ME-Project Report

about

Quad-rotors & Payload system dynamics and stability

By: Ran , Oct 2017

Abstract

This report is the summary of my research work as part of Master of Engineering in Autonomous Systems and Robotics (TASP).

In this research, I am investigating the stability of payload carried by a 2 quadrotors 'array', under certain conditions. This case is in interest because of possible payload delivery mission required by companies such 'Amazon' and others, to deliver a relatively big-size and heavy payload – a mission that is not always possible for 1 quadrotor alone.

I'll show the model equations of motion, analytical investigation and a numerical investigation results for comparison.

This report will show the system bifurcation structure , and highlight interesting parameters thresholds.

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1. Introduction

There is a growing amount of interest in the controlled autonomous behavior of collectively operating unmanned aerial vehicles. An example is an array of autonomous quadrotors which are under consideration for a variety of missions including surveillance [Acevedo et al., 2012], heavy payload delivery [Bernard et al, 2011], and assembly of structures [Kumar and Michael, 2012]. While there is an abundance of documentation of multi-agent behavior of very large groups (or swarms) such as flocks of birds and schools of fish which can quickly adapt to a complex terrain [Shklarsh et al, 2011] or environmental conditions [Elor and Bruckstein, 2011; Agmon et al, 2011], and an uncertain and changing environment in nature [Young et al, 2013], there is limited research on small size arrays of autonomous elements that continue to maneuver collectively under severe environmental conditions.

Documented research on single quadrotor dynamics, stability and control consists of rigid-body dynamical systems models augmented by angular rotor dynamics [Leishman 2006]. Investigations include nonlinear control for take-off, hovering and landing [Kendoul et al, 2007] and to overcome path following uncertainties [Raffo et al, 2010] and disturbances [Schoellig et al., 2012]. Control of aggressive maneuvers such as flying through a narrow gap [Mellinger et al, 2012] have been implemented, and investigations include robustness analysis applied to wind gusts [Alexis et al, 2011; Escareno et al, 2013], and control of cable suspended payload has been proposed [Sreenath et al, 2013].

Recently, there is increasing interest in payload carried by more than 1 vehicle, investigating better controls and better flight structure to get better performance for flight stability and distance [J. Enciu 2017].

Motivated by simulation studies of decision making in animal groups in motion, the stability of multiagent particle dynamical systems models have been analyzed to reveal cohesive behavior [Liu and Passino, 2005] and separation of fast and slow time scales reflecting a local bifurcation structure indicative of a compromise by individual elements with conflicting preferences [Nabet et al., 2009]. Symmetrical and asymmetrical bifurcations have been shown in a swarm robotics test bed [Garnier et al, 2013] and a noise intensity threshold was shown to govern swarm transition from a misaligned state into an aligned state [Mier et al, 2012]. Furthermore, nonlinear multi-agent swarm models exhibit existence of periodic limit-cycles culminating with non-stationary chaotic solutions [Das et al, 2012], and stochastic bifurcations [Ebeling and Schimansky-Geier, 2008].

In the light of the current scientific background the behavior in severe environmental conditions is yet unresolved. Thus, this paper’s aim is to derive a consistent dynamical systems model for a 2-element ‘array’ of quadrotors which can withstand severe and unsteady aerodynamic disturbances. Investigating the nonlinear array dynamics asymptotically and numerically, culminating with a system bifurcation structure highlighting parameters thresholds to stability.

This approach can help to bridge the gap between documented stable operations and large time-dependent perturbations expected in a changing non-stationary environment.

The paper contains : i) derivation of 2-element-with-payload system dynamic model ii) the model will be investigated via the asymptotic multiple-scales method to yield stability thresholds for synchronous and non-stationary dynamics, iii) numerical stability analysis of the dynamical system to validate the asymptotic stability thresholds, iv) conclusions and summary.

2. Nomenclature

i : index for object {1,2,p} regarding: quad #1, quad #2, Payload, or: cable #1, cable #2.

k_i : returning force constant of the linear spring i

$\mathrm{L0}_i$: free spring length (not loaded)

l_i : current length of the loaded spring

m_i : mass of object i

x_i, y_i : location of the i'th mass center, in inertial coor.system

θ_p : rotation angle around $\widehat{Z}_I$ axis, of the rigid body payload, relative to the Inertial frame.

w_p : geometric (half) length of the payload rigid body

h_p : geometric (half) height of the payload rigid body

R_p^I : rotation matrix from payload to Inertial coordinate frame

I_i : moment of inertia, around axis $\widehat{Z}_I$, for object i

L : Lagrangian of the system

T : kinetic energy

V : potential energy

u : air velocity in global framework, in X direction

v : air velocity in global framework, in Y direction

ρ : air density. Treated as constant

C_D : drag coefficient of the payload. Taken as equal for both directions of X, Y.

Acronyms

DOF : Degreed of Freedom

EOM : Equations of motion

3. Problem formulation and system dynamics

The examined system is composed of 2 units of quadrotors, and 1 rigid body payload which is connected to each of the quadrotors by cables connected to the anchor points.

The system is described and investigated in the 2D world.

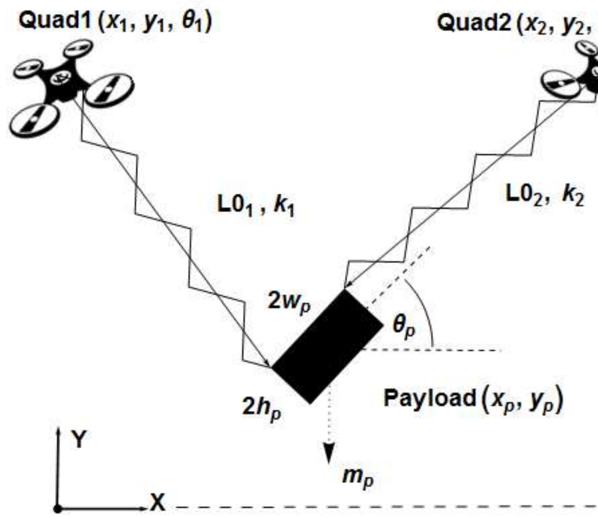


Figure 1 - system setup

model assumptions

Quadrotor:

1. Quad body and parts are **rigid**. *No* elasticity is included.
2. Geometry structure is **symmetrical** in relation to the principal axes. And the mass distribution is **uniform**. Hence the Inertia matrix is taken as pure diagonal.
3. **quads motion is treated as given**! Hence the quad pure dynamics and control are not considered here, and they are mentioned only to give a better perspective for the problem in hand.

Payload & cable construction:

4. The 'cable' which the payload is connected to, is modeled as a **lumped mass linear spring**, with initial length $\mathrm{L0}_i$, and a linear returning force coefficient k_i.
5. The cable is connected to the quadrotor exactly in its center of mass (C.G).
   There is no friction and moment actuated through those hanging points.
6. In the payload hanging points there is an arbitrary **structural dumping $\mathbf{c}_i$** (resultant of hanging point friction and spring dumping).
7. The payload is a **rectangular box**, similar to a 'CONEX' cargo container, characterized by width of (2 w_p), and height of (2 h_p), and with inertia matrix I_p.
8. There is no consideration in possible aerodynamic drag of the cables.

9. **Simplified aerodynamic forces** (lift and drag) on the payload are considered – will be addressed in the non-conservative forces section.

Coordinate systems, and general coordinates

I – inertial coordinates frame. It is the global reference point for the problem.

Its' axes are : $(\widehat{X_I}, \widehat{Y_I}, \widehat{Z_I})^T$

P – Payload coordinate frame. The origin is located at the C.G of that rigid body.

Since we are looking at 2D problem, for each system element we have 3D.O.F which are planar position and pitch angle. For the full 2D problem, we have the general coordinates as:

$$q = (x_1, y_1, \theta_1, x_2, y_2, \theta_2, x_p, y_p, \theta_p)^T \tag{1}$$

which is a 9 D.O.F problem.

<u>The problem's geometry</u>

Schematics of the system, in accordance with the nomenclature listed above:

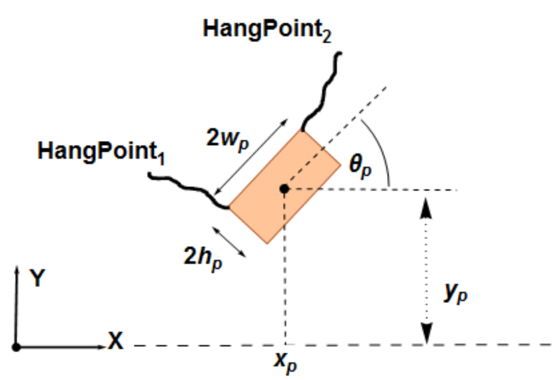


Figure 2 - detailed sketch for hanging points geometry

Where

$$\text{HangPoint}_1 = (x_p, y_p)^T + \mathrm{R}_p^I(-w_p\,, +h_p)^T \tag{2}$$

$$\text{HangPoint}_2 = (x_p, y_p)^T + \mathrm{R}_p^I(+w_p\,, +h_p)^T$$

$$\mathrm{R}_p^I = \begin{pmatrix} \cos(\theta_p) & -\sin(\theta_p) \\ \sin(\theta_p) & \cos(\theta_p) \end{pmatrix}$$

$2w_p$ is the width of the payload. $2h_p$ is the height of the payload.

