§2.4. Sequences and Summations

1. sequences

Def. A sequence is a function from a subset of the set of integers to a set S.

$$a_n \qquad n \text{ (Domain) } \geq 0$$

Ex $\quad a_n = \frac{1}{n}, \; n \geq 1 \quad (n \neq 0)$

$a_1 = \frac{1}{1} = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \; \ldots$

$a_n = 5n - 3, \quad a_0 = 5(0) - 3 = -3$

$\qquad\qquad\quad a_1 = 5(1) - 3 = 2$

$\qquad\qquad\quad a_2 = 5(2) - 3 = 7$

$\qquad\qquad\quad \vdots$

$a_n \Rightarrow n^{th}$ term of the sequence.

2. Arithmetic sequence

Ex. $\underbrace{3, 5}_{+2}, \underbrace{5, 7}_{+2}, \underbrace{7, 9}_{+2}, 11, \ldots$

2 : Common difference $= d$

Arithmetic Progression : $\underset{a_0}{a}, \underset{a_1}{a+d}, \underset{a_2}{a+2d}, \underset{a_3}{a+3d}, \ldots$

a is initial term, d = common difference.

Arithmetic Formula: (Explicit formula) $\quad a_n = a + n \cdot d \qquad n \geq 0$

or $\quad a_n = a + (n-1)d \qquad n \geq 1$

Ex, $3, 5, 7, 9, 11, \ldots$

$a = 3, \; d = 2$

$\boxed{n \geq 0 \quad a_n = 3 + 2n}$

$n \geq 1 \quad a_n = 3 + (n-1) \cdot 2 = 3 + 2n - 2 = 1 + 2n$

both are fine but we'll follow $n \geq 0$ because CS starts with 0^{th} term.

3. Geometric Sequence.

$a, a \cdot r, ar^2, ar^3, \ldots$

a = initial term, r = Common ratio.

Explicit formula for geometric seq. $\quad a_n = a \cdot r^n \qquad n \geq 0$

Ex. $a_n = 2 \cdot (5)^n$, Find First Five terms.

$a_0 = 2 \cdot (5)^0 = 2$

$a_1 = 2 \cdot (5)^1 = 10$

$a_2 = 2 \cdot (5)^2 = 50$

$a_3 = 2 \cdot (5)^3 = 250$

$a_4 = 2 \cdot (5)^4 = 1250$

$2, \underbrace{\ }_{\times 5} 10, \underbrace{\ }_{\times 5} 50, \underbrace{\ }_{\times 5} 250, \underbrace{\ }_{\times 5} 1250, \ldots$

Ex. Simple Interest: deposit \$1000, 2% / Year Interest, Put in for 5 Years.

$a = 1000$

$r = 1 + 2\% = 1.02$

Money deposit ↑ ↑ Interest

$a_n = a \cdot r^n \Rightarrow a_5 = 1000(1.02)^5 = \$1104.08$

1000 | 1000 + 1000 × 0.02 = $\boxed{1020}$ | 1020 + 1020 × 0.02 = $\boxed{1040.4}$

4. Recurrence Relation.

It is an equation that expresses a_n in terms of one or more of the previous terms of the sequence.

Ex. $a_n = a_{n-1} + 3$ $\quad \boxed{a_0 = 2}$ ⇒ Basic case is extremly important for Recursion!

$a_5 = a_4 + 3$ $\qquad a_5 = 14 + 3 = \boxed{17}$

$a_4 = a_3 + 3$ $\qquad a_4 = 11 + 3 = 14$

$a_3 = a_2 + 3$ $\qquad a_3 = 8 + 3 = 11$

$a_2 = a_1 + 3$ $\qquad a_2 = 5 + 3 = 8$

$a_1 = a_0 + 3 = 2 + 3 = 5$

Recurrence Relation: easier to find pattern but difficult to solve for Human brains!

Computer: Not going to find patterns, but calculate everything fast!

Recursion combines the Human brains and Computer.
(find patterns) (calculate)

Ex. 1, 2, 4, 7, 11, 16, ...
+1 +2 +3 +4 +5

(Not Arithmetic Not Geometric)

$a_0 = 1$
$a_1 = 1 + 1 = a_0 + 1 = 2$
$a_2 = 2 + 2 = a_1 + 2 = 4$
$a_3 = 4 + 3 = a_2 + 3 = 7$
$\vdots$

$$\boxed{a_n = a_{n-1} + n, \quad a_0 = 1}$$

Code it, and Computer will solve it for you.
a_{100} (for us take forever to calculate
computer : not even 1 sec)

Ex. $\boxed{a_n = a_{n-1} + 3 \quad a_0 = 2}$ Recurrence Formula.

