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A Mathematica program for the approximate analytical solution to a nonlinear undamped Duffing equation by a new approximate approach ☆

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Abstract

According to Mickens [R.E. Mickens, Comments on a Generalized Galerkin's method for non-linear oscillators, J. Sound Vib. 118 (1987) 563], the general HB (harmonic balance) method is an approximation to the convergent Fourier series representation of the periodic solution of a nonlinear oscillator and not an approximation to an expansion in terms of a small parameter. Consequently, for a nonlinear undamped Duffing equation with a driving force $B\cos(\omega x)$, to find a periodic solution when the fundamental frequency is identical to ω, the corresponding Fourier series can be written as

$$\tilde{y}(x) = \sum_{n=1}^{m} a_n \cos\bigl[(2n-1)\omega x\bigr].$$

How to calculate the coefficients of the Fourier series efficiently with a computer program is still an open problem. For HB method, by substituting approximation $\tilde{y}(x)$ into force equation, expanding the resulting expression into a trigonometric series, then letting the coefficients of the resulting lowest-order harmonic be zero, one can obtain approximate coefficients of approximation $\tilde{y}(x)$ [R.E. Mickens, Comments on a Generalized Galerkin's method for non-linear oscillators, J. Sound Vib. 118 (1987) 563]. But for nonlinear differential equations such as Duffing equation, it is very difficult to construct higher-order analytical approximations, because the HB method requires solving a set of algebraic equations for a large number of unknowns with very complex nonlinearities. To overcome the difficulty, forty years ago, Urabe derived a computational method for Duffing equation based on Galerkin procedure [M. Urabe, A. Reiter, Numerical computation of nonlinear forced oscillations by Galerkin's procedure, J. Math. Anal. Appl. 14 (1966) 107–140]. Dooren obtained an approximate solution of the Duffing oscillator with a special set of parameters by using Urabe's method [R. van Dooren, Stabilization of Cowell's classic finite difference method for numerical integration, J. Comput. Phys. 16 (1974) 186–192].

In this paper, in the frame of the general HB method, we present a new iteration algorithm to calculate the coefficients of the Fourier series. By using this new method, the iteration procedure starts with $a(x)\cos(\omega x) + b(x)\sin(\omega x)$, and the accuracy may be improved gradually by determining new coefficients $a_1, a_2, \ldots$ will be produced automatically in an one-by-one manner. In all the stage of calculation, we need only to solve a cubic equation. Using this new algorithm, we develop a Mathematica program, which demonstrates following main advantages over the previous HB method: (1) it avoids solving a set of associate nonlinear equations; (2) it is easier to be implemented into a computer program, and produces a highly accurate solution with analytical expression efficiently. It is interesting to find that, generally, for a given set of parameters, a nonlinear Duffing equation can have three independent oscillation modes. For some sets of the parameters, it can have two modes with complex displacement and one with real displacement. But in some cases, it can have three modes, all of them having real displacement. Therefore, we can divide the parameters into two classes, according to the solution property: there is only one mode with real displacement and there are three modes with real displacement. This program should be useful to study the dynamically periodic behavior of a Duffing oscillator and can provide an approximate analytical solution with high-accuracy for testing the error behavior of newly developed numerical methods with a wide range of parameters.

☆ This paper and its associated computer program are available via the Computer Physics Communications homepage on ScienceDirect (http://www.sciencedirect.com/science/journal/00104655).

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Program summary

Title of program: AnalyDuffing.nb
Catalogue identifier: ADWR_v1_0
Program summary URL: http://cpc.cs.qub.ac.uk/summaries/ADWR_v1_0
Program obtainable from: CPC Program Library, Queen's University of Belfast, N. Ireland
Licensing provisions: none
Computer for which the program is designed and others on which it has been tested: the program has been designed for a microcomputer and been tested on the microcomputer.
Computers: IBM PC
Installations: the address(es) of your computer(s)
Operating systems under which the program has been tested: Windows XP
Programming language used: Software Mathematica 4.2, 5.0 and 5.1
No. of lines in distributed program, including test data, etc.: 23 663
No. of bytes in distributed program, including test data, etc.: 152 321
Distribution format: tar.gz
Memory required to execute with typical data: 51 712 Bytes
No. of bits in a word:
No. of processors used: 1
Has the code been vectorized?: no
Peripherals used: no
Program Library subprograms used: no
Nature of physical problem: To find an approximate solution with analytical expressions for the undamped nonlinear Duffing equation with periodic driving force when the fundamental frequency is identical to the driving force.
Method of solution: In the frame of the general HB method, by using a new iteration algorithm to calculate the coefficients of the Fourier series, we can obtain an approximate analytical solution with high-accuracy efficiently.
Restrictions on the complexity of the problem: For problems, which have a large driving frequency, the convergence may be a little slow, because more iterative times are needed.
Typical running time: several seconds
Unusual features of the program: For an undamped Duffing equation, it can provide all the solutions or the oscillation modes with real displacement for any interesting parameters, for the required accuracy, efficiently. The program can be used to study the dynamically periodic behavior of a nonlinear oscillator, and can provide a high-accurate approximate analytical solution for developing high-accurate numerical method.

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Keywords: Undamped Duffing equation; Approximate analytical solution; Nonlinear second-order ordinary differential equation; High-accurate methods; Nonlinear oscillator; Harmonic balance method

1. Introduction

In the past several decades, the numerical solution for a second-order differential equation with periodic solutions has attracted great interest [3–14]. It is well known that Duffing equation is a mathematical model for describing a classical oscillator in a single-well or double-well potential with a periodic driving force, and has been widely investigated in chaotic phenomena [15,16]. In this paper the nonlinear undamped Duffing equation is considered, which can be written as

$$y''(x) = f\big(x, y(x)\big) \quad \text{with } y(x_0) = y_0 \text{ and } y'(x_0) = y'_0, \tag{1}$$

where

$$f\big(x, y(x)\big) = -\varepsilon y(x) - \psi y(x)^3 + B\cos(\omega x).$$

This equation with the parameters of $B = 0.002$, $\omega = 1.01$ and $\varepsilon = \psi == 1$ was first studied numerically by van Dooren [3] in 1974. He presented a modified Cowell's two-step method for (1), and got an approximate analytical solution by using Galerkin procedure developed by Urabe forty years ago [2,17]. The approximate analytical solution for (1) with this set of parameters can be written as

$$g(x) = \sum_{i=0}^{4} K_{2i+1}\cos\big[(2i+1)\omega x\big] \tag{2}$$

with

$$\{K_1, K_3, K_5, K_7, K_9\} = \{0.20017947753662, 2.46946143256 \times 10^{-4}, 3.0401499 \times 10^{-7}, 3.7435 \times 10^{-10}, 4.6 \times 10^{-13}\}.$$

We will call (1) with this set of parameters the Dooren problem, and (2) the Dooren solution for convenience. As to the terminology about the solution of (2), different authors adopted different words to express it. Such as 'exact' solution was used in [5–7,10], and 'periodic solution' was used in [3,4]. Obviously, it does not have the meaning of analytical in mathematics, but just for distinguishing from the numerically approximate solution. For convenience, in this paper, we use 'analytical' or 'approximate analytical' to describe such kind of solution as (2).

Since 1974, Dooren solution has been accepted by many authors as the touchstone to test the accuracy of newly developed numerical methods for Dooren problem. Dooren solution greatly impulsed the activities to search for a better numerical method for this nonlinear equation. Chawla and Neta developed a two-step four-order P-stable method [4], Jain developed a modified Stieffel–Bettis method [9], Ananthakrishnaiah, using multi-derivative, developed an adaptive method [5] and a P-stable Obrechkoff method [6], Simos developed a P-stable complete in phase trigonometrically-fitted Obrechkoff method of $O(h^{13})$ [7]. We studied the importance of the first-order derivative formula in the Obrechkoff method and introduced a high-accurate first-order derivative formula to improve the accuracy of the Obrechkoff method greatly [13], developed a new trigonometrically-fitting method [14], which simplified the coefficients of Simos Obrechkoff method [7] and improved its stability for small step size, and developed several Obrechkoff multi-step methods to increase the efficiency. In particular, the solution to the problem (1) has been greatly improved in the accuracy from, for step size of $\pi/5$, initially, 10^{-2} [8], 10^{-4} [6,10] to 10^{-6} [7], finally, to 10^{-9} [14].

