EOM =

$$\mathbf{D[\mathcal{X},\ \{t,\ 2\}]} == \left(\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \left(1 - \frac{1}{A}\right)\right).\mathcal{V}_1 + \left(\kappa \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \left(1 - \frac{1}{B}\mathcal{L}\right)\right).\mathcal{V}_2 - \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}\right) \text{ // Flatten;}$$

$$\{x_1[t] = y_1[t] = y_2[t] = 0,\ x_2[t] = 2\, w_p,\ \theta_p[t] \to 0\} : \left\{x_p[t] \to w_p,\ y_p[t] \to \frac{1}{2}\left(-2 - \gamma - 2\, h_p\right)\right\}$$

perturbations :

$$\text{EquilibiumPoinit} = \left\{\theta_{p0} \to 0,\ x_{p0} \to w_p,\ y_{p0} \to -\left(\frac{1}{2}\gamma + h_p + 1\right)\right\}$$

$$\text{GivenEquibPoints} = \{x_1[t] \to 0,\ y_1[t] \to 0,\ y_2[t] \to 0,\ x_2[t] \to 2\, w_p\}$$

```
perturb = {
   θp[t] → θp0 + δθ[t],
   xp[t] → xp0 + δx[t],
   yp[t] → yp0 + δy[t]
  }
perturbD2 = {
   D[θp[t], {t, 2}] → D[δθ[t], {t, 2}],
   D[xp[t], {t, 2}] → D[δx[t], {t, 2}],
   D[yp[t], {t, 2}] → D[δy[t], {t, 2}]
  }
```

$$\left\{\theta_{p0} \to 0,\ x_{p0} \to w_p,\ y_{p0} \to -1 - \frac{\gamma}{2} - h_p\right\}$$

$$\{x_1[t] \to 0,\ y_1[t] \to 0,\ y_2[t] \to 0,\ x_2[t] \to 2\, w_p\}$$

$$\{\theta_p[t] \to \theta_{p0} + \delta\theta[t],\ x_p[t] \to x_{p0} + \delta x[t],\ y_p[t] \to y_{p0} + \delta y[t]\}$$

$$\{\theta_p''[t] \to \delta\theta''[t],\ x_p''[t] \to \delta x''[t],\ y_p''[t] \to \delta y''[t]\}$$

D[𝒳, {t, 2}] /. perturbD2

$$\{\{\delta x''[t]\},\ \{\delta y''[t]\},\ \{\delta\theta''[t]\}\}$$

Aw = A /. nameChange

$$\sqrt{\left((\text{Sin}[\theta_p[t]]\, h_p + \text{Cos}[\theta_p[t]]\, w_p + x_1[t] - x_p[t])^2 + (-\text{Cos}[\theta_p[t]]\, h_p + \text{Sin}[\theta_p[t]]\, w_p + y_1[t] - y_p[t])^2\right)}$$

Bw = B /. nameChange

$$\sqrt{\left((\text{Sin}[\theta_p[t]]\, h_p - \text{Cos}[\theta_p[t]]\, w_p + x_2[t] - x_p[t])^2 + (-\text{Cos}[\theta_p[t]]\, h_p - \text{Sin}[\theta_p[t]]\, w_p + y_2[t] - y_p[t])^2\right)}$$

smallAngleRule = {Cos[δθ[t]] → 1, Sin[δθ[t]] -> δθ[t]}

$$\{\text{Cos}[\delta\theta[t]] \to 1,\ \text{Sin}[\delta\theta[t]] \to \delta\theta[t]\}$$

```
(v1 = 𝒱₁ /. nameChange /. perturb /. EquilibiumPoinit /. smallAngleRule) //
  TraditionalForm
(v2 = 𝒱₂ /. nameChange /. perturb /. EquilibiumPoinit /. smallAngleRule) //
  TraditionalForm
```

$$\begin{pmatrix} -\delta\mathrm{x}(t) + h_p\,\delta\theta(t) + x_1(t) \\ \frac{\gamma}{2} - \delta\mathrm{y}(t) + w_p\,\delta\theta(t) + y_1(t) + 1 \\ w_p\left(\frac{\gamma}{2} + h_p - \delta\mathrm{y}(t) - \delta\theta(t)\,(-w_p - \delta\mathrm{x}(t) + x_1(t)) + y_1(t) + 1\right) + h_p\left(-w_p - \delta\mathrm{x}(t) + x_1(t) + \delta\theta(t)\left(\frac{\gamma}{2} + h_p - \delta\mathrm{y}(t) + y_1(t) + 1\right)\right) \end{pmatrix}$$

$$\begin{pmatrix} -2\,w_p - \delta\mathrm{x}(t) + h_p\,\delta\theta(t) + x_2(t) \\ \frac{\gamma}{2} - \delta\mathrm{y}(t) - w_p\,\delta\theta(t) + y_2(t) + 1 \\ w_p\left(-\frac{\gamma}{2} - h_p + \delta\mathrm{y}(t) + \delta\theta(t)\,(-w_p - \delta\mathrm{x}(t) + x_2(t)) - y_2(t) - 1\right) + h_p\left(-w_p - \delta\mathrm{x}(t) + x_2(t) + \delta\theta(t)\left(\frac{\gamma}{2} + h_p - \delta\mathrm{y}(t) + y_2(t) + 1\right)\right) \end{pmatrix}$$

```
v1 /. GivenEquibPoints // Simplify
v2 /. GivenEquibPoints // Simplify
```

$$\left\{\{-\delta x[t] + h_p\,\delta\theta[t]\},\ \left\{1 + \frac{\gamma}{2} - \delta y[t] + w_p\,\delta\theta[t]\right\},\ \left\{\frac{1}{2}\left(2\,w_p^2\,\delta\theta[t] + w_p\,(2 + \gamma - 2\,\delta y[t] + 2\,\delta x[t]\,\delta\theta[t]) + h_p\,(-2\,\delta x[t] + (2 + \gamma + 2\,h_p - 2\,\delta y[t])\,\delta\theta[t])\right)\right\}\right\}$$

