MULTI AGENT SIMULATION TO STUDY THE EFFECTS OF LEADERSHIP IN A COLLECTIVE ACTION PROBLEM

Research internship report - MAP 594

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March 25, 2019 - July 26, 2019
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Abstract

Gathered in groups, individuals are constantly confronted with the phenomenon of decision-making: going hunting, finding a place to settle, sharing information...

Often, an individual will be the initiator of the decision making and will propose to other individuals to follow the action. We then say that this individual is a Leader. The others will comply or not. This choice to comply depends on many factors. Here, we consider that not all individuals have the same strength; that is, they can be more or less influential within the group. By adopting a mathematical perspective in the study of an Evolutionary Game Theory Model, we study how this disparity of strengths affects the outcome of the Public Good Game. Strengths also influence Decision-Making and Leadership.

Contents

Introduction

In this report, I present the work done during my internship at the Institute of Cognitive Sciences and Technologies (ISTC), member of the Consiglio Nazionale delle Ricerche (CNR) in Rome. I was followed by my supervisor, Vito TRIANNI, from the Laboratory of Autonomous Robotics and Artificial Life (LARAL), to whom I extend my sincere thanks. I am also grateful to Luis Alberto MARTINEZ VAQUERO, who wrote elements of code in PYTHON which provided a solid foundation for the computer implementations made during the internship.
My internship is a continuation of the one of Aurélien BROUILLAUD (X 2015), who was in Rome last year. I therefore relied on his work (1, 2), which was of great help in expanding mine.
The heading of my internship was *Multi agent simulation to study the effects of Leadership in a collective action problem*. It was therefore necessary to develop a Public Good Game (PGG), i.e. an Evolutionary Game Theory (EGT) model to combine two distinct notions in the game:

1. A difference in strengths within the population (individuals do no longer behave in a uniform way). In essence, we considered the existence of two different plausible behaviours: some people behave like strong ones, others like weak ones.

2. The emergence and existence of a Leader who will play a particular role within the group, by exerting influence on the other individuals in the group (he suggests that they act like himself and has a certain interest in the action being performed.).

The choice of the Public Good Game as the starting model is relevant because it is large enough to be developed and refined according to our needs. It allows a fairly effective mathematization of phenomena observed in living structures. We assume that these living structures are sufficiently advanced to be able to assess the social influence of others, in particular to identify a Leader within a group of individuals.

Our general problem will be to determine whether or not the exercise of Leadership promotes cooperation within populations whose individuals do not all behave in the same way.

We will therefore first focus on the concepts of Leadership and Decision-Making and then consider a Public Good Game for an uniform population before developing the model for several strengths. Finally, we will fine-tune our model in order to make it ever closer to phenomena observed in living structures.

1 Decision-Making and Leadership

The key to the emergence of leadership and followership is the need to coordinate (9). Animals living in groups (e.g. baboons (10)) need to coordinate the timing and nature of their activities. This requires that group members organize their movements and collectively agree on the coordination of important events. The stimulus for action is provided by a Leader.

1.1 What is a Leader ?

A Leader is an individual that will initiate the choice during a situation where Decision Making is required. It can initiate this choice for three different reasons :

1. It benefits from more information than the others (9, 5).
2. It has some notorious physical attributes (e.g. age, sex, size, weight...) (16).
3. It has a particular personal interest in imposing its preferences to the group (8).

1.1.1 Discussion about the power of the Leader

For sufficiently large groups, only a minor proportion of informed leaders is needed to achieve close to maximal accuracy (especially in homogeneous groups). Couzin showed that the larger the group the smaller the proportion of informed individuals needed to guide the group, and that only a very small proportion of informed individuals is required to achieve great accuracy. Groups can make consensus decisions, even though informed individuals are not aware whether they are in a majority or minority, how is the quality of their information compares with that of others, or even whether there are any other informed individuals (5). However, where conflict does occur, theoretical models and experiments predict these groups to almost always decide in favour of the majority preference. Thus, if only a few leaders possess valuable information in the first place, just one more informed leader can have a decisive role in the collective action of the entire group.

The evolution of cooperation remains a puzzling as the collective action can be easily undermined by free-riders, i.e. individuals who reap the benefits from others' efforts without contributing to the common interest. One potential solution to the free-rider problem is punishment (banish them from the group or withhold them from the gain). We will come back to this later in this report.

1.1.2 What characterizes a Leader ?

In (10), the Author considered a matrix M of departures of baboons from the group composed of the relative frequency of following for each dyad during successful departures (i.e. where all adults and their offspring joined the movement). For each dyad, say i and j :

$$M_{i,j} = \frac{\text{nb. events wh. } i \hookrightarrow j \; + \; \text{nb. events wh. } j \hookrightarrow i}{\text{total nb. of departures}}$$

To characterize the importance of the leadership of each individual living in the group, eigenvector centrality is used to measure the centrality of individuals in the network, ranging each baboon from 0 (least central) to 1 (most central). Their results suggest that a *follow-a-friend* rule, where friendships (social affiliations) are indexed by grooming interactions and spatial associations, satisfactorily explains both the outcome and process of baboon collective departures from sleeping sites.

When individuals with high eigenvector centrality in the grooming or spatial network initiate a departure, many individuals tend to follow, and the group collectively moves away. In contrast, when individuals that are peripheral to these networks initiate a departure, they are rarely followed, and the group does not depart. Thus, an individual with high eigenvector centrality can be called a Leader whereas an individual with low eigenvector centrality can be called a Follower.

De facto, some mechanisms are involved so individuals may recognize and monitor who are the leaders (i.e. who to follow) ; and who are the followers. Therefore, how do the individuals do to recognize their rank and the rank of others in the group ?

To observe the gradation of forces in a population, aggression is studied within a group of captive parakeets (6). An aggression can occur when an individual desires a place like a tree branch to another. Individuals can use localized patterns in the aggression network to learn the relative ranks of others. These signals of rank strongly correlate with decisions to aggress. Over time, feedback between knowledge and behavior leads to the emergence of strategic aggression: individuals focus their aggression on those nearby in rank.

Hobson summarizes his results on the rank of individuals in three points, for individuals possessing social memory and social inference:

1. High influence of the rank on Decision Making;
2. Both levels of aggression and subsets of the full network (network motifs in the form of aggression chains) provide signals of rank;
3. Network motifs predict behavior.

1.1.3 Which individual can stand out as a Leader?

We repeat an evolutionary game where, in each round after the first, with a certain probability (the player's intrinsic leadership), the individual chooses its own preferred option, whereas with the complementary probability, it copies the most recent choice of another, randomly chosen group member. In other terms, this probability is the probability of following.

In all cases (8), starting with a monomorphic population in which all individuals had an intrinsic leadership of 0.5, selection leads rapidly (within a few hundred time steps) to evolutionary branching and polymorphism. For most group sizes there is a single branching event that gives rise to a stable dimorphism, featuring extreme leaders (with values of intrinsic leadership very close to 1), who almost always choose their own preferred option, and extreme followers (with values of intrinsic leadership close to 0), who almost always copy the choices of others.

Leaders do well when they interact with a group consisting mainly of followers, because the latter consent to their preferences, leading to effective coordination on the course of action favored by the Leader. At the same time, however, leaders do poorly when they interact with a group consisting mainly of other leaders, because the group then fails to coordinate, each individual sticking stubbornly to their own preferred option.

The same individual may emerge as a Leader when interacting with one set of partners, but as a Follower in a different social context.

1.1.4 What happens during the Decision-Making process?

Although the role of Leader is undeniable, the decision is collective so we have to be interested in the subject of the will of the group. It may override dominant preferences when the consensus of subordinates is sufficiently great (16, 15). Specific behavioural mechanisms are used for voting during Decision Making. When a certain threshold $M_{\mathcal{B}}$ is reached, the decision is made.

The following basic function describes a quorum responses mathematically:

$$p_{\mathcal{O}} = \frac{A^m}{A^m + B^m}$$

whereby $p_{\mathcal{O}}$ is the probability that an animal will choose a particular option, A is the number of animals which have already chosen the option, B is the threshold quorum at which the response steeply increases and m ($m \geq 1$) determines the steepness of the response.

A quorum response leads indeed to higher collective decision accuracy than does solitary decision-making. However, this higher accuracy comes at the price of a slower overall decision speed. Usually decision speed increases with group size (or the number of informed individuals).

The study of animal departure requires a temporal sub-study. An individual has to be prepared to depart during a time window around its optimal time T_A. The strategy (R_A, W_A) defines this time window as ($T_A - R_A$ to $T_A + W_A$) for animal A. If animal B departs before $T_A - R_A$, animal A will not depart. If animal B departs within animal A's time window, then animal A departs together with animal B. If animal B has not departed before time $T_A + W_A$, animal A departs at time $T_A + W_A$. Animal B then either follows or stays behind, according to its own time window ($T_B - R_B$ to $T_B + W_B$). Thus, depending on its strategy (R_A, W_A), animal A might pay an early or late departure cost and might receive grouping benefit.

What is the best strategy (R^*, W^*) at which the net expected gains to an individual are maximal ? The model predicts for the majority of biologically relevant parameter values that the best strategy for an animal is to be ready to join the other individual in foraging before its own optimal time (i.e. $R^* > 0$), but not necessarily to wait beyond that optimal time (i.e. $W^* = 0$) unless waiting is cheap. Thus, the pair-synchronization model predicts, like the Leader - Follower model, that a group of two should change activity synchronously when the first individual does so, and that the decision is, strictly speaking, a shared decision (4).

Thereafter, we will consider that the decision making following the initiation of an action is made in a shared form.

1.2 Choice of the Leader

From now on, we must determine a Leader within each group. Several methods are possible:

- The choice of the Leader is uniform within the group.
- The choice of the Leader depends on its type of behaviour (or *social rank*).

