Chapter 2

Basic Operations

Overview

SQI_{SIGN} involves the connection between two mathematical concepts that seem unrelated:

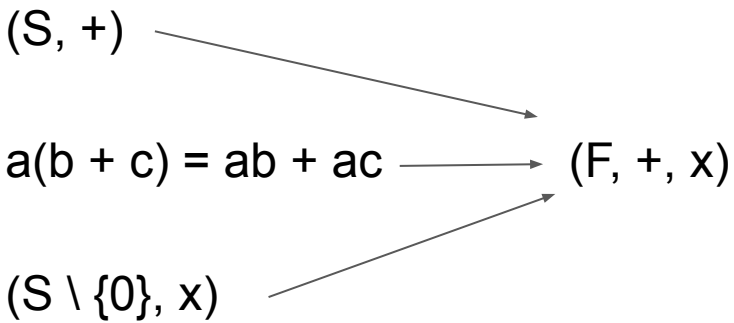
1) Isogenies between supersingular elliptic curves over finite fields
2) Maximal orders and ideals of quaternion algebras

Section 2.1 - Finite Fields

A field is a relatively well-equipped algebraic structure. As a review, we can build up to it from more fundamental structures

Abelian group

Field



Example

- An example is $\mathbb{Q}$, the rational numbers (set of fractions with integer numerator and denominator (den $\neq 0$))
- Closure: Sum or multiplication of rational numbers is rational
- Associativity: A sequence of addition operations can be done in any order (similarly for multiplication)
- Identity: Additive identity: 0; Multiplicative identity: 1
- Inverse: Can simply negate a number to get its additive inverse; Can flip a fraction to get its multiplicative inverse (remember 0 is not included here)
- Commutativity: Can add or multiply two rationals in any order

Finite fields

We care about fields with a finite number of elements. All such fields are of prime power order (powers of a single prime). Specifically we care about F_p (operations are mod p) and F_{p^2} which both have characteristic p.

Ex. $F_p = (\mathbb{Z}_p, +, x)$ has characteristic p because $\underbrace{1 + 1 + 1 + \ldots + 1}_{\text{p times}} = 0 \pmod p$

We will consider $p = 3 \pmod 4$

Quadratic residue in F_p

- Useful for later definitions throughout the paper
- A number congruent to a perfect square: $b^2 = a \pmod p$
- Can test if ‘a’ is a perfect square in F_p by raising both sides by (p - 1) / 2
- $b^{p-1} = a^{(p-1)/2} \pmod p = 1 \pmod p$ (by Fermat’s little theorem)
- If this equation is not true, then ‘a’ wasn’t a perfect square to begin with

F_q for $q = p^2$

If F_p was analogous to the real numbers, then $F_{p\hat{}2}$ is analogous to the complex numbers. With $i^2 + 1 = 0$, elements of $F_{p\hat{}2}$ are of the form $a_0 + a_1 i$ for a_0, a_1 in F_p

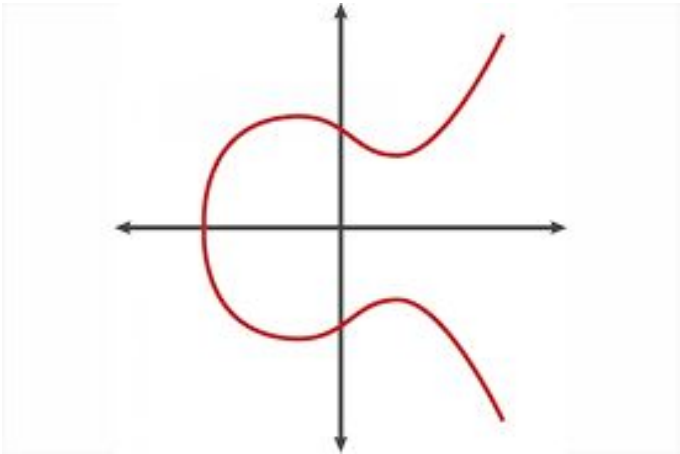
It's additive and multiplicative operations are the familiar ones when working with complex numbers. For instance, the multiplicative inverse of a + bi is:

$$\frac{1}{a + bi} \longrightarrow \frac{1}{a + bi}\frac{(a-bi)}{(a-bi)} \longrightarrow \frac{a-bi}{a^2 + b^2}$$