$$\left\{\{-\delta x[t] + h_p\,\delta\theta[t]\},\ \left\{\frac{1}{2}\,(2 + \gamma - 2\,\delta y[t] - 2\,w_p\,\delta\theta[t])\right\},\ \left\{w_p^2\,\delta\theta[t] - \frac{1}{2}\,w_p\,(2 + \gamma - 2\,\delta y[t] + 2\,\delta x[t]\,\delta\theta[t]) + \frac{1}{2}\,h_p\,(-2\,\delta x[t] + (2 + \gamma + 2\,h_p - 2\,\delta y[t])\,\delta\theta[t])\right\}\right\}$$

```
(*D[Aw,xp[t]]
 D[Aw,yp[t]]
 D[Aw,θp[t]]*)
temp = {xp[t] -> xp0, yp[t] → yp0, θp[t] → θp0};
"derivatives of 'A' in the 0 point:"
D[Aw^2, xp[t]] /. temp /. EquilibiumPoinit
D[Aw^2, yp[t]] /. temp /. EquilibiumPoinit
D[Aw^2, θp[t]] /. temp /. EquilibiumPoinit
```

```
"derivatives of 'B' in the 0 point:"
D[Bw^2, x_p[t]] /. temp /. EquilibiumPoinit
D[Bw^2, y_p[t]] /. temp /. EquilibiumPoinit
D[Bw^2, θ_p[t]] /. temp /. EquilibiumPoinit
```

derivatives of 'B' in the 0 point:

$$-2\,(-2\,w_p + x_2[t])$$

$$-2\left(1 + \frac{\gamma}{2} + y_2[t]\right)$$

$$2\,h_p\,(-2\,w_p + x_2[t]) - 2\,w_p\left(1 + \frac{\gamma}{2} + y_2[t]\right)$$

```
n = 1; Ataylored = Series[Aw /. GivenEquibPoints,
   {x_p[t], x_p0, n}, {y_p[t], y_p0, n}, {θ_p[t], θ_p0, n}] /. EquilibiumPoinit
n = 1; Series[Bw, {x_p[t], x_p0, n}, {y_p[t], y_p0, n}, {θ_p[t], θ_p0, n}] /. EquilibiumPoinit
% // Simplify // TraditionalForm
```

$$\Bigg(\Bigg(\frac{1}{2}\sqrt{(\gamma+2)^2} + \frac{(\gamma+2)\,w_p\,\theta_p(t)}{\sqrt{(\gamma+2)^2}} + O\big(\theta_p(t)^2\big)\Bigg) +$$

$$\left(\frac{\gamma}{2} + h_p + y_p(t) + 1\right)\left(\frac{-\gamma-2}{\sqrt{(\gamma+2)^2}} + O\big(\theta_p(t)^2\big)\right) + O\left(\left(\frac{\gamma}{2} + h_p + y_p(t) + 1\right)^2\right)\Bigg) + (x_p(t) - w_p)$$

$$\Bigg(\Bigg(-\frac{2\,h_p\,\theta_p(t)}{\sqrt{(\gamma+2)^2}} + O\big(\theta_p(t)^2\big)\Bigg) + \Bigg(-\frac{4\,h_p\,\theta_p(t)}{(\gamma+2)\sqrt{(\gamma+2)^2}} + O\big(\theta_p(t)^2\big)\Bigg)\left(\frac{\gamma}{2} + h_p + y_p(t) + 1\right) + O\left(\left(\frac{\gamma}{2} + h_p + y_p(t) + 1\right)^2\right)\Bigg) + O\Big($$

$$(x_p(t) - w_p)^2\Big)$$

```
Ataylored = 1 + γ/2 - δy[t] + w_p δθ[t]
Btaylored = 1 + γ/2 - δy[t] - w_p δθ[t]
𝒱taylored_1 = v1 /. GivenEquibPoints
𝒱taylored_2 = v2 /. GivenEquibPoints
```

$$1+\frac{\gamma}{2}-\delta y[t]+w_p\,\delta\theta[t]$$

$$1+\frac{\gamma}{2}-\delta y[t]-w_p\,\delta\theta[t]$$

$$\Big\{\{-\delta x[t]+h_p\,\delta\theta[t]\},\ \Big\{1+\frac{\gamma}{2}-\delta y[t]+w_p\,\delta\theta[t]\Big\},\ \Big\{w_p\Big(1+\frac{\gamma}{2}+h_p-\delta y[t]-(-w_p-\delta x[t])\,\delta\theta[t]\Big)+h_p\Big(-w_p-\delta x[t]+\Big(1+\frac{\gamma}{2}+h_p-\delta y[t]\Big)\,\delta\theta[t]\Big)\Big\}\Big\}$$

$$\Big\{\{-\delta x[t]+h_p\,\delta\theta[t]\},\ \Big\{1+\frac{\gamma}{2}-\delta y[t]-w_p\,\delta\theta[t]\Big\},\ \Big\{w_p\Big(-1-\frac{\gamma}{2}-h_p+\delta y[t]+(w_p-\delta x[t])\,\delta\theta[t]\Big)+h_p\Big(w_p-\delta x[t]+\Big(1+\frac{\gamma}{2}+h_p-\delta y[t]\Big)\,\delta\theta[t]\Big)\Big\}\Big\}$$

$$\texttt{EOM(*=D[}\mathcal{X}\texttt{,\{t,2\}]==}\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&-\alpha\end{pmatrix}\left(1-\frac{1}{A}\right)\right).\mathcal{V}_1+\left(\kappa\begin{pmatrix}1&0&0\\0&1&0\\0&0&-\alpha\end{pmatrix}\left(1-\frac{1}{B}\mathcal{L}\right)\right).\mathcal{V}_2-\begin{pmatrix}0\\\gamma\\0\end{pmatrix}\texttt{*)}$$

```
  (*/.perturbD2*) // TraditionalForm
```

