CSCI 220 | Spring 2022

Discrete Structure

Conditional Probability

Discrete Mathematics and its Application

Section 7.2, 7.3

Probability Exercise

$2^3 = 8$

- Suppose that we flip a coin three times, and all eight possibilities are equally likely. Moreover, suppose we know that the event F, that the first flip comes up tails, occurs. Given this information, what is the probability of the event E, that an odd number of tails appears?

F : T X X

E : odd # of T

1T or 3T

THH
THT
TTH
TTT

$$P(E \mid T) = \frac{2}{4} = \frac{1}{2}$$

Conditional Probability

- Let E and F be events with $p(F) > 0$. The conditional probability of E given F, denoted by $p(E \mid F)$, is defined as $p(E \mid F) = \frac{p(E \cap F)}{p(F)}$. ← $\frac{|E \cap F|}{|F|}$

Given

$$P(E \overset{\text{and}}{\cap} F) = P(E) * P(F \mid E)$$
$$= P(F) * P(E \mid F)$$

Exercises on Conditional Probability

- A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

$2^4 = 16$

$$P(\overbrace{\text{at least two consecutive 0s.}}^{E} \mid \overbrace{0XXX}^{F}) = \frac{5}{8}$$

$0XXX$

$00XX$ $01XX$

$2^2 = 4$ $+$ 0100

1

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{5/16}{8/16} = \frac{5}{8}$$

Exercises on Conditional Probability

- What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl. (Note that BG represents a family with an older boy and a younger girl while GB represents a family with an older girl and a younger boy.)

$$P(2B \mid \geq 1B) = \frac{1}{3}$$

$$P(2B) = \frac{1}{4}$$

$$P(2B \mid \geq 1B) \neq P(2B)$$

Exercises on Conditional Probability

- Suppose a coin is flipped three times, as described in the introduction to our discussion of Links conditional probability. Does knowing that the first flip comes up tails (event F) alter the probability that tails come up an odd number of times (event E)? In other words, is it the case that $p(E|F) = p(E)$?

$$p(E|F) = 2/4 = \frac{1}{2}$$

$$p(E) = \frac{1\text{H or }3\text{H}}{8} = \frac{3+1}{8} = \frac{4}{8} = \frac{1}{2}$$

$$p(E|F) = p(E)$$

Independence

- The events E and F are independent if and only if $p(E \cap F) = p(E)p(F)$.

$$P(E \cap F) = P(E|F) * P(F)$$
$$= P(F|E) * P(E)$$

$$\frac{P(E \cap F)}{P(E)} = P(F|E) = P(F)$$

$$\frac{P(E \cap F)}{P(F)} = P(E|F) = P(E)$$

Probability Exercises on Independence

- Suppose E is the event that a randomly generated bit string of length four begins with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

E: 1XXX $\leftarrow P(E) = \frac{1}{2}$

F: even # of 1. $\leftarrow P(F) = \frac{\overset{1}{C(4,0)} + \overset{6}{C(4,2)} + \overset{1}{C(4,4)}}{16} = \frac{8}{16} = \frac{1}{2}$

$E \cap F$: 1XXX and even # of 1 $\leftarrow P(E \cap F) = \frac{C(3,1) + C(3,3)}{16} = \frac{3+1}{16} = \frac{4}{16} = \frac{1}{4}$

$P(E \cap F) = P(E) \cdot P(F)$

$\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$ ✓

E and F are independent.

Probability Exercises on Independence

- Assume, as in pervious example, that each of the four ways a family can have two children is equally likely. Are the events E, that a family with two children has two boys, and F, that a family with two children has at least one boy, independent?

BB, BG, GB, GG.

$P(2B) = \frac{1}{4}$

$P(\geq 1B) = \frac{3}{4}$

$\frac{1}{4} * \frac{3}{4} = \frac{3}{16} \neq \frac{1}{4}$

$P(2B \cap \geq 1B) = \frac{1}{4}$

E and F are not independent.

$P(2B \mid \geq 1B) = \frac{1}{3} \neq P(2B)$

Probability Exercises on Independence

- Are the events E, that a family with three children has children of both sexes, and F, that this family has at most one boy, independent? Assume that the eight ways a family can have three children are equally likely.

E : both B and G

F : ≤ 1 B.
0 or 1 B.

$P(E) = \frac{6}{8} = \frac{3}{4}$

1B2G or 2B1G
3 + 3 = 6

$8 - \overset{\text{ALLB}}{1} - \overset{\text{ALLG}}{1} = 6$.

