Enhancement Seminar
on

Nonlinear bending and buckling analysis of laminated and sandwich curved panels in hygrothermal environment

by
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- Now a days, structural components are trending from traditional material to advanced structural materials, require high specific strength, low specific density and other properties.
- Composite material one such type of advanced structural materials, is widely in various engineering fields like aerospace and aeronautical, mechanical, marine and naval, etc.

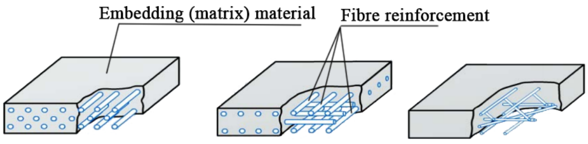

Figure: Fiber reinforced composite material

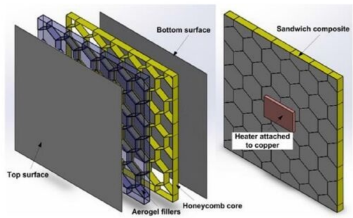

Figure: Sandwich composite material

- However, laminated and sandwich composites are weak in shear due to their low shear modulus compared to extensional rigidity which makes shear deformation a prominent factor during external influence.
- A clear understanding on shear deformation and structural response is required to achieve the full range of capabilities on the exemplary performance of composite structure.

Introduction (cont.)

- Only material with specific strength and stiffness is not sufficient, material in a proper geometric shape is necessary to utilize the benefit.
- Among various geometric shape, shell structure resist the external applied loads with optimum use of material. By virtue of the curved shape, shell structure withstand both membrane and bending forces in compare to only bending forces in plate structure, which makes them useful for utilizing various purposes and large space.
- From application point of view, shell is the most commonly used geometric shape in many engineering and industrial structure to meet the required demand for applications, like wings and fuselage of airplanes, exteriors of rockets, missiles, tanks, pressure vessels, fluid reservoir, containers of liquids, pipes, submarine hulls, ships hulls, hydro-space, concrete roofs, buildings and many other structure.

- In general, Light-weight structures usually thin-walled are very much used in the aircraft industries.
- But, thin-walled structure undergo large deformation when subjected to external aerodynamic load and liable to buckle under the influence of compression load during flight operation.
- Thus, structural stability and large bending analysis gives a safety margin for operation, and plays an important role in design and analysis.

To illustrate the importance of non-linear bending and stability analysis in low safety-margin-structure, a comparison is considered. For Example : When a car-driver finds a visible buckle in a fender of his car and he knows that this buckle has not yet deteriorated the safety of his car. But, same buckle is not tolerable in the aircraft due to there less margin of safety in compare to car, building and like same.

- Moreover, load carrying ability of structure augmented by visible buckles or after buckling depends upon on the post-buckling behavior of the structure.
- Thus, post buckling is also important for buckled or geometrical imperfect structure for further strengthening of the structure. The post-buckling behavior is not only tell about the load carrying capacity after buckling, but it also influence the magnitude of buckling load and it is very useful for design and manufacturing purpose.

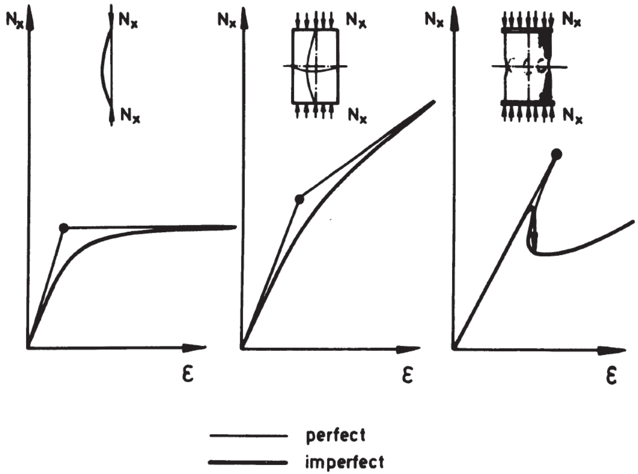

Figure: Buckling and post-buckling behavior of imperfect structures [1].

- Considering all the aspect of curved panel, theoretically buckling load of the perfect shell must not be considered as the load carrying capacity and actual load in the most unfavorable case can as low as 20% of the theoretical buckling load for the perfect shell due to surpass effect of imperfection.
- Higher factor of safety has to be prescribed for curved panel, than to a flat plate which carry several times of the buckling load after buckling.

Motivation

1. Usage of composite material in day to day life is increasing constantly due to their low specific weight with equivalent required stiffness.
2. A clear understanding of structural response of these structure under various operating condition is mandatory for wide range of applicability.
3. However, the thin shell structure underwent large deformation under the influence of external load or elevated hygrothermal atmosphere. So, it is necessary to consider the non-linearity during design and analysis.
4. A clear understanding of geometrically non-linear analysis of composite structure, which possesses complex response during mechanical and thermal load is necessary. The same non-linear behavior is exhibit during buckling due to in-plane mechanical and thermal load.

- These interesting behavior of composite shell panel has motivated to carry out the further research work on this area.
- In next section, Literature on large bending deformation of the composite plate subjected to the mechanical load is reviewed.

Nonlinear Bending analysis of Plate due to transverse mechanical load

Observations and remarks from the Literature survey

Nonlinear Bending analysis of laminated composite plate due to transverse mechanical load

Sabir AB and Lock AC (1972) [2]

- investigated geometrically nonlinear response of cylindrical shells under lateral loading.

Bathe and Bolourchi (1980)[3]

- applied the degenerated shell element for geometric and material nonlinear analysis for both total and updated Lagrangian formulation.
- a variable node shell element is developed to be employed as transition element which has a compatibility with both shell and solid shell element.