$$\begin{pmatrix}x_p''(t)\\y_p''(t)\\\theta_p''(t)\end{pmatrix}=\left(\begin{array}{r}(\sin(\theta_p(t))\,h_p+\cos(\theta_p(t))\,l_p+x_1(t)-x_p(t))\\-\gamma+\left(1-\dfrac{1}{\sqrt{(\sin(\theta_p(t))\,h_p+\cos(\theta_p(t))\,l_p+x_1(t)-x_p(t))^2+(-\cos}}\right.\\-\alpha\,(l_p\,(\cos(\theta_p(t))\,(y_1(t)-y_p(t))-\sin(\theta_p(t))\,(x_1(t)-x_p(t)))+h_p\,(\cos(\theta_p(t))\,(x_1(t)-x_p(t))+\sin(\theta_p(t))\,(y_1(t)-y_p(t)))\end{array}\right.$$

$$\texttt{(*}\left(\texttt{EOMrephrase=D[}\mathcal{X}\texttt{,\{t,2\}]==}\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&-\alpha\end{pmatrix}\left(\frac{A-1}{A}\right)\right).\mathcal{V}_1+\left(\kappa\begin{pmatrix}1&0&0\\0&1&0\\0&0&-\alpha\end{pmatrix}\left(\frac{B-\mathcal{L}}{B}\right)\right).\mathcal{V}_2-\begin{pmatrix}0\\\gamma\\0\end{pmatrix}\ \texttt{(*//Flatten*)}\right)\texttt{//Simplify//TraditionalForm*)}$$

```
(EOMrephrase = D[𝒳, {t, 2}] A B ==
    B (({{1, 0, 0}, {0, 1, 0}, {0, 0, -α}}) (A - 1)).𝒱1 + A (κ {{1, 0, 0}, {0, 1, 0}, {0, 0, -α}} (B - ℒ)).𝒱2 - A B {0, γ, 0}) // TraditionalForm
```

$$\left(\begin{array}{ll} \sqrt{(\sin(\theta_p(t))\, h_p + \cos(\theta_p(t))\, l_p + x_1(t) - x_p(t))^2 + (-\cos(\theta_p(t))\, h_p + \sin(\theta_p(t))\, l_p + y_1(t) - y_p(t))^2} & \sqrt{(\sin(\theta_p(t))\, h_p - \cos(\theta_p(t))\, l_p} \\ \sqrt{(\sin(\theta_p(t))\, h_p + \cos(\theta_p(t))\, l_p + x_1(t) - x_p(t))^2 + (-\cos(\theta_p(t))\, h_p + \sin(\theta_p(t))\, l_p + y_1(t) - y_p(t))^2} & \sqrt{(\sin(\theta_p(t))\, h_p - \cos(\theta_p(t))\, l_p} \\ \sqrt{(\sin(\theta_p(t))\, h_p + \cos(\theta_p(t))\, l_p + x_1(t) - x_p(t))^2 + (-\cos(\theta_p(t))\, h_p + \sin(\theta_p(t))\, l_p + y_1(t) - y_p(t))^2} & \sqrt{(\sin(\theta_p(t))\, h_p - \cos(\theta_p(t))\, l_p} \end{array}\right.$$

```
(EOMLinearized = (D[𝒳, {t, 2}] /. perturbD2) Ataylored Btaylored ==
    (Btaylored {{1, 0, 0}, {0, 1, 0}, {0, 0, -α}} (Ataylored - 1)).𝒱taylored1 +
     (Ataylored κ {{1, 0, 0}, {0, 1, 0}, {0, 0, -α}} (Btaylored - ℒ)).𝒱taylored2 -
     Ataylored Btaylored {0, γ, 0}) // TraditionalForm
```

$$\begin{pmatrix} \left(\frac{\gamma}{2} - \delta y(t) - w_p\, \delta\theta(t) + 1\right)\left(\frac{\gamma}{2} - \delta y(t) + w_p\, \delta\theta(t) + 1\right)\delta x''(t) \\ \left(\frac{\gamma}{2} - \delta y(t) - w_p\, \delta\theta(t) + 1\right)\left(\frac{\gamma}{2} - \delta y(t) + w_p\, \delta\theta(t) + 1\right)\delta y''(t) \\ \left(\frac{\gamma}{2} - \delta y(t) - w_p\, \delta\theta(t) + 1\right)\left(\frac{\gamma}{2} - \delta y(t) + w_p\, \delta\theta(t) + 1\right)\delta\theta''(t) \end{pmatrix} = \left(\begin{array}{r} \\ -\gamma\left(\frac{\gamma}{2} - \delta y(t) - w_p\, \delta \right. \\ -\alpha\left(\frac{\gamma}{2} - \delta y(t) - w_p\, \delta\theta(t) + 1\right)\left(\frac{\gamma}{2} - \delta y(t) + w_p\, \delta\theta(t)\right)\left(w_p\left(\frac{\gamma}{2} + \right.\right. \end{array}\right.$$

```
temp3 = EOMLinearized //. (1 + γ/2 → γ12) //. (-1 - γ/2 → -γ12)
```

```
{{(γ12 - δy[t] - w δθ[t]) (γ12 - δy[t] + w δθ[t]) δx''[t]},
  {(γ12 - δy[t] - w δθ[t]) (γ12 - δy[t] + w δθ[t]) δy''[t]},
  {(γ12 - δy[t] - w δθ[t]) (γ12 - δy[t] + w δθ[t]) δθ''[t]}} ==
 {{(γ12 - δy[t] - w δθ[t]) (γ/2 - δy[t] + w δθ[t]) (-δx[t] + hp δθ[t]) +
    κ (-ℒ + γ12 - δy[t] - w δθ[t]) (γ12 - δy[t] + w δθ[t]) (-δx[t] + hp δθ[t])},
  {-γ (γ12 - δy[t] - w δθ[t]) (γ12 - δy[t] + w δθ[t]) +
    κ (-ℒ + γ12 - δy[t] - w δθ[t]) (γ12 - δy[t] + w δθ[t]) (γ12 - δy[t] - wp δθ[t]) +
    (γ12 - δy[t] - w δθ[t]) (γ/2 - δy[t] + w δθ[t]) (γ12 - δy[t] + wp δθ[t])},
  {-α (γ12 - δy[t] - w δθ[t]) (γ/2 - δy[t] + w δθ[t])
     (wp (γ12 + hp - δy[t] - (-wp - δx[t]) δθ[t]) +
       hp (-wp - δx[t] + (γ12 + hp - δy[t]) δθ[t])) - α κ (-ℒ + γ12 - δy[t] - w δθ[t])
     (γ12 - δy[t] + w δθ[t]) (wp (-γ12 - hp + δy[t] + (wp - δx[t]) δθ[t]) +
       hp (wp - δx[t] + (γ12 + hp - δy[t]) δθ[t]))}}
```

```
neglectedCombinations =
 {(*δy[t] δx''[t]→0, δy[t] δy''[t]→0,*)a_[t]^2 → 0, a_[t]^3 → 0, a_[t] b_[t] → 0}
{a_[t]^2 → 0, a_[t]^3 → 0, a_[t] b_[t] → 0}
```

```
temp5 = (EOMLinearized // Expand) //. neglectedCombinations
(temp6 = Collect[temp5, {δx[t], δy[t], δθ[t]}, Simplify]) // TraditionalForm
((temp6) // Expand // Simplify) // TraditionalForm
```

