Blade Element Theory

- Momentum theory gives rapid and simple method to estimate of necessary Power.
- This approach is sufficient to size a rotor (i.e. select the disk area) for a given power plant (engine), and a given gross weight.
- This approach is not <u>adequate</u> for designing the rotor.

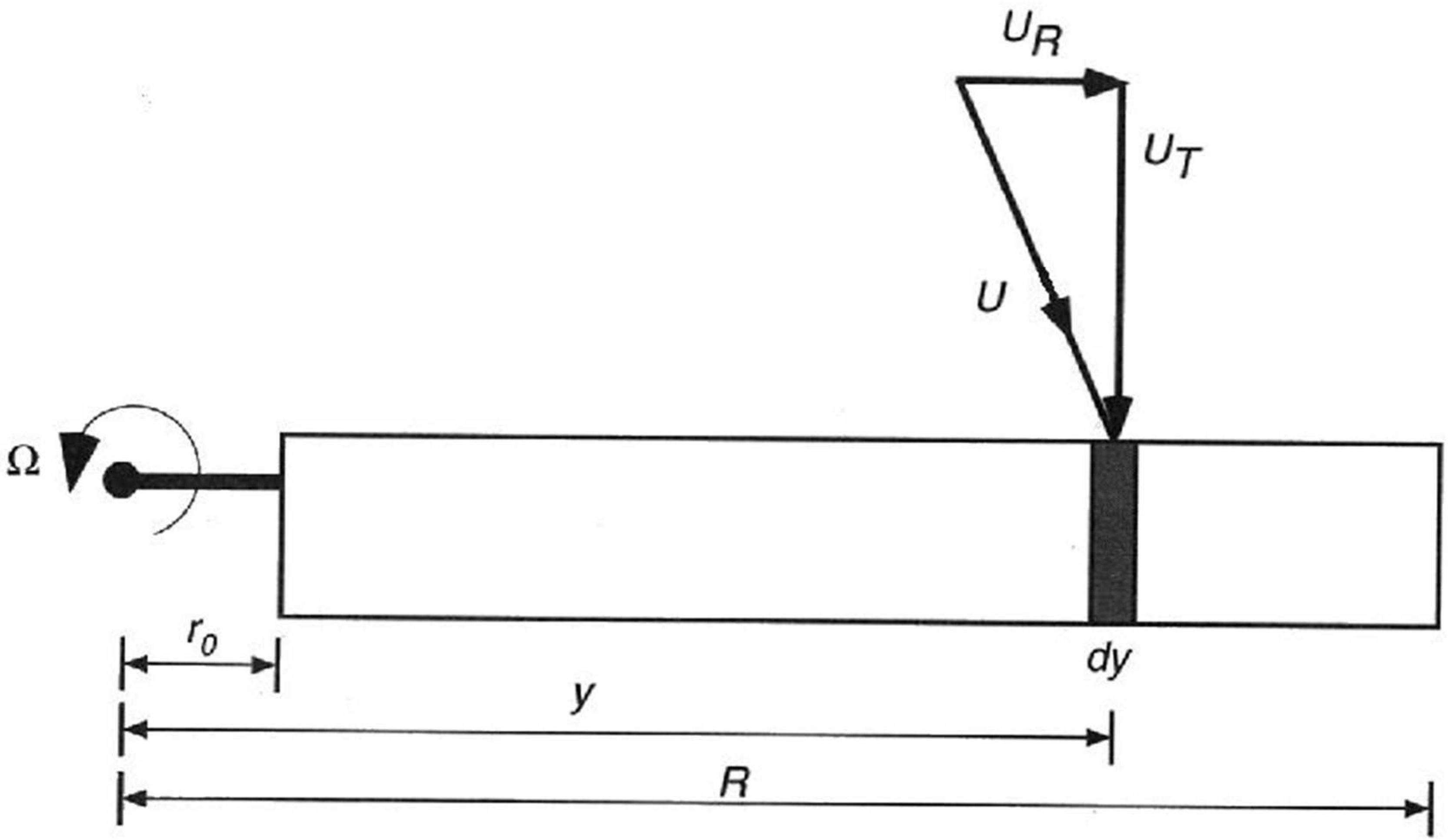
Blade Element Theory

- The momentum theory does not take into account
  - Number of blades
  - Airfoil characteristics (lift, drag, angle of zero lift)
  - Blade planform (taper, sweep, root cut-out)
  - Blade twist distribution
  - Compressibility effects

