CSCI 220 | Spring 2022

Discrete Structure

Discrete Probability

Discrete Mathematics and its Application

Section 7.1, 7.2

Probability Exercises:

- What is the probability of rolling an even number on a fair 6-sided dice?

$\{1, 2, 3, 4, 5, 6\}$

$$\frac{3}{6} = \frac{1}{2}$$

- What is the probability of getting all heads on flipping a fair coin three times?

H H H

H or T H or T H or T

$= 2^3$

$$\frac{1}{2^3} = \frac{1}{8}$$

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Finite Probability

- If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the probability of E is $p(E) = \frac{|E|}{|S|}$.

Probability Exercise:

- What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

$$P(d_1 + d_2 = 7) = \frac{6}{6 * 6} = \boxed{\frac{1}{6}}$$

$d_1 \quad d_2$

d_1	d_2
1	6
2	5
3	4
4	3
5	2
6	1

Probability Exercise:

- There is a lottery that award the prize to a person who correctly choose a set of six numbers out of the first 40 positive integers. What is the probability that a person wins the prize?

$$\frac{1}{C(40,6)}$$

Probability Exercise:

- What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers 1, 2,…, 50 if the ball selected is not returned to the bin before the next ball is selected?

$\frac{1}{C(50,5)}$ $\frac{1}{P(50,5)}$ $= \frac{1}{50} * \frac{1}{49} * \frac{1}{48} * \frac{1}{47} * \frac{1}{46}$

Order doesn't matter. $\frac{1}{C(50,5)} = \frac{5}{50} * \frac{4}{49} * \frac{3}{48} * \frac{2}{47} * \frac{1}{46}$

$\{11, 4, \cancel{17}, 39, 23\}$

Probability Exercise:

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$$\frac{1}{50} * \frac{1}{50} * \frac{1}{50} * \frac{1}{50} * \frac{1}{50} = \frac{1}{50^5}$$

Order doesn't matter:

$$\frac{5}{50} * \frac{4}{50} * \frac{3}{50} * \frac{2}{50} * \frac{1}{50} = \frac{5!}{50^5}$$

Probability Exercise:

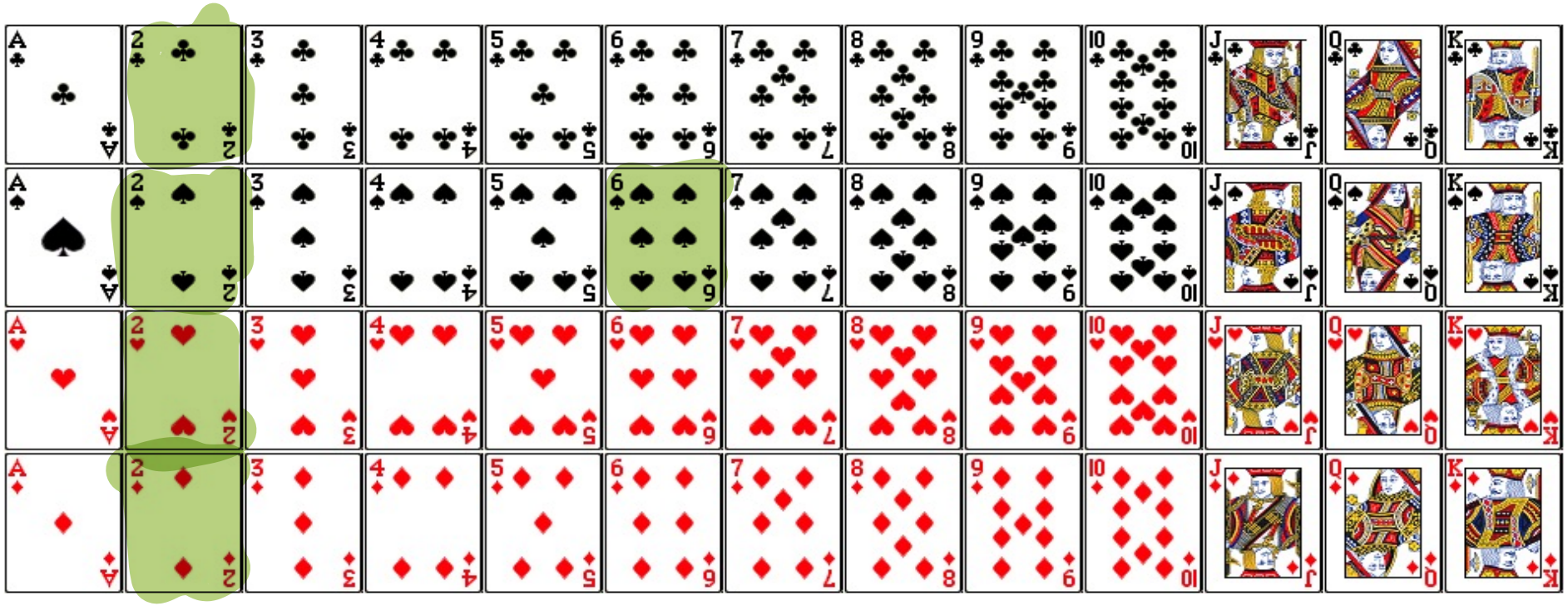
- Find the probability that a hand of five cards in poker contains four cards of one kind.

