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MAT 120 **Final Exam**

Show relevant work, where appropriate, answers without support may receive little or no credit.

Total: 109 points

1. A collection of logical operators is said to be functionally complete if every compound statement is logically equivalent to one involving only these operators. The operator "↓" is defined as NOR (i.e. Not OR, $p \downarrow q = \neg(p \vee q)$). Show that the operator "↓" is functionally complete by expressing the following statements using only "↓." (12 points)

$\sim p \Leftrightarrow p \downarrow p$ $\qquad p \downarrow p \equiv \neg(p \vee p) \overset{\text{Idemp.}}{\equiv} \neg p$

$p \vee q \Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)$ $\qquad (p \downarrow q) \downarrow (p \downarrow q) \equiv$

$\neg((p \downarrow q) \vee (p \downarrow q))$

$\overset{\text{Idemp.}}{\equiv} \neg(p \downarrow q) \equiv p \vee q$

$p \wedge q \Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)$ $\qquad (p \downarrow p) \downarrow (q \downarrow q) \equiv \neg p \downarrow \neg q$

$\equiv \neg(\neg p \vee \neg q)$

$\overset{\text{DM}}{\equiv} p \wedge q$

$p \to q \Leftrightarrow [(p \downarrow p) \downarrow q] \downarrow ((p \downarrow p) \downarrow q)]$

$(p \downarrow p) \equiv \neg p$ so $[\neg p \downarrow q] \downarrow [\neg p \downarrow q]$

$\equiv \neg[(\neg p \downarrow q) \vee (\neg p \downarrow q)] \overset{\text{Idemp.}}{\equiv} \neg(\neg p \downarrow q)$

$\equiv \neg p \vee q \overset{\text{CE}}{\equiv} p \to q$

$U = \{1, 2, 3, \ldots\}$

2. **Determine the truth value of each of the following statements if the universe of discourse is the set of positive integers. Justify your answers! (9 points)**

   a) $\forall x \exists y (x^2 = y)$

True, b/c every integer has a square that is also an integer. For ~~any~~ x chosen you can find a $y \in \mathbb{Z}^+$. For example, $3^2 = 9$, $5^2 = 25$, $10^2 = 100$, ...

   b) $\forall x \exists y (y^2 = x)$

False, b/c not every integer has a square root that is an integer (not every integer is a perfect square). For example if $x = 3$, then $y^2 = 3$, $y = \pm\sqrt{3} \notin \mathbb{Z}^+$.

   c) $\forall x \exists y (x - y = 1)$

~~True, b/c for every x, if $y = x - 1$, then $x - (x-1) \stackrel{?}{=} 1 \to 1 \stackrel{\checkmark}{=} 1$ is satisfied.~~

False, b/c for example if $x = 1$, only $y = 0$ satisfies but $0 \notin \mathbb{Z}^+$.

3. **Find the sum of the series (6 points)**

$$\sum_{n=0}^{\infty} \frac{5^n + 2^{n+1}}{7^{n-1}}$$

$$\sum_{n=0}^{\infty} \frac{5^n}{7^{n-1}} + \sum_{n=0}^{\infty} \frac{2^{n+1}}{7^{n-1}} = \sum_{n=0}^{\infty} \frac{5^n}{7^n(7^{-1})} + \sum_{n=0}^{\infty} \frac{2^n(2)}{7^n(7^{-1})} =$$

$$= 7\sum_{n=0}^{\infty} \left(\frac{5}{7}\right)^n + 14\sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n = 7\left(\frac{1}{1-\frac{5}{7}}\right) + 14\left(\frac{1}{1-\frac{2}{7}}\right) =$$

$$7\left(\frac{7}{2}\right) + 14\left(\frac{7}{5}\right) = \frac{49}{2} \cdot \frac{245}{10} + \frac{98}{5} \cdot \frac{196}{10} = \frac{441}{10}$$

4. For the following recurrence relations with the given initial conditions (12 points)
   a) Use iterative approach find $a_n = 3a_{n-1} - 8, a_0 = 10$
   b) Use mathematical induction to prove the formula obtained in a) is correct.

(a) $a_n = 3(3a_{n-2} - 8) - 8 = 3(3[3a_{n-3} - 8] - 8) - 8 = 3^3 a_{n-3} - [8 + 8(3) + 8(3)^2]$

$= \ldots = 3^n a_{n-n} - 8[1 + 3 + 3^2 \ldots + 3^{n-1}] = 3^n(a_0) - 8\left(\frac{1-3^n}{1-3}\right) =$

$3^n(10) - 4(3^n - 1) = 6 \cdot 3^n + 4 = 2 \cdot 3^{n+1} + 4$

MI proof

Basic step: $n = 0 \to a_0 \overset{\text{left}}{=} 10$ $\quad$ $\overset{\text{right}}{2 \cdot 3^{0+1} + 4} = 6 + 4 = 10 \to$ Left = right

So the basic step is proved

Inductive step: Assume $a_k = 2 \cdot 3^{k+1} + 4$

Is $a_{k+1} \overset{?}{=} 2 \cdot 3^{k+2} + 4$ true?