The Lagrangian of the system

By definition of the Lagrangian $L = T - V$, we construct the system Lagrangian:

$$L = T_{\text{quad\#1}} + T_{\text{quad\#2}} + T_{\text{payload}} - \left(V_{\text{quad\#1}} + V_{\text{quad\#2}} + V_{\text{payload}} + V_{\text{spring\#1}} + V_{\text{spring\#2}}\right) \qquad (3)$$

Where

$$T_{\text{payload}} = \frac{1}{2} m_p\left(\dot{x_p}^2 + \dot{y_p}^2\right) + \frac{1}{2} I_p \dot{\theta_p}^2 \qquad V_{\text{payload}} = g m_p y_p \qquad V_{\text{spring\#1}} = \frac{1}{2} k_1 (\mathrm{L}_1 - \mathrm{L0}_1)^2$$

$$V_{\text{quad\#1}} = g m_1 y_1$$

$$T_{\text{quad\#1}} = \frac{1}{2} m_1\left(\dot{x_1}^2 + \dot{y_1}^2\right) + \frac{1}{2} I_1 \dot{\theta_1}^2 \qquad V_{\text{spring\#2}} = \frac{1}{2} k_2 (\mathrm{L}_2 - \mathrm{L0}_2)^2$$

$$V_{\text{quad\#2}} = g m_2 y_2$$

$$T_{\text{quad\#2}} = \frac{1}{2} m_2\left(\dot{x_2}^2 + \dot{y_2}^2\right) + \frac{1}{2} I_2 \dot{\theta_2}^2$$

$$\mathrm{L}_1 = \sqrt{\left(w_p \cos(\theta_p) + h_p \sin(\theta_p) - (x_p - x_1)\right)^2 + \left(-h_p \cos(\theta_p) + w_p \sin(\theta_p) - (y_p - y_1)\right)^2}$$

$$\mathrm{L}_2 = \sqrt{\left(-w_p \cos(\theta_p) + h_p \sin(\theta_p) - (x_p - x_2)\right)^2 + \left(-h_p \cos(\theta_p) - w_p \sin(\theta_p) - (y_p - y_2)\right)^2}$$

The Lagrange equations will be calculated using the next equation (while using the common fact that V is not dependent on $\dot{q_i}$ for mechanical systems):

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q_i}}\right) - \frac{\partial L}{\partial q_i} = \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q_i}}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \qquad ; \qquad i = 1 \div 9 \qquad (4)$$

Q_i – are the non-conservative forces.

Note: If I would consider the inner dynamics of each of the quads and their controllers – I would need to consider at least those 9 Equations Of Motion.

Because this is a 'Research project' and not a full research work - I will focus my desired analysis on the payload stability, and I will limit myself to deal with $\begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix}$, while $(x_1, y_1, \theta_1, x_2, y_2, \theta_2)$ are treated as given.

The dimensional 9 EOM are described in Appendix A for reference.

non-conservative forces

we consider the next general forces (for the payload-related general coordinates):

- *structural dumping*

As described briefly in the assumptions section above. It is related to the relative movement of the payload to each of the quadrotors, and will be written as:

$$Q_c = -\begin{pmatrix} c_x(\dot{x}_p - \dot{x}_1) + c_x(\dot{x}_p - \dot{x}_2) \\ c_y(\dot{y}_p - \dot{y}_1) + c_y(\dot{y}_p - \dot{y}_2) \\ c_\theta \dot{\theta}_p \end{pmatrix} \quad ; \quad c_i > 0 \tag{5}$$

- *aerodynamic forces*

While the desire is to use simple terms for the aerodynamic contribution, I will assume just drag forces in x,y directions. Furthermore, I'll assume C_D which is like ones' of 'CONEX' cargo container.

$$\begin{aligned} F_{x_p} &= \tfrac{1}{2}\rho \dot{x}_p^{\,2}(2h_p)C_{F_{x_\alpha}}\theta_p \sim \rho\, h_p\, C_D(\dot{x}_p - u)^2 \\ F_{y_p} &= \tfrac{1}{2}\rho \dot{y}_p^{\,2}(2w_p)C_{F_{y_\alpha}}(\pi - \theta_p) \sim \rho\, w_p\, C_D\,(\dot{y}_p - v)^2 \end{aligned} \tag{6}$$

$$C_D = 1.1$$

$$Q_{Aero} = \begin{pmatrix} -F_x \\ -F_y \\ 0 \end{pmatrix}$$

u,v are the air velocities in the relevant directions.

Equations of Motion (dimensional)

As mentioned above, from now on we deal with only 3D.O.F – for the general coordinates of the payload.

From (4), using (3),(5),(6), and rearranging, we can get the *dimensional* equations of motion for the payload:

$$\begin{pmatrix} m_p\ddot{x}_p = k_1\dfrac{\mathrm{dx}_1\left(\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}-\mathrm{L0}_1\right)}{\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}} + k_2\dfrac{\mathrm{dx}_2\left(\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}-\mathrm{L0}_2\right)}{\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}} - c_x(\dot{x}_p - \dot{x}_1) - c_x(\dot{x}_p - \dot{x}_2) - \rho C_D h_p \dot{x}_p^{\,2} \\ m_p\ddot{y}_p = -k_1\dfrac{\mathrm{dy}_1\left(\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}-\mathrm{L0}_1\right)}{\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}} - k_2\dfrac{\mathrm{dy}_2\left(\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}-\mathrm{L0}_2\right)}{\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}} - g m_p - c_y(\dot{y}_p - \dot{y}_1) - c_y(\dot{y}_p - \dot{y}_2) - \rho C_D w_p \dot{y}_p^{\,2} \\ I_p\ddot{\theta}_p = k_1\dfrac{\mathrm{term1}\left(\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}-\mathrm{L0}_1\right)}{\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}} - k_2\dfrac{\mathrm{term2}\left(\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}-\mathrm{L0}_2\right)}{\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}} - c_\theta\dot{\theta}_p \end{pmatrix} \tag{7}$$

Where:

$$\begin{pmatrix} \mathrm{dx}_1 = w_p\,\mathrm{c}(\theta_p) + h_p\,\mathrm{s}(\theta_p) - x_p + x_1 \\ \mathrm{dy}_1 = h_p\,\mathrm{c}(\theta_p) - w_p\,\mathrm{s}(\theta_p) + y_p - y_1 \\ \mathrm{dx}_2 = -w_p\,\mathrm{c}(\theta_p) + h_p\,\mathrm{s}(\theta_p) - x_p + x_2 \\ \mathrm{dy}_2 = h_p\,\mathrm{c}(\theta_p) + w_p\,\mathrm{s}(\theta_p) + y_p - y_2 \\ \mathrm{term1} = h_p\big(\mathrm{c}(\theta_p)\,(x_p - x_1) + (y_p - y_1)\,\mathrm{s}(\theta_p)\big) + w_p\Big(\big(y_p - y_1\big)\,\mathrm{c}(\theta_p) + \mathrm{s}(\theta_p)(x_1 - x_p)\Big) \\ \mathrm{term2} = h_p\big(\mathrm{c}(\theta_p)\,(x_2 - x_p) + (y_2 - y_p)\,\mathrm{s}(\theta_p)\big) + w_p\Big(\big(y_p - y_2\big)\,\mathrm{c}(\theta_p) + \mathrm{s}(\theta_p)\,(x_2 - x_p)\Big) \end{pmatrix} \tag{8}$$

* *note*: 'c' is for Cos(), 's' is for Sin()

Non-dimensional Equations of Motion

Scaling the dimensional variables according to these relations:

$$\tilde{x_p} = \frac{x_p}{\mathrm{L0}_1} \;;\; \tilde{y_p} = \frac{y_p}{\mathrm{L0}_1} \;;\; \tau = t\,\omega_s \;;\; \omega_s^2 = \frac{k_1}{m_p} \tag{9}$$

$$\tilde{h_p} = \frac{h_p}{\mathrm{L0}_1} \;;\; \tilde{w_p} = \frac{w_p}{\mathrm{L0}_1}$$

This results also in:

$$\tilde{\mathrm{dx}_i} = \frac{\mathrm{dx}_i}{\mathrm{L0}_1} \;;\mathrm{\tilde{dy}}_i = \frac{\mathrm{dy}_i}{\mathrm{L0}_1} \;;\; \tilde{\mathrm{term}}_i = \frac{\mathrm{term}_i}{\mathrm{L0}_1{}^2} \;;\; \tilde{F_i} = \frac{F_i}{\mathrm{L0}_1^3\omega_s^2} \;;\; \tilde{\dot{x}_i c_i} = \frac{\dot{x}_i c_i}{\mathrm{L0}_1\omega_s}$$

So **the non-dimensional equations** can be written in matrix form in the following way:

$$\ddot{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \mathrm{DX}_1\,\mathcal{V}_1 + \kappa \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \mathrm{DX}_2\,\mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} - \mathcal{A} - \mathcal{D} \tag{10}$$

Where $(all\ \tilde{}\ notations\ are\ removed\ for\ brevity)$:

$$\mathcal{X} = \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix} \;;\; \mathrm{DX}_1 = 1 - \frac{1}{\sqrt{\mathrm{dx}_1^2 + \mathrm{dy}_1^2}} \;;\; \mathrm{DX}_2 = 1 - \frac{1}{\sqrt{\mathrm{dx}_2^2 + \mathrm{dy}_2^2}}\mathcal{L}$$

$$\mathcal{V}_1 = \begin{pmatrix} \mathrm{dx}_1 \\ \mathrm{dy}_1 \\ \mathrm{term1} \end{pmatrix} \;;\; \mathcal{V}_2 = \begin{pmatrix} \mathrm{dx}_2 \\ \mathrm{dy}_2 \\ \mathrm{term2} \end{pmatrix}$$

$$\mathcal{A} = Q_{\mathrm{Aero}} * (\mathrm{L0}_1^3\omega_s^2) \;;\; \mathcal{D} = \begin{pmatrix} (\mathrm{L0}_1\omega_s) & 0 & 0 \\ 0 & (\mathrm{L0}_1\omega_s) & 0 \\ 0 & 0 & (\omega_s) \end{pmatrix} Q_c$$

$$\mathcal{L} = \frac{\mathrm{L0}_2}{\mathrm{L0}_1} \;;\; \kappa = \frac{k_2}{k_1} \;;\; \gamma = \frac{g m_p}{\mathrm{L0}_1 k_1} \;;\; \alpha = \frac{k_1}{I_p \omega_s{}^2}$$

further analysis assumptions

in order to simplify the analysis continuation and to focus on the method, I'll assume the geometry of the system is symmetric. Hence: $\boldsymbol{\mathcal{L}} = \mathbf{1}$**,** $\boldsymbol{\kappa}$**=1** from here next.