Kim (1999) [4]

- presented nonlinear finite element analysis of composite panel; in this an eight-node shell element with six degrees of freedom per node is used.

Nonlinear Bending analysis of laminated composite plate due to transverse mechanical load (cont.)

X. Zhao, et al.(2008) [5]

- has analyzed the geometrically nonlinear behavior of shells and plates using linearly conforming radial point interpolation method.

Riberio et al. (2010) [6]

- applied the p-version of FEM on deep cylindrical shell using shallow shell element for transient and vibration analysis. Geometrical nonlinearity due to large deflection is considered, occur in shallower shells and thinner deep shells.
- it is observed that computation cost of p-version FEM is lower computation cost than regular h-FEM due to finer mesh require to capture the curvature in the shell.

Patel (2015)[7]

- analyzed the nonlinear finite element bending of composite shell panel subjected to transverse load using degenerated shell element with five degree of freedom.
- Total Lagrangian approach is used for formulation in conjunction of Green-Lagrange strain displacement relationship with arc-length technique.
- However, results are compared up to limit point in load-displacement curve.

Hughes and Liu (1981) [8]

- presented a nonlinear finite element model for the three dimensional quasi static analysis of shell structures.
- They considered large strains and rotation effects in their analysis.
- Same work was extended to two-dimensional shell [9].

Nonlinear Bending analysis of laminated composite plate due to transverse mechanical load* (cont.)

Zhang and Kim (2006) [10]

- developed a 4-noded quadrilateral finite element model using FSDT, based on von-Karman's large deflection theory with total Lagrangian approach.

Cetkovic and Vuksanovic (2011) [11]

- developed a geometrically non-linear laminated finite element based on layer-wise displacement field of Reddy.
- Their formulation considered the Green-Lagrange small strain relation and non-linear system of equations are solved by direct iteration procedure.

Kundu et al. (2007) [12]

- studied the effect of elevated hygrothermal environment on bending behavior of doubly curved laminated shell using FEM. A FSDT is employed with von-Karman geometrically non-linear relation.
- A total Lagrangian approach in-conjunction with Arc-length method is used to trace the equilibrium path of shell panel.
- It was observed that residual stresses due to elevate temperature and moisture induce large bending deformation.

Nonlinear Bending analysis of laminated composite plate due to transverse mechanical load* (cont.)

Patel (2015) [7]

- applied the same degenerated technique to predict the non-linear behavior of the composite shell subjected to transverse load using FEM utilizing FSDT.
- A total Lagrangian approach is utilized to formulate the problem considering Green-Lagrange strain-displacement relationship with arc-length technique. However, results are compared up to limit point in the load-displacement curve.

Fares (1999) [13]

- presented a refined single-layered, first-order laminates model using a mixed variational approach which incorporates the von-Karman type of geometric non-linearity in his formulation.

Urthaler and Reddy (2008) [14]

- investigated the nonlinear bending analysis of laminated composite plate using a mixed variational formulation of the FSDT in which displacement and bending moments are treated as independent fields.
- The finite element consists of eight degrees of freedom per node with five normal degree of freedom and three bending moments.

Dash and Singh (2010) [15]

- proposed a penalty based nine-noded isoparametric finite element with 10 degrees of freedom per node.
- Their formulation is based on higher order shear deformation theory (HSDT) based on full Green-Lagrange type of nonlinearity.

Nonlinear Bending analysis of laminated composite plate due to transverse mechanical load* (cont.)

Shukla and Nath (2000) [16]

- presented an analytical solution based on Chebyshev polynomial for geometrically non-linear bending problem of the laminated composite plate which undergo moderately large deformation.
- Influence of boundary conditions is also obtained, and their method was found to be less expensive.

Andakhshideh et al. (2009) [17]

- applied the generalized differential quadrature (GDQ) method to obtained linear and nonlinear response of laminated composite plate considering von-Karman nonlinearity using FSDT.

Singh and Shukla (2012) [18]

- applied the mesh-free method to calculate the non-linear flexural response of laminated using HSDT with von-Karman non-linear kinematics.

Kapoor and Kapania (2012) [19]

- analyzed the geometrically non-linear behavior of laminated composite plates using non-uniform rational B-splines (NURBS) based isogeometric finite element analysis.
- Their formulation considered the von-Karman type of non-linearity with FSDT to model the deformation of the plate.

Tran et al. (2015) [20]

- presented a numerical approach based on isogeometric analysis (IGA) and HSDT for geometrically nonlinear (von-Karman) analysis of laminated composite plates.

Nonlinear Bending analysis of sandwich composite plate due to transverse mechanical load* (cont.)

Marcinowski (2003) [21]

- applied the degenerated shell element for full geometrically non-linear bending analysis of sandwich structure. However, the proposed formulation does not consider for the local deformation of the face sheets.

Madhukar and Singha (2013) [22]

- studied the behavior of geometrically non-linear analysis of soft core sandwich plate using higher order finite element model, considering the effect of transverse shear and normal deformation in the formulation.
- It is observed that the effect of shear and normal deformation increases with increase in thickness-to-span ratio and core-to-face thickness ratio of the sandwich plate.

Elmalich and Rabinovitch (2013) [23]

- A specially tailored geometrically nonlinear finite element based on higher-order sandwich plate theory was developed
- This procedure accounts for the von-Karman type of geometrical nonlinearity in the face sheet and considers a shear and through-thickness deformation of the core.
- This study also reveals an interesting physical phenomenon, like localized diagonal wrinkling patterns, evolved due to small-scale effect and coupling between local and global response in the sandwich plate.

