(described in 18/09/17)

(modified on 3/10/17)

Non-dimensional Equations of motion are:

$$\ddot{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \left(1 - \tfrac{1}{A}\right) \mathcal{V}_1 + \kappa \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \left(1 - \tfrac{1}{B}\mathcal{L}\right) \mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} \tag{1}$$

Where:

$$\mathcal{X} = \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix}$$

$$A = \sqrt{\left(h_p \sin(\theta_p) + l_p \cos(\theta_p) - x_p + x_1\right)^2 + \left(-h_p \cos(\theta_p) + l_p \sin(\theta_p) - y_p + y_1\right)^2}$$

$$B = \sqrt{\left(h_p \sin(\theta_p) - l_p \cos(\theta_p) - x_p + x_2\right)^2 + \left(-h_p \cos(\theta_p) - l_p \sin(\theta_p) - y_p + y_2\right)^2}$$

$$\mathcal{V}_1 = \begin{pmatrix} \sin(\theta_p)\, h_p + \cos(\theta_p)\, l_p + x_1 - x_p \\ -\cos(\theta_p)\, h_p + \sin(\theta_p)\, l_p + y_1 - y_p \\ l_p \left(\cos(\theta_p)\,(y_1 - y_p) - \sin(\theta_p)\,(x_1 - x_p)\right) + h_p \left(\cos(\theta_p)\,(x_1 - x_p) + \sin(\theta_p)\,(y_1 - y_p)\right) \end{pmatrix}$$

$$\mathcal{V}_2 = \begin{pmatrix} \sin(\theta_p)\, h_p - \cos(\theta_p)\, l_p + x_2 - x_p \\ -\cos(\theta_p)\, h_p - \sin(\theta_p)\, l_p + y_2 - y_p \\ h_p \left(\cos(\theta_p)\,(x_2 - x_p) + \sin(\theta_p)\,(y_2 - y_p)\right) + l_p \left(\sin(\theta_p)\,(x_2 - x_p) + \cos(\theta_p)\,(y_p - y_2)\right) \end{pmatrix}$$

$$\alpha = \frac{m_p \mathrm{L0}_1{}^2}{I_p} = \frac{3\mathrm{L0}_1{}^2}{\left(w_p{}^2 + h_p{}^2\right)} \quad *$$

$$\kappa = \frac{k_2}{k_1}$$

$$\mathcal{L} = \frac{\mathrm{L0}_2}{\mathrm{L0}_1}$$

$$\gamma = \frac{g}{\mathrm{L0}_1} \frac{m_p}{k_1}$$

$*\quad (assuming\ Inertia\ of\ a\ rectangular\ payload)$

All dimensional quantities are originally normalized by :

$$\omega_s{}^2 = \frac{k_1}{m_p}$$

A symmetrical case is considered and it is:

$$k_2 \to k_1 \quad ; \quad \mathrm{L0}_2 \to \mathrm{L0}_1$$

*Note: typical values that are intended to be considered are :

$k_1 \to 200,\ \mathrm{L0}_1 \to 2$

$m_p \to 2,\ h_p \to 0.1,\ \ w_p \to 1,$

$g \to 9.81$

$x_1[t] \to 0$
$y_1[t] \to 0$
$x_2[t] \to 2w_p$
$y_2[t] \to y_1[t]$

Equilibrium point is :

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{w_p} \\ -(\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\gamma} + \boldsymbol{h_p} + \mathbf{1}) \\ \mathbf{0} \end{pmatrix}$$

Linearizing about that equilibrium point:

$$x_p \to \delta\mathrm{x} + x_{p_0}$$

$$y_p \to \delta\mathrm{y} + y_{p_0}$$

$$\theta_p \to \delta\theta + \theta_{p_0}$$

…

After all calculations we can describe **the linearized equations as:**

$\mathbf{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F}$

Where :

$$\boldsymbol{M} = \begin{pmatrix} (\gamma+2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\gamma+2) \end{pmatrix}$$

$$\boldsymbol{F} = \{0\}, \boldsymbol{C} = \{0\}$$

$$\boldsymbol{K} = \begin{pmatrix} 2\gamma & 0 & -2\gamma h_p \\ 0 & 2 & 0 \\ -2\alpha\gamma h_p & 0 & \alpha\left(2\gamma h_p^2 + \gamma(\gamma+2)h_p + 2(\gamma+2)w_p^2\right) \end{pmatrix}$$

The natural frequencies can be calculated from :

$$\mathrm{Det}[K - \omega^2 M] = 0$$

For the y component we get :

$$\omega_y{}^2 = 2$$

Other frequencies are:

$$\omega_{1,2} = \pm\frac{\sqrt{\frac{2\gamma}{\rho} + \alpha\gamma h_p + \frac{2\alpha\gamma h_p^2}{\rho} + 2\alpha w_p^2 \pm \frac{\sqrt{-4\rho\left(2\alpha\gamma^2 h_p + 4\alpha\gamma w_p^2\right) + \left(-2\gamma - \alpha\gamma\rho h_p - 2\alpha\gamma h_p^2 - 2\alpha\rho w_p^2\right)^2}}{\rho}}}{\sqrt{2}}$$

It can be shown (detailed later) that , for certain parameters, we get :

$$\omega_1 \sim \boldsymbol{\omega_x} < \sqrt{2} \quad ; \quad \omega_2 \sim \boldsymbol{\omega_\theta} \gg \omega_y = \sqrt{2}$$

For the case of $\boldsymbol{\omega_\theta} \gg \boldsymbol{\omega_y}$, I would like to neglect θ by setting $\ddot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}} \cong \mathbf{0} \to \ \theta = \theta_0 + \delta\theta$, where $\boldsymbol{\theta_0} = \mathbf{0}$. *It will remain true for whole the rest of the problem.*

Back to the starting 3D.O.F E.O.M (1) I will isolate $\theta = \delta\theta$ using the 3rd equation, in order to get $\theta = f(x, y)$ and use it back in 1st two equations of x,y. and by that getting 2D.O.F dynamics to investigate asymptotically.