Moreover I_p (around the payloads' Z axis, in center of gravity) is the one relevant for a rectangular shape which is:

$$I_p = \frac{1}{3} m_p \left({h_p}^2 + {w_p}^2 \right) \quad \rightarrow \quad \alpha = \frac{3}{\left(\left(\frac{hp}{w_p} \right)^2 + 1 \right)} \left(\frac{\mathrm{L0}_1}{w_p} \right)^2 \tag{11}$$

I'll check only for the case where quadrotors inputs are:

$$x_1 = y_1 = 0 \quad ; \quad x_2 = 2w_p \,, y_2 = 0 \tag{12}$$

Limiting case test of elastic pendulum is shown in Appendix 1.

4. Equilibrium analysis

Equilibrium points:

We find the equilibrium possible states by setting all time derivatives to zero ($\dot{\mathrm{q}} = \ddot{q} = 0$).

It results with all accelerations, aerodynamic forces, and dumping forces and moments are zeroed.

The solutions are:

$$\boldsymbol{\mathcal{X}_1} = \begin{pmatrix} \boldsymbol{w_p} \\ -\left(\frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{\gamma} + \boldsymbol{h_p} + \mathbf{1} \right) \\ \mathbf{0} \end{pmatrix} \quad ; \quad \mathcal{X}_2 = \begin{pmatrix} w_p \\ -\left(\frac{1}{2} \gamma + h_p - 1 \right) \\ 0 \end{pmatrix} \tag{13}$$

- From now on I'll relate only to $\boldsymbol{\mathcal{X}_1}$ because this is the case of the quads can take the payload. The 2nd case can be related to similar case of inverted pendulum.

Linearizing about that equilibrium point

The goal now is to find the natural frequencies of the system. it is done by looking on the effect of small deviations around the equilibrium point, while the aerodynamic and dumping forces are zeroed. Using perturbations syntax:

$$x_p \rightarrow \delta \mathrm{x} + x_{p_0} \quad ; \quad y_p \rightarrow \delta \mathrm{y} + y_{p_0} \quad ; \quad \theta_p \rightarrow \delta \theta + \theta_{p_0} \tag{14}$$

Detailed derivation of the equations is in *Appendix B*.

The linearized equation , in B7, lead to the following linearized natural frequencies:

$$\omega_y{}^2 = 2 \quad ; \qquad \omega_1{}^2 = \frac{\zeta-\sqrt{\beta}}{2(2+\gamma)(h_p^2+w_p^2)} \;;\; \omega_2{}^2 = \frac{\zeta+\sqrt{\beta}}{2(2+\gamma)(h_p^2+w_p^2)} \tag{15}$$

Where

$$\beta = -24\gamma(2+\gamma)(h_p^2+w_p^2)(\gamma h_p + 2w_p^2) + \left(3\gamma(2+\gamma)h_p + 8\gamma h_p^2 + 4(3+2\gamma)w_p^2\right)^2$$

$$\zeta = 6\gamma h_p + 3\gamma^2 h_p + 8\gamma h_p^2 + 12 w_p^2 + 8\gamma w_p^2$$

It can be verified that for $h_p, w_p, \gamma > 0$ we always get $\beta > 0$, $\zeta > 0$,and $\boldsymbol{\omega_i}^{\mathbf{2}}$>0 .

Reducing order of the system dynamics to 2DOF

It can be shown (detailed in *Appendix C*) that, for certain parameters, we can get :

$$\omega_1 \sim \boldsymbol{\omega_x} < \sqrt{2} \quad ; \quad \omega_2 \sim \boldsymbol{\omega_\theta} \gg \omega_y = \sqrt{2} \tag{16}$$

While ω_θ is larger than ω_y we can neglect the dynamics of θ and relate only to the dynamics of general coordinates of x, y.

We return to the full non-dimensional equations of motion and set $\boldsymbol{\ddot{\theta}} = \boldsymbol{\dot{\theta}} \cong \mathbf{0} \rightarrow \theta = \theta_0 + \delta\theta$ where $\boldsymbol{\theta_0} = \mathbf{0}$.

Looking at the 3rd equation from (10):

$$0 = \alpha\ \mathrm{DX}_1\ \mathcal{V}_{13} - \alpha\ \mathrm{DX}_2\ \mathcal{V}_{23} \tag{17}$$

and term for $\theta_p = f(h_p, w_p, x_i, y_i)$ $, where\ \ i = 1,2,p$ can be extracted.

further details are elaborated in *Appendix D*.

the resultant 2DOF equations (of x_p, y_p) are:

$$\ddot{\mathcal{X}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(1-\frac{1}{A}\right)\mathcal{V}_1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(1-\frac{1}{B}\right)\mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \end{pmatrix} - C\,\dot{\mathcal{X}} - \begin{pmatrix} Fx \\ Fy \end{pmatrix} \tag{18}$$

Where:

$$\mathcal{X} = \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

$$A = \sqrt{(h_p\theta_p + w_p - x_p + x_1)^2 + (-h_p + \theta_p w_p - y_p + y_1)^2}$$

$$B = \sqrt{(h_p\theta_p - w_p - x_p + x_2)^2 + (-h_p - \theta_p w_p - y_p + y_2)^2}$$

$$\mathcal{V}_1 = \begin{pmatrix} w_p + x_1 - x_p + h_p\theta_p \\ -h_p + y_1 - y_p + w_p\theta_p \end{pmatrix} \;;\; \mathcal{V}_2 = \begin{pmatrix} -w_p + x_2 - x_p + h_p\theta_p \\ -h_p + y_2 - y_p - w_p\theta_p \end{pmatrix} \;;\; C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

$$\theta_p = \frac{a_{12}b_{11}(\sqrt{a_{11}} - a_{11}) + a_{11}b_{12}(\sqrt{b_{11}} - b_{11})}{a_{12}^2 b_{11} + a_{11}b_{12}^2 - a_{12}b_{12}\left(\sqrt{a_{11}} - a_{11} + \sqrt{b_{11}} - b_{11}\right) - b_{11}c_{11}(\sqrt{a_{11}} - a_{11}) - a_{11}c_{12}(\sqrt{b_{11}} - b_{11})}$$

$$\begin{aligned}
a_{11} &= \left(-h_p + y_1 - y_p\right)^2 + \left(w_p + x_1 - x_p\right)^2 \\
a_{12} &= h_p(x_1 - x_p) + w_p(y_1 - y_p) \\
b_{11} &= \left(-h_p + y_2 - y_p\right)^2 + \left(-w_p + x_2 - x_p\right)^2 \\
b_{12} &= h_p\left(x_2 - x_p\right) - w_p(y_2 - y_p) \\
c_{11} &= h_p\left(y_1 - y_p\right) - w_p(x_1 - x_p) \\
c_{12} &= h_p(y_2 - y_p) + w_p(x_2 - x_p)
\end{aligned}$$

5. Asymptotic analysis

Taking the last equations of (18) and expand by Taylor series about the nominal case of hovering.

The equations to investigate are:

6. Numerical analysis

Taking the 3DOF equations from (10), and numerically integrating them for several test cases.

I intend to find the critical parameters that will cause the system to move from 1 fixed-point solution to limit-cycle solution, and to chaos behavior.

I'll use the equilibrium point found in (13) as a reference point, to describe the Initial Conditions relative to it.

The parameters of the system are:

$g, m_p. k_1, \mathrm{L0}_1, w_p, h_p, c_x, c_y, c_\theta, \rho, C_D$

x_1, y_1, x_2, y_2 and $v_y, \epsilon\mathrm{F}, \Omega_y$, for platform movement description by $y = v_y + \epsilon\mathrm{F}\ \mathrm{Sin}\left[\frac{\Omega_y t(*}{\omega_s *)}\right]$

I'd like to test:

- Dynamics solution to case without non-conservative forces, but with oscillating y_i.
  ϵF will be varied and the system response will be tested.
- Dynamics solution when quadrotors are moving upwards with oscillations, and there are non-conservative forces active.

Constant parameters are taken as:

$g = 9.81,\ m_p = 1,\ k_1 = 200,\ \mathrm{L0}_1 = 1,\ \rho = 1.225,$

$w_p = 2,\ h_p = 1$ (w,h are in non-dimensional units)

The 3 equations of motion are manipulated and changed to 6 equations of order 1.

