Charlet Antoine

Lab 2 of models of epidemics

FMNS, March 2024

Subtask 1:

Description:

As I explained in my lab 1 work, my model already had a non-constant population $N(t)$. Obviously, the equation $N(t) = S(t) + I_1(t) + I_2(t) + R(t)$ remains true.$\Lambda = \mu N$ is our constant recruitment rate; more specifically, in our case, It is our birth rate (recruitment rates can mean something else like a person entering the group of sexually active persons for instance) and newborn babies fell in the susceptible group(the disease is not hereditary i.e. not vertical). μ is our mortality rate and It is the same mortality rate for all groups in our model.γ is the rate of the infected getting vaccinated(this parameter comes from the fact that our model was inspired by a S I V model that was itself based upon a S I R model). β is our infection rate. I_1 and I_2 are our infected without and with vaccination groups, R is the recovered group. Our model is based off the (2.23) model in P. 73: BRAUER, FRED - MATHEMATICAL MODELS IN EPIDEMIOLOGY.-SPRINGER (2019).

Model from the book:

$$(2.23)\begin{cases} S'(t) = \Lambda - \dfrac{\beta S\, I_1}{N} - \mu S \\ I_1'(t) = \dfrac{\beta S\, I_1}{N} - (\mu + \gamma) I_1 \\ I_2'(t) = \gamma I_1 - \mu I_2 \\ N'(t) = S'(t) + I_1'(t) + I_2'(t) \end{cases}$$

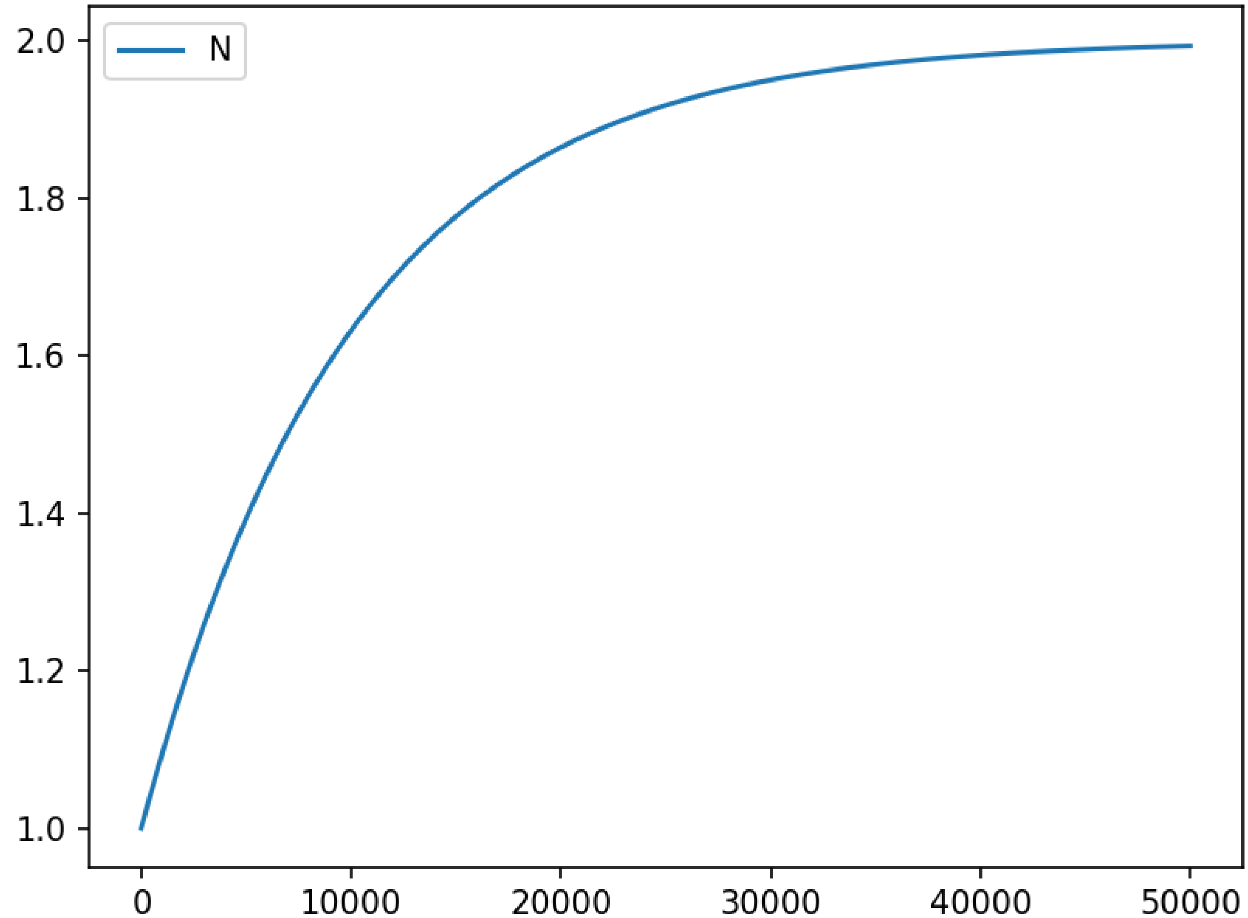
Model I used for lab 1:

$$\begin{cases} S'(t) = \Lambda - \dfrac{\beta S\, I_1}{N} - \mu S \\ I_1'(t) = \dfrac{\beta S\, I_1}{N} - (\mu + \gamma) I_1 - \alpha_1 * I_1 \\ I_2'(t) = \gamma I_1 - \mu I_2 - \alpha_2 * I_2 \\ R'(t) = \alpha_1 * I_1 + \alpha_2 * I_2 - \mu R \\ N'(t) = S'(t) + I_1'(t) + I_2'(t) + R'(t) \end{cases}$$

1.

$$\begin{aligned} N'(t) &= S'(t) + I_1'(t) + I_2'(t) + R'(t) \\ &= \Lambda - \frac{\beta S\, I_1}{N} - \mu S + \frac{\beta S\, I_1}{N} - (\mu + \gamma) I_1 - \alpha_1 * I_1 + \gamma I_1 - \mu I_2 - \alpha_2 * I_2 + \alpha_1 * I_1 + \alpha_2 * I_2 - \mu R \\ &= \Lambda - \mu(S + I_1 + I_2 + R) \\ &= \Lambda - \mu N \end{aligned}$$

2. We get $\Lambda(\mathrm{N}) = \Lambda$ in question 1. because the population was already non-constant. In class, you said to me that I could use the population instead of plotting a constant function. $\forall \mathrm{N}_0, \lim_{t\to\infty} \mathrm{N}(t) = \frac{\Lambda}{\mu} = 2$ which is weird if I take a lot of people for N_0.