The procedure is like this :

From 3rd equation of (1) :

$0 = (-\alpha)\left(1 - \frac{1}{A}\right)\mathcal{V}_{13} + \kappa(-\alpha)\left(1 - \frac{1}{B}\mathcal{L}\right)\mathcal{V}_{23}$, where $\kappa = \mathcal{L} = 1$

we write it as :

$$\mathbf{0} = \big(\mathbf{B}\,(\mathbf{A} - \mathbf{1})\big)\boldsymbol{\mathcal{V}}_{\mathbf{13}} + \mathbf{A}\,\big((\mathbf{B} - \mathbf{1})\big)\boldsymbol{\mathcal{V}}_{\mathbf{23}} \qquad \mathbf{(2)}$$

In order to extract θ_p we **use first order approximation** :

$$\cos(\theta_p) \to 1 \ ; \ \ \sin(\theta_p) \to \theta_p \qquad (3)$$

to get:

$$\mathcal{V}_{13} = a_{12} + c_{11}\theta_p \quad ; \quad \mathcal{V}_{23} = b_{12} + c_{12}\theta_p \qquad (4)$$

And **use Taylor series** (around $\theta_0 = 0$) for A,B :

$$A = \sqrt{a_{11}} + \frac{a_{12}}{\sqrt{a_{11}}}\,\theta_p \quad ; \quad B = \sqrt{b_{11}} + \frac{b_{12}}{\sqrt{b_{11}}}\theta_p \qquad (5)$$

Where :

$$\begin{pmatrix} a_{11} \\ a_{12} \\ b_{11} \\ b_{12} \\ c_{11} \\ c_{12} \end{pmatrix} = \begin{pmatrix} \left(-h_p + y_1 - y_p\right)^2 + \left(w_p + x_1 - x_p\right)^2 \\ h_p(x_1 - x_p) + w_p(y_1 - y_p) \\ \left(-h_p + y_2 - y_p\right)^2 + \left(-w_p + x_2 - x_p\right)^2 \\ h_p\left(x_2 - x_p\right) - w_p(y_2 - y_p) \\ h_p\left(y_1 - y_p\right) - w_p(x_1 - x_p) \\ h_p(y_2 - y_p) + w_p(x_2 - x_p) \end{pmatrix} = f(h_p, w_p, y_p, y_1, y_2, x_p, x_1, x_2)$$

Setting (4,5) into (2) **and eliminating 2nd and 3rd order elements of $\boldsymbol{\theta_p}$,** getting the relation :

$$\theta_p = \frac{a_{12}b_{11}(\sqrt{a_{11}} - a_{11}) + a_{11}b_{12}(\sqrt{b_{11}} - b_{11})}{a_{12}^2 b_{11} + a_{11}b_{12}^2 - a_{12}b_{12}\left(\sqrt{a_{11}} - a_{11} + \sqrt{b_{11}} - b_{11}\right) - b_{11}c_{11}(\sqrt{a_{11}} - a_{11}) - a_{11}c_{12}(\sqrt{b_{11}} - b_{11})} \qquad \mathbf{(6)}$$

θ_p *is therefore a function of* $\{h_p, w_p, x_i, y_i\}$; $i = 1,2,p$

Next step is :

Using (3) for the 1st two equations of (1) and getting new equations for a 2D.O.F problem, to asymptotically investigate :

Non-dimensional 2D.O.F Equations of motion are:

$$\ddot{\mathcal{X}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \frac{1}{A}\right) \mathcal{V}_1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \frac{1}{B}\right) \mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \end{pmatrix} - C\,\dot{\mathcal{X}} + \begin{pmatrix} Fx \\ Fy \end{pmatrix} \qquad (1)$$

Where:

$$\mathcal{X} = \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

$$A = \sqrt{(h_p\theta_p + w_p - x_p + x_1)^2 + (-h_p + \theta_p w_p - y_p + y_1)^2}$$

$$B = \sqrt{(h_p\theta_p - w_p - x_p + x_2)^2 + (-h_p - \theta_p w_p - y_p + y_2)^2}$$

$$\mathcal{V}_1 = \begin{pmatrix} w_p + x_1 - x_p + h_p\theta_p \\ -h_p + y_1 - y_p + w_p\theta_p \end{pmatrix}$$

$$\mathcal{V}_2 = \begin{pmatrix} -w_p + x_2 - x_p + h_p\theta_p \\ -h_p + y_2 - y_p - w_p\theta_p \end{pmatrix}$$

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

$$F_x, F_y \ are\ the\ aerodynamic\ drag\ forces$$

$$\theta_p\ is\ taken\ from\ (6) above\ as\ function\ of\ \{h_p, w_p, x_i, y_i\}\ ;\ \ i = 1,2,p$$