ME-Project Report

about

Quad-rotors & Payload system dynamics and stability

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Introduction

In this paper, I will describe the dynamical system of 2 Quad-Rotor (aka quad) units, utilizing a common payload.

The problem formulation assumes 2D framework. The more general 3D case is not treated here.

The quads motion is treated as system inputs, and not discussed here by itself. I will discuss the payloads' dynamics and stability.

The investigation work flow will be:

1. The dynamic equations of the quads and payload will be described, and some limiting cases will be shown to verify the model.
    a. Coordinates definition in inertial frame
    b. Lagrangian term composition
    c. Deriving the equations of motion without non-conservative forces
    d. Verify result with limiting cases of:
        i. Elastic pendulum
    e. Find natural frequency, from equilibrium state
    f. Referring to non-conservative forces (and moments)
    g. Move to non-dimensional terms (by length and time scales)
    h. Define the treated maneuver in the problem (hover, translation of payload from points A to B)
2. Characterize the problem with certain parameters. Such as $(k_i, \mathrm{L0}_i, \mathrm{m}_p)$ and initial conditions and maneuvers (x_i, y_i)
3. The next step in this work will be to analyze the equations by Multiple Scales method or Averaging method.

A representative diagram for the system is shown here:

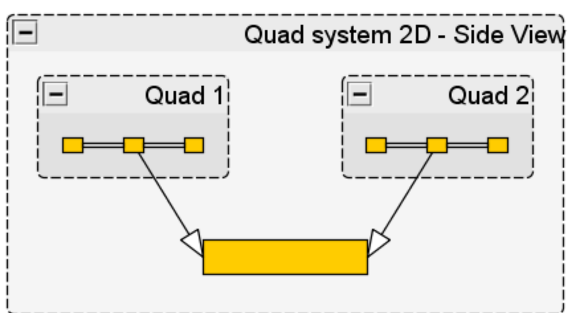
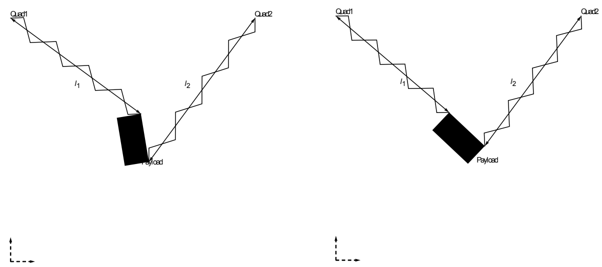


Figure 1 - system view

And some more possible ‘screenshots’ of possible system states:



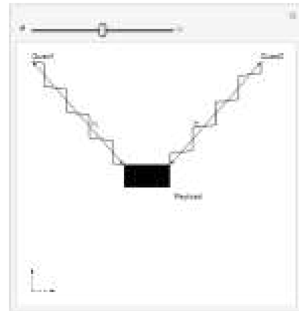


Figure 2 – θ_p = -73deg

Figure 3 - θ_p = 0 deg

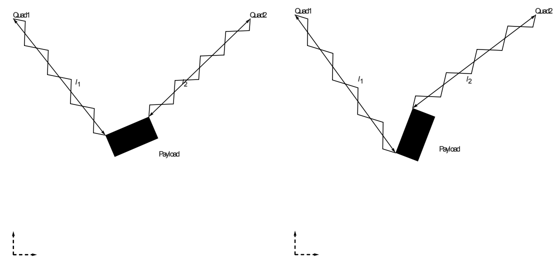


Figure 4 - θ_p = 73deg

Nomenclature

i : index for object {1,2,p} regarding: quad #1, quad #2, Payload, or: cable #1, cable #2.

θ_p : rotation angle around $\widehat{Z_I}$ axis, of the rigid body payload, relative to the Inertial frame.

k_i : spring i constant

$\mathrm{L0}_i$: spring length when not loaded ~~in equilibrium~~ (length of the free-load spring $\cancel{+\frac{m_p g}{k_i}}$)

l_i : current length of the loaded spring

l_p : geometric length of the payload rigid body

h_p : geometric height of the payload rigid body

R_p^I : rotation matrix of payload relative to Inertial coordinate frame

R_I^p : rotation matrix from Inertial to payload coordinate frame

m_i : mass of object i

I_i : moment of inertia , around axis $\widehat{Z_I}$, for object i

L : Lagrangian of the system

T : kinetic energy

V : potential energy

The system dynamics

The examined system is composed of 2 units of quadrotors, and 1 payload which is connected to each of the quadrotors. And by that it is connecting between the 2 quads.