1.2.1 Uniform choice of the Leader

In some groups, an individual may wish to initiate a Decision-Making process (and thus become a Leader) because it has information that others do not have. This characteristic therefore does not depend on its type of behaviour. Consequently, the choice of Leader can be uniform among the group. We note I the individual, L the Leader and $N \in \mathbb{N}^*$ the size of the group.

$$\mathbb{P}(\mathrm{I} = \mathrm{L}) = \frac{1}{N}$$

If we consider that the population is composed of two types (see below, in section 3, page 19), which we call Weak $\mathcal{W}$ and Strong $\mathcal{S}$, we note $N_{\mathcal{W}}$ the number of weak individuals and $N_{\mathcal{S}}$ the number of strong individuals. We then have the distribution:

$$N = \langle N_{\mathcal{W}}; N_{\mathcal{S}} \rangle$$

Thus, we do have:

$$\begin{cases} \mathbb{P}\left(\mathrm{L} \in \mathcal{W}\right) & = \frac{N_{\mathcal{W}}}{N} \\ \mathbb{P}\left(\mathrm{L} \in \mathcal{S}\right) & = \frac{N_{\mathcal{S}}}{N} \end{cases}$$

1.2.2 Choice of the Leader depending on Type

In other groups, the Decision-Making process is more naturally initiated by individuals whose strength is recognized within individuals. It is therefore a question of preferring a strong individual as a Leader, where a weak individual will not be privileged.

We are consequently interested in a model where the Leader is no longer chosen in an equiprobable way within the group, but in a way that depends on his type (Weak $\mathcal{W}$ and Strong $\mathcal{S}$).

We introduce a parameter δ_{Leader} called strength of leadership, which defines the probabilities of being a Leader according to the behaviour of the individual:

$$\begin{cases} \mathbb{P}\left(\mathrm{I}^{\mathcal{W}}=\mathrm{L}\right) & \propto \frac{1}{1+e^{\delta_{\text{Leader}}}} \\ \mathbb{P}\left(\mathrm{I}^{\mathcal{S}}=\mathrm{L}\right) & \propto \frac{1}{1+e^{-\delta_{\text{Leader}}}} = \frac{e^{\delta_{\text{Leader}}}}{1+e^{\delta_{\text{Leader}}}} \end{cases}$$

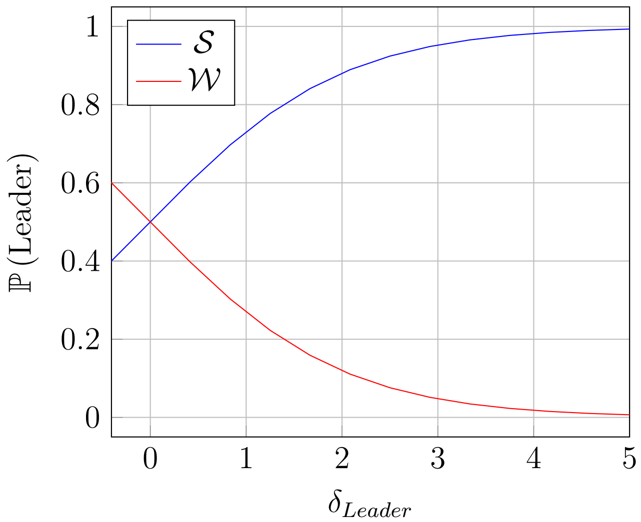


Figure 1: Being a Leader depends on type

It seems natural that a strong individual is more inclined to be a Leader, so we deduce that: $\delta_{\text{Leader}} \geq 0$.

2 Leadership in a uniform population

This section includes some results found by Aurélien BROUILLAUD (X 2015), who was an intern at the CNR the previous year. It is also based on the key book of the lesson ECO 555 : *Bases Mathématiques de la Théorie des Jeux (Mathematical foundations of Game Theory)* (11).

2.1 Uniform Public Good Game

2.1.1 Non repeated Prisoner's Dilemma Game

Two players are offered a certain reward R for mutual cooperation, and a penalty P for mutual defection. If one player cooperates while the other defects, then the cooperator gets the lowest payoff S while the defector gains the highest payoff T. We have :

$$T > R > P > S$$

Each player disposes of two strategies : COOPERATE $\mathcal{C}$ or DEFECT $\mathcal{D}$. The payoff matrix for this 2-players game is given by a 2×2 matrix, denoting the two possible strategies of each player :

	$\mathcal{C}$	$\mathcal{D}$
$\mathcal{C}$	$R; R$	$S; T$
$\mathcal{D}$	$T; S$	$P; P$

The player is encouraged to be a defector, because Game Theory predicts that the Nash Equilibrium of the Prisoner's Dilemma Game is : $NE_{PDG} = \{\mathcal{D}; \mathcal{D}\}$. Therefore, the payoff of each player is P. Thus, in the non-repeated Prisoner's Dilemma, defectors dominate cooperators (3, 11).

2.1.2 Non repeated Generalized Prisoner's Dilemma Game

We play a Public Good Game, also called Generalized Prisoner Dilemma, with an uniform population.

The population is composed of N individuals who can play one of the two following strategies : COOPERATE $\mathcal{C}$ or DEFECT $\mathcal{D}$. A cooperator gives a contribution to the group which costs him c, whereas a defector does not. The total contribution is multiplied by a ratio $r \geq 1$ and shared equally between the N members of the group. Let's assume there is $N_{\mathcal{C}}$ cooperators and $N_{\mathcal{D}}$ defectors in group, i.e.: $N = \langle N_{\mathcal{C}}; N_{\mathcal{D}} \rangle$. We have the payoffs for a cooperator and a defector given by :

$$\begin{cases} \Pi^{\mathcal{C}} & = r\frac{N_{\mathcal{C}}}{N}c - c \\ \Pi^{\mathcal{D}} & = r\frac{N_{\mathcal{C}}}{N}c \end{cases}$$

The payoff of a defector is always greater than the one of a cooperator. However, if nobody cooperates, each individual will have a null payoff, whereas if the whole population cooperates, each individual will gain $c(r-1)$.

We therefore observe that this game proposes a discussion on the duality of personal interest and collective interest.

2.1.3 Non repeated Generalized Prisoner's Dilemma Game with threshold

Game Theory often consider a N-individuals game with a threshold M less than the total group is required to produce benefits. Increasing participation leads to increasing productivity. Let $\Theta_M(\cdot)$ be a M-indexed function which is the null function below M and the identity above. The function $\Theta_3(\cdot)$ is shown below.

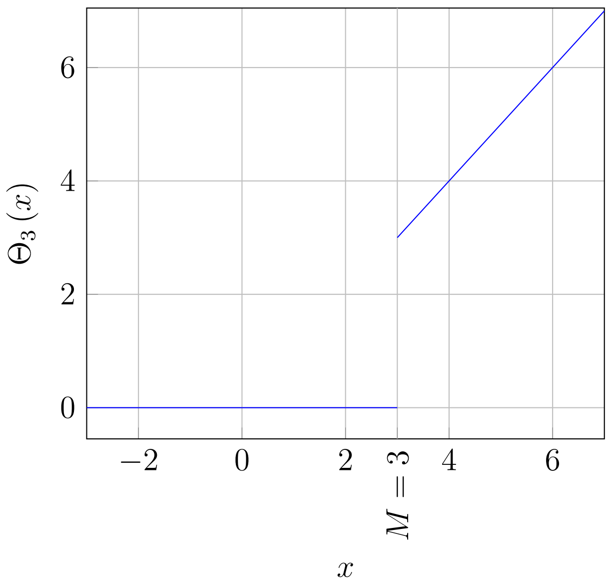


Figure 2: Θ_3 function

We have, in the case of the Generalized Prisoner's Dilemma :

$$\begin{cases} \Pi^{\mathcal{C}} = r\frac{N_{\mathcal{C}}}{N}c \times \Theta\left(rN_{\mathcal{C}}c - M\right) - c \\ \Pi^{\mathcal{D}} = r\frac{N_{\mathcal{C}}}{N}c \times \Theta\left(rN_{\mathcal{C}}c - M\right) \end{cases}$$

Pacheco studied such an N-person stag hunt dilemma (14). In infinite populations this leads to a rich dynamics that admits multiple equilibria. Scenarios of defector dominance, pure coordination or coexistence may arise simultaneously.

2.1.4 Repeated Generalized Prisoner's Dilemma

The Evolutionary Game Theory will focus on the evolution of the game after a large number of steps (or repetitions). It is a question of seeing which strategies tend to prevail and which strategies tend to disappear, when the process will reach a stable state. Evolutionary game theory is an essential component of a mathematical and computational approach to biology. Roughly speaking, game theory is the mathematical toolbox for methodological individualism, the systematic attempt to found social theory on the actions and needs of individual agents (12).

We will repeat the same game many times and look at the expectations of the payoffs according to the strategies adopted. In a group of size N, the payoffs of a focal individual are given in average by :

$$\begin{cases} \Pi^{\mathcal{C}} &= \sum_{k=0}^{N-1} \mathbb{P}_{\mathcal{C}}(N_{\mathcal{C}} = k+1) r \frac{(k+1)}{N} c - c \\ \Pi^{\mathcal{D}} &= \sum_{k=0}^{N-1} \mathbb{P}_{\mathcal{D}}(N_{\mathcal{C}} = k) r \frac{k}{N} c \end{cases}$$

Note that we have : $(\forall k \in [0, N-1])\ \mathbb{P}_{\mathcal{C}}(N_{\mathcal{C}} = k+1) = \mathbb{P}_{\mathcal{D}}(N_{\mathcal{C}} = k)$. So $\Pi^{\mathcal{C}} - \Pi^{\mathcal{D}} = \frac{r}{N} - c$. Hence, if $\frac{r}{N} > c$, cooperators invade the population through generations, whereas if $\frac{r}{N} < c$, defectors invade (13, 1).