Poker Card

4 of a kind. 5th card

$$\frac{13 \times 48}{C(52, 5)}$$

- 52 cards:
  - 2 colors – black and red;
  - 4 suits – club, spade, heart, and diamond
  - 13 kinds – A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, and K



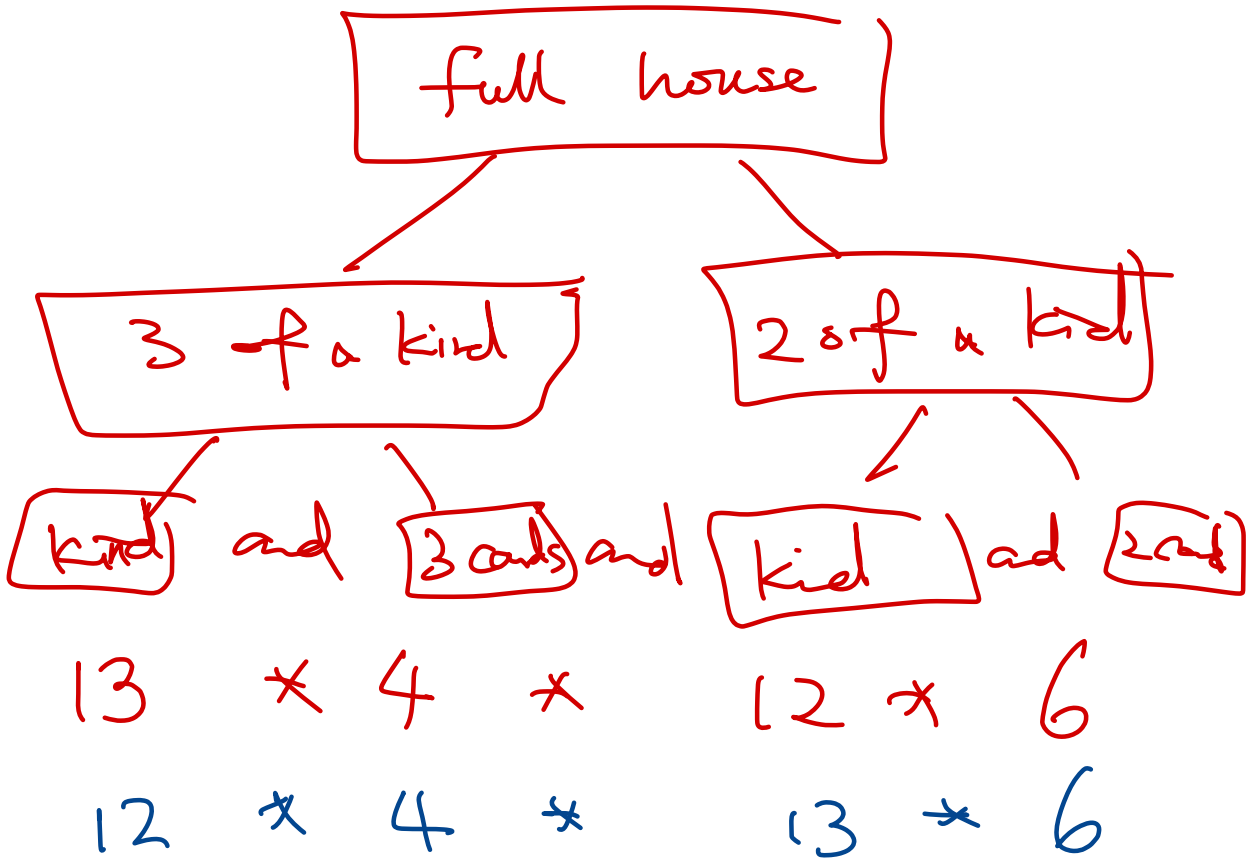
Probability Exercise:

- What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind?

5 cards.

kind C(4,3) kind C(4,2)

$$\frac{13 * 4 * 12 * 6}{C(52, 5)}$$



Poker Card

- 52 cards:
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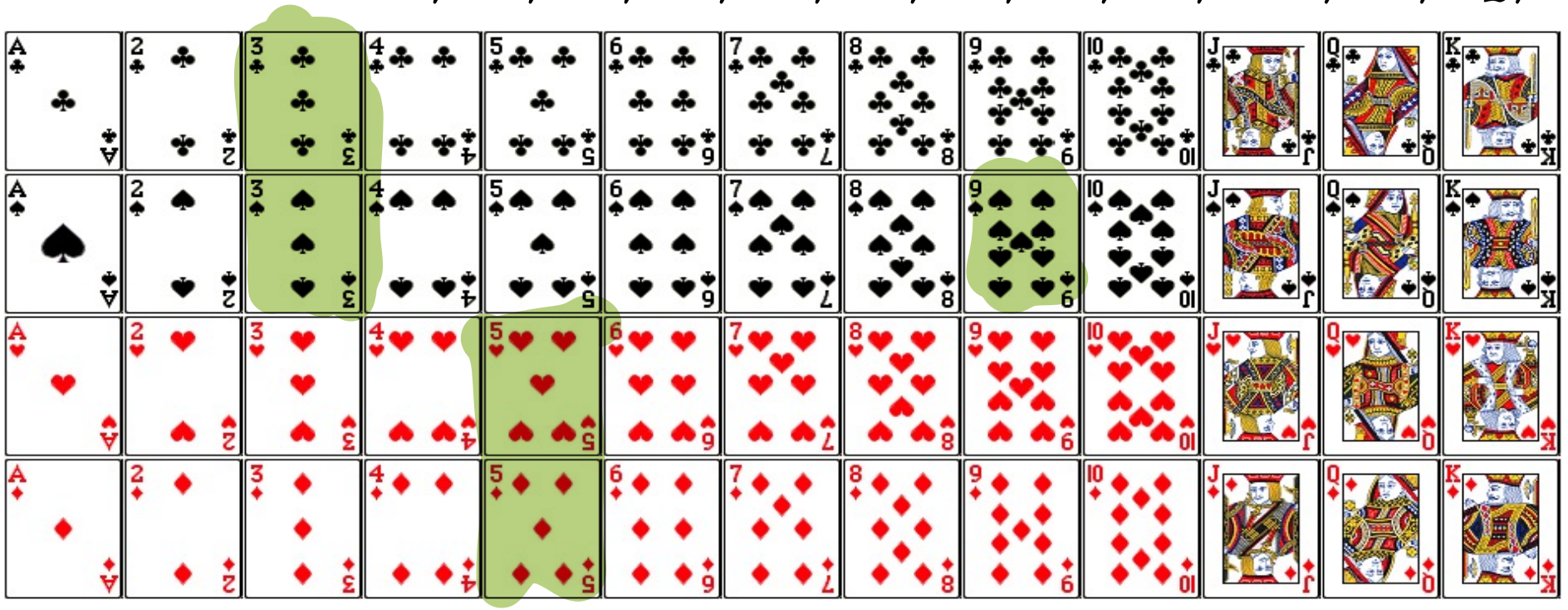

