Name: ____________________

Mat 120 **EXAM II**

Show relevant work, where appropriate, answers without support may receive little or no credit.

Total: 105 points

1. (11 points) Use the set operations or Venn diagram to solve the following question.

   In a survey of 60 people, it was found that:

   25 read *Newsweek* magazine, 26 read *Time,* 26 read *Fortune*

   9 read *Newsweek* and *Fortune,* 11 read *Newsweek* and *Time,* 8 read *Fortune* and *Time*

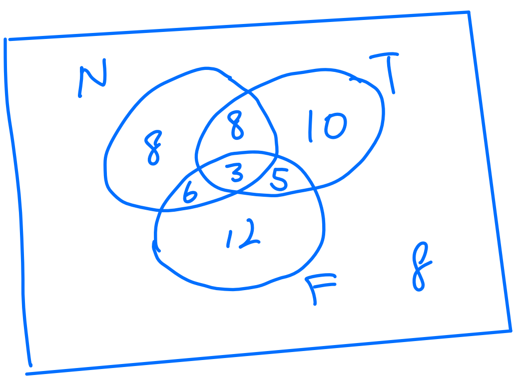
   3 read all

   Use Venn Diagram and label the numbers inside (5 points)

   (a) (2 point) Find the number of people who read none of these three magazines 8

   (b) (2 point) Find the number of people who read exactly one magazine. 30

   (c) (2 point) Find the number of who read at least one of the three magazines. 52



1−13

2,3,5,7,11,13

2,4,6,8,10,12

2. Let $U = \{n \in Z^+ \mid n \leq 13\}$ be the universal set and let $A = \{n \in U \mid n\ is\ prime\}$, $B = \{n \in U \mid n\ is\ even\}$, and $C = \{n \in U \mid n < 7\}$. List all of the elements in the following sets. (12 points)

1−6

a) $A \cap B$

$\{2\}$

b) $A - C$

$\{7, 11, 13\}$

c) $B \cup \bar{C}$ $\bar{C}: \{7, 8, 9, 10, 11, 12, 13\}$

$\{2, 4, 6, 7, 8, 9, 10, 11, 12, 13\}$

d) $\overline{A \cup B \cup C}$

$A \cup B \cup C: \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13\}$

$\overline{A \cup B \cup C} = \{9\}$

3. (12 points) Suppose that: $g: A \to B$ and $f: B \to C$ where $A = B = C = \{1,2,3,4\}$, $g = \{(1,4), (2,1), (3,1), (4,2)\}$, and $f = \{(1,3), (2,2), (3,4), (4,2)\}$ (i.e. f(1) =3, f(2)=2, f(3)=4, f(4)=2…)

a) Find $f \circ g$

$f \circ g = \{(1,3), (2,2), (4,2), (3,3)\}$

$f(g(1)) = f(4) = 2$ $f(g(3)) = f(1) = 3$

$f(g(2)) = f(1) = 3$ $f(g(4)) = f(2) = 2$

b) Does $g^{-1} \circ f$ exist? If it exists, find it. If it doesn't exist, explain why.