Recurren Relation: a_{100} (Long time to solve)

Arithemetic seq : $2, 5, 8, 11, \cdots$
$\xrightarrow{+3}$

$a_n = a_0 + n \cdot d = 2 + 3n$ (Explicit Formula)

$a_{100} = 2 + 3(100) = 302$

Recurrence Formula vs. Explicit formula.
(pattern easy to find but difficult to solve any term we want in the seq. | difficult to find formula but easy to solve any term we want in the seq.)

page 169. Iteration : Recurren Fomula $\xrightarrow[\text{Iteration}]{}$ Explicit formula.

① Forward substitution (Start with a_0 or a_1)

② Backward substitution. (Start with a_n)

Ex. $a_n = 2a_{n-1} - 3$, $a_0 = -1$ $f(n)$

use iterative approach to find Explicit formula (closed formula).

Forward substitution: $a_0 = -1$

More important

$a_1 = 2a_0 - 3 = \boxed{2(-1) - 3} = -5$ -3

$a_2 = 2\boxed{a_1} - 3 = 2[2(-1) - 3] - 3$

$= 2^2(-1) - 2\cdot 3 - 3$ $-2^1\cdot 3 - 3$

$= -13$

$a_3 = 2a_2 - 3 = 2[2^2(-1) - 2\cdot 3 - 3] - 3$

$= 2^3(-1) - 2^2\cdot 3 - 2\cdot 3 - 3$ $-2^2\cdot 3 - 2\cdot 3 - 3$

$\vdots$

$a_n = 2^n(-1) - 2^{n-1}\cdot 3 - \cdots\cdots - 2^2\cdot 3 - 2\cdot 3 - 3$

$= 2^n(-1) - \underbrace{[3 + 2\cdot 3 + 2^2\cdot 3 + 2^3\cdot 3 + \cdots + 2^{n-1}\cdot 3]}_{\text{Geometric series. (Add the terms in the sequence)}}$

5. Series. (Summation)

Ex. Geometric sequence: $3, 2\cdot 3, 2^2\cdot 3, 2^3\cdot 3, \cdots\cdots$

Geometric series (summation): $3 + 2\cdot 3 + 2^2\cdot 3 + 2^3\cdot 3 + \cdots$

Formula for Geometric Series:

$$S_n = a_0 + a_1 + a_2 + a_3 + \cdots + a_n$$

(the sum of the first $(n+1)$ terms)

Geometric seq.: $a, ar, ar^2, ar^3, ar^4, \cdots a\cdot r^n$

① $S = a + ar + ar^2 + ar^3 + \cdots + ar^n$

Multiply by r $\quad r\cdot S = r(a + ar + ar^2 + \cdots\cdots ar^n)$

② $rS = ar + ar^2 + ar^3 + \cdots\cdots ar^{n+1}$

$$② - ①\quad \begin{aligned} rs &= \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots \cancel{\ } + ar^{n+1} \\ - \quad s &= a + \cancel{ar} + \cancel{ar^2} + \cancel{\cdots} + \cancel{ar^n} \end{aligned}$$

$$rs - s = ar^{n+1} - a$$

$$\frac{S(r-1)}{r-1} = \frac{a(r^{n+1}-1)}{r-1}$$

Geometric series Formula :

$$\boxed{S = \frac{a \cdot (r^{n+1}-1)}{r-1}}$$

Note: $n+1$ in the formula is just the number of terms.

$a_0 - a_6$: r^7

$a_1 - a_6$: r^6

Geometric

$a_0, a_1, \cdots a_6$ → 7 terms

$$S = \frac{a_0 \cdot (r^7-1)}{r-1}$$

$a_1, a_2, \cdots a_6$ 6 terms

$$S = \frac{a_1 (r^6-1)}{r-1}$$

Back to Iterative approach $a_n = 2a_{n-1} - 3$

$$a_n = 2^n(-1) - \underbrace{[2^0 \cdot 3 + 2 \cdot 3 + 2^2 \cdot 3 + 2^3 \cdot 3 + \cdots + 2^{n-1} \cdot 3]}_{\text{Geometric series}}$$

$a = 3$, $r = 2$ # of terms $= n$

$$a_n = 2^n(-1) - \frac{3(2^n - 1)}{2-1}$$

$$\begin{aligned} a_n &= 2^n(-1) - 3 \cdot (2^n - 1) \\ &= \boxed{2^n(-1)} - \boxed{2^n \cdot 3} + 3 \\ &= 2^n(-1-3) + 3 \\ &= 2^n(-4) + 3 \\ &= 2^n \cdot (-2^2) + 3 \end{aligned}$$

$$\boxed{a_n = -2^{n+2} + 3}$$ Explicit formula.

$a_0 = -2^2 + 3 = -4 + 3 = -1$ (check)