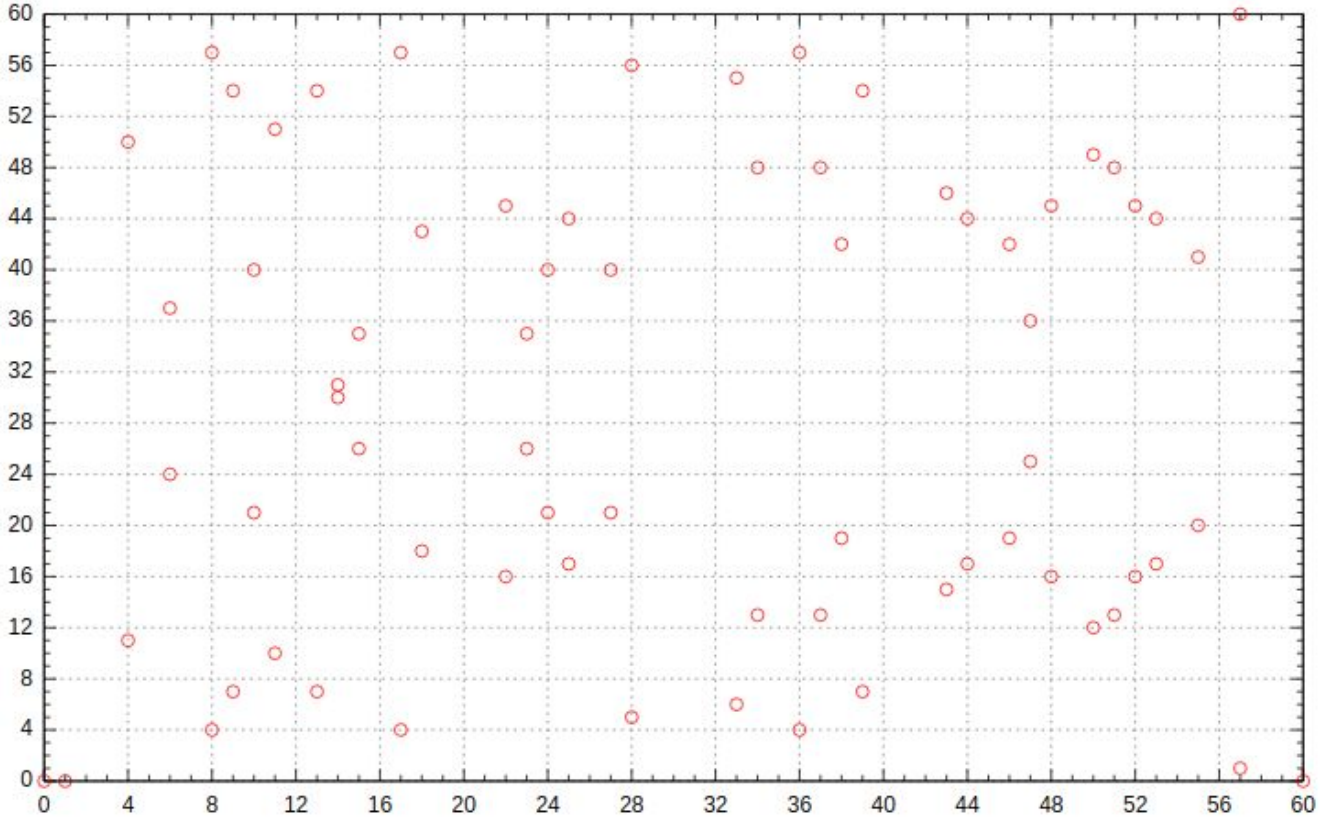
Section 2.2 - Elliptic Curves

- Montgomery elliptic curves are curves of the form: $By^2 = x^3 + Ax^2 + x$
- We also importantly consider a “point at infinity”, ∞, to be part of the curve
- Two curves are isomorphic if there is a bijective (one-to-one correspondence) mapping between them of the form $(x, y) \rightleftharpoons (D(x + R), Cy)$
- They are “quadratic twists” of one another if $C = \sqrt{(B/B')}$ and are isomorphic if B/B' is a perfect square

Elliptic curve over the real numbers



An elliptic curve over the finite field, F_{61}



Supersingular meaning

If the number of solutions to the curve is congruent to 1 mod char(F_q), it is called supersingular.

