Total : 109 pts        1 cheat sheet is allowed.

8 Questions + Bonus

1. The logic connection that we have never learned. ( use it + equivalences to represent for some logical statements)

$\boxed{\oplus}$

Ex. $P \boxed{\leftrightarrow} Q = \neg(P \oplus Q)$

2. Nested Quantifiers and Determine T or F. Explain.

3. Sum of Series ( Geometric Series)

4. 1) Use Iterative approach to find Explicit formula

2). Mathematical Induction to prove the Explicit formula

5. Modular Congruent   # 17 page 258

$a \equiv 4 \pmod{13}$   $b \equiv 9 \pmod{13}$ ,   $c \equiv a^2+b^2 \pmod{13}$

6. Euclidean Algorithm & Linear Combination. to find the $\gcd = as + bt$.

$\gcd(s,t)$

** 7. proof : choose 4 out of 6.

Direct pf , Indirect pf, Induction < weak Induction / Strong Induction.

8. T/F.

Ex. $n! \leq n^n$ for any integer $n \geq 1$

Pf. Basic step: $n=1$ , $1! \leq 1^1$ ✓

Inductive step: suppose that $P(k) = $ True for $k \geq 1$.

$k! \leq k^k$

We need to prove that $p(k+1)$ is true!

$k! \leq k^k$ by inductive Hypothesis

$(k+1) \cdot k! \leq (k+1) \cdot k^k$   by Multiply by $(k+1)$

$(k+1)! \leq (k+1) \cdot k^k < (k+1) \cdot (k+1)^k$   b/c $k < k+1$

goal: $p(k+1)$

$\boxed{(k+1)! \leq (k+1)^{k+1}}$ ?

$= (k+1)(k)(k-1) \cdots 2 \cdot 1$   $(k+1)^k \cdot (k+1)^1$

$= \boxed{(k+1) \cdot k!}$

$\boxed{\boxed{k!} \leq k^k}$

$(k+1) \cdot k! \leq (k+1) \cdot k^k$

$(k+1)! \leq (k+1) \cdot k^k < (k+1)(k+1)^k$

b/c $k < k+1$

$(k+1)! \le (k+1)^{k+1}$

$(k+1)! \boxed{?} (k+1)^{k+1}$

$(k)!(k+1) \boxed{?} (k+1)^k \cdot (k+1)$

$(k)! \boxed{?} (k+1)^k$

$(k)! \boxed{\le} k^k < (k+1)^k$

known

$\le$

unknown

verify

Extra credit, part 2.

h). $6 \mid 7^{2n} - 19^n \quad \forall n \in \mathbb{Z}^+$

pf by induction: Basic step : $n=1$ $\quad 7^{2(1)} - 19^1 = 7^2 - 19 = 49 - 19 = 30$

$6 \mid 30$ ✓

Inductive step : Assume $P(k) = \text{True}$, $6 \mid 7^{2k} - 19^k$, $k \in \mathbb{Z}^+$

Try to prove that $P(k+1)$ True.

$7^{2(k+1)} - 19^{k+1}$

$= 7^{2k+2} - 19^{k+1}$

$= 7^{2k} \cdot 7^2 - 19^k \cdot 19$

$= 7^{2k} \cdot 49 - 19^k \cdot 19$

$= 7^{2k} \cdot (19+30) - 19^k \cdot 19$

$= 7^{2k} \cdot 19 + 7^{2k} \cdot 30 - 19^k \cdot 19$

$= 19(7^{2k} - 19^k) + 7^{2k} \cdot 30$

Goal: $6 \mid 7^{2(k+1)} - 19^{k+1}$

$7^{2(k+1)} - 19^{k+1}$

$= 7^{2k+2} - 19^{k+1} = \boxed{6 \cdot M}$

IH: $\boxed{7^{2k} - 19^k} = 6 \cdot P$

$7^{2k} \cdot 7^2 - 19^k \cdot 19$

$= 7^{2k} \cdot 49 - 19^k \cdot 19$

$19 + 30$

$= 7^{2k} \cdot 19 + 7^{2k} \cdot 30 - 19^k \cdot 19$

$= \underbrace{(7^{2k} - 19^k)}_{\text{IH}} \cdot 19 + \underbrace{7^{2k} \cdot 30}_{\text{Divisible by 6}}$

Divisible by 6

by inductive hypothesis $6 \mid 7^{2k} - 19^k \Rightarrow 7^{2k} - 19^k = 6 \cdot P$, $P \in \mathbb{Z}$

$7^{2k} \cdot 30$ since $6 \mid 30$, then by theorem 1 of divisibility

$6 \mid 7^{2k} \cdot 30$.

