HW §2.3

\# 15. $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ $\left\langle \begin{matrix} \mathbb{Z}^+ \\ 0 \\ \mathbb{Z}^- \end{matrix}\right.$ onto

a). T $m+n$ let $m=0$, $\underbrace{F(m,n)}_{\in\mathbb{Z}} = 0+n = \underset{\in\mathbb{Z}}{n}$

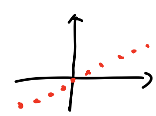
b) F $f(m,n) = \underset{\geq 0}{m^2} + \underset{\geq 0}{n^2} \geq 0$ $\mathbb{Z}^-$ doesn't get satisfied

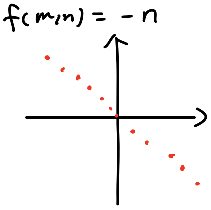
c) T $f(m,n) = m$

d). F $f(m,n) = |n| \geq 0$ $f(m,n) \geq 0$ $\mathbb{Z}^-$

e) T $f(m,n) = m-n$ let $n=0$ $f(m,n) = m$

or let $m=0$ $f(m,n) = -n$





\#25. by def (strictly increasing) : $x_1 < x_2$, $f(x_1) < f(x_2)$ strictly decreasing def $x_1 < x_2$, $f(x_1) > f(x_2)$

iff : ① if $f(x)$ is strictly increasing, then $\frac{1}{f(x)}$ strictly decreasing

② if $\frac{1}{f(x)}$ is strictly decreasing, then $f(x)$ is strictly increasing.

①

$\Rightarrow$ by def. let $x_1, x_2 \in \mathbb{R}$. $f(x_1), f(x_2) > 0$

strictly increasing $\underset{}{\overset{x_1 < x_2}{f(x_1) < f(x_2)}}$

$$\frac{1}{f(x_1)} > \frac{1}{f(x_2)}$$

Ex. $f(x_1)=3$, $f(x_2)=5$

$\frac{1}{f(x_1)} = \frac{1}{3} < f(x_2) = \frac{1}{5}$

(Conversely)

② $\Leftarrow$ by def. $x_1 < x_2$

$f(x_1)f(x_2)\,\frac{1}{f(x_1)} > \frac{1}{f(x_2)}\,f(x_1)f(x_2)$ $f(x_1)\cdot f(x_2) > 0$

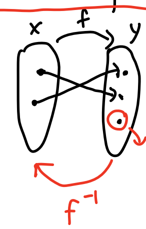
$f(x_2) > f(x_1)$

$\boxed{x_1 < x_2,\ f(x_1) < f(x_2)}$ strictly increasing.

§ 2.3.

Assuming this onto.

1. Inverse Function. ($\boxed{1-1 \text{ correspondence}}$: Bijection)



in domain but not get covered in f^{-1}

It is NOT a function!

Ex.

let $f: \mathbb{Z} \to \mathbb{Z}$ $\quad f(x) = 2x+1$ odd integers (Not onto)

Is invertible? Not invertible.

$f: \mathbb{R} \to \mathbb{R}$ $\quad y = 2x+1$

① switch x, y

$x = 2y+1$

② solve for y.

$\frac{x-1}{2} = y = f^{-1}(x)$

$\begin{cases} f(x) = y \\ f^{-1}(y) = x \end{cases}$

$f^{-1}(a) = b$

Ex. $f(3) = 5$

$f^{-1}(5) = 3$

$f^{-1}(x) = y$

$f(x)$ $\quad$ x in the original function.

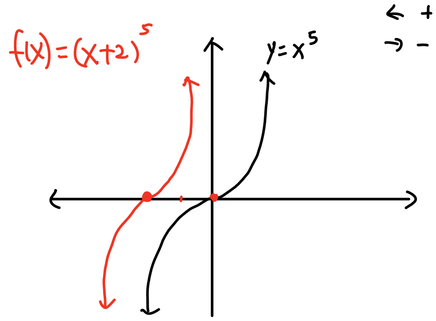
2. Compositions of functions.

Ex. $f(x) = (x+2)^5$

pre-calc: Transformation : $x^5 \longrightarrow (x+2)^5$

$\leftarrow +$ shift left by 2 units.

