

(described in 18/09/17)

(modified on 3/10/17)

Non-dimensional Equations of motion are:

$$\ddot{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \left(1 - \frac{1}{A}\right) \mathcal{V}_1 + \kappa \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \left(1 - \frac{1}{B}\right) \mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} \quad (1)$$

Where:

$$\mathcal{X} = \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix}$$

$$A = \sqrt{(h_p \sin(\theta_p) + l_p \cos(\theta_p) - x_p + x_1)^2 + (-h_p \cos(\theta_p) + l_p \sin(\theta_p) - y_p + y_1)^2}$$

$$B = \sqrt{(h_p \sin(\theta_p) - l_p \cos(\theta_p) - x_p + x_2)^2 + (-h_p \cos(\theta_p) - l_p \sin(\theta_p) - y_p + y_2)^2}$$

$$\mathcal{V}_1 = \begin{pmatrix} \sin(\theta_p) h_p + \cos(\theta_p) l_p + x_1 - x_p \\ -\cos(\theta_p) h_p + \sin(\theta_p) l_p + y_1 - y_p \\ l_p (\cos(\theta_p) (y_1 - y_p) - \sin(\theta_p) (x_1 - x_p)) + h_p (\cos(\theta_p) (x_1 - x_p) + \sin(\theta_p) (y_1 - y_p)) \end{pmatrix}$$

$$\mathcal{V}_2 = \begin{pmatrix} \sin(\theta_p) h_p - \cos(\theta_p) l_p + x_2 - x_p \\ -\cos(\theta_p) h_p - \sin(\theta_p) l_p + y_2 - y_p \\ h_p (\cos(\theta_p) (x_2 - x_p) + \sin(\theta_p) (y_2 - y_p)) + l_p (\sin(\theta_p) (x_2 - x_p) + \cos(\theta_p) (y_p - y_2)) \end{pmatrix}$$

$$\alpha = \frac{m_p L0_1^2}{I_p} = \frac{3L0_1^2}{(w_p^2 + h_p^2)} \quad *$$

$$\kappa = \frac{k_2}{k_1}$$

$$\mathcal{L} = \frac{L0_2}{L0_1}$$

$$\gamma = \frac{g}{L0_1} \frac{m_p}{k_1}$$

* (assuming Inertia of a rectangular payload)

All dimensional quantities are originally normalized by :

$$\omega_s^2 = \frac{k_1}{m_p}$$

A symmetrical case is considered and it is:

$$k_2 \rightarrow k_1 \quad ; \quad L_{02} \rightarrow L_{01}$$

*Note: typical values that are intended to be considered are :

$$k_1 \rightarrow 200, L_{01} \rightarrow 2$$

$$m_p \rightarrow 2, h_p \rightarrow 0.1, w_p \rightarrow 1,$$

$$g \rightarrow 9.81$$

$$x_1[t] \rightarrow 0$$

$$y_1[t] \rightarrow 0$$

$$x_2[t] \rightarrow 2w_p$$

$$y_2[t] \rightarrow y_1[t]$$

Equilibrium point is :

$$\mathcal{X} = \begin{pmatrix} w_p \\ -(\frac{1}{2}\gamma + h_p + 1) \\ 0 \end{pmatrix}$$

Linearizing about that equilibrium point:

$$x_p \rightarrow \delta x + x_{p_0}$$

$$y_p \rightarrow \delta y + y_{p_0}$$

$$\theta_p \rightarrow \delta \theta + \theta_{p_0}$$

...

After all calculations we can describe **the linearized equations as:**

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}$$

Where :

$$\mathbf{M} = \begin{pmatrix} (\gamma + 2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\gamma + 2) \end{pmatrix}$$

$$\mathbf{F} = \{\mathbf{0}\}, \mathbf{C} = \{\mathbf{0}\}$$

$$\mathbf{K} = \begin{pmatrix} 2\gamma & 0 & -2\gamma h_p \\ 0 & 2 & 0 \\ -2\alpha\gamma h_p & 0 & \alpha(2\gamma h_p^2 + \gamma(\gamma + 2)h_p + 2(\gamma + 2)w_p^2) \end{pmatrix}$$

The natural frequencies can be calculated from :

$$\text{Det}[\mathbf{K} - \omega^2 \mathbf{M}] = 0$$

For the y component we get :

$$\omega_y^2 = 2$$

Other frequencies are:

$$\omega_{1,2} = \pm \sqrt{\frac{\frac{2\gamma}{\rho} + \alpha\gamma h_p + \frac{2\alpha\gamma h_p^2}{\rho} + 2\alpha w_p^2 \pm \sqrt{-4\rho(2\alpha\gamma^2 h_p + 4\alpha\gamma w_p^2) + (-2\gamma - \alpha\gamma\rho h_p - 2\alpha\gamma h_p^2 - 2\alpha\rho w_p^2)^2}}{\rho}} \sqrt{2}$$

It can be shown (detailed later) that , for certain parameters, we get :

$$\omega_1 \sim \omega_x < \sqrt{2} \quad ; \quad \omega_2 \sim \omega_\theta \gg \omega_y = \sqrt{2}$$

For the case of $\omega_\theta \gg \omega_y$, I would like to neglect θ by setting $\ddot{\theta} = \dot{\theta} \cong 0 \rightarrow \theta = \theta_0 + \delta\theta$, where $\theta_0 = 0$. It will remain true for whole the rest of the problem.