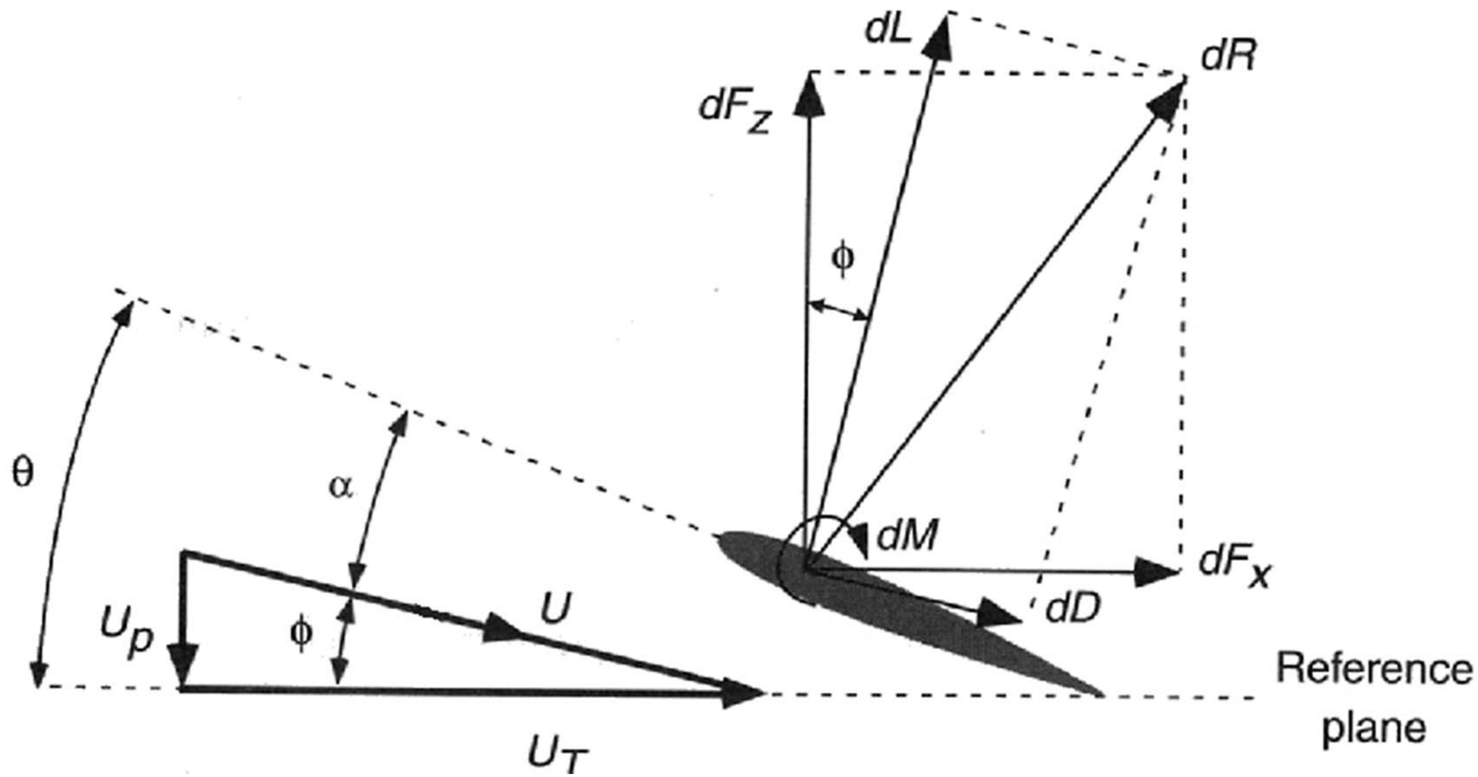
Blade Element Theory

- Blade Element Theory (BET) was first proposed by Drzwiecki in 1892 for the analysis of airplane propeller.
- BET assumes that each blade section acts as a two dimensional airfoil to produce aerodynamic forces
- The blade is then divided in non-interacting sections where all the computations are performed using 2-D aerodynamics
- An integration over the blade length gives the total thrust and total power

BET Model



BET Model



- The in plane Velocity $U_T = \Omega y$
- The out of plane Velocity $U_P = V_C + v_i$
- Therefore the total velocity is $U = \sqrt{U_T^2 + U_P^2}$

BET model

- The relative inflow angle:

$$\phi = \tan^{-1}\left(\frac{U_P}{U_T}\right)$$

- If the blade element has a pitch angle of θ, the effective angle of attack is:

$$\alpha = \theta - \phi = \theta - \tan^{-1}\left(\frac{U_P}{U_T}\right)$$

BET model

- The incremental lift per unit span:

$$dL = \tfrac{1}{2}\rho U^2 c C_l dy$$

- The incremental drag per unit span:

$$dD = \tfrac{1}{2}\rho U^2 c C_d dy$$

- Or in quantities parallel and perpendicular to the rotor disk plane:

$$\begin{cases} dF_z = dL\cos\phi - dD\sin\phi \\ dF_x = dL\sin\phi + dD\cos\phi \end{cases}$$

BET model

- We can then calculate the Thrust:

$$dT = N_b dF_z$$

- The Torque

$$dQ = N_b dF_x y$$

- The Power

