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# A unified buckling formulation for linear and nonlinear analysis of laminated plates using penalty based $C^0$ FEM-HSDT Model

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***Abstract*—**
The effects of the method used to model stress stiffening nonlinearity and the type of nonlinear methodology on the buckling analysis of laminated composite plates are investigated in this study. Stress stiffening is modeled via the use of von Kármán and Green-Lagrange strain-displacement relations, and nonlinear-eigenvalue approach and nonlinear static response are compared to assess their reliability and accuracy. A $C^0$ finite element (FE) plate model that is based on Reddy's third-order shear deformation theory (TSDT) via unified $C^1$ higher-order shear deformation model is used in the analyses. The governing equations are derived using the principle of virtual displacement and solved using the tangent-based arc-length method in conjunction with a simple branch switching technique. The broader applicability of the nonlinear buckling approach is demonstrated and the limited range of application of the nonlinear eigenvalue approach is highlighted. The critical buckling loads obtained using Green-Lagrange nonlinearity are more conservative than those obtained using von Kármán nonlinearity.

***Keywords*—Buckling analysis; Green-Lagrange nonlinearity; Higher-order shear deformation theory; Finite element method**

## I. INTRODUCTION

Fiber-reinforced laminated composite materials have seen a considerable rise in attention owing to their characteristic like high specific strength and stiffness, tailoring properties through ply orientation and thickness, and wide knowledge base to understand its behavior [1]. The use of plate-like laminated composite structures is very common in engineering applications, especially in aerospace industry. Generally, these structures have a tendency to suffer structural instability when subjected to inplane compressive loads [2]. Further, independent of their thicknesses, they are susceptible to transverse shear deformation due to their low transverse to inplane moduli ratio. These behaviors make it imperative to include the effects of transverse shear deformation in the design and analysis of laminated composite plates [3].

Equivalent plate models are the most preferred model for the design and stability analysis of laminated composite structures. They are computationally cheaper than other contemporary models and can produce reliable results with acceptable accuracy [4]. Equivalent plate models manifest as different plate theories, namely, classical laminate plate theory (CLPT), first-order shear deformation theory (FSDT), and higher-order shear deformation theory (HSDT) [5]. The HSDTs that assume zero traction at the top and bottom surfaces of the plate yield accurate analysis. A generalized $C^1$ HSDT model is employed in this study such that a third-order shear deformation theory (TSDT) and a wide range of non-polynomial shear deformation theories (NPSDTs) can be easily represented.

The governing equations resulting from the equivalent plate models are solved analytically or numerically with the $C^0$ finite element method (FEM) being the most preferred numerical solution methodology due to its wide range of applicability and strong mathematical foundation. Thus, the present study is based on $C^0$ FEM in which $C^1$ requirement of HSDT is satisfied by using a penalty approach because the Lagrange multiplier approach is computationally expensive and difficulty to implement.

The literature is replete with linear studies [6]–[11] that predict the buckling strength of composite plates subjected to inplane mechanical loads. Stress-stiffening in majority of these studies [6]–[9] is modeled solely through von Kármán type strain-displacement relationship. In particular, Dennis and Palazotto [13] underscore the significance of incorporating inplane Green-Lagrange strain to enhance the precision of predictions of the buckling strength of plates. Shufrin and Eisenberger [14] observe a 7.5 % reduction in the buckling load of rectangular plates when nonlinear curvature terms are included in the strain expression. Similarly, Ruocco and Minutolo [15], [16] conclude that models that include von Kármán nonlinearity tend to overestimate the critical load compared to those employing nonlinear inplane strain.

In linear buckling analysis, the critical buckling load or load multiplier is typically obtained by solving an eigenvalue problem using assumed uniform stress distribution. This is a simplified assumption because the true distribution of stress is influenced by various factors. To capture the realistic stress distribution for an eigenvalue analysis, a two-stage analysis is generally performed: firstly, a static analysis is performed under inplane loads with pre-buckling boundary conditions, and secondly, an eigenvalue problem is solved with buckling boundary conditions. In this context, Nima and Ganesan [17] investigate the effect of different pre-buckling boundary conditions on buckling strength. Patel and Sheikh [12] suggest the use of the same boundary conditions in the two-stage analysis. The effects of loading conditions, i.e., stress loading due to the applied load and strain loading due to the constrained displacements, are studied by Prajapati et al. [18]. They recommend the consideration of stress loading condition, i.e., applied loads, for more reliable and conservative estimate of the buckling strength of plate structures under inplane compression.