By theorem 1 $\quad 6 \mid 19(7^{2k} - 19^k) + 7^{2k} \cdot 30 = 6 \mid 7^{2(k+1)} - 19^{k+1}$

d). $f_n \ge \left(\frac{3}{2}\right)^{n-2}$ for all $n \in \mathbb{Z}^+$

left $\qquad$ right

Pf Basic step: $n=1$ $f_1 = 1$ $(\frac{3}{2})^{1-2} = (\frac{3}{2})^{-1} = (\frac{2}{3})$

$1 \geq (\frac{2}{3})$ ✓

Inductive step. Assume $p(1) \wedge p(2) \wedge p(3) \wedge \cdots p(k)$ $k \in \mathbb{Z}^+$
$= \text{true}$

$p(k)$: $f_k \geq (\frac{3}{2})^{k-2}$ $(f_k = f_{k-1} + f_{k-2})$

$p(k-1)$: $f_{k-1} \geq (\frac{3}{2})^{k-1-2} = (\frac{3}{2})^{k-3}$ $(f_{k-1} = f_{k-2} + f_{k-3})$

Goal: $f_{k+1} \geq (\frac{3}{2})^{k+1-2} = (\frac{3}{2})^{k-1}$

$f_{k-1} + f_k \geq (\frac{3}{2})^{k-3} + (\frac{3}{2})^{k-2}$

$\geq (\frac{3}{2})^{k-3}$ $\geq (\frac{3}{2})^{k-2}$

$(\frac{3}{2})^{k-1}\underbrace{\left[(\frac{3}{2})^{-2} + (\frac{3}{2})^{-1}\right]}_{\geq 1}$

$\geq (\frac{3}{2})^{k-1}$

$a > b > c$
$\Rightarrow a > c$

Try to prove $p(k+1)$ is true.

$f_{k+1} = f_{k-1} + f_k$ by the definition of Fibonacci #.

$f_{k+1} = f_{k-1} + f_k \geq (\frac{3}{2})^{k-3} + (\frac{3}{2})^{k-2}$ by the IH.

$= (\frac{3}{2})^{k-1} \cdot \left[(\frac{3}{2})^{-2} + (\frac{3}{2})^{-1}\right]$

$= (\frac{3}{2})^{k-1} \cdot \left[(\frac{2}{3})^2 + (\frac{2}{3})^1\right]$

$= (\frac{3}{2})^{k-1} \left[\frac{4}{9} + \frac{2 \cdot 3}{3 \cdot 3}\right]$

$= (\frac{3}{2})^{k-1} \cdot \left[\frac{4}{9} + \frac{6}{9}\right] = (\frac{3}{2})^{k-1} \cdot \boxed{\left[\frac{10}{9}\right]}^{\geq 1} \geq (\frac{3}{2})^{k-1} \cdot \boxed{1}$

$f_{k+1} \geq (\frac{3}{2})^{k-1}$.

c). $f_n = \frac{1}{\sqrt{5}}(p^n - q^n)$ $p = \frac{1+\sqrt{5}}{2}$ $q = \frac{1-\sqrt{5}}{2}$

$\Rightarrow$ $p^2 = p+1$ $q^2 = q+1$

left: $p^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{(1+\sqrt{5})^2}{2^2} = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{2(3+\sqrt{5})}{4} = \frac{3+\sqrt{5}}{2}$

right $p+1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{1+\sqrt{5}}{2} + \frac{2}{2} = \frac{3+\sqrt{5}}{2}$

left = right ✓ $p^2 = p+1$. (same for $q^2 = q+1$)

Pf. Basic step $n=0$ left $f_0 = 0$
right: $\frac{1}{\sqrt{5}} \cdot (p^0 - q^0) = \frac{1}{\sqrt{5}}(1-1) = 0$ ✓

Inductive step: Assume $p(0) \wedge p(1) \wedge p(2) \wedge \cdots p(k) = \text{True}$

$p(k)$ : $f_k = \frac{1}{\sqrt{5}} \cdot (p^k - q^k)$

$P(k-1): \quad f_{k-1} = \frac{1}{\sqrt{5}}(p^{k-1} - q^{k-1})$

Try to find $P(k+1)$

left

$$f_{k+1} = f_k + f_{k-1} \quad \text{By the def of Fibonacci \#.}$$

$$= \frac{1}{\sqrt{5}}(p^k - q^k) + \frac{1}{\sqrt{5}}(p^{k-1} - q^{k-1}) \quad \text{by IH}$$

$$= \frac{1}{\sqrt{5}}(p^k - q^k + p^{k-1} - q^{k-1})$$

$$= \frac{1}{\sqrt{5}}[(p^k + p^{k-1}) - (q^k + q^{k-1})]$$

$$= \frac{1}{\sqrt{5}}[p^{k-1}\cdot(p+1) - q^{k-1}(q+1)]$$

$p^2 = p+1 \qquad q^2 = q+1$

$$= \frac{1}{\sqrt{5}}[p^{k-1}\cdot p^2 - q^{k-1}\cdot q^2] = \frac{1}{\sqrt{5}}\cdot[p^{k+1} - q^{k+1}] \;. \quad \blacksquare$$

$P(k+1):$

$$f_{k+1} = \frac{1}{\sqrt{5}}(p^{k+1} - q^{k+1})$$

$f_k + f_{k-1}$

IH IH