$$dP = N_b dF_x \Omega y$$

- Remember N_b is the number of blades

BET model

- And we can relate all three with C_l and C_d

$$\begin{cases} dT = N_b\left(dL\cos\phi - dD\sin\phi\right) \\ dQ = N_b\left(dL\sin\phi + dD\cos\phi\right)y \\ dP = N_b\left(dL\sin\phi + dD\cos\phi\right)\Omega y \end{cases}$$

BET model assumptions

- The following assumptions are valid within the helicopter aerodynamics

$$U_T >> U_P \Rightarrow U = \sqrt{U_P^2 + U_T^2} \approx U_T$$

$$\phi \approx 0 \Rightarrow \begin{cases} \phi = \tan^{-1}\left(U_P / U_T\right) \approx U_P / U_T \\ \sin\phi = \phi \\ \cos\phi = 1 \end{cases}$$

$$dD << dL \Rightarrow dD \sin\phi \approx dD\phi \approx 0$$

Basic Equations

- The expression for Thrust, Torque and Power are:

$$\begin{cases} dT = N_b\left(dL\cos\phi - dD\sin\phi\right) = N_b\left(dL\right) \\ dQ = N_b\left(dL\sin\phi + dD\cos\phi\right)y = N_b\left(dL\phi + dD\right)y \\ dP = N_b\left(dL\sin\phi + dD\cos\phi\right)\Omega y = N_b\left(dL\phi + dD\right)\Omega y \end{cases}$$

- Let's now nondimensionalize using for length R and for speed $V_{tip}=\Omega R$

Nondimensional form

- $r = y/R$
- $U_T/\Omega R = \Omega y / \Omega R = y/R = r$
- And the thrust, torque and power coefficients already defined:

$$dC_T = \frac{dT}{\rho A(\Omega R)^2}, dC_Q = \frac{dQ}{\rho A(\Omega R)^2 R}, dC_P = \frac{dP}{\rho A(\Omega R)^3}$$

