# Bell's Violation on Photon Added Two-mode Coherent State

**Dissertation submitted to Madurai Kamaraj University in partial fulfillment of the requirement for the degree of**

**Master of Science**
**In**
**Physics**

*Submitted by*


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**June, 2021**

# DECLARATION

I hereby declare that the work presented in this M.Sc., thesis entitled **Bell's Violation of Photon Added Two-mode Coherent State** has been carried out by me and further declare that it has not been submitted earlier in part or in whole to any University or Institution for the award of any Degree or Diploma. The work embodied in this thesis has been carried out at Department of Theoretical Physics, Madurai Kamaraj University(MKU), Madurai under the supervision of **Dr. A. BASHERRUDIN MAHMUD AHMED**, during the Period of 2020-2021. The extent of information derived from the existing literature has been indicated in the body of the thesis at appropriate places giving the source of information.

**Suriyaprasanth S**

Place: Madurai

Date:

# CERTIFICATE

This is to certify that the work reported in this thesis entitled **Bell's Violation of Photon Added Two-mode Coherent State** has been carried out by **Mr. Suriyaprasanth S** in partial fulfilment of the requirement for the award of the degree of **Master of Science (M.Sc.) in Physics** (2019-2021 batch). The work has been carried out in School of Physics at Madurai Kamaraj University, under the guidance of **Dr.A.BASHERRUDIN MAHMUD AHMED**, Department of Theoretical Physics, Madurai Kamaraj University, Madurai - 625021.

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# Contents

# List of Figures

# ABSTRACT

We study the nonclassical features of the chosen coherent state by converting it into a quantum state, adding some number of photons to each mode. This quantum state is called as Superposition of photon added two mode coherent states (PATMCS). The analysis is based on how the state behaves when the photons are added locally and nonlocally. The Nonclassical, Nonlocal and other Entanglement properties were also quantified for the both ways of photon addition, how the photon addition in- fluences the quantum properties of the state. The conclusion is that the state well behaves after photon addition, shown that is a nonclassical, nonlocal, and entangled state which also violates the $B_{chsh}$ inequality.

**Keywords:** Nonlocality, $B_{chsh}$ inequality, Coherent state, Nonclassical, Wigner function

# List of Abbreviations

| | |
|---|---|
| **WF** | **W**igner **F**unction |
| **CS** | **C**oherent **S**tate |
| **B**$_{chsh}$ | **C**lauser **H**orne **S**himony **H**olt |
| **EPR** | **E**instein-**P**odolsky-**R**osen |
| **QS** | **Q**uantum **S**tate |
| **QC** | **Q**uantum **C**orrelations |
| **QIT** | **Q**uantum **I**nformation **T**heory |
| **PATMCS** | **P**hoton-**A**dded **T**wo-**M**ode **C**oherent **S**tates |
| **LPATMCS** | **L**ocal **P**hoton-**A**dded **T**wo-**M**ode **C**oherent **S**tates |
| **NPATMCS** | **N**onlocal **P**hoton-**A**dded **T**wo-**M**ode **C**oherent **S**tates |

# Chapter 1

# Introduction

## 1.1 Quantum Information Theory

Here we transfer the Information through Quantum channels using Quantum information processing techniques. This field is interdisciplinary which involves quantum mechanics, computer science, information theory, philosophy and cryptography and other fields. Here the concept of superposition plays a major role. The applications of QIT are extensively applicable not only in Science but also in the field of Economics and Commerce.

One of the main applications are Quantum Computers which we rely on Computing large data sets, establishing secure connections, Optimization of Numerical algorithms, transferring huge number of visual data's, and Solving Problems which could not be solved by Classical computer.

## 1.2 QIT Prerequisites

### 1.2.1 Superposition

#### Classical View

In classical physics, the concept of **superposition** is described by the two sum of two physical quantities which gives a product which is different from the previous two. The concept of classical superposition principle is better understood when playing with waves which is shown in Figure 1.1.

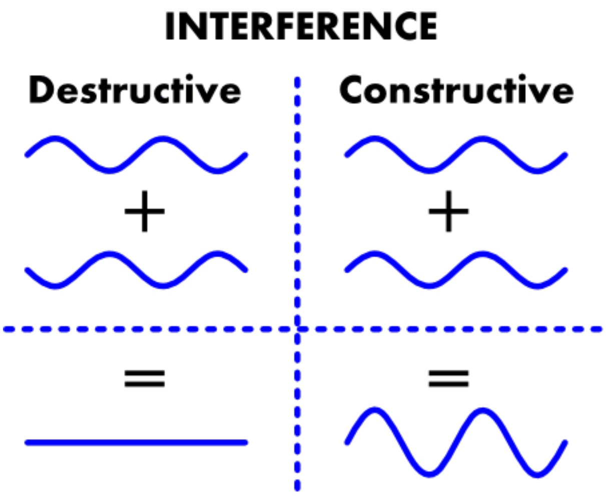


FIGURE 1.1: Examples of constructive and destructive interference due to the classical superposition principle.

Noise-canceling headphones use superposition by creating acoustic wave s with an equivalent magnitude because the incoming sound wave but with a frequency completely out of phase, thereby canceling the acoustic wave . This destructive interference is illustrated in Figure 1.1.

#### Quantum Mechanical View

When we deal with quantum systems,we often refer to quantum superposition. Some of the known quantum systems inclued small objects such as nuclei, electrons, elementary particles, and photons, for which dualnature and other nonclassical features are observed.

In Schrodinger's thought experiment, a cat is placed in a closed box with a radioactive isotope. As radioactive decay is a spontaneous process, it is impossible to predict when the nucleus decays. Hence, we cannot evidently prove that the cat is alive or dead without opening the box.

Now we can be say that the cat is both alive **AND** dead with some probability. That is, the cat is in a state of quantum superposition until you open the box and measure its state. When measured, the cat is either alive **OR** dead and we get one definite answer, said to be in a non-superposition state.

Quantum systems can be in a superposition state, and measuring the state of the system will destroy the superposition state,into a classical state. This might be hard to understand from a classical point of view, as we usually do not see quantum superposition in macroscopic objects. Einstein was really bothered by this feature of quantum systems. His friend, Abraham Pais, records: "I recall that during one walk, Einstein suddenly stopped, turned to me, and asked whether I really believed that the moon exists only when I look at it."

### 1.2.2 Qubit

In classical computers, information is represented as the binary digits 0 or 1, called bits.Every data in classical computers are encoded by bits, such as a e-book, an image, and a video., etc.

Quantum bits or qubits, are similar to bits in that there are two quantifyable states called the 0 and 1 states. qubits can also be in a superposition state of these 0 and 1 states, as shown in Figure 1.2.

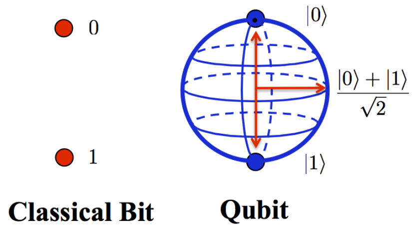


FIGURE 1.2: Classical bit & Qubit

In a classical computers the computations are done on 0 or 1 separately but when its on a quantum computer it could do it as a single operation as they are super imposed and do not require as many trails as a classical computer. This reduces the time spent on computing the problem, and more accurate than a classical machine. As qubit is both 0 and 1 at the same time(**'superposition'**), when measuring we get either 0 or 1.

### 1.2.3 Examples

- Chess-Rice Grain Puzzle
- Infinte Hotel Paradox

#### Chess-Rice Grain Puzzle

Once a King in ancient India wanted a new game, so he summoned his best game creator to make something new for him.

The man took some ample time and invented chess game. In order to reward him the king asked the man what he needs?. The Inventor's answer would shock you, As there are 64 squares in chess board he demanded one grain rice for first square in day 1 and wants to double it for the next day of the next square, and wants to do the same for the consecutive 64 squares as shown in Figure 1.3. At day one he has one grain of rice, at day two he as two grains, when its half a month he has a huge pile of rice. The number of rice grain he has is $64!$ at the end

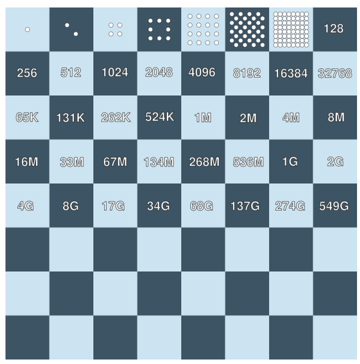


FIGURE 1.3: Chess-Rice Grain

of 64 days. Numbers like these are hard for classical computers to work with. So we need quantum computers compute these huge numbers.

**Infinite Hotel Paradox**

Assume there is an hotel with infinite rooms and infinite guests are arriving any moment, the hotel incharge does-not know which rooms is empty. If we give this problem to a classical computer, it opens each rooms and checks whether the room is empty(which is 0) in a full stretch or occupied(which is 1) in a full stretch.This would take ages to compute if its given to a classical computer. But in a quantum computer the computation could reduce to polynomial time.


FIGURE 1.4: Hilbert's hotel

Although it has been proved mathematically that Hilbert Hotel Paradox is finite, The hotel has rooms that are equal to countable infinity and at a certain case that it would be out of rooms for upcoming guests. But this is a different story out of the context.

## 1.3 Mathematical Representation of Qubit

### 1.3.1 Bloch Representation of a Qubit

The definition for a qubit has not changed lately, but people have named it's representation in lot of ways. Some call it as a Bloch Sphere, Poincare Sphere, Riemann Sphere. You can call it as you like depending on the analogy. I would rather stick with Bloch Sphere as it makes much sense in Visualizing. A single qubit can also be represented using an eigenspinor of a Pauli Matrix for a direction specified by its $\theta$ & $\phi$ in spherical coordinates direction.[3]

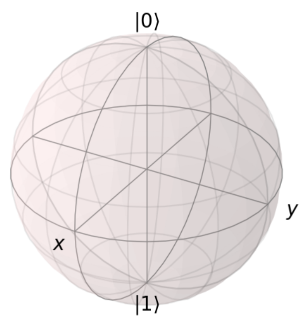


FIGURE 1.5: Qubit as a Bloch Sphere

The North and South poles of the sphere are chosen to corresponding basis vectors $|0\rangle$ and $|1\rangle$. In other words, it is said to be spin up and spin down states of an electron. This how a qubit is treated in the realm of quantum computing because of its unique property of superposition. The Figure shows an empty bloch sphere on the basis of $|0\rangle$ and $|1\rangle$.

### 1.3.2 Dirac bra-ket notation

In order to work with qubits, it is useful to know how one can express quantum mechanical states with mathematical formulas. Dirac or "bra-ket" notation is commonly used in quantum mechanics and quantum computing. A qubit $|\psi\rangle$, could be in a $|0\rangle$ or $|1\rangle$ state which is a superposition of both $|0\rangle$ and $|1\rangle$. This is written as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \tag{1.1}$$

with $\alpha$ and $\beta$ called the amplitudes of the states. Amplitudes are generally complex numbers (a special type of number used in mathematics and physics). However, to understand the meaning of amplitudes, we can just imagine the amplitudes as being ordinary (real) numbers. Amplitudes are used to represent all of the possible superpositions mathematically.

$$|cat\rangle = \alpha\,|alive\rangle + \beta\,|dead\rangle\,, \tag{1.2}$$

Amplitudes are very important because they tell us the probability of finding the particle in that specific state when performing a measurement. The probability of measuring the particle in state $|0\rangle$ is $|\alpha|^2$, and the probability of measuring the particle in state$|1\rangle$ is $|\beta|^2$. Since the total probability of observing all the states of the quantum system must add up to $100\%$, the amplitudes must follow this rule:

$$|\alpha|^2 + |\beta|^2 = 1, \tag{1.3}$$

This is called a normalization rule. The coefficients $\alpha$ and $\beta$ can always be rescaled by some factor to normalize the quantum state.