$g^{-1}(f(1)) = g^{-1}(3) = ?$ $g^{-1}(f(3)) = g^{-1}(4)$

g is Not 1−1 correspondence

$g^{-1}(f(2)) = g^{-1}(2)$ $g^{-1}(f(4)) = g^{-1}(2)$

c) Find $g \circ (g \circ g)$

$\{(1,1), (2,2), (3,2), (4,4)\}$

$g(g(g(1))) = g(g(4)) = g(2) = 1$

$g(g(g(2))) = g(g(1)) = g(4) = 2$

$g(g(g(3))) = g(g(1)) = g(4) = 2$

$g(g(g(4))) = g(g(2)) = g(1) = 4$

4. (13 points) Find the first five terms of the sequence a_n, $n \geq 0$

(a) $a_n = 2^n + (-2)^n$

$a_0 = 2^0 + (-2)^0 = 2$

$a_1 = 2^1 + (-2)^1 = 0$

$a_2 = 2^2 + (-2)^2 = 8$

$a_3 = 2^3 + (-2)^3 = 0$

$a_4 = 2^4 + (-2)^4 = 16+16 = 32$

(b) $a_n = \lfloor n/2 \rfloor + \lceil n/2 \rceil$

$a_0 = \lfloor 0 \rfloor + \lceil 0 \rceil = 0$

$a_1 = \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil = 1$

$a_2 = \lfloor 1 \rfloor + \lceil 1 \rceil = 2$

$a_3 = \lfloor \frac{3}{2} \rfloor + \lceil \frac{3}{2} \rceil = 1 + 2 = 3$

$a_4 = \lfloor 2 \rfloor + \lceil 2 \rceil = 4$

(c) $a_n = na_{n-1} + n^2 a_{n-2}, a_0 = 1, a_1 = 1$. (Find a_2, a_3, a_4)

$a_2 = 2a_1 + 2^2 a_0 = 2 + 4 = 6$

$a_3 = 3a_2 + 9a_1 = 3(6) + 9(1) = 27$

$a_4 = 4a_3 + 16a_2 = 4(27) + 16(6)$

$= 108 + 96$

$= 204$

5. (20 points) What are the values of these sums?

a) $\sum_{k=4}^{112} 2+\text{k}$

$112 - 4 + 1$
$= 108 + 1$
$= 109$

$= \sum_{k=4}^{112} 2 + \sum_{k=4}^{112} k$

$= 2(109) + \sum_{k=1}^{112} k - (1+2+3)$

$= 218 + \frac{112(113)}{2} - 6$

$= 218 + 6328 - 6$

$= 6540$

b) $\sum_{n=0}^{\infty} \frac{2^n+3^n}{5^n}$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

$$= \frac{1}{1-\frac{2}{5}} + \frac{1}{1-\frac{3}{5}}$$

$$= \frac{5}{3} + \frac{5}{2} = 5\left(\frac{5}{6}\right) = \frac{25}{6}$$

c) $\sum_{i=0}^{4} \sum_{j=2}^{5} i^3 j^2$

$$= \sum_{i=0}^{4} \left(4i^3 + 9i^3 + 16i^3 + 25i^3\right)$$

$$= \sum_{i=0}^{4} \left(54 i^3\right)$$

$$= 54\left(0^3 + 1^3 + 2^3 + 3^3 + 4^3\right)$$

$$= 54(1 + 8 + 27 + 64)$$

$$= 5400$$

d) $\sum_{j=0}^{9}(2^{j+1} - 2^j)$ (This is called the telescoping series. Hint: Instead of calculating all the numbers out, you may want to list all the terms first)

$$(\cancel{2^1} - 2^0) + (\cancel{2^2} - \cancel{2^1}) + (\cancel{2^3} - \cancel{2^2}) + \cdots (2^{10} - \cancel{2^9})$$

$$= 2^{10} - 2^0$$

$$= 1024 - 1 = 1023$$

e) $\sum_{i=2}^{9} 3(2^{i-1})$

$$\frac{a \cdot (r^{n+1} - 1)}{r-1}$$

$n+1 \rightarrow$ # of terms

$$= \frac{3 \cdot (2) \cdot [2^8 - 1]}{2-1}$$

$$= 1530$$

f) **Extra credits: (5 points)** Express $0.\overline{8}$ as a ratio of integers. ($0.\overline{8} = 0.8888888\ldots$ it is a repeated decimal number, but you can also rewrite $0.\overline{8} = 0.8 + 0.08 + 0.008 + \cdots$.)

$$0.\overline{8} = 0.8 + 0.8 \times \tfrac{1}{10} + 0.8 \times \left(\tfrac{1}{10}\right)^2 + \cdots$$
$$= \sum_{k=0}^{\infty} 0.8\left(\tfrac{1}{10}\right)^k$$
$$= \frac{0.8}{1-\frac{1}{10}} = \frac{8}{10-1} = \frac{8}{9}$$

g) **Extra credits: (5 points)** Simplify the following function to the simplest form.