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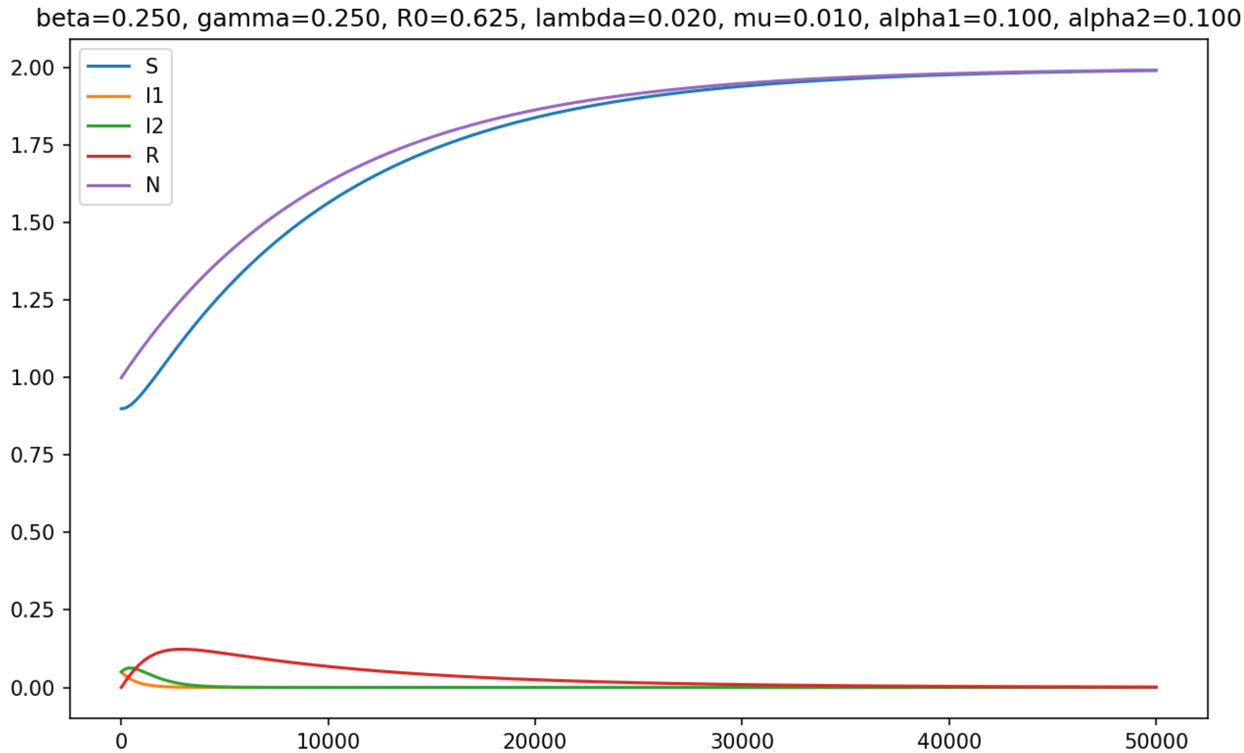
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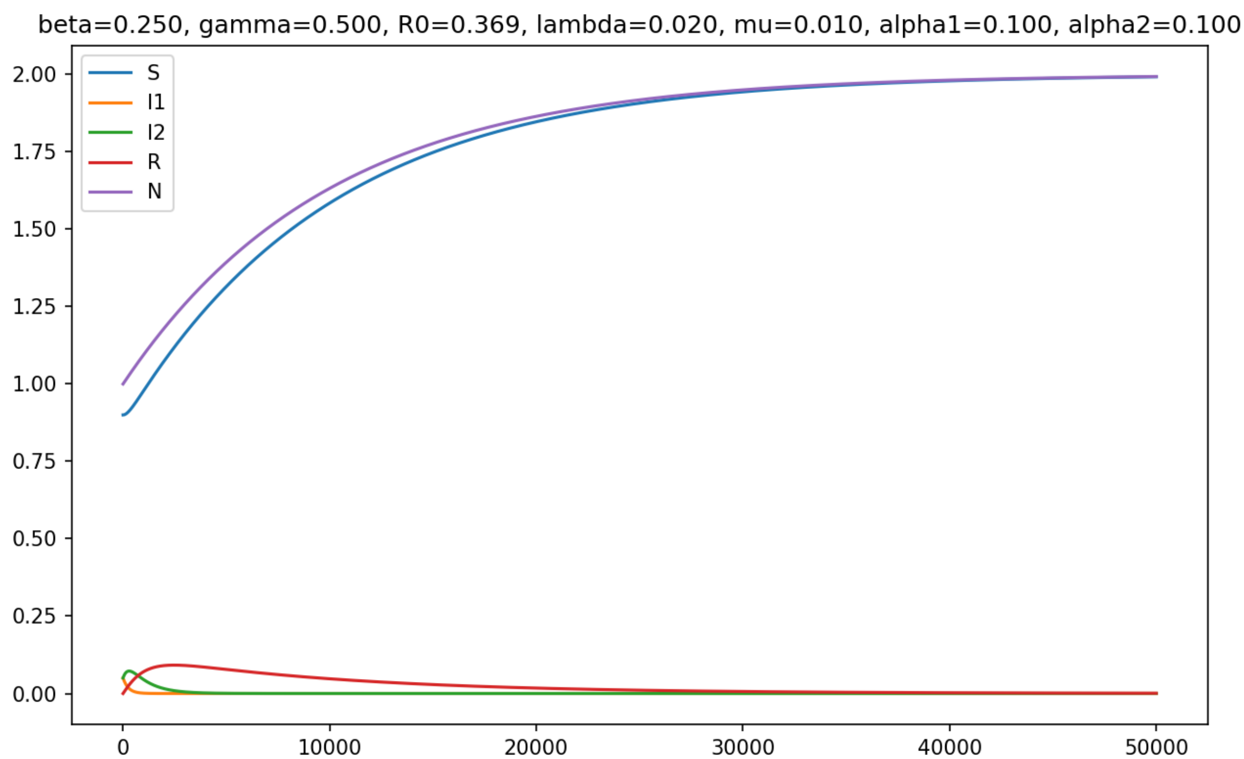