$\rightarrow -$



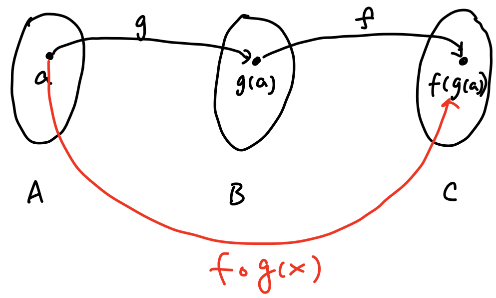
calculus: chain Rule: $f(x) = \underline{(\underline{x+2})^5}$ — Inside: $(x+2)$; Outside: $(\ x\ )^5$

Def: let g be a function from $A \to B$, let f be a function from $B \to C$.

Notation

$f \circ g(a) \quad a \in A$

$= f(g(a))$



Ex. If $f(x) = 2x+3$, $g(x) = x^2$

$$f \circ g(x) = f(g(x)) = f(x^2) = 2x^2+3$$

$$g \circ f(x) = g(f(x)) = g(2x+3) = (2x+3)^2$$

$\boxed{f(a)=b} \Rightarrow f^{-1}(b) = a$

precalc: ① $f^{-1}(f(x)) = f^{-1}(f(a)) = f^{-1}(b) = a$ $f(x)$ and $f^{-1}(x)$ are inverse to each other.

② $f(f^{-1}(x)) = f(f^{-1}(b)) = f(a) = b$

Ex. $f(x) = 2x+1$ ($R \to R$) $f^{-1}(y) = \frac{1}{2}y - \frac{1}{2}$ (previous example from inverse)

$f^{-1}(2) = \frac{1}{2}(2) - \frac{1}{2} = \frac{1}{2}$

$f(f^{-1}(2)) = f(\frac{1}{2}) = 2(\frac{1}{2})+1 = 1+1 = 2$

$f^{-1}(f(2)) = f^{-1}(5) = \frac{1}{2}(5) - \frac{1}{2} = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$ $f(2) = 2(2)+1 = 5$

Example to understand. (can't be the proof)

show $f(x)$ and $g(x)$ are inverse to each other

You need to show that $f(g(a)) = a$, $g(f(a)) = a$] $f(x)$ and $\boxed{g(x)}$ ($f^{-1}(x)$) are inverse to each other

$f(x) = 2x+1$ $g(x) = \frac{1}{2}x - \frac{1}{2}$

pf. $f(g(a)) = f(\frac{1}{2}a - \frac{1}{2}) = 2(\frac{1}{2}a - \frac{1}{2}) + 1 = a - 1 + 1 = a$

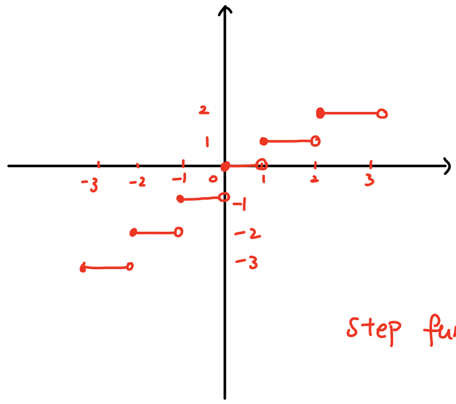
$g(f(a)) = g(2a+1) = \frac{1}{2}(2a+1) - \frac{1}{2} = a + \frac{1}{2} - \frac{1}{2} = a$

$f(x)$ and $g(x)$ are inverse to each other.

3. Floor and Ceiling Function

① Floor Function: $\lfloor x \rfloor$ assigns the real # x the largest integer that $\leq x$.

Ex. $\lfloor 4.1 \rfloor = 4$. $\lfloor -\frac{1}{2} \rfloor = -1$



$\lfloor 0 \rfloor = 0$

$x \in [0, 1)$ $\quad \lfloor 0.5 \rfloor = 0$

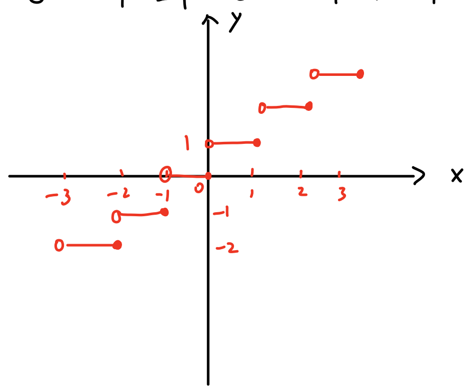
$x \in [1, 2)$ $\quad \lfloor 1.5 \rfloor = 1$

$x \in [-1, 0)$ $\quad \lfloor -0.5 \rfloor = -1$

Step function.

② Ceiling Function: $\lceil x \rceil$ assigns the real # x the smallest integer that $\geq x$.