- Now the inflow ratio is

$$\lambda = \frac{V_c + v_i}{\Omega R} = \frac{V_c + v_i}{\Omega y}\left(\frac{\Omega y}{\Omega R}\right) = \frac{U_P}{U_T}\left(\frac{y}{R}\right) \quad = \phi r$$

Thrust coefficient (incremental)

- Substituting the previous equations in the Thrust coefficient equation:

$$dC_T = \frac{N_b dL}{\rho A(\Omega R)^2} = \frac{N_b\left(\frac{1}{2}\rho {U_T}^2 c C_l dy\right)}{\rho A(\Omega R)^2}$$

$$= \frac{1}{2}\left(\frac{N_b c}{\pi R}\right) C_l \left(\frac{y}{R}\right)^2 d\left(\frac{y}{R}\right) = \frac{1}{2}\sigma C_l r^2 dr$$

Power coefficient (incremental)

- Using the same analysis for the Power coefficient

$$dC_P = dC_Q = \frac{dQ}{\rho A(\Omega R)^2 R} = \frac{N_b\left(\phi dL + dD\right)y}{\rho A(\Omega R)^2 R}$$

$$= \frac{1}{2}\sigma\left(\phi C_l + C_D\right)\left(\frac{y}{R}\right)^3 d\left(\frac{y}{R}\right)$$

$$= \frac{1}{2}\sigma\left(\phi C_l + C_D\right)r^3 dr$$

Total Thrust and Power

- To find the total blade contribution for Thrust and power we have on integrate between the root and tip of the blade

$$C_T = \tfrac{1}{2}\int_0^1 \sigma C_l r^2 dr = \tfrac{1}{2}\sigma\int_0^1 C_l r^2 dr$$

- If the blade is rectangular *c=const*
- For the torque and power coefficient

$$C_Q = C_P = \tfrac{1}{2}\sigma\int_0^1 (\phi C_l + C_d) r^3 dr = \tfrac{1}{2}\sigma\int_0^1 \left(\lambda C_l r^2 + C_d r^3\right) dr$$

Total Thrust and Power

- To evaluate the previous expressions we need:
- Inflow ratio $\lambda = \lambda(r)$
- Lift coefficient $C_l = C_l(\alpha, Re, M)$
- Drag coefficient $C_d = C_d(\alpha, Re, M)$
- AOA $\alpha = \alpha(V_C, \theta, v_i)$
- Induced Velocity $v_i = v_i(r)$

Numerical Solution needed

Approximations

- With certain assumptions and approximations it is possible to find closed form analytical solutions.
- The solutions are important because they serve to illustrate the fundamental form of the results in term of operational and geometric parameters of the rotor
- Let’s the assume a rectangular blade $c=const.$ From the definition $\sigma=const.$ too.

Thrust approximation

- From the Steady linearized aerodynamics:

$$C_l = C_{l_\alpha}\left(\alpha - \alpha_0\right) = C_{l_\alpha}\left(\theta - \phi - \alpha_0\right)$$

- We can consider C_{l_α} constant without serious loss of accuracy
- Let's also assume symmetric airfoils $\alpha_0 = 0$
- We can then write:

$$C_T = \tfrac{1}{2}\int_0^1 \sigma C_l r^2 dr = \tfrac{1}{2}\sigma C_{l\alpha}\int_0^1 \left(\theta - \phi\right) r^2 dr$$

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}\int_0^1 \left(\theta r^2 - \lambda r\right) dr$$

Untwisted Blades

- For a blade with zero twist $\theta = const. = \theta_0$.
- Let's also assume uniform inflow velocity, as assumed in the momentum theory $\lambda = const$.
- The Thrust coefficient can be written as:

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}\int_0^1\left(\theta_0 r^2 - \lambda r\right)dr = \tfrac{1}{2}\sigma C_{l\alpha}\left[\theta_0\tfrac{r^3}{3} - \lambda\tfrac{r^2}{2}\right]_0^1$$