$$\left\{\left\{\delta x''[t]+\gamma\,\delta x''[t]+\frac{1}{4}\gamma^2\,\delta x''[t]\right\},\ \left\{\delta y''[t]+\gamma\,\delta y''[t]+\frac{1}{4}\gamma^2\,\delta y''[t]\right\},\ \left\{\delta\theta''[t]+\gamma\,\delta\theta''[t]+\frac{1}{4}\gamma^2\,\delta\theta''[t]\right\}\right\}==$$

$$\left\{\left\{-\frac{1}{2}\gamma\,\delta x[t]-\frac{1}{4}\gamma^2\,\delta x[t]-\kappa\,\delta x[t]+\mathcal{L}\,\kappa\,\delta x[t]-\gamma\,\kappa\,\delta x[t]+\frac{1}{2}\mathcal{L}\,\gamma\,\kappa\,\delta x[t]-\frac{1}{4}\gamma^2\,\kappa\,\delta x[t]+\frac{1}{2}\gamma\,h_p\,\delta\theta[t]+\frac{1}{4}\gamma^2\,h_p\,\delta\theta[t]+\kappa\,h_p\,\delta\theta[t]-\mathcal{L}\,\kappa\,h_p\,\delta\theta[t]+\gamma\,\kappa\,h_p\,\delta\theta[t]-\frac{1}{2}\mathcal{L}\,\gamma\,\kappa\,h_p\,\delta\theta[t]+\frac{1}{4}\gamma^2\,\kappa\,h_p\,\delta\theta[t]\right\},\right.$$

$$\left\{-\frac{\gamma}{2}-\frac{\gamma^2}{2}-\frac{\gamma^3}{8}+\kappa-\mathcal{L}\,\kappa+\frac{3\,\gamma\,\kappa}{2}-\mathcal{L}\,\gamma\,\kappa+\frac{3\,\gamma^2\,\kappa}{4}-\frac{1}{4}\mathcal{L}\,\gamma^2\,\kappa+\frac{\gamma^3\,\kappa}{8}-\delta y[t]+\frac{1}{4}\gamma^2\,\delta y[t]-3\,\kappa\,\delta y[t]+2\,\mathcal{L}\,\kappa\,\delta y[t]-3\,\gamma\,\kappa\,\delta y[t]+\mathcal{L}\,\gamma\,\kappa\,\delta y[t]-\frac{3}{4}\gamma^2\,\kappa\,\delta y[t]+w_p\,\delta\theta[t]+\gamma\,w_p\,\delta\theta[t]+\frac{1}{4}\gamma^2\,w_p\,\delta\theta[t]-\kappa\,w_p\,\delta\theta[t]-\gamma\,\kappa\,w_p\,\delta\theta[t]-\frac{1}{4}\gamma^2\,\kappa\,w_p\,\delta\theta[t]\right\},$$