Ex. $\lceil 4.1 \rceil = 5 \qquad \lceil -\frac{1}{2} \rceil = 0 \qquad \lceil 7 \rceil = 7$



$x \in (0, 1] \qquad \lceil 0.5 \rceil = 1, \ \lceil 1 \rceil = 1$

Ex. Prove that if $x \in \mathbb{R}$, $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

#: 1) $x = 0.6$

$\lfloor 2x \rfloor = \lfloor 1.2 \rfloor = 1$

$\lfloor x + \frac{1}{2} \rfloor = \lfloor 0.6 + 0.5 \rfloor = \lfloor 1.1 \rfloor = 1$

2) $x = 0.4$

$\lfloor 2x \rfloor = \lfloor 0.8 \rfloor = 0$

$\lfloor x + \frac{1}{2} \rfloor = \lfloor 0.4 + 0.5 \rfloor = \lfloor 0.9 \rfloor = 0$

n.S $\quad$ Ex. 3.6 $\quad n = 3, \ S = 0.6$

$S \geq 0.5$ $\quad \lfloor 2s \rfloor$ will go to the next integer

$S < 0.5$ $\quad \lfloor 2s \rfloor$ will keep the same integer.

same thing for $\lfloor s + 0.5 \rfloor$

Pf. let $x = n + s$, $n \in \mathbb{Z}$, $0 \leq s < 1$

Ex. 2.7

$n = 2, \ s = 0.7$

$x = 2 + 0.7$

Two cases

case 1. $\boxed{0 \leq s < \frac{1}{2}}$

left: $\lfloor 2x \rfloor = \lfloor 2(n+s) \rfloor = \lfloor \underbrace{2n}_{\text{integer}} + 2s \rfloor = \lfloor 2n \rfloor + \lfloor 2s \rfloor$

$= 2n + 0$

$= 2n$

$0 \leq 2s < 1$

$\lfloor 2s \rfloor = 0$

$2 \cdot 0 \leq 2 \cdot s < 2 \cdot \frac{1}{2}$

$0 \leq 2s < 1$

right: $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

$= \lfloor n + s \rfloor + \lfloor n + s + \frac{1}{2} \rfloor$

$= \lfloor n \rfloor + \underbrace{\lfloor s \rfloor}_{0} + \lfloor n \rfloor + \underbrace{\lfloor s + \frac{1}{2} \rfloor}_{<1} = 0$

$= n + 0 + n + 0 = 2n$

$0 + \frac{1}{2} \leq s + \frac{1}{2} < \frac{1}{2} + \frac{1}{2}$

$\frac{1}{2} \leq s + \frac{1}{2} < 1$

left = right ✓

Case 2. $\frac{1}{2} \le s < 1$

left: $\lfloor 2x \rfloor = \lfloor 2(n+s) \rfloor = \lfloor 2n + 2s \rfloor = \lfloor 2n \rfloor + \lfloor \overbrace{2s}^{[1,2)} \rfloor$

$= 2n + 1$

$\frac{1}{2} \cdot 2 \le s \cdot 2 < 1 \cdot 2$

$1 \le 2s < 2$

$\ne \lfloor n \rfloor + \lfloor s \rfloor + \lfloor \frac{1}{2} \rfloor$

right: $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor n+s \rfloor + \lfloor n + \underbrace{s + \frac{1}{2}} \rfloor$

$= \lfloor n \rfloor + \underbrace{\lfloor s \rfloor}_{0} + \lfloor n \rfloor + \underbrace{\lfloor s + \frac{1}{2} \rfloor}_{1}$

$\frac{1}{2} + \frac{1}{2} \le s + \frac{1}{2} < 1 + \frac{1}{2}$

$1 \le s + \frac{1}{2} < 1.5$

$= n + 0 + n + 1 = 2n + 1$

left = right ∎

$= \lfloor 4 \rfloor + \lfloor 0.7 \rfloor$

$= \lfloor 4 + 0.2 + 0.5 \rfloor$

$\lfloor 4.2 + 0.5 \rfloor$

$\lfloor 4 + 0.7 \rfloor$

$= \lfloor 4 \rfloor + \lfloor 0.7 \rfloor$

$\lfloor 3 + \underbrace{0.6 + 0.5} \rfloor$

$\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$

$x = n + s \quad n \in \mathbb{Z},\ 0 \le s < 1$

3 cases

$0 \le s < \frac{1}{3}$

$\frac{1}{3} \le s < \frac{2}{3}$

$\frac{2}{3} \le s < 1$