With the development of new numerical method, Dooren solution can no longer satisfy the requirement for accuracy. Certainly, Galerkin method can have high-accurate solution by supposing that the m terms of a Fourier series to the approximation of $y(x)$ is big enough. With the same problem as HB method, it cannot avoid solving a set of associate nonlinear algebraic equations. Therefore, it is necessary to use interpolation to build a set of associate nonlinear equations for m unknowns and use Newton method to simplify such kind of equations. The Galerkin's procedure is time consuming and converges slowly, as HB method, it is not an efficient method for Duffing equation. We can also use the fitting method [12], the built function of *Mathematica* software, and high-accurate numerical solution, which is obtained by a high-accurate Obrechkoff method, to find a better analytical solution, which can be written as

$$g_0(x) = \sum_{n=1}^{6} K_n \cos\big[(2n-1)\omega x\big] \tag{3}$$

with the coefficients as

$$\{K_1, K_2, K_3, K_4, K_5, K_6\} = \{0.200179477536594, 2.469461432561 \times 10^{-4}, 3.0401498535 \times 10^{-7}, 3.74349 \times 10^{-10}, 4.6095 \times 10^{-13}, 5.67 \times 10^{-16}\}.$$

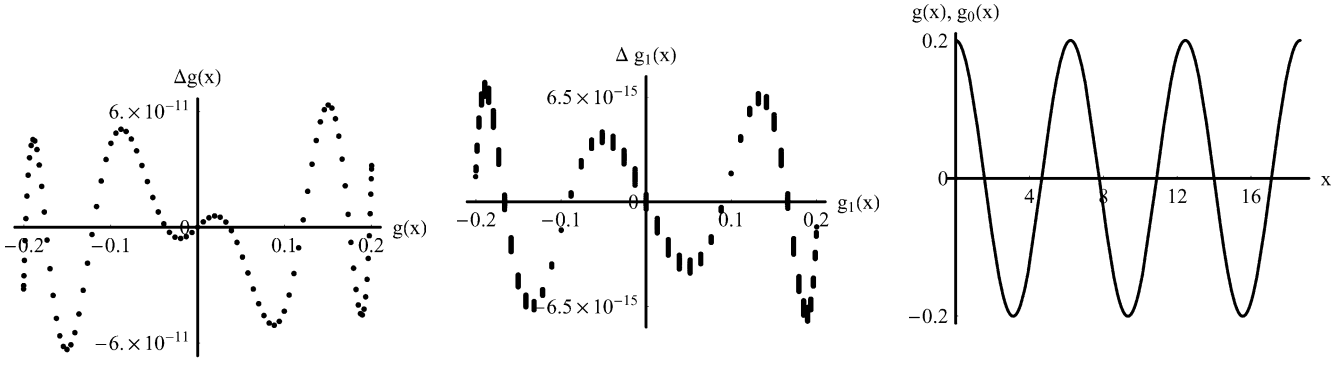
In order to compare the error behavior between $g(x)$ and $g_0(x)$, We will define the following deviation function for $g(x)$, which can be written as

$$\Delta g(x) = g''(x) + g(x) + g^3(x) - B\cos(\omega x). \tag{4}$$

Since $\Delta g(x)$ and $g(x)$ have the same period, it is convenient to plot $\Delta g(x)$ in the phase portrait as shown in Fig. 1(a) and (b). Both $g(x)$ and $g_0(x)$ change with x follow the same rule as shown in Fig. 1(c). These figures correspond to the initial values of

$$\begin{aligned} &g(0) = 0.200426728069 \quad \text{and} \quad g'(0) = 0, \\ &g_0(0) = 0.2004267280696459 \quad \text{and} \quad g_0'(0) = 0 \end{aligned} \tag{5}$$

with the maximum error of the $g(x)$ and $g_0(x)$ to be about 6×10^{-11} and 6.5×10^{-15}, respectively.



(a) Doreen solution

(b) Solution obtained by the fitting method

(c) The displacement of the oscillator

Fig. 1. (a) $\Delta g(x)$ as a function of $g(x)$; (b) $\Delta g_0(x)$ as a function of $g_0(x)$; (c) $g(x)$ and $g_0(x)$ are plotted as a function of x according to (2) and (3), respectively.

The high-accurate numerical solution is the objective for finding the high-accurate analytical solution by the fitting method of *Mathematica*. Hence, it is not convenient for the fitting method to get an analytical solution. Therefore, it is interesting and important to search for a better method for the high-accurate analytical solution.

When commenting to the paper by Chen [19], Mickens stressed that it is important to remember that this approximation (generalized HB) is to the convergent Fourier series representation of the solution and not an approximation to an expansion in terms of a small parameter ε [1].

Hence, to find a periodic solution for a nonlinear undamped Duffing equation with a driving force $B\cos(\omega x)$ with the required accuracy when the fundamental frequency is identical to ω, we can follow the general idea of Galerkin method [2,17] and HB method [18] and write the approximation $\tilde{y}(x)$ as a Fourier series as given in the abstract. In order to calculate the coefficients of the Fourier series efficiently with a computer program, we will develop a new method in this paper.

The purpose of this paper is to present a modified approximate method to obtain the analytical solution for the Duffing equation (1) for any interesting parameters with the desired accuracy. Based on this algorithm, we will develop a *Mathematica* program. This paper contains four sections, in the next section, the new approximate approach method for the Duffing equation will be derived; in the third section, the complete *Mathematica* program code will be presented and some related problems will be discussed, in the fourth section, some numerical illustrations will be given, and in the final section, the conclusion will be drawn.

2. Derivation

In this section, we have three subsections. In the first one, five basic assumptions will be given as the preliminaries, the next two are for the derivation of the recurrent formulas and the first approximation formulas.

2.1. Preliminaries

We have five assumptions about the analytical solution $z(x)$ to (1).

(a) We suppose that $z(x)$ can be expanded into a convergent series as

$$\{z_0(x), z_1(x), \ldots, z_n(x), \ldots\}, \tag{6}$$

which should satisfy the following requirement, if using (4) to find their deviation function

$$|\Delta z_0(x)| > |\Delta z_1(x)| > |\Delta z_2(x)| > \cdots > |\Delta z_n(x)| > \cdots \tag{7}$$

with

$$|\Delta z_\infty| \to 0. \tag{8}$$

We will discuss how to find z_ν by some special approximation method.

(b) There exist the following recurrent relations for the convergent series (6)

$$z_\nu(x) = z_{\nu-1}(x) + a_\nu(x)\cos(\omega x) + b_\nu(x)\sin(\omega x) \quad (\nu = 1, 2, \ldots) \tag{9}$$

with the following approximation for initialization

$$z_0(x) = a_0(x)\cos(\omega x) + b_0(x)\sin(\omega x) \tag{10}$$

which is the basic structure used in the averaging method [2].

(c) $z(x)$ is periodic, with the same periodicity as that of the driving force, $T = 2\pi/\omega$. So $z(x) = z(x + nT)$,

$$z_\nu(x) = z_\nu(x + nT), \qquad a_\nu(x) = a_\nu(x + nT) \quad \text{and} \quad b_\nu(x) = b_\nu(x + nT) \tag{11}$$

hold for any integer n.

(d) The solution of (1) is an even function. It can be easily verified that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(y(x) - y(-x)\big) + \varepsilon\big(y(x) - y(-x)\big) + \psi\big(y(x) - y(-x)\big)\big(y(x)^2 + y(-x)^2 + y(x)y(-x)\big) = 0. \tag{12}$$

Therefore, it can be valid only when $y(x) = y(-x)$. It follows that $z_\nu(x) = z_\nu(-x)$ is valid for $\nu = 0, 1, 2, \ldots$. From (9) we have

$$\big(a_\nu(x) - a_\nu(-x)\big)\cos(\omega x) + \big(b_\nu(x) + b_\nu(-x)\big)\sin(\omega x) = 0, \tag{13}$$

so that

$$a_\nu(x) = a_\nu(-x), \qquad b_\nu(x) = -b_\nu(-x) \quad \text{and} \quad b_\nu(0) = 0. \tag{14}$$

(e) We assume both $a_\nu(x)$ and $b_\nu(x)$ are real.

2.2. Recurrent formulas

From (9), the first-order derivative of $z_\nu(x)$ can be written as

$$z'_\nu(x) = z'_{\nu-1}(x) - \omega\big(a_\nu(x)\sin(\omega x) - b_\nu(x)\cos(\omega x)\big). \tag{15}$$

We use

$$a'_\nu(x)\cos(\omega x) + b'_\nu(x)\sin(\omega x) = 0, \tag{16}$$

as the auxiliary condition, which can guarantee no higher-order derivatives of $a_\nu(x)$ and $b_\nu(x)$ to be involved in the calculation. This formula is often used in many classical methods for the nonlinear problems, such as the averaging method [2]. In our method, $a_\nu(x)$ and $b_\nu(x)$ will be determined by the following procedure.

From (15), the second-order derivative of $z_\nu(x)$ is found to be

$$z''_\nu(x) = z''_{\nu-1}(x) - \omega^2\big(a_\nu(x)\cos(\omega x) + b_\nu(x)\sin(\omega x)\big) - \omega\big(a'_\nu(x)\sin(\omega x) - b'_\nu(x)\cos(\omega x)\big). \tag{17}$$

Substituting $z_\nu(x)$ and $z''_\nu(x)$ into (1) in terms of (9) and (17), we have

$$\begin{aligned} &z''_{\nu-1}(x) + \varepsilon z_{\nu-1}(x) - B\cos(\omega x) + (\varepsilon-\omega^2)\big(a_\nu(x)\cos(\omega x) + b_\nu(x)\sin(\omega x)\big) \\ &\quad + \psi\big(a_\nu(x)\cos(\omega x) + b_\nu(x)\sin(\omega x) + z_{\nu-1}(x)\big)^3 - \omega\big(a'_\nu(x)\sin(\omega x) - b'_\nu(x)\cos(\omega x)\big) = 0. \end{aligned} \tag{18}$$

We assume that $z_{\nu-1}(x)$ has been found at this stage. According to (2) and (3) and the evenness property, the form of $z_\nu(x)$ can be assumed as

$$z_\nu(x) = \sum_{k=1}^{m} \chi_{\nu,k}\cos\big((2k-1)\omega x\big) \tag{19}$$

for all positive integer values of ν. The validation of (19) can be found in [1].