### 1.3.3 Matrix Respresentation

When writing one qubit in a superposition $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, it is useful to use matrix algebra. In matrix representation, a qubit is written as a two-dimensional vector where the amplitudes are the components of the vector:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{1.4}$$

The states are $|0\rangle$ and $|1\rangle$ are usually represented as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.5}$$

## 1.4 Quantum Entanglement

This is a quantum mechanical phenomenon in which the quantum states of two or more particles has to be described with reference to each other, even though the individual objects may be spatially separated to any extent.

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through $n$, and system number $i$ is prepared in the state $|\psi_i\rangle$ then the joint state of the total system is

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle, \tag{1.6}$$

Entanglement is a uniquely quantum Mechanical resource that plays a key role in many of the most interesting applications of the quantum computation and quantum information; Entanglement is the iron to the classical world's bronze age.

### 1.4.1 Quantum Computation

The Quantum computations are can only be performed by the quantum computers. Qubits and quantum entanglement are the fundamental building blocks of quantum computer. A qubit can have 'N' States , Using the principle of superposistion of differnt $2^N$ states of binary numbers can be obtained.

By simultaneously superimposing the states, these states gets operated by Unitary Transformations. The entanglement property is the key for researchers to develop and design new applications in the field of QIT. The gateway to perform fast computation is superposition and entanglement, by this we can reduce the runtime of algorithms.The realm of quantum computation will replace the Classical computers in near future.

### 1.4.2 Quantum Optics

Quantum optics is a field of research that deals with the application of quantum mechanics to phenomena involving light and its interactions with matter. One of the main goals is to understand the quantum nature of information and to learn how to formulate, manipulate, and process it using physical systems that operate on quantum mechanical principles. Applied Quantum Optics is has wide areas of research such as Interferometry, Neutrino detection and Quantum Metrology., etc.

## 1.5 Mathematical Tools

Let us discuss the some of the mathematical tools that were used in this dissertation.

### 1.5.1 Coherent State

A coherent state is a quantum state with minimum uncertainty which is similar to the classical particle in the phase space. In other words, number state's superposistion or anhilation operator's eigen states $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$. The Coherent state is described by

$$|\alpha\rangle = e^{-|\alpha|/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\,, \tag{1.7}$$

$|n\rangle$ is the number state corresponds to the $|0\rangle\,, |1\rangle\,, |2\rangle$ and so on. The completeness relation is given by

$$\frac{1}{\pi} \int |\alpha\rangle \langle\alpha|\, d^2\alpha = 1^{\mathrm{a}} \tag{1.8}$$

$\alpha$ is a complex number and $|\alpha|$ is the coherent amplitude that tells the strength of photon in an optical source. The canonical coherent states are

$$\Delta x \Delta p = \frac{\hbar}{\sqrt{2}}, \tag{1.9}$$

The applications of coherent states are not only limited, but also applicable in Laser Physics and Mathematical Physics etc.

### 1.5.2 Laguerre polynomials

Laguerre polynomial of order *n* is defined as

$$L_n(z) = \sum_{a=0}^{n} \frac{n!(-z)^a}{(a!)(n-a)!} \tag{1.10}$$

From this we can write down the first few Laguerre polynomials $L_0(z) = 1$,$L_1(z) = -x + 1$,,$L_2(z) = (1/2!)x^2 - 4x + 2$,,$L_3(z) = -(1/3!)x^3 + 9x^2 - 18x + 6$. With the above mathematical tools we will proceed to the next chapter.

[a]See Appendix A

# Chapter 2

# Nonclassicality and Nonlocality

## 2.1 Nonclassical Properties

### 2.1.1 Q - Function

The Husimi distribution function is also called the Q-function. This is a similar distribution function like wigner function. It is a positive, semi-definite, semi-classical distribution function. The distribution function which helps in determining the anti-normally ordered correlation functions is the Q-Representation.

Q-function is the expectation value of the density operator $\rho$ in coherent state. It is defined as

$$Q(\gamma_1, \gamma_2) = \frac{1}{\pi^2} \langle \gamma_1, \gamma_2 | \rho | \gamma_1, \gamma_2 \rangle \tag{2.1}$$

Where $\rho = |\psi\rangle \langle\psi|$ and $\rho$ refers to the density operator of the two mode state and $|\gamma_1, \gamma_2\rangle = |\gamma_1\rangle |\gamma_2\rangle$. $Q(\gamma_1, \gamma_2)$ is proportional to the diagonal element of the density operator in the coherent state representation.

### 2.1.2 Mandel Q parameter

Mandel Q parameter is used to find the divergence of the Photon Number Statistics from Poissonian Statistics

$$(Q)_{Mandel} = \frac{\langle (\Delta\hat{n})^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle}; \quad (\Delta\hat{n})^2 = \left\langle \hat{n}^2 \right\rangle - \langle \hat{n} \rangle^2 \tag{2.2}$$

$$(Q)_{Mandel} = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle} - 1 \tag{2.3}$$

### 2.1.3 The Wigner-Weyl distribution

The WF is a quasiprobability distribution or often called as quantum phase space distribution. Earlier it was used as a measure to quantify coherence in optical fields. This WF plays a major role in the measurement of nonlocality of a quantum state.

Not all nonclassical states exhibit negative WF, but if WF's are negative for a state then its nonclassical.

$$W(\gamma_1, \gamma_2) = \frac{4N}{\pi^4} e^{2\left(|\gamma_1|^2+|\gamma_2|^2\right)} \int d^2 z_1 d^2 z_2 \\ \langle -z_1 - z_2|\rho|z_1 z_2\rangle \, e^{2\left(\gamma_1 z_1^* - \gamma_1^* z_1\right)} e^{2\left(\gamma_2 z_2^* - \gamma_2^* z_2\right)}, \tag{2.4}$$

Here $\rho = |\psi\rangle \langle\psi|$ which is the density operator. The Properties of density matrix are

1. $\rho$ is Hermitian, So the eigen values are real and eigenvectors are orthogonal
2. It is idempotent $\rho = \rho^2$
3. The $tr(\rho^2) = 1$

The CS are non-orthogonal and their overlap of the state $|\alpha\rangle$ and $|z\rangle$ is defined,

$$\langle\alpha|z\rangle = e^{-\frac{|z|^2}{2} - \frac{|\alpha|^2}{2} - \alpha^* z}, \tag{2.5}$$

The Properties of WF are

- $\int W(x,p)dp = |\psi(x)|^2$,
- $\int W(x,p)dx = |\psi(p)|^2$,
- $\int \int W(x,p)dxdp = 1$

this is obtained using completeness equation. WF is also very much suitable for explaining the property of entanglement.

## 2.2 Entanglement & Nonlocality

### 2.2.1 Einstein-Podolski-Rosen Paradox

The key points of this theory are

1. According to quantum mechanics, until the time of measurement, the particles do not have a definite quantum spin but are in a state of superposition.
2. When we measure the spin of particle A, We could be sure what particle B would yield.

3. This two points are valid even though these particles are spatially separated over any distance of the universe.

Einstein called this phenomenon as "**Sphuckhafte Ferwirklung** - Spooky action at a distance". He was also trying to save his Theory of Relativity, so he said there is something hidden by this action which lead to Hidden Variable Theory. Einstein could not accept the instantaneous nature of particles. He still argued that the quantum mechanics is incomplete.

### 2.2.2 John Stewart Bell- "The saviour of Quantum Physics"

John Bell was the first person to find a way to experiment EPR's paradox, followed by John Clauser who tested the paper.

Now let's setup a scenario, Charlie Prepares two particles. The way he prepares the particles doesn't matter here. Once after preparing the particles he sends one particle to Alice and another particle to Bob. After receiving the particle Alice does her measurement, remember she only has two types of Basis for Measurements. Alice chose measure the particle on one basis which was determines by using a coin flip, So did Bob. The Alice's measurements are by Physical properties which we label as $P_Q$ and $P_R$.

Bob's Measurements are by Physical Properties which we label as $P_S$ and $P_T$. Consider that measurements can have one of the outcomes $\pm 1$ for Alice and Bob. Experiment is setup in a way that Alice and Bob Measures the experiment in same instant. So that Alice's measurement cannot disturb Bob's and Vice versa, because physical influences cannot propagate faster than the speed of light. The Experimental setup is shown in Figure 2.1. These probabilities depend on how particles are prepared and also depend on the noise,

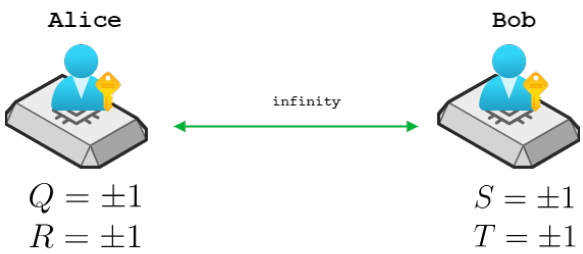


FIGURE 2.1: EPR Experimental setup

$$\mathbb{E}(QS) + \mathbb{E}(RS) + \mathbb{E}(RT) - \mathbb{E}(QT) \leq 2 \tag{2.6}$$

This equation (2.7) is called the *CHSH inequality* it belongs to a larger set called Bell's Inequality. Charlie prepares a quantum system of two qubit state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{2.7}$$

He Passes the first qubit to Alice and Second qubit to Bob.They perform Measurements of the following observables:

$$Q = Z_1 \quad S = \frac{-Z_2 - X_2}{\sqrt{2}} \tag{2.8}$$

$$R = X_1 \quad T = \frac{Z_2 - X_2}{\sqrt{2}} \tag{2.9}$$

Average values of these observables are written in Quantum Mechanical $\langle . \rangle$ notaion are;

$$\langle QS \rangle = \frac{1}{\sqrt{2}}; \langle RS \rangle = \frac{1}{\sqrt{2}}; \langle RT \rangle = \frac{1}{\sqrt{2}}; \langle QT \rangle = \frac{1}{\sqrt{2}} \tag{2.10}$$

Thus,

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2} \tag{2.11}$$

The key points of Bell's inequality are

1. The assumption that the physical properties $P_{Q,R,S,T}$
   have definite values Q,R,S,T which exist independent of observation.

2. The idea is coined as assumption of realism.

3. The assumption that Alice performing her measurement does not influence the result of Bob's measurement. This is sometimes known as the assumption of *locality*

But Bell's inequality disproved these assumption of local realism.