The system is described in the 2D world.

The used assumptions for the system analysis are:

Quadrotor:

1. Quad body and parts are **rigid**. *No* elasticity is considered.
2. Geometry structure is **symmetrical** in relation to the principal axes. And the mass distribution is **uniform**. Hence the Inertia matrix is taken as pure diagonal.
3. quads resultant motion is given!

Payload & cable construction:

4. The 'cable' which the payload is connected to is modeled as straight spring, with initial length $\mathrm{L0}_i$, and has no mass.
5. The cable is connected to the quadrotor exactly in its center of mass (C.G).
6. **No friction** nor moments are present in the spring connection points.
7. The payload is a rectangular box, characterized with I_p as it's inertia metrix.
8. Possible spring dumping might be considered in the follow up work. (it might be added as non-conservative force)
9. Aerodynamic forces (lift and drag) on the payload – can be addressed in the non-conservative forces.

Coordinate systems , State variables, and Rotation matrices

I – inertial coordinates frame. It is the global reference point for the problem.

Its' axes are : $\left(\widehat{X_I}, \widehat{Y_I}, \widehat{Z_I}\right)^T$

P – Payload coordinate frame. The origin is located at the C.G of that rigid body.

The total general coordinates are:

(1) $$q = \begin{pmatrix} x_1 \\ y_1 \\ \theta_1 \\ x_2 \\ y_2 \\ \theta_2 \\ x_p \\ y_p \\ \theta_p \end{pmatrix}, where\ the\ actual\ interesting\ ones\ are\ only \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix}$$

Meaning the problem is actually only 3 D.O.F.

<u>The problem geometry</u>

Schematics of the system, in accordance with the nomenclature listed above:

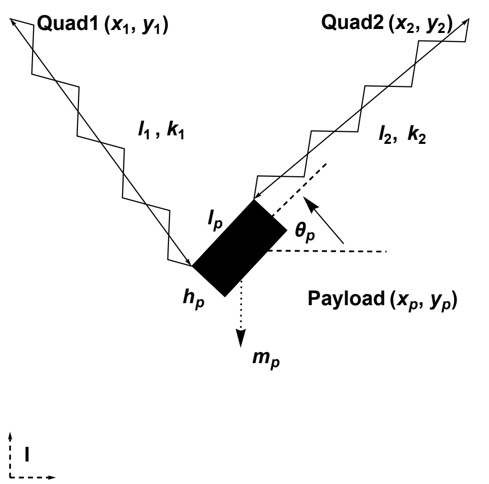


Figure 5

Where

$$\text{HangPoint1} = \text{PayloadCenterPos} + \text{Rp2I.}\left\{-\frac{1}{2}l_p, +\frac{1}{2}h_p\right\}$$

$$\text{HangPoint2} = \text{PayloadCenterPos} + \text{Rp2I.}\left\{+\frac{1}{2}l_p, +\frac{1}{2}h_p\right\}$$

<u>The Lagrangian of the system is:</u>

(2) $$L = T - V$$

(3) $$L = T_{\text{quad\#1}} + T_{\text{quad\#2}} + T_{\text{payload}} - \left(V_{\text{quad\#1}} + V_{\text{quad\#2}} + V_{\text{payload}} + V_{\text{spring\#1}} + V_{\text{spring\#2}}\right)$$

$$= \frac{1}{2} m_p \left(\left(x_p{}'\right)^2 + \left(y_p{}'\right)^2\right) + \frac{1}{2} m_1((x_1{}')^2 + (y_1{}')^2) + \frac{1}{2} m_2((x_2{}')^2 + (y_2{}')^2) + \frac{1}{2} i_{p,\text{zz}}\left(\theta_p{}'\right)^2$$
$$+ \frac{1}{2} i_{1,\text{zz}}(\theta_1{}')^2 + \frac{1}{2} i_{2,\text{zz}}(\theta_2{}')^2 - g m_p y_p - g m_1 y_1 - g m_2 y_2$$
$$- \frac{1}{2} k_1 \left(\sqrt{\left(\frac{1}{2} l_p \,\mathrm{c}(\theta_p) + \frac{1}{2} h_p \,\mathrm{s}(\theta_p) - x_p + x_1\right)^2 + \left(-\frac{1}{2} h_p \,\mathrm{c}(\theta_p) + \frac{1}{2} l_p \,\mathrm{s}(\theta_p) - y_p + y_1\right)^2} - \mathrm{L0}_1\right)^2$$
$$- \frac{1}{2} k_2 \left(\sqrt{\left(-\frac{1}{2} l_p \,\mathrm{c}(\theta_p) + \frac{1}{2} h_p \,\mathrm{s}(\theta_p) - x_p + x_2\right)^2 + \left(-\frac{1}{2} h_p \,\mathrm{c}(\theta_p) - \frac{1}{2} l_p \,\mathrm{s}(\theta_p) - y_p + y_2\right)^2} - \mathrm{L0}_2\right)^2$$