Nguyen et al. (2014) [24]

- presented a improved finite element computation model using a flat four-node element with smoothed strain for geometrically non-linear analysis of composite plate and shell structure.
- A von-Karman type of non-linearity in the framework of total Lagrangian approach utilizing the FSDT is considered, which provides less loss of accuracy in the distorted coarse meshes as integration is done on boundaries of smoothing element.

Observations and remarks from the Literature survey

- The existing literature mainly focuses on the analysis of composite plate, less work is done on the composite shell.
- Characteristic response of laminated composite plate have been dealt by many researcher. However, few studies have been conducted on sandwich structure due to high shear to longitudinal modulus ratio which leads to prominent transverse effect.
- The plate models, which satisfies zero transverse shear stress condition on the top and bottom of the plate for nonlinear problem is generally limited to Von-karman sense and is not reported adequately in Green-Lagrange sense.
- In general mathematical model are based on polynomial approximation; on the other hand non-polynomial based approximation perform better, however is less studied and thus in dept study is needed to explorer for better design, performance and safety of the structure.

Research objective and Scope

Non-linear bending and post-buckling analysis of laminated and sandwich curved panel in hygrothermal environment

Scope

- Flat Plate
  1. Validation and comparative study of linear bending and buckling using NPSDT
  2. Geometric non-linear analysis of laminated and sandwich composite plate under mechanical load using NPSDT
  3. Post buckling analysis of laminated plate under the influence of in-plane mechanical load using NPSDT
  4. Post buckling analysis of composite plate under hygrothermal environment using NPSDT
- Shell Panel
  1. Linear and Nonlinear bending analysis of various laminated curved panel
  2. Linear buckling and nonlinear buckling analysis of laminated and sandwich panel
  3. Nonlinear post buckling analysis of composite panel under hygrothermal environment.

- Present formulation consider both linear and non-linear linear strain, von-Karman type of geometrically nonlinearity is employed to model the behavior of laminated and sandwich composite plate.
- In this case, material exhibits within the linear elastic limit.

A rectangular laminated composite plate of dimension $a \times b \times h$, consist of n orthotropic ply stacked in particular orientation is considered. A schematic diagram of laminated composite plate in cartesian coordinate system $(X - Y - Z)$ is shown in Figure. 4.

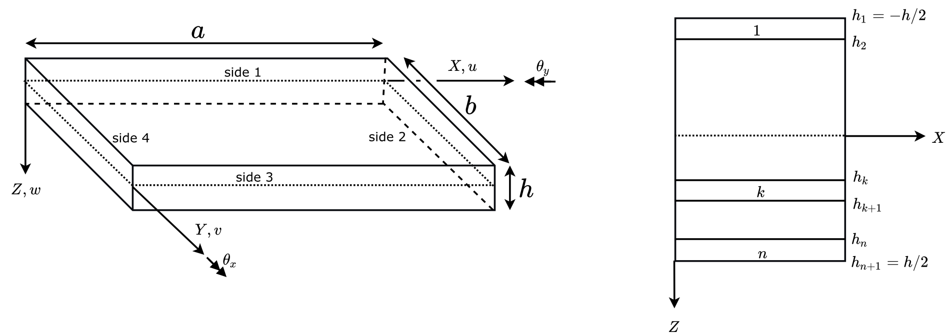

Figure: Schematic diagram of laminated composite plate

A non-polynomial based higher order displacement fields are considered for modeling of deformation of plate as represented as :

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) - z\frac{\partial w_0}{\partial x} + (g(z) + z\Omega)\,\phi_x(x,y) \\
v(x,y,z) &= v_0(x,y) - z\frac{\partial w_0}{\partial y} + (g(z) + z\Omega)\,\phi_y(x,y) \\
w(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{1}
$$

where u_0, v_0, w_0 are displacements along the mid plane of plate, ϕ_x, ϕ_y are the shear deformation at the mid plane. The functions $g(z)$ and Ω incorporate the nonlinear sense in transverse strain and represent the warping of cross-section perpendicular to mid-plane.

- Different nature of shear strain function, as proposed by many researchers [25, 26, 27, 28, 29, 30, 31], give rise to several type of shear deformation theory and among these three are listed in the Table. 1.

Table: Various shear strain function used in non-polynomial shear deformation theories

S. No.	$g(z)$	Ω	r	Abbreviation
1.	$sinh^{-1}\left(\frac{rz}{h}\right)$	$-\frac{2r}{h\sqrt{r^2+4}}$	3.0	*IHDT* (Inverse Hyperbolic) [27]
2.	$tanh\left(\frac{rz}{h}\right)$	$-\frac{r}{h}sech^2\left(\frac{r}{2}\right)$	2.5	*HDT* (Hyperbolic) [31]
3.	$\left(\frac{rz}{h}\right)^3 e^{\left(2\sqrt{\pi}\right)}$	$-\frac{3e^{2\sqrt{\pi}}r^3}{4h}$	0.95	*EDT* (Exponential)[31]

The state-of-strain $\{\varepsilon\}$ at a point correspond to non-polynomial displacement fields, in the cartesian coordinate, is expressed as

$$\{\varepsilon\}=\left\{\begin{array}{c}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\end{array}\right\}=\begin{bmatrix}\frac{\partial}{\partial x} & 0 & -z\frac{\partial^2}{\partial x^2} & (g(z)+z\Omega)\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y} & -z\frac{\partial^2}{\partial y^2} & 0 & (g(z)+z\Omega)\frac{\partial}{\partial y}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -2z\frac{\partial^2}{\partial x\partial y} & (g(z)+z\Omega)\frac{\partial}{\partial x} & (g(z)+z\Omega)\frac{\partial}{\partial y}\\ 0 & 0 & 0 & 0 & \left(\frac{\partial g}{\partial z}+\Omega\right)\\ 0 & 0 & 0 & \left(\frac{\partial g}{\partial z}+\Omega\right) & 0\end{bmatrix}\left\{\begin{array}{c}u_0\\ v_0\\ w_0\\ \phi_x\\ \phi_y\end{array}\right\} \quad (2)$$

