105 + 5 = 110

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EXAM II

Mat 120

Show relevant work, where appropriate, answers without support may receive little or no credit.

Good job!

Total: 105 points

1. (11 points) Use the set operations or Venn diagram to solve the following question.

   In a survey of 60 people, it was found that:

   25 read *Newsweek* magazine, 26 read *Time,* 26 read *Fortune*

   9 read *Newsweek* and *Fortune,* 11 read *Newsweek* and *Time,* 8 read *Fortune* and *Time*

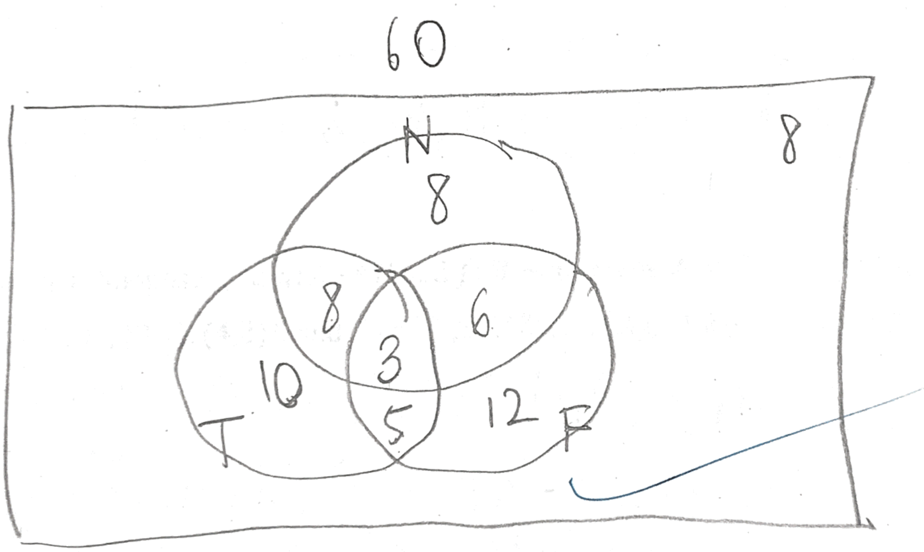
   3 read all

   Use Venn Diagram and label the numbers inside (5 points)

   (a) (2 point) Find the number of people who read none of these three magazines

   (b) (2 point) Find the number of people who read exactly one magazine.

   (c) (2 point) Find the number of who read at least one of the three magazines.



(a) 8

(b) 10+8+12 = 30

(c) 60 - 8 = 52

$U = \{1, 2, 3, -, 13\}$ $C = \{1, 2, 3, 4, 5, 6\}$ $A = \{2, 3, 5, 7, 11, 13\}$ $B = \{2, 4, 6, 8, 10, 12\}$

2. Let $U = \{n \in Z^+ \mid n \leq 13\}$ be the universal set and let $A = \{n \in U \mid n \text{ is prime}\}$, $B = \{n \in U \mid n \text{ is even}\}$, and $C = \{n \in U \mid n < 7\}$. List all of the elements in the following sets. (12 points)

a) $A \cap B$

$\{2\}$

b) $A - C$

$\{\cancel{2}, \cancel{3}, \cancel{5}, 7, 11, 13\} - \{1, 2, 3, 4, 5, 6\} = \{7, 11, 13\}$

c) $B \cup \bar{C}$

$\{2, 4, 6, 8, 10, 12\} \cup \{7, 8, 9, 10, 11, 12, 13\} = \{2, 4, 6, 7, 8, 9, 10, 11, 12, 13\}$

d) $\overline{A \cup B \cup C}$

$\overline{\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13\}} = \{9\}$

3. (12 points) Suppose that: $g: A \to B$ and $f: B \to C$ where $A = B = C = \{1,2,3,4\}$, $g = \{(1,4), (2,1), (3,1), (4,2)\}$, and $f = \{(1,3), (2,2), (3,4), (4,2)\}$ (i.e. f(1) =3, f(2)=2, f(3)=4, f(4)=2…)

$f \circ g = \{(1,2), (2,3), (3,3), (4,2)\}$

a) Find $f \circ g$

$f(g(1)) = f(4) = 2$ $f(g(3)) = f(1) = 3$

$f(g(2)) = f(1) = 3$ $f(g(4)) = f(2) = 2$

b) Does $g^{-1} \circ f$ exist? If it exists, find it. If it doesn't exist, explain why.