The Lagrange equations, without non-conservative forces, using (2) and the knowledge that V is not dependent on $\dot{q}_\iota$ for mechanical systems:

(4) $$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_\iota}\right) - \frac{\partial L}{\partial q_i} = \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_\iota}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \qquad ; \qquad i = 1 \div 3$$

(5a) $$k_1\left(\mathrm{r1x} - \frac{\mathrm{L0_1 r1x}}{\sqrt{\mathrm{r1x}^2 + \mathrm{r1y}^2}}\right) + k_2 \mathrm{r2x}\left(1 - \frac{\mathrm{L0}_2}{\sqrt{\mathrm{r2x}^2 + \mathrm{r2y}^2}}\right) - m_p x_p{}'' = 0$$

(5b) $$k_1\left(\mathrm{r1y} - \frac{\mathrm{L0_1 r1y}}{\sqrt{\mathrm{r1x}^2 + \mathrm{r1y}^2}}\right) + k_2\left(\mathrm{r2y} - \frac{\mathrm{L0_2 r2y}}{\sqrt{\mathrm{r2x}^2 + \mathrm{r2y}^2}}\right) - m_p\left(g + y_p{}''\right) = 0$$

(5c) $$\frac{1}{2}\left(\frac{k_1(\mathrm{dr1} + \mathrm{dr2})\left(\sqrt{\mathrm{r1x}^2 + \mathrm{r1y}^2} - \mathrm{L0}_1\right)}{\sqrt{\mathrm{r1x}^2 + \mathrm{r1y}^2}} + \frac{k_2(\mathrm{dr3} + \mathrm{dr4})\left(\sqrt{\mathrm{r2x}^2 + \mathrm{r2y}^2} - \mathrm{L0}_2\right)}{\sqrt{\mathrm{r2x}^2 + \mathrm{r2y}^2}} + 2 I_p \theta_p{}''\right) = 0$$

Where the inner terms are:

(6)
$$\begin{pmatrix} \frac{1}{2} l_p \, \mathrm{c}(\theta_p) + \frac{1}{2} h_p \, \mathrm{s}(\theta_p) - x_p + x_1 \to \mathrm{r1x} \\ -\frac{1}{2} h_p \, \mathrm{c}(\theta_p) + \frac{1}{2} l_p \, \mathrm{s}(\theta_p) - y_p + y_1 \to \mathrm{r1y} \\ -\frac{1}{2} l_p \, \mathrm{c}(\theta_p) + \frac{1}{2} h_p \, \mathrm{s}(\theta_p) - x_p + x_2 \to \mathrm{r2x} \\ -\frac{1}{2} h_p \, \mathrm{c}(\theta_p) - \frac{1}{2} l_p \, \mathrm{s}(\theta_p) - y_p + y_2 \to \mathrm{r2y} \\ l_p \left((y_1 - y_p) \, \mathrm{c}(\theta_p) + x_1 (-\mathrm{s}(\theta_p)) + x_p \, \mathrm{s}(\theta_p) \right) \to \mathrm{dr1} \\ h_p (x_1 \, \mathrm{c}(\theta_p) - x_p \, \mathrm{c}(\theta_p) + (y_1 - y_p) \, \mathrm{s}(\theta_p)) \to \mathrm{dr2} \\ h_p (x_2 \, \mathrm{c}(\theta_p) - x_p \, \mathrm{c}(\theta_p) + (y_2 - y_p) \, \mathrm{s}(\theta_p)) \to \mathrm{dr4} \\ l_p \left((y_p - y_2) \, \mathrm{c}(\theta_p) + x_2 \, \mathrm{s}(\theta_p) - x_p \, \mathrm{s}(\theta_p) \right) \to \mathrm{dr3} \end{pmatrix}$$

Those equations of motion are of the form $\ddot{X} = f(X)$, f is non-linear.