$$\left\{-\frac{1}{2}\alpha\,\gamma\,w_p-\frac{1}{2}\alpha\,\gamma^2\,w_p-\frac{1}{8}\alpha\,\gamma^3\,w_p+\alpha\,\kappa\,w_p-\mathcal{L}\,\alpha\,\kappa\,w_p+\frac{3}{2}\alpha\,\gamma\,\kappa\,w_p-\mathcal{L}\,\alpha\,\gamma\,\kappa\,w_p+\frac{3}{4}\alpha\,\gamma^2\,\kappa\,w_p-\frac{1}{4}\mathcal{L}\,\alpha\,\gamma^2\,\kappa\,w_p+\frac{1}{8}\alpha\,\gamma^3\,\kappa\,w_p+\frac{1}{2}\alpha\,\gamma\,h_p\,\delta x[t]+\frac{1}{4}\alpha\,\gamma^2\,h_p\,\delta x[t]+\alpha\,\kappa\,h_p\,\delta x[t]-\mathcal{L}\,\alpha\,\kappa\,h_p\,\delta x[t]+\alpha\,\gamma\,\kappa\,h_p\,\delta x[t]-\frac{1}{2}\mathcal{L}\,\alpha\,\gamma\,\kappa\,h_p\,\delta x[t]+\frac{1}{4}\alpha\,\gamma^2\,\kappa\,h_p\,\delta x[t]+\alpha\,w_p\,\delta y[t]+2\,\alpha\,\gamma\,w_p\,\delta y[t]+\frac{3}{4}\alpha\,\gamma^2\,w_p\,\delta y[t]-3\,\alpha\,\kappa\,w_p\,\delta y[t]+2\,\mathcal{L}\,\alpha\,\kappa\,w_p\,\delta y[t]-3\,\alpha\,\gamma\,\kappa\,w_p\,\delta y[t]+\mathcal{L}\,\alpha\,\gamma\,\kappa\,w_p\,\delta y[t]-\frac{3}{4}\alpha\,\gamma^2\,\kappa\,w_p\,\delta y[t]-\frac{1}{2}\alpha\,\gamma\,h_p\,\delta\theta[t]-\frac{1}{2}\alpha\,\gamma^2\,h_p\,\delta\theta[t]-\frac{1}{8}\alpha\,\gamma^3\,h_p\,\delta\theta[t]-\alpha\,\kappa\,h_p\,\delta\theta[t]+\mathcal{L}\,\alpha\,\kappa\,h_p\,\delta\theta[t]-\frac{3}{2}\alpha\,\gamma\,\kappa\,h_p\,\delta\theta[t]+\mathcal{L}\,\alpha\,\gamma\,\kappa\,h_p\,\delta\theta[t]-\frac{3}{4}\alpha\,\gamma^2\,\kappa\,h_p\,\delta\theta[t]+\frac{1}{4}\mathcal{L}\,\alpha\,\gamma^2\,\kappa\,h_p\,\delta\theta[t]-\frac{1}{8}\alpha\,\gamma^3\,\kappa\,h_p\,\delta\theta[t]-\frac{1}{2}\alpha\,\gamma\,h_p^2\,\delta\theta[t]-\frac{1}{4}\alpha\,\gamma^2\,h_p^2\,\delta\theta[t]-\alpha\,\kappa\,h_p^2\,\delta\theta[t]+\mathcal{L}\,\alpha\,\kappa\,h_p^2\,\delta\theta[t]-\alpha\,\gamma\,\kappa\,h_p^2\,\delta\theta[t]+\frac{1}{2}\mathcal{L}\,\alpha\,\gamma\,\kappa\,h_p^2\,\delta\theta[t]-\frac{1}{4}\alpha\,\gamma^2\,\kappa\,h_p^2\,\delta\theta[t]-\alpha\,w_p^2\,\delta\theta[t]-\alpha\,\gamma\,w_p^2\,\delta\theta[t]-\frac{1}{4}\alpha\,\gamma^2\,w_p^2\,\delta\theta[t]-\alpha\,\kappa\,w_p^2\,\delta\theta[t]-\alpha\,\gamma\,\kappa\,w_p^2\,\delta\theta[t]-\frac{1}{4}\alpha\,\gamma^2\,\kappa\,w_p^2\,\delta\theta[t]\right\}\right\}$$

$$\begin{pmatrix}\frac{1}{4}(\gamma+2)^2\,\delta x''(t)\\ \frac{1}{4}(\gamma+2)^2\,\delta y''(t)\\ \frac{1}{4}(\gamma+2)^2\,\delta\theta''(t)\end{pmatrix}=\left(\begin{matrix}\frac{1}{4}(\gamma+2)(\kappa\,\gamma+\gamma-2\\ \frac{1}{8}(\gamma(\kappa-1)-2(\mathcal{L}-1)\kappa)(\gamma+2)^2-\\ \frac{1}{8}\alpha(\gamma(\kappa-1)-2(\mathcal{L}-1)\kappa)\,w_p(\gamma+2)^2+\frac{1}{4}\alpha(\kappa\,\gamma+\gamma-2\,\mathcal{L}\,\kappa+2\,\kappa)\,h_p\,\delta x(t)(\gamma+2)-\frac{1}{4}\alpha(3\,\gamma(\kappa-1)\end{matrix}\right.$$

$$\left(\begin{matrix}\frac{1}{4}(\gamma+2)((\kappa\,\gamma+\gamma-2\,\mathcal{L}\,\kappa+2\,\kappa)(\delta x(t)-h_p\,\delta\theta(t))+(\gamma+2)\,\delta\\ -\frac{1}{8}(\gamma+2)(2(-3\,\kappa\,\gamma+\gamma+4\,\mathcal{L}\,\kappa-6\,\kappa-2)\,\delta y(t)+(\gamma+2)(\kappa\,\gamma-\gamma-2\,\mathcal{L}\,\kappa+2\,\kappa-\\ \frac{1}{8}(\gamma+2)\left(2(\gamma+2)\,\delta\theta''(t)-\alpha\left(-2(\gamma+2)(\kappa+1)\,\delta\theta(t)\,w_p^2+((\gamma+2)(\gamma(\kappa-1)-2(\mathcal{L}-1)\kappa)+(-6\,\gamma(\kappa-1)+8\,\mathcal{L}\,\kappa-12\,\kappa+\right.\right.\end{matrix}\right.$$

```
(temp7 = temp6 /. κ → 1 /. ℒ → 1 // Simplify) // TraditionalForm
```

$$\begin{pmatrix} \frac{1}{4}(\gamma+2)\,(2\,\gamma\,\delta x(t) - 2\,\gamma\,h_p\,\delta\theta(t) + (\gamma+2)\,\delta x''(t)) \\ \frac{1}{4}(\gamma+2)^2\,(2\,\delta y(t) + \delta y''(t)) \\ \frac{1}{4}(\gamma+2)\left(2\,\alpha\,\gamma\,\delta\theta(t)\,h_p^2 + \alpha\,\gamma\,((\gamma+2)\,\delta\theta(t) - 2\,\delta x(t))\,h_p + (\gamma+2)\left(2\,\alpha\,\delta\theta(t)\,w_p^2 + \delta\theta''(t)\right)\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

```
M x'' + C x' + K x == F
```

assuming for the 1st and 2nd equation : (γ + 2) != 0

```
X /. perturb
D[X, {t, 2}] /. perturbD2
```

$\{\{x_{p0} + \delta x[t]\}, \{y_{p0} + \delta y[t]\}, \{\Theta_{p0} + \delta\Theta[t]\}\}$

$\{\{\delta x''[t]\}, \{\delta y''[t]\}, \{\delta\Theta''[t]\}\}$

```
(Collect[temp7[[1]][[3]] / (1/4 (γ + 2)) // Expand, {δx[t], δy[t], δΘ[t]}, Simplify]) //
 TraditionalForm
```

$$\left\{-2\,\alpha\,\gamma\,h_p\,\delta x(t) + \alpha\,\delta\theta(t)\left(2\,\gamma\,h_p^2 + \gamma\,(\gamma+2)\,h_p + 2\,(\gamma+2)\,w_p^2\right) + (\gamma+2)\,\delta\theta''(t)\right\}$$