Plates do not usually lose their entire stability at the initial bifurcation load, but follow a secondary path, called post-buckling path, before terminating at the failure criteria. For such problems, a nonlinear analysis is essential to obtain the limit load, equilibrium path, and other details that a linear buckling analysis cannot provide. There are mainly two types of approaches for performing nonlinear buckling analysis: nonlinear eigenvalue approach and nonlinear (static) response approach (hereinafter called nonlinear buckling approach). In nonlinear eigenvalue approach, linear buckling analysis is augmented with nonlinear stiffness matrix. Most of the nonlinear buckling studies in the literature [6], [19]–[21] use von Kármán nonlinearity to capture stress stiffening with few studies [22]–[26] employing Green-Lagrange nonlinearity.

The buckling behavior of the composite plates is extensively studied using linear buckling analyses compared to nonlinear analyses. While stress stiffening is mainly modeled with only von Kármán nonlinearity, a comprehensive understanding of the consequences of using Green-Lagrange nonlinearity instead and the reliability of the nonlinear eigenvalue and nonlinear buckling approaches is lacking. The investigation reported in this paper is an attempt to fill this knowledge gap; the buckling response of laminated composite plates due to inplane mechanical loads using a unified formulation of a penalty based $C^0$ FEM-TSDT model is examined. A nine-node Lagrange finite element is employed, and the buckling response is obtained using the arc-length method in conjunction with a simple branch-switching technique [27]. The advantage of using Green-Lagrange nonlinearity over von Kármán nonlinearity is examined via validatory and comparative studies. The results indicate that the nonlinear buckling approach is more reliable. Further, the same boundary conditions must be used in the two-stage linear buckling analysis.

## II. Mathematical formulation

Consider an $n$ layered rectangular laminated composite plate with stacking sequence $(\theta_1/\theta_2/\theta_3/\theta_4/\ldots)$, length $a$, width $b$, and uniform thickness $h$. A global Cartesian coordinate system $xyz$ with $xy$ being the midplane of the laminated plate and $z$ pointing in the increasing ply direction is considered and shown in Fig. 1.

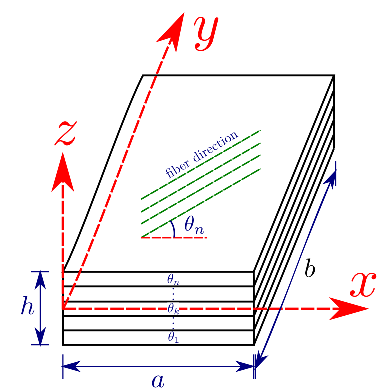



Figure. 1: Schematic diagram of laminated composite plate

### A. Higher-order shear deformation theory

The kinematics of the laminated composite plate is approximated by generalized $C^1$ HSDT. The displacement field variables for TSDT ($f(z) = z - 4z^3/3h^2$) are given as

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z\frac{\partial w_0}{\partial x} + f(z)\,\theta_x(x,y) \\ v(x,y,z) &= v_0(x,y) - z\frac{\partial w_0}{\partial y} + f(z)\,\theta_y(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

where $u$, $v$, $w$ are displacements in $x$, $y$, and $z$ directions, respectively; $u_0$, $v_0$, and $w_0$ are the corresponding midplane displacements; and $\theta_x$ and $\theta_y$ are the shear deformation of the normal to the midplane about $y$-axis and $x$-axis, respectively.

The above kinematics is made suitable for $C^0$ FEM by introducing artificial field variables ($\phi_x = -\frac{\partial w_0}{\partial x}$ $\phi_y = -\frac{\partial w_0}{\partial y}$) with new displacement field components $\boldsymbol{u} = \{u_0\ v_0\ w_0\ \phi_x\ \phi_y\ \theta_x\ \theta_y\}^T$, where $T$ denotes *transpose* of a matrix. The modified kinematics is rewritten as

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\phi_x + f(z)\,\theta_x(x,y) \\ v(x,y,z) &= v_0(x,y) + z\phi_y + f(z)\,\theta_y(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{2}$$