By setting

$\mathrm{X1} = x_p,\ \mathrm{X2} = \dot{X}1,$

$\mathrm{X3} = y_p,\ \mathrm{X4} = \dot{X}3,$

$\mathrm{X5} = \theta_p,\ \mathrm{X6} = \dot{X}5$

And solution will be integrated for $2T$, where $T = 2\pi/\Omega_y$ is the expected time period.

I will display graphs of time-series, phase-space, Poincare, and Bifurcation.

Study Case of conservative system

Using the next parameters:

$$c_x = c_y = c_\theta = C_D = 0$$

$$X1(0) = w_p + \delta x, \qquad \mathrm{X3}(0) = -\left(\frac{1}{2}\gamma + h_p + 1\right) + \delta y, \qquad \mathrm{X5}(0) = \delta\theta$$
$$X2(0) = \delta vx, \qquad \mathrm{X4}(0) = \delta vy, \qquad \mathrm{X6}(0) = \delta v\theta$$

$$x_1(t) = 0, \qquad y_1(t) = v_y + \epsilon\mathrm{F}\ \mathrm{Sin}(\Omega_y t)$$
$$x_2(t) \to 2w_p, \qquad y_2(t) \to y_1[t]$$

$$v_y = 0,\ \Omega_y = 2$$

I'll check for case of $\delta\theta = \delta v\theta = 0.01$, while the other δ are zeros.

$\epsilon\mathrm{F}$ will be 'swept' between 0.001 to 0.15.

Phase-space for $\epsilon\mathrm{F} = 0.0011$:

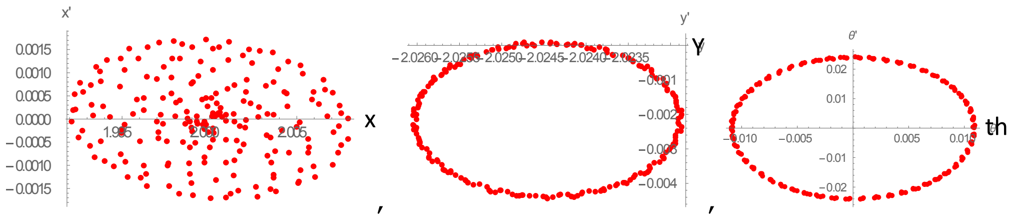


Figure 3 - phase space for eF=0.0011

Phase-space for $\epsilon\mathrm{F} = 0.08$:

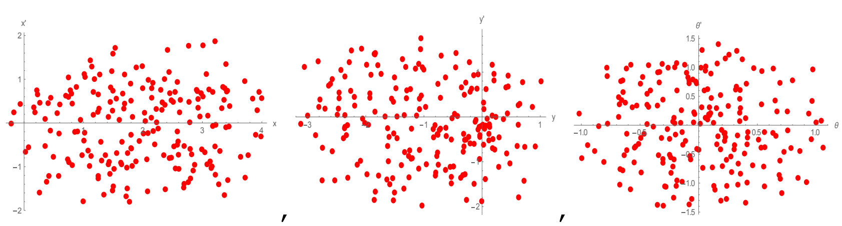
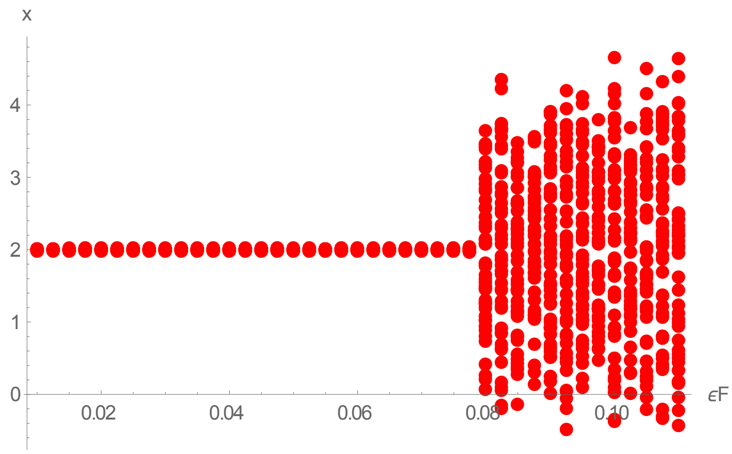


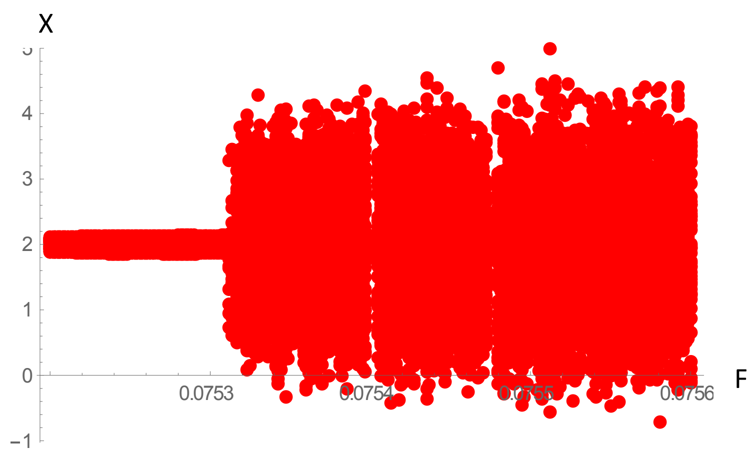
Figure 4 - - phase space for eF=0.08

We can see the system moved to chaos when $\epsilon\mathrm{F}$ grew-up towards 0.08 value.

Bifurcation graph for F from 0.001 to 0.15:



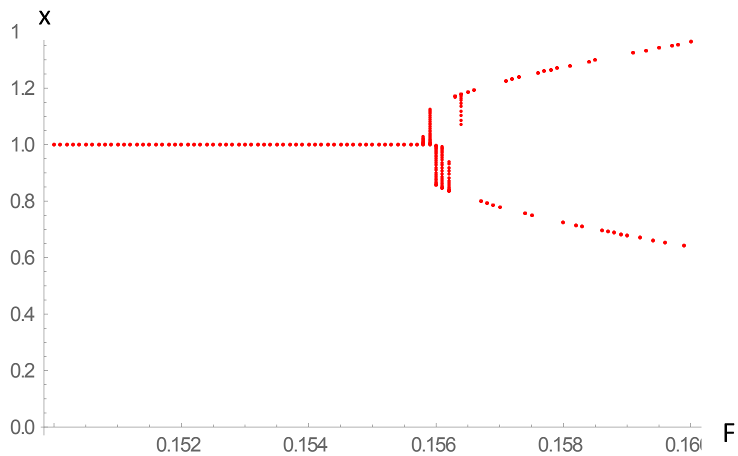
Trying to zoom on $(0.0752 \; to \; 0.0756)$ it gives:



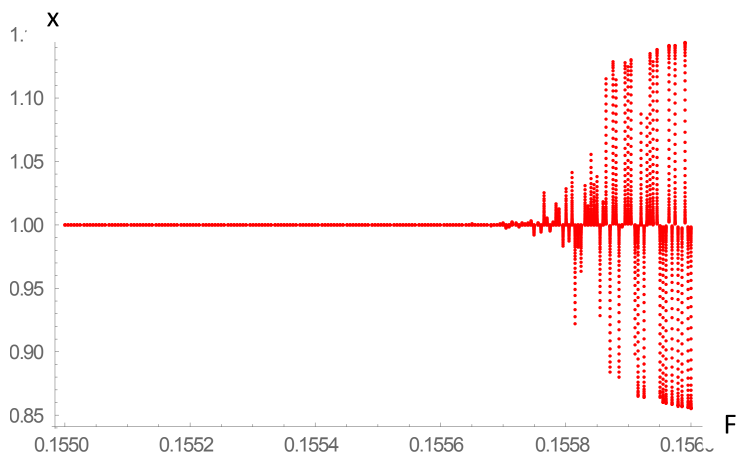
I didn't get the desired double-period visual.

For another case – different starting conditions for $\delta vx = 0.001$ and $\delta v\theta = 0$, I could find better visualization for the 1st bifurcation:

At around 0.156 we get:



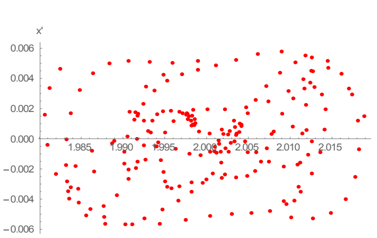
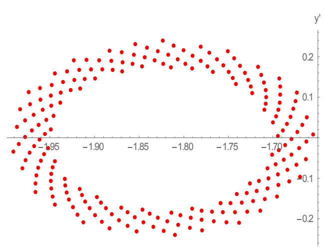
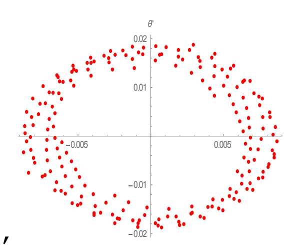
And closer:



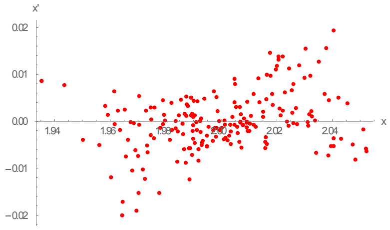
I could not get better more accurate results for this – probably very sensitive to numerics because of the small numbers dealing with.