No, b/c g^{-1} does not exist since it is not one-to-one (both 2 and 3 map to 1) and thus is not a bijection.

c) Find $g \circ (g \circ g)$

$g(g(g(1))) = g(g(4)) = g(2) = 1$

$g(g(g(2))) = g(g(1)) = g(4) = 2$

$g(g(g(3))) = g(g(1)) = g(4) = 2$

$g(g(g(4))) = g(g(2)) = g(1) = 4$

$g \circ (g \circ g) = \{(1,1), (2,2), (3,2), (4,4)\}$

4. (13 points) Find the first five terms of the sequence a_n, $n \geq 0$

(a) $a_n = 2^n + (-2)^n$

$a_0 = 2^0 + (-2)^0 = 1 + 1 = 2$

$a_1 = 2^1 + (-2)^1 = 2 - 2 = 0$

$a_2 = 2^2 + (-2)^2 = 4 + 4 = 8$

$a_3 = 2^3 + (-2)^3 = 8 - 8 = 0$

$a_4 = 2^4 + (-2)^4 = 16 + 16 = 32$

(b) $a_n = \lfloor n/2 \rfloor + \lceil n/2 \rceil$

$a_0 = \lfloor 0/2 \rfloor + \lceil 0/2 \rceil = 0 + 0 = 0$

$a_1 = \lfloor 1/2 \rfloor + \lceil 1/2 \rceil = 0 + 1 = 1$

$a_2 = \lfloor 2/2 \rfloor + \lceil 2/2 \rceil = 1 + 1 = 2$

$a_3 = \lfloor 3/2 \rfloor + \lceil 3/2 \rceil = 1 + 2 = 3$

$a_4 = \lfloor 4/2 \rfloor + \lceil 4/2 \rceil = 2 + 2 = 4$

(c) $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$. (Find a_2, a_3, a_4)

$a_0 = 1$

$a_1 = 1$

$a_2 = 2(1) + 2^2(1) = 2 + 4 = 6$

$a_3 = 3(6) + 3^2(1) = 18 + 9 = 27$

$a_4 = 4(27) + 4^2(6) = 108 + 96 = 204$

5. (20 points) What are the values of these sums?

a) $\sum_{k=4}^{112} 2+k$

$$\sum_{K=4}^{112} 2+K = \sum_{K=4}^{112} 2 + \sum_{K=4}^{112} k = 109(2) + \left[\sum_{K=1}^{112} k - \sum_{K=1}^{3} k\right]$$

$$= 218 + \left[\frac{112(113)}{2} - \frac{3(4)}{2}\right]$$

$$= 218 + 6328 - 6 = 6540$$

b) $\sum_{n=0}^{\infty} \frac{2^n+3^n}{5^n}$

$$\sum_{n=0}^{\infty} \frac{2^n+3^n}{5^n} = \sum_{n=0}^{\infty} \frac{2^n}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^n} = \sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n$$

$$= \frac{1}{1-\frac{2}{5}} + \frac{1}{1-\frac{3}{5}} = \frac{1}{3/5} + \frac{1}{2/5} =$$

$$\frac{\overset{10}{\cancel{5}}}{\underset{6}{\cancel{3}}} + \frac{\overset{15}{\cancel{5}}}{\underset{6}{\cancel{2}}} = \frac{25}{6}$$

c) $\sum_{i=0}^{4} \sum_{j=2}^{5} i^3 j^2$

$$\sum_{i=0}^{4} \sum_{j=2}^{5} i^3 j^2 = \sum_{i=0}^{4} \left[2^2 i^3 + 3^2 i^3 + 4^2 i^3 + 5^2 i^3\right]$$

