Hw §2.4

#9. d). $a_n = n a_{n-1} + n^2 \cdot a_{n-2}$ $\quad a_0 = 1,\ a_1 = 1$

$$a_2 = 2 \cdot a_1 + 2^2 \cdot a_0$$
$$= 2(1) + 4(1)$$
$$= 6$$

$$a_3 = 3a_2 + 3^2 a_1$$
$$= 3(6) + 9(1)$$
$$= 18 + 9$$
$$= 27$$

$$a_4 = 4a_3 + 4^2 a_2$$
$$= 4(27) + 16(6)$$
$$= 108 + 96$$
$$= 204$$

11. $a_n = 2^n + 5 \cdot 3^n$ $\quad n = 0, 1, 2, \ldots$

a) $a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 = 6$

$a_1 = 2^1 + 5 \cdot 3^1 = 2 + 15 = 17$

$a_2 = 2^2 + 5 \cdot 3^2 = 4 + 5 \cdot 9 = 49$

$a_3 = 2^3 + 5 \cdot 3^3 = 8 + 5(27) = 8 + 135 = 143$

$a_4 = 2^4 + 5 \cdot 3^4 = 16 + 5(81) = 16 + 405 = 421.$

c). $a_n = 5a_{n-1} - 6a_{n-2}$ $\quad n \geq 2.$

left: $a_n = \boxed{2^n + 5 \cdot 3^n}$

right: $5a_{n-1} - 6a_{n-2}$

$a_{n-1} = 2^{n-1} + 5 \cdot 3^{n-1}$

$a_{n-2} = 2^{n-2} + 5 \cdot 3^{n-2}$

$$\text{right: } 5\left(2^{n-1}+5\cdot 3^{n-1}\right) - 6\left(2^{n-2}+5\cdot 3^{n-2}\right)$$

$$= 5\cdot 2^{n-1} + 25\cdot 3^{n-1} - 6\cdot 2^{n-2} - 30\cdot 3^{n-2}$$

$$= \left(5\cdot 2^{n-1} - 6\cdot 2^{n-2}\right) + \left(25\cdot 3^{n-1} - 30\cdot 3^{n-2}\right)$$

$$= \left(5\cdot 2^{n}\cdot 2^{-1} - 6\cdot 2^{n}\cdot 2^{-2}\right) + \left(25\cdot 3^{n}\cdot 3^{-1} - 30\cdot 3^{n}\cdot 3^{-2}\right)$$

$$= 2^{n}\cdot\left(\frac{5}{2} - \frac{6}{4}\right) + 3^{n}\cdot\left(\frac{25}{3} - \frac{30}{9}\right)$$

$$= 2^{n}\underbrace{\left(\frac{5}{2} - \frac{3}{2}\right)}_{1} + 3^{n}\underbrace{\left(\frac{25}{3} - \frac{10}{3}\right)}_{\frac{15}{3}=5}$$

$$= 2^{n} + 5\cdot 3^{n}$$

left = right. $a_n = 5a_{n-1} - 6a_{n-2}$. $n \geq 2$.

Fibonacci Sequence.

$f_0, f_1, f_2, \dots$ $f_0 = 0$, $f_1 = 1$, the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

Fibonacci seq:

0, 1, 1, 2, 3, 5, 8, 13, ...

2. Summation.

Geometric Series:

$\boxed{a + ar + ar^2 + \dots + ar^n}$ $(n+1)$ terms.

$\longrightarrow$ # of terms.

$$S_n = \frac{a\cdot(r^{n+1} - 1)}{(r-1)}$$

$$S_n = \sum_{k=0}^{n} ar^k = a\cdot r^0 + ar^1 + ar^2 + \dots + ar^n$$

$$S = \sum_{k=0}^{\infty} ar^k = ar^0 + ar^1 + ar^2 + \cdots \quad \text{(infinite series)}$$

$$S_n = \frac{a \cdot (\boxed{r^{n+1}} - 1)}{r - 1}$$

$n \to \infty$, $n+1 \to \infty$ $\quad r^{n+1}$

Ex. $r = 2$, $2^{n+1} = 2^{\infty} = \infty$

$r = \frac{1}{2}$ $\quad (\frac{1}{2})^{n+1} = (\frac{1}{2})^{\infty} = 0$

$|r| < 1 \quad r^{n+1} \to 0 \quad S = \sum_{k=0}^{\infty} ar^k$

$$= \frac{a \cdot (0 - 1)}{r - 1}$$

$$= \frac{a(-1)}{r-1}$$

infinite Geometric series when $|r|<1$ $\quad \boxed{S = \frac{a}{1-r}} \Leftarrow$

Ex. $\sum_{k=0}^{\infty} 3 \cdot (\frac{1}{4})^k = \frac{3}{1 - \frac{1}{4}} = \frac{3}{\frac{3}{4}} = 3 \cdot \frac{4}{3} = 4$

$\frac{1}{4} < 1$

$$\sum_{k=2}^{\infty} 3 \cdot (\tfrac{1}{4})^k = \frac{3 \cdot (\frac{1}{4})^2}{1 - \frac{1}{4}}$$

$3 \cdot (\frac{1}{4})^2 + 3(\frac{1}{4})^3 + 3(\frac{1}{4})^4 + \cdots$

$= \underbrace{\boxed{3 \cdot (\tfrac{1}{4})^2}}_{a} + \underbrace{\boxed{3(\tfrac{1}{4})^2} (\tfrac{1}{4})}_{a \cdot r} + a \cdot r^2 + \cdots$

$$= \frac{3 \cdot (\frac{1}{16})}{\frac{3}{4}}$$

$$= \frac{3}{16} \cdot \frac{4}{3} = \frac{1}{4}$$

$$\sum_{k=0}^{\infty} \frac{4^{k-1}}{5^{k+2}} = \sum_{k=0}^{\infty} \frac{4^k \cdot 4^{-1}}{5^k \cdot 5^2} = \sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^k \cdot \frac{\frac{1}{4}}{25}$$

$$= \sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^k \cdot \frac{1}{100}$$

$$= \frac{\frac{1}{100}}{1 - \frac{4}{5}} = \frac{\frac{1}{100}}{\frac{1}{5}} = \frac{1}{100} \cdot \frac{5}{1} = \frac{1}{20}$$