2.2 Leadership

Different improvements can be proposed to this game (e.g. Threshold, Strengths). Here, we will introduce the notion of leadership, which the purpose of our work. An individual will be called upon to play the role of Leader and his action will take precedence over that of others.

The Leader is designated as a random member within the group (see 1.2.1, Page 13). Each individual can therefore be a Leader with a probability $\frac{1}{N}$. This individual will be the initiator and the other individuals will follow its strategy with a certain probability $p_F \in [0, 1]$ or play their own strategy with probability $1 - p_F$. Simulations show that simply designing a random player as Leader promotes the invasion of cooperators in the long term.

However, the random choice of the Leader raises questions and does not seem to be in line with reality, whether social, anthropological or biological. Indeed, the Leader often belongs to a strong class of individuals, and it seems to be necessary to introduce different strengths into our model. We will then focus on a heterogeneous population model (i.e. a population that is no longer uniform) to fine-tune the study of leadership in Section 3.

3 Leadership in a heterogeneous population

It appears that in some groups society is not egalitarian. Depending on age, gender, physical strength and other factors, individuals play different social roles. This is why it is appropriate to focus on the process of choosing and emerging Leadership in such situations of social differentiation in societies (7). We therefore consider a population of Z individuals, divided into N groups. There are two types of individuals in the population, which differ from their rank: the Weak and the Strong. Mathematically, each individual belongs to a set : $\mathcal{W}$ if it is weak ; $\mathcal{S}$ if it is strong.

The main idea of the model is to suggest that a strong individual will be more likely to initiate an action and make a decision by himself, while a weak individual is more likely to follow the movement and has little decision-making power. To schematize, one can imagine that a strong individual will have more of a leadership behaviour or a weak individual will have more of a follower behaviour. We had two distinct approaches:

1. We determine the precise number of strong and weak individuals in the population (Subsection 3.1).
2. We defined for each individual an independent probability law governing its type. The distribution of types of individuals in the population is therefore probabilistic (Subsection 3.2).

3.1 Fixed behaviours

We considered a Rectified Linear Unit function (ReLU) to make $p_{F_\mathcal{S}}$ evolve with $p_{F_\mathcal{W}}$.

$$\text{ReLU}^c(x) = \begin{cases} x - c \text{ if } x > c \\ 0 \text{ else} \end{cases}$$

Thus, we can imagine such an evolution for the function $p_{F_\mathcal{W}} \to p_{F_\mathcal{S}}\left(p_{F_\mathcal{W}}\right)$ (which is null until $p_{F_\mathcal{W}} = 0.6 = c$):

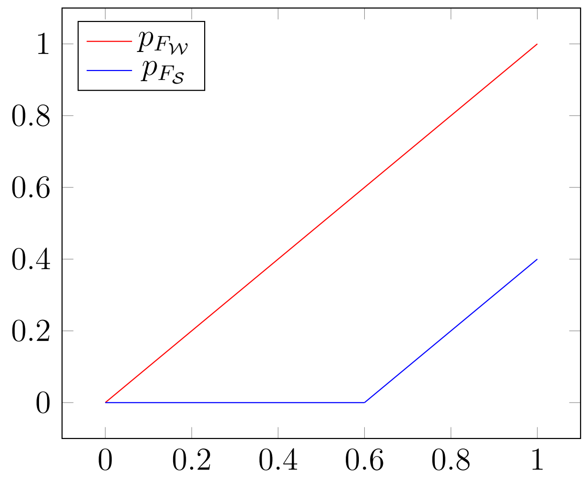


Figure 3: Rectified Linear Unit function with $c = 0.6$

We used , in this subsection, the following parameters :

Parameter	Notation	Value
Population	Z	200
Population of weak individuals	$Z_\mathcal{W}$	100
Population of strong individuals	$Z_\mathcal{S}$	100
Group size	N	20
Mutation coefficient	β	0.5
Error when trying to perform an action	ε	0.01

Weak and strong individuals are distributed in the same way within the groups. The results obtained in this case are linear, as we can see in Figure 4, where we are gradually growing $p_{F_\mathcal{S}}$.

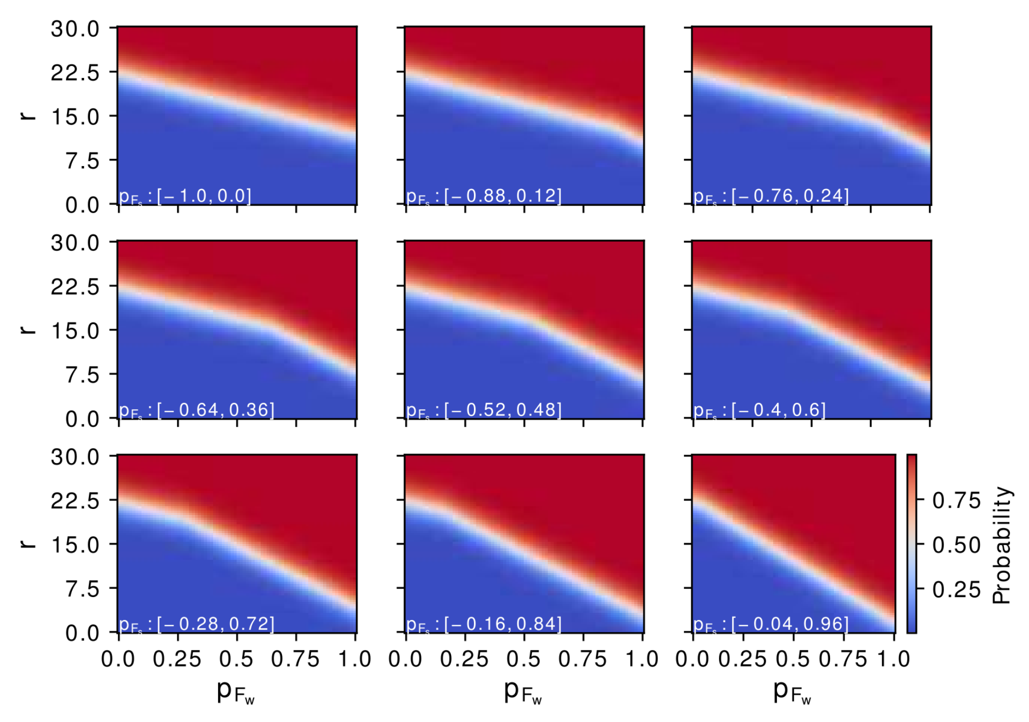


Figure 4: Heat maps with different vectors of $p_{F_{\mathcal{S}}}$

3.2 Probabilistic behaviours

From now on, each individual can behave according to the characteristics of a type : Weak $(\mathcal{W})$ Strong $(\mathcal{S})$. The behaviour of an individual I is governed by a probability $p_{\mathcal{S}} \in [0;1]$:

$$\begin{cases} \mathbb{P}(\mathrm{I} \in \mathcal{W}) & = p_{\mathcal{S}} \\ \mathbb{P}(\mathrm{I} \in \mathcal{S}) & = 1 - p_{\mathcal{S}} \end{cases}$$

Thus, the number of weak individuals in a group is a random variable that follows a binomial distribution : $N_{\mathcal{W}} \xrightarrow{\mathcal{L}} \mathcal{B}(N, p_{\mathcal{S}})$ (consequently, we have $N_{\mathcal{S}} \xrightarrow{\mathcal{L}} \mathcal{B}(N, 1-p_{\mathcal{S}})$).

Each individual can play two strategies : COOPERATE ($\mathcal{C}$) or DEFECT ($\mathcal{D}$). In the population, we have $Z_{\mathcal{C}}$ cooperators and $Z_{\mathcal{D}}$ defectors, with : $Z_{\mathcal{C}} + Z_{\mathcal{D}} = Z$. Thus, in a group, the probability to have $N_{\mathcal{C}}$ cooperators and $N_{\mathcal{D}}$ defectors is given by an hypergeometric distribution:

$$\mathbb{P}\left(N = \langle N_{\mathcal{C}}; N_{\mathcal{D}} \rangle\right) = \frac{\binom{Z_{\mathcal{C}}}{N_{\mathcal{C}}}\binom{Z_{\mathcal{D}}}{N_{\mathcal{D}}}}{\binom{Z}{N}}$$

The strategy of a cooperator will be cooperation and the one of a defector will be defection. However, an individual will not always play its own strategy (this can be empirically observed in living structures). Sometimes, the individual will mistakenly play the opposite strategy to its own. We then introduce a parameter ε that corresponds to the error when trying to perform an action. This parameter corresponds to the probability of playing according to the opposite strategy to its own. In the rest of the report, we will not necessarily always detail the case where a player does not play its strategy by a concern for clarity of purpose. However, this does not affect the understanding of the game. In practice, we take ε small:

$$\varepsilon = 0.01$$

To summarize, we can observe a total of eight profiles of individuals in the group, depending on their behaviour, strategy, and ability to play according this strategy. Here, each of the three elements depends on a probability: firstly the strength probability $p_{\mathcal{S}}$, secondly the hypergeometric probability $\mathbb{P}(N)$, thirdly the error probability ε.