$$= \sum_{i=0}^{4} \left[4i^3 + 9i^3 + 16i^3 + 25i^3\right]$$

$$= \sum_{i=0}^{4} 54i^3 = 54 \sum_{i=0}^{4} i^3 = 54\left(\frac{4(4+1)}{2}\right)^2 = 54(100) = 5400$$

d) $\sum_{j=0}^{9}(2^{j+1} - 2^j)$ (This is called the telescoping series. Hint: Instead of calculating all the numbers out, you may want to list all the terms first)

$$\sum_{j=0}^{9} 2^{j+1} - 2^j =$$

$$\begin{array}{r} \cancel{2} + \cancel{4} + \cancel{8} + \cdots + \cancel{512} + 1024 \\ -\ 1 + \cancel{2} + \cancel{4} + \cdots + \cancel{256} + \cancel{512} \\ \hline \end{array}$$

$$1024 - 1 = 1023$$

e) $\sum_{i=2}^{9} 3(2^{i-1})$

$$\sum_{i=2}^{9} (3)(2^{i-1}) = 3(2)(2^0 + 2^1 + \cdots + 2^7)$$

$$= 6\left[\frac{1-2^8}{1-2}\right] = 6\left[2^8 - 1\right] = 1530$$

f) **Extra credits: (5 points)** Express $0.\bar{8}$ as a ratio of integers. ($0.\bar{8} = 0.8888888\ldots$ it is a repeated decimal number, but you can also rewrite $0.\bar{8} = 0.8 + 0.08 + 0.008 + \cdots$.)

$$.\bar{8} = .8 + .08 + \cdots = .8\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$$
$$= .8\left(\frac{1}{1-\frac{1}{10}}\right) = .8\left(\frac{1}{9/10}\right) = .8\left(\frac{10}{9}\right) = \frac{8}{9}$$

+5

g) **Extra credits: (5 points)** Simplify the following function to the simplest form.

$$\frac{1}{x-1} + \frac{\sum_{k=0}^{2020}(k+1)x^k}{\sum_{k=0}^{2021} x^k}$$

$$\frac{1}{x-1} + \frac{\sum_{k=0}^{2020} kx^k + \sum_{k=0}^{2020} x^k}{\sum_{k=0}^{2021} x^k} = \frac{1}{x-1} + \frac{\sum_{k=0}^{2021} kx^k - 2021x^{2021} + \sum_{k=0}^{2021} x^k - x^{2021}}{\sum_{k=0}^{2021} x^k} =$$

Can't be canceled

$$\frac{\sum_{k=0}^{2021} kx^k}{x^k} + \sum_{k=0}^{2021}\frac{x^k}{x^k} - \frac{2022x^{2021}}{\sum_{k=0}^{2021} x^k} + \frac{1}{x-1} = \sum_{k=0}^{2021} k + \sum_{k=0}^{2021} 1 - \frac{2022x^{2021}}{\sum_{k=0}^{2021} x^k} + \frac{1}{x-1} =$$

$$\frac{1}{x-1} + \frac{2021(2022)}{2} + 2022(1) - \left(\frac{2022x^{2021}}{\frac{1-x^{2022}}{1-x}}\right) = \frac{1}{x-1} + 204\cancel{5}3 - \frac{(2022x^{2021})(1-x)}{1-x^{2022}}$$

6. (9 points) Which of the following functions are: one-to-one, onto, both, or neither. Justify your answers. ($\mathbb{Z}^2$ *means* $m \in \mathbb{Z}, n \in \mathbb{Z}$, $\mathbb{Z}$ *is the set of all integers.*)

a) $f: N \to N, f(x) = 4x + 1$

one-to-one b/c strictly increasing $(\forall x \forall y\, (x<y \to 4x+1 < 4y+1)$

Not onto b/c for instance $3 \in \mathbb{N}$ but $f(x) \neq 3$ for all x.