Bell's inequality in terms of WF is written as follows

$$B_{CHSH} = W(\chi_1, \chi_2) + W(\chi_1^{'}, \chi_2) + W(\chi_1, \chi_2^{'}) - W(\chi_1^{'}, \chi_2^{'}) \tag{2.12}$$

The maximum value of Bell's inequality is $2\sqrt{2}$, while its range is $-2 \leq B_{chsh} \leq 2$.

$$|B| \leq 2\sqrt{2} \tag{2.13}$$

### 2.2.3 EPR Correlation

According to EPR any inseparable two mode states must have total variance $< 2$. This is also one of the criteria to check the nonlocality.

$$EPR = \left\langle \Delta^2 X \right\rangle + \left\langle \Delta^2 P \right\rangle \ll 2 \tag{2.14}$$

### 2.2.4 Nonlocality

Before diving into Nonlocality, we need to know what is locality. Locality describes that a particle can only be influenced by its immediate surroundings. When a quantum state disobeys the Bell's inequality, that is said to be Bell's Violation or Bell's Nonlocality.

$$B_{CHSH} = W(\chi_1, \chi_2) + W(\chi_1^{'}, \chi_2) + W(\chi_1, \chi_2^{'}) - W(\chi_1^{'}, \chi_2^{'}) \tag{2.15}$$

$$|B| \geq 2\sqrt{2} \tag{2.16}$$

When a state is said to be nonlocal the WF should be strictly negative. Bell's inequality should be greater than $2\sqrt{2}$

If a Quantum State exhibits all the above properties, then the state is a perfect Nonlocal state.

# Chapter 3

# Local and Nonlocal PATMCS

## 3.1 Local-PATMCS

The Local PATMCS is defined as

$$|\phi\rangle = \frac{1}{\sqrt{N}}(a_1^{\dagger m} + a_2^{\dagger m})\,|\alpha_1, \alpha_2\rangle \tag{3.1}$$

for our convinience we rewrite as

$$|\phi\rangle = \frac{1}{\sqrt{N}}(a_1^{\dagger m}\,|\alpha_1\rangle\,|\alpha_2\rangle + a_2^{\dagger m}\,|\alpha_1\rangle\,|\alpha_2\rangle) \tag{3.2}$$

$$\langle\phi| = \frac{1}{\sqrt{N}}(\hat{a_1}^m\,\langle\alpha_1|\,\langle\alpha_2| + \hat{a_2}^m\,\langle\alpha_1|\,\langle\alpha_2|) \tag{3.3}$$

The CS $|\alpha_1, \alpha_2\rangle \equiv |\alpha_1\rangle\,|\alpha_2\rangle$. The state is also in a super posistion of $a_1^{\dagger m}\,|\alpha_1\rangle\,|\alpha_2\rangle$ and $a_2^{\dagger m}\,|\alpha_1\rangle\,|\alpha_2\rangle$.

The Normalization $N$ is found by $|N|^2\,\langle\phi|\phi\rangle = 1$; which we obtained is

$$\begin{aligned} N = m!(L_m(-|\alpha_1|^2) + L_m(-|\alpha_2|^2)) \\ +(\alpha_1\alpha_2^*)^m + (\alpha_1^*\alpha_2)^m \end{aligned} \tag{3.4}$$

## 3.2 Nonlocal-PATMCS

The Nonlocal PATMCS is defined as

$$|\psi\rangle = \frac{1}{\sqrt{N}}(a^{\dagger m} + b^{\dagger n})\,|\alpha, \beta\rangle \tag{3.5}$$

for our convinience we rewrite as

$$|\psi\rangle = \frac{1}{\sqrt{N}}(a^{\dagger m}|\alpha\rangle|\beta\rangle + b^{\dagger n}|\alpha\rangle|\beta\rangle) \tag{3.6}$$

$$\langle\psi| = \frac{1}{\sqrt{N}}(\hat{a}^{m}\langle\alpha|\langle\beta| + \hat{b}^{n}\langle\alpha|\langle\beta|) \tag{3.7}$$

The CS $|\alpha,\beta\rangle \equiv |\alpha\rangle|\beta\rangle$. The state is also in a super position of $a^{\dagger m}|\alpha\rangle|\beta\rangle$ and $b^{\dagger n}|\alpha\rangle|\beta\rangle$.

The Normalization $N$ is found by $|N|^2\langle\psi|\psi\rangle = 1$; which we obtained is

$$\begin{aligned} N = m!L_m(-|\alpha|^2) + n!L_n(-|\beta|^2) \\ +(\alpha^{*m}\beta^{n}) + (\beta^{*n}\alpha^{m}) \end{aligned} \tag{3.8}$$

My Prespective is that the both states that are described above are the same, what matters is the photon addition parameter. Local Photon addition means if the photon number has same order for all the modes in the state. Nonlocal Photon addition means if the photon number has unequal order for all the modes in the state. I have picturized it below,

# Chapter 4

# Nonclassical properties

On this chapter let us compare the Nonclassical featuers of both the states which we have seen it in the above chapter.

## 4.1 Nonclassicality

### 4.1.1 Q-Function

$$Q(\beta_A, \beta_B) = \frac{N_m^{-1}}{\pi^2} \langle \beta_A, \beta_B | \, \rho \, | \beta_A, \beta_B \rangle \tag{4.1}$$

From the figure 4.1 we can clearly see that the Q function decreases with increase in Photon added parameter m. For the values of $\alpha = e^{i\pi/3}$ and $\beta_B = 1/2$. The figure is plotted for different values of $m = 0, 1, 2, 3, 4, 5$.
Here, $\rho$ is the density operator of the two mode CS $|\beta_A, \beta_B\rangle \equiv |\beta_A\rangle \, |\beta_B\rangle$.

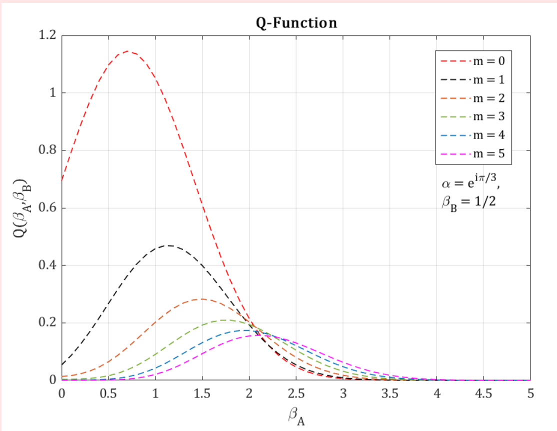


FIGURE 4.1: Q function for LPATMCS

$$Q(\chi_A, \chi_B) = \frac{N^{-1}}{\pi^2} \langle \chi_A, \chi_B | \rho | \chi_A, \chi_B \rangle \tag{4.2}$$

From the figure 4.2 we can clearly see that the Q function decreases with increase in Photon added parameter m. For the values of $\alpha = e^{i\pi/3}$ and $\chi_B = 1/2$. The figure is plotted for different values of m and n (nonlocal photon addition).

Here, $\rho$ is the density operator of the two mode CS $|\chi_A, \chi_B\rangle \equiv |\chi_A\rangle |\chi_B\rangle$.

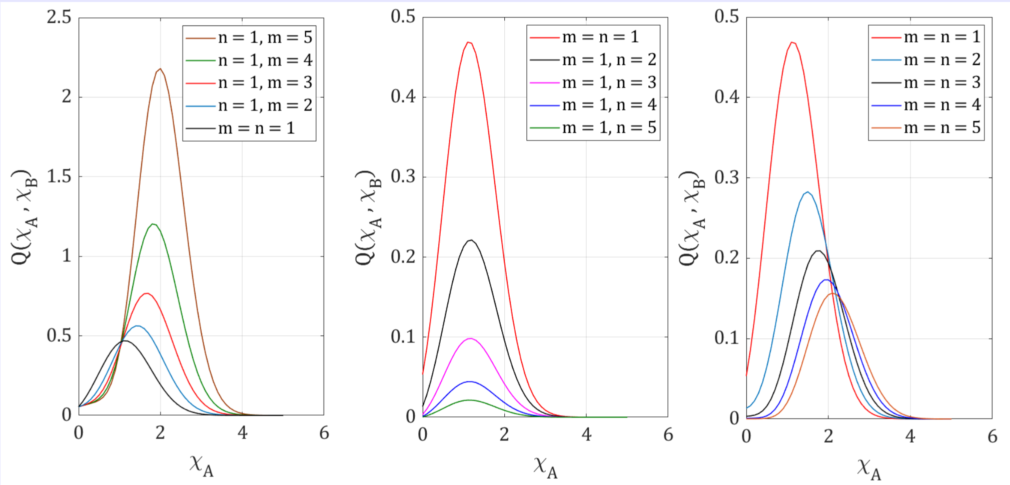


FIGURE 4.2: Q function for NPATMCS

On comparing the figures 4.1 and 4.2 it can be clearly seen that the Nonlocal-PATMCS is very much positive side,this is an evidence that shows the influence of photon addition parameter on the state. However when you increase the value of a mode A $> 2$ in both of the states, the distribution gradually becomes zero. This holds true for any order of photons irresopective of the state.

### 4.1.2 Mandel Q Parameter

We use mandel parameter to determine the deviation of PND from poissonian statistics. The Mandel parameter in antinormal order is defined as

$$Q_{a_i} = \frac{\left\langle \hat{a}_i^2 a_i^{\dagger 2} \right\rangle - \left\langle \hat{a}_i a_i^{\dagger} \right\rangle^2 - \left\langle \hat{a}_i a_i^{\dagger} \right\rangle}{\left\langle \hat{a}_i a_i^{\dagger} \right\rangle - 1} - 1; \quad i = 1, 2 \tag{4.3}$$

$$Q_{a_1} = \frac{\left\langle \hat{a}_1^2 a_1^{\dagger 2} \right\rangle - \left\langle \hat{a}_1 a_1^{\dagger} \right\rangle^2 - \left\langle \hat{a}_1 a_1^{\dagger} \right\rangle}{\left\langle \hat{a}_1 a_1^{\dagger} \right\rangle - 1} - 1; \tag{4.4}$$

$$Q_{a_2} = \frac{\left\langle \hat{a}_2^2 a_2^{\dagger 2} \right\rangle - \left\langle \hat{a}_2 a_2^{\dagger} \right\rangle^2 - \left\langle \hat{a}_2 a_2^{\dagger} \right\rangle}{\left\langle \hat{a}_2 a_2^{\dagger} \right\rangle - 1} - 1; \tag{4.5}$$