3.2.1 Probabilities to follow

First, we can consider a model where weak and strong individuals each have distinct probabilities of following the Leader, called $p_{F_{\mathcal{W}}}$ and $p_{F_{\mathcal{S}}}$ with the choice that $p_{F_{\mathcal{S}}} < p_{F_{\mathcal{W}}}$. For different ratios r, we obtain the following results as Heat Maps (with the convention of null probability when $p_{F_{\mathcal{S}}} > p_{F_{\mathcal{W}}}$, which explains the upper left triangle of uniform color). We used the following parameters:

Parameter	Symbol	Value
Population	Z	100
Group Size	N	9
Mutation coefficient	β	1.0
Error when trying to perform an action	ε	0.01

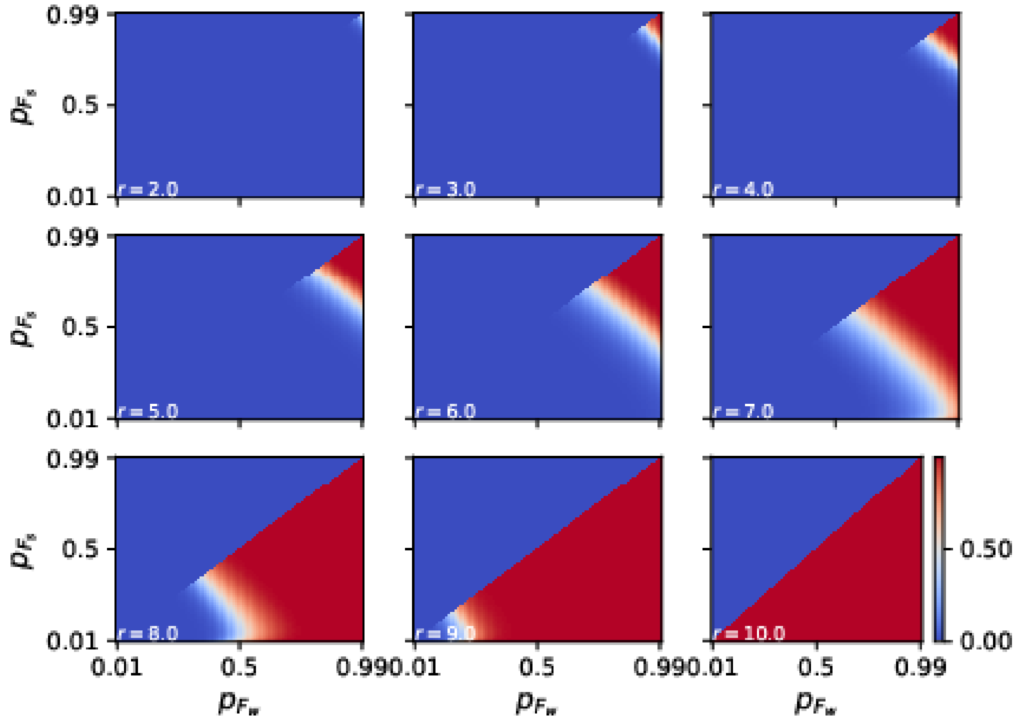


Figure 5: Heat Maps with different ratios r

We naturally notice that cooperation increases with the ratio r, and the probabilistic ability to follow the Leader for individuals (regardless of whether they are Weak or Strong).

Now, we will consider different probabilities to follow the Leader, depending on its strength, which seems more consistent with the reality of living structures.

The strength parameter f is defined as a real : $f \in \mathbb{R}^*$. We introduce variations of strength $\Delta f_{\bullet,\bullet}$ defined as follow :

$$\Delta f_{\mathcal{W},\mathcal{W}} = 0$$
$$\Delta f_{\mathcal{S},\mathcal{S}} = 0$$
$$\Delta f_{\mathcal{W},\mathcal{S}} = -\Delta f_{\mathcal{S},\mathcal{W}}$$

So we just have to introduce as a second term Δf which satisfies :

$$\Delta f = \Delta f_{\mathcal{W},\mathcal{S}} = -\Delta f_{\mathcal{S},\mathcal{W}}$$

The parameters f and Δf will help us define probabilities to follow using a Fermi distribution (introduced by Enrico Fermi and Paul Dirac in the context of Statistical Physics in 1926). An Individual of type $\mathcal{A}$ named $\mathrm{I}^{\mathcal{A}}$ will follow a Leader of type $\mathcal{B}$ named $\mathrm{L}^{\mathcal{B}}$ with probability $p_{F_{\mathcal{A},\mathcal{B}}}$, i.e. :

$$\mathbb{P}\left(\mathrm{I}^{\mathcal{A}} \hookrightarrow \mathrm{L}^{\mathcal{B}}\right) = p_{F_{\mathcal{A},\mathcal{B}}} = \frac{1}{1+\exp\left(-\beta_f(f+\Delta f_{\mathcal{A},\mathcal{B}})\right)}$$

We define a matrix $P_F = (P_{F\,\mathcal{A},\mathcal{B}})_{\mathcal{A},\mathcal{B}}$ as follow, with $\beta_f \in \mathbb{R}_+$:

$$P_F = \begin{pmatrix} P_{F_{\mathcal{W},\mathcal{W}}} & P_{F_{\mathcal{W},\mathcal{S}}} \\ P_{F_{\mathcal{S},\mathcal{W}}} & P_{F_{\mathcal{S},\mathcal{S}}} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\exp\left(-\beta_f f\right)} & \frac{1}{1+\exp\left(-\beta_f(f+\Delta f)\right)} \\ \frac{1}{1+\exp\left(-\beta_f(f-\Delta f)\right)} & \frac{1}{1+\exp\left(-\beta_f f\right)} \end{pmatrix}$$

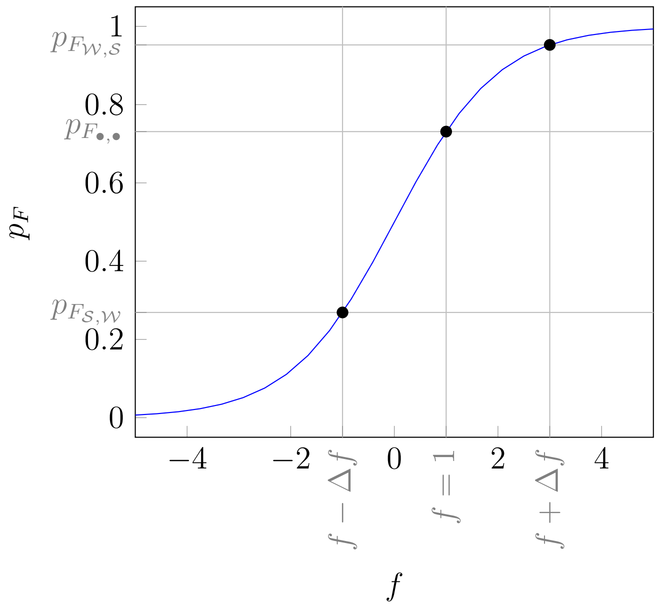


Figure 6: Probabilities to follow with $f=1$ and $\Delta f = 2$

We plotted the Fermi function for different values of β_f, showing how crucial is the influence of strength in Decision-Making.

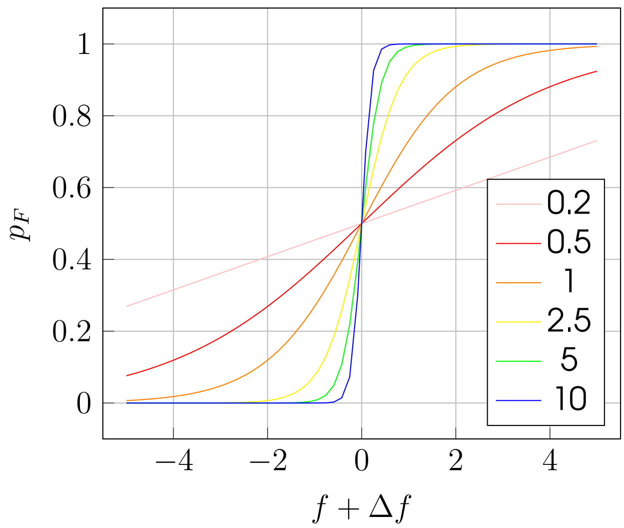


Figure 7: Sigmoids with different β_f

We see that the evolution of the probability depends on the product $\beta_f\,(f + \Delta f)$. In order to get away from a supernumerary parameter and without losing any generality, we choose to have $\beta_f = 1$. Here follows a 3D representation of the Fermi function with $\beta_f = 1$:

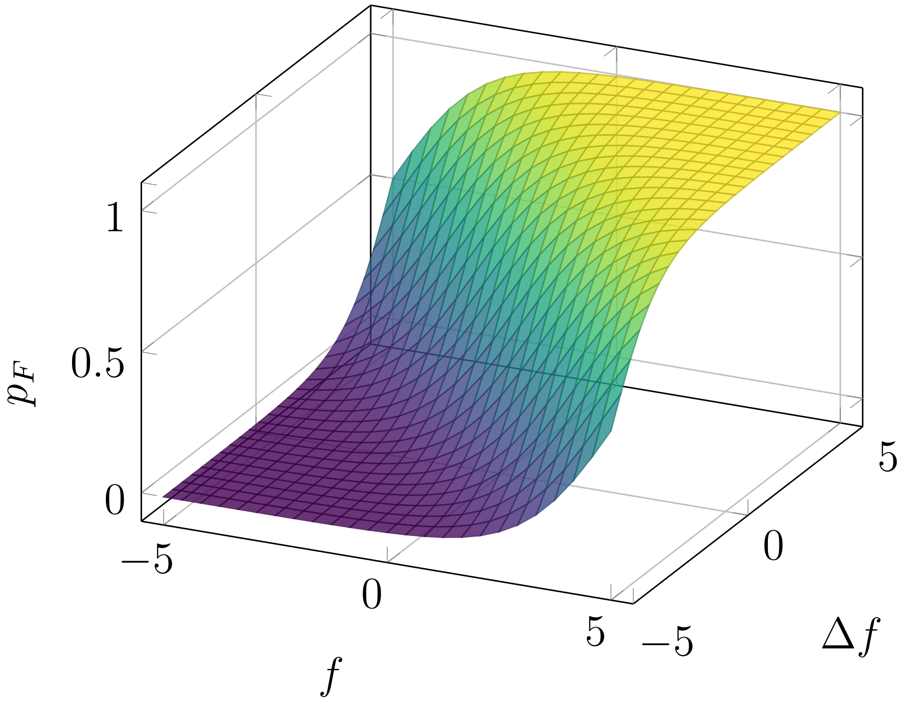


Figure 8: 3D representation of the Fermi function

The difference $\delta_{f,\Delta f}$ between the two extreme probabilities to follow is:

$$
\begin{aligned}
\delta_{f,\Delta f} &= p_{F_{\mathcal{W},\mathcal{S}}} - p_{F_{\mathcal{S},\mathcal{W}}} \\
&= \frac{1}{1+\exp\left[-\left(f+\Delta f\right)\right]} - \frac{1}{1+\exp\left[-\left(f-\Delta f\right)\right]} \\
&= 2\exp(-f) \cdot \frac{\sinh\left(\Delta f\right)}{1+2\exp(-f)\cdot\cosh(\Delta f)+\exp(-2f)}
\end{aligned}
$$