$a_{k+1} = 3 a_k - 8 = 3(2 \cdot 3^{k+1} + 4) - 8 = 2 \cdot 3^{k+2} + 12 - 8 = 2 \cdot 3^{k+2} + 4$

by inductive hypothesis

So $a_{k+1} = 2 \cdot 3^{k+2} + 4$

So the inductive step is proved

By the principle of MI, the statement is true.

5. (10 points) Suppose that a and b are integers, a ≡ 4(mod 7), and b ≡ 6(mod 7). Find the integer c with −6 ≤ c ≤ 6 such that (You may have multiple solutions.)

   a) c ≡ 3a (mod 7)

$a - 4 = 7k \to a = 7k + 4$, $b - 6 = 7p \to b = 7p + 6$

$c \equiv [3(7k+4)] \pmod 7$

$c \bmod 7 = (7(3k) + 12) \pmod 7$

$= 12 \bmod 7 = 5$

$c \bmod 7 = 5 \to c = 7m + 5 \to -6 \le c \le 6 \to c = 5, -2$

b) $c \equiv a^2 - b^2 \pmod{7}$

$c \equiv [(7k+4)^2 - (7p+6)^2] \pmod{7}$

$c \bmod 7 = [\cancel{7^2k^2} + \cancel{2(7k)(4)} + 16 - (\cancel{7^2p^2} + \cancel{2(7p)(6)} + 36] \pmod{7}$

$= -20 \bmod 7 = 1$

$c \bmod 7 = 1 \quad -6 \le c \le 6 \rightarrow c = 1, -6$

$c = 7l + 1$

6. (10 points)

a) Use the Euclidean Algorithm to find gcd(277,123).

$277 = 123(2) + 31$

$123 = 31(3) + 30$

$31 = 30(1) + \boxed{1} \rightarrow \gcd(227, 123) = 1$

$30 = 1(30) + 0$

b) Express gcd (277,123) as a linear combination of 277 and 123.

$1 = 31 - 30 = 31 - (123 - 31(3)) =$

$= 4(31) - 123 = 4 \cdot (277 - 123(2)) - 123$

$= 4 \cdot 277 - 9 \cdot 123$

$30 = 123 - 31(3)$

$31 = 277 - 123(2)$

7. Do any 4 of the following 6 problems. (ie. Prove or disprove.) Write down the question number that you choose. (40 points)

a) If a and b are integers and $a + b$ is even, then $a^2 + b^2$ is even.

Direct proof:

If $a+b$ is even $\to a+b = 2k \to (a+b)^2 = (2k)^2 \to a^2+2ab+b^2 = 4k^2$

$a^2+b^2 = 4k^2 - 2ab = 2(2k^2-ab) = 2M$ where $M = 2k^2-ab \in \mathbb{Z}$ since $k, a, b \in \mathbb{Z}$

So by the definition of even,

a^2+b^2 is even

So the statement is true.

b) $3|(n^3 + 3n^2 + 2n)$ for $n \geq 1$

Try proof by MI

Basic step: $n=1 \to 3 \overset{?}{|} (1^3+3(1)^2+2(1)) \to 3 \overset{?}{|} (1+3+2) \to 3|6$ ✓ So the basic step is proved

Inductive step: Assume $3|(k^3+3k^2+2k) \to 3m = k^3+3k^2+2k$

Is $3 \overset{?}{|} ((k+1)^3+3(k+1)^2+2(k+1))$ true?

$(k+1)^3+3(k+1)^2+2(k+1) = k^3+3k^2+3k+1+3(k^2+2k+1)+2(k+1) =$

$k^3+3k^2+3k+1+3k^2+6k+3+2k+2 = k^3+6k^2+11k+6 = \longrightarrow$ To the back

c) $log_2 10$ is irrational.

(b)

$= k^3 + 3k^2 + 2k + 3k^2 + 9k + 6$

$= 3m + 3(k^2 + 3k + 2)$

by inductive hypothesis

$= 3(m + k^2 + 3k + 2)$

$= 3N$ where $N = m + k^2 + 3k + 2 \in \mathbb{Z}$ since $m, k \in \mathbb{Z}$

which is divisible by 3,

so $3 \mid (k+1)^3 + 3(k+1)^2 + 2(k+1)$ ✓

So the inductive step is proved.

By the principle of MI, the statement is true.

0, 1, 1, 2, 3, 5, —

d) $\sum_{i=1}^{n} f_i = f_{n+2} - 1$, where f_i is the i^{th} Fibonacci number.

Try proof by SMI

Basic step: $n = 1 \rightarrow$ left: $\sum_{i=1}^{1} f_i = f_1 = 1$ | right: $f_{1+2} - 1 = f_3 - 1 = 2 - 1 = 1$

left = right ✓ So the basic step is proved

Inductive step: Assume $P(1) \wedge P(2) \wedge \ldots \wedge P(k-1) \wedge P(k)$ are true.