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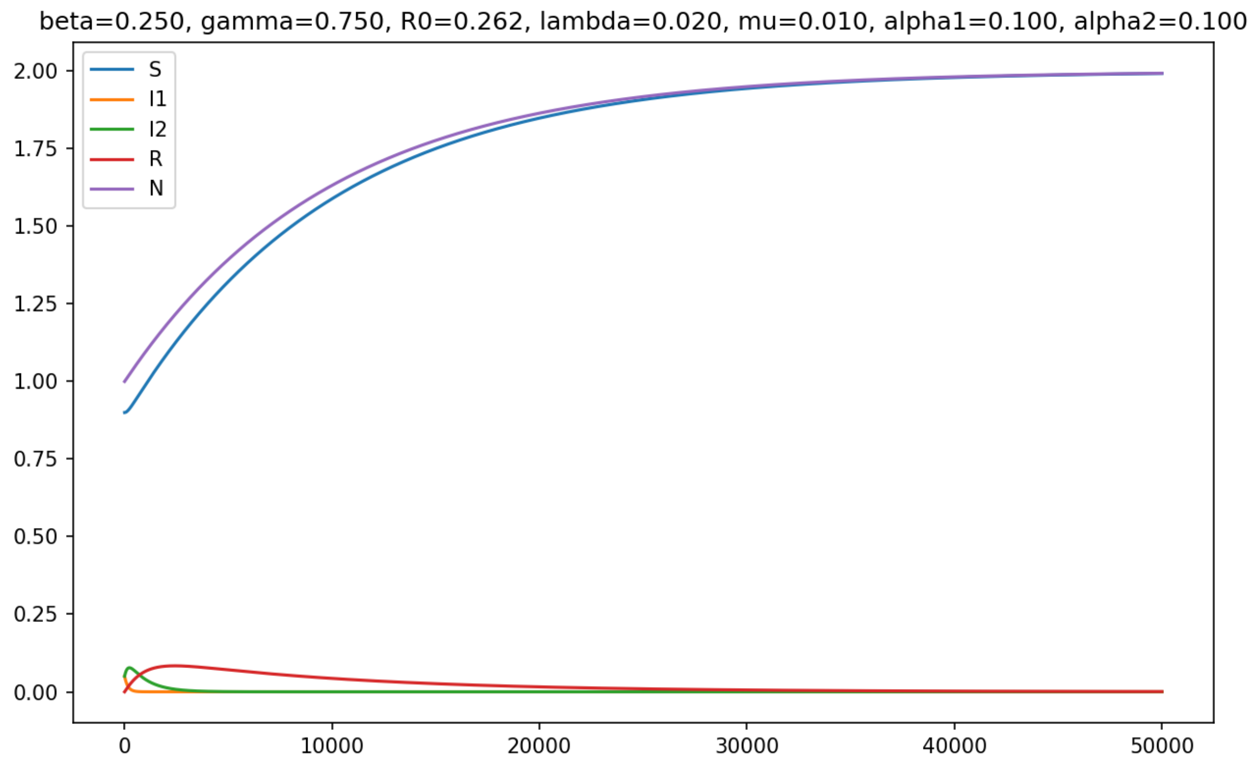
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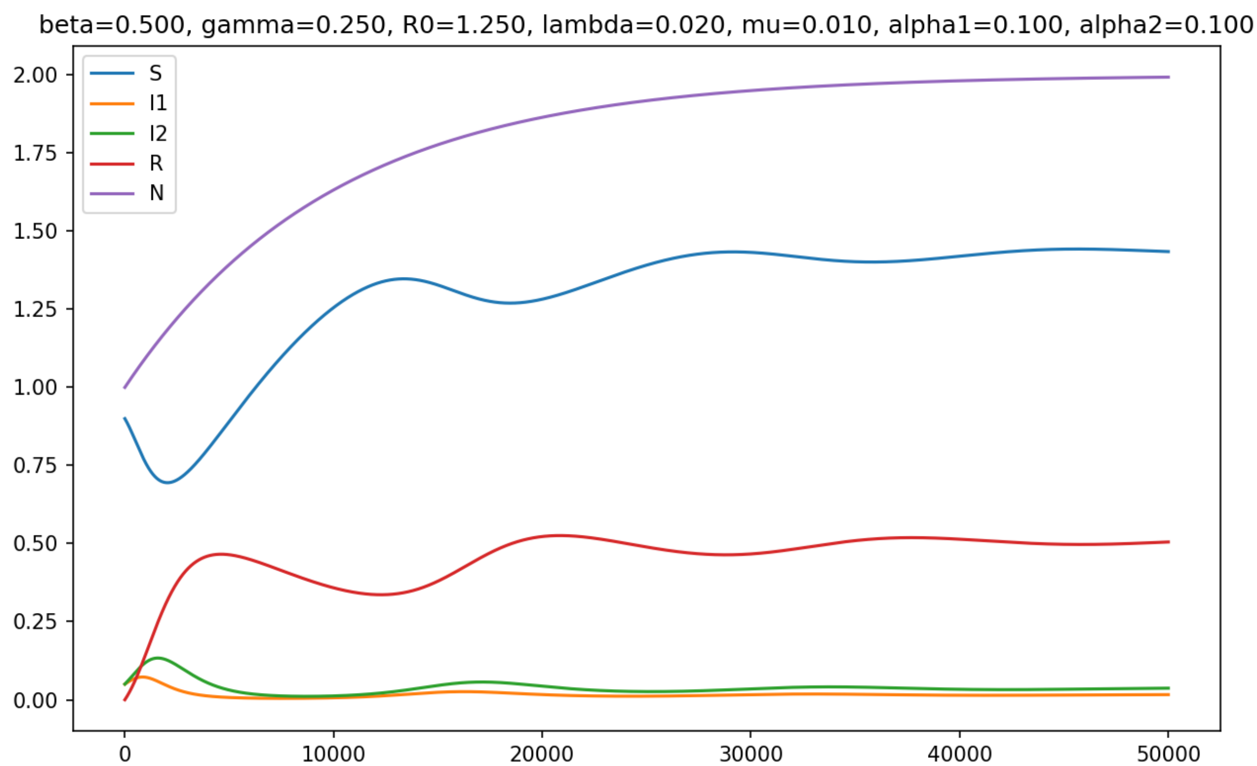
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