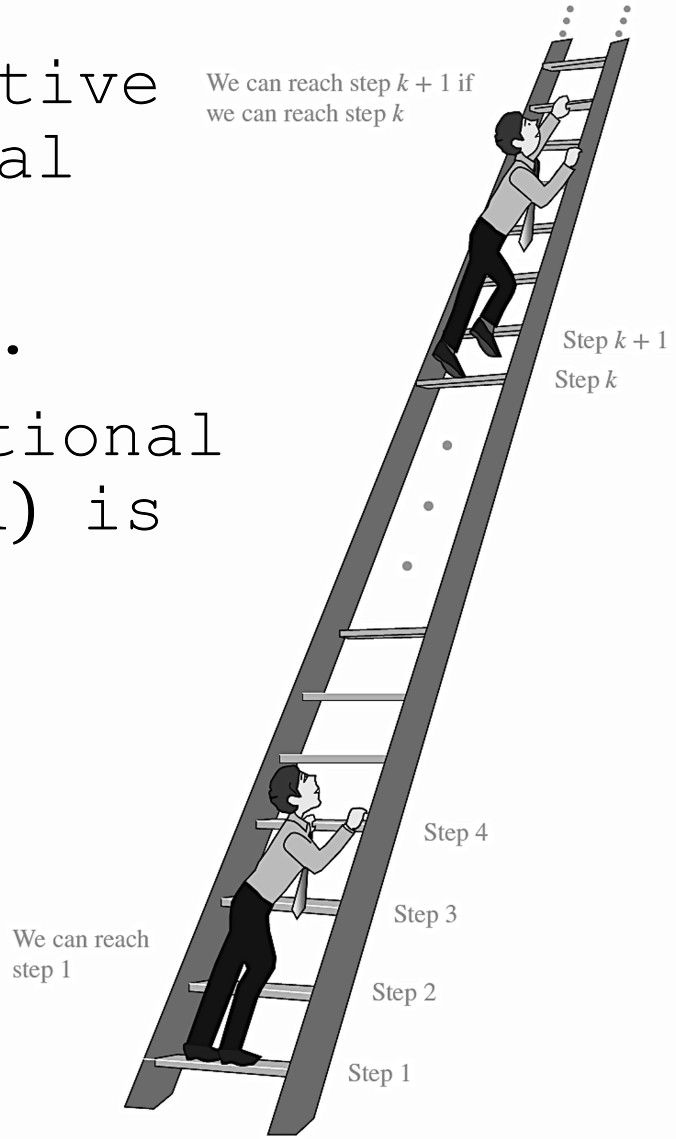
$\blacksquare$

Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.



Mathematical Induction

Prove that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for positive integer n.

Basis step : $n = 1$ $\quad 1 = \frac{1(1+1)}{2}$ ✓

$1 = 1$

Inductive step: IH: Assume $1+2+3+\dots+k = \frac{k(k+1)}{2}$ for $k \geq 1$.

WTS (want to show) $\quad 1+2+3+\dots+(k+1) = \frac{(k+1)((k+1)+1)}{2}$

$$\underbrace{1+2+3+\dots+k}+(k+1) = \frac{(k+1)(k+2)}{2}$$

by IH, $\quad \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

$$\frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

Mathematical Induction

Prove that $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ for positive integer n.

Basis Step: $n=1$: $1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6}$

$1 = 1$ ✓

Inductive Step: IH : Assume $1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$ for $k \geq 1$.

Want to show $1^2+2^2+3^2+\cdots+(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$.

$$\underbrace{1^2+2^2+3^2+\cdots+k^2}_{\text{by IH}}+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

by IH, $\left(\frac{k(k+1)(2k+1)}{6}+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}\right) * 6$

$$k(k+1)(2k+1)+6(k+1)^2 = (k+1)(k+2)(2k+3)$$

$$(k+1)(k(2k+1)+6(k+1)) = (k+1)[(k+2)(2k+3)]$$

$$(k+1)(2k^2+k+6k+6) = (k+1)(2k^2+4k+3k+6)$$

$$(k+1)(2k^2+7k+6) = (k+1)(2k^2+7k+6)$$ ✓ ∎

Mathematical Induction

Prove or disprove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = n^4 - n^3 + 1$ for positive integer n.

False.

