§ 1.7. proof

1. Direct proof.

$\forall x \quad \underbrace{P(x)}_{T} \xrightarrow[\text{Pfs}]{} Q(x)$ T

Even #: Def. if $n = 2\cdot k$, $k \in \mathbb{Z}$, then n is even.

odd #: Def if $n = 2k+1$, $k \in \mathbb{Z}$, then n is odd (When n divided by 2, the remainder is 1.)
or $n = 2k-1$

Ex. If n is an odd integer, then $\boxed{3n+7 \text{ is even.}}$ $(3n+7 = \boxed{2\cdot k})$

Pf. by def, $n = 2k+1$, $k \in \mathbb{Z}$

$$
\begin{aligned}
3n+7 &= 3(2k+1)+7 \\
&= 6k+3+7 \\
&= 6k+10 \\
&= 2(\underbrace{3k+5}_{\in \mathbb{Z}})
\end{aligned}
$$

Since $k \in \mathbb{Z}$, $3k+5 \in \mathbb{Z}$, Let $3k+5 = M$, $M \in \mathbb{Z}$

$3n+7 = 2\cdot M$, $M \in \mathbb{Z}$, by def. $3n+7$ is even. ▨

or QED

$2k+1$
$2k-1$

Ex. If n is even, then $-5n-3$ is odd.

Pf. by def $n = 2k$, $k \in \mathbb{Z}$

$$
\begin{aligned}
-5n-3 &= -5(2k)-3 \\
&= -10k-3
\end{aligned}
$$

① $-5n-3 = -10k-2-1$

$= 2(-5k-1)-1$

$k \in \mathbb{Z}$, $-5k-1 \in \mathbb{Z}$
let $M = -5k-1$, $M \in \mathbb{Z}$

$-5n-3 = 2M-1$, $M \in \mathbb{Z}$

By def, $-5n-3$ is odd. ▨.

② $-5n-3 = -10k-4+1$
$= 2(-5k-2)+1$

⋮

2. Proof by Contrapositive

Recall: $p \to q \equiv \neg q \to \neg p$

Instead of proving directly from p to q, we can prove the $\neg q$ to $\neg p$ is true!

Ex. let $x \in \mathbb{Z}$, If $5x-7$ is even, then x is odd.

direct pf. $5x-7 = 2k$

$+7 \quad +7$

$\frac{5x}{5} = \frac{2k}{5} + \frac{7}{5}$

$x = \frac{2}{5}k + \frac{7}{5}$?

Pf by contrapositive:

let $x \in \mathbb{Z}$, If x is even, then $5x-7$ is odd

let $x = 2k$, $k \in \mathbb{Z}$

$5x-7 = 5(2k)-7 = 10k-7$

$= 10k-8+1$

$= 2(5k-4)+1$

$k \in \mathbb{Z}$, $5k-4 \in \mathbb{Z}$, let $M = 5k-4$, $M \in \mathbb{Z}$

$5x-7 = 2M+1$, $M \in \mathbb{Z}$

by def, $5x-7$ is odd.

Pf by contrapositive shows that if $5x-7$ is even, then x is odd.

Direct pf. Let $5x-7 = 2k$, $k \in \mathbb{Z}$

$2(\quad)+1$

$x + 4x - 7 = 2k$

$-4x+7 \quad -4x+7$

$x = 2k - 4x + 7$

$+6+1$

$x = (2k-4x+6)+1$

$x = 2(k-2x+3)+1$

$k, x \in \mathbb{Z}$, $k-2x+3 = M \in \mathbb{Z}$

$x = 2M+1$, $M \in \mathbb{Z}$, x is odd. ▨

Ex. let $x \in \mathbb{Z}$, x^2 is odd $\underbrace{\overset{\longleftrightarrow}{\text{iff}}}_{\text{if and only if}}$ x is odd.

Note: $\longleftrightarrow$: Need to prove both directions!

Pf: $\rightarrow$ if x^2 is odd then x is odd.

Pf by Contrapositive: if x is even, then x^2 is even.

let $x = 2k$, $k \in \mathbb{Z}$

$x^2 = (2k)^2 = 4k^2 = 2 \cdot (2k^2)$ $\quad k \in \mathbb{Z}$, $2k^2 \in \mathbb{Z} = M$

$x^2 = 2M$, $M \in \mathbb{Z}$, x^2 is even.