By (16), (18) and (19), $a'_\nu(x)$ and $b'_\nu(x)$ can be separately expressed in terms of $a_\nu(x)$ and $b_\nu(x)$ as Fourier series such as

$$a'_\nu(x) = c_{\nu,0}(x) + c_{\nu,1}(x)\cos(2\omega x) + c_{\nu,2}(x)\cos(4\omega x) + \cdots + d_{\nu,1}(x)\sin(2\omega x) + d_{\nu,2}(x)\sin(4\omega x) + \cdots \tag{20}$$

and

$$b'_\nu(x) = \tilde{c}_{\nu,0}(x) + \tilde{c}_{\nu,1}(x)\cos(2\omega x) + \tilde{c}_{\nu,2}(x)\cos(4\omega x) + \cdots + \tilde{d}_{\nu,1}(x)\sin(2\omega x) + \tilde{d}_{\nu,2}(x)\sin(4\omega x) + \cdots, \tag{21}$$

where $c_{\nu,0}(x), c_{\nu,1}(x), \ldots, \tilde{c}_{\nu,0}(x), \tilde{c}_{\nu,1}(x), \ldots, d_{\nu,0}(x), d_{\nu,1}(x), \ldots, \tilde{d}_{\nu,0}(x), \tilde{d}_{\nu,1}(x), \ldots$, which are relevant to the coefficients of Fourier series, are only the functions of $a_\nu(x)$ and $b_\nu(x)$. If all $a_\nu(x)$ and $b_\nu(x)$ on the right-hand side of (20) and (21) are replaced by $a_\nu(0)$ and $b_\nu(0)$, the approximate expressions for $a'_\nu(x)$ and $b'_\nu(x)$ can be obtained only in terms of $a_\nu(0)$ and $b_\nu(0)$. The resultant approximate expressions, which are denoted as $\alpha'_\nu(x)$ and $\beta'_\nu(x)$, respectively, can be written as

$$\alpha'_\nu(x) = c_{\nu,0}(0) + c_{\nu,1}(0)\cos(2\omega x) + c_{\nu,2}(0)\cos(4\omega x) + \cdots + d_{\nu,1}(0)\sin(2\omega x) + d_{\nu,2}(0)\sin(4\omega x) + \cdots \tag{22}$$

and

$$\beta'_\nu(x) = \tilde{c}_{\nu,0}(0) + \tilde{c}_{\nu,1}(0)\cos(2\omega x) + \tilde{c}_{\nu,2}(0)\cos(4\omega x) + \cdots + \tilde{d}_{\nu,1}(0)\sin(2\omega x) + \tilde{d}_{\nu,2}(0)\sin(4\omega x) + \cdots. \tag{23}$$

By integrating (22) and (23) from 0 to x, we have two associated integral equations for $\alpha(x)$ and $\beta(x)$ as

$$\alpha_\nu(x) = \alpha_\nu(0) + \int_0^x \big(c_{\nu,0}(0) + c_{\nu,1}(0)\cos(2\omega t) + c_{\nu,2}(0)\cos(4\omega t) + \cdots + d_{\nu,1}(0)\sin(2\omega t) + d_{\nu,2}(0)\sin(4\omega t) + \cdots\big)\,\mathrm{d}t \tag{24}$$

and

$$\beta_\nu(x) = \beta_\nu(0) + \int_0^x \big(\tilde{c}_{\nu,0}(0) + \tilde{c}_{\nu,1}(0)\cos(2\omega t) + \tilde{c}_{\nu,2}(0)\cos(4\omega t) + \cdots + \tilde{d}_{\nu,1}(0)\sin(2\omega t) + \tilde{d}_{\nu,2}(0)\sin(4\omega t) + \cdots\big)\,\mathrm{d}t. \tag{25}$$

It is easy to see that $\alpha_\nu(x)$ and $\beta_\nu(x)$ preserve the same periodic property as $a_\nu(x)$ and $b_\nu(x)$. Based on the periodic property, (24) and (25) can be greatly simplified. Letting the integral domain to be $t \in (0, T)$ and noticing $\alpha_\nu(T) = \alpha_\nu(0)$ in (24) and $\beta_\nu(T) = \beta_\nu(0)$ in (25), we have

$$\int_0^T \big(c_{\nu,0}(0) + c_{\nu,1}(0)\cos(2\omega t) + c_{\nu,2}(0)\cos(4\omega t) + \cdots + d_{\nu,1}(0)\sin(2\omega t) + d_{\nu,2}(0)\sin(4\omega t) + \cdots\big)\,\mathrm{d}t = 0 \tag{26}$$

and

$$\int_0^T \big(\tilde{c}_{\nu,0}(0) + \tilde{c}_{\nu,1}(0)\cos(2\omega t) + \tilde{c}_{\nu,2}(0)\cos(4\omega t) + \cdots + \tilde{d}_{\nu,1}(0)\sin(2\omega t) + \tilde{d}_{\nu,2}(0)\sin(4\omega t) + \cdots\big)\,\mathrm{d}t = 0. \tag{27}$$

Since for any integer n, the following relationships

$$\int_0^T \cos(2n\omega t)\,\mathrm{d}t = 0 \quad \text{and} \quad \int_0^T \sin(2n\omega t)\,\mathrm{d}t = 0 \tag{28}$$

hold, then

$$\int_0^T c_{\nu 0}(0)\,\mathrm{d}t = 0 \quad \text{and} \quad \int_0^T \tilde{c}_{\nu 0}(0)\,\mathrm{d}t = 0. \tag{29}$$

Hence, $c_{\nu,0}$ and $\tilde{c}_{\nu,0}$ must be zero, from which $a_\nu(0)$ and $b_\nu(0)$ can be determined. Once $a_\nu(0)$ are known, according to (14), $b_\nu(0) = 0$, then, $a_\nu(0)$ and $b_\nu(0)$ will be substituted back into (24) and (25), from which $\alpha_\nu(x)$ and $\beta_\nu(x)$ can be easily found. Finally, $z_\nu(x)$ for $z(x)$ with the nth approximation can be found if $a_\nu(x)$ and $b_\nu(x)$ of (15) are replaced by $\alpha_\nu(x)$ and $\beta_\nu(x)$.

2.3. Formula for the first approximation

From the last subsection, it is known that $z_\nu(x)$ can be found from $z_{\nu-1}(x)$ by the recurrent formula. In the same way, $z_{\nu-1}(x)$ can be found from $z_{\nu-2}(x)$ and so on. Hence, we need firstly to know $z_0(x)$ before starting the iterative procedure with the recurrent formulas. We will consider $z_0(x)$ as the first approximation. From (10), (16) and (18) with $\nu = 0$, we have

$$a_0'(x)\cos(\omega x) + b_0'(x)\sin(\omega x) = 0 \tag{30}$$

and

$$\begin{aligned}&(\varepsilon - \omega^2)\big(a_0(x)\cos(\omega x) + b_0(x)\sin(\omega x)\big) + \psi\big(a_0(x)\cos(\omega x) + b_0(x)\sin(\omega x)\big)^3\\ &\quad - B\cos(\omega x) - \omega\big(a_0'(x)\sin(\omega x) - b_0'(x)\cos(\omega x)\big) = 0.\end{aligned} \tag{31}$$

From (31) and condition (30), $a_0'(x)$ and $b_0'(x)$ can be separately expressed in terms of $c_{0,0}(x)$, $c_{0,1}(x)$, $c_{0,2}(x)$, $d_{0,1}(x)$, $d_{0,2}(x)$ and $\tilde{c}_{0,0}(x)$, $\tilde{c}_{0,1}(x)$, $\tilde{c}_{0,2}(x)$, $\tilde{d}_{0,1}(x)$, $\tilde{d}_{0,2}(x)$ as

$$a_0'(x) = c_{0,0}(x) + c_{0,1}(x)\cos(2\omega x) + c_{0,2}(x)\cos(4\omega x) + d_{0,1}(x)\sin(2\omega x) + d_{0,2}(x)\sin(4\omega x) \tag{32}$$

and

$$b_0'(x) = \tilde{c}_{0,0}(x) + \tilde{c}_{0,1}(x)\cos(2\omega x) + \tilde{c}_{0,2}(x)\cos(4\omega x) + \tilde{d}_{0,1}(x)\sin(2\omega x) + \tilde{d}_{0,2}(x)\sin(4\omega x), \tag{33}$$

where

$$\begin{aligned}
c_{0,0}(x) &= \frac{1}{2\omega}\left(\varepsilon - \omega^2 + \frac{3}{4}\psi\big(a_0(x)^2 + b_0(x)^2\big)\right)b_0(x),\\
c_{0,1}(x) &= -\frac{1}{2\omega}\big(\varepsilon - \omega^2 + \psi b_0(x)^2\big)b_0(x),\\
c_{0,2}(x) &= \frac{1}{8\omega}\psi\big(b_0(x)^2 - 3a_0(x)^2\big)b_0(x),\\
d_{0,1}(x) &= \frac{1}{2\omega}\left(\left(\epsilon - \omega^2 + \frac{1}{2}\psi\big(a_0(x)^2 + 3b_0(x)^2\big)\right)a_0(x) - B\right),\\
d_{0,2}(x) &= \frac{1}{8\omega}\psi\big(a_0(x)^2 - 3b_0(x)^2\big)a_0(x),\\
\tilde{c}_{0,0}(x) &= -\frac{1}{2\omega}\left(\left(\varepsilon - \omega^2 + \frac{3}{4}\psi\big(a_0(x)^2 + b_0(x)^2\big)\right)a_0(x) - B\right),\\
\tilde{c}_{0,1}(x) &= -\frac{1}{2\omega}\big(\big(\varepsilon - \omega^2 + \psi a_0(x)^2\big)a_0(x) - B\big),\\
\tilde{c}_{0,2}(x) &= -\frac{1}{8\omega}\psi\big(a_0(x)^2 - 3b_0(x)^2\big)a_0(x),
\end{aligned}$$

$$\tilde{d}_{0,1}(x) = -\frac{1}{2\omega}\left(\varepsilon - \omega^2 + \frac{1}{2}\psi\left(3a_0(x)^2 + b_0(x)^2\right)\right)b_0(x),$$
$$\tilde{d}_{0,2}(x) = \frac{1}{8\omega}\psi\left(b_0(x)^2 - 3a_0(x)^2\right)b_0(x). \tag{34}$$

