CSCI 220 | Spring 2022

Discrete Structure

Growth of Functions

Discrete Mathematics and its Application

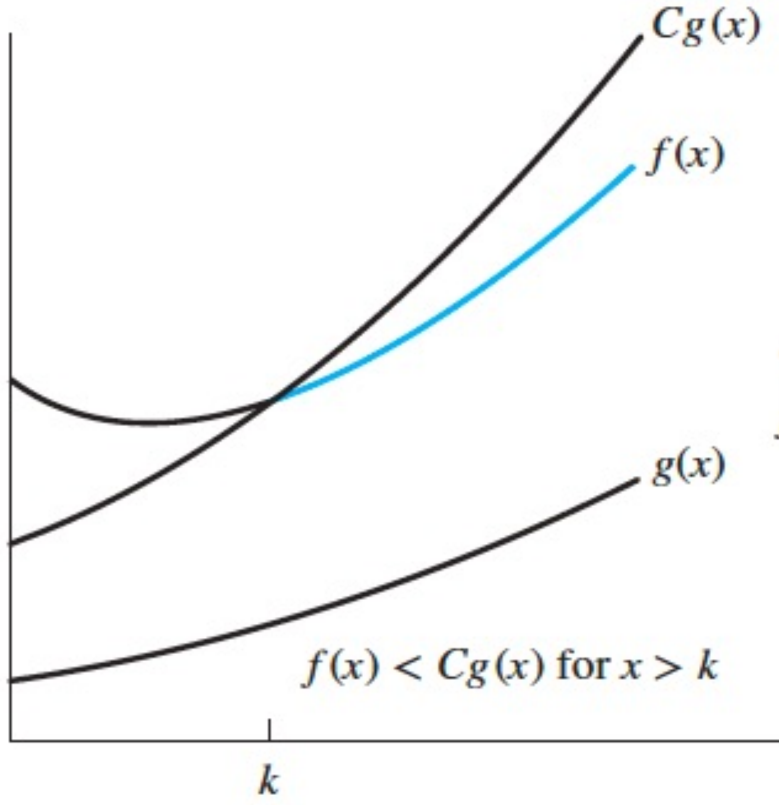
Section 3.2

Big-O Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-oh of $g(x)$."]



Big-O Notation

- Show that $\underbrace{x^2 + 2x + 1}_{f(x)} = O(\underbrace{x^2}_{g(x)})$.

$x^2 + 2x + 1 \le C \cdot x^2$ for $x > k$

$x^2 + 2x + 1 \le x^2 + 2x^2 + x^2 = 4x^2$ for $x > 1$.

$\left\{ \begin{array}{l} x^2 \le x^2 \\ x \le x^2 \\ 1 \le x^2 \\ x > 1 \end{array} \right.$

$\boxed{\begin{array}{c} C = 4 \\ k = 1 \end{array}}$

Big-O Notation

- Is $2x^2 - 10 = O(x^2)$?

$2x^2 - 10 \leq C x^2$

$2x^2 - 10 \leq 2x^2$ for $x > 1$

$\boxed{\begin{array}{l} C = 2 \\ k = 1 \end{array}}$

Yes

- Is $x^3 - 3x + 4 = O(x^2)$?

No!

$\frac{x^3 - 3x + 4}{x^2} \leq \frac{C x^2}{x^2}$

$\Rightarrow x - \frac{3}{x} + \frac{4}{x^2} \leq C$ for $x > k$

$\lim_{x \to \infty} x - \frac{3}{x} + \frac{4}{x^2} = \infty$

C DNE.

- Is $\sin x = O(1)$?

$|\sin(x)| \leq C \cdot 1 \leftarrow$

$-1 \leq \sin(x) \leq 1$

$\boxed{\begin{array}{l} C = 1 \\ k = 1 \end{array}}$

Yes

Big-O Notation

- If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is bounded, then $f(x) = O\big(g(x)\big)$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \neq \infty$$

$$\frac{f(x)}{g(x)} \leq \frac{C \cdot g(x)}{g(x)}$$

$$\frac{f(x)}{g(x)} \leq C \quad \text{for } \underset{\substack{\Downarrow \\ x\to\infty}}{x > k}$$

$\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is bounded.

Big-O Notation

- Is $\sqrt{x} = O(x^4)$?

$\sqrt{x} \le x^4$ for $x > 1$

$\boxed{\begin{array}{l} C = 1 \\ k = 1 \end{array}}$

Yes

or $\lim_{x\to\infty} \frac{\sqrt{x}}{x^4} = 0$

- Is $\sqrt{x} + \frac{1}{x} = O(2\sqrt{x})$?

$\lim_{x\to\infty} \frac{\sqrt{x} + \frac{1}{x}}{2\sqrt{x}} = \frac{1}{2}$

$\frac{1}{x} \to 0$

Yes

$\sqrt{x} + \frac{1}{x} \le C \cdot 2\sqrt{x}$

$\sqrt{x} + \frac{1}{x} \le \sqrt{x} + \sqrt{x} = 2\sqrt{x}$

for $x > 1$

$\boxed{\begin{array}{l} C = 1 \\ k = 1 \end{array}}$

- Is $\sqrt{x} = O\left(\frac{\sqrt{x}}{100!}\right)$?