In[23]:= `Quit[]`

In[1]:=

$$M = \begin{pmatrix} (\gamma+2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\gamma+2) \end{pmatrix}$$

$$K = \begin{pmatrix} 2\gamma & 0 & -2\gamma h_p \\ 0 & 2 & 0 \\ -2\alpha\gamma h_p & 0 & \alpha\left(2\gamma h_p^2 + \gamma(\gamma+2)h_p + 2(\gamma+2)w_p^2\right) \end{pmatrix}$$

Out[1]= $\{\{2+\gamma, 0, 0\}, \{0, 1, 0\}, \{0, 0, 2+\gamma\}\}$

Out[2]= $\left\{\{2\gamma, 0, -2\gamma h_p\}, \{0, 2, 0\}, \left\{-2\alpha\gamma h_p, 0, \alpha\left(\gamma(2+\gamma)h_p + 2\gamma h_p^2 + 2(2+\gamma)w_p^2\right)\right\}\right\}$

In[3]:=

$$\text{Solve}\left[\text{Det}[K - \omega^2 M] == 0\,(*/.(\gamma+2)\to\rho*)\ /.\ \alpha \to 3\,\frac{1}{w_p^2 + h_p^2}\ /.\ \gamma \to 3.448\ /.\ h_p \to 1\ /.\ w_p \to 1,\ \omega\right]\ //\ N\ //\ \text{Simplify}$$

Out[3]= $\{\{\omega \to -3.21491\}, \{\omega \to -1.0004\}, \{\omega \to 1.0004\}, \{\omega \to 3.21491\}, \{\omega \to -1.41421\}, \{\omega \to 1.41421\}\}$

$$(*\text{Solve}\left[\text{Det}[K-\omega^2 M]==0/.(\gamma+2)\to\rho/.\alpha\to 3\,\frac{1}{w_p^2+h_p^2},\omega\right]//\text{Simplify}*)$$

In[4]:= **Solve[Det[K - ω² M] == 0 /. (γ + 2) → ρ, ω] // Simplify**

Out[4]=
$$\left\{\left\{\omega \to -\sqrt{2}\right\}, \left\{\omega \to \sqrt{2}\right\}, \left\{\omega \to -\frac{1}{\sqrt{2}}\left(\sqrt{\left(\frac{1}{\rho}\left(2\,\gamma + \alpha\,\gamma\,\rho\,h_p + 2\,\alpha\,\gamma\,h_p^2 + 2\,\alpha\,\rho\,w_p^2 - \sqrt{\left(-8\,\alpha\,\gamma\,\rho\,\left(\gamma\,h_p + 2\,w_p^2\right) + \left(\alpha\,\gamma\,\rho\,h_p + 2\,\alpha\,\gamma\,h_p^2 + 2\,\left(\gamma + \alpha\,\rho\,w_p^2\right)\right)^2\right)}\right)\right)}\right)\right\},\right.$$
$$\left\{\omega \to \frac{1}{\sqrt{2}}\left(\sqrt{\left(\frac{1}{\rho}\left(2\,\gamma + \alpha\,\gamma\,\rho\,h_p + 2\,\alpha\,\gamma\,h_p^2 + 2\,\alpha\,\rho\,w_p^2 - \sqrt{\left(-8\,\alpha\,\gamma\,\rho\,\left(\gamma\,h_p + 2\,w_p^2\right) + \left(\alpha\,\gamma\,\rho\,h_p + 2\,\alpha\,\gamma\,h_p^2 + 2\,\left(\gamma + \alpha\,\rho\,w_p^2\right)\right)^2\right)}\right)\right)}\right)\right\},$$
$$\left\{\omega \to -\frac{1}{\sqrt{2}}\left(\sqrt{\left(\frac{1}{\rho}\left(2\,\gamma + \alpha\,\gamma\,\rho\,h_p + 2\,\alpha\,\gamma\,h_p^2 + 2\,\alpha\,\rho\,w_p^2 + \sqrt{\left(-8\,\alpha\,\gamma\,\rho\,\left(\gamma\,h_p + 2\,w_p^2\right) + \left(\alpha\,\gamma\,\rho\,h_p + 2\,\alpha\,\gamma\,h_p^2 + 2\,\left(\gamma + \alpha\,\rho\,w_p^2\right)\right)^2\right)}\right)\right)}\right)\right\},$$
$$\left.\left\{\omega \to \frac{1}{\sqrt{2}}\left(\sqrt{\left(\frac{1}{\rho}\left(2\,\gamma + \alpha\,\gamma\,\rho\,h_p + 2\,\alpha\,\gamma\,h_p^2 + 2\,\alpha\,\rho\,w_p^2 + \sqrt{\left(-8\,\alpha\,\gamma\,\rho\,\left(\gamma\,h_p + 2\,w_p^2\right) + \left(\alpha\,\gamma\,\rho\,h_p + 2\,\alpha\,\gamma\,h_p^2 + 2\,\left(\gamma + \alpha\,\rho\,w_p^2\right)\right)^2\right)}\right)\right)}\right)\right\}\right\}$$

(*M= $\begin{pmatrix} (\gamma+2) & 0 \\ 0 & (\gamma+2) \end{pmatrix}$

K= $\begin{pmatrix} 2\ \gamma & -2\ \gamma\ h_p \\ -2\ \alpha\ \gamma\ h_p\ \alpha & \left(2\ \gamma\ h_p^2+\gamma\ (\gamma+2)\ h_p+2\ (\gamma+2)\ w_p^2\right) \end{pmatrix}$