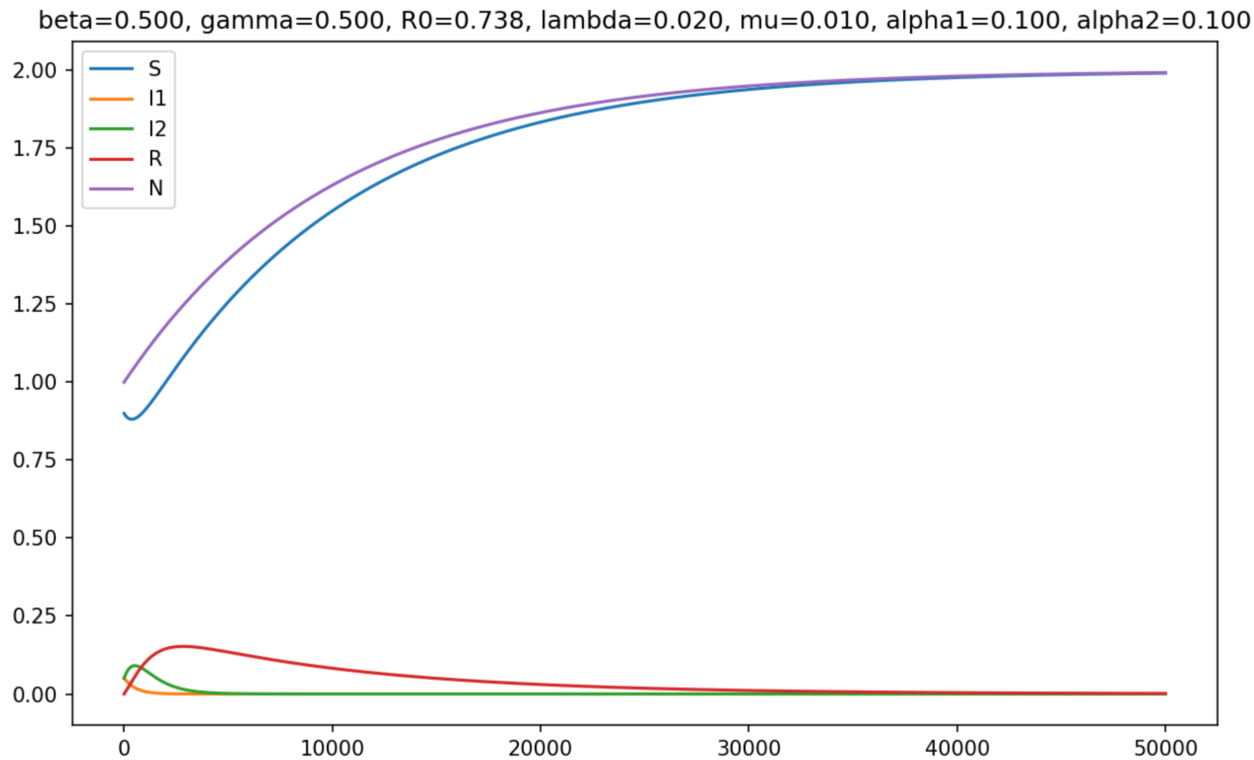
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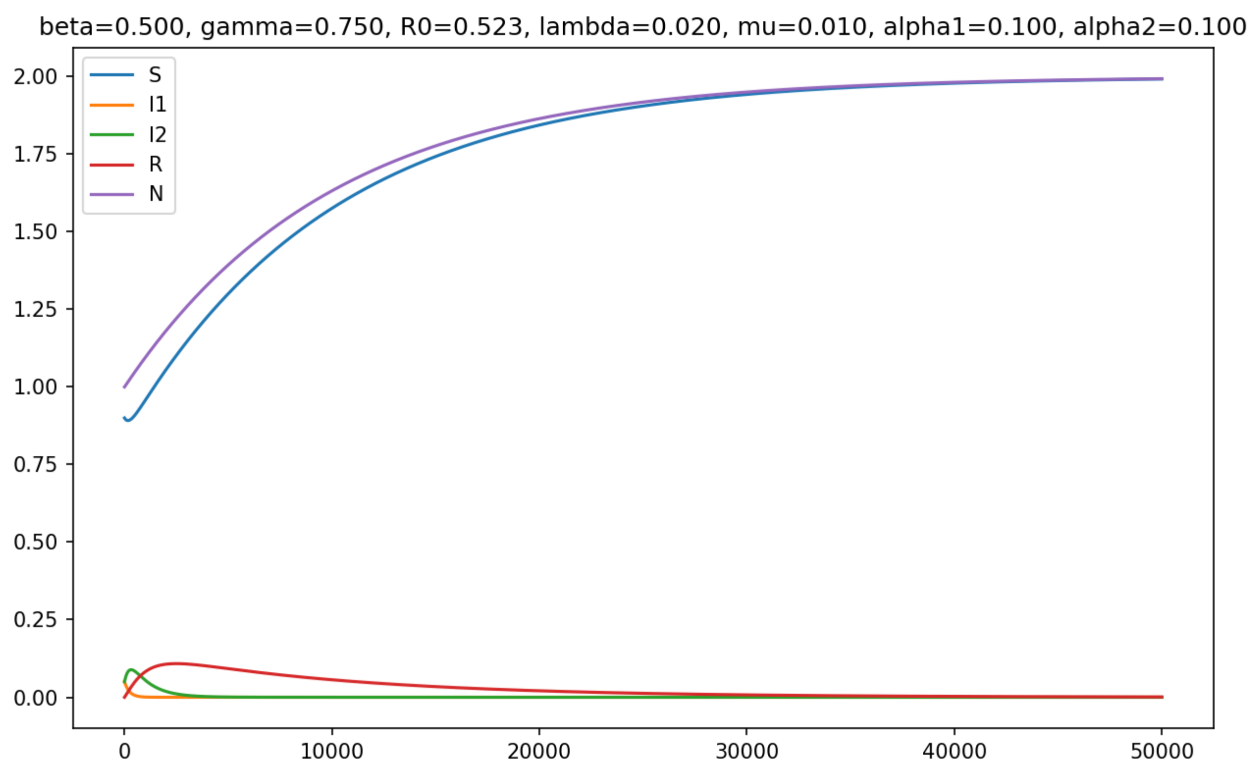
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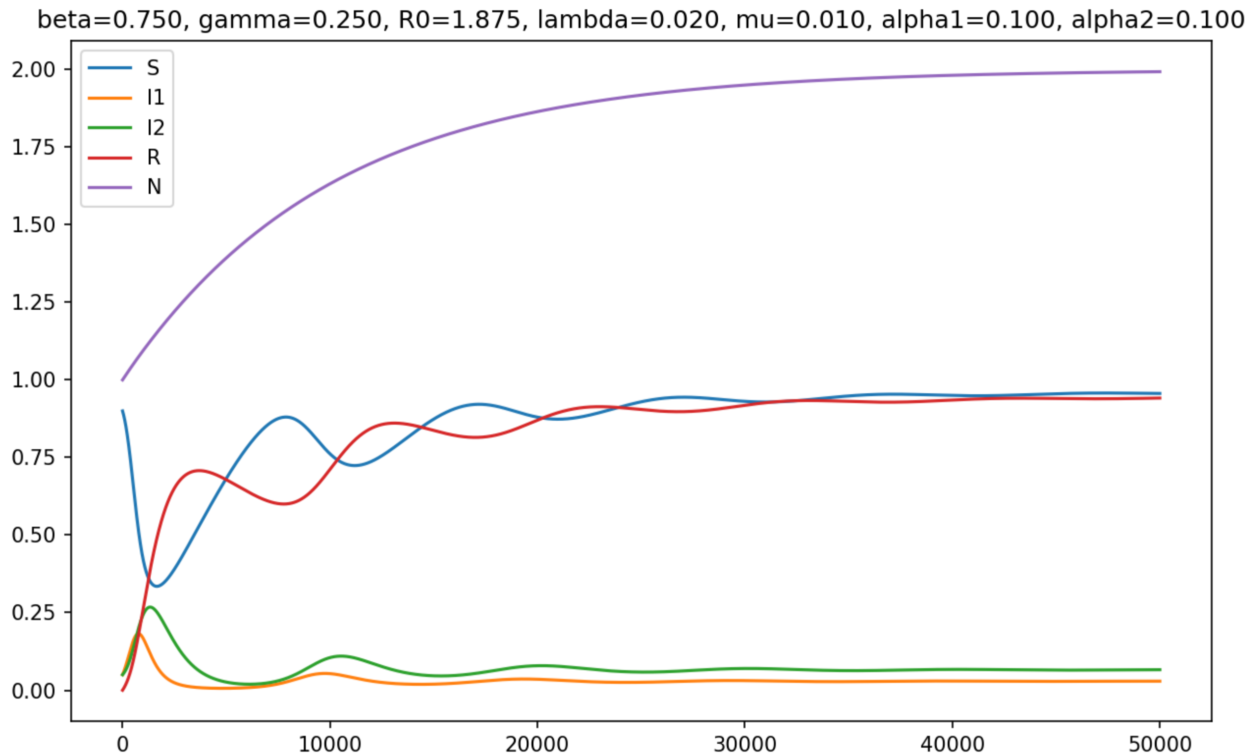