If all $a_0(x)$ and $b_0(x)$ on the right-hand side of (34) are replaced by $a_0(0)$ and $b_0(0)$, we can integrate analytically the resultant expressions of $a_0'(x)$ and $b_0'(x)$. Then the approximate expressions of $a_0(x)$ and $b_0(x)$ are represented by $\alpha_0(x)$ and $\beta_0(x)$, respectively, which can be written as

$$\alpha_0(x) = \alpha_0(0) + \int_0^x \left(c_{0,0}(0) + c_{0,1}(0)\cos(2\omega t) + c_{0,2}(0)\cos(4\omega t) + d_{0,1}(0)\sin(2\omega t) + d_{0,2}(0)\sin(4\omega t)\right)\mathrm{d}t \tag{35}$$

and

$$\beta_0(x) = \beta_0(0) + \int_0^x \left(\tilde{c}_{0,0}(0) + \tilde{c}_{0,1}(0)\cos(2\omega t) + \tilde{c}_{0,2}(0)\cos(4\omega t) + \tilde{d}_{0,1}(0)\sin(2\omega t) + \tilde{c}_{0,2}(0)\sin(4\omega t)\right)\mathrm{d}t. \tag{36}$$

Applying (35) and (36) at $x = 2\pi/\omega$, $c_{0,0}$ and $\tilde{c}_{0,0}$ must be zero, so we have

$$\left(\varepsilon - \omega^2 + \frac{3}{4}\psi\left(a_0(0)^2 + b_0(0)^2\right)\right)b_0(0) = 0 \tag{37}$$

and

$$\left(\varepsilon - \omega^2 + \frac{3}{4}\psi\left(a_0(0)^2 + b_0(0)^2\right)\right)a_0(0) - B = 0. \tag{38}$$

From (37), we have

$$b_0(0) = 0 \quad \text{or} \quad \varepsilon - \omega^2 + \frac{3}{4}\psi\left(a_0(0)^2 + b_0(0)^2\right) = 0. \tag{39}$$

If the second equation of (39) is right, then it yields that B is zero, which is impossible. So the second one of (39) is invalid. It further confirms that the conclusion for $b_\nu(0) = 0$ in (14) is right. Substituting $b_0(0) = 0$ into (38), we have a cubic equation for $a_0(0)$ as

$$a_0(0)^3 + pa_0(0) + q = 0 \tag{40}$$

with

$$p = \frac{4(\varepsilon - \omega^2)}{3\psi} \quad \text{and} \quad q = -\frac{4B}{3\psi}. \tag{41}$$

We define

$$\Delta = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = \frac{4}{9\psi^2}\left(B^2 - \frac{16}{81\psi}\left(\omega^2 - \omega_0^2\right)^3\right), \tag{42}$$

where $\omega_0^2 = \varepsilon$ represents the natural frequency, from which we can determine the property of the roots from the parameters of (1).

If $\Delta > 0$, (40) has one real and two complex roots, and $\Delta < 0$, (40) has three real roots. In this paper we only consider the cases with $a_\nu(0)$ being real.

Since $\psi z(x)^3$ represents a quartic potential, $\psi > 0$. We can divide the parameters into two cases, according to the property of the roots, Δ greater or less than zero.

For Case 1, when $B^2 > 16(\omega^2 - \omega_0^2)^3/(81\psi)$, that is $\Delta > 0$, there are one real root and two complex ones. $\alpha_0(0)$ can be written as

$$\alpha_0(0) = \left(-\frac{1}{2}q + \sqrt{\Delta}\right)^{1/3} + \left(-\frac{1}{2}q - \sqrt{\Delta}\right)^{1/3} \quad \text{and} \quad \beta_0(0) = 0, \tag{43}$$

respectively.

For Case 2, when $B^2 < 16(\omega^2 - \omega_0^2)^3/(81\psi)$, that is $\Delta < 0$, so there are three real roots. $\alpha_0(0)$ can be written as

$$\left\{{}^1\alpha_0(0), {}^2\alpha_0(0), {}^3\alpha_0(0)\right\} = \sqrt{-\frac{4p}{3}}\left\{\cos\frac{\theta}{3}, \cos\frac{2\pi + \theta}{3}, \cos\frac{4\pi + \theta}{3}\right\}, \tag{44}$$

respectively, where $\theta = \arctan -q/2\sqrt{-(p/3)^3}$.

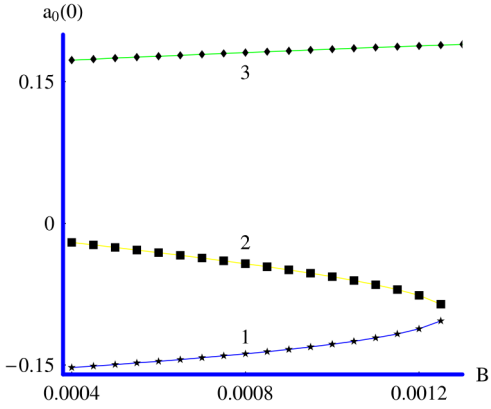


Fig. 2. The real roots of $a_0(0)$ are plotted as a function of the parameter B in the range of $\{0.0004, 0.0013\}$ according to (43) and (44), when the other parameters are fixed with $\omega = 1.01$; $\varepsilon = 1$ and $\psi = 1$.

It is interesting to see that the number of real roots for $a_0(x)$ changes with the amplitude of the driving force and jumps from one to three, when B decreases below 0.00126 (see Fig. 2).

How many undamped modes with real amplitude for a Duffing oscillator can we obtain due to the appearance of thee real roots, $\{{}^1\alpha_0(0), {}^2\alpha_0(0), {}^3\alpha_0(0)\}$? This is an interesting problem, we will discuss it in the next subsection. In the following discussion, we consider the case with only one real solution.

Since $b_0(0) = 0$, only $d_{0,1}$, $d_{0,2}$, $\tilde{c}_{0,1}$ and $\tilde{c}_{0,2}$ are not zero in (34), the approximate expressions of (32) and (33) can be reduced as

$$\alpha_0'(x) = \frac{(B - (\varepsilon - \omega^2)a_0(0))}{6\omega}\big(-\sin(2\omega x) + \sin(4\omega x)\big) \tag{45}$$

and

$$\beta_0'(x) = -\frac{(B - (\varepsilon - \omega^2)a_0(0))}{6\omega}\big(\cos(2\omega x) + \cos(4\omega x)\big). \tag{46}$$

Next, by integrating (45) and (46) analytically, $\alpha_0(x)$ and $\beta_0(x)$ can be obtained as the first approximation

$$\alpha_0(x) = \alpha_0(0) + \int_0^x \alpha_0'(t)\,\mathrm{d}t = a_0(0) + \frac{(B - (\varepsilon - \omega^2)a_0(0))}{24\omega^2}\big(2\cos(2\omega x) - \cos(4\omega x)\big) \tag{47}$$

and

$$\beta_0(x) = \beta_0(0) + \int_0^x \beta_0'(t)\,\mathrm{d}t = -\frac{(B - (\varepsilon - \omega^2)a_0(0))}{24\omega^2}\big(2\sin(2\omega x) + \sin(4\omega x)\big). \tag{48}$$

Then, according to (10), we can use the results (47) and (48) to obtain $z_0(x)$ for the first approximation, with $a_0(x) = \alpha_0(x)$ and $b_0(x) = \beta_0(x)$, which can be written as

$$z_0(x) = \alpha_0(x)\cos(\omega x) + \beta_0(x)\sin(\omega x). \tag{49}$$

2.4. Number of approximate analytical solutions

From the discussion of Section 2.3, for the parameters of Case 2, there should be three real roots as shown in (44), ${}^1\alpha_0(0)$, ${}^2\alpha_0(0)$, ${}^3\alpha_0(0)$. These three roots will result in three analytical solutions ${}^1z_0(x)$, ${}^2z_0(x)$, ${}^3z_0(x)$ for the first-order approximation. If using ${}^1z_0(x)$ to find the analytical solutions for the second-order approximation, we will encounter the same situation and obtain three real roots, which are denoted as ${}^{1,1}\alpha_1(0)$, ${}^{1,2}\alpha_1(0)$, ${}^{1,3}\alpha_1(0)$ and three analytical solutions for the second approximation as ${}^{1,1}z_1(x)$, ${}^{1,2}z_1(x)$, ${}^{1,3}z_1(x)$. Hence, we can obtain nine solutions for the second-order approximation, if ${}^1z_0(x)$, ${}^2z_0(x)$ and ${}^3z_0(x)$ are used in turn.

