CSCI 220 | Spring 2022

Discrete Structure

Integer Representations and Divisibility Rules

Discrete Mathematics and its Application

Section 4.2

Representations of Integers

Let b be an integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form $n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$, where k is a nonnegative integer, $a_0, a_1, \ldots, a_k$ are nonnegative integers less than b, and $a_k \neq 0$.

Examples of Expended Forms

- Find the expended form of 12034 in base 10.

$$12034 = 1\cdot 10^4 + 2\cdot 10^3 + 0\cdot 10^2 + 3\cdot 10 + 4$$

$$(\underset{4}{1}\underset{3}{2}\underset{2}{0}\underset{1}{3}\underset{0}{4})_{10}$$

- Find the expended form of 46 in base 2.

$$2^6=64 \quad 2^5=32 \quad 2^4=16 \quad 2^3=8 \quad 2^2=4 \quad 2^1=2 \quad 2^0=1$$

$$46 = 1\cdot 2^5 + 0\cdot 2^4 + 1\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2 + 0$$

$$(\underset{5}{1}\underset{4}{0}\underset{3}{1}\underset{2}{1}\underset{1}{1}\underset{0}{0})_2$$

$$\begin{array}{r} 46 \\ -32 \\ \hline 14 \\ -8 \\ \hline 6 \\ -4 \\ \hline 2 \\ -2 \\ \hline 0 \end{array}$$

Representations of Integers

Let b be an integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form $n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$, where k is a nonnegative integer, $a_0, a_1, \ldots, a_k$ are nonnegative integers less than b, and $a_k \neq 0$.

- The representation of n above is called **the base $\boldsymbol{b}$ expansion of $\boldsymbol{n}$**. The base b expansion of n is denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$.

Examples of Integer Representations

- What is the decimal expansion of the number with binary expansion $(1\,0101\,1111)_2$?

$$\begin{matrix} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{matrix}$$

$$1\cdot 2^8 + 0\cdot 2^7 + 1\cdot 2^6 + 0\cdot 2^5 + 1\cdot 2^4 + 1\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2 + 1$$

$$256 + 0 + 64 + 0 + 16 + 8 + 4 + 2 + 1 = (351)_{10}$$

- What is the $\underbrace{\text{decimal}}$ expansion of the number with $\underbrace{\text{octal}}_{\text{base } 8}$ expansion $(7016)_{\underline{\underline{8}}}$? → base 10

$$\begin{matrix} 3 & 2 & 1 & 0 \end{matrix}$$

$$7\cdot 8^3 + 0\cdot 8^2 + 1\cdot 8 + 6$$

$$7\cdot 512 + 0 + 8 + 6 = (3598)_{10}$$

Examples of Integer Representations

- What is the decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

base 16

$$\underset{4}{2}\underset{3}{A}\underset{2}{E}\underset{1}{0}\underset{0}{B}$$

0
1
2
⋮
9
10 - A
11 - B
12 - C
13 - D
14 - E
15 - F

$$2\cdot 16^4 + 10\cdot 16^3 + 14\cdot 16^2 + 0\cdot 16 + 11$$

$$= 175627$$

Base Conversion

- Find the decimal expansion of $(12034)_{10}$.

$$12034 = 10(1203) + 4$$

$$1203 = 10(120) + 3$$

$$120 = 10(12) + 0$$

$$12 = 10(1) + 2$$

$$1 = 10(0) + 1$$

stop

$$(12034)_{10}$$

Base Conversion

- Find the ~~binary~~ base 2 expansion of $(46)_{10}$.

$$46 = 2(23) + 0$$
$$23 = 2(11) + 1$$
$$11 = 2(5) + 1$$
$$5 = 2(2) + 1$$
$$2 = 2(1) + 0$$
$$1 = 2(0) + 1$$

$$(101110)_2$$

Base Conversion

To construct the base b expansion of an integer n.

- First, divide n by b to obtain a quotient and remainder, that is, $n = bq_0 + a_0,\ 0 \leq a_0 < b$.
- The remainder, a_0, is the rightmost digit in the base b expansion of n. Next, divide q_0 by b to obtain
$$q_0 = bq_1 + a_1,\ 0 \leq a_1 < b.$$
- We see that a_1 is the second digit from the right in the base b expansion of n. Continue this process, successively dividing the quotients by b, obtaining additional base b digits as the remainders.
- This process terminates when we obtain a quotient equal to zero. It produces the base b digits of n from the right to the left.