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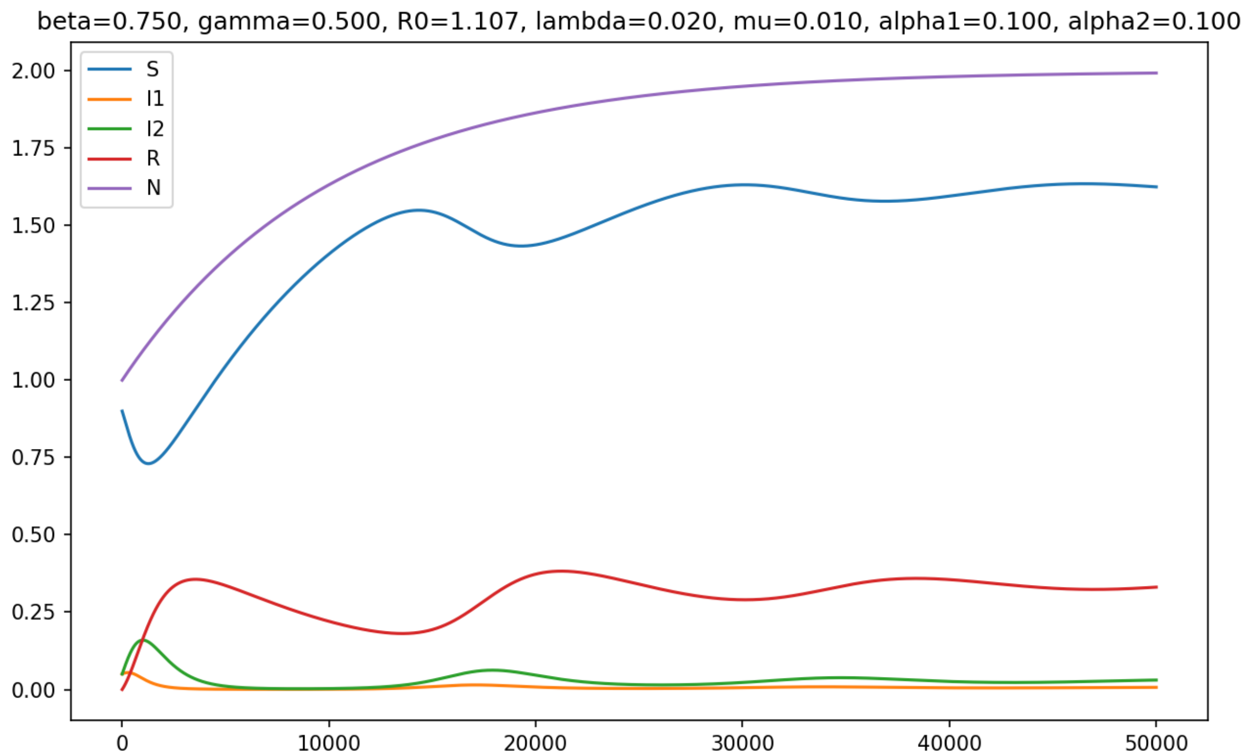
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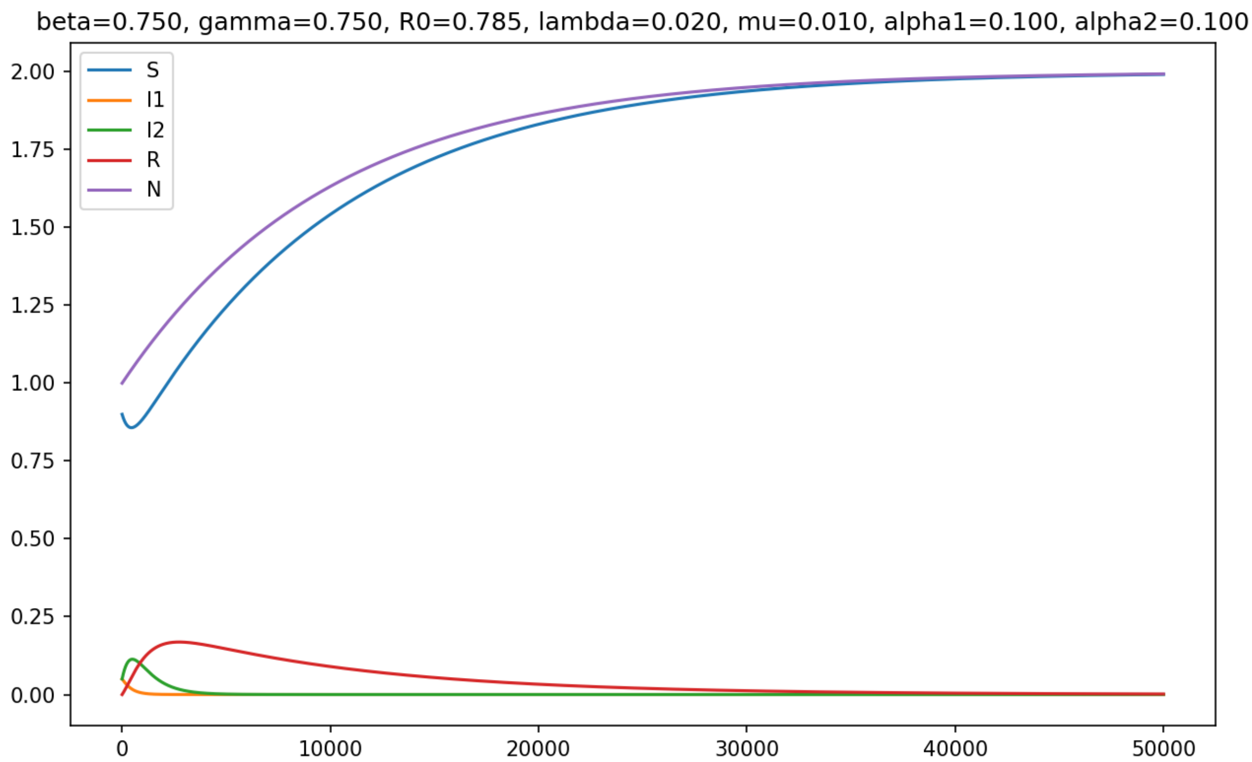