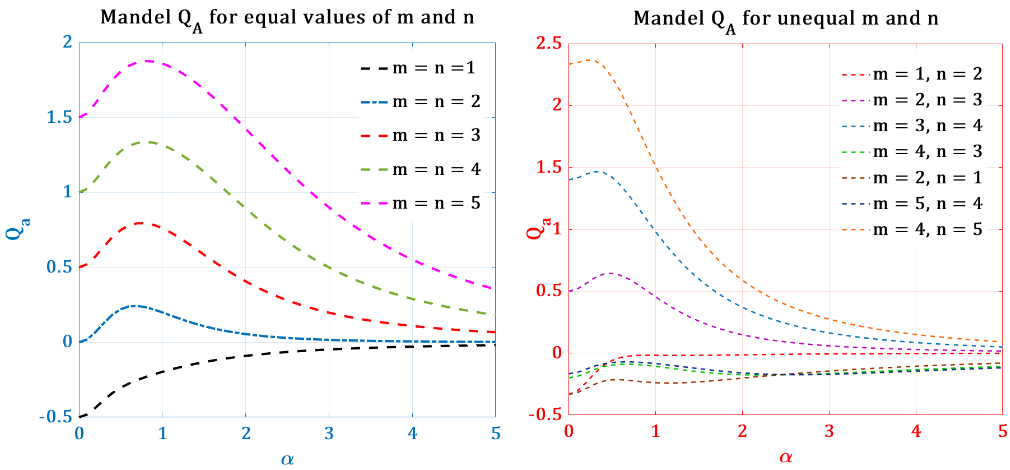


FIGURE 4.3: Mandel Q parameter plot for Nonlocal PATMCS

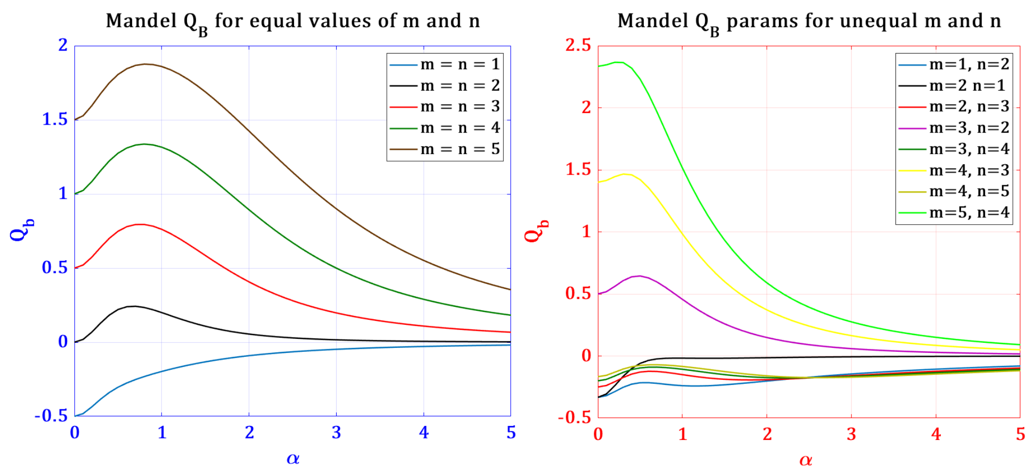


FIGURE 4.4: Mandel Q parameter plot for Nonlocal PATMCS

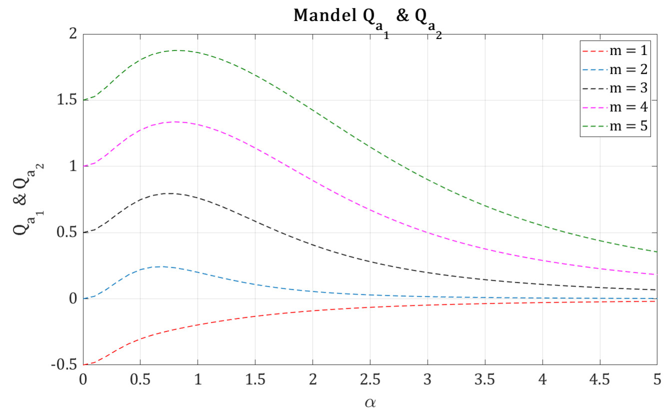


FIGURE 4.5: Mandel Q parameter plot for Local PATMCS

### 4.1.3 Wigner Function

Let us see the behaviour of the local photon addition on a quantum state. Just picking some ingredients (3.2), (3.3), (3.4) and adding flavours into the dish (A.5). We have obtained

$$\begin{aligned} W(\gamma_1,\gamma_2) = (-1)^m \frac{4}{\pi^2} N_m^{-1} e^{-2|\gamma_1-\alpha_1|^2-2|\gamma_2-\alpha_2|^2} \\ [m! L_m(|2\gamma_1-\alpha_1|^2)+L_m(|2\gamma_2-\alpha_2|^2)]+ \\ (\alpha_1^*-2\gamma_1^*)^m(2\gamma_2-\alpha_2)^m+(2\gamma_2^*-\alpha_2^*)^m(\alpha_1-2\gamma_1)^m \end{aligned} \tag{4.6}$$

from the above equation we can say that the equation is negative. My statement could also be false, we now plot the equation over phase space to check the statement.

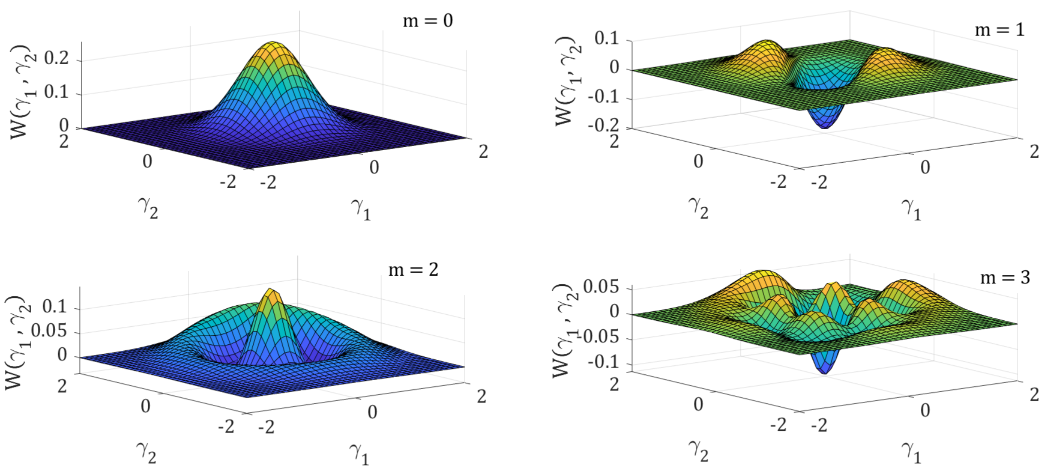


FIGURE 4.6: Wigner function for m = 0,1,2,3 by setting the values of $\alpha_1 = (1/2)e^{i\pi/3}$&$\alpha_2 = (1/4)e^{i\pi/4}$

See when the photon number m is 0 the equation exhibits gaussian distribution, which is true as we know that when there are no photons added, the state is a pure classical state. When the photon number m is 1. we could see that the equation shows some negativity, this indication is enough to prove this state as a nonclassical one. But we cannot be sure so we tend to increase the photon addition to 2 and 3 as shown in the figure 4.6, you can see that when m is 3, it still goes negative. From the previous interpretations made, it can be clearly seen that this state is a nonclassical. There are also other resource theoretical based measures to interpret how much nonclassical a particluar state is, but we wont be digging much about these. Let's discuss how the state behaves after nonlocal photon addition.

Substituting the eqn(A.3),(A.4) in (A.5) we get

$$\begin{aligned} W(\chi_1,\chi_2) &= \frac{4N^{-1}}{\pi^2} e^{-2|\chi_1-\alpha|^2-2|\chi_2-\beta|^2} \\ &[(-1)^m(m!L_m(|2\chi_1-\alpha|^2)+(1)^n(2\chi_1-\alpha)^m(2\chi_2^*-\beta^*)^n) \\ &+(-1)^n(n!L_n(|2\chi_2-\beta|^2)+(1)^m(2\chi_1^*-\alpha^*)^m(2\chi_2-\beta)^n)] \end{aligned} \tag{4.7}$$

From the figure 4.7 it is evidently true that shown below that NPATMCS behaves like LPATMCS by just setting the photon numbers parameter equal.

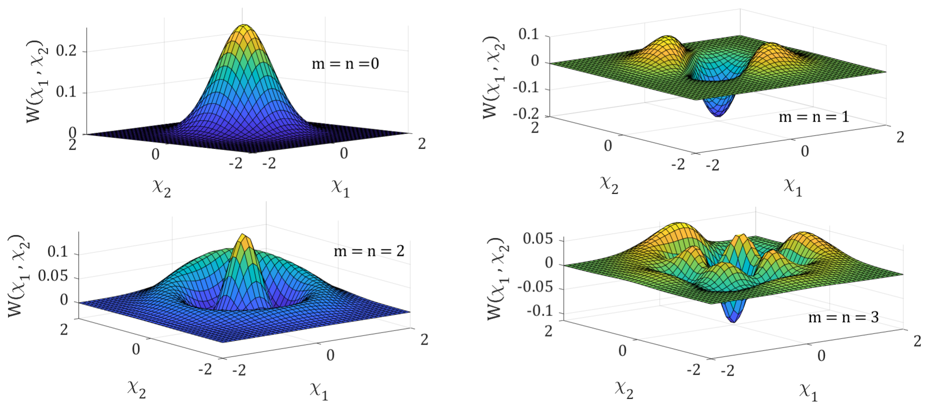


FIGURE 4.7: shows the plot for WF with Photon numbers m and n with equal values by setting the values of $\alpha = (1/2)e^{i\pi/3}$&$\beta = (1/4)e^{i\pi/4}$

Now we set any of two parameters to 0 and then vary the other photon number. In the figure 4.8 we have set m as 0 and ploting for any values of n we see a little nonclassicality when n is set to 3.

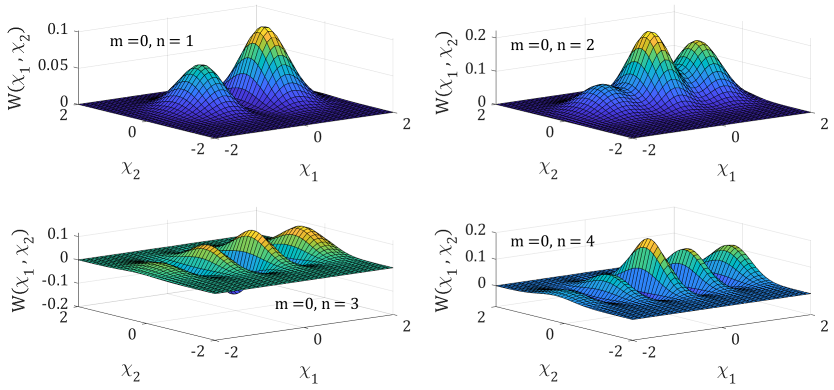


FIGURE 4.8: shows the plot for WF with setting Photon numbers m and n as n = 0 and m = 1,2,3,4 the values of $\alpha = (1/2)e^{i\pi/3}$&$\beta = (1/4)e^{i\pi/4}$

In the figure 4.9 we have set n as 0 and ploting for any values of n we see a little nonclassicality when m is set to 3.

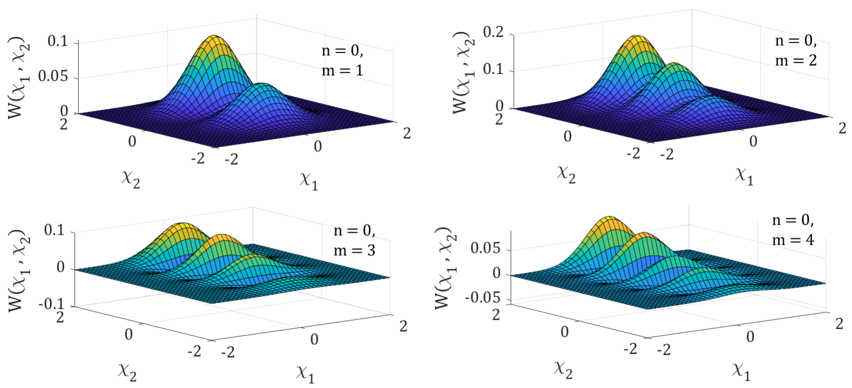


FIGURE 4.9: shows the plot for WF with setting Photon numbers m and n as m = 0 and n = 1,2,3,4 the values of $\alpha = (1/2)e^{i\pi/3}$&$\beta = (1/4)e^{i\pi/4}$

# Chapter 5

# Nonlocal properties

### 5.0.1 Bell's Inequality on Local PATMCS

The plot for equation (A.88) is shown in figure (5.1) setting the $\gamma_1$and $\gamma_2 = 0$ , photon number $m = 1$ and $\alpha_1 = \alpha_2 = 0.1$. It is clearly seen that the plot is above 2 so the state is nonlocal.