- In Equation. 2, presence of second order derivative operator require C^1continuity of field variable, specifically w_0. For analytical Navier type of solution, C^1continuity is easily satisfied, however numerical method like finite element method used Lagrangian element that gives at most C^0continuity of field variables.
- The reduction of C^1 to C^0 continuity requirement in finite element method using Lagrangian element is achieved by imposing an artificial constraint. The constraint treats $\partial w_0/\partial x$ and $\partial w_0/\partial y$ as separate parameter θ_x and θ_y respectively. So, after incorporating the above constraint, the finite element model contain seven field variables.

$$\left(\frac{\partial w_0}{\partial x}=\theta_x\right),\text{and}\left(\frac{\partial w_0}{\partial y}=\theta_y\right) \quad (3)$$

- Consequence of this artificial constraint has been taken care in strain energy expression as a penalty and detail implementation is given in next section.

The state-of-strain $\{\varepsilon\}$ at a point, considering von-Karman non-linearity, correspond to non-polynomial displacement fields, in the cartesian coordinate, is expressed as

$$\{\varepsilon\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\ 0 \\ 0 \end{Bmatrix} \tag{4}$$

The total strain vector $\{\varepsilon\}$ is the sum of the linear strain vector $\{\varepsilon_{\boldsymbol{L}}\}$ and non-linear strain vector $\{\varepsilon_{\boldsymbol{NL}}\}$, and can be written as

$$\{\varepsilon\} = \{\varepsilon_{\boldsymbol{L}}\} + \{\varepsilon_{\boldsymbol{NL}}\}$$

Again

$$\{\varepsilon_{\boldsymbol{L}}\} = \left\{ \begin{array}{c} \varepsilon_{\boldsymbol{m}} \\ \beta_1 \end{array} \right\} + z \left\{ \begin{array}{c} \boldsymbol{k}_1 \\ 0 \end{array} \right\} + \left\{ \begin{array}{c} g(z)\boldsymbol{k}_2 \\ g'(z)\beta_2 \end{array} \right\} \tag{5}$$

in which

$$\varepsilon_{\boldsymbol{m}} = \left\{ \begin{array}{c} u_{0,x} \\ v_{0,y} \\ u_{0,y} + v_{0,x} \end{array} \right\};\ \boldsymbol{k}_1 = \left\{ \begin{array}{c} \Omega\phi_{x,x} - \theta_{x,x} \\ \Omega\phi_{y,y} - \theta_{y,y} \\ \Omega(\phi_{x,y} + \phi_{y,x}) - (\theta_{x,y} + \theta_{y,x}) \end{array} \right\}$$

$$\boldsymbol{k}_2 = \left\{ \begin{array}{c} \phi_{x,x} \\ \phi_{y,y} \\ \phi_{x,y} + \phi_{y,x} \end{array} \right\};\ \beta_1 = \left\{ \begin{array}{c} w_{0,y} - \theta_y + \Omega\phi_y \\ w_{0,x} - \theta_x + \Omega\phi_x \end{array} \right\};\ \beta_2 = \left\{ \begin{array}{c} \phi_y \\ \phi_x \end{array} \right\}$$

and the non-linear strain vector $\{\varepsilon_{\boldsymbol{NL}}\}$ can be written as

$$\{\varepsilon_{\boldsymbol{NL}}\} = \frac{1}{2} \left\{ \begin{array}{c} w_{,x}^2 \\ w_{,y}^2 \\ 2w_{,xy} \end{array} \right\} = \frac{1}{2}\boldsymbol{A}_\theta \theta$$

where $\boldsymbol{A}_\theta = \begin{bmatrix} w_{0,x} & 0 \\ 0 & w_{0,y} \\ w_{0,y} & w_{0,x} \end{bmatrix}$ and $\theta = \left\{ \begin{array}{c} w_{0,x} \\ w_{0,y} \end{array} \right\}$

In equivalent single layer (ESL) theories, stress-strain relation (constitutive relation) for an arbitrary k^{th} orthotropic layer in structural reference system (global coordinate) for plane stress problem is given by the following relation

$$\{\sigma\}^{(k)} = \left[\bar{\boldsymbol{Q}}\right]^{(k)} \{\varepsilon\}^{(k)} \tag{6}$$

which can be elaborately written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}^{(k)} = \left[\mathscr{R}^{(k)}\right] \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & Q_{16} \\ Q_{21} & Q_{22} & 0 & 0 & Q_{26} \\ 0 & 0 & Q_{44} & Q_{45} & 0 \\ 0 & 0 & Q_{54} & Q_{55} & 0 \\ Q_{61} & Q_{62} & 0 & 0 & Q_{66} \end{bmatrix} \left[\mathscr{R}^{(k)}\right]^T \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}^{(k)} \tag{7}$$

where, $\{\sigma\}$, $\{\varepsilon\}$ and $\left[\bar{Q}\right]$ are stress vector, strain vector and the material constants in the global coordinate, respectively and $[\mathscr{R}]$ is the local-to-global transformation matrix [32]. Due to symmetry in the orthotropic material $Q_{21} = Q_{12}$, $Q_{61} = Q_{16}$, $Q_{62} = Q_{26}$, and $Q_{54} = Q_{45}$.
Each material matrix coefficient can be expressed as

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}},\ Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}},\ Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}},\ Q_{66} = G_{12},\ Q_{44} = G_{23},\ Q_{55} = G_{13}$$

Longitudinal and transverse direction are defined by 1 and 2 respectively. Above, E_1 and E_2 are the Young's modulus; G_{12}, G_{23}, G_{13} are the shear modulus; and ν_{12} and ν_{21} are major and minor Poisson ratios.