### B. Strain-displacement relation

The Green-Lagrange nonlinearity strain vector $\boldsymbol{\epsilon}$ at a generic point $(x, y, z)$ may be expressed as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon_l} + \frac{1}{2}\boldsymbol{\epsilon_{nl}} \tag{3}$$

where $\epsilon_l$ and $\frac{1}{2}\epsilon_{nl}$ denote the linear and nonlinear component of the strain vector $\epsilon$, respectively. By using the expression of the modified displacement relations Eq. (2), the linear strain vector $\epsilon_l$ and nonlinear strain vector $\epsilon_{nl}$ can be separated into inplane strain vector and transverse strain vector as

$$\begin{aligned}\boldsymbol{\epsilon}_l &= \left\{ \begin{array}{c} \boldsymbol{\epsilon}_{lb} \\ \boldsymbol{\epsilon}_{ls} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{Z}_{lb}\hat{\boldsymbol{\epsilon}}_{lb} \\ \boldsymbol{Z}_{ls}\hat{\boldsymbol{\epsilon}}_{ls} \end{array} \right\} = \boldsymbol{Z}_l\hat{\boldsymbol{\epsilon}}_l \\ \boldsymbol{\epsilon}_{nl} &= \left\{ \begin{array}{c} \boldsymbol{\epsilon}_{nlb} \\ \boldsymbol{\epsilon}_{nls} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{Z}_{nlb}\hat{\boldsymbol{\epsilon}}_{nlb} \\ \boldsymbol{Z}_{nls}\hat{\boldsymbol{\epsilon}}_{nls} \end{array} \right\} = \boldsymbol{Z}_{nl}\hat{\boldsymbol{\epsilon}}_{nl}\end{aligned} \tag{4}$$

The components of these matrices are presented in the Ref. [1].

### C. Constitutive relation

The constitutive relation for an arbitrary $k^{\text{th}}$ orthotropic layer of laminated plate with zero transverse normal stress condition may be expressed as

$$\boldsymbol{\sigma} = \bar{\boldsymbol{Q}}^{(k)}\boldsymbol{\epsilon} = \left[T^{(k)}_{\text{trans}}\right]\boldsymbol{Q}^{(k)}\left[T^{(k)}_{\text{trans}}\right]^T\boldsymbol{\epsilon} \tag{5}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$, and $\bar{\boldsymbol{Q}}$ are stress, strain, and constitutive matrix in the global Cartesian coordinate system, respectively; $\boldsymbol{Q}^{(k)}$ denotes the material matrix in the local Cartesian coordinate system. All other relevant details are discussed in the Ref. [1].

### D. Variational principle

For admissible virtual displacement $\delta\{u, v, w\}$, the principle of virtual work for the given system may be written as

$$\begin{aligned}&\int_V [\ \delta\{\boldsymbol{\epsilon}\}^T\{\boldsymbol{\sigma}\} + \delta\left(\frac{\partial w_0}{\partial x} + \phi_x\right)^T\gamma\left(\frac{\partial w_0}{\partial x} + \phi_x\right) \\ &\quad +\delta\left(\frac{\partial w_0}{\partial y} + \phi_y\right)^T\gamma\left(\frac{\partial w_0}{\partial y} + \phi_y\right)\ ]\,dV \\ &\quad = \int_s (\delta u_0 n_x + \delta v_0 n_y)\,P_i ds\end{aligned} \tag{6}$$

The first term in the left-hand side of the equation represents the virtual strain energy; the second and third terms together represent virtual strain energy due to artificial constraints with $\gamma$ being the penalty parameter. The right-hand side terms represent the virtual work done by the inplane mechanical load.

### E. Finite element formulation

The plate displacement vector $\boldsymbol{u}$ at arbitrary point $(x, y)$ is discretized using nine-noded isoparametric Lagrange elements as $\boldsymbol{u} = \sum_{i=1}^{9} I_7 N_i \boldsymbol{q}_i$ where $I_7$ denotes a $7\times7$ identity matrix, and $\boldsymbol{q}_i$ represents the nodal displacement vector $\boldsymbol{q}_i$=$\{u_{0i}$, $v_{0i}$, $w_{0i}$, $\phi_{xi}$, $\phi_{yi}$, $\theta_{xi}$, $\theta_{yi}\}^T$ corresponding to $i^{\text{th}}$ node with $N_i(\xi, \eta)$ shape function.