Base Conversion

- Find the octal expansion of $(12345)_{10}$.

base 8

$$12345 = 8(1543) + 1$$
$$1543 = 8(192) + 7$$
$$192 = 8(24) + 0$$
$$24 = 8(3) + 0$$
$$3 = 8(0) + 3$$

$$(30071)_8$$

Base Conversion

- Find the $\underbrace{\text{hexadecimal}}_{\text{base } 16}$ expansion of $(177130)_{10}$.

$$177130 = 16(11070) + 10 \quad A$$
$$11070 = 16(691) + 14 \quad E$$
$$691 = 16(43) + 3 \quad 3$$
$$43 = 16(2) + 11 \quad B$$
$$2 = 16(0) + 2 \quad 2$$

stop

$$(2B3EA)_{16}$$

4 3 2 1 0

Divisibility Rule

$12345 \equiv 15 \equiv 6 \equiv 0 \pmod 3$

- Divisibility Rule for 3 : Check sum of the digits.

ex: 12345, $1+2+3+4+5=15$ $\quad 3 \mid 15 \rightarrow 3 \mid 12345$.

Prove that $3 \mid N$, $N = (a_n a_{n-1} \cdots a_1 a_0)_{10} \rightarrow 3 \mid a_n + a_{n-1} \cdots + a_1 + a_0$

$N \equiv 0 \pmod 3 \rightarrow a_n + a_{n-1} + \cdots + a_1 + a_0 \equiv 0 \pmod 3$

$N \equiv a_n + a_{n-1} + \cdots + a_1 + a_0 \pmod 3$

$N \equiv a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots \overset{a_2 \cdot 10^2}{} + a_1 \cdot 10 + a_0 \pmod 3$

$N \equiv a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 \pmod 3$

$$\boxed{\begin{aligned} &10^k \equiv 1 \pmod 3 \\ &10^k \equiv 1^k \equiv 1 \pmod 3 \end{aligned}}$$

Divisibility Rule

- Divisibility Rule for 9 : Check the sum of the digits.

Divisibility Rule

- Divisibility Rule for 11 : Check alternated sum of digits.

Prove that $11 \mid N$, $N = (a_n a_{n-1} \cdots a_1 a_0)_{10}$ $\rightarrow$ $11 \mid a_n - a_{n-1} \cdots - a_1 + a_0$

$\Updownarrow$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $\Updownarrow$

$N \equiv 0 \pmod{}$ $\rightarrow$ $a_n - a_{n-1} \cdots - a_1 + a_0 \equiv 0 \pmod{}$

$$N \equiv a_n \; a_{n-1} \cdots - a_1 + a_0 \pmod{11}$$

$$N \equiv a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \pmod{11}$$

$$N \equiv a_n - a_{n-1} \cdots + a_2 - a_1 + a_0 \pmod{11}$$

$10^k \pmod{11}$

$10^k \equiv (-1)^k \pmod{11}$

$\equiv \underset{k \text{ even}}{1}$ or $\underset{k \text{ odd}}{-1}$

Divisibility Rule

- Divisibility Rule for 11 :

$11 \mid 12345$

$12345 \equiv 1-2+3-4+5 \equiv 3 \pmod{11}$

$1-2+3-4+5 = 3$ $\quad 11 \nmid 3 \to 11 \nmid 12345$

$-1+2-3+4-5 = -3$ $\quad 11 \nmid -3 \to 11 \mid 12345$

123453 $\quad -1+2-3+4-5+3 = 0$ $\quad 11 \mid 0 \to 11 \mid 123453$

Divisibility Rule

- Divisibility Rule for 2 : Check last digit.

  5

  10

$$N = 10 \cdot q + r \, , \quad 0 \le r < 10 .$$

last digit.

$$N \equiv 10q + r \pmod 2$$

$$\equiv 0 \cdot q + r$$

$$N \equiv r \pmod 2$$

Divisibility Rule

- Divisibility Rule for 4: Check last two digits.

  8: Check last three digits

$$N = 100 \cdot q + r, \quad 0 \le r < 100$$

last digit.

$$N \equiv 100q + r \pmod{4}$$

$$0q + r$$

$$N \equiv r \pmod{4}$$

Divisibility Rules

- Divisibility Rule for 2: ✓
- Divisibility Rule for 3: ✓
- Divisibility Rule for 4: ✓
- Divisibility Rule for 5: ✓
- Divisibility Rule for 6: Combine the rule for 2 and 3
- Divisibility Rule for 7:
- Divisibility Rule for 8: ✓
- Divisibility Rule for 9: ✓
- Divisibility Rule for 10: ✓
- Divisibility Rule for 11: ✓
- Divisibility Rule for 12: Combine the rule for 3 and 4.