Here follows two representations (3D and 2D) of this difference $\delta_{f,\Delta f}$:

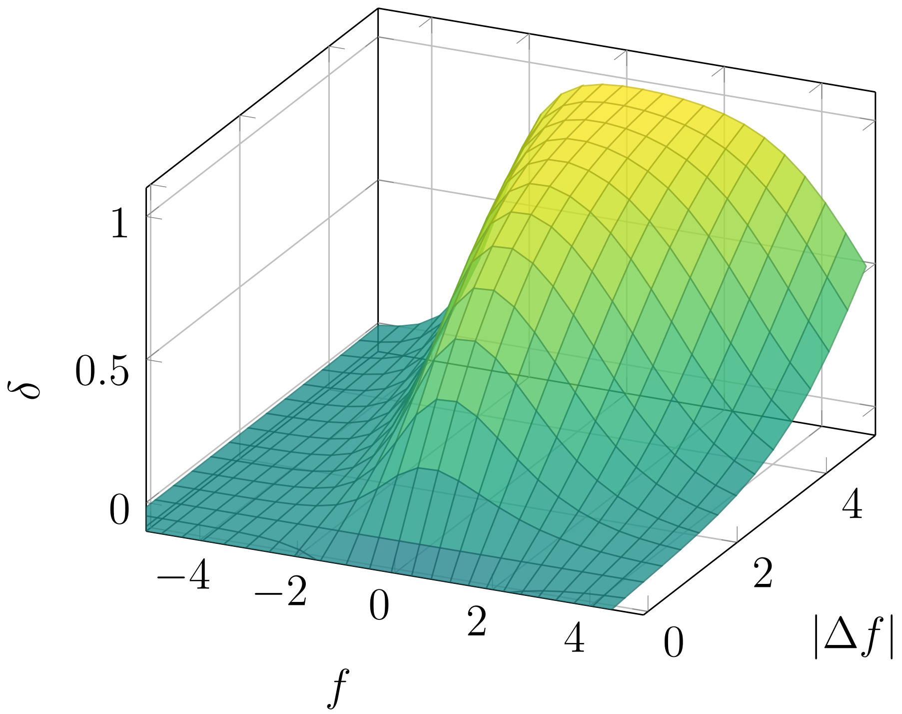


Figure 9: 3D representation of $\delta_{f,\Delta f}$

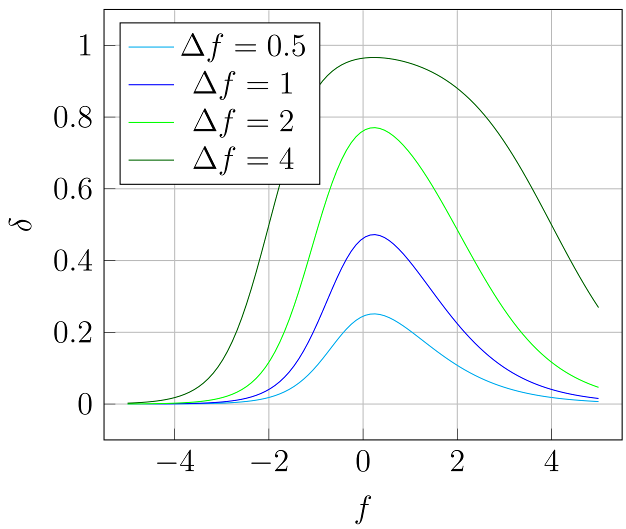


Figure 10: Representation of $\delta_{f,\Delta f}$ for different values of Δf

This is interesting to have an idea of this function because it is a good help to choose the experimental parameters for the numerical analysis.

Hence we have :

$$p_F = \frac{1}{1 + \exp(-f \mp \Delta f)} \Longleftrightarrow f \pm \Delta f = \log\left(\frac{p_F}{1 - p_F}\right)$$

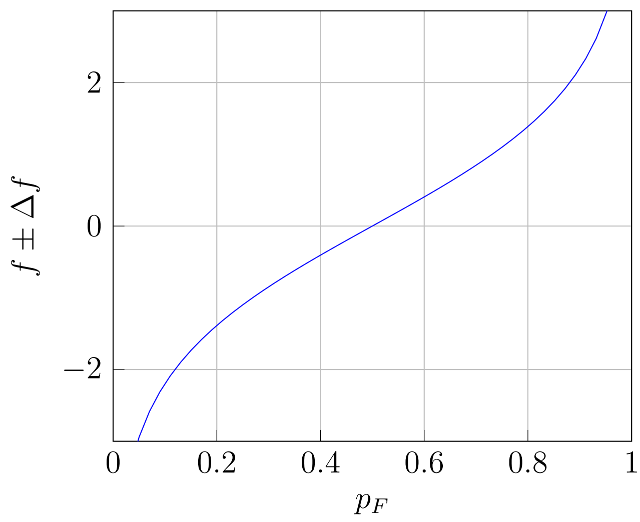


Figure 11: Relationship between $f \pm \Delta f$ and p_F

3.2.2 Process of the game

The Leader will play its own strategy with probability $1 - \varepsilon$. The remaining individuals will either follow the strategy of the Leader, or play their own strategy. The action of following the Leader depends on the types of the Leader and of the plausible Follower, that we will discuss in the Subsection 3.2.1, in which we define the probabilities to follow the Leader.

To illustrate the process, let's suppose that the Leader is a strong defector. He will cooperate with probability ε. The actions of the remaining individuals can be distinguish on their strategy :

- The $N_{\mathcal{C}}$ cooperators are distributed as k weak (and $N_{\mathcal{C}} - k$ strong) individuals with probability $\binom{N_{\mathcal{C}}}{k} p_{\mathcal{S}}^k (1 - p_{\mathcal{S}})^{N_{\mathcal{C}} - k}$. Thus, k of them will follow the Leader (and play his strategy) with probability $p_{F_{\mathcal{W},\mathcal{S}}}$ or play their own strategy (COOPERATE) with probability

$1 - p_{F_{\mathcal{W},\mathcal{S}}}$. The $N_{\mathcal{C}} - k$ other cooperators will follow the Leader (and play his strategy) with probability $p_{F_{\mathcal{S},\mathcal{S}}}$ or play their own strategy with probability $1 - p_{F_{\mathcal{S},\mathcal{S}}}$.

- The $N_{\mathcal{D}} - 1$ remaining defectors are distributed as l weak (and $N_{\mathcal{D}} - 1 - l$ strong) individuals with probability $\binom{N_{\mathcal{D}} - 1}{l} p_{\mathcal{S}}^l (1 - p_{\mathcal{S}})^{N_{\mathcal{D}}-1-l}$. Thus, l of them will follow the Leader (and play his strategy) with probability $p_{F_{\mathcal{W},\mathcal{S}}}$ or play their own strategy (DEFECT) with probability $1 - p_{F_{\mathcal{W},\mathcal{S}}}$. The $N_{\mathcal{D}} - 1 - l$ other defectors will follow the Leader (and play his strategy) with probability $p_{F_{\mathcal{S},\mathcal{S}}}$ or play their own strategy with probability $1 - p_{F_{\mathcal{S},\mathcal{S}}}$.

The process is essentially the same for the other three cases of Leaders.

Hence, we must consider the payoff for each possible distribution of types within the strategies (we use a binomial distribution), then for each distribution of cooperators and defectors within the group (we use an hypergeometric distribution).