We use the same set of parameters as used by Dooren [3], with $B = 0.0012$, $\omega = 1.01$ and $\varepsilon = \psi = 1$ to calculate the analytical solutions for the first and second approximation. After analyzing the wave form, we find that these nine solutions are not independent. The following rules are useful for the calculation of the high-order approximation.

(1) There are only three independent analytical solutions, which can be differentiable by comparing with ${}^1z_0(x)$, ${}^2z_0(x)$ and ${}^3z_0(x)$, so there are only three undamped oscillation modes.

(2) ${}^{1}z_0(x)$, ${}^{1,1}z_1(x)$, ${}^{1,2}z_1(x)$ and ${}^{1,3}z_1(x)$ describe the same undamped modes. From their deviation functions, $\Delta({}^{1,2}z_1(x))$ and $\Delta({}^{1,3}z_1(x))$ are nearly of the same order as $\Delta({}^{1}z_0(x))$. However, $\Delta({}^{1,1}z_1(x))/\Delta({}^{1}z_0(x)) \approx 1.7 \times 10^{-3}$. So only ${}^{1,1}z_1(x)$ is needed.
(3) In the same way, ${}^{2,2}z_1(x)$ and ${}^{3,3}z_1(x)$ are needed for the second and third undamped modes.

We can apply these rules to all the high-order approximation.

3. Mathematica program

In this section, we present a *Mathematica* program to calculate the approximate analytical solutions of the Duffing equation (1) by using our new algorithm. It contains two subsections: in the first one, the outline, the modified formulas are introduced to reduce the symbol calculation in the program; the flowcharts for the program are shown and in Section 3.2, the complete program codes are given.

3.1. Outline

If $x \to \omega x$ is applied to the analytical solution (2), then $\cos(n\omega x)$ can be simplified to $\cos(nx)$. In this way, we need to replace $y(\frac{x}{\omega})$, $\frac{\mathrm{d}}{\mathrm{d}x}y(\frac{x}{\omega})$ and $\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(\frac{x}{\omega})$ by $\eta(x)$, $\omega\frac{\mathrm{d}}{\mathrm{d}x}\eta(x)$ and $\omega^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\eta(x)$, respectively, which can greatly reduce the CPU time due to less symbol manipulation.

Assuming that (9) has been iterated $(\nu-1)$ times, then $z_{\nu-1}(x)$, which is represented by $\xi(x)$ in the program, is known. Following (19), $\xi(x)$ can be written as

$$\xi(x) = \sum_{k=1}^{m_s} \gamma(k)\cos\bigl((2k-1)x\bigr). \tag{50}$$

The next step is to calculate $z_\nu(x)$, which is represented by $\eta(x)$ in the program. Through the recurrent formula for (9), $\eta(x)$ can be expressed as

$$\eta(x) = a(x)\cos(x) + b(x)\cos(x) + \xi(x). \tag{51}$$

Considering the auxiliary condition of (16) and the transform rule introduced at the beginning of this subsection, the first- and the second-order derivatives for $\eta(x)$ of (51) can be written as

$$\eta_1(x) = \omega\bigl(-a(x)\sin(x) + b(x)\cos(x) + \xi'(x)\bigr) \tag{52}$$

and

$$\eta_2(x) = \omega^2\bigl(-\bigl(a(x)\cos(x) + b(x)\sin(x)\bigr) - \bigl(a'(x)\sin(x) - b'(x)\cos(x)\bigr) + \xi''(x)\bigr), \tag{53}$$

respectively.

Since $\eta(x)$ is an approximate solution of the undamped Duffing equation, we can define a deviation function such as

$$qs(x) = \eta_2(x) + \varepsilon\eta(x) + \psi\eta(x)^3 - B\cos(x) \tag{54}$$

which should be zero.

Considering the auxiliary condition again, $a'(x)$ can be obtained by substituting $\sin(x)b'(x)$ to $\cos(x)qs(x)$ by $-a'(x)\cos(x)$, and $b'(x)$ can be obtained by substituting $\cos(x)a'(x)$ into $\sin(x)qs(x)$ by $-b'(x)\sin(x)$. Since $a'(x)$ and $b'(x)$ will be used several times, we store them into τ_1 and τ_2, respectively, such that, by using the replacing rule in *Mathematica*

$$\tau_1 = \frac{qs(x)}{\omega\sin(x)}/.\bigl\{a'(t) \to 0\bigr\}, \qquad \tau_2 = -\frac{qs(x)}{\omega\cos(x)}/.\bigl\{b'(t) \to 0\bigr\}. \tag{55}$$

In order to expand $a'(t)$ and $b'(t)$ into Fourier series, we need to manipulate the right-hand side of τ_1 and τ_2 in (55) by using the built-in function **TrigReduce** before putting them into the memory.

From (55), we can obtain all the Fourier coefficients $c_{\nu,0}$, $c_{\nu,k}$, $\tilde{c}_{\nu,0}$, $\tilde{c}_{\nu,k}$, $d_{\nu,k}$, $\tilde{d}_{\nu,k}$ defined by (20) by and (21) and store them in

$$\begin{aligned} &\mu_1, c_1(1), c_1(2), \ldots, d_1(1), d_1(2), \ldots, \\ &\mu_2, c_2(1), c_2(2), \ldots, d_2(1), d_2(2), \ldots, \end{aligned} \tag{56}$$

respectively.

By using the built-in function **Coefficient** and replacing rule, we have

$$\mu_1 = \tau_1/.\{\cos[x\_] \to 0, \sin[x\_] \to 0\},$$
$$\mu_2 = \tau_2/.\{\cos[x\_] \to 0, \sin[x\_] \to 0\} \tag{57}$$

and

$$\textbf{Table}\big[\{c_1(k), d_1(k)\} = \textbf{Coefficient}\big[\tau_1, \{\cos(2kt), \sin(2kt)\}\big],\ \{k, 1, ms\}\big], \tag{58}$$
$$\textbf{Table}\big[\{c_2(k), d_2(k)\} = \textbf{Coefficient}\big[\tau_2, \{\cos(2kt), \sin(2kt)\}\big],\ \{k, 1, ms\}\big]. \tag{59}$$

From the discussion about (37) and (38), it is known that μ_2 provides a cubic equation for $a_\nu(0)$, which is represented by α. Three roots for α are obtained by built-in function **Solve** and stored into q_1, q_2 and q_3, respectively, such as

$$\{q_1, q_2, q_3\} = \alpha/.\,\textbf{Solve}[\mu_2 = 0, \alpha]. \tag{60}$$

From $\{q_1, q_2, q_3\}$, we can determine the number of real roots to be obtained. If only one is real, it is stored in $\kappa[1]$ and set $N_s = 1$. If all of them are real, they are stored in $\kappa[1]$, $\kappa[2]$ and $\kappa[3]$ and set $N_s = 3$. This is important to determine how the accuracy of the approximate solution will be improved for the high-order harmonics.

By considering (24) and (25) and after integrating, then, according to the number of the solutions N_s, we can obtain $a(x)\cos(x) + b(x)\sin(x)$, which is stored in $t[j]$ as

$$t[j] = \cos[x]\kappa[j] + \sum_{k=1}^{m_s} \frac{1}{2k\omega}\Big\{\big(c_1(k)\cos(t) + c_2(k)\sin(t)\big)\sin(2kt) + \big(d_1(k)\cos(t) + d_2(k)\sin(t)\big)\cos(2kt)\Big\}/.\{\alpha \to \kappa[j]\}. \tag{61}$$

Finally, $\eta(x) = a(x)\cos(x) + b(x)\cos(x) + \xi(x)$ is obtained, which is stored in $tq[j]$.

The Mathematica program contains a preliminary which contains five predefined functions, two modules and a main program. The flowcharts of the main program and the modules are given in Figs. 3–6.

3.2. Program codes

3.2.1. Initialization

```
Off [ General :: "spell1" ] ; Off [ General :: "spell" ] ;
<< Graphics‘MultipleListPlot’
```

3.2.2. Preliminary

The formulas which are used repeatedly are defined beforehand.

```
ζ [t_] := Sum [γ [k] Cos[ (2 k − 1) t ], {k, 1, ms} ] ;
η [t_] := a [t] Cos[ t ] + b [ t ] Sin[ t ] + ζ [ t ] ;
η1 [t_] := −ω ( a [t] Sin[ t ] − b [t] Cos[ t ] − ζ′ [t] ) ;
qs [t_] := ω η1′[t] + ϵ η [t] + ψ η [t]^3 − B Cos[ t ] ;
G [t_] := g″ [t] + ϵ0 g [t] + ψ0 g [t]^3 − B0 Cos[ ω0 t ]
```

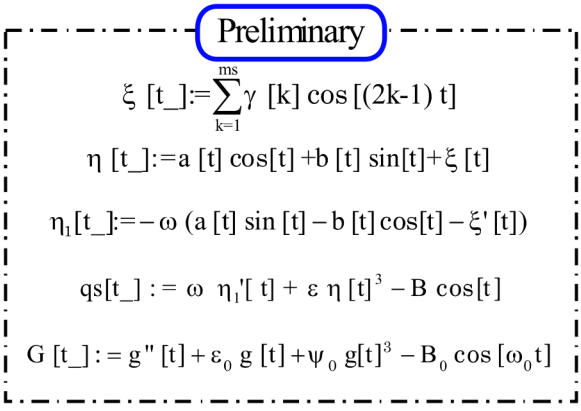


Fig. 3. The five functions are defined as the preliminary.