The in-plane forces, moments and shear forces are defined as

$$\begin{Bmatrix} N_{ij} \\ M_{ij} \\ P_{ij} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{ij} \begin{Bmatrix} 1 \\ z \\ g(z) \end{Bmatrix} dz \text{ and } \begin{bmatrix} Q_y & R_y \\ Q_x & R_x \end{bmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} 1 & g'(z) \end{pmatrix} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz \tag{8}$$

Substituting Eq. 7 into Eq. 8, stress resultants are obtained as

$$\hat{\sigma} = \begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \\ \boldsymbol{P} \\ \boldsymbol{Q} \\ \boldsymbol{R} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{E} & 0 & 0 \\ \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{F} & 0 & 0 \\ \boldsymbol{E} & \boldsymbol{F} & \boldsymbol{H} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{A}^s & \boldsymbol{B}^s \\ 0 & 0 & 0 & \boldsymbol{B}^s & \boldsymbol{D}^s \end{bmatrix} \begin{Bmatrix} \varepsilon_L \\ \kappa_1 \\ \kappa_2 \\ \beta_1 \\ \beta_2 \end{Bmatrix} = \hat{\boldsymbol{D}}\hat{\varepsilon}$$

where

$$A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} = \int_{-h/2}^{h/2} \left(1, z, z^2, g(z), zg(z), g^2(z)\right) \bar{Q}_{ij} dz, \quad i,j = 1,2,6$$

$$A_{ij}^s, B_{ij}^s, D_{ij}^s = \int_{-h/2}^{h/2} \left(1, g'(z), g'(z)\right)^2 \bar{Q}_{ij} dz, \quad i,j = 4,5$$

and the generalized strain vector, $\hat{\varepsilon}$ is divided into linear and non-linear strain components $\hat{\varepsilon}_L = \begin{bmatrix} \varepsilon_L & \kappa_1 & \kappa_2 & \beta_1 & \beta_2 \end{bmatrix}^T$ and $\hat{\varepsilon}_{NL} = \begin{bmatrix} \varepsilon_{NL} & 0 & 0 & 0 & 0 \end{bmatrix}^T$ respectively.

$$\hat{\varepsilon} = \hat{\varepsilon}_{\boldsymbol{L}} + \frac{1}{2}\hat{\varepsilon}_{\boldsymbol{NL}} \tag{9}$$

Lagrange equation for conservative system

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \{\dot{q}\}}\right) - \frac{\partial U}{\partial \{q\}} + \frac{\partial W}{\partial \{q\}} - \frac{\partial U_\gamma}{\partial \{q\}} = 0 \tag{10}$$

K : Kinetic energy of the system
U : Total strain energy of the system
W : Work done by external forces
U_γ: strain energy due to imposition of artificial constraint

Finite element discretization

Element selection

- Nine node iso-parametric element with 7 DOFs at each node of the element

$$\{q\} = \sum_{i=1}^{9} N_i \{q_i\}; \quad x = \sum_{i=1}^{9} N_i x_i; \quad y = \sum_{i=1}^{9} N_i y_i \tag{11}$$

- Where, $\{q\}$ is the generalized field variables and $\{q_i\}$ is the corresponding nodal field variables associated with i^{th} node, x and y are the generalized spatial coordinate and x_i and y_i are the coordinate values of the corresponding i^{th} node.

$$\{q_i\} = \{u_{0i}, v_{0i}, w_{0i}, \phi_{xi}, \phi_{yi}, \theta_{xi}, \theta_{yi}\}$$

- 7 dof $\times$ 9 node $=$ 63 DOFs per element.

Governing equation

Substituting Eq. 11 into Eq. 9, the generalized strains can be rewritten as

$$\hat{\varepsilon} = \sum_{A=1}^{9} \left(\boldsymbol{B}_A^L + \frac{1}{2} \boldsymbol{B}_A^{NL} \right) \boldsymbol{q}_A \tag{12}$$

where

$$\boldsymbol{B}_A^L = \begin{bmatrix} (\boldsymbol{B}_A^m)^T & \left(\boldsymbol{B}_A^{b1}\right)^T & \left(\boldsymbol{B}_A^{b2}\right)^T & (\boldsymbol{B}_A^{s1})^T & (\boldsymbol{B}_A^{s2})^T \end{bmatrix}^T$$

in which

$$\boldsymbol{B}_A^m = \begin{bmatrix} N_{A,x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_{A,y} & 0 & 0 & 0 & 0 & 0 \\ N_{A,y} & N_{A,x} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{B}_A^{b2} = \begin{bmatrix} 0 & 0 & 0 & N_{A,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{A,y} & 0 & 0 \\ 0 & 0 & 0 & N_{A,y} & N_{A,x} & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{B}_A^{b1} = \begin{bmatrix} 0 & 0 & 0 & \Omega N_{A,x} & 0 & -N_{A,x} & 0 \\ 0 & 0 & 0 & 0 & \Omega N_{A,y} & 0 & -N_{A,y} \\ 0 & 0 & 0 & \Omega N_{A,y} & \Omega N_{A,x} & -N_{A,y} & -N_{A,x} \end{bmatrix}$$

$$\boldsymbol{B}_A^{s1} = \begin{bmatrix} 0 & 0 & N_{A,y} & 0 & \Omega N_A & 0 & -N_A \\ 0 & 0 & N_{A,x} & \Omega N_A & 0 & -N_A & 0 \end{bmatrix}, \boldsymbol{B}_A^{s2} = \begin{bmatrix} 0 & 0 & 0 & 0 & N_A & 0 & 0 \\ 0 & 0 & 0 & N_A & 0 & 0 & 0 \end{bmatrix}$$