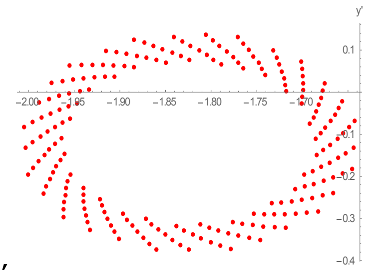
Testing for maneuver of lift with oscilations: $y_{1,2}(t) = v_y + \epsilon\text{F Sin}(\Omega_y t)$; $v_y = 0.05$; $\Omega_y = 2$, dumping is active with $c_i = 0.05$, no drag yet ($C_D = 0$).

Phase-space graphs of x-x', y-y', $\theta - \theta'$ correspondently , for $\epsilon\text{F} = 0.001$:

 ,  , 

for $\epsilon F = 0.06$:



,


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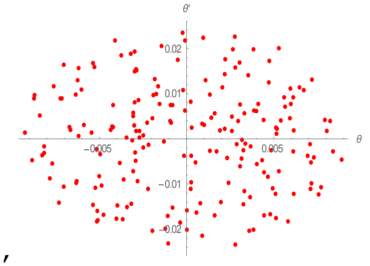
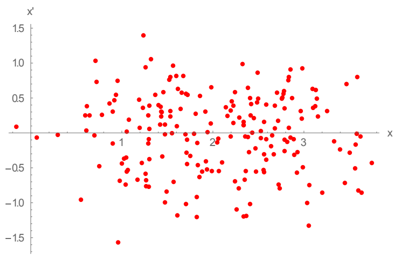
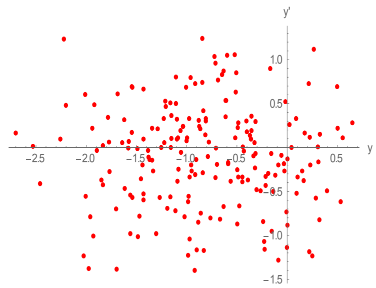
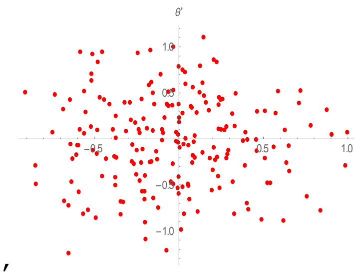


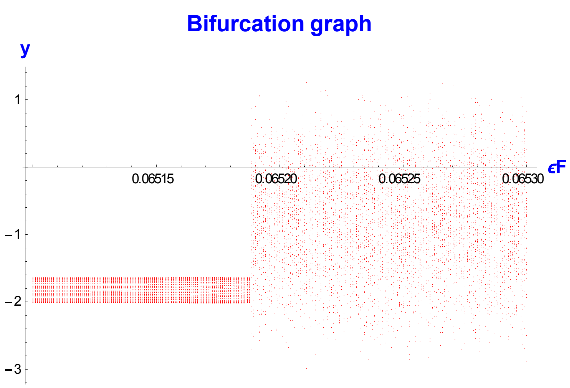
for $\epsilon F = 0.07$:



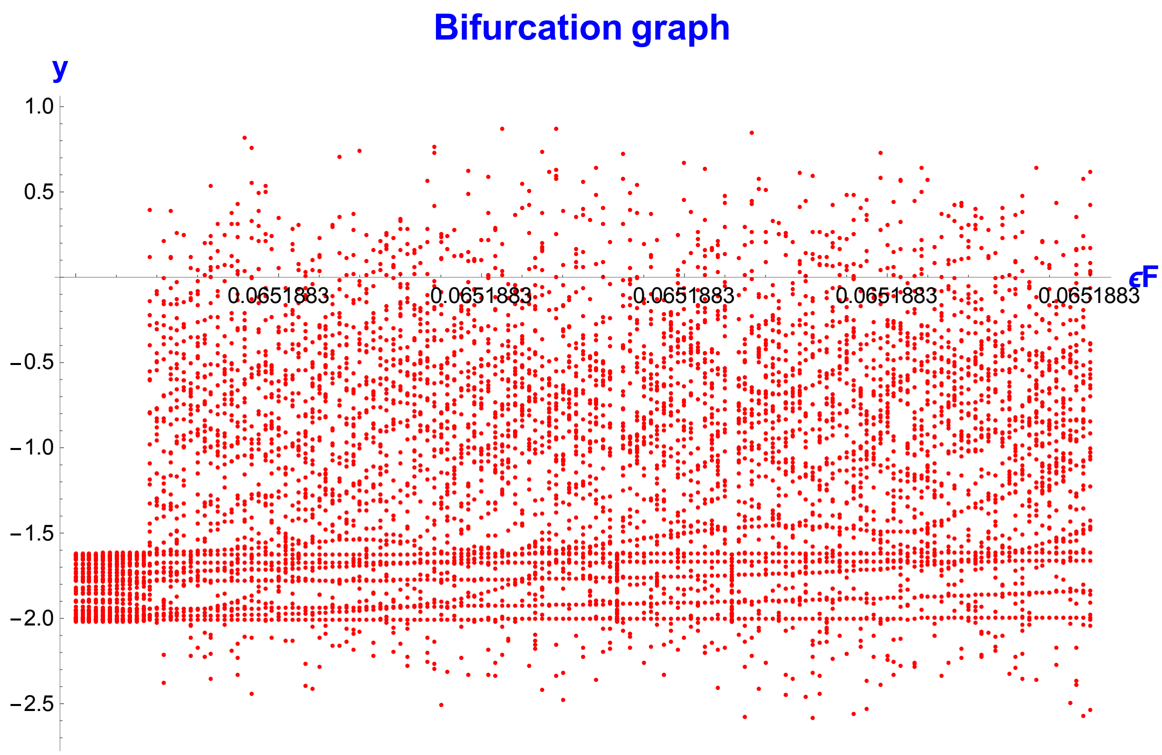
,


,


The relevant bifurcation graph is:



And zooming to $\epsilon F = 0.0651882974\ between\ 0.06518829765$:



7. Discussion

Numerical simulations were conducted for the given non-dimensional dynamics in (10).

Tested for two cases of :

- Vertical oscillation while hanging near equilibrium point, with initial position deviation. No drag forces and dumping moments were activated. Graphs were presented for this case.
- Vertical oscillations while moving upwards in constant speed , with dumping active, but no drag forces (actually drag forces made the numeric simulation run with 'stiff' warning).
  Relevant graphs were presented here as well.

It is noted that when activated with vertical nominal velocity the magnitude of the oscillating force, that effected the start of chaos behavior was lower in magnitude than the 1st case (~0.065188297 instead of ~0.0753).

Summary

I described the 2D dynamics of system of 2 quadrotors and 1 connected rigid body payload.

I verified against limiting cases of:

1. elastic pendulum

Non-dimensional equations were submitted, and equilibrium analysis was done.

Soon I will do the asymptotic analysis in order to find the stability criteria for the system.

Numerical analysis was done for specific test cases. It can be done for much more versatile range of the system parameters, in order to get better view about the systems' behavior and critical boundaries.

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Appendices

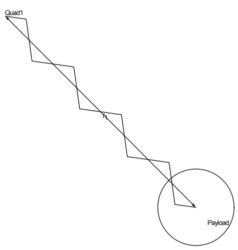
Appendix 1 –Limiting case dynamics – elastic pendulum

Reminding about the full problem equations of motion, from (10):

$$\ddot{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \mathrm{DX}_1\, \mathcal{V}_1 + \kappa \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \mathrm{DX}_2\, \mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} - \mathcal{A} - \mathcal{D}$$

When looking on elastic pendulum for lumped mass, we can assume:

(1) $w_p \to 0\,, h_p \to 0$ for the lumped mass (hence θ_p doesn't matter any more)
(2) $k_2 \to 0$ for the connection, only to the first base, and not the 2nd one ($\kappa \to 0$)
(3) Arbitrarily I will assume $x_1 \to 0,\ y_1 \to 0$ which means also the 1st base is static



The equations of motion become:

$$x_p \left(\frac{1}{\sqrt{x_p^2 + y_p^2}} - 1 \right) = x_p{}'' \tag{1.1}$$

$$y_p \left(\frac{1}{\sqrt{x_p^2 + y_p^2}} - 1 \right) - \frac{g}{\mathrm{L0}_1} \frac{m_p}{k_1} = y_p{}'' \tag{1.2}$$

Finding the equilibrium point - we set the derivatives of x_p, y_p to 0 :

$$\left\{ \begin{matrix} \boldsymbol{x_p} = \boldsymbol{0} \\ \boldsymbol{y_p} = -1 - \frac{g}{\mathrm{L0}_1} \frac{m_p}{k_1} \end{matrix} \right\}, \left\{ \begin{matrix} x_p = 0 \\ y_p = 1 - \frac{g}{\mathrm{L0}_1} \frac{m_p}{k_1} \end{matrix} \right\} \tag{1.3}$$

The 1st option is the relevant one, for the considered state of $y_p < 0$.