Back to the starting 3D.O.F E.O.M (1) I will isolate $\theta = \delta\theta$ using the 3rd equation, in order to get $\theta = f(x, y)$ and use it back in 1st two equations of x,y. and by that getting 2D.O.F dynamics to investigate asymptotically.

The procedure is like this :

From 3rd equation of (1) :

$$0 = (-\alpha) \left(1 - \frac{1}{A}\right) \mathcal{V}_{13} + \kappa(-\alpha) \left(1 - \frac{1}{B}\right) \mathcal{V}_{23}, \text{ where } \kappa = \mathcal{L} = 1$$

we write it as :

$$0 = (\mathbf{B}(\mathbf{A} - \mathbf{1}))\mathcal{V}_{13} + \mathbf{A}((\mathbf{B} - \mathbf{1}))\mathcal{V}_{23} \quad (2)$$

In order to extract θ_p we use first order approximation :

$$\cos(\theta_p) \rightarrow 1 ; \quad \sin(\theta_p) \rightarrow \theta_p \quad (3)$$

to get:

$$\mathcal{V}_{13} = a_{12} + c_{11}\theta_p ; \quad \mathcal{V}_{23} = b_{12} + c_{12}\theta_p \quad (4)$$

And use Taylor series (around $\theta_0 = 0$) for A,B :

$$A = \sqrt{a_{11}} + \frac{a_{12}}{\sqrt{a_{11}}} \theta_p ; \quad B = \sqrt{b_{11}} + \frac{b_{12}}{\sqrt{b_{11}}} \theta_p \quad (5)$$

Where :

$$\begin{pmatrix} a_{11} \\ a_{12} \\ b_{11} \\ b_{12} \\ c_{11} \\ c_{12} \end{pmatrix} = \begin{pmatrix} (-h_p + y_1 - y_p)^2 + (w_p + x_1 - x_p)^2 \\ h_p(x_1 - x_p) + w_p(y_1 - y_p) \\ (-h_p + y_2 - y_p)^2 + (-w_p + x_2 - x_p)^2 \\ h_p(x_2 - x_p) - w_p(y_2 - y_p) \\ h_p(y_1 - y_p) - w_p(x_1 - x_p) \\ h_p(y_2 - y_p) + w_p(x_2 - x_p) \end{pmatrix} = f(h_p, w_p, y_p, y_1, y_2, x_p, x_1, x_2)$$

Setting (4,5) into (2) and eliminating 2nd and 3rd order elements of θ_p , getting the relation :

$$\theta_p = \frac{a_{12}b_{11}(\sqrt{a_{11}} - a_{11}) + a_{11}b_{12}(\sqrt{b_{11}} - b_{11})}{a_{12}^2b_{11} + a_{11}b_{12}^2 - a_{12}b_{12}(\sqrt{a_{11}} - a_{11} + \sqrt{b_{11}} - b_{11}) - b_{11}c_{11}(\sqrt{a_{11}} - a_{11}) - a_{11}c_{12}(\sqrt{b_{11}} - b_{11})} \quad (6)$$

θ_p is therefore a function of $\{h_p, w_p, x_i, y_i\}$; $i = 1, 2, p$

Next step is :

Using (3) for the 1st two equations of (1) and getting new equations for a 2D.O.F problem, to asymptotically investigate :

Non-dimensional 2D.O.F Equations of motion are:

$$\ddot{\mathcal{X}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \frac{1}{A}\right) \mathcal{V}_1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \frac{1}{B}\right) \mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \end{pmatrix} - C \dot{\mathcal{X}} + \begin{pmatrix} F_x \\ F_y \end{pmatrix} \quad (7)$$

Where:

$$\mathcal{X} = \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

$$A = \sqrt{(h_p \theta_p + w_p - x_p + x_1)^2 + (-h_p + \theta_p w_p - y_p + y_1)^2}$$

$$B = \sqrt{(h_p \theta_p - w_p - x_p + x_2)^2 + (-h_p - \theta_p w_p - y_p + y_2)^2}$$

$$\mathcal{V}_1 = \begin{pmatrix} w_p + x_1 - x_p + h_p \theta_p \\ -h_p + y_1 - y_p + w_p \theta_p \end{pmatrix}$$

$$\mathcal{V}_2 = \begin{pmatrix} -w_p + x_2 - x_p + h_p \theta_p \\ -h_p + y_2 - y_p - w_p \theta_p \end{pmatrix}$$

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

F_x, F_y are the aerodynamic drag forces

θ_p is taken from (6) above as function of $\{h_p, w_p, x_i, y_i\}$; $i = 1, 2, p$

F_x, F_y will be considered as result of drag on 'CONEX' cargo container payload. No sling cables drag will be considered :

$$F_{x_p} = \frac{1}{2} \rho \dot{x}_p^2 (2h_p) C_{F_{x\alpha}} \theta_p \sim \rho h_p C_D \dot{x}_p^2$$

$$F_{y_p} = \frac{1}{2} \rho \dot{y}_p^2 (2w_p) C_{F_{y\alpha}} (\pi - \theta_p) \sim \rho w_p C_D \dot{y}_p^2$$

$$C_D = 1.1$$