b) $g: \mathbb{R} \to \mathbb{R}, g(x) = 2x + 1$

One-to-one b/c strictly increasing $(\forall x \forall y\, (x<y \to 2x+1 < 2y+1))$

Onto b/c the whole of $\mathbb{R}$ (the codomain) is reached ($x = \frac{g(x)-1}{2}$ maps to every real $g(x)$)

c) $p: \mathbb{Z}^2 \to \mathbb{Z}, p(x,y) = x^2 + y^2$

Not one-to-one b/c for instance $x=2, y=3$ maps to $p = 2^2+3^2 = 13$ but $x=-2$ and $y=-3$ also maps to $p = (-2)^2 + (-3)^2 = 13$

So $\forall x \forall y\, (f(x) = f(y) \to x = y)$ is false. Check back of page

Not onto b/c $p=3$ is in ~~co~~domain $\mathbb{Z}$ is not in the range
b/c 3 cannot be written as the sum of 2 integer squares.

7. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 4A + 7I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (Recall: $A^2 = A \times A$, I is the identity matrix.) (8 points)

$A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$ $4A = \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix}$ $7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$A^2 + 4A + 7I = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8. (10 points) Use the iterative approach to find the solution to the recurrence relation with the given initial condition.

$$a_n = 3a_{n-1} + 4, a_0 = 1$$

$a_n = 3a_{n-1} + 4, a_0 = 1$

$= 3(3(a_{n-2}) + 4) + 4 = 3^2 a_{n-2} + 4(1+3)$

$= 3(3(3(a_{n-3}) + 4) + 4) + 4 = 3^3 a_{n-3} + 4(1+3+3^2)$

$=$

$= 3^n a_{n-n} + 4(1+3+3^2+ \cdots + 3^{n-1})$

$= 1(3^n) + 4 \sum_{k=0}^{n-1} 3^k$

$= 3^n + 4\left(\frac{1-3^n}{1-3}\right) = 3^n + 4\left(\frac{3^n - 1}{2}\right) = 3^n + 2(3^n) - 2$

$= 3^n(1+2) - 2$

$= 3^n(3) - 2$

$= 3^{n+1} - 2$

9. True or False. (Explanations are not required) (10 points)

a) $f(x) = x^3 + 1$ *is a bijection from* **R** *to* **R**

True

b) $f(x) = \cos(x)$ *is a onto function but not one – to – one function for* $0 \leq x \leq 2\pi, -1 \leq f(x) \leq 1$.

True

c) $f(x) = \tan(x)$ *is a one – to – one but not onto for* $-\frac{\pi}{2} < x < \frac{\pi}{2}, f(x) \in R$.

False

d) Let $f: R \to R$ and let $f(x) > 0$ *for all* $x \in R$. *If* $f(x)$ *is strictly increasing then* $g(x) = \frac{1}{f(x)}$ *is strictly decreasing.*

True

e) Assume that A is a subset of some underlying universal set U. Then $A \cup U = U, A \cap U = A$ and $\emptyset - A = \emptyset$.

True

f) $\{0\} \subset \emptyset$

False

g) $\{\emptyset\} \subseteq \{\emptyset\}$

True

h) $2 \in \{\{2\}, \{\{2\}\}\}$

False

i) $A = \{a, b, c, d\}, B = \{y, z\}$, *then* $A \times B = B \times A$

False

j) If matrix $\begin{bmatrix} x & 1 & 3 \\ 1 & 0 & 3a-b \\ a+b & 4 & y \end{bmatrix}$ is a symmetric matrix, then $a = \frac{7}{4}, b = \frac{5}{4}$

$$\begin{bmatrix} x & 1 & 3 \\ 1 & 0 & 3a-b \\ a+b & 4 & y \end{bmatrix}^T$$

$a + b = 3$

$3a - b = 4$

$4a = 7$

$\overline{4} \quad \overline{4}$

$a = \frac{7}{4}$ ✓

$\frac{7}{4} + b = 3 \frac{12}{4}$

$-\frac{7}{4} \quad -\frac{7}{4}$

$b = \frac{5}{4}$ ✓

→ True