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}\left[\frac{\theta_0}{3} - \frac{\lambda}{2}\right]$$

Uniform inflow

- Let's use the result from the momentum theory

$$\lambda_i = \lambda_h = \sqrt{C_T / 2}$$

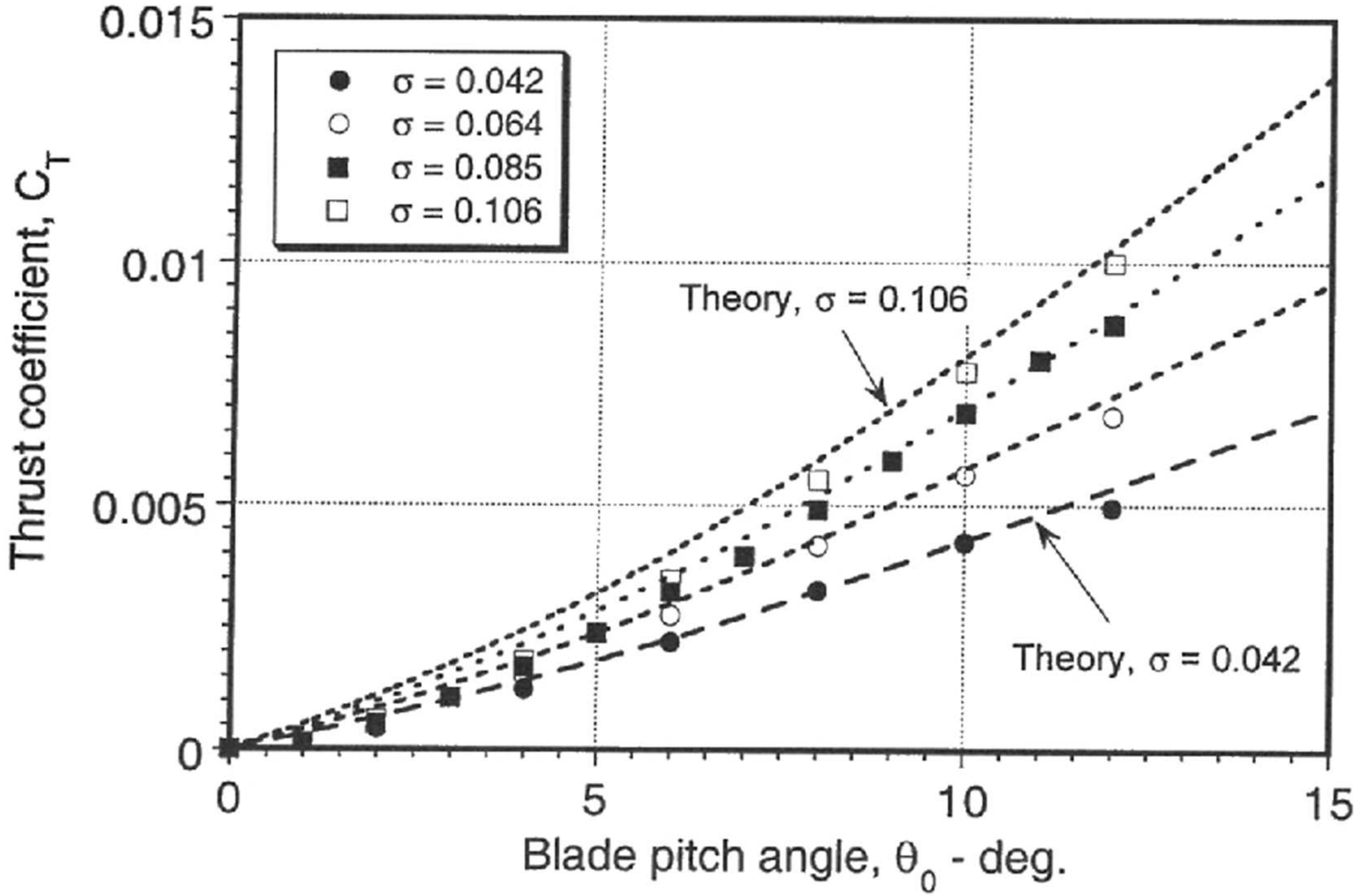
- So the thrust coefficient is:

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}\left[\frac{\theta_0}{3} - \frac{1}{2}\sqrt{\frac{C_T}{2}}\right]$$

- And we can calculate the pitch angle

$$\theta_0 = \frac{6C_T}{\sigma C_{l_\alpha}} + \frac{3}{2}\sqrt{\frac{C_T}{2}}$$

Untwisted Blades, Uniform inflow



Linearly Twisted Blades, Uniform inflow

- Let's now assume that we have a linear twist, common practice in helicopter blades:

$$\theta(r) = \theta_0 + r\theta_{tw}$$

- Substituting in the C_T equation:

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}\int_0^1 \left(\left(\theta_0 + r\theta_{tw}\right)r^2 - \lambda r\right)dr =$$

$$= \tfrac{1}{2}\sigma C_{l\alpha}\left[\frac{\theta_0}{3} + \frac{\theta_{tw}}{4} - \frac{\lambda}{2}\right]$$

Linearly Twisted Blades, Uniform inflow

- If the reference blade pitch angle is taken a 3/4 - radius ($\theta_{0.75}$) then

$$\theta(r) = \theta_{0.75} + (r - 0.75)\theta_{tw}$$

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}\int_0^1 \left(\left(\theta_{0.75} + (r-0.75)\theta_{tw}\right)r^2 - \lambda r\right)dr =$$

$$= \tfrac{1}{2}\sigma C_{l\alpha}\int_0^1 \left(\theta_{0.75}r^2 + \theta_{tw}r^3 - 0.75\theta_{tw}r^2 - \lambda r\right)dr =$$

$$= \tfrac{1}{2}\sigma C_{l\alpha}\left[\frac{\theta_{0.75}}{3} + \frac{\theta_{tw}}{4} - \frac{\theta_{tw}}{4} - \frac{\lambda}{2}\right] = \tfrac{1}{2}\sigma C_{l\alpha}\left[\frac{\theta_{0.75}}{3} - \frac{\lambda}{2}\right]$$

- Same result as for the constant pitch blade

Power approximations

- We have seen that the incremental power coefficient (that is equal to the torque coefficient):

$$dC_P = \tfrac{1}{2}\sigma\left(\phi C_l + C_d\right)r^3 dr = \tfrac{1}{2}\sigma\left(\lambda C_l r^2 + C_d r^3\right)dr =$$

$$= \tfrac{1}{2}\sigma\lambda C_l r^2 dr + \tfrac{1}{2}\sigma C_d r^3 dr =$$

$$= dC_{P_i} + dC_{P_0}$$

- Remembering that

$$dC_{P_i} = \lambda dC_T \Rightarrow \qquad dC_P = \lambda dC_T + dC_{P_0}$$

Power approximations

- Therefore the total power:

$$C_P = \int_{r=0}^{r=1} \lambda dC_T + \int_0^1 \tfrac{1}{2}\sigma C_d r^3 dr = \lambda C_T + \tfrac{1}{8}\sigma C_{d_0}$$

- Assuming uniform inflow and $C_d = C_{d0} = const.$
- Using once more the inflow expression obtained in hover:

$$C_P = \frac{C_T^{3/2}}{\sqrt{2}} + \tfrac{1}{8}\sigma C_{d_0}$$

- Expression already obtained in the momentum theory

FM for BET

$$FM = \frac{C_{P\,ideal}}{C_{P_{real}}} = \frac{\lambda C_T}{\lambda C_T + \sigma C_{d0}/8}$$