3.2.3 Process equations setting

We would recall the following matrix $P_F = (P_{F\,\mathcal{A},\mathcal{B}})_{\mathcal{A},\mathcal{B}}$:

$$P_F = \begin{pmatrix} p_{F_{\mathcal{W},\mathcal{W}}} & p_{F_{\mathcal{W},\mathcal{S}}} \\ p_{F_{\mathcal{S},\mathcal{W}}} & p_{F_{\mathcal{S},\mathcal{S}}} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\exp\left(-\beta_f f\right)} & \frac{1}{1+\exp\left(-\beta_f (f+\Delta f)\right)} \\ \frac{1}{1+\exp\left(-\beta_f (f-\Delta f)\right)} & \frac{1}{1+\exp\left(-\beta_f f\right)} \end{pmatrix}$$

Hence, we define the **benefit for weak leader** $B_{\mathcal{W}}^{\varepsilon}$ and the **benefit for strong leader** $B_{\mathcal{S}}^{\varepsilon}$ as follow :

$$
\begin{aligned}
B_{\mathcal{W}}^{\varepsilon} &= \frac{r}{N} \times \Bigg[\frac{k_{\mathcal{W}}}{N_{\mathcal{W}}} \times \Bigg(1 - \varepsilon + \left(1 - p_{F_{\mathcal{W}, \mathcal{W}}}\right) \times \left(\left(k_{\mathcal{W}} - 1\right) \left(1 - \varepsilon\right) + \left(N_{\mathcal{W}} - k_{\mathcal{W}}\right) \varepsilon \right) \\
&+ \left(1 - p_{F_{\mathcal{S}, \mathcal{W}}}\right) \times \left(k_{\mathcal{S}} \left(1 - \varepsilon\right) + \left(N_{\mathcal{S}} - k_{\mathcal{S}}\right) \varepsilon \right) \\
&+ p_{F_{\mathcal{W}, \mathcal{W}}} \left(N_{\mathcal{W}} - 1\right) \left(\left(1 - \varepsilon\right)^2 + \varepsilon^2 \right) + p_{F_{\mathcal{W}, \mathcal{W}}} \times N_{\mathcal{S}} \left(\left(1 - \varepsilon\right)^2 + \varepsilon^2 \right) \Bigg) \\
&+ \left(1 - \frac{k_{\mathcal{W}}}{N_{\mathcal{W}}}\right) \times \Bigg(\varepsilon + \left(1 - p_{F_{\mathcal{W}, \mathcal{W}}}\right) \times \left(k_{\mathcal{W}} \left(1 - \varepsilon\right) + \left(N_{\mathcal{W}} - k_{\mathcal{W}} - 1\right) \varepsilon \right) \\
&+ \left(1 - p_{F_{\mathcal{S}, \mathcal{W}}}\right) \times \left(k_{\mathcal{S}} \left(1 - \varepsilon\right) + \left(N_{\mathcal{S}} - k_{\mathcal{S}}\right) \varepsilon \right) \\
&+ p_{F_{\mathcal{W}, \mathcal{W}}} \left(N_{\mathcal{W}} - 1\right) \left(2\varepsilon \left(1 - \varepsilon\right)\right) + p_{F_{\mathcal{S}, \mathcal{W}}} N_{\mathcal{S}} \left(2\varepsilon \left(1 - \varepsilon\right)\right) \Bigg) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
B_{\mathcal{S}}^{\varepsilon} &= \frac{r}{N} \times \Bigg[\frac{k_{\mathcal{S}}}{N_{\mathcal{S}}} \times \Bigg(1 - \varepsilon + \left(1 - p_{F_{\mathcal{W}, \mathcal{S}}}\right) \times \left(k_{\mathcal{W}} \left(1 - \varepsilon\right) + \left(N_{\mathcal{W}} - k_{\mathcal{W}}\right) \varepsilon \right) \\
&+ \left(1 - p_{F_{\mathcal{S}, \mathcal{S}}}\right) \times \left(\left(k_{\mathcal{S}} - 1\right) \left(1 - \varepsilon\right) + \left(N_{\mathcal{S}} - k_{\mathcal{S}}\right) \varepsilon \right) \\
&+ p_{F_{\mathcal{W}, \mathcal{S}}} \times N_{\mathcal{W}} \left(\left(1 - \varepsilon\right)^2 + \varepsilon^2 \right) + p_{F_{\mathcal{S}, \mathcal{S}}} \times \left(N_{\mathcal{S}} - 1\right) \left(\left(1 - \varepsilon\right)^2 + \varepsilon^2 \right) \Bigg) \\
&+ \left(1 - \frac{k_{\mathcal{S}}}{N_{\mathcal{S}}}\right) \times \Bigg(\varepsilon + \left(1 - p_{F_{\mathcal{W}, \mathcal{S}}}\right) \times \left(k_{\mathcal{W}} \left(1 - \varepsilon\right) + \left(N_{\mathcal{W}} - k_{\mathcal{W}}\right) \varepsilon \right) \\
&+ \left(1 - p_{F_{\mathcal{S}, \mathcal{S}}}\right) \times \left(k_{\mathcal{S}} \left(1 - \varepsilon\right) + \left(N_{\mathcal{S}} - k_{\mathcal{S}} - 1\right) \varepsilon \right) \\
&+ p_{F_{\mathcal{W}, \mathcal{S}}} N_{\mathcal{W}} \left(2\varepsilon \left(1 - \varepsilon\right)\right) + p_{F_{\mathcal{S}, \mathcal{S}}} \left(N_{\mathcal{S}} - 1\right) \left(2\varepsilon \left(1 - \varepsilon\right)\right) \Bigg) \Bigg]
\end{aligned}
$$

These two similar formulas determine the benefit of the group according to the type of Leader and the number and type of individuals who will cooperate among all individuals in the group, divided into weak and strong.

Then, we define the **benefit of the group** B^{ε} with :

$$
B^{\varepsilon} = \mathbb{E}\left(B_{\bullet}^{\varepsilon}\right) = \frac{N_{\mathcal{W}} f_{\mathcal{W}}}{N_{\mathcal{W}} f_{\mathcal{W}} + N_{\mathcal{S}} f_{\mathcal{S}}} B_{\mathcal{W}}^{\varepsilon} + \frac{N_{\mathcal{S}} f_{\mathcal{S}}}{N_{\mathcal{W}} f_{\mathcal{W}} + N_{\mathcal{S}} f_{\mathcal{S}}} B_{\mathcal{S}}^{\varepsilon}
$$

To have an equation with the cost, we need to define the probabilities to follow considered in this equation $\widetilde{p_{F_{\mathcal{W}}}}$ and $\widetilde{p_{F_{\mathcal{S}}}}$ with :

$$\begin{cases} \widetilde{p_{F_\mathcal{W}}} = \frac{N_\mathcal{W}}{N} p_{\mathcal{W},\mathcal{W}} + \frac{N_\mathcal{S}}{N} p_{\mathcal{W},\mathcal{S}} \\ \widetilde{p_{F_\mathcal{S}}} = \frac{N_\mathcal{W}}{N} p_{\mathcal{S},\mathcal{W}} + \frac{N_\mathcal{S}}{N} p_{\mathcal{S},\mathcal{S}} \end{cases}$$

Now we can write the equation of the cost. With two kinds of individual, strong ($\mathcal{S}$) and weak ($\mathcal{W}$) we would have this equation for the cost $C^{\varepsilon}_{k,i}$:

$$\begin{aligned} C^{\varepsilon}_{k,i} &= \frac{1}{N}\Bigg((1-2\varepsilon)i+\varepsilon\Bigg) + \left(1-\frac{1}{N}\right)\Bigg(\frac{N_w}{N}\Bigg((1-\widetilde{p_{F_\mathcal{W}}})\Bigg((1-2\varepsilon)i+\varepsilon\Bigg) \\ &\quad + \widetilde{p_{F_\mathcal{W}}}\Bigg((1-\varepsilon)\left((1-2\varepsilon)\frac{k-i}{N-1}+\varepsilon\right) + \varepsilon\left((1-2\varepsilon)\frac{N-k-1+i}{N-1}+\varepsilon\right)\Bigg)\Bigg) \\ &\quad + \frac{N_s}{N}\Bigg((1-\widetilde{p_{F_\mathcal{S}}})\Bigg((1-2\varepsilon)i+\varepsilon\Bigg) + \widetilde{p_{F_\mathcal{S}}}\Bigg((1-\varepsilon)\left((1-2\varepsilon)\frac{k-i}{N-1}+\varepsilon\right) \\ &\quad + \varepsilon\left((1-2\varepsilon)\frac{N-k-1+i}{N-1}+\varepsilon\right)\Bigg)\Bigg)\Bigg) \end{aligned}$$

3.2.4 Results

We no longer wished to choose the Leader in a uniform way, but rather to select him according to is strength parameter. Thus, we have introduced a strength indicator δ_{Leader} allowing the choice of the Leader: The probabilities of an Individual to be a Leader depends on its type and is given by a Fermi Distribution. We choosed the convention:

$$\delta_{\text{Leader}} = \Delta f \ (= \texttt{delta})$$

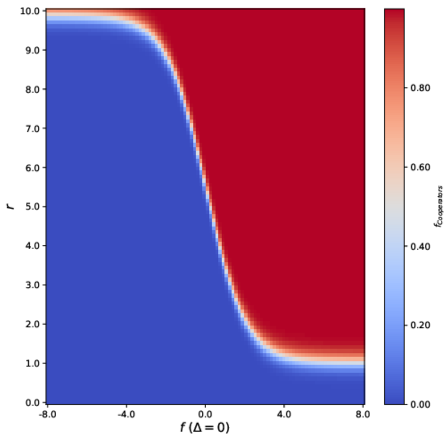
We performed our simulations with the following parameters:

Parameter	Symbol	Value
Population	Z	100
Probability for an Individual to be Weak	$p_\mathcal{S}$	0.5
Group Size	N	9
Error when trying to perform an action	ε	0.01
Mutation coefficient	β	1.0
Inflection parameter in the Fermi distribution	β_f	1.0

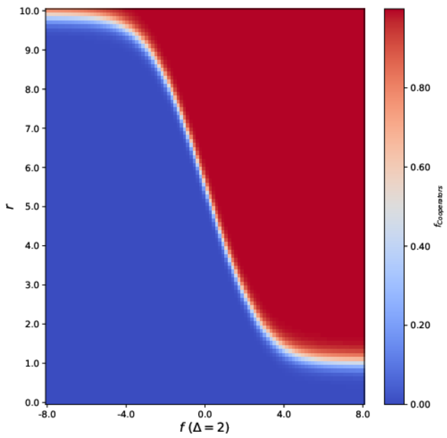
We then varied the following parameters in these ranges:

Parameter	Symbol	Value
Strength parameter	f	$[-8\ ;\ 8]$
Variation of strength	Δf	$[0\ ;\ 10]$
Ratio	r	$[0\ ;\ 10]$

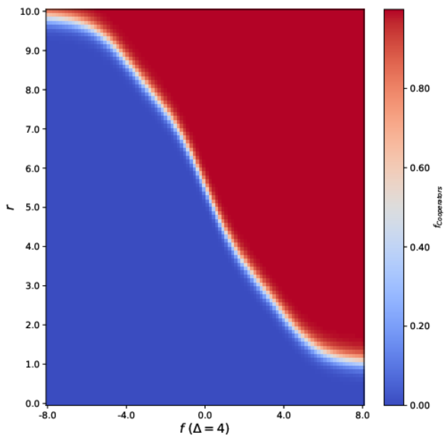
First of all, here are the Heat Maps obtained for different $\Delta f \in [0.0, 2.0, 4.0, 8.0]$.