Then, the interpolated displacement vector is substituted into Eq. (4) and the resulting generalized strain vector $\hat{\boldsymbol{\epsilon}}$ can be rewritten in terms of elemental strain-displacement matrix $\boldsymbol{B}$ and elemental displacement vector $\boldsymbol{q}$ as

$$\begin{aligned}\hat{\boldsymbol{\epsilon}}_{lj} = \boldsymbol{B}^L\boldsymbol{q};\ \ \hat{\boldsymbol{\epsilon}}_{nlj} = \boldsymbol{B}^{NL}\boldsymbol{q} \\ \boldsymbol{q} = \left\{\ \boldsymbol{q}_1^T \quad \boldsymbol{q}_2^T \quad \cdots \quad \boldsymbol{q}_8^T \quad \boldsymbol{q}_9^T\ \right\}^T\end{aligned} \tag{7}$$

The detailed expressions of $\boldsymbol{B}^L$ and $\boldsymbol{B}^{NL}$ are discussed in the Ref. [1].

### F. System of equations

By using the interpolated field variables and strain-displacement matrices ($B^L$ and $B^{NL}$) in Eq. (6), and applying the variational method concept, the set of governing equations for the nonlinear buckling analysis of the composite plate under inplane mechanical loads is obtained as

$$(\boldsymbol{K} + \gamma\boldsymbol{K}_\gamma)\,\boldsymbol{q} = \boldsymbol{F}_P \tag{8}$$

where details of stiffness matrix $\boldsymbol{K}$, penalty stiffness matrix $\boldsymbol{K}_\gamma$ and force vector $\boldsymbol{F}_P$ are given in the Ref. [1].

### G. Solution procedure

The nonlinear global system of equations that results after assembly of the element equations and the imposition of suitable boundary conditions are solved using a tangent based arc-length iterative method [28]. Both relative displacement norm $\frac{||\Delta\boldsymbol{q}||}{||\boldsymbol{q}||} < \beta$, and residual normal criterion are used for convergence with error tolerance $\beta = 10^{-2}$.

The above procedure is limited to a single solution path. However, in the case of the buckling analysis of perfect plates, a primary solution path is bifurcated into two stable symmetric post-buckling solution paths which are traced using a simple branch switching technique [27] in this study. Only a single post-buckling solution path is generally traced for laminated composite plates due to symmetry. The nonlinear eigenvalue approach is also used to trace the post-buckling solution path as explained in Ref. [29].

## III. Results and discussions

The present numerical investigation is focused on the nonlinear buckling analysis of composite plates under uniform uniaxial inplane line loads, i.e, $P_i = P$, that are applied normal to the midplane of the plate. The post-buckling responses of the laminated plates are determined using both nonlinear eigenvalue and nonlinear buckling approaches. First, a linear buckling analysis is performed using either pre-buckling approach or linear buckling approach to initiate the nonlinear eigenvalue method. Pre-buckling boundary conditions and buckling boundary conditions are respectively used in the pre-buckling approach and linear buckling approach to perform linear static analysis. A linear eigenvalue analysis is performed with buckling boundary conditions in the second stage to obtain the initial buckling load. Then, a nonlinear eigenvalue problem with augmented nonlinear stiffness is solved to obtain the post-buckling solution paths.

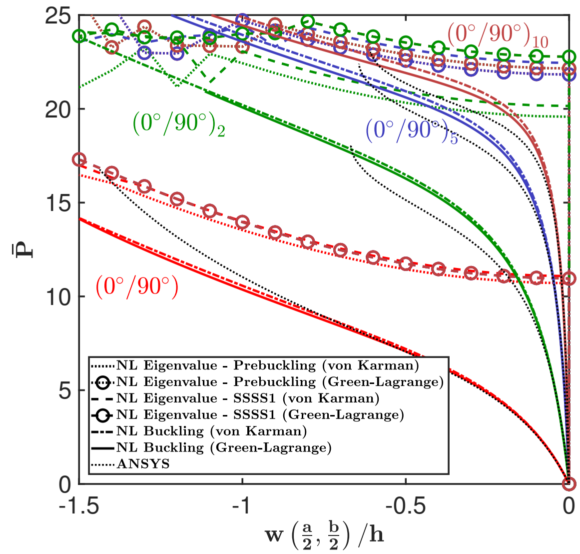
A selective integration rule is used in the numerical integration, i.e., $3 \times 3$ Gauss-Legendre quadrature rule for linear bending stiffness and force vector and a $2 \times 2$ Gauss-Legendre quadrature rule for remaining matrices. The plate is discretized by $12 \times 12$ mesh. The penalty parameter $\gamma = 10^9$. The kinematics constraints of the simply-supported plate are $u_0 = w_0 = \phi_x = \theta_x = 0$ along $y = 0$, $b$ and $v_0 = w_0 = \phi_y = \theta_y = 0$ along $x = 0$, $a$. Note that $u_0$ and $v_0$ are not constrained in pre-buckling boundary conditions. The material properties used for buckling problems are tabulated in Table I.

