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MAT 120 **Final Exam**

Show relevant work, where appropriate, answers without support may receive little or no credit.

Total: 109 points

1. A collection of logical operators is said to be functionally complete if every compound statement is logically equivalent to one <u>involving only these operators</u>. The operator "↓" is defined as NOR (i.e. Not OR, $p{\downarrow}q = \neg(p \vee q)$). Show that the operator "↓" is functionally complete by expressing the following statements using only "↓." (12 points) $\neg p \wedge \neg p = \neg p$

$\sim p \Leftrightarrow$ $p \downarrow p$

$p \vee q \Leftrightarrow$ $(p\downarrow q)\downarrow(p\downarrow q)$ $\neg\neg(p\vee q) = \neg(p\downarrow q)$

$p \wedge q \Leftrightarrow$ $(p\downarrow p)\downarrow(q\downarrow q)$ $\neg(\neg p \vee \neg q)$

$= (\neg p)\downarrow(\neg q)$

$= (p\downarrow p)\downarrow(q\downarrow q)$

$p \rightarrow q \Leftrightarrow$ $[(p\downarrow p)\downarrow q]\downarrow[(p\downarrow p)\downarrow q]$ $\neg p \vee q$

$\neg\neg(\neg p \vee q)$

$= \neg(\neg p\downarrow q)$

$= (\neg p\downarrow q)\downarrow(\neg p\downarrow q)$

$= [(p\downarrow p)\downarrow q]\downarrow[(p\downarrow p)\downarrow q]$.

2. Determine the truth value of each of the following statements if the universe of discourse is the set of positive integers. Justify your answers! (9 points)

a) $\forall x \exists y (x^2 = y)$

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b) $\forall x \exists y (y^2 = x)$

F $x = 2, y^2 = 2. \quad y = \sqrt{2} \notin \mathbb{Z}$

c) $\forall x \exists y (x - y = 1)$

F $x = y + 1$

$x = 0, y = -1$

3. Find the sum of the series (6 points)

$$\sum_{n=0}^{\infty} \frac{5^n + 2^{n+1}}{7^{n-1}}$$

$$= \sum_{n=0}^{\infty} \frac{5^n}{7^n \cdot 7^{-1}} + \sum_{n=0}^{\infty} \frac{2^n \cdot 2}{7^n \cdot 7^{-1}}$$

$$= 7 \cdot \frac{1}{1 - \frac{5}{7}} + 14 \cdot \frac{1}{1 - \frac{2}{7}}$$

$$= \frac{441}{10}$$

4. For the following recurrence relations with the given initial conditions (12 points)
   a) Use iterative approach find $a_n = 3a_{n-1} - 8, a_0 = 10$
   b) Use mathematical induction to prove the formula obtained in a) is correct.

$a_n = 3a_{n-1} - 8$

$= 3(3a_{n-2} - 8) - 8$

$= 3^2 a_{n-2} - 3\times 8 - 8$

$= 3^3 a_{n-3} - 3^2\times 8 - 3\times 8 - 8$

$\vdots$

$= 3^n a_0 - 8(1+3+3^2+\cdots 3^{n-1})$

$= 3^n \cdot 10 - 8\cdot \frac{(1-3^n)}{1-3}$

$= 3^n \cdot 10 + 4(1-3^n)$

$= 6\cdot 3^n + 4$

$= 2\cdot 3^{n+1} + 4$

Basic step

pf. $a_0 = 10$

$a_n = 2\cdot 3^{0+1} + 4 = 10$

Inductive step

Assume $P(k) =$ True

$a_k = 2\cdot 3^{k+1} + 4$

Try to prove $a_{k+1} = 2\cdot 3^{k+2} + 4$

$a_{k+1} = 3a_k - 8$

$= 3\cdot(2\cdot 3^{k+1} + 4) - 8$

$= 2\cdot 3^{k+2} + 4$

$\frac{3}{4}$

$\boxed{3}$

5. (10 points) Suppose that a and b are integers, a ≡ 4(mod 7), and b ≡ 6(mod 7). Find the integer c with −6 ≤ c ≤ 6 such that (You may have multiple solutions.)

a) c ≡ 3a (mod 7)

$a - 4 = 7k \quad k \in \mathbb{Z}, \qquad b - 6 = 7p, \quad p \in \mathbb{Z}$

$a = 4 + 7k \qquad b = 6 + 7p$

$3a = 3(4+7k).$

$= 12 + 21k$

$c \equiv (12+21k) \pmod 7$

$c \bmod 7 = (12+21k) \bmod 7$

$c \bmod 7 = 5$

$c = 7x + 5$

$x = 0, \quad c = 5$

$x = -1 \quad c = -2$

b) $c \equiv a^2 - b^2 \pmod 7$

$c \bmod 7 = (a^2 - b^2) \bmod 7$

$= [(4+7k)^2 - (6+7p)^2] \bmod 7$

$= [16 + \cancel{2 \cdot 4 \cdot 7k} + \cancel{49k^2} - 36 \mp \cancel{2 \cdot 6 \cdot 7p} - \cancel{49p^2}] \bmod 7$