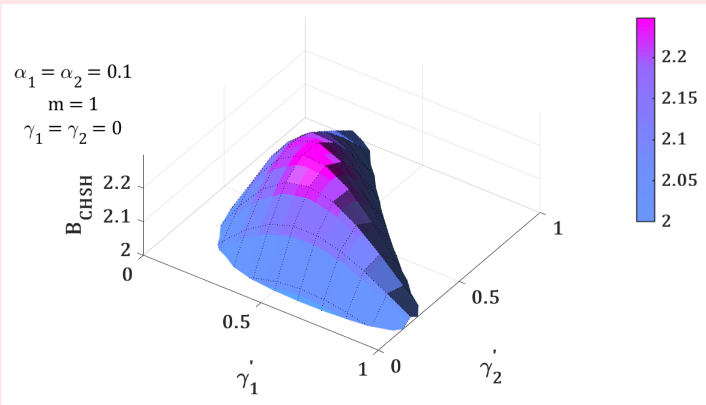


FIGURE 5.1: Bell CHSH inequality plot for Local PATMCS

$$|B| = \frac{\pi^2}{4}\left|W(\gamma_1, \gamma_2) + W(\gamma_1', \gamma_2) + W(\gamma_1, \gamma_2') - W(\gamma_1', \gamma_2')\right| \geq 2 \tag{5.1}$$

Here we set the variables $\gamma_1 \& \gamma_2 = 0$ and plot the equation with respect to $\gamma_1^{'} \& \gamma_2^{'}$.

$$\begin{aligned} W(\gamma_1, \gamma_2) = (-1)^m \frac{4}{\pi^2} N_m^{-1} e^{-2|\gamma_1 - \alpha_1|^2 - 2|\gamma_2 - \alpha_2|^2} \\ [m! L_m(|2\gamma_1 - \alpha_1|^2) + L_m(|2\gamma_2 - \alpha_2|^2)] + \\ (\alpha_1^* - 2\gamma_1^*)^m (2\gamma_2 - \alpha_2)^m + (2\gamma_2^* - \alpha_2^*)^m (\alpha_1 - 2\gamma_1)^m \end{aligned} \tag{5.2}$$

$$
\begin{aligned}
W(\gamma_1^{`}, \gamma_2) = (-1)^m \frac{4}{\pi^2} N_m^{-1} e^{-2|\gamma_1^{`}-\alpha_1|^2 - 2|\gamma_2-\alpha_2|^2} \\
[m! L_m(|2\gamma_1^{`} - \alpha_1|^2) + L_m(|2\gamma_2 - \alpha_2|^2)] + \\
(\alpha_1^* - 2\gamma_1^{*'})^m (2\gamma_2 - \alpha_2)^m + (2\gamma_2^* - \alpha_2^*)^m (\alpha_1 - 2\gamma_1^{`})^m
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
W(\gamma_1, \gamma_2^{`}) = (-1)^m \frac{4}{\pi^2} N_m^{-1} e^{-2|\gamma_1-\alpha_1|^2 - 2|\gamma_2^{`}-\alpha_2|^2} \\
[m! L_m(|2\gamma_1 - \alpha_1|^2) + L_m(|2\gamma_2^{`} - \alpha_2|^2)] + \\
(\alpha_1^* - 2\gamma_1^*)^m (2\gamma_2^{`} - \alpha_2)^m + (2\gamma_2^{*`} - \alpha_2^*)^m (\alpha_1 - 2\gamma_1)^m
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
W(\gamma_1^{`}, \gamma_2^{`}) = (-1)^m \frac{4}{\pi^2} N_m^{-1} e^{-2|\gamma_1^{`}-\alpha_1|^2 - 2|\gamma_2^{`}-\alpha_2|^2} \\
[m! L_m(|2\gamma_1^{`} - \alpha_1|^2) + L_m(|2\gamma_2^{`} - \alpha_2|^2)] + \\
(\alpha_1^* - 2\gamma_1^{`*})^m (2\gamma_2^{`} - \alpha_2)^m + (2\gamma_2^{*`} - \alpha_2^*)^m (\alpha_1 - 2\gamma_1^{`})^m
\end{aligned}
\tag{5.5}
$$

Plug the equations (5.2), (5.3), (5.4) and (5.5) in (A.88) and Plot it to get figure (5.1). Here we have observed the behaviour of the bell-inequality for different photon number, you can see that after some values of $\alpha$ the nonlocality vanishes.

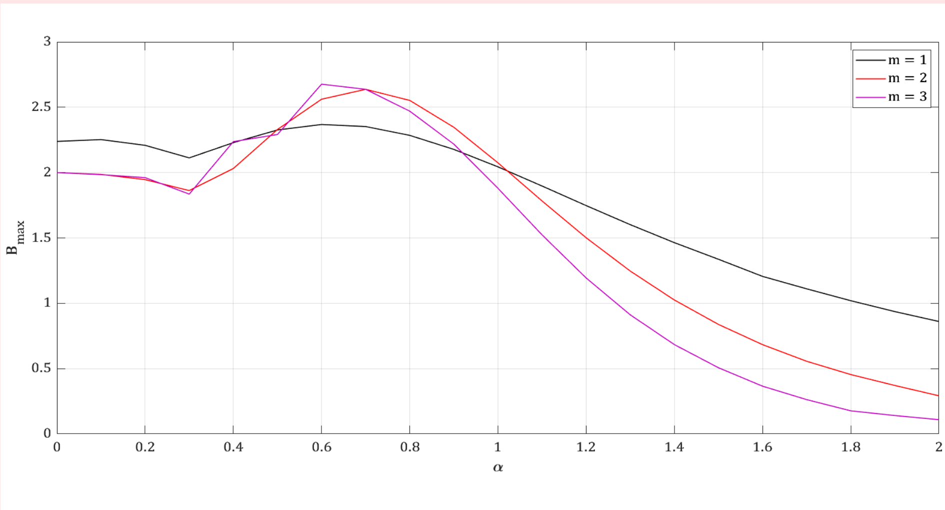


FIGURE 5.2: Bell Violation Plot for Local PATMCS

When nonlocality vanishes and the state acts as they are local, for further increasing the values of $\alpha$ the inequality of the state collapses to zero.

### 5.0.2 Bell's Violation on Nonlocal PATMCS

To minimize the equation A.88 using fminsearch algorithm in Matlab with random initial guesses the plot is obtained

$$|B| = \frac{\pi^2}{4}\Big|W(\chi_1,\chi_2) + W(\chi_1',\chi_2) + W(\chi_1,\chi_2') - W(\chi_1',\chi_2')\Big| \geq 2 \tag{5.6}$$

$$\begin{gathered} W(\chi_1',\chi_2) = \frac{4N^{-1}}{\pi^2} e^{-2\left|\chi_1'-\alpha\right|^2 - 2|\chi_2-\beta|^2} \\ [(-1)^m m! L_m(\left|2\chi_1' - \alpha\right|^2)(2\chi_1' - \alpha)^m (2\chi_2^* - \beta^*)^n \\ +(-1)^n n! L_n(|2\chi_2 - \beta|^2)(2\chi_1'^* - \alpha^*)^m (2\chi_2 - \beta)^n] \end{gathered} \tag{5.7}$$

$$\begin{gathered} W(\chi_1,\chi_2') = \frac{4N^{-1}}{\pi^2} e^{-2|\chi_1-\alpha|^2 - 2\left|\chi_2'-\beta\right|^2} \\ [(-1)^m m! L_m(|2\chi_1 - \alpha|^2)(2\chi_1 - \alpha)^m (2\chi_2'^* - \beta^*)^n \\ +(-1)^n n! L_n(\left|2\chi_2' - \beta\right|^2)(2\chi_1^* - \alpha^*)^m (2\chi_2' - \beta)^n] \end{gathered} \tag{5.8}$$

$$\begin{gathered} W(\chi_1',\chi_2') = \frac{4N^{-1}}{\pi^2} e^{-2\left|\chi_1'-\alpha\right|^2 - 2\left|\chi_2'-\beta\right|^2} \\ [(-1)^m m! L_m(\left|2\chi_1' - \alpha\right|^2)(2\chi_1' - \alpha)^m (2\chi_2'^* - \beta^*)^n \\ +(-1)^n n! L_n(\left|2\chi_2' - \beta\right|^2)(2\chi_1'^* - \alpha^*)^m (2\chi_2' - \beta)^n] \end{gathered} \tag{5.9}$$

Plug equations (4.7),(5.7),(5.8) and (5.9) in (5.6). The procedure for the calculation of nonlocality is same as the previous chapter.

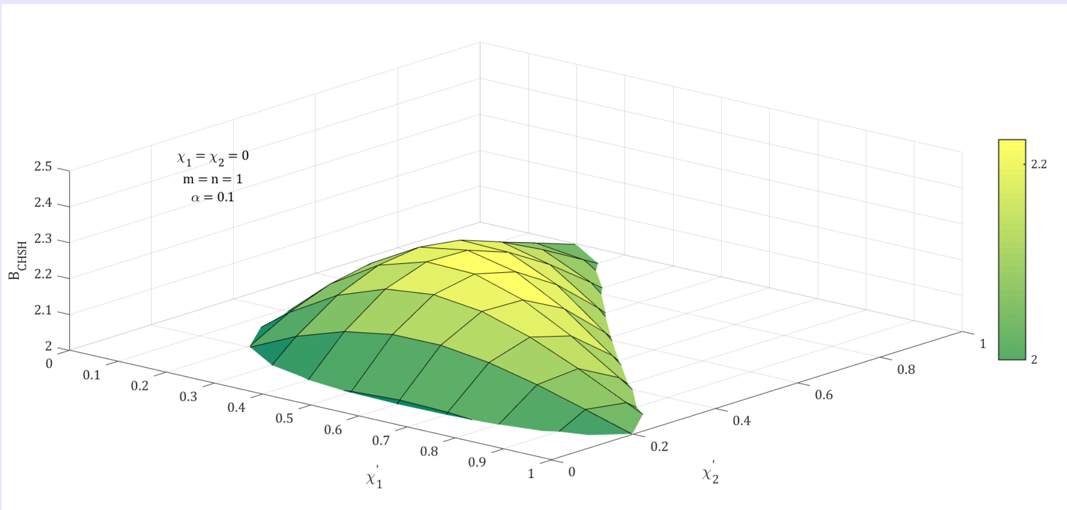


FIGURE 5.3: $Bell_{CHSH}$ inequality function plot for Nonlocal PATMCS

by setting up $\chi_1 = \chi_2 = 0$ and $\alpha = \beta = 0.1$ and Photon number $m = n = 1$ from the fig it can be seen that the state is nonlocal.