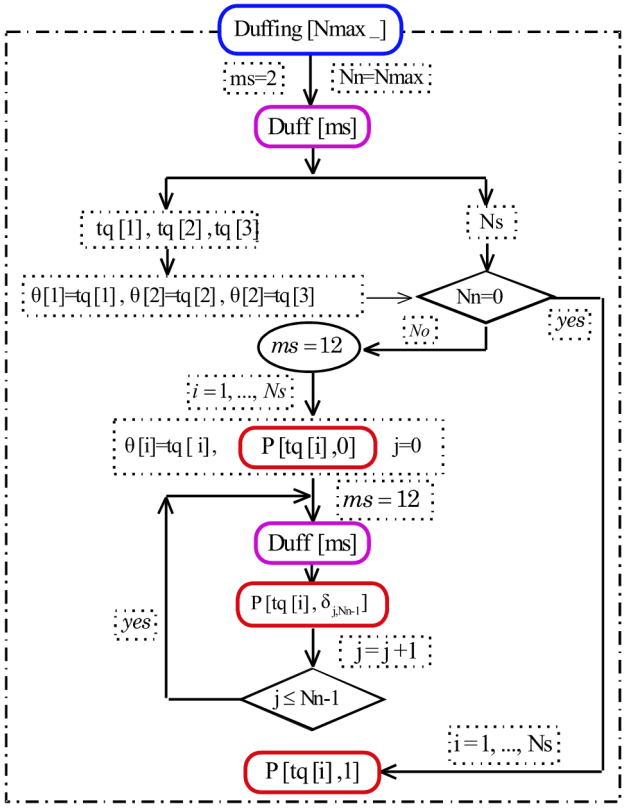


Fig. 4. The flowchart of main program Duffing.

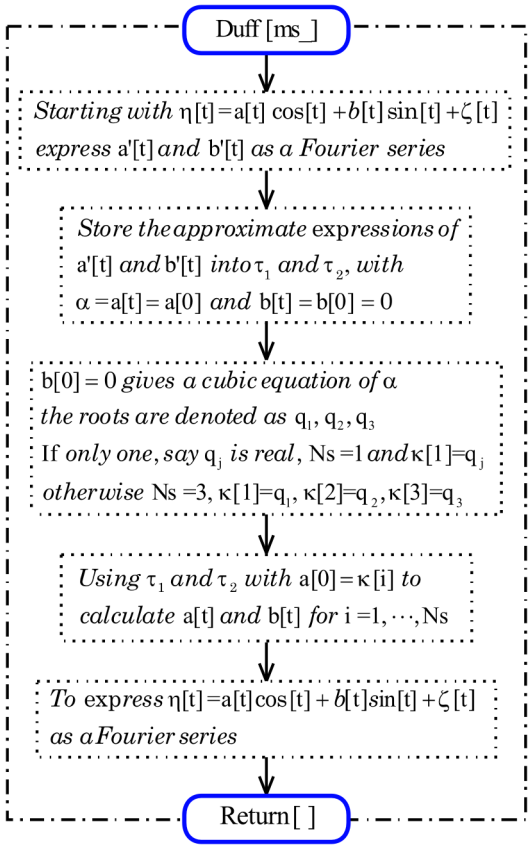


Fig. 5. The flowchart of module Duff.

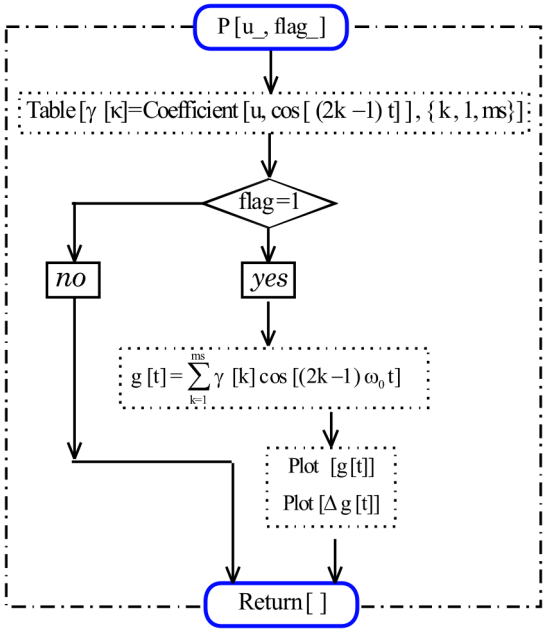


Fig. 6. The flowchart of module P.

3.2.3. *Main Program Duffing*

According to the input parameters, **Duffing** calls **Duff** for the first time to judge how many analytical solutions need to be calculated. Then the information of ${}^{i}z_0(x)$ is stored in $\theta[i]$. For every ${}^{i}z_0(x)$, **Duffing** calls **Duff** the required number of times to obtain the analytical solution.

```
Duffing[Nmax_] := (
  ds = {B → B0, ω → ω0, ϵ → ϵ0, ψ → ψ0};
  Nn = Nmax;
  ms = 2;
  Table[γ[k] = 0, {k, 1, ms}];
  Duff[ms];
  θ[1] = tq[1]; θ[2] = tq[2]; θ[3] = tq[3];
  If[Nn == 0, Do[P[tq[kj], 1], {kj, 1, Ns}],
  ms = 12;
  Do[
    tq[kj] = θ[kj]; P[tq[kj], 0];
    Do[Duff[ms]; P[tq[kj], KroneckerDelta[j, Nn − 1]], {j, 0, Nn − 1}]
  {kj, 1, Ns}]]);
```

3.2.4. Module P

Module **P**[u _, flag_] is used to deal with the calculation result $\eta(x)$ from **Duff**, which is to update the coefficients of $\xi(x)$ and plot the final results according to the flag.

```
P[u_, flag_] := (
  Table[γ[k] = Coefficient[u, Cos[(2k − 1)t], 1], {k, 1, ms}];
  If[flag == 1,
    g[t_] := Chop[Sum[γ[k]Cos[(2k − 1)ω0t], {k, 1, ms}], 10^−16];
    Print[{kj, Nn, g[t]}];
    data = Table[{g[t], G[t]}, {t, 0, 2π / ω0, π / (50ω0)}];
    MultipleListPlot[data, PlotJoined → True];
    Plot[g[t], {t, 0, 6π / ω0}];
  ]);
```

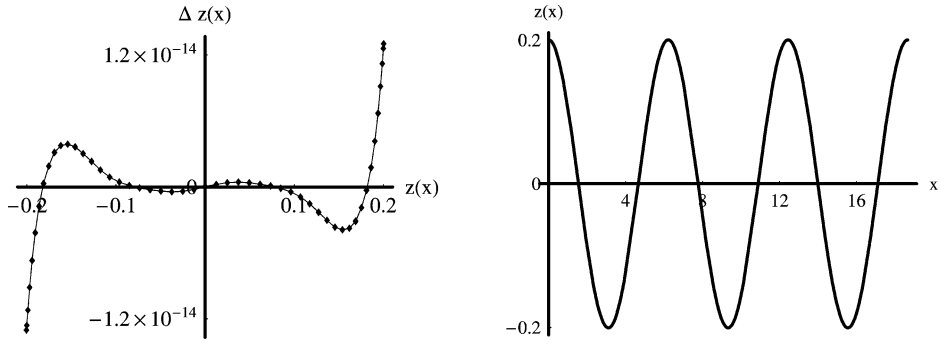
3.2.5. Module Duff

Module **Duff** is used to calculate $\eta(x)$ according to the input $\xi(x)$.

```
Duff[ms_] := (
  {τ1, τ2} = Collect[TrigReduce[qs[t] / ω{Sin[t], −Cos[t]}],
  {Cos[x_], Sin[x_]}, Simplify] /.
  {a[t] → α, b[t] → 0, a′[t] → 0, b′[t] → 0};
  {μ1, μ2} = {τ1, τ2} /.{Cos[x_] → 0, Sin[x_] → 0};
  Table[{{c1[k], d1[k]}, {c2[k], d2[k]}} =
    {Coefficient[τ1, {Cos[2kt], Sin[2kt]}],
     Coefficient[τ2, {Cos[2kt], Sin[2kt]}]}, {k, 1, ms}];
  {q1, q2, q3} = Chop[α /. Solve[(μ2 /. ds) == 0, α], 10^−20];
  Ns = 0;
  Do[If[Im[qi] == 0, Ns = Ns + 1; κ[Ns] = qi], {i, 1, 3}];
  Table[
    t[j] =
    Chop[TrigReduce[Cos[t]κ[j]+
    Sum[1 / (2kω)((c1[k]Cos[t] + c2[k]Sin[t])Sin[2kt]−
    (d1[k]Cos[t] + d2[k]Sin[t])Cos[2kt]), {k, 1, ms}]], 10^−20] /.
    {α → κ[j]} /. ds, {j, 1, Ns}];
  Table[tq[j] = Collect[t[j] + ζ[t] /. ds, Cos[x_], Simplify], {j, 1, Ns}];
Return[Ns])
```

4. Numerical illustration

The analytical solutions for the three sets of parameters are obtained by the program.



(a) $\Delta z(x)$ as a function of $z(x)$ (b) $z(x)$ as a function of x

Fig. 7. The plots are shown for the output from **Duffing[5]** with $B = 0.002$; $\omega = 1.01$; $\varepsilon = 1$ and $\psi = 1$.