$= [-20] \bmod 7$

$$\begin{array}{r} -3 \\ 7\overline{)-20} \\ -21 \\ \hline 1 \end{array}$$

$c \bmod 7 = 1$

$C = 7\cancel{k} + 1$

$C = 1$

$C = -6$

6. (10 points)

a) Use the Euclidean Algorithm to find gcd(277,123).

gcd (277, 123)

$277 = 123 \times 2 + 31$

$123 = 31 \times 3 + 30$

$31 = 30 \times 1 + \boxed{1}$

$30 = 1 \times 30 + 0$

$31 = 277 - 123 \times 2$

$30 = 123 - 31 \times 3$

$1 = 31 - 30$

b) Express gcd (277,123) as a linear combination of 277 and 123.

~~1 = 277~~

$1 = 31 - (123 - 31 \times 3)$

$= -123 + 31 \times 4$

$= -123 + (277 - 123 \times 2) \times 4$

$= -123 + 277 \times 4 - 123 \times 8$

$= -123 \times 9 + 277 \times 4$

7. Do any 4 of the following 6 problems. (ie. Prove or disprove.) Write down the question number that you choose. (40 points)

a) If a and b are integers and $a + b$ is even, then $a^2 + b^2$ is even.

let $(a+b) = 2k$, $k \in \mathbb{Z}$

$$(a+b)^2 = (2k)^2$$
$$a^2+2ab+b^2 = 4k^2$$
$$a^2+b^2 = 4k^2-2ab$$
$$= 2(2k^2-ab)$$

$k \in \mathbb{Z}$, $a, b \in \mathbb{Z}$ $\quad \therefore a^2+b^2$ is even.

b) $3|(n^3 + 3n^2 + 2n)$ for $n \geq 1$

Basic step $n=1$ $\quad n^3+3n^2+2n = 1+3+2=6 \quad 3|6.$

Inductive step $P(k)$ is true $3 | k^3+3k^2+2k$

To prove $P(k+1)$

$$(k+1)^3+3(k+1)^2+2(k+1)$$
$$= k^3+3k^2+3k+1 + 3k^2+6k+3+2k+2$$
$$= (k^3+3k^2+2k) + (3k^2+9k+6)$$

$3 | k^3+3k^2+2k \quad 3 | 3(k^2+3k+2)$ ✓

c) $log_2 10$ is irrational.

Assume $\log_2 10$ is rational

$\log_2 10 = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$, a, b are simplified form

$$2^{\frac{a}{b}} = 10$$
$$2^a = 10^b$$
$$\underbrace{2^a}_{\text{even}} = \underbrace{2^b \cdot 5^b}_{\text{odd}}$$

Impossible.

d) $\sum_{i=1}^{n} f_i = f_{n+2} - 1$, where f_i is the i^{th} Fibonacci number.

Basic step. $f_1 = 1$ $f_3 - 1 = \cancel{2} - 1 = 1$

Inductive step. $P(1) \wedge P(2) \wedge \cdots \wedge P(k) = \text{True}$

$$\sum_{i=1}^{k+1} f_i = \sum_{i=1}^{k} f_i + f_{k+1}$$

$$= f_{k+2} - 1 + f_{k+1}$$

$$= f_{k+3} - 1$$

e) $2n + 3 \leq 2^n, \forall n \geq 4$, n is an integer.

Basic step $n = 4$ left: $8 + 3 = 11$ $11 \leq 16$

right: $2^4 = 16$

Inductive step: Assume $P(k) = \text{True}$

$2k + 3 \leq 2^k$

Try to prove $P(k+1)$

$$\boxed{\begin{aligned} 2(k+1) + 3 &\leq 2^{k+1} \\ 2k + 2 + 3 &\leq 2^k \cdot 2 \\ 2k + 3 &\leq 2^k \\ 2 &\leq 2^k \end{aligned}}$$

$2(k+1) + 3$

$= 2k + 2 + 3$

$= 2k + 3 + 2 \leq 2^k + 2 < 2^k + 2^k$ $k \geq 4$

$= 2 \cdot 2^k = 2^{k+1}$

$2(k+1) + 3 \leq 2^{k+1}$ ☒

f) $\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+a_n \;=S_n$

I. Find an expression for a_n, the n[th] term of the series. Include the domain for n

II. Find an expression for S_n, the sum of the series

III. Prove the formula for the sum using the method of induction.

(I). $a_n = \frac{1}{n} - \frac{1}{n+1}$ $\quad n \geq 1$

or $a_n = \frac{1}{n+1} - \frac{1}{n+2}$ $\quad n \geq 0$

II. $(1-\cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}) + \cdots + (\cancel{\tfrac{1}{n}} - \frac{1}{n+1})$

$= 1 - \frac{1}{n+1} = \frac{n}{n+1} = S_n$

Basic step

III $S_1 = \frac{1}{2}$ $\quad 1 - \frac{1}{2} = \frac{1}{2}$ ✓

Inductive step:

Assume $P(k) =$ True

$S_k = \frac{k}{k+1}$

Try to prove $S_{k+1} = \frac{k}{k+1} + \left(\frac{1}{k+1} - \frac{1}{k+2}\right)$

$= 1 - \frac{1}{k+2} = \frac{k+1}{k+2}$

$\boxed{}$

8. True or False. (The reason is not necessary) (10 points)

F a) $((p \to \neg q) \wedge q) \to \neg p$ is a tautology.

T b) $\forall x \in Z^+[x \neq 0 \to \exists y \in Q^+(xy = 2)]$

F c) $f(x) = x^3 + 5$ *is bijection. from Z to Z.*

F d) *If* $A = \{1,2,3,4,5\}, B = \{x \mid x \text{ is an integer and } x^2 \leq 25\}$, *then* $B \subseteq A$

T e) $(123 \bmod 19 + 342 \bmod 19) \bmod 19 = 9$

T f) If p and q are primes (>2), then $pq + 1$ is never primes.

F g) 14, 17, 85 are pairwise relatively prime

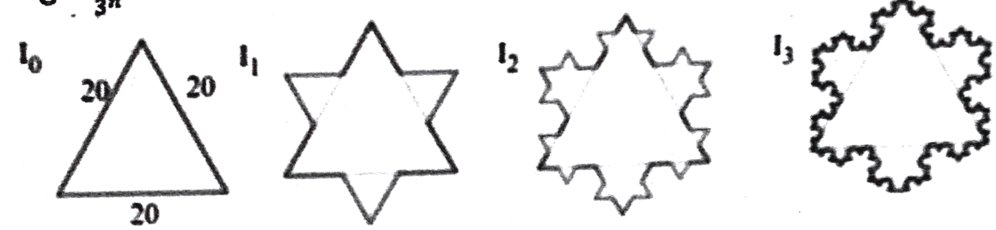
F h) $(763)_8 + (147)_8 = (1032)_8$

F i) If A is a 6 x 4 matrix and B is a 4 x 5 matrix, then AB has 16 entries.

F j) The premises "Some math majors left the campus for the weekend" and "All seniors left the campus for the weekend" imply the conclusion "Some seniors are math majors."

Bonus (10 points)

Let I_0 be an equilateral triangle with sides of length 20. The figure I_1 is obtained by replacing the middle third of each side of I_0 by a new outward equilateral triangle with sides of length $\frac{20}{3}$. The process is repeated where I_{n+1} is obtained by replacing the middle third of each side of I_n by a new outward equilateral triangle with sides of length $\frac{20}{3^n}$.



a) Let P_n be the perimeter of I_n. Find the explicit expression of P_n

b) Let A_n be the area of I_n. Find the explicit expression of A_n

a). $P_0 = 20 \times 3$

$P_1 = \left(\frac{20}{3}\right) \times 4 \times 3$

$P_2 = \left(\frac{20}{3^2}\right) \times 4^2 \times 3$

$P_n = \left(\frac{20}{3^n}\right) \times 4^n \times 3$

$= 60 \times \left(\frac{4}{3}\right)^n$

$n \to \infty \quad P_n \to \infty$

b) $A_0 = \frac{1}{2} \times 20 \times 10\sqrt{3} = 100\sqrt{3}$

$A_1 = 100\sqrt{3} + \frac{\sqrt{3}}{4}\left(\frac{20}{3}\right)^2 \cdot 3$

$A_2 = 100\sqrt{3} + \frac{\sqrt{3}}{4}\left(\frac{20}{3}\right)^2 \cdot 3 + \frac{\sqrt{3}}{4}\left(\frac{20}{3^2}\right)^2 \cdot 4 \cdot 3$

$\vdots$

$A_n = 100\sqrt{3} + \sum_{k=1}^{n} \frac{\sqrt{3}}{4}\left(\frac{20}{3^k}\right)^2 \cdot 4^{k-1} \cdot 3$

$\lim_{n\to\infty} A_n = 100\sqrt{3} + \sum_{n=1}^{\infty} 75\sqrt{3} \left(\frac{4}{9}\right)^n$

$= 100\sqrt{3} + \dfrac{75\sqrt{3} \cdot \frac{4}{9}}{1 - \frac{4}{9}} = 160\sqrt{3}$