$C(13,2) = \frac{13 \cdot 12}{2}$

two pairs

2 kind | 2 | 2

$C(13,2)\,C(4,2)\,C(4,2)$

Two Pairs

2 of a kind | 2 of a kind | 1 of a kind

$\frac{13 * 6 * 12 * 6}{2} * 11 * 4$

$\frac{\frac{13 \cdot 12}{2} \cdot 6 \cdot 6 \cdot 11 \cdot 4}{C(52,5)}$

Probability Exercise:

- What is the probability of getting at least one heads on flipping a fair coin four times?

$$\frac{15}{16}$$

$$\frac{\overset{1H}{4} + \overset{2H}{6} + \overset{3H}{4} + \overset{4H}{1}}{16} = \frac{15}{16}$$

$$P(\geq 1\,H) = 1 - P(0\,H)$$

$$= 1 - \frac{1}{16}$$

$$= \frac{15}{16}$$

Probabilities of Complements

- Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by $p(\overline{E}) = 1 - p(E)$.

Probability Exercises:

- A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

$$P(\geq 1 \text{ 0's}) = 1 - P(\underset{(\text{All 1's})}{\text{No 0's}})$$

$$= \boxed{1 - \frac{1}{2^{10}}}$$

- A sequence of 10 bits is randomly generated. What is the probability that at least two of these bits is 0?

$$P(\geq 2 \text{ 0's}) = 1 - P(\text{No 0's}) - P(\text{one 0's})$$

$$= 1 - \frac{1}{2^{10}} - \frac{10}{2^{10}}$$

Probability Exercise:

- A sequence of 10 bits is randomly generated. What is the probability that it starts with 11 or ends with 000?
- $11xxxxxxxx \rightarrow 2^8$

  $xxxxxxx000 \rightarrow 2^7$

  over count $11xxxxx000 \rightarrow 2^5$

$$\frac{2^8 + 2^7 - 2^5}{2^{10}}$$

Probabilities of Unions of Events

- Let E_1 and E_2 be events in the sample space S.

  Then $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$.

Exercises of Assigning Probabilities

- What probabilities should be assigned to these outcomes when the coin is biased so that heads come up twice as often as tails?

$P(H) = 2P(T)$

$P(H) + P(T) = 1$

$2P(T) + P(T) = 1$

$3P(T) = 1$

$P(T) = \frac{1}{3}$ $\quad P(H) = \frac{2}{3}$

Exercises of Assigning Probabilities

- Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

$P(1) = P(2) = P(4) = P(5) = P(6) = \frac{1}{7}$

$P(3) = 2P(1) = \frac{2}{7}$

$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$

$P(1) + P(1) + 2P(1) + P(1) + P(1) + P(1) = 1$

$7P(1) = 1$

$P(1) = \frac{1}{7}$

$P(\text{odd})$

$= P(1) + P(3) + P(5)$

$= \frac{1}{7} + \frac{2}{7} + \frac{1}{7}$

$= \frac{4}{7}$

Exercises of Assigning Probabilities

- A coin is biased so that the probability of heads is 2/3. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

$P(H) = \frac{2}{3}$

$P(T) = \frac{1}{3}$

$$P(4\,H) = C(7,4) \cdot \left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{3}\right)^3$$

Bernoulli Trials

- The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is $C(n, k)p^k q^{n-k}$.

Exercises on Bernoulli Trials

- Suppose that the probability that a 0 bit is generated is 0.9, that the probability that a 1 bit is generated is 0.1, and that bits are generated independently. What is the probability that exactly eight 0 bits are generated when 10 bits are generated?