$n = 1$ $\quad 1^3 = 1^4 - 1^3 + 1$

$1 = 1$ ✓

$n = 2$ $\quad 1^3 + 2^3 = 2^4 - 2^3 + 1$

$1 + 8 = 16 - 8 + 1$

$9 = 9$ ✓

$n = 3$ $\quad 1^3 + 2^3 + 3^3 = 3^4 - 3^3 + 1$

$1 + 8 + 27 = 81 - 27 + 1$

$36 \neq 55$

Assume $1^3 + 2^3 + \cdots + k^3 = k^4 - k^3 + 1, \ldots$

$\underbrace{1^3 + 2^3 + 3^3 + \cdots + k^3}_{} + (k+1)^3 = (k+1)^4 - (k+1)^3 + 1$

by I H $\quad k^4 - k^3 + 1 + (k+1)^3 =$

$\neq$

Mathematical Induction $b^{x+y} = b^x \cdot b^y$

Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Prove that $7^{n+2} + 8^{2n+1} \equiv 0 \pmod{57}$ for $n \geq 0$.

Basis step: $n=0$ $\quad 7^{0+2} + 8^{2(0)+1} \equiv 7^2 + 8 \equiv 49 + 8 \equiv 57 \equiv 0 \pmod{57}$ ✓

Inductive step: IH: Assume $7^{k+2} + 8^{2k+1} \equiv 0 \pmod{57}$

WTS $7^{(k+1)+2} + 8^{2(k+1)+1} \equiv 0 \pmod{57}$

$$7^{(k+1)+2} + 8^{2(k+1)+1} \equiv 7^{k+3} + 8^{2k+3} \equiv 7^{k+2} \cdot 7 + 8^{2k+1} \cdot 8^2 \pmod{57}$$

$$\equiv 7^{k+2} \cdot 7 + 8^{2k+1} \cdot 64 \equiv 7^{k+2} \cdot 7 + 8^{2k+1} \cdot 7$$

$$\equiv 7 \cdot (\underbrace{7^{k+2} + 8^{2k+1}}_{\text{by IH}}) \equiv 7 \cdot 0 \equiv 0 \pmod{57}$$

Mathematical Induction

Prove that $f_1 + f_2 + f_3 + \ldots + f_n = f_{(n+2)} - 1$ for positive integer n.

Let f_n be the nth Fibonacci number. $f_1 = 1,\ f_2 = 1,\ f_n = f_{n-1} + f_{n-2}$.

$f_0 = 0,\ f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5,\ f_6 = 8,\ f_7 = 13, \ldots$

Basis step: $n = 1$.

$f_1 = f_{1+2} - 1$
$f_1 = f_3 - 1$
$1 = 2 - 1$
$1 = 1$ ✓

Inductive step: IH: Assume $f_1 + f_2 + f_3 + \cdots + f_k = f_{k+2} - 1$ for $k \geq 1$.

WTS $f_1 + f_2 + \cdots + f_{k+1} = f_{(k+1)+2} - 1$

$\underbrace{f_1 + f_2 + \cdots + f_k}_{\text{by IH}} + f_{k+1} = f_{k+3} - 1$

$\underbrace{f_{k+2} - 1 + f_{k+1}}_{\text{by def.}} = f_{k+3} - 1$

$f_{k+3} - 1 = f_{k+3} - 1$ ✓

Mathematical Induction

Show that $2^n < n!$ for integer $n \geq 4$.

$n! = n(n-1) \cdots 2 \cdot 1$ ex: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

Basis step: $n = 4$ $2^4 < 4!$
$16 < 24$ ✓

Inductive step: IH: Assume $2^k < k!$ for $k \geq 4$.

WTS. $2^{k+1} < (k+1)!$

$2^k \cdot 2 < (k+1) \cdot k!$

$2^k < k!$ by IH $2 < k+1$ by IH, $k \geq 4$

$$2^{k+1} = 2^k \cdot 2 \underbrace{<}_{\text{by IH}} k! \cdot 2$$

by IH
$k \geq 4$

$< k! \cdot (k+1)$

$= (k+1)!$

Mathematical Induction

FToA.

Show that if n is an integer greater than 1, then n can be written as a prime or product of primes.

