§ 2.3 Functions

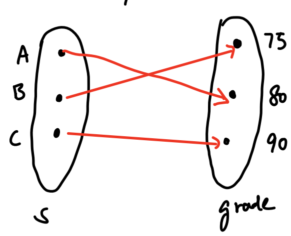
1. Functions

<u>Def.</u> let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of set B. assigned by the function f to the element a of A.

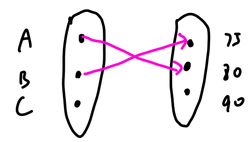
$$f : A \to B$$

Ex. Students : A , B, C
grade : 80 , 75, 90

Mapping / Transformation

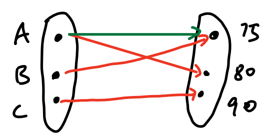


① in the student set, all of students have grades.



(Not a function. b/c C doesn't have a grade)

② for each student, he/she will get only one grade for the test.



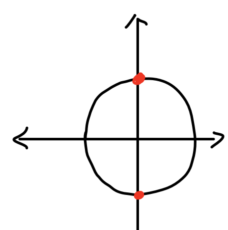
A : 75, 80
(impossible)

(Not a function)

Ex. Circle equation : $x^2 + y^2 = 1$

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1-x^2}$$



let $x = 0$, $y = \pm\sqrt{1} = \pm 1$ (Not a function)

2. <u>Def.</u> If f is a function from A to B, we say that A is the domain of f. and B is codomain of f.

$$f(a) = b$$

a: pre-image of f
b: Image

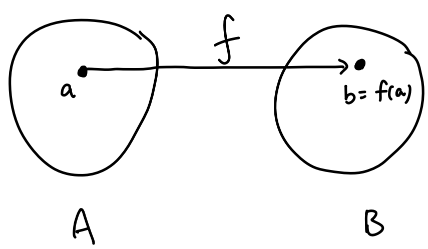
Range of f is the set of all images of elements of A.

Ex. students : A, B, C $\Leftarrow$ Domain.

grades : 0 − 100 $\Leftarrow$ Co domain.

Assigned grades : 80, 75, 90 $\Leftarrow$ Range.

Mapping



If f is a function from A to B,
We say that f maps A to B.

3. one-to-one (1-1) and onto function

Vertical line test : Function ?

Horizontal line test : 1-1 ?

① 1-1 Def : A function is said to be 1-1, or an injection iff $f(a) = f(b)$ implies that $a = b$.

for all a and b in the domain of f, A.

$a, b \in A$

① $\forall a \forall b [(f(a) = f(b)) \rightarrow (a = b)]$

② $\equiv \forall a \forall b [(a \neq b) \rightarrow (f(a) \neq f(b))]$

Ex. Determine whether the function $f(x) = x+1$ from the set of real # to itself is 1-1.

$f : R \rightarrow R$

$f(x) = x+1$

Pf : ① by def. $f(a) = f(b) \rightarrow (a = b)$ ✓

let $a, b \in R$, $f(a) = f(b)$

$a+1 = b+1$

$a = b$ ∎

② by contrapositive of def $a \neq b \rightarrow f(a) \neq f(b)$

let $a, b \in R$, $a \neq b$

Add 1 $\quad a+1 \neq b+1$

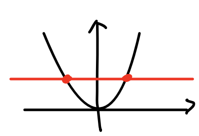
$f(a) \neq f(b)$ ✓

∎.

✓

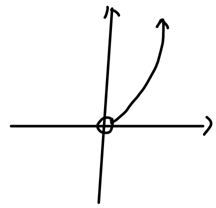
Ex. $f(x) = x^2 \quad Z \rightarrow Z$

Not 1-1, b/c $x = 1, x = -1 \quad y = 1$

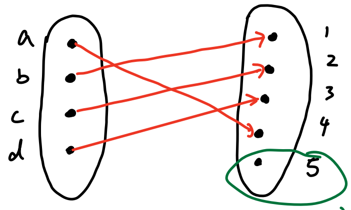


Horizontal line test.