TABLE. I: Material properties of laminated composite plates

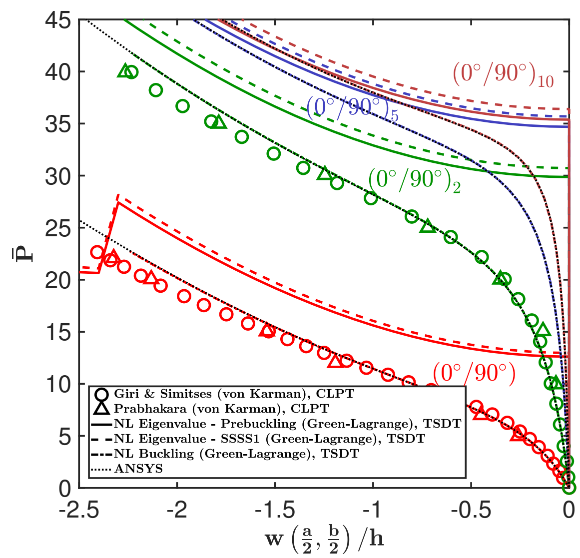
| **Parameters** | **MM1** [5], [30], [31] | **MM2** [5] |
|---|---|---|
| $E_1$ (GPa) | $25E_2$ or Specified | $40E_2$ or Specified |
| $E_2$ (GPa) | 1 | 1 |
| $G_{12}$ (GPa) | $0.5E_2$ | $0.6E_2$ |
| $G_{13}$ (GPa) | $0.5E_2$ | $0.6E_2$ |
| $G_{23}$ (GPa) | $0.2E_2$ | $0.5E_2$ |
| $\nu_{12}$ | 0.25 | 0.25 |

*1) Simply supported anti-symmetric cross-ply laminated plate:* In this numerical example, two simply supported (SSSS) square laminated $(0°/90°)_n$ plates with $a/h = 10$ and 100 are considered. The values of the material constants correspond to MM1 [31] with $E_1/E_2 = 40$. The same problem for thin plate ($a/h = 100$) is studied by Giri and Simitses [31], and Prabhakara [30] using CLPT in conjunction with von Kármán nonlinearity. Fig. 2 shows the plots of the normalized deflection of the center of the plate $w\left(\frac{a}{2}, \frac{b}{2}\right)/h$ against the normalized load parameter $\bar{P} = Pb^2/E_2h^3$ for $n = 1, 2, 5$, and 10. The present FEM-TSDT results are obtained via both nonlinear eigenvalue and nonlinear buckling approaches using both von Kármán and Green-Lagrange nonlinearities. It is observed from the nonlinear buckling approach response that anti-symmetric cross-ply laminated plates do not exhibit bifurcation buckling due to the presence of bending-stretching coupling. This is also verified by ANSYS solutions. The nonlinear eigenvalue approach is not a reliable approach for buckling analysis of a general laminated plate; the nonlinear buckling approach must be used instead. Further, the solution paths obtained using Green-Lagrange nonlinearity are found to be conservative in comparison to the solution paths obtained with von Kármán nonlinearity.

*2) Effect of fiber orientation on nonlinear buckling response:* In this section, the influence of fiber orientation on the post-buckling response of angle-ply laminated plates is studied. For this, a thin ($a/h = 100$) simply supported (SSSS) square angle-ply $(\theta/-\theta)$ laminated plate with material properties MM2 [5] is considered. The present FEM-TSDT solution paths are obtained by considering only von Kármán nonlinearity because the plate is thin ($a/h = 100$). Fig. 3 depicts the plots of the normalized deflection $w_{\text{max}}/h$ against the normalized load $\hat{P} = \bar{P}/\bar{P}_{\text{critical}}$ for different ply orientations using both nonlinear eigenvalue and nonlinear buckling approaches. It is observed that angle-ply composite plates do not exhibit pure bifurcation. They initially bend and then follow a secondary path. Further, post-buckling deflection profile depends upon fiber orientation as plates with layups $(45°/-45°)$ , $(60°/-60°)$, and $(75°/-75°)$ buckle with second buckling mode while plates with layups $(15°/-15°)$ and $(30°/-30°)$ buckle in the first mode shape. The initial