Solve[Det[$K-\omega^2 M$ **/. (γ+2)→ρ] == 0, ω] *)**

In[3]:=
```
variables = {ω1^2 → 1/(2 ρ) (2 γ + α γ ρ hp + 2 α γ hp^2 + 2 α ρ wp^2 -
      Sqrt[(-8 α γ ρ (γ hp + 2 wp^2) + (α γ ρ hp + 2 α γ hp^2 + 2 (γ + α ρ wp^2))^2)]),
   ω2^2 → 1/(2 ρ) (2 γ + α γ ρ hp + 2 α γ hp^2 + 2 α ρ wp^2 +
      Sqrt[(-8 α γ ρ (γ hp + 2 wp^2) + (α γ ρ hp + 2 α γ hp^2 + 2 (γ + α ρ wp^2))^2)]),
   α → 3 1/(wp^2 + hp^2),
   ρ -> (γ + 2)}
term1 = ω1^2 //. variables(*//.variables*)
term2 = ω2^2 //. variables(*//.variables*)
```

Out[3]=
$$\left\{\omega_1^2\to\frac{1}{2\rho}\left(2\gamma+\alpha\gamma\rho h_p+2\alpha\gamma h_p^2+2\alpha\rho w_p^2-\sqrt{\left(-8\alpha\gamma\rho\left(\gamma h_p+2w_p^2\right)+\left(\alpha\gamma\rho h_p+2\alpha\gamma h_p^2+2\left(\gamma+\alpha\rho w_p^2\right)\right)^2\right)}\right),\ \omega_2^2\to\frac{1}{2\rho}\left(2\gamma+\alpha\gamma\rho h_p+2\alpha\gamma h_p^2+2\alpha\rho w_p^2+\sqrt{\left(-8\alpha\gamma\rho\left(\gamma h_p+2w_p^2\right)+\left(\alpha\gamma\rho h_p+2\alpha\gamma h_p^2+2\left(\gamma+\alpha\rho w_p^2\right)\right)^2\right)}\right),\ \alpha\to\frac{3}{h_p^2+w_p^2},\ \rho\to2+\gamma\right\}$$

Out[4]=
$$\frac{1}{2(2+\gamma)}\left(2\gamma+\frac{3\gamma(2+\gamma)h_p}{h_p^2+w_p^2}+\frac{6\gamma h_p^2}{h_p^2+w_p^2}+\frac{6(2+\gamma)w_p^2}{h_p^2+w_p^2}-\sqrt{\left(-\frac{24\gamma(2+\gamma)\left(\gamma h_p+2w_p^2\right)}{h_p^2+w_p^2}+\left(\frac{3\gamma(2+\gamma)h_p}{h_p^2+w_p^2}+\frac{6\gamma h_p^2}{h_p^2+w_p^2}+2\left(\gamma+\frac{3(2+\gamma)w_p^2}{h_p^2+w_p^2}\right)\right)^2\right)}\right)$$

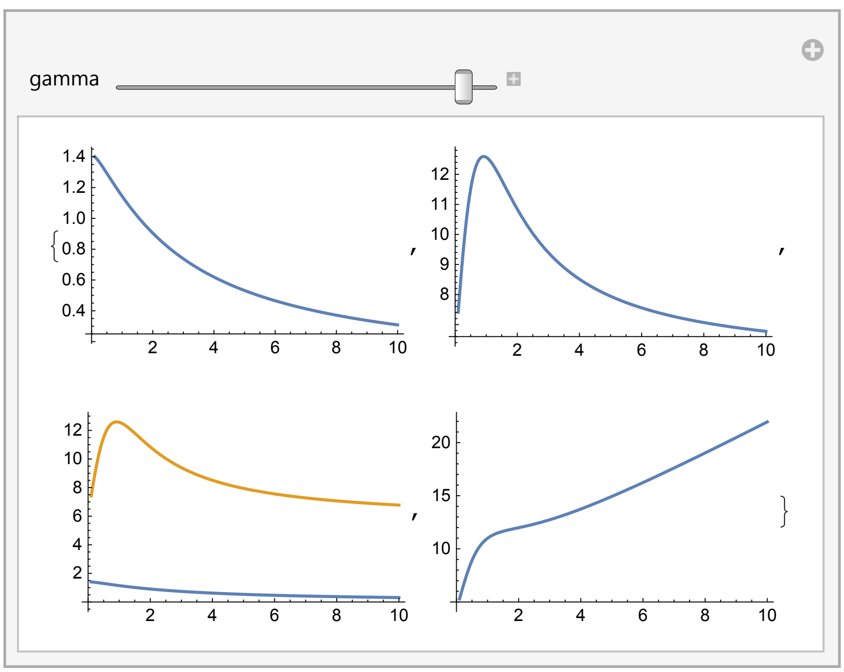
Out[5]=
$$\frac{1}{2(2+\gamma)}\left(2\gamma+\frac{3\gamma(2+\gamma)h_p}{h_p^2+w_p^2}+\frac{6\gamma h_p^2}{h_p^2+w_p^2}+\frac{6(2+\gamma)w_p^2}{h_p^2+w_p^2}+\sqrt{\left(-\frac{24\gamma(2+\gamma)\left(\gamma h_p+2w_p^2\right)}{h_p^2+w_p^2}+\left(\frac{3\gamma(2+\gamma)h_p}{h_p^2+w_p^2}+\frac{6\gamma h_p^2}{h_p^2+w_p^2}+2\left(\gamma+\frac{3(2+\gamma)w_p^2}{h_p^2+w_p^2}\right)\right)^2\right)}\right)$$

In[6]:=
```
plotsss[wp_, max_, isMesh_] :=
  {myTitle = (*"ωi^2*)" as func of hp & γ for constant wp";
   Plot3D[term1 /. {wp → wp}, {hp, 0.1, max/10}, {γ, 0.1, max},
    PlotLabel → myTitle + " ω1^2", AxesLabel → {"hp", "γ"},
    ColorFunction → "Rainbow", Mesh → isMesh, ImageSize → Medium],
   Plot3D[term2 /. {wp → wp}, {hp, 0.1, max/10}, {γ, 0.1, max},
    PlotLabel → myTitle + "ω2^2 ", AxesLabel → {"hp", "γ"},
    ColorFunction → "Rainbow", Mesh → isMesh, ImageSize → Medium]}

(*Manipulate[plotsss[wp,100,True],{{wp,1},0.1,10}]*)
(* add cross section , and tooltip to indicate values of h,γ,
w and ω^2 . and sync axis and views angles for both plots *)
(* special color for spline lines of h=0.1,1, 10 *)
```