$\lim_{x\to\infty} \frac{\cancel{\sqrt{x}} \cdot 100!}{\frac{\cancel{\sqrt{x}}}{\cancel{100!}} \cdot \cancel{100!}} = 100!$

Yes.

Big-O Notation

- Is $1 + 2 + 3 + \cdots + n = O(n^2)$?

$$n^2 = n \cdot n = \underbrace{n + n + n + \cdots + n}_{n \text{ times}}$$

$$1 + 2 + 3 + \cdots + n \le n + n + n + \cdots + n = n^2$$

$$\boxed{\begin{matrix} C = 1 \\ k = 1 \end{matrix}}$$ Yes.

- Is $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = O(n)$?

$$n = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ times}}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le 1 + 1 + 1 + \cdots + 1 = n$$

$$\boxed{\begin{matrix} C = 1 \\ k = 1 \end{matrix}}$$ Yes

- Is $n! = O(n^n)$?

$$n! = n(n-1)(n-2)\cdots(2)(1) \le n \cdot n \cdot n \cdots n = n^n$$

$$\boxed{\begin{matrix} C = 1 \\ k = 1 \end{matrix}}$$ Yes.

Big-O Notation

- Arrange the following functions in a list so that each function is big-O of the next function.
  - ~~n~~
  - 2^n
  - ~~$\log n$~~
  - $n!$
  - ~~$n \log n$~~
  - ~~1~~
  - ~~n^2~~

slow → fast

1

$\log n$

n

$n \log n$

n^2

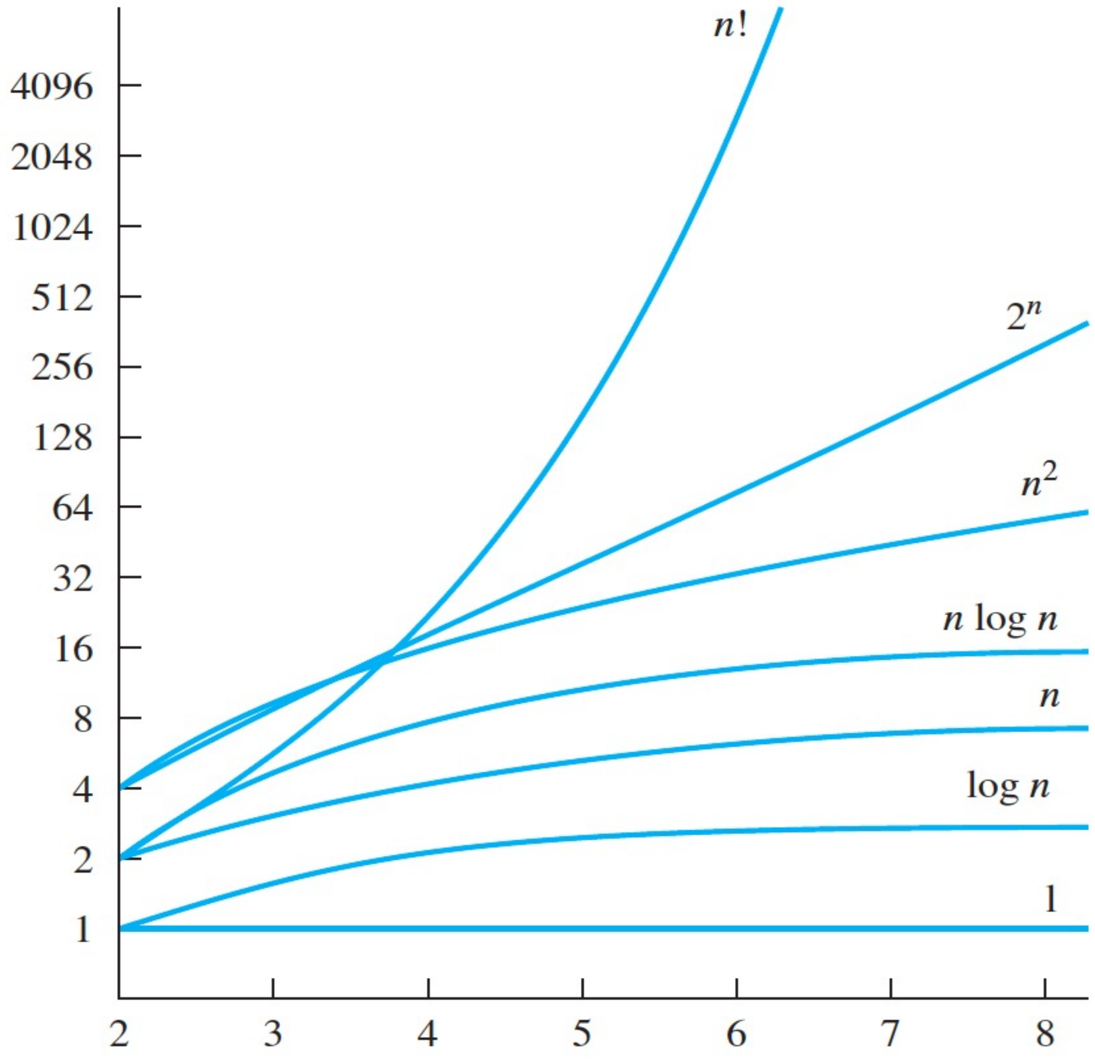
2^n

$n!$

Growth of functions $(\log x)^{1000} < x^{0.001}$

- Constant
- Logarithm
- Polynomial
- Exponential
- Factorial

n^n



Little-o Notation

- If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, then $f(x) = o(g(x))$.