We can see that in some cases, $R_0 < 1$ so pretty much nothing happens. In other cases, S oscillates because the disease outbreaks each time less powerful than the other

The influx and outflux can be written as follow:

$$\mathcal{F}(I_1, I_2) = \left(\frac{\beta S\, I_1}{N}, \gamma\, I_1\right) \; and \; \nu(I_1, I_2) = ((\mu + \gamma + \alpha_1) I_1, (\mu + \alpha_2) * I_2)$$

$$F = \left.\begin{bmatrix} \frac{d\mathcal{F}_1}{dI_1} & \frac{d\mathcal{F}_1}{dI_2} \\ \frac{d\mathcal{F}_2}{dI_1} & \frac{d\mathcal{F}_2}{dI_2} \end{bmatrix}\right|_{X_0}$$

$$= \left.\begin{bmatrix} \frac{\beta S}{N} & 0 \\ \gamma & 0 \end{bmatrix}\right|_{X_0}$$

$$N = \left.\begin{bmatrix} \frac{d\nu_1}{dI_1} & \frac{d\nu_1}{dI_2} \\ \frac{d\nu_2}{dI_1} & \frac{d\nu_2}{dI_2} \end{bmatrix}\right|_{X_0}$$

$$= \left.\begin{bmatrix} \mu + \gamma + \alpha_1 & 0 \\ 0 & \mu + \alpha_2 \end{bmatrix}\right|_{X_0}$$

$$We\ get\ F = \begin{bmatrix} \frac{\beta S_0}{N_0} & 0 \\ \gamma & 0 \end{bmatrix} \; and \; N = \begin{bmatrix} \mu + \gamma + \alpha_1 & 0 \\ 0 & \mu + \alpha_2 \end{bmatrix}$$

$$The\ next\ generation\ matrix\ G\ is\ G = FN^{-1} = \frac{1}{(\mu + \gamma + \alpha_1)(\mu + \alpha_2)} \begin{bmatrix} \frac{\beta S_0}{N_0} & 0 \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} \mu + \alpha_2 & 0 \\ 0 & \mu + \gamma + \alpha_1 \end{bmatrix}$$

$$= \frac{1}{(\mu + \gamma + \alpha_1)(\mu + \alpha_2)} \begin{bmatrix} (\mu + \alpha_2) \frac{\beta S_0}{N_0} & 0 \\ (\mu + \alpha_2)\gamma & 0 \end{bmatrix}$$

$$The\ reproductive\ number\ is\ \Re_0 = \rho(G)$$

$$which\ means\ that\ the\ reproductive\ number\ is\ the\ maximum\ real\ part\ of\ the\ eigenvalues\ of\ G.$$

$$Tr(G) = \frac{1}{(\mu + \gamma + \alpha_1)(\mu + \alpha_2)}\left((\mu + \alpha_2)\frac{\beta S_0}{N_0} + 0\right) = \frac{\beta S_0}{N_0(\mu + \gamma + \alpha_1)}$$

$$\text{and Det(G)} = \frac{1}{(\mu + \gamma + \alpha_1)(\mu + \alpha_2)}\left((\mu + \alpha_2)\frac{\beta S_0}{N_0} * 0 - 0 * (\mu + \alpha_2)\gamma\right) = 0$$

$$\lambda_{1,2} = \frac{Tr(G)}{2} \pm \sqrt[2]{(\frac{Tr(G)}{2})^2 - Det(G)}$$

$$= \frac{\beta S_0}{2N_0(\mu + \gamma + \alpha_1)} \pm \sqrt[2]{(\frac{\beta S_0}{2N_0(\mu + \gamma + \alpha_1)})^2 - 0}$$

$$= (\frac{\beta S_0}{N_0(\mu + \gamma + \alpha_1)}, 0)$$

$$\textit{Our reproductive number is } \Re_0 = \rho(G) = \frac{\beta S_0}{N_0(\mu + \gamma + \alpha_1)}.$$

4. In question 1., you could have expected that I would chose $\Lambda(N) \neq \Lambda = constant$ but since It is the model I chosed for lab 1, I will stick to It. In my model, pretty much everything is N dependent so I need to find N_∞ first, but the linearization analysis is still applicable as a result.