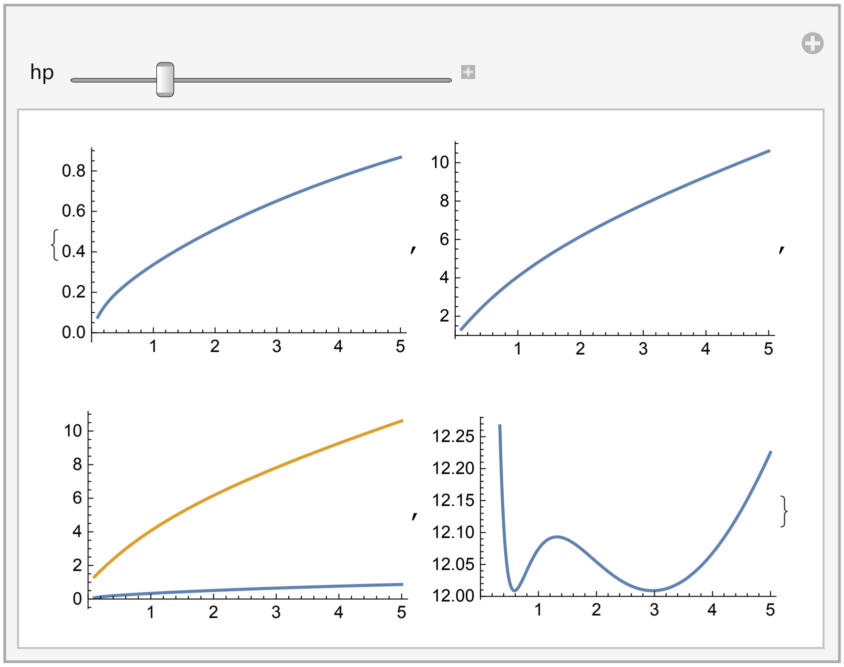
```
(*Manipulate[plotsss[wp,5,True],{{wp,1},0.1,10}]*)

(* add swipe animation across each axis to show influence of γ,h on ω1 , ω2 *)
(* should be γ important to ω1 , h important to ω2 *)
```

In[7]:=
```
Manipulate[
 {Plot[(term1 /. wp → 1(*/.hp→hp*) /. γ → gamma), {hp, 0.1, 10}, PlotRange → All],
  Plot[(term2 /. wp → 1(*/.hp→hp*) /. γ → gamma), {hp, 0.1, 10}, PlotRange → All],
  Plot[{(term1 /. wp → 1(*/.hp→hp*) /. γ → gamma),
    (term2 /. wp → 1(*/.hp→hp*) /. γ → gamma)}, {hp, 0.1, 10}, PlotRange → All],
  Plot[(term2/term1 /. wp → 1(*/.hp→hp*) /. γ → gamma),
   {hp, 0.1, 10}, PlotRange → All]}
 , {{gamma, 1}, 0.1, 5}]
Manipulate[
 {Plot[(term1 /. wp → 1 /. hp → hp(*/.γ→gamma*)), {γ, 0.1, 5}],
  Plot[(term2 /. wp → 1 /. hp → hp(*/.γ→gamma*)), {γ, 0.1, 5}],
  Plot[{(term1 /. wp → 1 /. hp → hp(*/.γ→gamma*)),
    (term2 /. wp → 1 /. hp → hp(*/.γ→gamma*))}, {γ, 0.1, 5}],
  Plot[(term2/term1 /. wp → 1 /. hp → hp(*/.γ→gamma*)), {γ, 0.1, 5}]}
 , {{hp, 1}, 0.1, 10}]
```

Out[7]=



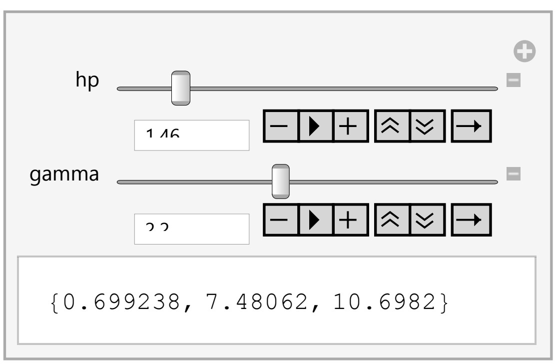
Out[8]=



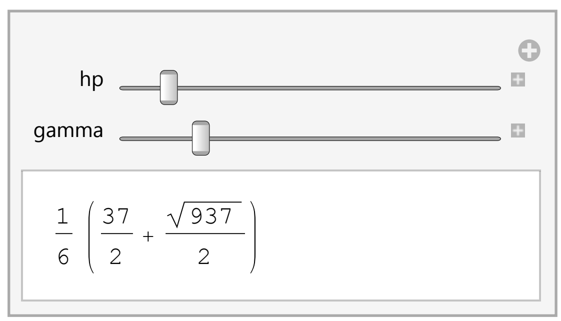
In[18]:=

```
{Manipulate[
  {(term1 /. w_p → 1 /. h_p → hp /. γ → gamma), (term2 /. w_p → 1 /. h_p → hp /. γ → gamma),
    (term2 / term1 /. w_p → 1 /. h_p → hp /. γ → gamma)} // N,
  {{hp, 1}, 0.1, 10}, {{gamma, 1}, 0.1, 5}],
 Manipulate[(term2 /. w_p → 1 /. h_p → hp /. γ → gamma),
  {{hp, 1}, 0.1, 10}, {{gamma, 1}, 0.1, 5}]}
```

Out[18]= {



,



}

In[10]:=

```
2 / 6.
```

Out[10]= 0.333333

In[9]:=

```
1 / 9 // N
```

Out[9]= 0.111111