$x^2 = o(x^3)$

Little-o Notation

- Is $2x^2 - 10 = o(x^2)$?

$$\lim_{x \to \infty} \frac{2x^2 - 10}{x^2} = 2 \neq 0$$ No

- Is $\sqrt{x} = o(x^4)$?

$$\lim_{x \to \infty} \frac{\sqrt{x}}{x^4} = 0$$ Yes

- Is $\sqrt{x} = o\left(\frac{\sqrt{x}}{100!}\right)$?

$$\lim_{x \to \infty} \frac{\sqrt{x}}{\frac{\sqrt{x}}{100!}} = 100! \neq 0$$ No

Little-o Notation

- Is $3^n = o(4^n)$?

$\left(\frac{3}{4}\right)^n$

$$\lim_{n\to\infty} \frac{3^n}{4^n} = 0$$ Yes

- Is $2^n = o\left(n^3\right)$?

$$\lim_{n\to\infty} \frac{2^n}{n^3} = \infty$$ No

- Is $\log n = o(\sqrt{n})$?

$$\lim_{n\to\infty} \frac{\log n}{\sqrt{n}} = 0$$ Yes

- Is $n! = o(n^n)$?

$1 \; (\le 1) \; (\le 1) \; \cdots$

$1 \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{2}{n}\right)\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} \frac{n\cdot(n-1)(n-2)\cdots 2\cdot 1}{n\cdot n\cdot n \cdots n\cdot n} = 0$$ Yes.

Big-O / Little-o Exercise

- Find example of $f(x)$ such that $\underbrace{f(x) = O(x)}_{\lim_{x\to\infty} \frac{f(x)}{x} \neq \infty}$, but $\underbrace{f(x) \neq o(x)}_{\lim_{x\to\infty} \frac{f(x)}{x} \neq 0}$.

(bounded)

$f(x) = x$ (any linear function works).

$$\lim_{x\to\infty} \frac{x}{x} = 1$$

Big-O / Little-o Exercise

- Find example of $f(x)$ such that $f(x) = o(x^2)$, but $f(x) \neq O(x)$.

$f(x) = x \log x$

$f(x) = x\sqrt{x}$

$\lim_{x\to\infty} \frac{f(x)}{x^2} = 0$, $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$

$f(x) \neq O(x)$, $f(x) = o(x)$?

$\lim_{x\to\infty} \frac{f(x)}{x} = \infty$, $\lim_{x\to\infty} \frac{f(x)}{x} = 0$

$f(x) = o(g(x))$, $g(x) = o(f(x))$?

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$

No such $f(x)$ and $g(x)$.

the two conditions can not be satisfied at the same time.

Big-Omega Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Omega(g(x))$ if there are constants C and k with C positive such that

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$.

[This is read as "$f(x)$ is big-Omega of $g(x)$."]

$x^2+2x+1 = \Omega(x^2)$

$x^2+2x+1 \geq Cx^2$ for $x > k$

$x^2+2x+1 \geq x^2$ for $x > 1$

$\boxed{C=1 \quad k=1}$

Big-Omega Notation

$f(x) = \omega(g(x))$ $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Omega(g(x))$ if there are constants C and k with C positive such that

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$.

[This is read as "$f(x)$ is big-Omega of $g(x)$."]

$x^2+2x+1 = \Omega(x^2)$

- If $\lim_{x\to\infty} \frac{g(x)}{f(x)}$ is bounded, then $f(x) = \Omega(g(x))$.

$\lim_{x\to\infty} \frac{x^2}{x^2+2x+1} = 1$

Big-Theta Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$. When $f(x)$ is $\Theta(g(x))$, we say that f is big-Theta of $g(x)$, that $f(x)$ is of *order* $g(x)$, and that $f(x)$ and $g(x)$ are of the *same order*.

$\exists\ c_1,\ c_2,\ k$ such that

$$c_1 \cdot g(x) \leq f(x) \leq c_2 \cdot g(x) \quad \text{for } x > k.$$

$$x^2 + 2x + 1 = \Theta(x^2)$$

Big-Theta Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$. When $f(x)$ is $\Theta(g(x))$, we say that f is big-Theta of $g(x)$, that $f(x)$ is of *order* $g(x)$, and that $f(x)$ and $g(x)$ are of the *same order*.

- If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is equal to a nonzero constant, then $f(x) = \Theta\big(g(x)\big)$.