Basis step: $n=2$ $\quad 2=2$ prime ✓

$2 \le j \le k$

$2=2$
$3=3$
$4=2\cdot2$
$5=5$
$6=2\cdot3$
$7=7$
$8=2\cdot2\cdot2$

Inductive step: IH: Assume j = prime or product of primes, $k>1$, $k \geq 2$.

WTS: $k+1$ = prime or product of prime.

C1: $k+1$ is prime $\quad k+1 = k+1$ prime ✓

C2: $k+1$ is not a prime, it's a composite.

$k+1 = a\cdot b \quad a,b \neq 1, k+1, \quad 1 < a,b < k+1 \Rightarrow 2 \le a,b \le k.$

by IH: a = prime or product of primes, b = prime or product of primes

$k+1 = a\cdot b$ = (prime or product of primes)(prime or product of primes) = product of prime.

Strong Induction

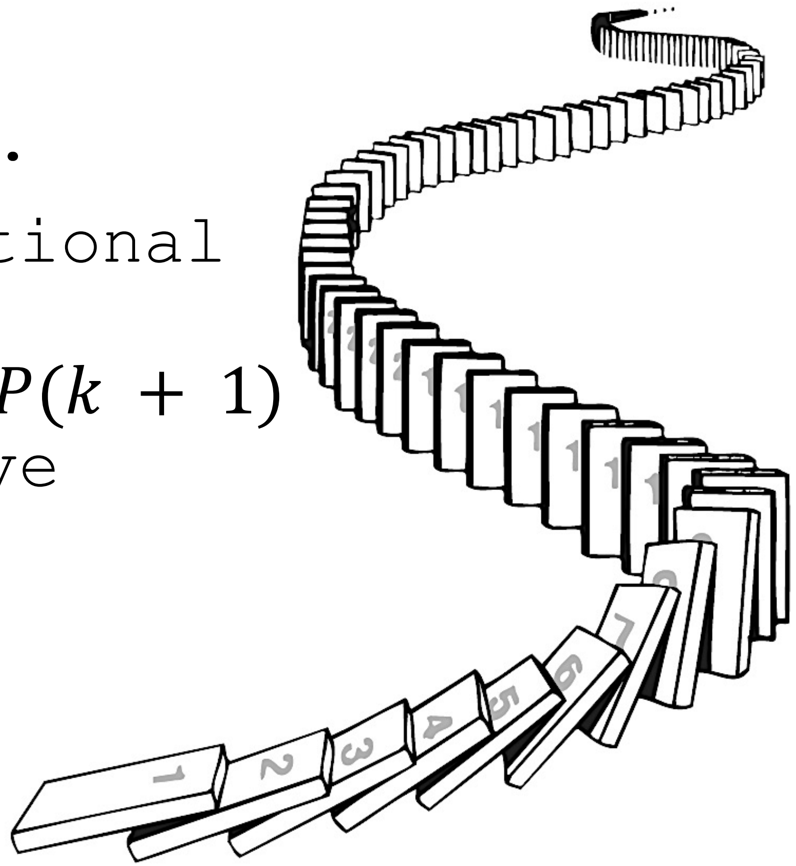
To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement

$$[P(1) \overset{\text{and}}{\wedge} P(2) \overset{\text{and}}{\wedge} \cdots \overset{\text{and}}{\wedge} P(k)] \rightarrow P(k+1)$$

is true for all positive integers k.



Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Mathematical Induction

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

VS

Strong Induction

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for all positive integers k.

Mathematical Induction

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

By Mathematical Induction ($P(k) \to P(k+1)$):

Basis Step: $n = 12$: 12 cents postage can be formed by three 4-cent stamps.

Inductive Step: IH: Assume k cents postage can be formed using just 4-cent and 5-cent stamps, $k \geq 12$.

Want to show $k+1$ cents postage can be formed using just 4-cent and 5-cent stamps.

By IH, we know k cents postage can be formed using just 4-cent and 5-cent stamps.

<u>case 1</u>: There are at least one 4 cent stamp used in k cents postage.
Exchange one 4-cent stamp to a 5-cent stamp. to form $k+1$ cents postage.