- High solidity σ (lot of blades, wide-chord, large blade area) leads to higher Power consumption, and lower FM.
- Low drag Airfoils leads to higher FM

Average Lift coefficient

- The average Lift coefficient is defined to give the same thrust coefficient when the blade is operating at the same local lift coefficient (optimum rotor):

$$C_T = \tfrac{1}{2}\int_0^1 \sigma r^2 C_l dr = \tfrac{1}{2}\int_0^1 \sigma r^2 \overline{C_l} dr = \tfrac{1}{6}\sigma \overline{C_l}$$

- Or $\overline{C_l} = 6\dfrac{C_T}{\sigma}$

- Typically $\overline{C_l}$ is found to be on the range of 0.5 to 0.8.

FM for Average Lift Coefficient

- We can now introduce the last expression on the FM equation already obtained:

$$FM = \frac{\lambda C_T}{\lambda C_T + \sigma C_{d0}/8} = \frac{1}{1+\sigma C_{d0}/\left(8\, C_T\, \lambda\right)} =$$

$$\frac{1}{1+\sigma C_{d0}/\left(\frac{8}{6}\,\sigma \overline{C}_L\, \lambda\right)} = \frac{1}{1+\frac{3}{4}\left[\left(C_{d0}/\overline{C}_l\right)/\,\lambda\right]} =$$

- FM is maximized if $\left(C_{d0}/\overline{C}_l\right)$ is minimized

Tip Loss factor

- We can assume that the outer portion of the blade (R-R_e=R-BR) does not produce lift. Therefore the thrust coefficient is:

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}\int_0^B \left(\theta r^2 - \lambda r\right)dr$$

- For a untwisted blade ($\theta=\theta_0$):

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}B^2\left[\frac{\theta_0 B}{3}-\frac{\lambda}{2}\right]$$

Tip Loss factor

- For a twisted blade ($\theta=\theta_{tip}/r$):

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}\int_0^B \left(\theta_{tip} r - \lambda r\right) dr = \tfrac{1}{4}\sigma C_{l\alpha} B^2 \left[\theta_{tip} - \lambda\right]$$

- For B between 0.95 and 0.98 we can calculate a 6 to 10% reduction in rotor thrust.
- Let's now assume that instead of having the blade tip not carrying any lift, let's see it's effect of the induced inflow velocity:

$$v_h = \sqrt{\frac{T}{2\rho A_e}} = \sqrt{\frac{T}{2\rho\left(AB^2\right)}} = \frac{1}{B}\sqrt{\frac{T}{2\rho A}}$$

Tip Loss factor

- Since the influence is a increase of λ by B^{-1} we can substitute in the equations obtained for no tip losses:
  - Untwisted blades and uniform inflow

$$C_T = \tfrac{1}{2}\sigma C_{l\alpha}\left[\frac{\theta_0}{3} - \frac{\lambda}{2B}\right]$$

  - Twisted blades and uniform inflow

$$C_T = \tfrac{1}{4}\sigma C_{l\alpha}\left[\theta_{tip} - \frac{\lambda}{B}\right]$$

Tip Loss factor

- Comparing with the results obtained earlier we see that these overpredict the effect of tip losses.
- Performing the same calculation for the power coefficient

$$\begin{cases} \theta = \theta_0 \Rightarrow C_P = \dfrac{\sigma C_{l_\alpha}}{2} \dfrac{\lambda}{B} \left[\dfrac{\theta_0}{3} - \dfrac{\lambda}{2B} \right] + \dfrac{1}{8} \sigma C_{d_0} \\ \theta = \theta_{tip} / r \Rightarrow C_P = \dfrac{\sigma C_{l_\alpha}}{4} \left[\dfrac{\lambda}{B} \left(\theta_{tip} - \dfrac{\lambda}{B} \right) \right] + \dfrac{1}{8} \sigma C_{d_0} \end{cases}$$

Tip Loss factor