$\leftarrow$ if x is odd then x^2 is odd.

Direct Pf. let $x = 2k+1$, $k \in \mathbb{Z}$

$$x^2 = (2k+1)^2 = (2k+1)(2k+1)$$
$$= 4k^2 + 4k + 1$$
$$= 2\underbrace{(2k^2+2k)} + 1$$

$k \in \mathbb{Z}$, $2k^2+2k \in \mathbb{Z}$ let $2k^2+2k = M$, $M \in \mathbb{Z}$

$x^2 = 2M+1$, $M \in \mathbb{Z}$, x^2 is odd. ∎

Ex let $x \in \mathbb{Z}$, If $5x-7$ is odd, then $9x+2$ is even.

$5x-7$ is odd $\xrightarrow{\text{lemma}}$ $9x+2$ is even

$\rightarrow x$ is even / odd? ✓ $\boxed{??}$

observation! (Not Pf)

try: $5x-7$ is odd

$x=1$, $5(1)-7 = -2$ even

$x=2$, $5(2)-7 = 3$ odd ✓

$x=3$, $5(3)-7 = 8$

$x=4$, $5(4)-7 = 13$ ✓

? Lemma (try to prove) If $5x-7$ is odd, then x is even.

Pf by contrapositive: if x is odd, then $5x-7$ is even.

$x = 2k+1$, $k \in \mathbb{Z}$

$$5x-7 = 5(2k+1)-7$$
$$= 10k+5-7$$
$$= 10k-2 = 2(5k-1)$$

since $k \in \mathbb{Z}$, $5k-1 \in \mathbb{Z}$, let $5k-1 = M$, $M \in \mathbb{Z}$

$5x-7 = 2M$, $M \in \mathbb{Z}$, $5x-7$ is even.

Lemma! if $5x-7$ is odd, then x is even:

Prove that if x is even, then $9x+2$ is even

Direct pf: let $x = 2k$, $k \in \mathbb{Z}$

$$9x+2 = 9(2k)+2 = 18k+2$$
$$= 2(9k+1), \quad k \in \mathbb{Z}, \; 9k+1 = M \in \mathbb{Z}$$

$9x+2 = 2M$, $M \in \mathbb{Z}$

$9x+2$ is even.

We proved that if $5x-7$ is odd, then $9x+2$ is even. ▨.

Note: Build the Lemma, Need to be proved!

Alternative pf: Direct pf

$5x-7 = 2k+1 \quad k \in \mathbb{Z}$

$+4x$

$5x+4x-7 = 4x+2k+1$

$+9$

$9x-7+9 = 4x+2k+1+9$

$9x+2 = 4x+2k+10$

$9x+2 = 2(2x+k+5)$

$x, k \in \mathbb{Z}$, $2x+k+5 \in \mathbb{Z}$ let $2x+k+5 = M$, $M \in \mathbb{Z}$

$9x+2 = 2 \cdot M$, $M \in \mathbb{Z}$

$9x+2$ is even. ▨

3. Proof by Contradiction.

$\forall x \; p(x) \to \boxed{Q(x)}$

Assume Conclusion is False

Use False Conclusion to verify $p(x)$.

if we get the opposite $p(x)$, Assumption is wrong!

$\to$ Conclusion is true!

Ex. let $x \in \mathbb{Z}$, $\boxed{\text{if } 5x-7 \text{ is even}}$, then $\boxed{x \text{ is odd.}}$ ⇐

Pf by Contradiction: Assume x is even. $\neg$

Reach the contradiction (Statement is good! prove it)

didn't reach the Contradiction (Statement is Not good. Counter-example disprove it)

$x = 2k, \ k \in \mathbb{Z}$

$5x-7 = 5(2k)-7 = 10k-7$

$= 10k-8+1$

$= 2(5k-4)+1$

$k \in \mathbb{Z}, \ 5k-4 \in \mathbb{Z}$ let $5k-4 = M, \ M \in \mathbb{Z}$

$5x-7 = 2M+1, \ M \in \mathbb{Z}$

by def. $5x-7$ is odd. but it Contradicts that $(5x-7)$ is even.

Which is the Contradiction, the assumption is wrong.

x has to be odd. ▨.

Note: Show that ⋯ / prove that ⋯ } statement True!

Prove or $\underbrace{\text{disprove}}_{\text{try to find counter example}}$ ⋯ (Statement ?)

Start with