Comparison Between FEniCS and PINNs

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Modeling & Simulation 2

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2D Poisson Equation

Consider a bounded domain Ω in 2D, defined as $\Omega = [a, b] \times [c, d]$, where a, b, c, and d are the bounds of the domain. The Poisson equation is given by:

$$\begin{aligned} \Delta u(x, y) &= f(x, y) \text{ for } (x, y) \in \Omega \\ u(x, y) &= g(x, y) \text{ for } (x, y) \in \partial\Omega \end{aligned} \tag{1}$$

In this case:

$$\begin{aligned} f(x, y) &= -\sin(\pi x)\sin(\pi y) \\ g(x, y) &= 0 \\ \Omega &= [0, 1] \times [0, 1] \end{aligned}$$

Use of Poisson Equation

- Physics: In areas such as electrostatics, where it describes the electric potential in a region with a given charge distribution. It is also used in fluid dynamics to model potential flow problems.
- Image Processing: In image processing tasks such as image inpainting and image denoising. It can be used to reconstruct missing or corrupted parts of an image based on the known information.
- Computational Geometry: Is employed in mesh generation and surface reconstruction algorithms. It helps to compute smooth surfaces that interpolate scattered data points or to generate meshes with desirable properties.

Variational Formulation

Consider the followings spaces:

$$V = \{v \in H^1(\Omega) : v|_{\partial\Omega} = g\}$$
$$\hat{V} = \{v \in H_0^1(\Omega) : v|_{\partial\Omega} = 0\}$$

Since $g(x, y) = 0$, then $V = \hat{V}$. Let $u, v \in V$, integrating over Ω:

$$\int_\Omega \Delta u v \, dx = \int_\Omega f v \, dx$$

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Variational Formulation

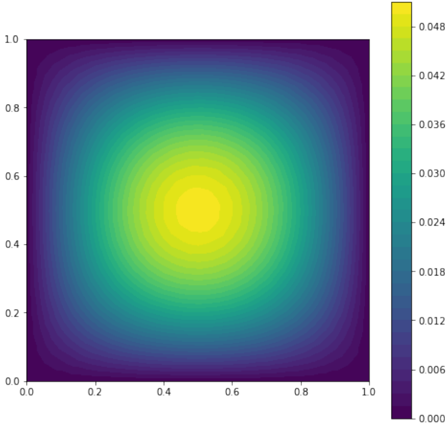
Applying Green's theorem on the left-hand side, we obtain that:

$$\int_\Omega \Delta u v \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v \, dS - \int_\Omega \nabla u \nabla v \, dx$$

Since $v|_{\partial\Omega} = 0$, the surface integral is 0. Therefore the variational formula is given by:

$$-\int_\Omega \nabla u \nabla v \, dx = \int_\Omega f v \, dx$$

Poisson Equation Approximation Using FEniCS



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Residual and Loss Functional

The method constructs an approximation $u_\theta(x, y) \approx u(x, y)$ of the solution of (1), where $u_\theta : \Omega \to \mathbb{R}$ denotes a function realized by a neural network with parameters θ. The solution for the PDE (1) is based on the residual of a given neural network approximation $u_\theta(x, y)$.

$$r_\theta := \Delta u(x, y) - f(x, y) \tag{2}$$

The PINN approach for the solution of the PDE (1) now proceeds by minimization of the loss functional

$$\phi_\theta(X) := \phi_\theta^r(X^r) + \phi_\theta^b(X^b) \tag{3}$$

where X denotes the collection of training data and the loss function ϕ_θ contains the following terms:

Residual and Loss Functional

- the mean squared residual:

$$\phi_\theta^r(X^r) := \frac{1}{N_r} \sum_{i=1}^{N_r} |r_\theta(x_i^r, y_i^r)|^2$$

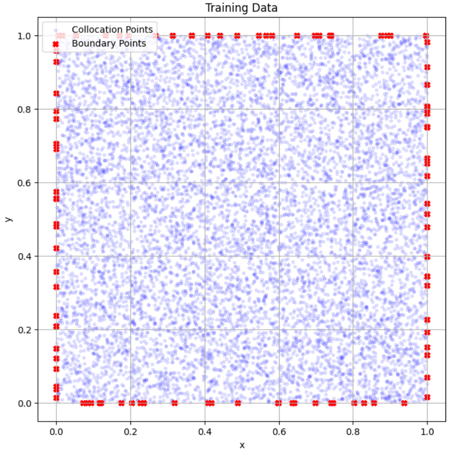
in a number of collocation points $X^r := \{(x_i^r, y_i^r)\}_{i=1}^{N_r} \subset \Omega$ and r_θ is the physics-informed neural network.