Governing equation (cont.)

while non-linear strain matrix, $\boldsymbol{B}_A^{NL}$is still dependent upon displacement gradient

$$\boldsymbol{B}_A^{NL}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{A}_\theta \\ 0 \end{bmatrix} \boldsymbol{B}_A^{g} \tag{13}$$

where $\boldsymbol{B}_A^{g} = \begin{bmatrix} 0 & 0 & N_{A,x} & 0 & 0 & 0 & 0 \\ 0 & 0 & N_{A,y} & 0 & 0 & 0 & 0 \end{bmatrix}$ and penalty condition in Eq. 3 can be written as

$$\frac{\partial w_0}{\partial y} - \theta_y = \boldsymbol{B}_A^{\gamma 1}\boldsymbol{q}_A \text{ and } \frac{\partial w_0}{\partial x} - \theta_x = \boldsymbol{B}_A^{\gamma 2}\boldsymbol{q}_A \tag{14}$$

Substituting Eqs. 11, 12, 13, 14 in Eq. 10, and equation of equilibrium can be obtained in the following matrix form

$$\left[\boldsymbol{K}(\boldsymbol{q}) + \gamma \boldsymbol{K}_\gamma\right]\{\boldsymbol{q}\} = \{\boldsymbol{F}\} \tag{15}$$

where $\boldsymbol{K}$, $\boldsymbol{K}_\gamma$, $\boldsymbol{F}$and $\boldsymbol{q}$ are the elemental stiffness, penalty stiffness matrices, force and displacement vector contain unknown degree of freedom, respectively.

$$\boldsymbol{K}(\boldsymbol{q}) = \int_\Omega \left(\boldsymbol{B}^L + \boldsymbol{B}^{NL}\right)^T \hat{\boldsymbol{D}} \left(\boldsymbol{B}^L + 0.5\boldsymbol{B}^{NL}\right) d\Omega$$

$$\boldsymbol{K}_\gamma = \left(\left(\boldsymbol{B}^{\gamma 1}\right)^T \boldsymbol{B}^{\gamma 1} + \left(\boldsymbol{B}^{\gamma 2}\right)^T \boldsymbol{B}^{\gamma 2}\right) d\Omega$$

$$\boldsymbol{F} = \int_\Omega \left\{ \begin{matrix} 0 & 0 & N_A & 0 & 0 & 0 & 0 \end{matrix} \right\}^T P(x,y)\, d\Omega$$

where P is the transverse mechanical load and elemental load vector, $\{\boldsymbol{F}\}$ consist 63 components, non-zero at w degree of freedom.

Solution technique - Newton-Raphson

In non-linear analysis, the unknown solution is obtained by various iteration methods [33, 34, 35, 36, 37, 38], the residual force vector, $\{\mathbb{R}\}$ is introduced to represent the error in approximation and which tends to zero during each iteration as solution approaches to exact value.
From Eq. 15, the residual force vector, $\{\mathbb{R}\}$ at i^{th} iteration can be defined as follows:

$$\{\mathbb{R}_i\} = \left[\boldsymbol{K}_i(\boldsymbol{q}_i) + \gamma \boldsymbol{K}_\gamma\right]\{\boldsymbol{q}_i\} - \{\boldsymbol{F}_i\}$$

To make $\{\mathbb{R}_{\boldsymbol{i}}\}$ equal to zero within the tolerance limit of $1/10^6$. An improved solution $\left\{\boldsymbol{q}_{i+1}\right\}$ can be obtained as

$$\left\{\boldsymbol{q}_{i+1}\right\} = \{\boldsymbol{q}_i\} + \Delta\{\boldsymbol{q}_i\}$$

where $\Delta\{\boldsymbol{q}_i\}$ is the incremental displacement vector and calculated as

$$\Delta\{\boldsymbol{q}_i\} = -\{\mathbb{R}_i\}/[\boldsymbol{K}_T]$$

in which $[\boldsymbol{K}_T]$ is called tangent stiffness matrix and computed by following Newton-Raphson method, i.e, $[\boldsymbol{K}_T] = \partial\mathbb{R}_i/\partial\boldsymbol{q}_i$.

Solution technique - Newton-Raphson (cont.)

More elaborately, tangent stiffness matrix can be expressed as

$$[\boldsymbol{K}_T] = [\boldsymbol{K}_L] + [\boldsymbol{K}_{NL}] + [\boldsymbol{K}_\sigma]$$

herein, $[\boldsymbol{K}_L]$, $[\boldsymbol{K}_{NL}]$, $[\boldsymbol{K}_\sigma]$ are the linear, non-linear and geometric stiffness matrix, respectively

$$[\boldsymbol{K}_L] + [\boldsymbol{K}_{NL}] = \int_\Omega \left(\boldsymbol{B}^L + \boldsymbol{B}^{NL}\right)^T \hat{\boldsymbol{D}} \left(\boldsymbol{B}^L + \boldsymbol{B}^{NL}\right) d\Omega$$

$$[\boldsymbol{K}_\sigma] = \int_\Omega (\boldsymbol{B}^\sigma)^T \left[\boldsymbol{N}_i^0\right] (\boldsymbol{B}^\sigma)\, d\Omega$$

in which $\left[\boldsymbol{N}_i^0\right] = \begin{bmatrix} N_{xx}^0 & N_{xy}^0 \\ N_{xy}^0 & N_{yy}^0 \end{bmatrix}$ is a matrix related to in-plane forces and $\boldsymbol{B}^\sigma$is same as $\boldsymbol{B}^g$ in Eq. 13.