Rearranging some more, we can write the equations as :

(7)
$$\boldsymbol{k_1 \frac{(a - \mathrm{L0}_1)}{a} \begin{pmatrix} \mathrm{r1x} \\ \mathrm{r1y} \\ \mathrm{c1} \end{pmatrix} + k_2 \frac{(b - \mathrm{L0}_1)}{b} \begin{pmatrix} \mathrm{r2x} \\ \mathrm{r2y} \\ \mathrm{c2} \end{pmatrix} + \begin{pmatrix} 0 \\ -m_p g \\ 0 \end{pmatrix} = \begin{pmatrix} m_p & 0 & 0 \\ 0 & m_p & 0 \\ 0 & 0 & -I_p \end{pmatrix} \begin{pmatrix} x_p''[t] \\ y_p''[t] \\ \theta_p''[t] \end{pmatrix}}$$

While the additional simplifications are :

(8)
$$\begin{pmatrix} \sqrt{\mathrm{r1x}^2 + \mathrm{r1y}^2} \to a \\ \sqrt{\mathrm{r2x}^2 + \mathrm{r2y}^2} \to b \\ \frac{1}{2} (\mathrm{dr1} + \mathrm{dr2}) \to \mathrm{c1} \\ \frac{1}{2} (\mathrm{dr3} + \mathrm{dr4}) \to \mathrm{c2} \end{pmatrix}$$

Test for limiting case

Limiting case test of elastic pendulum is shown in Appendix 1.

Non-dimensional equations

Using the next conversions:

(9) $\tilde{y_p}[t] = y_p[t]/\mathrm{L0}_1$, or for any other of the lengths variables $(x_p, \mathrm{r1x}, \mathrm{r1y}, \mathrm{r2x}, \mathrm{r2y}, h_p, l_p)$

And $\tilde{c_i}[t] = c_i[t]/\mathrm{L0}_1{}^2$

$t = \tau/\omega_s$, where $\boldsymbol{\omega_s}^2 = \frac{\boldsymbol{k_1}}{\boldsymbol{m_p}} [\frac{g}{l} = \frac{1}{s^2}]$

From (7) we get:

(10)
$$\begin{pmatrix} \mathrm{r1x\,L0}_1 \frac{k_1}{m_p} + \mathrm{r2x\,L0}_1 \frac{k_2}{m_p} - \mathrm{r1x\,L0}_1 \frac{k_1}{m_p}\frac{\mathrm{L0}_1}{a} - \mathrm{r2x\,L0}_1 \frac{k_2}{m_p}\frac{\mathrm{L0}_2}{b} == (\mathrm{L0}_1 \omega_s{}^2)\ddot{x_p} \\ \mathrm{L0}_1 \frac{\mathrm{r1y}\,k_1}{m_p} + \mathrm{L0}_1 \frac{\mathrm{r2y}\,k_2}{m_p} - \mathrm{L0}_1 \frac{\mathrm{r1y}\,k_1 \mathrm{L0}_1}{m_p a} - \mathrm{L0}_1 \frac{\mathrm{r2y}\,k_2 \mathrm{L0}_2}{m_p b} - g == (\mathrm{L0}_1 \omega_s{}^2)\ddot{y_p} \\ \mathrm{L0}_1{}^2 \mathrm{c1}\,k_1 + \mathrm{L0}_1{}^2 \mathrm{c2}\,k_2 - \mathrm{L0}_1{}^2 \frac{\mathrm{c1}\,k_1 \mathrm{L0}_1}{a} - \mathrm{L0}_1{}^2 \frac{\mathrm{c2}\,k_2 \mathrm{L0}_2}{b} == (-I_p \omega_s{}^2)\ddot{\theta_p} \end{pmatrix}$$

(now all variables are non-dimensional variables. For simplicity – the notation $\mathrm{x} \to \tilde{x}$ is not changed.)

