Test Flight Problem Set Solutions

April 11, 2021

A1. False. If $n \geq 2$, then for any m, we have that $3m + 5n \geq 12$ (since $3m+5n \geq 3m+10 \geq 12$). Thus, the only way to find such a solution for n in the natural numbers would be when $n = 1$. Substituting, we have $3m + 5 \cdot 1 = 3m + 5 = 12$, or $3m = 7$. But since there is no natural number m satisfying this equation, then the result is proved.

A2. True. Let the consecutive five integers be written like: $n-2,\ n-1,\ n,\ n+1,\ n+2$. Summing these integers, we have $5n$, which is divisible by 5. The result is proved.

A3. True. Rewrite n^2+n+1, as $n \cdot (n+1)+1$. If n is even, then $n+1$ is odd. If n is odd, then $n+1$ is even. In either case, $n \cdot (n+1)$ is even because the product of an even and odd number is even. Rewrite $n \cdot (n+1) + 1$ as $2k+1$, which is odd. The result is proved.

A4. From the remainder theorem: if a, b are integers with $b > 0$, then there exist unique integers q, r such that$a = bq + r$ and $0 \leq r < b$. If we let $b = 4$ (and $n = q$), then $a = 4n + r$ with $0 \leq r < 4$. If $r = 0$ or 2, then $a = 4n$ or $a = 4n + 2$, which are even natural numbers. If $r = 1$ or 3, then $a = 4n + 1$ or $a = 4n + 3$, which are odd natural numbers. Since a is any odd natural number, satisfying the antecedent, it must be of one of the following forms $a = 4n + 1$ or $a = 4n + 3$. The result is proved.

A5. From the remainder theorem: if a, b are integers with $b > 0$, then there exist unique integers q, r such that$a = bq+r$ and $0 \leq r < b$. If $b = 3$, then $a = 3q + r$ with $0 \leq r < 3$. Expanding out (and letting $n = a$), then $n = 3q$, or $n = 3q + 1$, or $n = 3q + 2$. Write n, $n+2$, and $n+4$ in these forms: n is either $3q$, $3q+1$, or $3q+2$. $n+2$ is either $3q+2$, $3q+3$, or $3q+4$. $n+4$ is either $3q+4$, $3q+5$, or $3q+6$. But in each of the forms, there exists an element which is divisible by 3 i.e. if n, $3|3q$ and if $n+2$, $3|(3q+3)$, and if $n+4$, $3|(3q+6)$. The result is proved.

A6. Prove this by contradiction, assume there exists $n > 3$, such that n, $n+2$, and $n+4$ are prime. But from the proof of #5, we have shown that one of n, $n+2$, $n+4$ must be divisible by 3. And since $n>3$, 3 isn't prime. Then, one of n, $n+2$, $n+4$ is not prime. The result is proved.

A7. Let the sum, $2+2^2+2^3+...+2^n$, be denoted by S. Multiplying by 2, then $2S = 2^2+2^3+...+2^{n+1}$. Substracting S from $2S$, we get $S = 2^{n+1}-2$, which is what we wanted to prove.

A8. Assume that for any given $\epsilon > 0$, there exists an n where for all $m \geq n$, $|a_m - L| < \epsilon$. The statement that Ma_n tends to ML as n tends to infinity is the same as for any given $\epsilon_1 > 0$, there exists an n where all $m \geq n$, $|Ma_m - ML| < \epsilon_1$. This simplifies to $|M|a_m - L| \iff M|a_m - L| < \epsilon_1 \iff |a_m - L| < \frac{\epsilon_1}{M}$. This is true if $\epsilon_1/M = \epsilon$, and find such an n. Since this can always be done, the result is proved.

A9. Let $A_n = (0, 1/n)$. We have A_n which is a subset of A_1 since $(0, 1/n)$ is a subset of $(0,1)$. Suppose that x is an element of $(0,1)$. We can always find a natural number m such that $1/m < x$. That means that x is not an element of A_m. Hence, x is not an element of the intersection of A_n where n is a natural number. Since we can always find this number m, we must necessarily have that intersection of A_n is empty. The result is proved.

A10. Let $A_n = [0, 1/n)$. We may write this set as $0 \cup B_n$, where $B_n = (0, 1/n)$. The intersection of A_n for n in the natural numbers may thus be written as $0 \cup (\cap B_n)$. We proved in #9 that intersection of all B_n is the empty set, we have that $0 \cup \emptyset = \{0\}$. The result is proved.