$|r| > 1 \quad \sum_{k=0}^{\infty} a \cdot r^k \Rightarrow \infty$ or $-\infty$ (Can't be determined)

$r = 1 \quad a, ar, ar^2, ar^3, ar^4, \ldots$

$\Rightarrow a, a, a, a, a, \ldots$

Ex. $\sum_{k=0}^{30} 5 = \underbrace{5+5+5+\cdots+5}_{31\text{ terms}} = 5\times 31 = 155$

$\sum_{k=4}^{21} 5 = \underbrace{5+5+\cdots+5}_{18} = 5\times 18 = 90$ # of terms: End term − Starting term + 1

21 − 4 + 1

$r=1$ $\sum_{k=0}^{n} a = a\cdot\underbrace{(n+1)}_{n+1\text{ terms.}}$

Ex. $\sum_{k=1}^{4} (5k^2 - 6k)$

Method 1 plug in.

$[5(1)^2 - 6(1)] + [5(2)^2 - 6(2)] + [5(3)^2 - 6(3)] + [5(4)^2 - 6(4)] = \square$

Method 2. $\sum_{j=0}^{n} (ax_j + by_j) = \sum_{j=0}^{n} ax_j + \sum_{j=0}^{n} by_j$

$= \boxed{a\cdot\sum_{j=0}^{n} x_j + b\cdot\sum_{j=0}^{n} y_j}$

$\sum_{k=1}^{4} (5k^2 - 6k)$

$= \sum_{k=1}^{4} 5k^2 - \sum_{k=1}^{4} 6k$

$= 5\cdot\sum_{k=1}^{4} k^2 - 6\cdot\sum_{k=1}^{4} k$

$= 5\cdot(1^2+2^2+3^2+4^2) - 6(1+2+3+4)$

$= 5\cdot(1+4+9+16) - 6(10)$

$= 5\cdot(30) - 60 = 150 - 60 = \boxed{90}$

$\sum_{k=1}^{10} 5k^2 - 6k = 5\sum_{k=1}^{10} k^2 - 6\cdot\sum_{k=1}^{10} k$

$= 5(1^2+2^2+\cdots+10^2) - 6(1+2+\cdots+10)$

Some useful summation formula.

① $\sum_{k=1}^{n} k = 1+2+3+\cdots+n.$ $\quad 1+2+3+\cdots+100$

$= (1+n)\cdot\frac{n}{2}$ or $\frac{n(n+1)}{2}$

$1+100=101$
$2+99=101$
$3+98=101$ } 50 pairs
$\vdots$

$101 \times 50 = 5050$

② $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)\cdot(2n+1)}{6}$

③ $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$

④ $\sum_{k=0}^{n} a\cdot r^k \quad \left(\begin{matrix} r\neq 0 \\ r\neq 1\end{matrix}\right) = \frac{a\cdot(r^{n+1}-1)}{r-1}$

$\sum_{k=0}^{\infty} a\cdot r^k \quad (|r|<1) = \frac{a}{1-r}$

If k start with other #s, can't use the formula.

$\sum_{k=1} k^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum k = \frac{n(n+1)}{2}$

go back to solve the example

$$\sum_{k=1}^{10} 5k^2 - 6k = 5\sum_{k=1}^{10} k^2 - 6\cdot\sum_{k=1}^{10} k$$

$$= 5(1^2+2^2+\cdots+10^2) - 6(1+2+\cdots+10)$$

$25\times7=175$

$$= 5\cdot\frac{\cancel{10}^{5}(11)(\cancel{21}^{7})}{\cancel{6}\,\cancel{3}\,1} - 6\cdot\frac{\cancel{10}^{5}\cdot(11)}{\cancel{2}\,1}$$

175×11

$$= \boxed{5\cdot5}\,11\cdot\boxed{7} - 6\cdot5\cdot11$$

1 9 2 5

$$= 1925 - 330$$

$$= 1595$$

Ex $\sum_{k=5}^{10} k^2 = \sum_{k=1}^{10} k^2 - \sum_{k=1}^{4} k^2 = 385 - \frac{\cancel{4}^{2}(5)\cdot(\cancel{9}^{3})}{\cancel{6}\,\cancel{3}\,1}$

$= 385 - 30$

$= 355$

$\underbrace{\underbrace{1, 2, 3, 4,}_{\sum_1^4} 5, 6, 7, 8, 9, 10}_{\sum_1^{10}}$

#17. a) $a_n = 3a_{n-1}$ $\boxed{a_0 = 2}$

Backward substitution:

$$
\begin{aligned}
a_n &= \boxed{3a_{n-1}} \\
&= 3\cdot(3a_{n-2}) \\
&= \boxed{3^2 a_{n-2}} \\
&= 3^2\cdot(3a_{n-3}) \\
&= \boxed{3^3 a_{n-3}} \\
&= 3^4 a_{n-4} \\
&\ \vdots \\
&= 3^n a_{n-n} \\
&= 3^n\cdot a_0
\end{aligned}
$$

Explicit formula $\boxed{a_n = 2\cdot 3^n}$

check.

$a_0 = 2$ $a_1 = 6$, $a_2 = 18$, ...

$a_2 = 2\cdot 3^2 = 2\cdot 9 = 18$