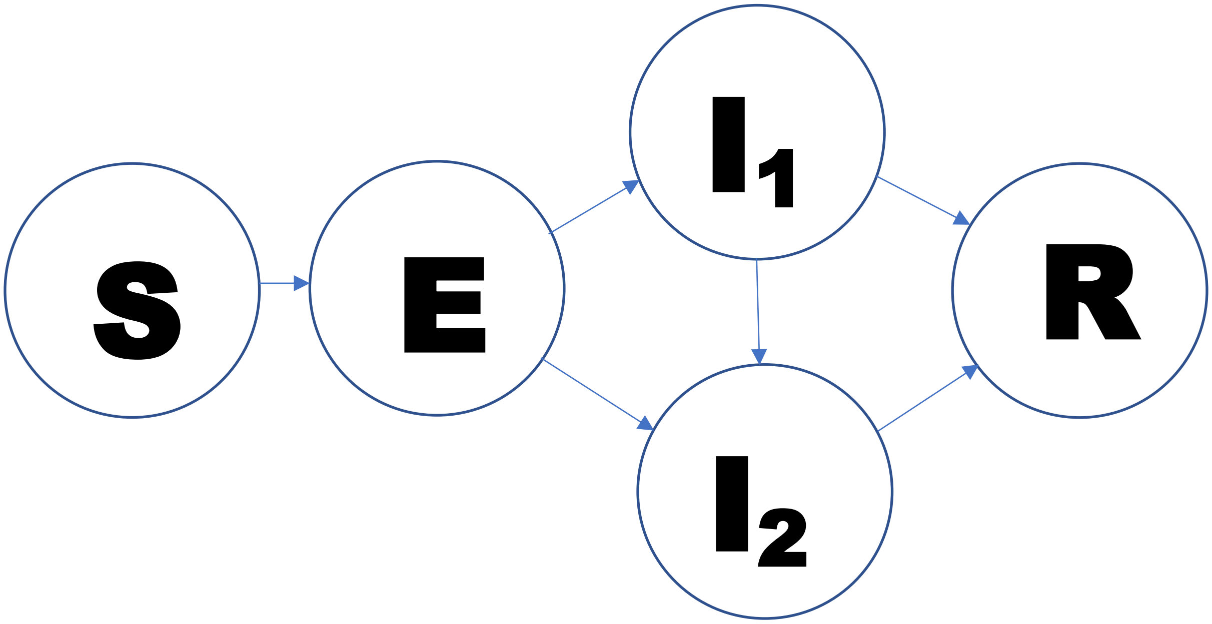
Subtask 2:

1.

$$\begin{cases} S'(t) = \Lambda - \frac{\beta SE}{N} - \mu S \\ E'(t) = \frac{\beta S\, E}{N} - (\kappa_1 I_1 + \kappa_2 I_2 + \mu)E \\ I_1'(t) = \kappa_1 I_1 E - (\mu + \gamma)I_1 - \alpha_1 * I_1 \\ I_2'(t) = \kappa_2 I_2 E + \gamma I_1 - \mu I_2 - \alpha_2 * I_2 \\ R'(t) = \alpha_1 * I_1 + \alpha_2 * I_2 - \mu R \\ N'(t) = S'(t) + I_1'(t) + I_2'(t) + R'(t) + E'(t) \end{cases}$$

There is a mathematical difference since some of $E(t)$ directly goes to I_2.

2. I won't repeat everything since most was explain in the description in subtask 1. The only difference is that people that get infected enter the E class but do not feel sick right away.



3. The influx and outflux can be written as follow:

$$\mathcal{F}(I_1, I_2) = (\kappa_1 I_1 \mathrm{E}, \kappa_2 I_2 \mathrm{E} + \gamma\, I_1)\ and\ \mathrm{v}(I_1, I_2) = ((\mu\ +\ \gamma +\ \alpha_1) I_1, (\mu + \alpha_2) * I_2)$$

$$F \quad = \quad \left.\begin{bmatrix} \frac{d\mathcal{F}_1}{dI_1} & \frac{d\mathcal{F}_1}{dI_2} \\ \frac{d\mathcal{F}_2}{dI_1} & \frac{d\mathcal{F}_2}{dI_2} \end{bmatrix}\right|_{X_0}$$

$$= \left.\begin{bmatrix} \kappa_1 \mathrm{E} & 0 \\ \gamma & \kappa_2 \mathrm{E} \end{bmatrix}\right|_{X_0}$$

$$\mathrm{N} \quad = \quad \left.\begin{bmatrix} \frac{d\mathrm{v}_1}{dI_1} & \frac{d\mathrm{v}_1}{dI_2} \\ \frac{d\mathrm{v}_2}{dI_1} & \frac{d\mathrm{v}_2}{dI_2} \end{bmatrix}\right|_{X_0}$$

$$= \left.\begin{bmatrix} \mu\ +\ \gamma +\ \alpha_1 & 0 \\ 0 & \mu + \alpha_2 \end{bmatrix}\right|_{X_0}$$

$$We\ get\ F = \begin{bmatrix} \kappa_1 \mathrm{E} & 0 \\ \gamma & \kappa_2 \mathrm{E} \end{bmatrix}\ and\ \mathrm{N} = \begin{bmatrix} \mu\ +\ \gamma +\ \alpha_1 & 0 \\ 0 & \mu + \alpha_2 \end{bmatrix}$$

$$The\ next\ generation\ matrix\ G\ is\ G = F\mathrm{N}^{-1} \quad = \quad \frac{1}{(\mu\ +\ \gamma +\ \alpha_1)(\mu + \alpha_2)} \begin{bmatrix} \kappa_1 \mathrm{E} & 0 \\ \gamma & \kappa_2 \mathrm{E} \end{bmatrix} \begin{bmatrix} \mu + \alpha_2 & 0 \\ 0 & \mu\ +\ \gamma +\ \alpha_1 \end{bmatrix}$$

$$= \frac{1}{(\mu\ +\ \gamma +\ \alpha_1)(\mu + \alpha_2)} \begin{bmatrix} \kappa_1 \mathrm{E}(\mu + \alpha_2) & 0 \\ \gamma(\mu + \alpha_2) & \kappa_2 \mathrm{E}(\mu\ +\ \gamma +\ \alpha_1) \end{bmatrix}$$

$$The\ reproductive\ number\ is\ \Re_0 = \rho(G)$$

$$which\ means\ that\ the\ reproductive\ number\ is\ the\ maximum\ real\ part\ of\ the\ eigenvalues\ of\ G.$$