- the mean squared misfit with respect to the boundary conditions:

$$\phi_\theta^b(X^b) := \frac{1}{N_b} \sum_{i=1}^{N_b} |u_\theta(x_i^b, y_i^b) - u_b(x_i^b, y_i^b)|^2$$

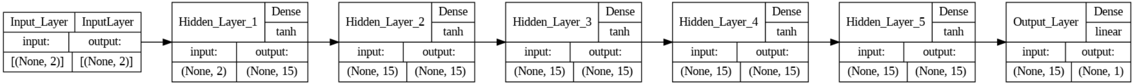
in a number of points $X^b := \{(x_i^b, y_i^b)\}_{i=1}^{N_b} \subset \partial\Omega$.

Training Data

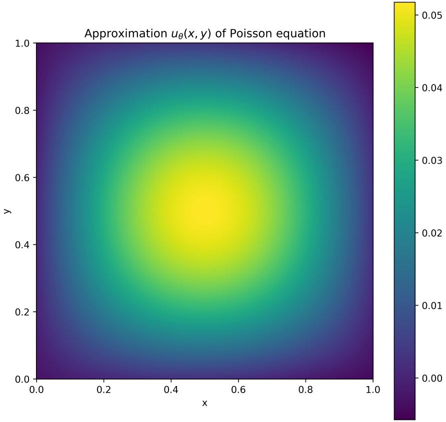


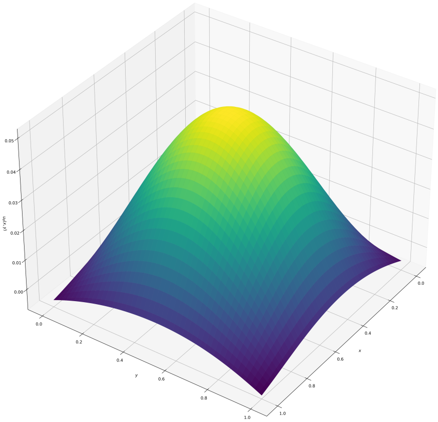
Neural Network Architecture

The deep neural network has the following structure: the input layer, followed by 5 fully connected layers each containing 15 neurons and each followed by a hyperbolic tangent activation function and one output layer. This setting results in a network containing 1021 trainable parameters (first hidden layer: $2 \cdot 15 + 15 = 45$; 4 intermediate layers: each $15 \cdot 15 + 15 = 240$; output layer: $15 \cdot 1 + 1 = 16$).

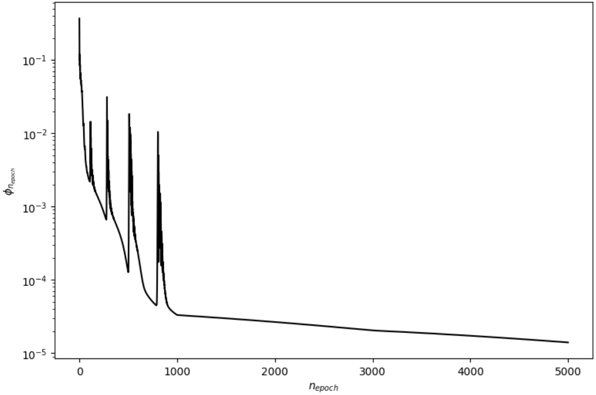


Poisson Equation Approximation Using PINNs

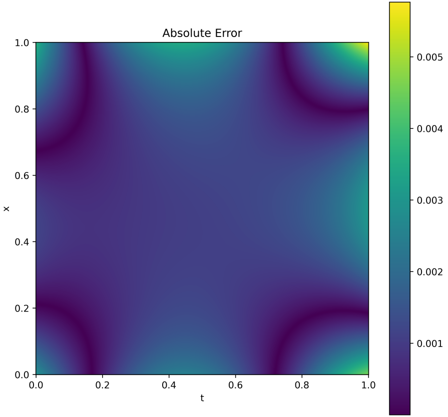




Loss Function



Absolute Error



References I

J. Blechschmidt and O. G. Ernst, “Three Ways to Solve Partial Differential Equations with Neural Networks — A Review ”, arXiv [math.NA]. 2021.