$P(F) = \frac{1+3}{8} = \frac{4}{8} = \frac{1}{2}$

$P(E \cap F) = \frac{3}{8} \overset{?}{=} P(E) \cdot P(F) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$

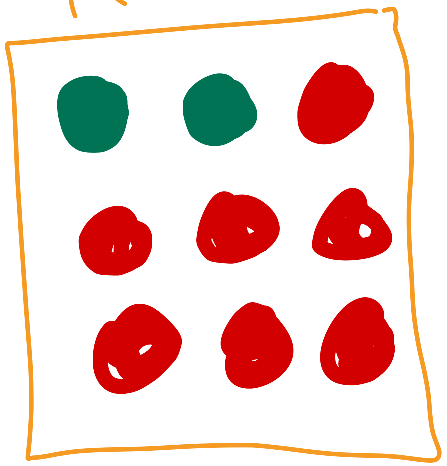
2G 1B

E and F are independent.

Probability Exercise

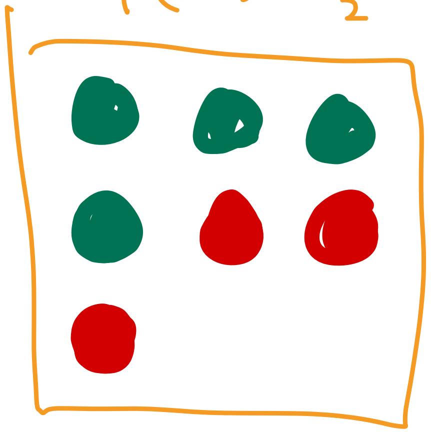
- We have two boxes. The first contains two green balls and seven red balls; the second contains four green balls and three red balls. Bob selects a ball by first choosing one of the two boxes at random. He then selects one of the balls in this box at random. If Bob has selected a red ball, what is the probability that he selected a ball from the first box?

$P(B1) = \frac{1}{2}$



B 1

$P(B2) = \frac{1}{2}$



B2

$$P(B1 \mid R) = \frac{P(B1 \cap R)}{P(R)}$$

$$= \boxed{\frac{\frac{7}{9} \cdot \frac{1}{2}}{\frac{7}{9} \cdot \frac{1}{2} + \frac{3}{7} \cdot \frac{1}{2}}}$$

$P(R \mid B1)\, P(B1)$ $\quad$ $P(R \mid B2) \cdot P(B2)$

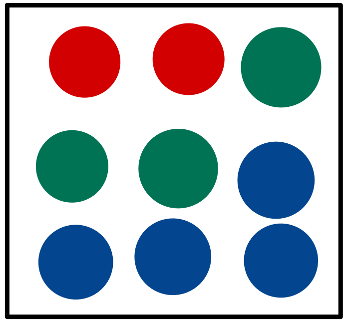
$P(R \cap B1)$ $\quad$ $P(R \cap B2)$

Bayes' Theorem

- Suppose that E and F are events from a sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$.

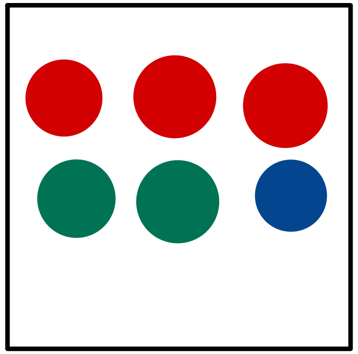
  Then $p(F \mid E) = \frac{p(E|F)p(F)}{p(E|F)p(F)+p(E|\overline{F})p(\overline{F})}$.

B1:



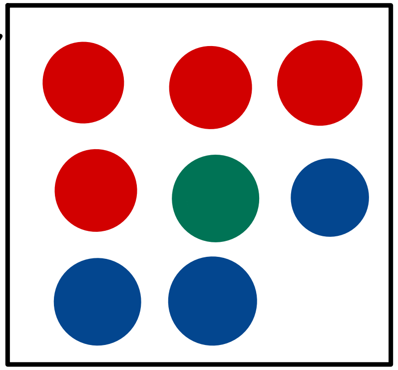
$P(B1) = \frac{1}{6}$

B2:



$P(B2) = \frac{1}{2}$

B3:



$P(B3) = \frac{1}{3}$

$$P(B1 \mid R) = \frac{P(B1 \cap R)}{P(R)}$$

$$= \frac{\frac{2}{9} * \frac{1}{6}}{\frac{2}{9} \cdot \frac{1}{6} + \frac{3}{6} * \frac{1}{2} + \frac{4}{8} * \frac{1}{3}}$$

Probability Exercises on Bayes’ Theorem

- Suppose that one person in 100,000 has a particular rare disease for which there is a fairly accurate diagnostic test. This test is correct 99.0% of the time when given to a person selected at random who has the disease; it is correct 99.5% of the time when given to a person selected at random who does not have the disease. Given this information, can we find
  a) the probability that a person who tests positive for the disease has the disease?
  b) the probability that a person who tests negative for the disease does not have the disease?
- Should a person who tests positive be very concerned that he or she has the disease?

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