In particular, $P(k) \Rightarrow \sum_{i=1}^{k} f_i = f_{k+2} - 1 \rightarrow f_{k+2} = \sum_{i=1}^{k} f_i + 1$

$P(k-1) \Rightarrow \sum_{i=1}^{k-1} f_i = f_{k+1} - 1 \rightarrow f_{k+1} = \sum_{i=1}^{k-1} f_i + 1$

Is $P(k+1)$ true? i.e. is $\sum_{i=1}^{k+1} f_i \overset{?}{=} f_{k+3} - 1$ true?

$f_{k+3} = f_{k+2} + f_{k+1} = \left(\left(\sum_{i=1}^{k} f_i\right) + 1\right) + \left(\left(\sum_{i=1}^{k-1} f_i\right) + 1\right) =$ to back

By defn. of Fibonacci

by inductive hypothesis

e) $2n + 3 \leq 2^n, \forall n \geq 4$, n is an integer.

Try proof by MI

Basic step: $n = 4 \rightarrow$ left: $2(4) + 3 = 11$ | right: $2^4 = 16$ $\rightarrow$ left < right ✓

So the basic step is proved

Inductive step: Assume $2k + 3 \leq 2^k$

Is $2(k+1) + 3 \overset{?}{\leq} 2^{k+1}$ true?

$2(k+1) + 3 = 2k + 5$

$= 2k + 3 + 2 \leq 2^k + 2^1$

by inductive hypothesis

$= 2^k + 2^1$

$2^{k+1} = 2^k \cdot 2 = 2^k + 2^k \geq 2^k + 2^3$

$\geq 2^k + 2^1$

So $2(k+1) + 3 \leq 2^k + 2^1 \leq 2^{k+1}$

So $2(k+1) + 3 \leq 2^{k+1}$

So the inductive step is proved.

By the principle of MI, the inequality is true.

(d)

$$= f_1 + f_2 + f_3 + \cdots + f_k + 1 + f_1 + f_2 + \cdots + f_{k-1} + 1$$

$$= f_1 + [f_2 + f_1] + [f_3 + f_2] + \cdots + [f_k + f_{k-1}] + 2$$

$$= 1 + 2 + f_3 + f_4 + \cdots + f_{k+1}$$

$$= 1 + 1 + f_3 + f_4 + \cdots + f_{k+1} + 1$$

$$= f_1 + f_2 + f_3 + f_4 + \cdots + f_{k+1} + 1$$

$$= \sum_{i=1}^{k+1} f_i + 1$$

So $f_{k+3} = \sum_{i=1}^{k+1} f_i + 1 \Rightarrow \sum_{i=1}^{k+1} f_i = f_{k+3} - 1$ ✓

So the inductive step is proved

So by the principle of SMI, the statement is proved.

f) $\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+a_n = S_n$

I. Find an expression for a_n, the n[th] term of the series. Include the domain for n

II. Find an expression for S_n, the sum of the series

III. Prove the formula for the sum using the method of induction.

8. True or False. (The reason is not necessary) (10 points)

a) $((p \to \neg q) \wedge q) \to \neg p$ is a tautology.

False

b) $\forall x \in Z^+[x \neq 0 \to \exists y \in Q^+(xy = 2)]$

True

c) $f(x) = x^3 + 5$ *is bijection. from Z to Z.*

False

d) *If* $A = \{1,2,3,4,5\}, B = \{x \mid x$ *is an integer and* $x^2 \leq 25\}$, *then* $B \subseteq A$

False

e) $(123 \bmod 19 + 342 \bmod 19) \bmod 19 = 9$

True

f) If p and q are primes (>2), then $pq + 1$ is never primes.

True

g) 14, 17, 85 are pairwise relatively prime

False

h) $(763)_8 + (147)_8 = (1032)_8$

False

i) If A is a 6 x 4 matrix and B is a 4 x 5 matrix, then AB has 16 entries.

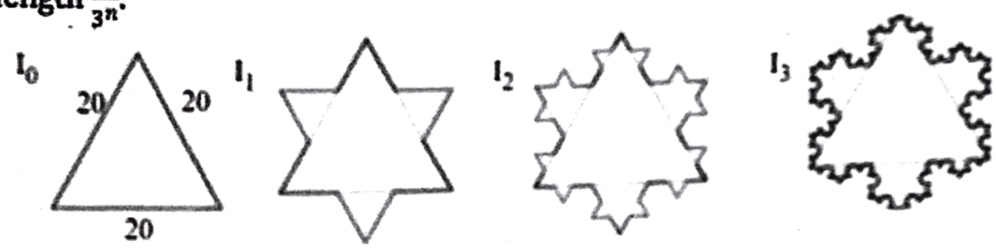
False

j) The premises "Some math majors left the campus for the weekend" and "All seniors left the campus for the weekend" imply the conclusion "Some seniors are math majors."

False

Bonus (10 points)

Let I_0 be an equilateral triangle with sides of length 20. The figure I_1 is obtained by replacing the middle third of each side of I_0 by a new outward equilateral triangle with sides of length $\frac{20}{3}$. The process is repeated where I_{n+1} is obtained by replacing the middle third of each side of I_n by a new outward equilateral triangle with sides of length $\frac{20}{3^n}$.



a) Let P_n be the perimeter of I_n. Find the explicit expression of P_n

b) Let A_n be the area of I_n. Find the explicit expression of A_n