The iteration is repeated until the displacement error between two consecutive iteration reduced to desired error tolerance, $\mathbb{N}$.

For present case, $\mathbb{N} = 10^{-6}$is considered.

$$\frac{\| \boldsymbol{q}_{i+1} - \boldsymbol{q}_i \|}{\| \boldsymbol{q}_i \|} < \mathbb{N}$$

Boundary Condition

- Simply supported condition for cross-ply plate ($SSSS1$)
  - Edge parallel to x-axis : $u_o = w_o = \phi_x = \theta_x = 0$
  - Edge parallel to y-axis : $v_o = w_o = \phi_y = \theta_y = 0$
- Simply supported condition for angle-ply plate($SSSS2$)
  - Edge parallel to x-axis : $v_o = w_o = \phi_x = \theta_x = 0$
  - Edge parallel to y-axis : $u_o = w_o = \phi_y = \theta_y = 0$
- Clamped condition ($CCCC$)
  - All sides : $u_o = v_0 = w_o = \phi_x = \phi_y = \theta_x = \theta_y = 0$

Material Properties

The material properties used for the analyses are :

1. *Material I* [20] : $E_1/E_2 = 40;\ G_{12} = G_{13} = 0.6E_2;\ G_{23} = 0.5E_2;\ \nu_{12} = 0.25$
2. *Material II* [39]: The following properties of sandwich material are
   1. Face sheets : $E_1 = 25E_2,\ G_{12} = G_{13} = 0.5E_2,\ G_{23} = 0.2E_2,$
      $\nu = 0.25,\ E_2 = 10^6\text{psi}$
   2. Core : $E_1 = E_2 = 0.04 \times 10^6\text{psi},\ G_{13} = G_{23} = 0.06 \times 10^6\text{psi},$
      $G_{12} = 0.016 \times 10^6\text{psi},\ \nu = 0.25$

Antisymmetric simply supported under UDL transverse load

- *Material I* [20] : $E_1/E_2 = 40$; $G_{12} = G_{13} = 0.6E_2$; $G_{23} = 0.5E_2$; $\nu_{12} = 0.25$
- Dimension: $a/h = 10$, $a = b = 1$
- Non-dimensional parameter : $\bar{P} = \frac{Pa^4}{E_2 h^3}$, $\bar{w} = \frac{w}{h}$
- Equilibrium path for this problem is traced by Newton-Raphson method.

$[0^0/90^0]_N$ Antisymmetric cross-ply simply supported under UDL transverse load

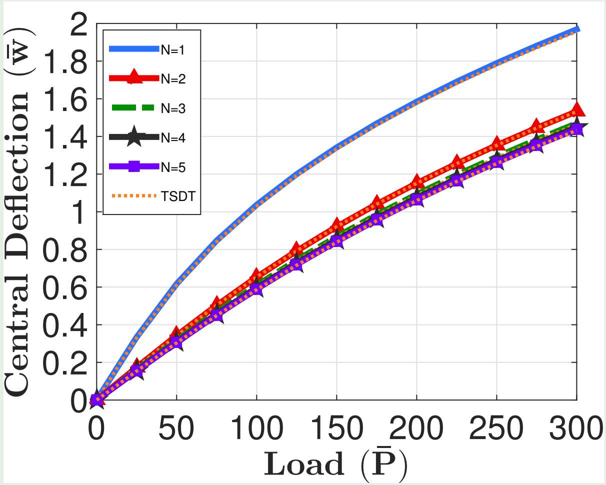

Figure: Effect of number of layers on deflection of the $[0/90]_N$ laminated composite plate.

- Same total thickness with increase in number of layers, N the plate becomes stiffer with reduction in deflection.
- Reduction in the non-linear effect on the laminated plate.
- Load-deflection curve becomes closer to straight line as the linear solution.
- Results obtained from present finite element code are compared with available TSDT results [20].

$[-\theta/\theta/-\theta/\theta]$ Anti-symmetric angle-ply simply supported under UDL transverse load

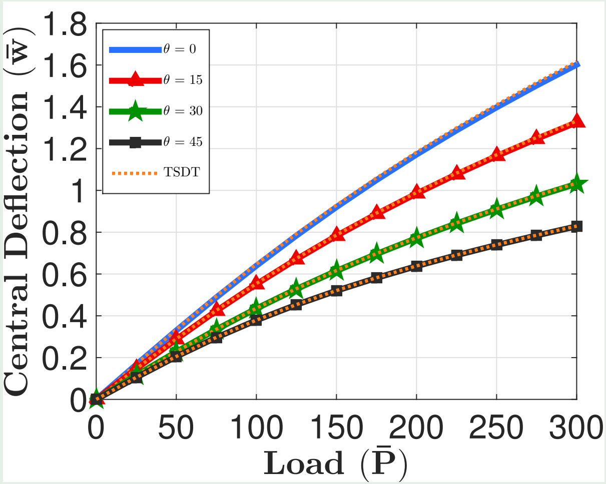

Figure: Effect of fiber orientation angle on load–deflection curves of the angle-ply $[-\theta/\theta/-\theta/\theta]$ plate.