We can also note that if considering $k_1 \to Inf\ : y_{p_{equib}} = -\mathrm{L0}_1$, which fits to a problem of a simple pendulum, hanged on a rod and not a spring.

assumption is y_p>0 fits to $y_p \to \frac{k_1 \mathrm{L0}_1 - g m_p}{k_1} = \mathrm{L0}_1 - \frac{g m_p}{k_1}$, so $\mathrm{L0}_1 > \frac{g m_p}{k_1}$ otherwise it means the spring k_1 is to small and weak.

assumption is y_p<0 fits to $y_p \to -\left(1 + \frac{g}{\mathrm{L0}_1} \frac{m_p}{k_1}\right)$

Linearization around the equilibrium point

1st order linearization is : $\sqrt{f[x,y]} = \sqrt{f[0,0]} + \frac{f^{(1,0)}[0,0]}{2\sqrt{f[0,0]}} x + \frac{f^{(0,1)}[0,0]}{2\sqrt{f[0,0]}} y + O[x]^2 + O[y]^2$

Near equilibrium:

$$x_p = x_{p_0} + \delta x_p \; ; \;\; y_p = y_{p_0} + \delta y_p$$

$$\sqrt{x_p^2 + y_p^2} = \sqrt{x_{p_0}^2 + y_{p_0}^2} + \frac{2x_{p_0}}{2\sqrt{x_{p_0}^2 + y_{p_0}^2}} \delta x_p + \frac{2y_{p_0}}{2\sqrt{x_{p_0}^2 + y_{p_0}^2}} \delta y_p$$

Testing for the 1st equilibrium point of:

$$y_{p_0} = -\left(1 + \frac{g}{\mathrm{L0}_1} \frac{m_p}{k_1}\right) = -(1+\gamma) \;\; ; \quad x_{p_0} = 0 \quad ; \quad \gamma > 0$$

$$\rightarrow \qquad \sqrt{x_p^2 + y_p^2} = \pm(y_{p_0} + \delta y_p) \tag{1.4}$$

The equations are written as:

$$\left(\cancel{x_{p_0}} + \delta x_p\right)\left(1 \pm (y_{p_0} + \delta y_p)\right) = \mp(y_{p_0} + \delta y_p)\delta \ddot{x}_p$$

$$\left(y_{p_0} + \delta y_p\right)\left(1 \pm \left(y_{p_0} + \delta y_p\right)\right) = \mp(y_{p_0} + \delta y_p)(\gamma + \delta \ddot{y}_p)$$

⇨ Using (1.4) and neglecting small terms such as $(\delta {x_p}^2)$, $(\delta \ddot{y}_p \delta x_p)$:

$$(\delta x_p)(1 + y_{p_0}) = -\delta \ddot{x}_p (y_{p_0}) \tag{1.5}$$

$$\left(y_{p_0}\right)(1 + y_{p_0}) + (\delta y_p)(1 + 2y_{p_0} + \gamma) = -(\delta \ddot{y}_p)(y_{p_0}) - \gamma(y_{p_0})$$

⇨ Using the relation from equilibrium above $\left(\pm 1 - y_{p_0}\right) = \gamma$ to eliminate o(1) terms

⇨

$$\begin{pmatrix} \delta \ddot{x}_p \\ \delta \ddot{y}_p \end{pmatrix} + \begin{pmatrix} \frac{1}{1+1/\gamma} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta x_p \\ \delta y_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.6}$$

It is equivalent to the matrix notation :

$$M\ddot{x} + C\dot{x} + Kx == F \tag{1.7}$$

Where :

$$M = \text{IdentityMatrix}_{2x2} \quad ; \quad C = \{0\} \quad ; \quad K = \begin{pmatrix} \frac{1}{1+1/\gamma} & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Where we know to find the natural frequencies by the requirement of:

$$det|K - \omega^2 M| = 0 \tag{1.8}$$

It is an equation of 4th order for ω , the relevant solutions are:

$${\omega_x}^2 = \frac{1}{1+1/\gamma} < 1 \quad ; \quad {\omega_y}^2 = 1 \tag{1.9}$$

$$rescaling\ result\ for\ comparison\ to\ literature:$$

$$\omega_y \omega_S = \sqrt{\frac{k_1}{m_p}}$$

$\omega_x \omega_S = \sqrt{\frac{\gamma}{1+\gamma}}\sqrt{\frac{k_1}{m_p}} = \sqrt{\frac{gk_1}{k_1 \mathrm{L0}_1 + gm_p}}$, where **expecting** $\omega_x \omega_S = \sqrt{\gamma}\sqrt{\frac{k_1}{m_p}} = \sqrt{\frac{g}{\mathrm{L0}_1}}$

If reformatting the above to $\sqrt{\frac{g}{\mathrm{L0}_1 + \frac{gm_p}{k_1}}}$ it is equivalent to the natural frequency of simple pendulum with a constant length of $\mathrm{L0}_1 + \frac{gm_p}{k_1}$!

Extra Limiting cases:

When $k_1 \to \infty$ it affects $\gamma \to 0\ and$ we get $\omega_y = 0$ and $\omega_x = \sqrt{\frac{g}{\mathrm{L0}_1}}$ which is similar to $\omega_{\text{simple\_pendulum}}$

Appendix A –Equations of Motion for 9D.O.F case

From (4), using (3), we get the *dimensional* equations of motion for the 2 quads and the payload:

$$
\left(
\begin{array}{c}
m_1\ddot{x}_1 = -k_1 \dfrac{\mathrm{dx}_1\left(\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}-\mathrm{L0}_1\right)}{\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}} \\
m_1\ddot{y}_1 = k_1 \dfrac{\mathrm{dy}_1\left(\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}-\mathrm{L0}_1\right)}{\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}} - m_1 g \\
I_1\ddot{\theta}_1 = 0 \\
m_2\ddot{x}_2 = -k_2 \dfrac{\mathrm{dx}_2\left(\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}-\mathrm{L0}_2\right)}{\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}} \\
m_2\ddot{y}_2 = \dfrac{\mathrm{dy}_2 k_2\left(\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}-\mathrm{L0}_2\right)}{\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}} - m_2 g \\
I_2\ddot{\theta}_2 = 0 \\
m_p\ddot{x}_p = k_1 \dfrac{\mathrm{dx}_1\left(\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}-\mathrm{L0}_1\right)}{\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}} + k_2 \dfrac{\mathrm{dx}_2\left(\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}-\mathrm{L0}_2\right)}{\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}} \\
m_p\ddot{y}_p = -k_1 \dfrac{\mathrm{dy}_1\left(\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}-\mathrm{L0}_1\right)}{\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}} - k_2 \dfrac{\mathrm{dy}_2\left(\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}-\mathrm{L0}_2\right)}{\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}} - g m_p \\
I_p\ddot{\theta}_p = k_1 \dfrac{\mathrm{term1}\left(\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}-\mathrm{L0}_1\right)}{\sqrt{\mathrm{dx}_1^2+\mathrm{dy}_1^2}} - k_2 \dfrac{\mathrm{term2}\left(\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}-\mathrm{L0}_2\right)}{\sqrt{\mathrm{dx}_2^2+\mathrm{dy}_2^2}}
\end{array}
\right)
$$

* no non-conservative forces are included here.

Where:

$$
\left(
\begin{array}{c}
\mathrm{dx}_1 = w_p\,\mathrm{c}(\theta_p) + h_p\,\mathrm{s}(\theta_p) - x_p + x_1 \\
\mathrm{dy}_1 = h_p\,\mathrm{c}(\theta_p) - w_p\,\mathrm{s}(\theta_p) + y_p - y_1 \\
\mathrm{dx}_2 = -w_p\,\mathrm{c}(\theta_p) + h_p\,\mathrm{s}(\theta_p) - x_p + x_2 \\
\mathrm{dy}_2 = h_p\,\mathrm{c}(\theta_p) + w_p\,\mathrm{s}(\theta_p) + y_p - y_2 \\
\mathrm{term1} = h_p\left(x_1(-\mathrm{c}(\theta_p)) + x_p\,\mathrm{c}(\theta_p) + (y_p - y_1)\,\mathrm{s}(\theta_p)\right) + w_p\left((y_p - y_1)\,\mathrm{c}(\theta_p) + x_1\,\mathrm{s}(\theta_p) - x_p\,\mathrm{s}(\theta_p)\right) \\
\mathrm{term2} = h_p\left(x_2\,\mathrm{c}(\theta_p) - x_p\,\mathrm{c}(\theta_p) + (y_2 - y_p)\,\mathrm{s}(\theta_p)\right) + w_p\left((y_p - y_2)\,\mathrm{c}(\theta_p) + x_2\,\mathrm{s}(\theta_p) - x_p\,\mathrm{s}(\theta_p)\right)
\end{array}
\right)
$$

Appendix B –Linearization, and natural frequencies

In this section, I'll linearize the equations of motion, and find the natural frequencies of the free non forced system.

Reminding the non-dimensional equations from (10):

$$\ddot{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \mathrm{DX}_1\, \mathcal{V}_1 + \kappa \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \mathrm{DX}_2\, \mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} - \mathcal{A} - \mathcal{D}$$

We set

$$x_p = \delta\mathrm{x} + x_{p_0} \quad ; \quad y_p = \delta\mathrm{y} + y_{p_0} \quad ; \quad \theta_p = \delta\theta + \theta_{p_0} \tag{B1}$$

$$\ddot{x_p} = \delta\ddot{\mathrm{x}} \quad ; \quad \ddot{y_p} = \delta\ddot{\mathrm{y}} \quad ; \quad \ddot{\theta_p} = \delta\ddot{\theta}$$

While

$$x_{p_0} = w_p \quad ; \quad y_{p_0} = -\left(\frac{1}{2}\gamma + h_p + 1\right) \quad ; \quad \theta_{p_0} = 0 \tag{B2}$$

I'll manipulate the equations before further calculation:

Defining :

$$A = -\frac{1}{(\mathrm{DX}_1 - 1)} = \sqrt{\mathrm{dx}_1^2 + \mathrm{dy}_1^2} \quad ; \quad B = -\frac{1}{(\mathrm{DX}_2 - 1)} = \sqrt{\mathrm{dx}_2^2 + \mathrm{dy}_2^2} \tag{B3}$$

And eliminating $\mathcal{A}, \mathcal{D}\ to\ 0$ for this analysis, getting:

$$\ddot{\mathcal{X}} A\, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} (A-1)\, B\, \mathcal{V}_1 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} A\, (B-1)\, \mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} A\, B \tag{B4}$$

Later on I'll use small angles rounding:

$$\cos(\delta\theta) \to 1, \sin(\delta\theta) \to \delta\theta \tag{B5}$$

and neglecting multiplications of the small perturbations elements (i.e. $\delta\mathrm{y}\delta\theta$, $\delta\theta^2$, etc.).

