§ 1.7. HW # 27.

Pf. Assume there is a rational root r

by def of rational #: $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

simplifiest form

$r^3 + r + 1 = 0$

left: $(\frac{a}{b})^3 + (\frac{a}{b}) + 1 = 0$

$\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0$ multiply by b^3

⇓ $a^3 + ab^2 + b^3 = 0$

Case 1. if $a = 0$

$a^3 + ab^2 + b^3 = 0 + 0 + b^3 \neq 0$

$b \neq 0$, $b^3 \neq 0$

this is a Contradiction.

Case 2. if $a \neq 0$

$a^3 \neq 0$, $ab^2 \neq 0$, $b^3 \neq 0$

$a^3 + ab^2 + b^3 \neq 0$

this is a Contradiction.

there is no rational root for $r^3 + r + 1 = 0$

$a = 2k$

$a^3 = (2k)^3 = 8k^3 = 2 \cdot (4k^3)$

textbook:

Case 1. a is even, b is odd.

a^3 is even , ab^2 = even

b^3 is odd

even + even + odd = odd ≠ 0

Case 2. a is odd, b is odd

a^3 is odd ab^2 is odd

b^3 is odd.

$\underbrace{\underbrace{\text{odd} + \text{odd}}_{\text{even}} + \text{odd}}_{\text{odd}} \neq 0$

⋮

$\mathbb{Z}$:

even	odd

−	0	+

§1.8 Proof by Strategies.

1. proof by cases.

If we can prove each case is true, then the whole thing is true!

Ex. If $n \in \mathbb{Z}$, then $n^2 + 3n + 5$ is odd. $= 2M + 1$

Pf. Case 1. let n is even

$n = 2k$, $k \in \mathbb{Z}$

$n^2 + 3n + 5 = (2k)^2 + 3(2k) + 5$

$$= 4k^2 + 6k + \underset{\wedge}{5}$$

$$= \underbrace{4k^2 + 6k + 4} + 1$$

$$= 2(2k^2+3k+2)+1$$, since $k \in \mathbb{Z}$, $2k^2+3k+2 = M \in \mathbb{Z}$

$= 2M+1$ is odd.

case 2. let n is odd.

$n = 2k+1$, $k \in \mathbb{Z}$

$$n^2+3n+5 = (2k+1)^2+3(2k+1)+5$$

$$= 4k^2+4k+1+6k+3+5$$

$$= \underbrace{4k^2+10k+8}+1$$

$$= 2(2k^2+5k+4)+1$$ $k \in \mathbb{Z}$, $2k^2+5k+4 = M \in \mathbb{Z}$

$= 2M+1$. is odd

▨

Ex. Prime numbers: an integer greater than 1 whose only factors are 1 and itself.

$\Rightarrow \overset{\text{even}}{2}, \underbrace{3, 5, 7, 11, 13, 17, 19, 23, 29, \cdots}_{\text{odd}}$

Belief: there are infinitely many prime numbers.

Need to prove it!

Contradiction! Assume there are finite prime numbers.

"Complete" list for the prime #s. : 2, 3, 5, 7, 11, ··· N

where "N" is the $\boxed{\text{"largest"}}$ prime #. ⟷ Strategy: try to find another $\underbrace{\text{prime \#}}_{x}$ that is $> N$.

prime ✓

Composite

$$X = \underbrace{\underbrace{2 \times (3 \times 5 \times 7 \times 11 \times \cdots \times N)}_{\text{Composite \# and Even!}} + 1}_{\text{odd}}$$

odd (there is a possibility that x is prime and x > N)

Case 1. if x is prime #.

then $x > N$ it contradicts to N being largest prime #.

Case 2. if x is a composite # (You can find other factors) → are prime if factor completely.

$$18 < \begin{matrix} 2 \\ 9 < \begin{matrix} 3 \\ 3 \end{matrix} \end{matrix}$$

$18 = 2(3^2)$

① $$\frac{X}{2} = \frac{\overbrace{(2)(3)(5)\cdots}\,(N) + \textcircled{1}}{2}$$

$$\begin{array}{r} 0 \\ 2\overline{)\,1} \\ \underline{0} \\ 1 \end{array} \Rightarrow \text{remainder} = 1$$

$$\frac{X}{2} = \underbrace{\frac{(\cancel{2})(3)(5)\cdots(N)}{\cancel{2}}}_{\text{integer}} + \underbrace{\frac{1}{2}}_{\text{remainder}=1} \Rightarrow \text{remainder} = 1$$

$$\frac{X}{3} = \frac{(2)(3)(5)\cdots(N)+1}{3} = \underbrace{\frac{(2)(\cancel{3})\cdots(N)}{\cancel{3}}}_{r=0} + \underbrace{\frac{1}{3}}_{r=1} \Rightarrow \text{remainder} = 1$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\frac{X}{N} = \frac{(2)(3)\cdots(\cancel{N})+1}{N} = \underbrace{\frac{(2)(3)\cdots(\cancel{N})}{\cancel{N}}}_{r=0} + \underbrace{\frac{1}{N}}_{r=1} \Rightarrow \text{remainder} = 1$$

It reaches the contradiction, X is not a composite, x is prime.

(When x divided by any of $\{2,3,5,\cdots N\}$, remainder $=0$)
but this is not the case, we have $r=1$ for "all" possible prime #s.

$X > N$ since $x = 2\cdot3\cdot5\cdots N + 1$, then there is

No Complete list of prime #s.

$\Rightarrow$ there are infinitely many prime #s. ▨

2. Backward Reasoning (Mostly in inequality)

Ex. If $x, y \in \mathbb{R}$, then $\boxed{\frac{1}{3}x^2 + \frac{3}{4}y^2 \geq xy.}$

If n is even, then n^2 is even
Forward reasoning (direct proof)

Backward reasoning: (Not an official pf)

$$\frac{1}{3}x^2 + \frac{3}{4}y^2 \geq xy$$
$$-xy \qquad -xy$$

$$12\left(\frac{1}{3}x^2 + \frac{3}{4}y^2 - xy\right) \geq (0)\,12$$

$$\Rightarrow 4x^2 + 9y^2 - 12xy \geq 0 \ ✓$$

$$(2x)^2 - 12xy + (3y)^2 \geq 0$$

$$(2x - 3y)^2 \ \boxed{\geq 0} \Leftarrow \text{the start for my proof.}$$

$a = 2x$
$b = 3y$
$2ab = 2\cdot2x\cdot3y$
$= 12xy$

$(a+b)^2 = (a+b)(a+b)$

formula: $a^2 + 2ab + b^2$

$\Rightarrow (a-b)^2 = a^2 - 2ab + b^2$

⇒ Use backward reasoning to do the official pf. .

⇒ Pf. Since $x, y, \in \mathbb{R}$. $(2x-3y)^2 \geq 0$

Divide by 12

$$\frac{4x^2}{12} - \frac{12xy}{12} + \frac{9y^2}{12} \geq \frac{0}{12}$$

$$\frac{x^2}{3} - xy + \frac{3y^2}{4} \geq 0$$

$$+xy \qquad\qquad +xy$$

$$\frac{x^2}{3} + \frac{3y^2}{4} \geq xy$$

▨.

Ex. If $x, y \in \mathbb{R}^+$, $\frac{(x+y)}{2} > \sqrt{xy}$