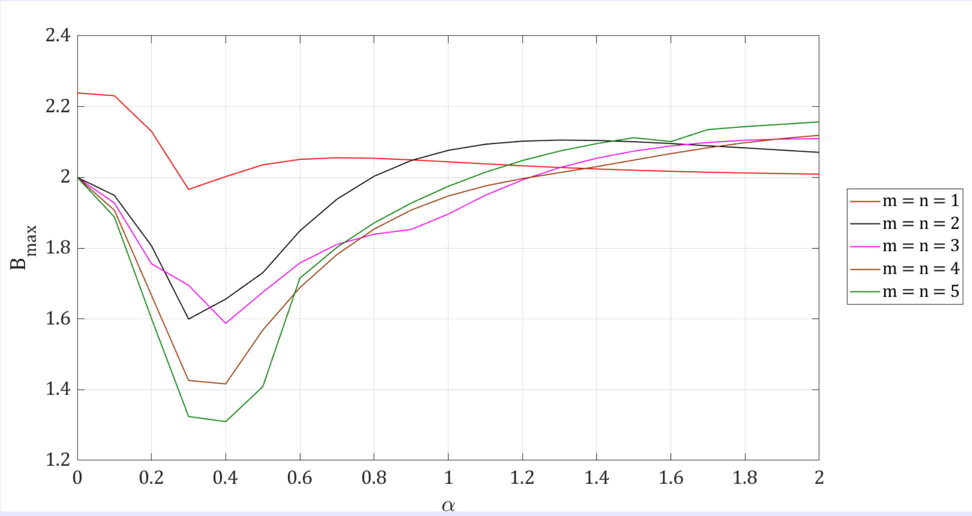


FIGURE 5.4: Minimization plot for $Bell_{CHSH}$ inequality

# Chapter 6

# Results and Disscussion

We conclude that the quantum states defined below are Nonclassical and Nonlocal

$$|\phi\rangle = \frac{1}{\sqrt{N}}(a_1^{\dagger m} + a_2^{\dagger m})\,|\alpha_1, \alpha_2\rangle \tag{6.1}$$

$$|\psi\rangle = \frac{1}{\sqrt{N}}(a^{\dagger m} + b^{\dagger n})\,|\alpha, \beta\rangle \tag{6.2}$$

- The test for nonclassicality is wigner function, the negative wigner function indicates the state is nonclassical.
- Then here comes the test for nonlocality, the minimum requirement that is necessary for a state to exhibit nonlocality is nonclassicality.
- There are some spin offs, A classical state can never be nonlocal. So we add photons to a classical state to change them into nonclassical. And not all nonclassical states can exhbit nonlocality. Yes these statements are kind of unambigous.
- So, how do i prove that my claim is cap? That's the niche part where we would write a code and try to solve it with a computer.
- The plots of the equation (4.6) and (4.7) shows that for some permutations of photon parameters, the plots lie on the negative region. This indicates that they are nonclassical.
- For Bell's Violation the plots should be greater than 2 for some combinations of $\alpha$ and photon number, thus we witness nonlocality. If they are below 2 then they are exhibiting locality.

# Appendix A

# Wigner Function

The Photon Added Two Mode Coherent State is defined as

$$|\psi\rangle = \frac{1}{\sqrt{N}}(a^{\dagger m} + b^{\dagger n})\,|\alpha, \beta\rangle \tag{A.1}$$

for our convinience we rewrite as

$$|\psi\rangle = \frac{1}{\sqrt{N}}(a^{\dagger m}\,|\alpha\rangle\,|\beta\rangle + b^{\dagger n}\,|\alpha\rangle\,|\beta\rangle) \tag{A.2}$$

$$\langle\psi| = \frac{1}{\sqrt{N^*}}(\hat{a}^m\,\langle\alpha|\,\langle\beta| + \hat{b}^n\,\langle\alpha|\,\langle\beta|) \tag{A.3}$$

The CS $|\alpha, \beta\rangle \equiv |\alpha\rangle\,|\beta\rangle$. The state is also in a super position of $a^{\dagger m}\,|\alpha\rangle\,|\beta\rangle$ and $b^{\dagger n}\,|\alpha\rangle\,|\beta\rangle$.

The Normalization $N_m$ is found by $|N|^2\,\langle\psi|\psi\rangle = 1$; which we obtained is

$$N = m!L_m(-|\alpha|^2) + n!L_n(-|\beta|^2) + (\alpha^{*m}\beta^n) + (\beta^{*n}\alpha^m) \tag{A.4}$$

The General Formula for Wigner function is

$$\begin{gathered} W(\gamma_1, \gamma_2) = \frac{4N}{\pi^4} e^{2\left(|\gamma_1|^2 + |\gamma_2|^2\right)} \int d^2 Z_1 d^2 Z_2 \\ \langle -Z_1 - Z_2|\rho|Z_1 Z_2\rangle\, e^{2\left(\gamma_1 Z_1^* - \gamma_1^* Z_1\right)} e^{2\left(\gamma_2 Z_2^* - \gamma_2^* Z_2\right)}, \end{gathered} \tag{A.5}$$

Here $\rho = |\psi\rangle\,\langle\psi|$ which is the density matrix.

$$\begin{aligned} \rho = N^{-1}[a^{\dagger m}\,|\alpha\rangle\,\langle\alpha|\,\hat{a}^m \quad & |\beta\rangle\,\langle\beta| \\ + |\alpha\rangle\,\langle\alpha| \quad & b^{\dagger n}\,|\beta\rangle\,\langle\beta|\,\hat{b}^n \\ + |\alpha\rangle\,\langle\alpha|\,\hat{a}^m \quad & b^{\dagger n}\,|\beta\rangle\,\langle\beta| \\ + a^{\dagger m}\,|\alpha\rangle\,\langle\alpha| \quad & |\beta\rangle\,\langle\beta|\,\hat{b}^n] \end{aligned} \tag{A.6}$$

Now plug equation (A.6) into (A.5) we get,

$$\begin{aligned}W(\gamma_1,\gamma_2) = \frac{4N^{-1}}{\pi^4}\int d^2Z_1 d^2Z_2 \\ [\langle -Z_1|\, a^{\dagger m}\,|\alpha\rangle\,\langle\alpha|\,\hat{a}^m\,|Z_1\rangle \quad \langle -Z_2|\beta\rangle\,\langle\beta|Z_2\rangle + \\ \langle -Z_1|\alpha\rangle\,\langle\alpha|Z_1\rangle \quad \langle -Z_2|\, b^{\dagger n}\,|\beta\rangle\,\langle\beta|\,\hat{b}^n\,|Z_2\rangle + \\ \langle -Z_1|\alpha\rangle\,\langle\alpha|\,\hat{a}^m\,|Z_1\rangle \quad \langle -Z_2|\, b^{\dagger n}\,|\beta\rangle\,\langle\beta|Z_2\rangle + \\ \langle -Z_1|\, a^{\dagger m}\,|\alpha\rangle\,\langle\alpha|Z_1\rangle \quad \langle -Z_2|\beta\rangle\,\langle\beta|\,\hat{b}^n\,|Z_2\rangle] \\ e^{2(|\gamma_1|^2+|\gamma_2|^2)}e^{2\left(\gamma_1 Z_1^*-\gamma_1^* Z_1\right)}e^{2\left(\gamma_2 Z_2^*-\gamma_2^* Z_2\right)}\end{aligned} \tag{A.7}$$

Now we split the integrals,

$$= \frac{4N^{-1}}{\pi^4} I \tag{A.8}$$

Where $I = [I_1 I_2 + I_4 I_3 + I_6 I_5 + I_8 I_7]$

$$\begin{aligned}I_1 &= \int d^2Z\,\langle -Z|\, a^{\dagger m}\,|\alpha\rangle\,\langle\alpha|\,\hat{a}^m\,|Z\rangle\, e^{2|\gamma_1|^2} + 2(\gamma_1 Z^* - \gamma_1^* Z) \\ I_2 &= \int d^2Z\,\langle -Z|\beta\rangle\,\langle\beta|Z\rangle\, e^{2|\gamma_2|^2} + 2(\gamma_2 Z^* - \gamma_2^* Z) \\ I_3 &= \int d^2Z\,\langle -Z|\alpha\rangle\,\langle\alpha|Z\rangle\, e^{2|\gamma_1|^2} + 2(\gamma_1 Z^* - \gamma_1^* Z) \\ I_4 &= \int d^2Z\,\langle -Z|\, b^{\dagger n}\,|\beta\rangle\,\langle\beta|\,\hat{b}^n\,|Z\rangle\, e^{2|\gamma_2|^2} + 2(\gamma_2 Z^* - \gamma_2^* Z) \\ I_5 &= \int d^2Z\,\langle -Z|\alpha\rangle\,\langle\alpha|\,\langle\alpha|\,\hat{a}^m\,|Z\rangle\, e^{2|\gamma_1|^2} + 2(\gamma_1 Z^* - \gamma_1^* Z) \\ I_6 &= \int d^2Z\,\langle -Z|\, b^{\dagger n}\,|\beta\rangle\,\langle\beta|Z\rangle\, e^{2|\gamma_2|^2} + 2(\gamma_2 Z^* - \gamma_2^* Z) \\ I_7 &= \int d^2Z\,\langle -Z|\, a^{\dagger m}\,|\alpha\rangle\,\langle\alpha|Z\rangle\, e^{2|\gamma_1|^2} + 2(\gamma_1 Z^* - \gamma_1^* Z) \\ I_8 &= \int d^2Z\,\langle -Z|\beta\rangle\,\langle\beta|\,\langle\beta|\,\hat{b}^n\,|Z\rangle\, e^{2|\gamma_2|^2} + 2(\gamma_2 Z^* - \gamma_2^* Z)\end{aligned} \tag{A.9}$$

Solving the Integrals separately and plugging into (A.8).
Similar Integrals are

$$I_1 = I_4, I_2 = I_3, I_5 = I_8, I_6 = I_7$$

Now Take $I_1$

$$= \int d^2Z\,\langle -Z|\, a^{\dagger m}\,|\alpha\rangle\,\langle\alpha|\,\hat{a}^m\,|Z\rangle\, e^{2|\gamma_1|^2} + 2(\gamma_1 Z^* - \gamma_1^* Z) \tag{A.10}$$

Here

$$\langle -Z| \, a^{\dagger m} \, |\alpha\rangle = (-Z^{*m}); \langle\alpha| \, \hat{a}^m \, |Z\rangle = (Z^m) \tag{A.11}$$

$$\langle -Z|\alpha\rangle = e^{\dfrac{-|Z|^2 - |\alpha|^2}{2} - Z^*\alpha}; \langle\alpha|Z\rangle = e^{\dfrac{-|Z|^2 - |\alpha|^2}{2} + Z\alpha^*} \tag{A.12}$$

$$= e^{2|\gamma_1|^2}(-1^m)\int d^2Z|Z|^{2m}e^{-|Z|^2-|\alpha|^2-Z^*\alpha-\alpha^*Z+2\gamma_1Z^*-2\gamma_1^*Z} \tag{A.13}$$