4.1. The first numerical example

The first example is for the Dooren problem. We use our program for 6 iterated times. The input is given as follows:

$B_0 = 0.002$; $\omega_0 = 1.01$; $\epsilon_0 = 1$; $\psi_0 = 1$;
Timing[Duffing[5]]

The analytical function $z(x)$ is given below

$$\begin{aligned} z(x) &= 0.200179\,\mathrm{Cos}[1.01x] + 0.000246946\,\mathrm{Cos}[3.03x] \\ &\quad + 3.04015\times 10^{-7}\,\mathrm{Cos}[5.05x] + 3.74349\times 10^{-10}\,\mathrm{Cos}[7.07t] \\ &\quad + 4.60964\times 10^{-13}\,\mathrm{Cos}[9.09x] + 5.68\times 10^{-16}\,\mathrm{Cos}[11.11x]. \end{aligned} \tag{62}$$

The CPU time is 1.531 s.

$\Delta z(x)$ as a function of $z(x)$ and $z(x)$ as a function of x are plotted in Fig. 7(a) and (b), respectively. Comparison with Fig. 1 shows that our program can get a better analytical solution easily.

4.2. The second numerical example

The second example corresponds to a smaller value of B_0 so that it can have three independent solutions according to (42) with $\Delta > 0$. The subroutine Duffing is iterated 5 times. The input is given as follows:

$B_0 = 0.0012$; $\omega_0 = 1.01$; $\epsilon_0 = 1$; $\psi_0 = 1$;
Timing[Duffing[4]]

The analytical function $z(x)$ is given below:

(a) for the first solution, the analytical solution is

$$\begin{aligned} z(t) &= -0.11164\,\mathrm{Cos}[1.01t] - 0.0000426175\,\mathrm{Cos}[3.03t] \\ &\quad - 1.6277\times 10^{-8}\,\mathrm{Cos}[5.05t] - 6.21531\times 10^{-12}\,\mathrm{Cos}[7.07t] - 2.37\times 10^{-15}\,\mathrm{Cos}[9.09t]; \end{aligned} \tag{63}$$

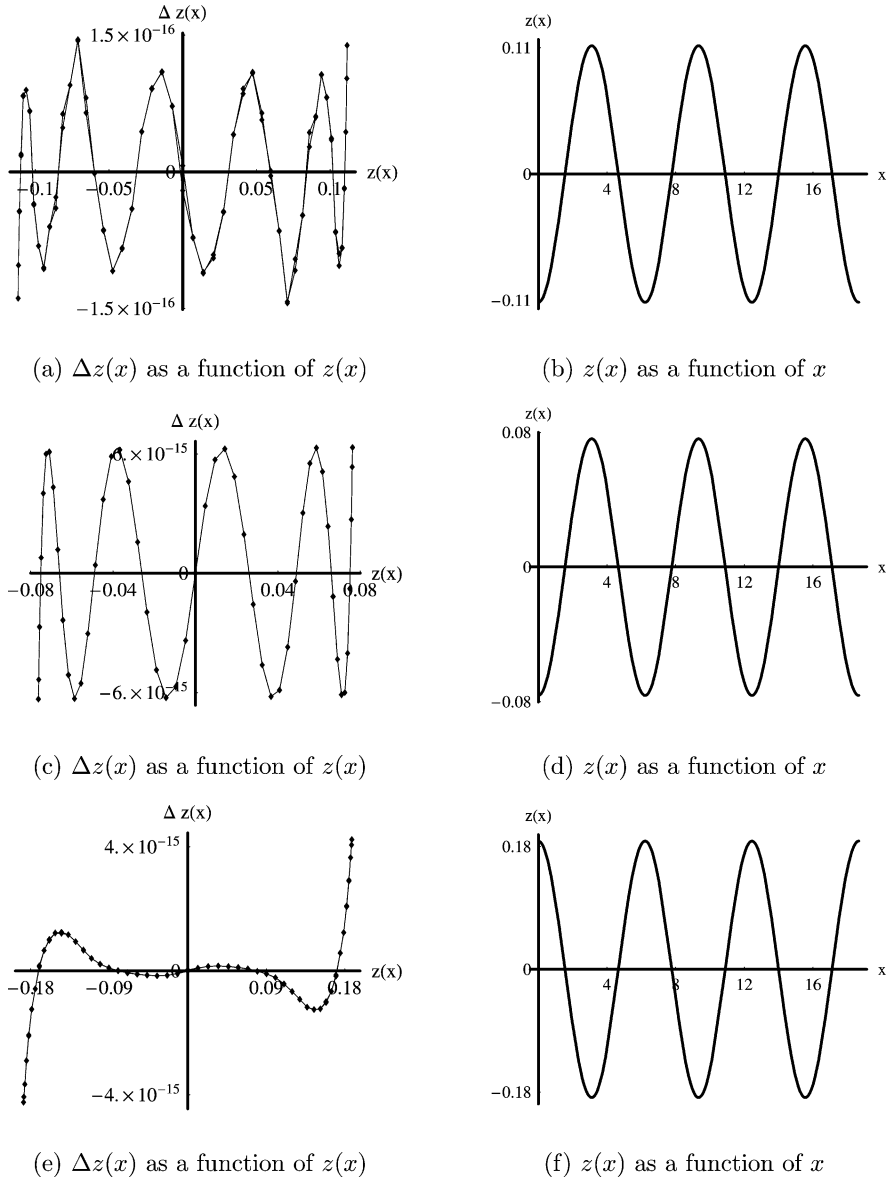
(b) for the second solution, the analytical solution is

$$\begin{aligned} z(t) &= -0.0762394\,\mathrm{Cos}[1.01t] - 0.0000135563\,\mathrm{Cos}[3.03t] \\ &\quad - 2.41314\times 10^{-9}\,\mathrm{Cos}[5.05t] - 4.29423\times 10^{-13}\,\mathrm{Cos}[7.07t]; \end{aligned} \tag{64}$$

(c) for the third solution, the analytical solution is

$$\begin{aligned} z(t) &= 0.18783\,\mathrm{Cos}[1.01t] + 0.000203822\,\mathrm{Cos}[3.03t] \\ &\quad + 2.20822\times 10^{-7}\,\mathrm{Cos}[5.05t] + 2.39268\times 10^{-10}\,\mathrm{Cos}[7.07t] \\ &\quad + 2.59065\times 10^{-13}\,\mathrm{Cos}[9.09t] + 2.72\times 10^{-16}\,\mathrm{Cos}[11.11t]. \end{aligned} \tag{65}$$

The total CPU time is 3.984 s.



(a) $\Delta z(x)$ as a function of $z(x)$ (b) $z(x)$ as a function of x

(c) $\Delta z(x)$ as a function of $z(x)$ (d) $z(x)$ as a function of x

(e) $\Delta z(x)$ as a function of $z(x)$ (f) $z(x)$ as a function of x

Fig. 8. The plots are shown for the output from **Duffing[4]** with $B = 0.0012$; $\omega = 1.01$; $\varepsilon = 1$ and $\psi = 1$. There are three solutions: (a) and (b) are for the first one; (c) and (d) are for the second one and (e) and (f) for the third one.

From Fig. 8(b), (d) and (f), it is easy to see that these solutions really describe three different vibration modes. Since (1) is a nonlinear equation, any one of these solutions cannot be expressed linearly in terms of the other two.

4.3. The third numerical example

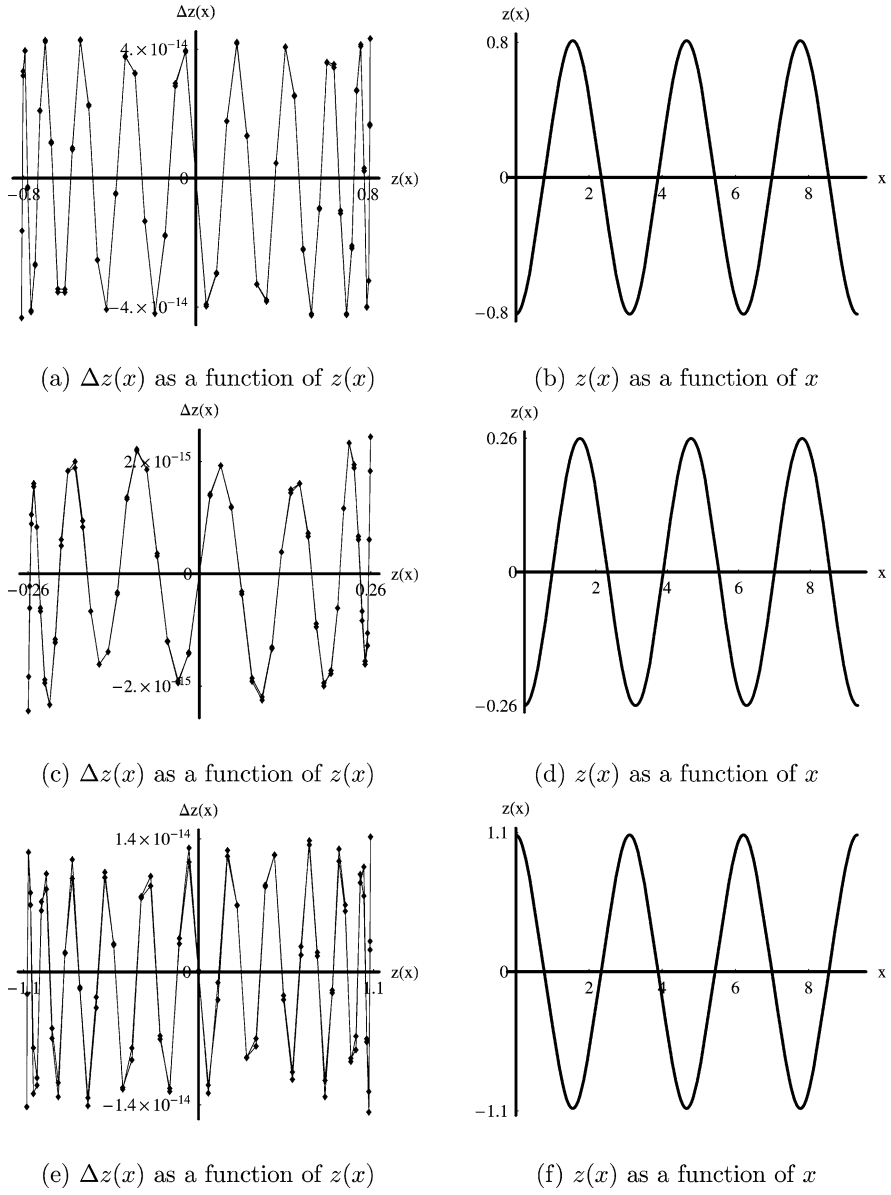
The last example also has three independent solutions but with a larger driving frequency. The subroutine Duffing is iterated 15 times. The input is given as follows:

$B_0 = 0.5$; $\omega_0 = 2.02$; $\epsilon_0 = 2$; $\psi_0 = 3$;
Timing[Duffing[14]]

The analytical function $z(x)$ and CPU time are given below:

(a) for the first one, the analytical solution is

$$z(t) = -0.797615\,\mathrm{Cos}[2.02t] - 0.0119534\,\mathrm{Cos}[6.06t] - 0.00017881\,\mathrm{Cos}[10.1t] - 2.66617 \times 10^{-6}\,\mathrm{Cos}[14.14t]$$



(a) $\Delta z(x)$ as a function of $z(x)$ (b) $z(x)$ as a function of x

(c) $\Delta z(x)$ as a function of $z(x)$ (d) $z(x)$ as a function of x

(e) $\Delta z(x)$ as a function of $z(x)$ (f) $z(x)$ as a function of x

Fig. 9. The plots are shown for the output from **Duffing[14]** with $B = 0.5$; $\omega = 2.02$; $\varepsilon = 2$ and $\psi = 3$. There are three solutions: (a) and (b) are for the first one; (c) and (d) are for the second one and (e) and (f) for the third one.

$$- 3.97411 \times 10^{-8}\,\mathrm{Cos}[18.18t] - 5.92331 \times 10^{-10}\,\mathrm{Cos}[22.22t]$$
$$- 8.82836 \times 10^{-12}\,\mathrm{Cos}[26.26t] - 1.31581 \times 10^{-13}\,\mathrm{Cos}[30.3t] - 1.96112 \times 10^{-15}\,\mathrm{Cos}[34.34t]; \quad (66)$$

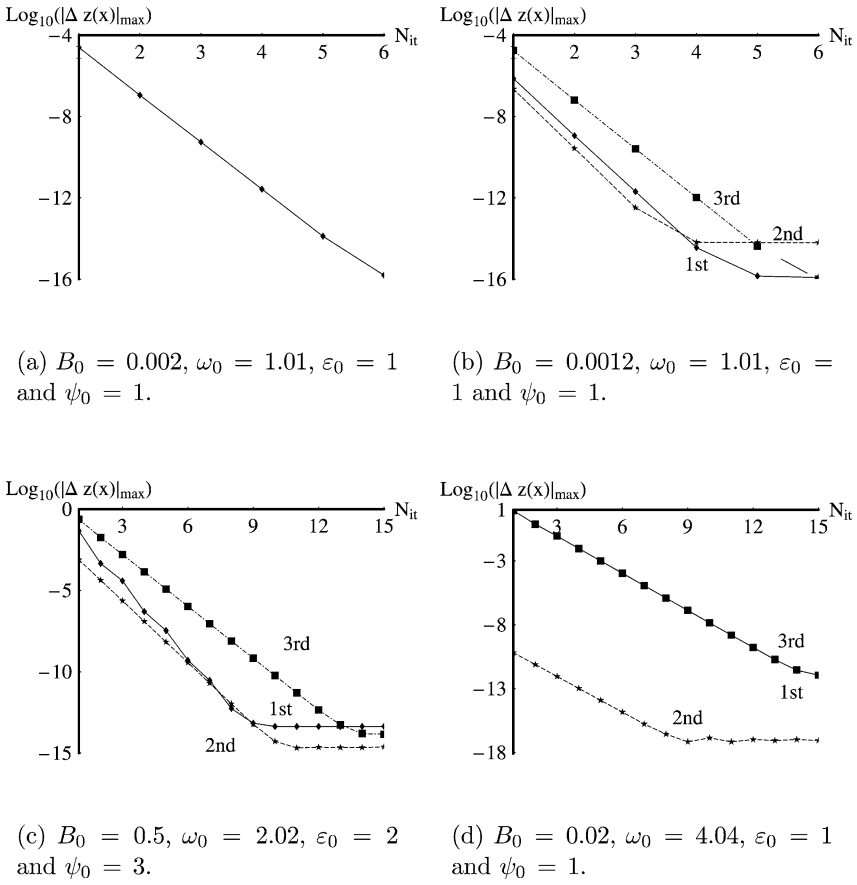
(b) for the second one, the analytical solution is

$$z(t) = -0.2592\,\mathrm{Cos}[2.02t] - 0.000379439\,\mathrm{Cos}[6.06t]$$
$$- 5.76106 \times 10^{-7}\,\mathrm{Cos}[10.1t] - 8.6678 \times 10^{-10}\,\mathrm{Cos}[14.14t]$$
$$- 1.30267 \times 10^{-12}\,\mathrm{Cos}[18.18t] - 1.95733 \times 10^{-15}\,\mathrm{Cos}[22.22t]; \quad (67)$$

(c) for the third one, the analytical solution is

$$z(t) = 1.05053\,\mathrm{Cos}[2.02t] + 0.0292924\,\mathrm{Cos}[6.06t]$$
$$+ 0.00078726\,\mathrm{Cos}[10.1t] + 0.0000212137\,\mathrm{Cos}[14.14t]$$
$$+ 5.71764 \times 10^{-7}\,\mathrm{Cos}[18.18t] + 1.54112 \times 10^{-8}\,\mathrm{Cos}[22.22t]$$
$$+ 4.15397 \times 10^{-8}\,\mathrm{Cos}[26.26t] + 1.11967 \times 10^{-11}\,\mathrm{Cos}[30.3t]$$
$$+ 3.018 \times 10^{-13}\,\mathrm{Cos}[34.34t] + 8.13484 \times 10^{-15}\,\mathrm{Cos}[38.38t] + 2.1927 \times 10^{-16}\,\mathrm{Cos}[42.42t]. \quad (68)$$

The total CPU time is 25.422 s. The plots are shown in Fig. 9.



(a) $B_0 = 0.002$, $\omega_0 = 1.01$, $\varepsilon_0 = 1$ and $\psi_0 = 1$.

(b) $B_0 = 0.0012$, $\omega_0 = 1.01$, $\varepsilon_0 = 1$ and $\psi_0 = 1$.

(c) $B_0 = 0.5$, $\omega_0 = 2.02$, $\varepsilon_0 = 2$ and $\psi_0 = 3$.

(d) $B_0 = 0.02$, $\omega_0 = 4.04$, $\varepsilon_0 = 1$ and $\psi_0 = 1$.

Fig. 10. The plots are shown for the relation between the logarithm deviation function for an analytical solution $\mathrm{Log}_{10}(|\Delta z(x)|_{\max})$ as a function of times for four sets of parameters with various driving frequencies.

The convergence speed to the analytical solution by the approximate method for various driving frequencies and other parameters is represented in Fig. 10. It is found that the convergence speed is dependent on the frequency, the larger the frequency is, the slower the convergence rate.

All computation were carried out on an IBM PC-AT with AMD Athlon XP 2800+, 512M DDR memory.

5. Conclusion

In the frame of generalized HB method, we report a new iterative algorithm to calculate the coefficients of the Fourier series which represents the approximations of the periodic solution of an nonlinear undamped Duffing oscillator when the fundamental frequency is identical to that of the periodic driving force. Based on this algorithm, the high-order terms of the Fourier series will be produced automatically with the corresponding increase in accuracy. The method does not need to solve associate nonlinear equations for a large number of unknowns, so the interpolation and Newton method is totally avoided. In all the stages of the calculation only several cubic equations need to be solved. From this cubic equation, the oscillation modes can be fully determined. We develop a Mathematica program called **AnalyDuffing** to implement this algorithm and use this program to demonstrate the accuracy and efficiency of the new computational method. Applying this program with any interesting parameters, we can easily obtain the approximate analytical solution with the desired accuracy. It is interesting to find that, generally, a nonlinear Duffing equation can have three independent oscillation modes. This program should be useful to study the dynamical behavior of a Duffing oscillator and to asses the error behavior of a highly accurate numerical integral method.

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