Further setup brings:

(11)
$$\boldsymbol{A} \begin{pmatrix} \mathbf{r1x} \\ \mathbf{r1y} \\ \mathbf{-E\,c1} \end{pmatrix} + \boldsymbol{B\,k} \begin{pmatrix} \mathbf{r2x} \\ \mathbf{r2y} \\ \mathbf{-E\,c2} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -\boldsymbol{D} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \ddot{\boldsymbol{x_p}} \\ \ddot{\boldsymbol{y_p}} \\ \ddot{\boldsymbol{\theta_p}} \end{pmatrix}$$

While noting 3 more non-dimensional terms :

(12) $\left(1 - \frac{\mathrm{L0}_1}{a}\right) \to A$; $\left(1 - \frac{\mathrm{L0}_2}{b}\right) \to B$; $\frac{k_2}{k_1} \to k$; $\frac{g}{\mathrm{L0}_1}\frac{1}{\omega_s{}^2} \to D$; $\frac{\mathrm{L0}_1{}^2 k_1}{I_p \omega_s{}^2} \to E$

Equilibrium check

This is the time where all state variables derivatives are zeroed. And especially relevant here:

$$x_p''[t] \to 0, \qquad y_p''[t] \to 0, \qquad \theta_p''[t] \to 0$$

Which gives

(13)
$$\boldsymbol{A} \begin{pmatrix} \mathbf{r1x} \\ \mathbf{r1y} \\ \mathbf{-E\,c1} \end{pmatrix} + \boldsymbol{B\,k} \begin{pmatrix} \mathbf{r2x} \\ \mathbf{r2y} \\ \mathbf{-E\,c2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{D} \\ \mathbf{0} \end{pmatrix}$$

And we can extract the 3 variables from those 3 equations:

(14)
$$x_p, y_p, \theta_p = f(x_1, y_1, x_2, y_2) + f'(k_1, k_2, l_p, h_p, \mathrm{L0}_1, \mathrm{L0}_2)$$

Example:

Let's say $y_1 = y_2,\ \ k_1 = k_2,\ \ \mathrm{L0}_1 = \mathrm{L0}_2$ (a symmetrical case):

I'll expect $\theta_p = 0,\ \ x_p = \frac{x_1+x_2}{2},\ \ \frac{y_p}{\mathrm{L0}_1} = -(1 + \frac{1}{2}D)$.

But it's hard to verify it analytically from (13).

Non-conservative general forces of the problem can be

1. Aerodynamic lift and drag = function of payload velocity and orientation relative to the surrounding air.
2. Dumping force in parallel to the spring tension force. Can be described according to (Kelvin-Voigt) model.

Treated maneuvers in the problem

1. hover
2. translation of payload from points A to B, in a straight line. With equal or different quads heights.

Trajectory can be described for example as:

(a) $\tau = 0$: $\ddot{y} = 1\,m/s^2$ until $y_1 = y_2 = 10\ \mathrm{L0}_1$
(b) $\ddot{y} = -1\,m/s^2$ until $\dot{y_1} = \dot{y_2} = 0$
(c) $\ddot{x_1} = \ddot{x_2} = 1\,m/s^2$ until $\dot{x_1} = \dot{x_2} = 2\,m/s$

(*) disterbunce can be input by $x_1 += 5\ \mathrm{L0}_1$ over $\frac{1}{100\sqrt{\omega_s}}\ [sec]$

equilibrium analysis

hover with wind force on payload vs specified motion

4 asymptotic analysis

for selected limiting cases that reveal a Hopf bifurcation and/or an orbital instability

5 numerical analysis

for asymptotic validation vs general maneuver

6 discussion

Summary

I described the 2D dynamics of system of 2 quadrotors and 1 connected rigid body payload.

I verified against limiting cases of:

1. elastic pendulum

Non-dimensional equations were submitted.

References

1. Dynamics Modeling and Control of a Quadrotor with Swing Load
2. flyingmachinearena Publications - http://flyingmachinearena.org/research/
3. TI tutorial : https://training.ti.com/webinar-how-extend-flight-time-and-battery-life-quadcopters-and-industrial-drones
4. Online course : https://www.coursera.org/learn/robotics-flight/
5. S. BOUABDALLAH , “design and control of quadrotors with application to autonomous flying”, THÈSE NO 3727 (2007), Lausanne, EPFL
6. http://lib.physcon.ru/file?id=bee776e3b376
7. http://www.cmi.ac.in/~ravitej/lab/param_res.pdf
8.