We need to now solve the above equation in polar coordinates

$$Z = re^{i\theta}, Z^* = re^{-i\theta}; A = (2\gamma_1 - \alpha), A^* = (2\gamma_1^* - \alpha^*)$$

$$= e^{2|\gamma_1|^2-|\alpha|^2}(-1^m)\int |Z|^{2m}e^{-|Z|^2-Z(2\gamma_1^*-\alpha^*)+Z^*(2\gamma_1-\alpha)}d^2Z \tag{A.14}$$

$$= e^{2|\gamma_1|^2-|\alpha|^2}(-1^m)\int |Z|^{2m}e^{-|Z|^2-zA^*+Z^*A}d^2Z \tag{A.15}$$

Here

$$A = |A|e^{i\phi}, A^* = |A|e^{-i\phi}, dz^2 = rdrd\theta$$

Now converting (A.15) to polar form

$$= e^{2|\gamma_1|^2-|\alpha|^2}(-1^m)\int r^{2m+1}e^{-r^2-r|A|e^{i\theta}e^{-i\phi}-r|A|e^{-i\theta}e^{i\phi}}drd\theta \tag{A.16}$$

$$= e^{2|\gamma_1|^2-|\alpha|^2}(-1^m)\int_0^\infty r^{2m+1}\int_0^{2\pi} e^{-r^2-r|A|[e^{i(\theta-\phi)}-e^{-i(\theta-\phi)}]}drd\theta \tag{A.17}$$

we know that,

$$\int_0^{2\pi} e^{i(\theta-\phi)} - e^{-i(\theta-\phi)} = \int_0^{2\pi} 2isin(\theta-\phi) \tag{A.18}$$

plug this into equation (A.17) we get,

$$= e^{2|\gamma_1|^2-|\alpha|^2}(-1^m)\int_0^\infty r^{2m+1}\int_0^{2\pi} e^{-r^2-2ir|A|sin(\theta-\phi)}drd\theta \tag{A.19}$$

we also know that

$$\int_0^{2\pi} 2ir|A|sin(\theta-\phi)d\theta = 2\pi J_0(2|A|r) \tag{A.20}$$

$$
\begin{aligned}
&= e^{2|\gamma_1|^2-|\alpha|^2}(-1^m)\int_0^\infty r^{2m+1}e^{-r^2}2\pi J_0(2|A|r)dr \\
&= e^{2|\gamma_1|^2-|\alpha|^2}(-1^m)2\pi\int_0^\infty r^{2m+1}e^{-r^2}J_0(2|A|r)dr
\end{aligned} \tag{A.21}
$$

Using the definite integral we knew,

$$\int_0^\infty e^{-x^2}x^{2m+\mu+1}J_\mu(2x\sqrt{Z})dx = \frac{n!}{2}e^{-Z}Z^{\mu/2}L_n^\mu(Z) \tag{A.22}$$

for our case $\mu = 0, m = n, x = r, Z = |A|^2$

$$= (-1^m)e^{2|\gamma_1|^2-|\alpha|^2}\frac{2\pi}{2}m!e^{-|A|^2}(|A|^2)^{0/2}L_m(|A|^2) \tag{A.23}$$

$$e^{-|A|} = e^{-4|\gamma_1|^2-|\alpha|^2+2\gamma_1\alpha^*+2\gamma_1^*\alpha} \tag{A.24}$$

plug (A.24) in (A.24) we get,

$$= (-1^m)m!e^{2|\gamma_1|^2-|\alpha|^2}e^{-4|\gamma_1|^2-|\alpha|^2+2\gamma_1\alpha^*+2\gamma_1^*\alpha}L_m(|2\gamma_1-\alpha|^2) \tag{A.25}$$

$$= (-1^m)m!e^{-2|\gamma_1|^2-2|\alpha|^2+2\gamma_1\alpha^*+2\gamma_1^*\alpha}L_m(|2\gamma_1-\alpha|^2) \tag{A.26}$$

$$I_1 = (-1^m)m!e^{-2|\gamma_1-\alpha|^2}L_m(|2\gamma_1-\alpha|^2) \tag{A.27}$$

We will get a similar result if we solve $I_4$ so i write it down as

$$I_4 = (-1^n)n!e^{-2|\gamma_2-\beta|^2}L_n(|2\gamma_2-\beta|^2) \tag{A.28}$$

Now we take $I_2$

$$I_2 = \int d^2Z\, \langle -Z|\beta\rangle\, \langle\beta|Z\rangle\, e^{2|\gamma_2|^2} + 2(\gamma_2 Z^* - \gamma_2^* Z) \tag{A.29}$$

$$= e^{2|\gamma_2|^2}\int e^{-|Z|^2-|\beta|^2+\beta^*Z-\beta Z^*+2\gamma_2Z^*-2\gamma_2^*Z}d^2Z \tag{A.30}$$

$$= e^{2|\gamma_2|^2}\int e^{-|Z|^2-|\beta|^2+Z^*(2\gamma_2-\beta)-Z(2\gamma_2^*-\beta^*)}d^2Z \tag{A.31}$$

Here

$$A = (2\gamma_2-\beta), A^* = (2\gamma_2^*-\beta^*)$$

$$= e^{2|\gamma_2|^2}\int e^{-|Z|^2-|\beta|^2}e^{Z^*A-zA^*}d^2Z \tag{A.32}$$

we solve this integral in cartesian coordinates

$$Z = Z_x + iZ_y, Z^* = Z_x - iZ_y$$

$$= e^{2|\gamma_2|^2} \int dZ_x dZ_y e^{-Z_x^2 - Z_y^2 - (Z_x + iZ_y)A^* + (Z_x - iZ_y)A} \tag{A.33}$$

$$= e^{2|\gamma_2|^2} \int dZ_x e^{-Z_x^2 - Z_x A^* + Z_x A \times \frac{2}{2}} \int dZ_y e^{-Z_y^2 - iZ_y A^* - iZ_y A \times \frac{2}{2}} \tag{A.34}$$

$$= e^{2|\gamma_2|^2 - |\beta|^2} \int dZ_x e^{-Z_x^2 - Z_x \frac{(A^* - A)^2}{2}} \int dZ_y e^{-Z_y^2 - iZ_y \frac{(A^* + A)^2}{2}} \tag{A.35}$$

we now take

$$\frac{(A^* - A)^2}{2} = M; i\frac{(A^* + A)^2}{2} = P$$

$$= e^{2|\gamma_2|^2 - |\beta|^2} \int dZ_x e^{-(Z_x + M)^2 + M^2} \int dZ_y e^{-(Z_y + P)^2 + P^2} \tag{A.36}$$

Here

$$X^2 = (Z_x + M)^2; Y^2 = (Z_y + P)^2; dX = dZ_x; dY = dZ_y$$

$$= e^{2|\gamma_2|^2 - |\beta|^2} e^{M^2 + P^2} \int dX e^{-X^2} \int dY e^{-Y^2} \tag{A.37}$$

As the above integral is Gaussian

$$\int dX e^{-X^2} = \sqrt{\pi}$$

$$= e^{2|\gamma_2|^2 - |\beta|^2} e^{M^2 + P^2} \sqrt{\pi}\sqrt{\pi} \tag{A.38}$$

$$M^2 + P^2 = |A|^2 \tag{A.39}$$

$$= -[(2\gamma_2 - \beta)(2\gamma_2^* - \beta^*)] \tag{A.40}$$

$$= -4|\gamma_2|^2 - |\beta|^2 + 2\gamma_2\beta^* + 2\gamma_2^*\beta \tag{A.41}$$

$$I_2 = \pi e^{-2|\gamma_2 - \beta|} \tag{A.42}$$

As the Integral $I_2 \& I_3$ are similar

$$I_3 = \pi e^{-2|\gamma_1 - \alpha|} \tag{A.43}$$

Now we take $I_5$

$$I_5 = \int d^2Z \langle -Z|\alpha\rangle \langle\alpha| \langle\alpha| \hat{a}^m |Z\rangle e^{2|\gamma_1|^2} + 2(\gamma_1 Z^* - \gamma_1^* Z) \tag{A.44}$$

$$= e^{2|\gamma_1|^2} \int Z^m \langle -Z|\alpha\rangle \langle\alpha|Z\rangle e^{2(\gamma_1 Z^* - \gamma_1^* Z)} d^2Z \tag{A.45}$$

$$= e^{2|\gamma_1|^2 - |\alpha|^2} \int |Z|^m e^{-|Z|^2 - Z(2\gamma_1^* - \alpha^*) + Z^*(2\gamma_1 - \alpha)} d^2Z \tag{A.46}$$

$$= e^{2|\gamma_1|^2-|\alpha|^2}(-1^m)\int |Z|^m e^{-|Z|^2-zA^*+Z^*A} d^2Z \tag{A.47}$$

$$Z = re^{i\theta}, Z^* = re^{-i\theta}; A = (2\gamma_1 - \alpha), A^* = (2\gamma_1^* - \alpha^*)$$

Here

$$A = |A|e^{i\phi}, A^* = |A|e^{-i\phi}, dz^2 = rdrd\theta$$

Now we change it to polar coordianates

$$= e^{2|\gamma_1|^2-|\alpha|^2}\int r^{m+1}e^{im\theta}e^{-r^2}e^{-re^{i\theta}|A|e^{-i\phi}+re^{-i\theta}|A|e^{i\phi}}drd\theta \tag{A.48}$$

$$= e^{2|\gamma_1|^2-|\alpha|^2}\int_0^\infty\int_0^{2\pi} r^{m+1}e^{im\theta}e^{-r^2}e^{-r|A|[e^{i(\theta-\phi)}-e^{-i(\theta-\phi)}]}drd\theta \tag{A.49}$$

$$\int_0^{2\pi} e^{i(\theta-\phi)} - e^{-i(\theta-\phi)} = \int_0^{2\pi} 2isin(\theta-\phi) \tag{A.50}$$

$$= e^{2|\gamma_1|^2-|\alpha|^2}\int_0^\infty r^{m+1}e^{-r^2}\int_0^{2\pi} e^{2i|A|sin(\theta-\phi)+im\theta}d\theta \tag{A.51}$$

$$= \int_0^{2\pi} e^{2i|A|sin(\theta-\phi)}e^{im\theta}\frac{e^{im\phi}}{e^{im\phi}}d\theta \tag{A.52}$$

$$= e^{-im\phi}\int_0^{2\pi} e^{2i|A|sin(\theta-\phi)}e^{im(\phi-\theta)}d\theta \tag{A.53}$$

Assume $\phi - \theta = \Theta$

$$= e^{im\phi+im\frac{\pi}{2}}\int_0^\infty e^{2i|A|sin(\theta-\frac{\pi}{2}+\frac{\pi}{2}-\phi)}e^{im\Theta}d\theta \tag{A.54}$$