- As seen, the plate obtains the lowest transverse displacement $\bar{w}$ at fiber angle $\theta = 45^0$.
- Again, results obtained from present finite element code are compared with available TSDT results [20].
- Collectively, IHSDT has no significant effect on non-linear bending response of laminated plate.

Anti-symmetric cross-ply with clamped boundary condition under both SSL and UDL transverse load

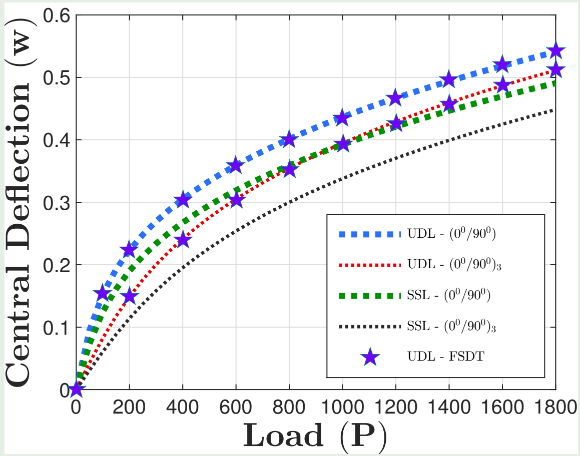

Figure: Nonlinear response of anti-symmetric cross ply laminated plate subjected to clamped boundary condition under SSL and UDL transverse load.

- $h = 0.3,\ a = b = 12\,\text{inch}$
- $Material - I$
- Reference : FSDT [32]

$(0^0/C/0^0)$ simply supported under UDL transverse load

- *Material II* [39]: The following properties of sandwich material are
  1. Face sheets : $E_1 = 25E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $\nu = 0.25$, $E_2 = 10^6$psi
  2. Core : $E_1 = E_2 = 0.04 \times 10^6$psi, $G_{13} = G_{23} = 0.06 \times 10^6$psi, $G_{12} = 0.016 \times 10^6$psi, $\nu = 0.25$
- Dimension : $a/h = 10, \text{and} 100$
- Non-dimensional parameter : $\bar{w} = \frac{w}{h}$; $\bar{P} = \frac{P}{E_2}\left(\frac{a}{h}\right)^4$
- Equilibrium path for this problem is traced by Newton-Raphson method.

$(0^0/C/0^0)$ simply supported under UDL transverse load

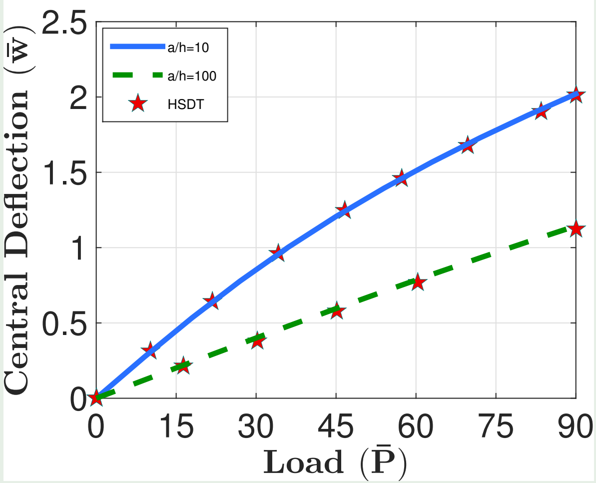

Figure: Displacement vs load for simply supported $(SSSS1)$ square sandwich $(0^0/C/0^0)$ plate under uniformly distributed transverse load.

- Present IHSDT results coincide with HSDT results [40].
- Similar to laminated plate, IHSDT does not show any significant change in bending response of sandwich composite plate.

Anti-symmetric cross-ply with clamped boundary condition under UDL transverse load

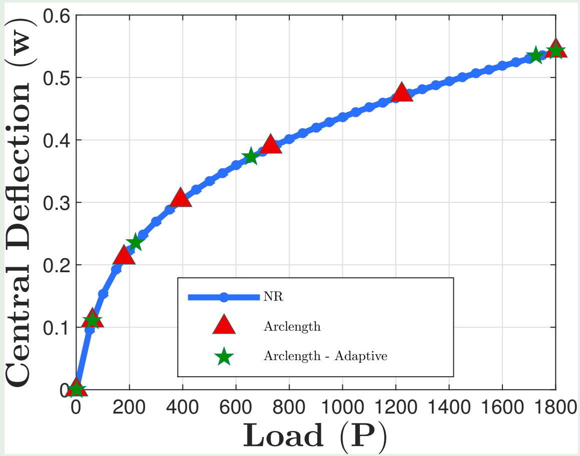

Figure: Nonlinear response of anti-symmetric cross ply $(0^0/90^0)$ laminated plate subjected to clamped boundary condition under UDL transverse load.

- $h = 0.3,\ a = b = 12\,\text{inch}$
- $Material - I$

- Non-linear finite element formulation is developed for laminated and sandwich composite plate.
- Non-linear equilibrium path is traced with Newton-Raphson method.
- Non-linear response obtained with IHSDT and TSDT are same.
- No significant change is observed for the results of IHSDT as observed in the linear analysis using same.

Work done so far

Linear bending and buckling analysis of laminated composite plate using various NPSDTs

Geometrical non-linear bending analysis of laminated and sandwich plate using NPSDT

Work to be completed

Post buckling analysis of laminated and sandwich plate under the influence of in-plane mechanical load using FSDT and NPSDT.

Post buckling analysis of laminated and sandwich composite plate under hygrothermal environment using NPSDT

Linear and non-linear bending analysis of various laminated and sandwich curved panel due to external mechanical load

Linear and non-linear buckling analysis of laminated and sandwich panel due to mechanical load

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THANK YOU

Queries ?