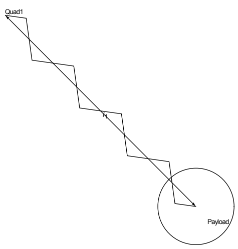
Appendix 1 –Limiting case dynamics – elastic pendulum

Reminding about the full problem equations of motion, from (7):

$$k_1 b(a-\mathrm{L0}_1)\begin{pmatrix}\mathrm{r1x}\\ \mathrm{r1y}\\ \mathrm{c1}\end{pmatrix}+k_2 a(b-\mathrm{L0}_1)\begin{pmatrix}\mathrm{r2x}\\ \mathrm{r2y}\\ \mathrm{c2}\end{pmatrix}+\begin{pmatrix}0\\ -m_p gab\\ 0\end{pmatrix}=ab\begin{pmatrix}m_p & 0 & 0\\ 0 & m_p & 0\\ 0 & 0 & -I_p\end{pmatrix}\begin{pmatrix}x_p''[t]\\ y_p''[t]\\ \theta_p''[t]\end{pmatrix}$$

When looking on elastic pendulum for lumped mass, we can assume:

(1) $l_p \to 0\,, h_p \to 0$ for the lumped mass (hence θ_p doesn't matter any more)
(2) $k_2 \to 0$ for the connection, only to the first base, and not the 2nd one
(3) Arbitrarily I will assume $x_1[t] \to 0, y_1[t] \to 0$ which means also the 1st base is static



The equations of motion become:

(4) $$k_1 x_p\left(\frac{\mathrm{L0}_1}{\sqrt{x_p^2+y_p^2}}-1\right)=m_p x_p''$$

(5) $$k_1 y_p\left(\frac{\mathrm{L0}_1}{\sqrt{x_p^2+y_p^2}}-1\right)=m_p\left(g+y_p''\right)$$

Finding the equilibrium point - we set the derivatives to 0 ($x_p''[t] \to 0, y_p''[t] \to 0$) :

(6) $$\left\{\begin{matrix}\boldsymbol{x_p[t]\to 0}\\ \boldsymbol{y_p[t]\to \frac{-k_1\mathrm{L0}_1-gm_p}{k_1}}\end{matrix}\right\},\left\{\begin{matrix}x_p[t]\to 0\\ y_p[t]\to \frac{k_1\mathrm{L0}_1-gm_p}{k_1}\end{matrix}\right\}$$

(if considering the non-dimensional variables, as described in the sections above, we can write $\tilde{y_p}[t] \to -1-\frac{g}{\mathrm{L0}_1}\frac{m_p}{k_1}$)

The 1st option is the relevant one, for the considered state of $y_p < 0$.

We can also note that if considering $k_1 \to Inf$: $y_{p_{equib}} = -\mathrm{L0}_1$, which fits to a problem of a simple pendulum, hanged on a rod and not a spring.

assumption is $y_p>0$ fits to $y_p[t] \to \frac{k_1\mathrm{L0}_1-gm_p}{k_1}=\mathrm{L0}_1-\frac{gm_p}{k_1}$, so $\mathrm{L0}_1 > \frac{gm_p}{k_1}$ otherwise it means the spring k_1 is to small and weak.

assumption is $y_p<0$ fits to $y_p[t] \to -\frac{k_1\mathrm{L0}_1-gm_p}{k_1}=-\mathrm{L0}_1-\frac{gm_p}{k_1}=-(\mathrm{L0}_1+\frac{gm_p}{k_1})$

Linearization around the equilibrium point

Get dimensionless variables by the relations:

$$x_p = \frac{x_p}{\mathrm{L0}_1}\ ;\ y_p = \frac{y_p}{\mathrm{L0}_1}\ ;\ t = \tau \frac{m_p}{k_1}$$

The non-dimensional equations are:

(7) $$x_p\left(\frac{1}{\sqrt{x_p^2+y_p^2}} - 1\right) = x_p''$$

(8) $$y_p\left(\frac{1}{\sqrt{x_p^2+y_p^2}} - 1\right) - \frac{g}{\mathrm{L0}_1}\frac{m_p}{k_1} = y_p''$$

1st order linearization is : $\sqrt{f[x,y]} = \sqrt{f[0,0]} + \frac{f^{(1,0)}[0,0]}{2\sqrt{f[0,0]}}x + \frac{f^{(0,1)}[0,0]}{2\sqrt{f[0,0]}}y + O[x]^2 + O[y]^2$

Near equilibrium:

$$x_p = x_{p_0} + \delta \mathrm{x}_p\ ;\ \ y_p = y_{p_0} + \delta \mathrm{y}_p$$

$$\sqrt{x_p^2 + y_p^2} = \sqrt{x_{p_0}^2 + y_{p_0}^2} + \frac{2x_{p_0}}{2\sqrt{x_{p_0}^2 + y_{p_0}^2}}\delta \mathrm{x}_p + \frac{2y_{p_0}}{2\sqrt{x_{p_0}^2 + y_{p_0}^2}}\delta \mathrm{y}_p$$

