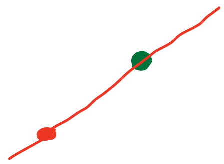
§ 1.7. Proof

1. Differences between Axiom, Theorem, Corollary, lemma

1) Axiom: a true mathematical statement whose truth is accepted without proof.

Ex. To get a unique line, we'll need two different points.



2) Theorem: A true mathematical statement whose truth can be verified (proved) also use it for something significant or interesting.

Ex.

4-Color theorem: (textbook. page 763. Theorem 1)

3) Corollary: A Mathematical result that can be deducted from, and is thereby a consequence of some earlier results

$1+1=2, \quad 2+2=4, \quad 3+3=6, \quad \Rightarrow \quad n+n=2n$

4). Lemma: A Mathematical result that is useful in establishing the truth of some other result.

$\boxed{A} \longrightarrow \boxed{\text{lemma}} \longrightarrow \boxed{B}$

2. Counter example (disprove the statement)

$(\forall x)\ p(x) \to Q(x) = \text{True}$ which means $p(x) \to Q(x) = T$ for each $x \in D$

if $p(x) \to Q(x) = \text{False}$ for at least 1 $x \in D$, then $\forall x\ p(x) \to Q(x) = F$.

Ex. ① If $x \in \mathbb{R}$, then $(x^2-1)^2 > 0$

False. $x = 1$ or $x = -1$ $\Rightarrow (x^2-1)^2 = 0$

② $p(x) = 3x^2 - 4x + 1$ is even where $x \in \mathbb{Z}^+$

$p(1) = 3 - 4 + 1 = 0$ even

$p(2) = 3(2)^2 - 4(2) + 1 = 12 - 8 + 1 = 5$ odd!

Counter example $\Rightarrow$ $p(x) = 3x^2 - 4x + 1$ is even where $x \in \mathbb{Z}^+$ is False!

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1 case = T can NOT prove the whole statement.

3. Trivial v.s. Vacuous.

1) Trivial pf $\forall x\ P(x) \to Q(x)$ $Q(x) = \text{True}$. $? \to T = T$

Ex. let $n \in \mathbb{Z}$, if $\underbrace{n^3 > 0}_{?}$, then $\underbrace{3 \text{ is odd}}_{T}$. $? \to T = \text{True}$

By trivial pf, the statement is true.

2). Vacuous. pf $\forall x\ P(x) \to Q(x)$ $P(x) = \text{False}$ $F \to ? = T$

Ex. let $n \in \mathbb{Z}$, if $\underbrace{3 \text{ is even}}_{F}$, then $\underbrace{n^3 > 0}_{?}$. $F \to ? = T$

By vacuous ~~pf~~, the statement is true.

3. Direct Proof

$\forall x\ P(x) \to Q(x)$, Assume $P(x) = \text{True}$ for some arbitrary x.,

show $Q(x) = \text{True}$ for this element x.

*** Assume you are familiar with the following properties:

① The negative of every integer is still an integer.

Ex. $x \in \mathbb{Z}$, $-x \in \mathbb{Z}$

② The sum(difference) of every two integers is integer.

Ex. $x + 7$ $x \in \mathbb{Z}$, $7 \in \mathbb{Z} \Rightarrow x + 7 \in \mathbb{Z}$.

③ The product of every two integers is an integer.

Ex. $3 \in \mathbb{Z}$, $x \in \mathbb{Z}$, $\Rightarrow 3 \cdot x \in \mathbb{Z}$