Remember that quads locations are also given, as in (12).

I would like to develop the A, B terms by a Taylor series expansion.

$$\mathrm{A} = \tfrac{2+\gamma}{2} - \delta\mathrm{y} + \delta\theta w_p \qquad \text{(B6)}$$

$$\mathrm{B} = \tfrac{2+\gamma}{2} - \delta\mathrm{y} - \delta\theta w_p$$

$\mathcal{V}_i$ after small angle assumption is:

$$\mathcal{V}_1 = \begin{pmatrix} -\delta\mathrm{x} + h_p\delta\theta \\ -1 - \dfrac{\gamma}{2} + \delta\mathrm{y} - w_p\delta\theta \\ -\dfrac{1}{2}(2+\gamma)w_p + h_p\delta\mathrm{x} + w_p\delta\mathrm{y} + \left(-\dfrac{1}{2}(2+\gamma)h_p - h_p^2 - w_p^2\right)\delta\theta \end{pmatrix}$$

$$\mathcal{V}_2 = \begin{pmatrix} -\delta\mathrm{x} + h_p\delta\theta \\ -1 - \dfrac{\gamma}{2} + \delta\mathrm{y} + w_p\delta\theta \\ -\dfrac{1}{2}(2+\gamma)w_p - h_p\delta\mathrm{x} + w_p\delta\mathrm{y} + \left(\dfrac{1}{2}(2+\gamma)h_p + h_p^2 + w_p^2\right)\delta\theta \end{pmatrix}$$

After implementing (B6) in (B4), and eliminating more small elements multiplications:

$$\begin{pmatrix} \dfrac{1}{4}(\gamma+2)^2\delta\mathrm{x}''(t) \\ \dfrac{1}{4}(\gamma+2)^2\delta\mathrm{y}''(t) \\ \dfrac{1}{4}(\gamma+2)^2\delta\theta''(t) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2}\gamma(\gamma+2)h_p\delta\,\theta(t) - \dfrac{1}{2}\gamma(\gamma+2)\delta\,\mathrm{x}(t) \\ -\dfrac{1}{2}(\gamma+2)^2\delta\,\mathrm{y}(t) \\ \dfrac{3\gamma(\gamma+2)h_p\delta\,\mathrm{x}(t)}{2\left(h_p^2+w_p^2\right)} - \dfrac{3(\gamma+2)\left(2\gamma h_p^2 + \gamma(\gamma+2)h_p + 2(\gamma+2)w_p^2\right)\delta\,\theta(t)}{4\left(h_p^2+w_p^2\right)} \end{pmatrix} \qquad \text{(B7)}$$

Or setting in the matrix form of $Mx'' + Cx' + Kx = F$ which gives:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\tfrac{1}{4}(\gamma+2)^2 \qquad ; \qquad F = \{0\}\ ;\ C = \{0\} \qquad \text{(B8)}$$

$$K = \begin{pmatrix} \tfrac{1}{2}\gamma(\gamma+2) & 0 & -\tfrac{1}{2}\gamma(\gamma+2)h_p \\ 0 & \tfrac{1}{2}(\gamma+2)^2 & 0 \\ -\tfrac{3\gamma(\gamma+2)h_p}{2\left(h_p^2+w_p^2\right)} & 0 & \tfrac{3(\gamma+2)\left(2\gamma h_p^2+\gamma(\gamma+2)h_p+2(\gamma+2)w_p^2\right)}{4\left(h_p^2+w_p^2\right)} \end{pmatrix}$$

Where the natural frequencies are calculated from:

$$\mathbf{Det}\left(\boldsymbol{K} - \boldsymbol{\omega}^2\boldsymbol{M}\right) = \mathbf{0} \quad => \quad \omega_{1,2,3} \qquad \text{(B9)}$$

For the y component we get uncoupled frequency as:

$$\boldsymbol{\omega_y}^{\mathbf{2}} = \mathbf{2} \tag{B10}$$

For θ, x components we get coupled frequencies as :

$$\boldsymbol{\omega_1}^{\mathbf{2}} = \frac{\zeta - \sqrt{\beta}}{2(2+\gamma)(h_p^2 + w_p^2)} \quad ; \quad \boldsymbol{\omega_2}^{\mathbf{2}} = \frac{\zeta + \sqrt{\beta}}{2(2+\gamma)(h_p^2 + w_p^2)} \tag{B11}$$

Where

$$\beta = -24\gamma(2+\gamma)(h_p^2 + w_p^2)(\gamma h_p + 2w_p^2) + \left(3\gamma(2+\gamma)h_p + 8\gamma h_p^2 + 4(3+2\gamma)w_p^2\right)^2$$

$$= 24\gamma^2(2+\gamma)h_p^3 + 64\gamma^2 h_p^4 + 24\gamma(6+5\gamma+\gamma^2)h_p w_p^2 + 16(3+\gamma)^2 w_p^4 + \gamma h_p^2\left(9\gamma(2+\gamma)^2 + 16(6+5\gamma)w_p^2\right) > 0$$

$$\zeta = 6\gamma h_p + 3\gamma^2 h_p + 8\gamma h_p^2 + 12 w_p^2 + 8\gamma w_p^2$$

Conditions for $\boldsymbol{\omega_i}^{\mathbf{2}}$>0 :

$$\beta > 0 \ , \quad \zeta - \sqrt{\beta} > 0 \tag{B12}$$

For $h_p, w_p, \gamma > 0$ it is straight forward that also $\beta, \zeta > 0$.

$\zeta^2 > \beta$ gives $\gamma(2+\gamma)(h_p^2 + w_p^2)(\gamma h_p + 2w_p^2) > 0$ which is also always true.

Appendix C –system parameters vs. natural frequencies

As continuation to Appendix B natural frequencies – I'll show here the change of $\boldsymbol{\omega_y}, \boldsymbol{\omega_x}, \omega_\theta$ and their relations as functions of the parameters γ, h_p, w_p.

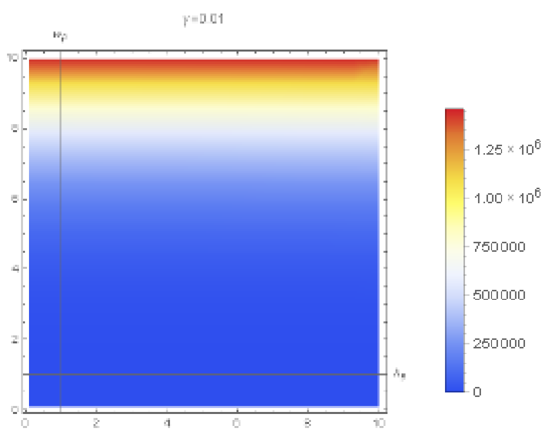
The assumed characteristic values range of those variables are:

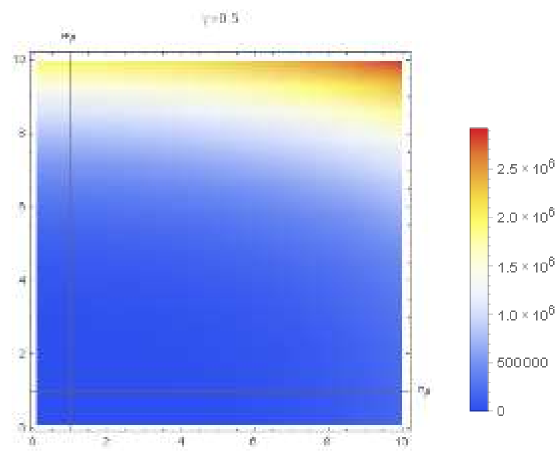
$$h_p, w_p = 0.1\ to\ 10 \quad ; \quad \gamma = 0.01\ to\ 0.9 \quad ^*$$

$$*\ in\ accordance\ with\ estimation\ of\ L0_1\ as\ 0.1\ to\ 10\ meters$$

If referring to β from (B11) we see it should be a big number in order to get $\frac{\omega_2}{\omega_1} \sim \frac{\omega_\theta}{\omega_x}$ >>1.