<u>case 2</u>: There are no 4-cent stamp used in k cents postage, which it's formed by only 5-cent stamps.
Since $k \geq 12$, at least three 5-cent stamps are used in k cents postage
Exchange three 5-cent stamps to four 4-cent stamp to form $k+1$ cents postage.
$(15 \to 16)$

Either ways, you can form $k+1$ cents postage from k cents postage. $\blacksquare$

Mathematical Induction

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

By Strong Induction:

Basis Step: $n=12$: 12 cents postage can be formed by three 4-cent stamps.

$n=13$: 13 cents postage can be formed by two 4-cent and one 5-cent stamps

$n=14$: 14 cents postage can be formed by one 4-cent and two 5-cent stamps.

$n=15$: 15 cents postage can be formed by three 5-cent stamps.

Inductive Step: IH: Assume j cents postage can be formed using just 4-cent and 5-cent stamps, $12 \le j \le k$, $k \ge 15$.

Want to show $k+1$ cents postage can be formed using just 4-cent and 5-cent stamps.

To form $k+1$ cents postage, we can look up how we form $(k+1)-4 = k-3$ cents postage, then adding a 4-cent stamp to $k-3$ cents postage, we can get $k+1$ cents postage.

Since $k \ge 15$, $k-3 \ge 15-3=12$, then $12 \le k-3 \le k$, by IH, $k-3$ can be formed by just 4-cent and 5-cent stamps. Then add a 4-cent stamp to it, we got $k+1$ cents postage.

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Mathematical Induction

Prove that $f_n < 2^n$ for positive integer n.

Let f_n be the nth Fibonacci number. $f_1 = 1,\ f_2 = 1,\ f_n = f_{n-1} + f_{n-2}$.

Basis step: $n = 1$, $f_1 < 2^1$

$1 < 2$ ✓

Inductive step: IH: Assume $f_k < 2^k$ for $k \geq 1$.

Want to show $f_{k+1} < 2^{k+1}$.

$$f_{k+1} = f_k + \underbrace{f_{k-1} \leq f_k}_{\substack{f_{k-1} \leq f_k \\ \text{by def}}} + f_k = 2 \cdot \underbrace{f_k < 2 \cdot 2^k}_{\text{by IH}} = 2^{k+1}$$

$$f_{k+1} < 2^{k+1}$$

$\boxed{}$

Mathematical Induction

Show that $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = (f_{2n})^2$ for positive integer n.

Let f_n be the nth Fibonacci number. $f_1 = 1,\ f_2 = 1,\ f_n = f_{n-1} + f_{n-2}$.

Basis Step: $n = 1$ $\quad f_0 f_1 + f_1 f_2 = (f_2)^2$ $\qquad f_0 = 0$

$$0 \cdot 1 + 1 \cdot 1 = 1^2$$

$$1 = 1 \checkmark$$

Inductive Step: IH: Assume $f_0 f_1 + f_1 f_2 + \cdots + f_{2k-1} f_{2k} = (f_{2k})^2$ for $k \geq 1$.

Want to show $f_0 f_1 + f_1 f_2 + \cdots + f_{2(k+1)-1} f_{2(k+1)} = (f_{2(k+1)})^2$

$$\underbrace{f_0 f_1 + f_1 f_2 + \cdots + f_{2k-1} f_{2k}}_{\text{by IH,}} + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} = f_{2k+2}^2$$

$$f_{2k}^2 + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} = f_{2k+2}^2$$

$$f_{2k}(f_{2k} + f_{2k+1}) + f_{2k+1} f_{2k+2} = f_{2k+2}^2$$

$$f_{2k} f_{2k+2} + f_{2k+1} f_{2k+2} = f_{2k+2}^2$$

$$f_{2k+2}(f_{2k} + f_{2k+1}) = f_{2k+2}^2$$

$$f_{2k+2} f_{2k+2} = f_{2k+2}^2 \checkmark$$

Mathematical Induction Model

To prove $P(n)$ is true for $n \geq b$ by induction:

Basis step: $n = b$ Show $P(b)$ is true.

Inductive step: IH: Assume $P(K)$, for $K \geq b$.

write the $P(K)$ statement

Want to show $P(K+1)$. write the $P(K+1)$ statement.

① Rewrite $P(K+1)$ so you can see $P(K)$ as part of $P(K+1)$.

by IH, ② Apply IH. Normally you will substitute the $P(K)$.

③ Show that $P(K+1)$ is true.

Mathematical Induction