$$\frac{1}{x-1} + \frac{\sum_{k=0}^{2020}(k+1)x^k}{\sum_{k=0}^{2021} x^k}$$

$$\frac{1}{x-1} + \frac{1+2x+3x^2+\cdots 2021x^{2020}}{1+x+x^2+x^3+\cdots+x^{2021}}$$

$$= \frac{(1+x+x^2+\cdots+x^{2021}) + (x-1)(1+2x+3x^2+\cdots 2021x^{2020})}{(x-1)(1+x+\cdots+x^{2021})}$$

$$= \frac{1+x+x^2+\cdots+x^{2021} + x+2x^2+3x^3+\cdots 2021x^{2021} - (1+2x+\cdots+2021x^{2020})}{(\cancel{x+x^2+x^3+\cdots}+x^{2022}) - (1+\cancel{x+\cdots+x^{2021}})}$$

$$= \frac{(\cancel{1+2x+3x^2+\cdots}\,2022x^{2021}) - (\cancel{1+2x+\cdots+2021x^{2020}})}{x^{2022}-1} = \frac{2022x^{2021}}{x^{2022}-1}$$

6. (9 points) Which of the following functions are: one-to-one, onto, both, or neither. Justify your answers. ($\mathbb{Z}^2$ *means* $m \in \mathbb{Z}, n \in \mathbb{Z}, \mathbb{Z}$ *is the set of all integers*.)

a) $f: N \to N, f(x) = 4x + 1$

Not onto $y=2$

one-to-one

b) $g: \mathbb{R} \to \mathbb{R}, g(x) = 2x + 1$

bijection

c) $p: \mathbb{Z}^2 \to \mathbb{Z}, p(x, y) = x^2 + y^2$

Not onto $p(x,y) = -1$

Not one-to-one $x=1, y=-1$

$x=-1, y=1$

7. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 4A + 7I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (Recall: $A^2 = A \times A$, I is the identity matrix.) (8 points)

$A^2 = A \cdot A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$

$4A = 4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix}$

$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$\begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8. (10 points) Use the iterative approach to find the solution to the recurrence relation with the given initial condition.

$$a_n = 3a_{n-1} + 4, a_0 = 1$$

$a_n = 3a_{n-1} + 4$

$= 3(3a_{n-2} + 4) + 4$

$= 3^2 a_{n-2} + 3 \cdot 4 + 4$

$= 3^3 a_{n-3} + 3^2 \cdot 4 + 3 \cdot 4 + 4$

$\vdots$

$= 3^n \cdot a_0 + 3^{n-1} \cdot 4 + 3^{n-2} \cdot 4 + \cdots + 3 \cdot 4 + 4$

$= 3^n(1) + \dfrac{4 \cdot (3^n - 1)}{(3-1)}$

$= 3^n + 2(3^n - 1)$

$= 3^n + 2 \cdot 3^n - 2$

$= 3^{n+1} - 2$

9. True or False. (Explanations are not required) (10 points)

a) $\text{f(x)} = x^3 + 1$ *is a bijection from* $\boldsymbol{R}$ *to* $\boldsymbol{R}$ T

b) $\text{f(x)} = \cos(x)$ *is a onto function but not one − to − one function for* $0 \leq x \leq 2\pi, -1 \leq f(x) \leq 1$.

T

c) $\text{f(x)} = \tan(x)$ *is a one − to − one but not onto for* $-\frac{\pi}{2} < x < \frac{\pi}{2}, f(x) \in R$.

F

d) Let $\text{f}: \text{R} \to \text{R}$ and let $\text{f(x)} > 0$ *for all* $x \in R$. *If* $f(x)$ *is strictly increasing then* $g(x) = \frac{1}{f(x)}$ *is strictly decreasing.*

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e) Assume that A is a subset of some underlying universal set U. Then $\text{A} \cup \text{U} = \text{U}$, $\text{A} \cap \text{U} = \text{A}$ and $\emptyset - \text{A} = \emptyset$. T

f) $\{0\} \subset \emptyset$

F

g) $\{\emptyset\} \subseteq \{\emptyset\}$ T

h) $2 \in \{\{2\}, \{\{2\}\}\}$

F

i) $A = \{a, b, c, d\}, B = \{y, z\}$, *then* $A \times B = B \times A$

F

j) If matrix $\begin{bmatrix} x & 1 & 3 \\ 1 & 0 & 3a - b \\ a + b & 4 & y \end{bmatrix}$ is a symmetric matrix, then $a = \frac{7}{4}, b = \frac{5}{4}$ T

$3 = a + b$
$4 = 3a - b$

$a = \frac{7}{4}$ $b = \frac{5}{4}$