Consider $\phi - \theta + \pi/2 = \Theta$

$$= e^{im\phi+im\frac{\pi}{2}}\int_0^\infty r^{m+1}e^{-r^2}\int_0^{2\pi} e^{im\Theta}e^{2i|A|cos(\Theta-\frac{\pi}{2})}d\Theta \tag{A.55}$$

$$\begin{aligned}\int_0^{2\pi} e^{2i|A|cos(\Theta-\frac{\pi}{2})}e^{im\Theta}d\Theta &= 2\pi i^m J_m(x)\\ &= 2\pi i^m J_m(2|A|r)\end{aligned} \tag{A.56}$$

$$= e^{im\phi}i^m[2\pi\int_0^\infty r^{m+1}e^{-r^2}J_m(2|A|r)dr] \tag{A.57}$$

$$\begin{gathered}\int_0^\infty x^{\nu+1}e^{-\alpha x^2}J_\nu(\beta x)dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}}e^{-\beta^2/4\alpha}\\ \nu = m, x = r, \alpha = 1, \beta = 2|A|\end{gathered} \tag{A.58}$$

$$= i^m 2\pi e^{2|\gamma_1|^2-|\alpha|^2} e^{im\phi+im\pi/2} (\frac{2^m|A|^m}{2^{m+}} e^{-(4|A|^2/4)} \tag{A.59}$$

$$= i^{2m} \pi e^{2|\gamma_1|^2-|\alpha|^2} e^{im\phi} |A|^m e^{-|A|^2} \tag{A.60}$$

$$A^m = e^{im\phi}|A|^m, A^m = (2\gamma_1 - \alpha)^m \tag{A.61}$$

$$= (-1)^m \pi A^m e^{-|2\gamma_1-\alpha|^2} \tag{A.62}$$

$$I_5 = (-1)^m \pi (2\gamma_1 - \alpha)^m e^{-|2\gamma_1-\alpha|^2} \tag{A.63}$$

$$I_8 = (-1)^n \pi (2\gamma_2 - \alpha)^n e^{-|2\gamma_2-\beta|^2} \tag{A.64}$$

Now we take $I_7$

$$I_7 = \int d^2Z \langle -Z|\alpha\rangle \langle\alpha| \langle\alpha| \hat{a}^{\dagger m} |Z\rangle e^{2|\gamma_1|^2} + 2(\gamma_1 Z^* - \gamma_1^* Z) \tag{A.65}$$

$$= (-1)^m e^{2|\gamma_1|^2} \int Z^{*m} \langle -Z|\alpha\rangle \langle\alpha|Z\rangle e^{2(\gamma_1 Z^* - \gamma_1^* Z)} d^2Z \tag{A.66}$$

$$= (-1)^m e^{2|\gamma_1|^2-|\alpha|^2} (-1^m) \int Z^{*m} e^{-|Z|^2 - Z(2\gamma_1^*-\alpha^*) + Z^*(2\gamma_1-\alpha)} d^2Z \tag{A.67}$$

$$= (-1)^m e^{2|\gamma_1|^2-|\alpha|^2} (-1^m) \int Z^{*m} e^{-|Z|^2 - zA^* + Z^*A} d^2Z \tag{A.68}$$

$$Z = re^{i\theta}, Z^* = re^{-i\theta}; A = (2\gamma_1 - \alpha), A^* = (2\gamma_1^* - \alpha^*)$$

Here

$$A = |A|e^{i\phi}, A^* = |A|e^{-i\phi}, dz^2 = rdrd\theta$$

Now we change it to polar coordianates

$$= (-1)^m e^{2|\gamma_1|^2-|\alpha|^2} \int r^{m+1} e^{-im\theta} e^{-r^2} e^{-re^{i\theta}|A|e^{-i\phi} + re^{-i\theta}|A|e^{i\phi}} drd\theta \tag{A.69}$$

$$= (-1)^m e^{2|\gamma_1|^2-|\alpha|^2} \int_0^\infty \int_0^{2\pi} r^{m+1} e^{-im\theta} e^{-r^2} e^{-r|A|[e^{i(\theta-\phi)} - e^{-i(\theta-\phi)}]} drd\theta \tag{A.70}$$

$$\int_0^{2\pi} e^{i(\theta-\phi)} - e^{-i(\theta-\phi)} = \int_0^{2\pi} 2isin(\theta-\phi) \tag{A.71}$$

$$= (-1)^m e^{2|\gamma_1|^2-|\alpha|^2} \int_0^\infty r^{m+1} e^{-r^2} \int_0^{2\pi} e^{2i|A|sin(\theta-\phi) - im\theta} d\theta \tag{A.72}$$

$$= \int_0^{2\pi} e^{2i|A|sin(\theta-\phi)} e^{-im\theta} \frac{e^{-im\phi}}{e^{-im\phi}} d\theta \tag{A.73}$$

$$= e^{-im\phi} \int_0^{2\pi} e^{2i|A|sin(\theta-\phi)} e^{im(\phi-\theta)} d\theta \tag{A.74}$$

$$= e^{-im\pi+im\frac{\pi}{2}} \int_0^{2\pi} e^{2i|A|sin(\phi-\frac{\pi}{2}+\frac{\pi}{2}-\theta)} e^{im(\phi-\theta-\pi/2)} d\theta \tag{A.75}$$

Consider $\theta - \phi + \pi/2 = \Theta$

$$= e^{-im\phi+im\frac{\pi}{2}} \int_0^{\infty} r^{m+1} e^{-r^2} \int_0^{2\pi} e^{im\Theta} e^{2i|A|cos(\Theta-\frac{\pi}{2})} d\Theta \tag{A.76}$$

$$\begin{aligned} \int_0^{2\pi} e^{2i|A|cos(\Theta-\frac{\pi}{2})} e^{im\Theta} d\Theta &= 2\pi i^m J_m(x) \\ &= 2\pi i^m J_m(2|A|r) \end{aligned} \tag{A.77}$$

$$= e^{-im\phi}(-1)^m i^m 2\pi \int_0^{\infty} r^{m+1} e^{-r^2} J_m(2|A|r) dr \tag{A.78}$$

$$\begin{aligned} \int_0^{\infty} x^{\nu+1} e^{-\alpha x^2} J_\nu(\beta x) dx &= \frac{\beta^\nu}{(2\alpha)^{\nu+1}} e^{-\beta^2/4\alpha} \\ \nu = m, x = r, \alpha &= 1, \beta = 2|A| \end{aligned} \tag{A.79}$$

$$= (-1)^m i^m 2\pi e^{2|\gamma_1|^2-|\alpha|^2} e^{-im\phi+im\pi/2} (\frac{2^m|A|^m}{2^{m+}} e^{-(4|A|^2/4)} \tag{A.80}$$

$$= (-1)^m i^{2m} \pi e^{2|\gamma_1|^2-|\alpha|^2} e^{-im\phi} |A|^m e^{-|A|^2} \tag{A.81}$$

$$A^{*m} = e^{-im\phi}|A|^m, A^{*m} = (2\gamma_1^* - \alpha^*)^m \tag{A.82}$$

$$= (1)^m \pi A^{*m} e^{-|2\gamma_1-\alpha|^2} \tag{A.83}$$

$$I_7 = (1)^m \pi (2\gamma_1^* - \alpha^*)^m e^{-|2\gamma_1-\alpha|^2} \tag{A.84}$$

$$I_6 = (1)^n \pi (2\gamma_2^* - \beta^*)^n e^{-|2\gamma_2-\beta|^2} \tag{A.85}$$

$$\begin{aligned} I = \pi^2 e^{-2|\gamma_1-\alpha|^2-2|\gamma_2-\beta|^2} \\ [(-1)^m m! L_m(|2\gamma_1-\alpha|^2) + \\ (-1)^n n! L_n(|2\gamma_2-\beta|^2) + \\ (-1)^m (1)^n (2\gamma_1-\alpha)^m (2\gamma_2^*-\beta^*)^n + \\ (-1)^n (1)^m (2\gamma_1^*-\alpha^*)^m (2\gamma_2-\beta)^n] \end{aligned} \tag{A.86}$$

After plugging the integrals we have obtained the wigner function

$$\begin{aligned} W(\gamma_1,\gamma_2) = \frac{4N^{-1}}{\pi^2} e^{-2|\gamma_1-\alpha|^2-2|\gamma_2-\beta|^2} \\ [(-1)^m m! L_m(|2\gamma_1-\alpha|^2) + \\ (-1)^n n! L_n(|2\gamma_2-\beta|^2) + \\ (-1)^m (1)^n (2\gamma_1-\alpha)^m (2\gamma_2^*-\beta^*)^n + \\ (-1)^n (1)^m (2\gamma_1^*-\alpha^*)^m (2\gamma_2-\beta)^n] \end{aligned} \tag{A.87}$$

And then we find the bell's inequality by substituting the value on to the below equation by setting the primes as the variables and non primes as a constant number.

$$|B| = \frac{\pi^2}{4}\Big|W(\gamma_1,\gamma_2) + W(\gamma_1',\gamma_2) + W(\gamma_1,\gamma_2') - W(\gamma_1',\gamma_2')\Big| \geq 2 \tag{A.88}$$

The Matlab Code for Bell's Inequality is given below,

```
tic
clear
ncut = 11 ;
m = 1 ;
    %ALPHA VALUES%
alpha1 = 0.1;
alpha2 = 0.1;
a1 = alpha1;
a2 = alpha2;
    %MOD ALPHA%

m1 = a1.*conj(a1);
m2 = a2.*conj(a2);
    %GAMMA VALUES
g1 = 0;
g2 = 0;

for nloop=1:ncut
    g1p = 0+ (nloop-1.0)/10;
    x(nloop) = g1p ;
for ploop=1:ncut
    g2p = 0 + (ploop-1.0)/10;
    y(ploop) = g2p ;

    %NORMALIZATION
N1 = factorial(m).*(Funlaguerre(m, -abs(m1)^2)+Funlaguerre(m,
   -abs(m2)^2));
N2 = (a1.^m.*conj(a2).^m) + (a2.^m.*conj(a1).^m);
N = 1/(N1 + N2);
 %mini terms
s1 = exp(-2.*abs(g1-a1)^2);
s2 = exp(-2.*abs(g2-a2)^2);
s1p = exp(-2.*abs(g1p-a1)^2);
s2p = exp(-2.*abs(g2p-a2)^2);
      %laguerre's function
F1 = Funlaguerre(m, abs(2.*g1-a1)^2);
F2 = Funlaguerre(m, abs(2.*g2-a2)^2);
F1p = Funlaguerre(m, abs(2.*g1p-a1)^2);
```

```
F2p = Funlaguerre(m, abs(2.*g2p-a2)^2);

b1 = (-2.*g1+a1)^m;
b2 = (2.*g2-a2)^m;
b1c = (-2.*conj(g1)+conj(a1))^m;
b2c = (2.*conj(g2)-conj(a2))^m;

p1 = (-2.*g1p+a1)^m;
p2 = (2.*g2p-a2)^m;
p1c = (-2.*conj(g1p)+conj(a1))^m;
p2c = (2.*conj(g2p)-conj(a2))^m;
%%    wigner 1    %%
A = (-1^m);
t1 = A.*s1.*s2.*((factorial(m).*(F1+F2))+b1c.*b2+b2c.*b1);
w1 = (4.*N)/(pi.*pi).*(t1);
%    wigner2    %
t2 = A.*s1p.*s2.*((factorial(m).*(F1p+F2))+p1c.*b2+p1.*b2c);
w2 =(4.*N)/(pi.*pi).*(t2);
%   wigenr3   %
t3 =  A.*s1.*s2p.*((factorial(m).*(F1+F2p))+b1c.*p2+b1.*p2c);
w3 = (4.*N)/(pi.*pi).*(t3);
% wigner4 %
t4 = A.*s1p.*s2p.*((factorial(m).*(F1p+F2p))+p1c.*p2+p1.*p2c);
w4 = (4.*N)/(pi.*pi).*(t4);
%Bell chsh inequality %
B = abs((pi.*pi)/4.*(w1+w2+w3-w4));
z(nloop,ploop) = real(B);
end
end
surf(x,y,z)
runtime = toc
```

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