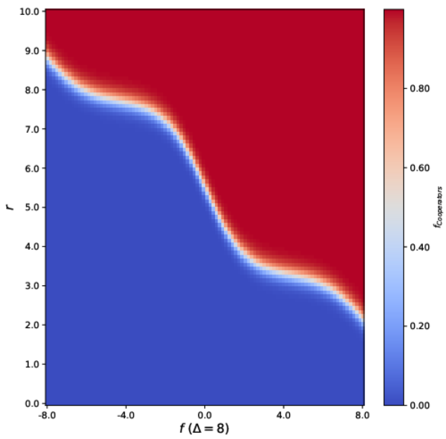
$\Delta f = 0.0$



$\Delta f = 2.0$



$\Delta f = 4.0$

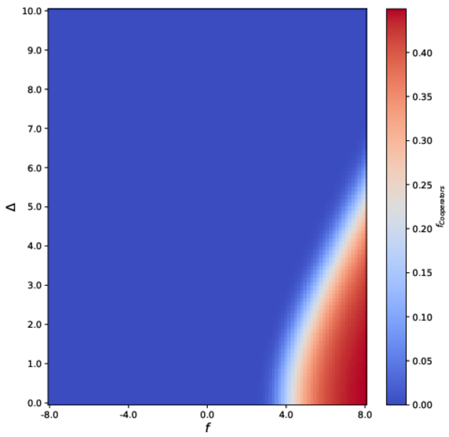


$\Delta f = 8.0$

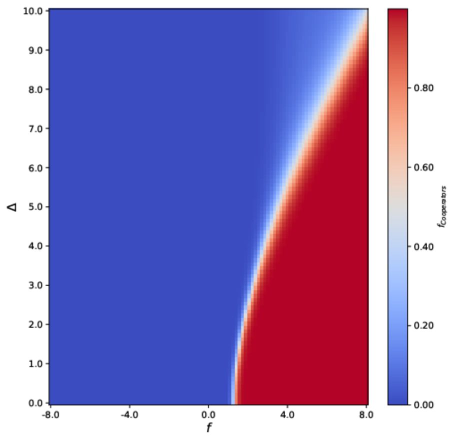
Figure 12: Heat Maps with different Δf

We also plotted heat maps by varying the ratio parameter:

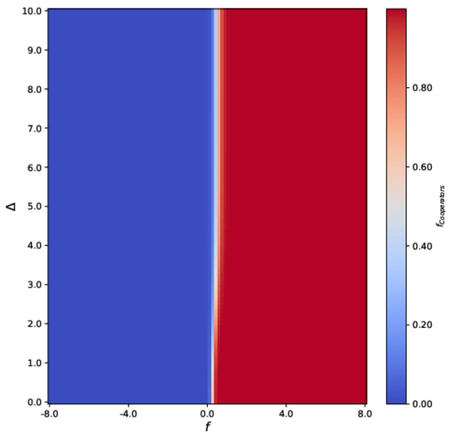
$$r \in [1.0, 3.0, 5.0, 7.0, 8.0, 9.0]$$



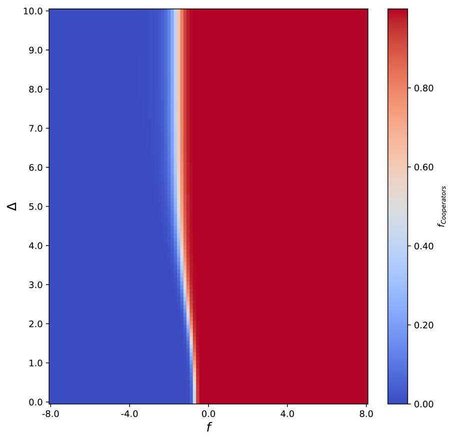
$r = 1.0$



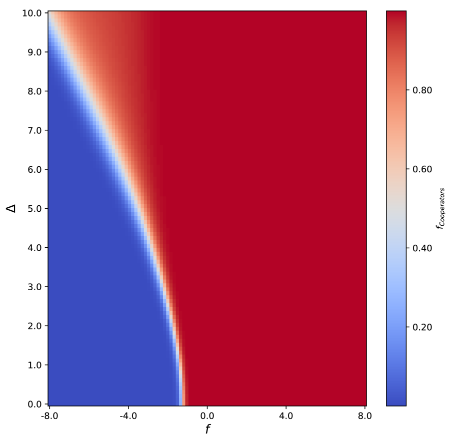
$r = 3.0$



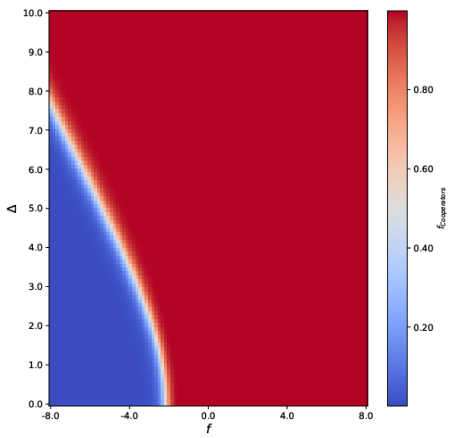
$r = 5.0$



$r = 7.0$



$r = 8.0$



$r = 9.0$

Figure 13: Heat Maps with different r

3.2.5 Analysis of the results

The red zones of the Heat Maps are zones of full cooperation ($f_C = 1$), the blue ones are zones of full defection ($f_C = 0$). The transition zones, going from shades of blue to shades of red, are due to the mutation parameter β and mean that, under these conditions, one strategy does not totally prevail over the other.

First, we check that the results obtained for $\Delta f = 0$ match those of the uniform population (for $\Delta f = 0$, we observe a sigmoid that separates the areas of cooperation and defection. At the probabilities to follow scale, this sigmoid becomes an affine line), which is consistent with what we expected. Thus, Δf may actually be characterized as an indicator of heterogeneity of behaviours in the population.

For low ratios, cooperation emerges in a *participatory democratic society*, which mathematically translates into low values of Δf and high values for f. Thus, we are getting closer to the uniform case (low Δf) and individuals must have a high tendency to follow the Leader for such a phenomenon to emerge (large f).

For large ratios, the observed phenomenon is different. Cooperation appears more frequently, the cases where it does not emerge is the ones of a society close to uniformity (low values for Δf) and where individuals are reluctant to follow the Leader (low values of f).

For intermediate ratios, the influence of Δf is secondary: it is the f factor that will essentially determine the emergence of cooperation. Cooperation is possible when this factor is high enough, i.e. when individuals are inclined to follow the Leader.

Generally, it can be said that **Leadership promotes cooperation** in relation to a Public Good Game without Leadership. The emergence of a single Leader is therefore beneficial to the group. Naturally, individuals must also be inclined to follow the Leader, otherwise its role is useless.

3.3 Game with Threshold

From now on, we would like to transcribe the idea of a game where the group must cooperate sufficiently so that the action can be beneficial. The concept of threshold is therefore the tool introduced. A parameter $M \in \mathbb{R}_+^*$ called threshold must be reached in order for individuals to obtain the expected benefit. $\Theta_M(\cdot)$ is the M-indexed function described in 2.1.3 (null below M and identity above). The payoff function is therefore given by:

$$\begin{cases} \Pi^{\mathcal{C}} &= \Theta_M \left(\sum\limits_{k=0}^{N-1} \mathbb{P}_{\mathcal{C}}(N_{\mathcal{C}} = k+1) r \frac{(k+1)}{N} c \right) - c \\ \Pi^{\mathcal{D}} &= \Theta_M \left(\sum\limits_{k=0}^{N-1} \mathbb{P}_{\mathcal{D}}(N_{\mathcal{C}} = k) r \frac{k}{N} c \right) \end{cases}$$

We still have that $(\forall k \in [0, N-1])\ \ \mathbb{P}_{\mathcal{C}}(N_{\mathcal{C}} = k+1) = \mathbb{P}_{\mathcal{D}}(N_{\mathcal{C}} = k)$

It is usual to introduce a threshold in order to promote cooperation: individuals know that they must be sufficient cooperators to be able to expect a non-zero benefit. However, they know that they risk heavy losses if so few individuals cooperate.