Testing for the 1st equilibrium point of:

(9) $$y_{p_0} \to -\left(1 + \frac{g}{\mathrm{L0}_1}\frac{m_p}{k_1}\right) = -(1+G)\ ;\ \ x_{p_0} \to 0$$

(10) setting : $G \to \frac{g}{\mathrm{L0}_1}\frac{m_p}{k_1}$

(11) $\rightarrow$ $$\sqrt{x_p^2 + y_p^2} = \pm(y_{p_0} + \delta \mathrm{y}_p)$$

The equations are written as:

(10a) $$\left(\cancel{x_{p_0}} + \delta \mathrm{x}_p\right)\left(1 \pm (y_{p_0} + \delta \mathrm{y}_p)\right) = \mp(y_{p_0} + \delta \mathrm{y}_p)\delta \ddot{\mathrm{x}}_p$$

(10b) $$\left(y_{p_0} + \delta \mathrm{y}_p\right)\left(1 \pm \left(y_{p_0} + \delta \mathrm{y}_p\right)\right) = \mp(y_{p_0} + \delta \mathrm{y}_p)\left(\frac{g}{\mathrm{L0}_1}\frac{m_p}{k_1} + \delta \ddot{\mathrm{y}}_p\right)$$

⇨ Using (9) and , for the 1st equib. point, neglecting small terms such as $(\delta \mathrm{x}_p{}^2)$, $(\delta \ddot{\mathrm{y}}_p \delta \mathrm{x}_p)$:

$$(\delta \mathrm{x}_p)(1 + y_{p_0}) = -\delta \ddot{\mathrm{x}}_p(y_{p_0})$$

$$\left(y_{p_0}\right)\left(1 + y_{p_0}\right) + \left(\delta \mathrm{y}_p\right)\left(1 + 2y_{p_0} + G\right) = -\left(\delta \ddot{\mathrm{y}}_p\right)\left(y_{p_0}\right) - (G)(y_{p_0})$$

- ⇨ Using the relation from equilibrium above $\left(\pm 1 - y_{p_0}\right) = G$ to eliminate o(1) terms
- ⇨

(12) $$\begin{pmatrix} \delta\ddot{x}_p \\ \delta\ddot{y}_p \end{pmatrix} + \begin{pmatrix} \frac{G}{1+G} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta x_p \\ \delta y_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It is equivalent to the matrix notation :

(13) $$M\ddot{x} + C\dot{x} + Kx == F$$

Where :

$$M = \mathrm{IdentityMatrix}_{2x2}$$

$$C = \{0\}$$

$$K = \begin{pmatrix} \dfrac{G}{1+G} & 0 \\ 0 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Where we know to find the natural frequencies by the requirement of:

(14) $$det|K - \omega^2 M| = 0$$

It is an equation of 4th order for ω , the relevant solutions are:

(15) $${\omega_x}^2 = \frac{G}{1+G} \quad ; \quad {\omega_y}^2 = 1$$

$$rescaling\ result\ for\ comparison\ to\ literature:$$

$$\omega_y \omega_S = \sqrt{\frac{k_1}{m_p}}$$

$\omega_x \omega_S = \sqrt{\frac{G}{1+G}}\sqrt{\frac{k_1}{m_p}} = \sqrt{\frac{gk_1}{k_1 \mathrm{L0}_1 + gm_p}}$, where **expecting** $\omega_x \omega_S = \sqrt{G}\sqrt{\frac{k_1}{m_p}} = \sqrt{\frac{g}{\mathrm{L0}_1}}$

If reformatting the above to $\sqrt{\frac{g}{\mathrm{L0}_1 + \frac{gm_p}{k_1}}}$ it is equivalent to the natural frequency of simple pendulum with a constant length of $\mathrm{L0}_1 + \frac{gm_p}{k_1}$!

Extra Limiting cases:

When $k_1 \to \infty$ it affects $G \to 0\ and$ we get $\omega_y, \omega_x = \omega_S, 0$ which is similar to $\omega_{\text{simple\_pendulum}} = \sqrt{\frac{g}{\text{L}0_1}}$.

<u>Stability</u>

From other references in literature we can note that (15) is relevant to a stable dynamic behavior.