(a) $a/h = 10$




(b) $a/h = 100$

Figure. 2: Nonlinear response of anti-symmetric cross-ply laminated plate under uniform uniaxial in-plane loading

buckling strength $\bar{P}_{\text{critical}} = P_{\text{critical}}b^2/E_2h^3$ obtained using the linear buckling approach for the following lamination layups $(15°/-15°)$ , $(30°/-30°)$, $(45°/-45°)$ , $(60°/-60°)$ and $(75°/-75°)$ is 26.5240, 19.7367, 33.0916, 41.5325, and 26.8976, respectively. It is observed from Fig. 3 that the true buckling load for the angle-ply laminates is different from the value predicted by the nonlinear eigenvalue approach. The mode shapes shown in the figure are obtained using the nonlinear buckling approach. It is concluded that the nonlinear buckling approach is superior for buckling analysis of angle-ply laminated composite plates.

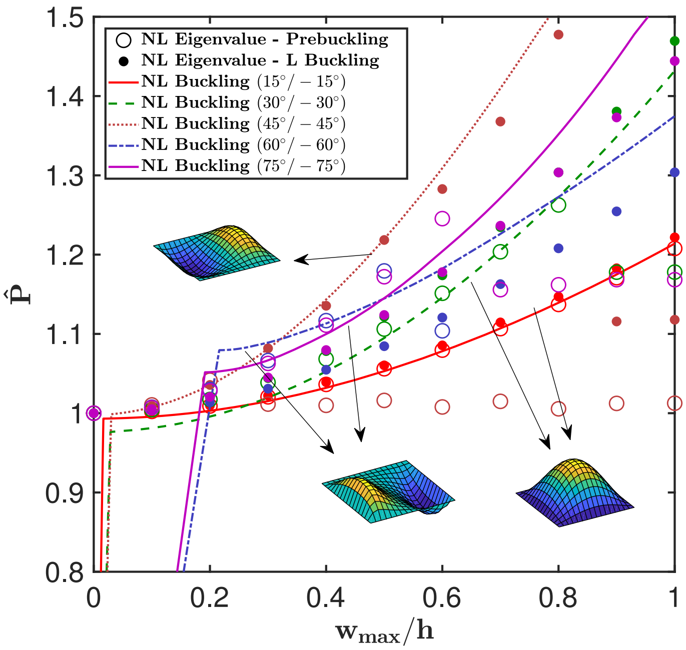



Figure. 3: Effect of fiber orientation on post-buckling characteristics of square $(\theta°/-\theta°)$ plate under uniaxial uniform load. The modes are obtained using the nonlinear buckling approach

## IV. Conclusion

A comparative study of nonlinear eigenvalue and nonlinear buckling approaches for buckling analysis of laminated composite plates using a penalty based $C^0$ FEM-TSDT plate model is presented in this paper. Both von Kármán and Green-Lagrange nonlinearities are employed to investigate the influence of stress stiffening on the buckling characteristics of laminated composite plates.

It is observed that nonlinear buckling approach in conjunction with Green-Lagrange nonlinearity can accurately predict the buckling loads along with pre-buckling and post-buckling solution paths for laminated composite plates. Further, it is superior to the nonlinear eigenvalue approach. The use of the Green-Lagrange nonlinearity over von Kármán nonlinearity is recommended for better design and stability analysis of moderately thick composite plates because it includes the effects of inplane displacements in the stress stiffening which is essential for compressive loading.

Further, anti-symmetric cross-ply laminated composite plates do not exhibit bifurcation buckling, rather, bending is prevalent due to the presence of bending-extension coupling. Whereas, angle-ply laminated composite plates do not exhibit pure bifurcation, i.e., bending-buckling response, because of extension-shear coupling.

The proposed FEM-TSDT model is based on a unified formulation that can be easily extended to any $C^1$ HSDT by using an appropriate transverse shear function, $f(z)$, and boundary conditions.

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