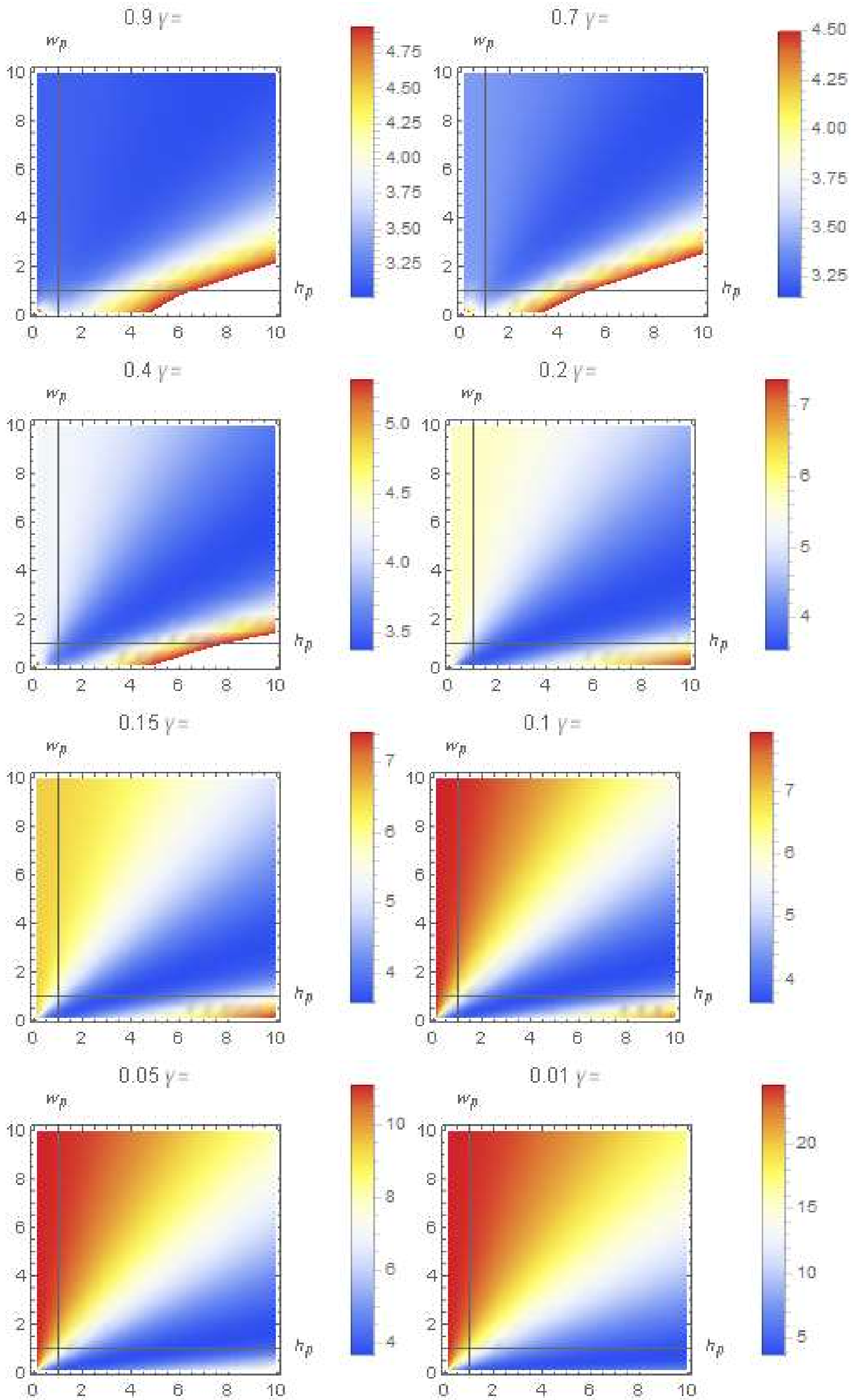
β behavior can be seen by:





It is expected that for $\gamma < 0.1$, for $w_p > 9$, without any seen dependency on h_p, we'll get high ratio between the frequencies.

Verifying for the relation itself $\Omega = \frac{\omega_\theta}{\omega_x}$ we note few working areas:



It is noted that for small γ we get high values of Ω $(higher\ then\ 10)$.

Numeric examples :

γ	w_p	h_p	ω_θ	ω_x	Ω
0.01	**10**	**2**	$\mathbf{2.40}$	$\mathbf{0.0997}$	$\mathbf{24.08}$
0.25	10	2	2.41	0.47	5.12
0.01	2	2	1.738	0.0995	17.47
0.05	2	2	1.764	0.218	8.08

For given $w_p \gg h_p$ it is noted that γ affects, almost, only on ω_x.

For given $w_p \ll h_p$, Ω is not high enough to neglect θ as part of the system degrees of freedom.

Appendix D – reducing system's DOF

For the case of $\boldsymbol{\omega_\theta} \gg \boldsymbol{\omega_y}$, I would like to neglect θ by setting $\ddot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}} \cong \mathbf{0} \rightarrow \theta = \theta_0 + \delta\theta$, where $\boldsymbol{\theta_0} = \mathbf{0}$.

Back to the starting 3DOF EOM (10) I will isolate $\theta = \delta\theta$ using the 3rd equation, in order to get $\theta = f(x,y)$ and use it back in 1st two equations of x,y. and by that getting 2DOF dynamics to investigate asymptotically.

__The procedure is:__

From 3rd equation of (10):

$0 = (-\alpha)\left(1 - \frac{1}{A}\right)\mathcal{V}_{13} + \kappa(-\alpha)\left(1 - \frac{1}{B}\mathcal{L}\right)\mathcal{V}_{23}$, where $\kappa = \mathcal{L} = 1$

we write it as:

$$\mathbf{0} = \big(\mathbf{B}\,(\mathbf{A}-\mathbf{1})\big)\boldsymbol{\mathcal{V}}_{\mathbf{13}} + \mathbf{A}\,\big((\mathbf{B}-\mathbf{1})\big)\boldsymbol{\mathcal{V}}_{\mathbf{23}} \qquad \textbf{(D1)}$$

In order to extract θ_p we **use first order approximation**:

$$\cos(\theta_p) \rightarrow 1 \;\; ; \;\; \sin(\theta_p) \rightarrow \theta_p \qquad \text{(D2)}$$

to get:

$$\mathcal{V}_{13} = a_{12} + c_{11}\theta_p \quad ; \quad \mathcal{V}_{23} = b_{12} + c_{12}\theta_p \qquad \text{(D3)}$$

And **use Taylor series** (around $\theta_0 = 0$) for A,B :

$$A = \sqrt{a_{11}} + \frac{a_{12}}{\sqrt{a_{11}}}\,\theta_p \quad ; \quad B = \sqrt{b_{11}} + \frac{b_{12}}{\sqrt{b_{11}}}\theta_p \qquad \text{(D4)}$$

Where :

$$\begin{pmatrix} a_{11} \\ a_{12} \\ b_{11} \\ b_{12} \\ c_{11} \\ c_{12} \end{pmatrix} = \begin{pmatrix} \left(-h_p + y_1 - y_p\right)^2 + \left(w_p + x_1 - x_p\right)^2 \\ h_p(x_1 - x_p) + w_p(y_1 - y_p) \\ \left(-h_p + y_2 - y_p\right)^2 + \left(-w_p + x_2 - x_p\right)^2 \\ h_p\left(x_2 - x_p\right) - w_p(y_2 - y_p) \\ h_p\left(y_1 - y_p\right) - w_p(x_1 - x_p) \\ h_p(y_2 - y_p) + w_p(x_2 - x_p) \end{pmatrix} = f(h_p, w_p, y_p, y_1, y_2, x_p, x_1, x_2)$$

Setting (4,5) into (2) **and eliminating 2[nd] and 3[rd] order elements of $\boldsymbol{\theta_p}$,** getting the relation :

$$\theta_p = \frac{a_{12}b_{11}(\sqrt{a_{11}} - a_{11}) + a_{11}b_{12}(\sqrt{b_{11}} - b_{11})}{a_{12}^2 b_{11} + a_{11}b_{12}^2 - a_{12}b_{12}\left(\sqrt{a_{11}} - a_{11} + \sqrt{b_{11}} - b_{11}\right) - b_{11}c_{11}(\sqrt{a_{11}} - a_{11}) - a_{11}c_{12}(\sqrt{b_{11}} - b_{11})} \qquad (\boldsymbol{D5})$$

θ_p $is\ therefore\ a\ function\ of\ \{h_p, w_p, x_i, y_i\}\ ;\ \ i = 1,2,p$

<u>*Next step is :*</u>

Using (3) for the 1[st] two equations of (1) and getting new equations for a 2D.O.F problem, to asymptotically investigate :

Non-dimensional 2D.O.F Equations of motion are:

$$\ddot{\mathcal{X}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(1 - \tfrac{1}{A}\right)\mathcal{V}_1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(1 - \tfrac{1}{B}\right)\mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \end{pmatrix} - C\,\dot{\mathcal{X}} - \begin{pmatrix} Fx \\ Fy \end{pmatrix}\dot{\mathcal{X}}^2 \qquad \text{(D6)}$$

Where:

$$\mathcal{X} = \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

$$A = \sqrt{(h_p\theta_p + w_p - x_p + x_1)^2 + (-h_p + \theta_p w_p - y_p + y_1)^2}$$

$$B = \sqrt{(h_p\theta_p - w_p - x_p + x_2)^2 + (-h_p - \theta_p w_p - y_p + y_2)^2}$$

$$\mathcal{V}_1 = \begin{pmatrix} w_p + x_1 - x_p + h_p\theta_p \\ -h_p + y_1 - y_p + w_p\theta_p \end{pmatrix}\ ;\ \ \mathcal{V}_2 = \begin{pmatrix} -w_p + x_2 - x_p + h_p\theta_p \\ -h_p + y_2 - y_p - w_p\theta_p \end{pmatrix}\ ;$$

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \qquad ; \qquad \begin{pmatrix} Fx \\ Fy \end{pmatrix} = \begin{pmatrix} \rho C_D h_p \\ \rho C_D w_p \end{pmatrix}$$