$\mathbb{R}^+ \to \mathbb{R}$
$\mathbb{Z}^+ \to \mathbb{Z}$] $f(x) = x^2$ 1-1.

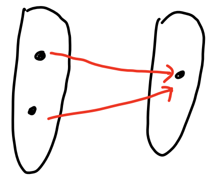


1-1 Mapping.

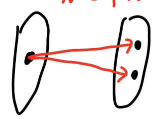


1-1

Not 1-1



Not function.



Not onto.

② Onto function : A function f from A to B is called onto, or a surjection iff every element $b \in B$, there is an element $a \in A$ with $f(a) = b$.

$$\forall y \exists x \, (f(x) = y)$$

Ex. $f(x) = x^2$ $\mathbb{Z} \to \mathbb{Z}$ onto ?

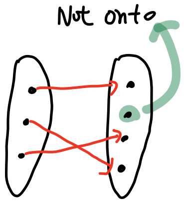
No! $y = 2, 3, 5, \cdots$ $y = 2,$ $x = \pm\sqrt{2} \notin \mathbb{Z}$ Not onto!

$y = -1, -2, -3, -4, \cdots$ $x^2 = -1$ $x = i \notin \mathbb{Z}$ $\mathbb{C} \to \mathbb{R}$ (Not required)

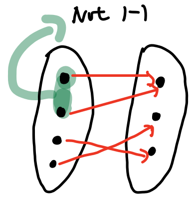
$f(x) = x^2$ $\mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, onto!

Summary

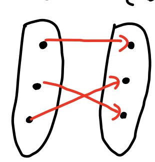
1-1
Not onto



onto
Not 1-1



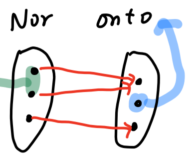
one-to-one
and onto (bijection)



$y = x + 1$ $\mathbb{Z} \to \mathbb{Z}$

$y - 1 = x \to$ onto!

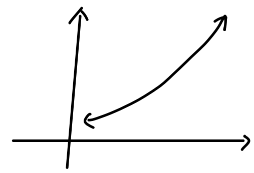
Neither one-to-one
Nor onto



4. Increasing and Decreasing functions.

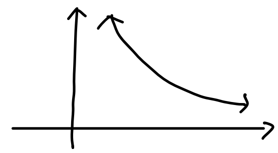
Increasing: $\forall x \forall y \, (x<y \rightarrow f(x) \leq f(y))$

Strictly increasing: $\forall x \forall y \, (x<y \rightarrow f(x) < f(y))$



Decreasing: $\forall x \forall y \, (x<y \rightarrow f(x) \geq f(y))$

Strictly decreasing: $\forall x \forall y \, (x<y \rightarrow f(x) > f(y))$



5. Inverse Functions

Def. let f be a 1-1 correspondence $(A \rightarrow B)$. The inverse function of f is the function that assigns to an element b belonging to set B the unique element a in A s.t $f(a)=b$. The inverse $f^{-1}(b)=a$

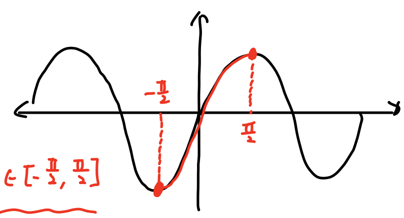
Note: only when f is bijection, you can find the inverse.

$f(x) = \sin x$

$y = \sin x$

$x = \sin^{-1}(y)$

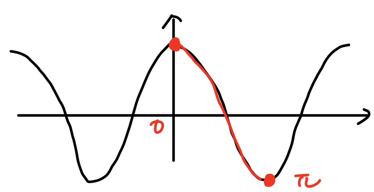
$x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$R \rightarrow R$. $y=\sin(x)$ is not 1-1

But we want to have the inverse b/c we want to get the angle.

Control the domain.



$y = \cos x$ $\quad x = \cos^{-1}(y)$

$x \in [0, \pi]$