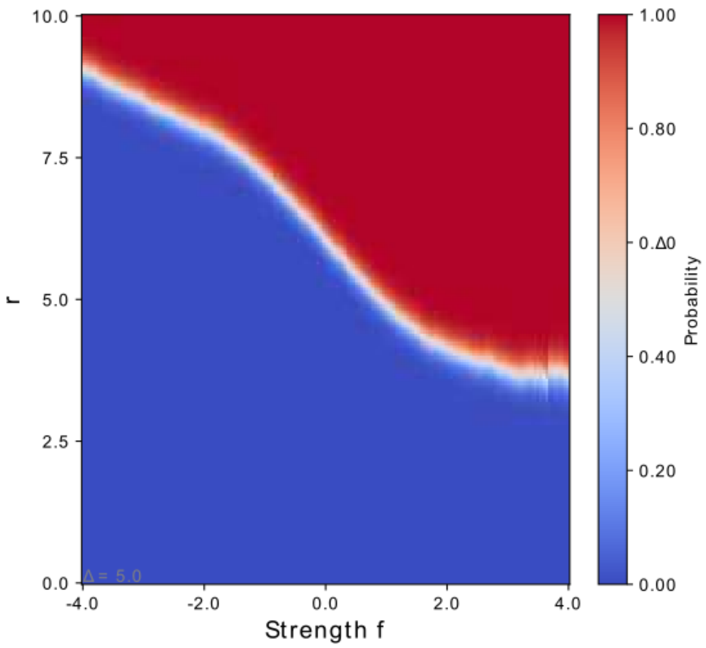


Figure 14: Heat Map with a threshold $M = 6$

The setting of a threshold significantly affects the results obtained. The first observation is that the threshold favours cooperation in cases where individuals are reluctant to follow the Leader (when the f values are low), but that it is hindered when individuals have a strong inclination to follow the Leader. A threshold therefore smoothes the bias between the ability to follow the leader or not, the appearance of a threshold seems almost similar to a decrease in the δ parameter that we chose equal to 1.

In addition, the threshold will affect the linear aspect of the results and relatively neglected parameters such as the mutation parameter β and the fact of making a mistake when performing an action ε will further affect the game.

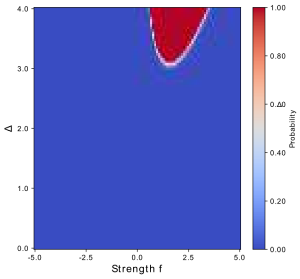
3.4 Risky Leadership & Punishment

The model presented in this subsection is influenced by (7). The main idea is the observation that the Leader of a group is the one who initiates the action. It is therefore in his natural interest that other individuals cooperate with the action. In addition, an individual will also more likely behave as a Leader if it obtains benefit from this situation.

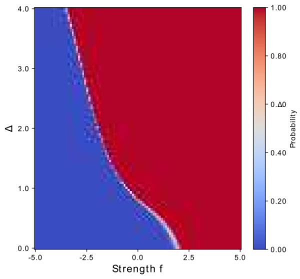
We will therefore introduce a notion of risk. If there are not enough people cooperating and the threshold M is not reached, the Leader will have to repay part of what the cooperators have invested. On the other hand, if the threshold M is reached, the Leader can expect a higher gain than other cooperators, by punishing defectors.

Hence, we introduce two new parameters:

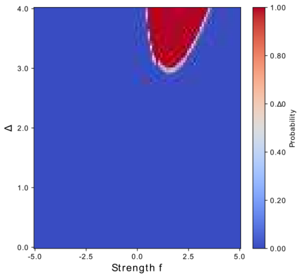
- The punishment for the Leader p_{L} that the Leader has to pay to the other contributors if the threshold is not reached. In this case, the payoff of the Leader will be, if it cooperates, $-c - \frac{N_{\mathcal{C}}-1}{N_{\mathcal{C}}}p_{\mathrm{L}}$ or $-p_{\mathrm{L}}$ if it defects. The payoff for the other Contributors will be $-c + \frac{1}{N_{\mathcal{C}}}p_{\mathrm{L}}$.
- The punishment for Defectors $p_{\mathcal{D}}$. If the threshold M is reached, the payoff for a Defector will be $r\frac{N_{\mathcal{C}}}{N}c - p_{\mathcal{D}}$ and the payoff of the Leader, if it cooperates, will be $r\frac{N_{\mathcal{C}}}{N}c - c + N_{\mathcal{D}} \cdot p_{\mathcal{D}}$ or $\frac{N_{\mathcal{C}}}{N}c - (N_{\mathcal{D}}-1)p_{\mathcal{D}}$ if it defects. The idea is to have : $0 \leq p_{\mathcal{D}} \leq M$.



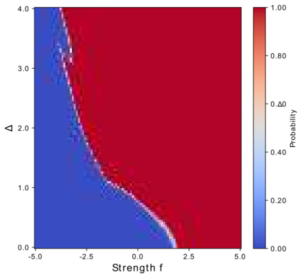
$M = 0\ ;\ r = 4.0$



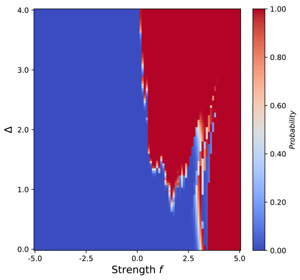
$M = 0\ ;\ r = 6.0$



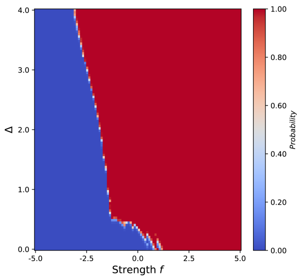
$M = 2\ ;\ r = 4.0$



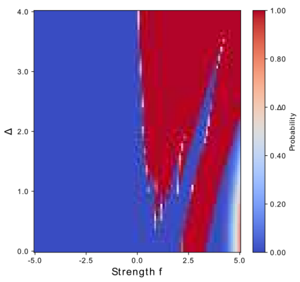
$M = 2\ ;\ r = 6.0$



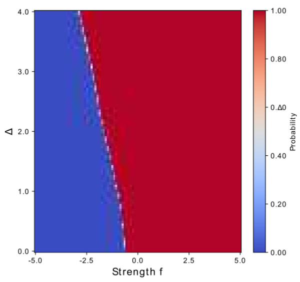
$M = 4\ ;\ r = 4.0$



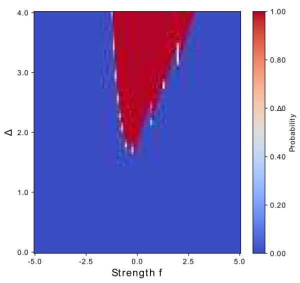
$M = 3\ ;\ r = 6.0$



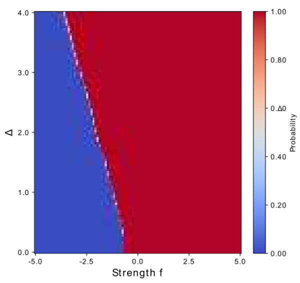
$M = 6\ ;\ r = 4.0$



$M = 4\ ;\ r = 6.0$



$M = 8\ ;\ r = 4.0$



$M = 6\ ;\ r = 6.0$

Figure 15: Heat Maps with different thresholds

The question of how the Leader is chosen remains. In (7), a system of taxes and penalties is implemented and is more complex than the model proposed here (three strategies cohabit). The individuals in the group then choose the most suitable Leader.

Conclusion

Leadership promotes cooperation

The overall conclusion of this report is that Leadership promotes cooperation. The influence of a Leader is therefore beneficial in living structures with the ability to organize around a group dynamic. In other words, the individuals considered must be able to recognize their different social status and identify an individual likely to be a Leader and then choose to follow him (or not).

We can fine-tune our conclusion by noting that the promotion of Leadership is more favored in hierarchical societies (different strengths of influence) than in a uniform society. Our work focused on a two-level hierarchy, i.e. where individuals had behaviors corresponding to a weak or strong type.

Thus, it might be interesting to study the case of a society with a hierarchy at more levels, or even a society where each individual has a specific strength (at least within the group, i.e. N types of strength). Nevertheless, it seems that such a model presents real computational and computational issues.

Lastly, we must note that the presence of a Leader does not drastically change the outcome of the game, even if the role it will have to play is relatively important. We remain essentially close to the Public Good Game without a Leader. Leadership must therefore be seen as a setting that fosters cooperation, an invective to contribute to the Common Good, as long as it does not radically change the outcome of the game (although it substantially affects its process).

Ultimately, an open-ended perspective of the subject may be to consider multiple Leaders, as suggested hereunder.

Multiple Leaders

At the end of this report, we propose a draft model that was not studied during the internship. This is a model with the possibility of several leaders per group. Such a model could echo many cases that can be observed in the living world, for example in (5).

We assume that each individual is I characterized by three parameters:

- Its probability of being a Leader $p_{\mathrm{L}}^{\mathrm{I}} \in [0,\ 1]$;
- Its strength as a Leader (or *Leadership*) $f_{\mathrm{L}}^{I} \in \mathbb{R}_{+}$;
- Its probability of following a Leader $p_{\mathrm{F}}^{\mathrm{I}} \in [0,\ 1]$.

Each individual has one of the two following strategies :

- COOPERATE $\mathcal{C}$;
- DEFECT $\mathcal{D}$.

The process would be the following :

1. Each individual I is a Leader with probability $p_{\mathrm{L}}^{\mathrm{I}}$. A Leader plays its own strategy. There may be none, one or more Leaders.
2. The remaining individuals are Followers with probability $p_{\mathrm{F}}^{\mathrm{I}}$. In the case where there is no Leader, we assume individuals cannot be Followers.
3. An individual who is not Follower is playing its own strategy.
4. A Follower will follow the same strategy as the Leader it is following. In order to determine which Leader the Follower follows, let us name $\mathrm{L}_1, \ldots, \mathrm{L}_K$ the K Leaders. For $i \in [1, \ldots, K]$, a Follower follows the Leader L_i with the following probability :

$$\frac{f_{\mathrm{L}}^{\mathrm{L}_i}}{\sum_{k \in [1,\ldots,K]} f_{\mathrm{L}}^{\mathrm{L}_k}}$$

We believe that we can expect results halfway between the uniform case and the case of a single Leader, because here, Leaders can have opposite strategies and will thus mitigate the exercise of the influence of the other.

List of Figures

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