Recall we care about p = 3 (mod 4) and F_{p^2}

In this case, if B = 1, the curve has exactly $(p + 1)^2$ points

Ex. B = 1, p = 7, $(7 + 1)^2 = 64 \bmod 7 = 1 \pmod 7$ ✓

Group structure of elliptic curves

Collectively supplementing the points on the elliptic curve with a binary addition operation yields an abelian group!

Define the additive identity to be the point at infinity, ∞. So, for a point P on the elliptic curve, $P + \infty = \infty + P = P$.

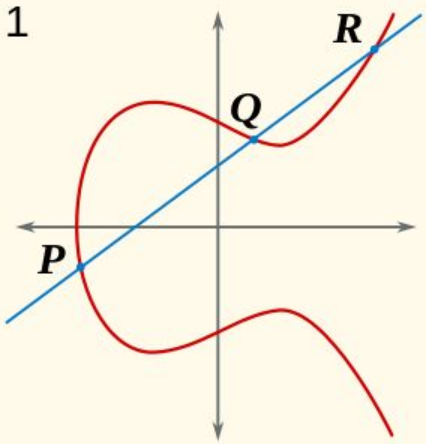
To add two points on the curve, draw a line of intersection through the points. The third point of intersection is defined as the negation of the sum of the points. This is because we define the sum of three points of intersection on a line as ∞.

To get the correct sum, reflect this point over the x-axis.

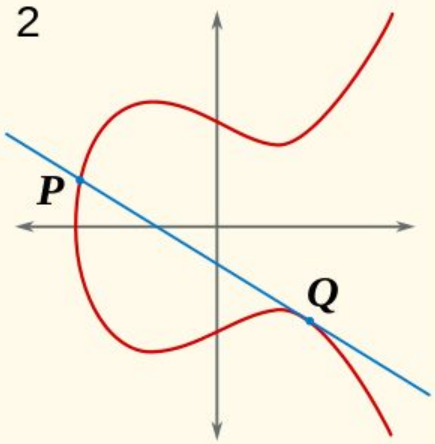
Note the third point may be ∞ if the line is vertical

Elliptic Curves

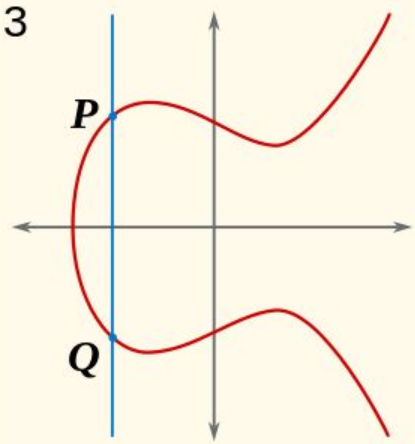
Point addition



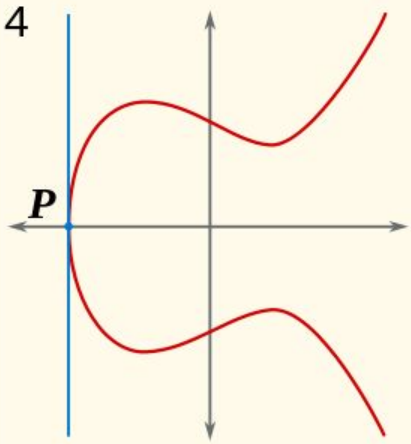
$P + Q + R = 0$



$P + Q + Q = 0$



$P + Q + 0 = 0$



$P + P + 0 = 0$

Abelian Group properties hold

Closure: The sum of two points on the curve is also a point on the curve

Associativity: Harder to show but is true

Identity: ∞

Inverse: Point reflected over x-axis is a point’s inverse

Commutativity: The order in which you add points doesn’t matter

Discrete logarithm problem

Scalar multiplication can be defined using repeated point doubling and addition

It is denoted: [k]P

Since we are dealing with repeated applications of an abelian group's operation, we can naturally define a discrete log problem:

Given a point P and a scalar multiple [k]P, find k

Now knowing how "strangely" addition works and how unrelated the sum of points seem to the original points, you can imagine how difficult solving the discrete log problem will be for large k

Solvability

For certain points, this is difficult for classical computers (they basically need to do an exhaustive search)

But for other choices of points it is efficiently solvable

However, this is the background required to understand the paper’s work with elliptic curves