$$Tr(G) = E\frac{\kappa_1(\mu + \alpha_2) + \kappa_2(\mu\ +\ \gamma +\ \alpha_1)}{(\mu\ +\ \gamma +\ \alpha_1)(\mu + \alpha_2)}$$

$$\text{and Det(G)} = \ \kappa_1 \kappa_2 E^2$$

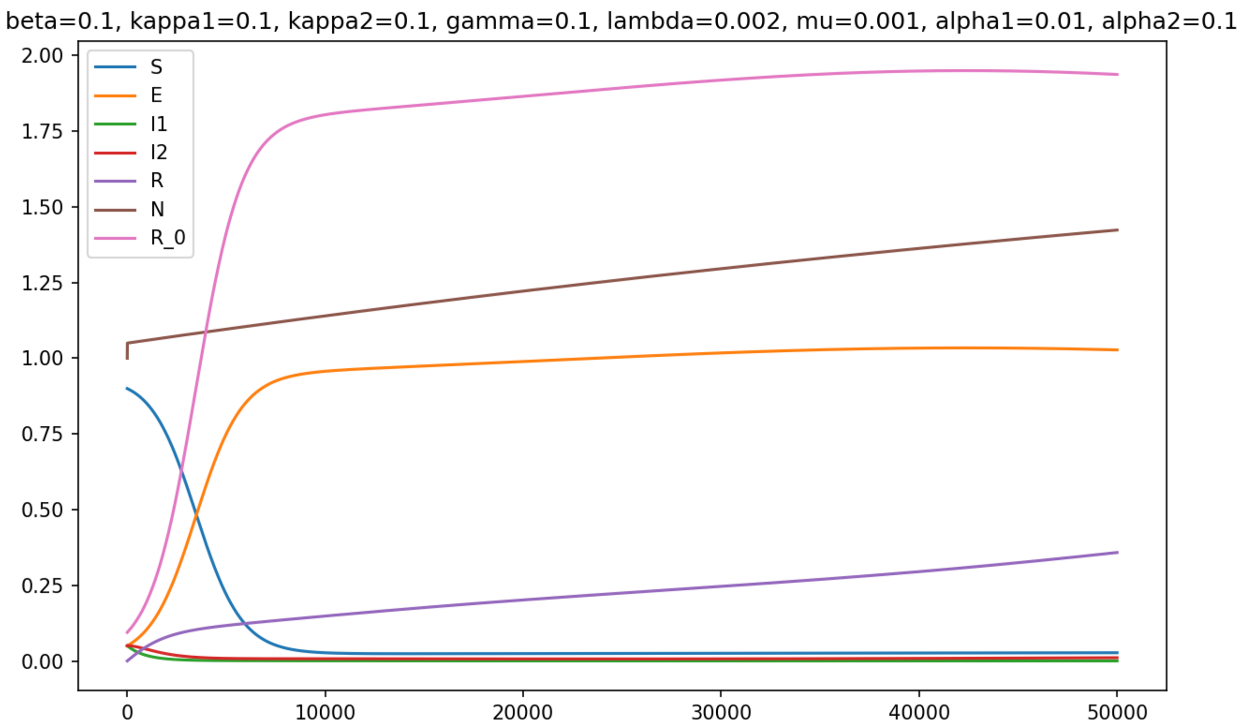
$$\lambda_{1,2} = \frac{Tr(G)}{2} \pm \sqrt[2]{\left(\frac{Tr(G)}{2}\right)^2 - Det(G)}$$

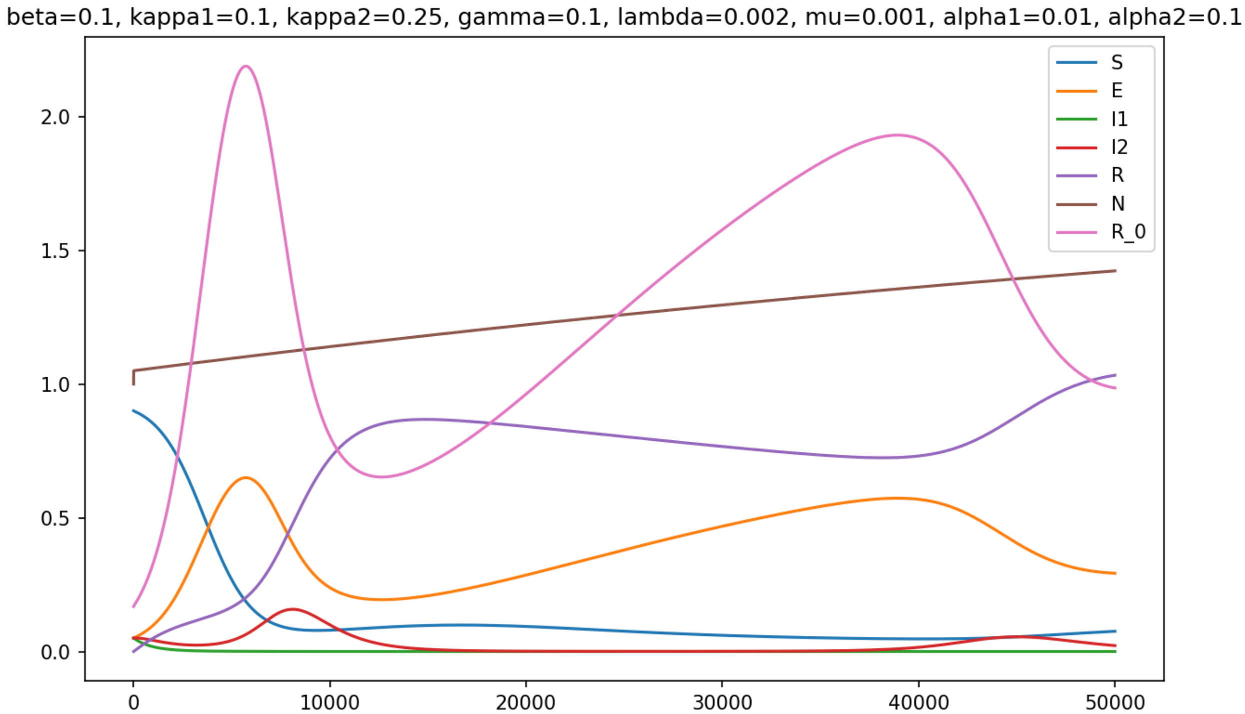
$$= E\frac{\kappa_1(\mu+\alpha_2)+\kappa_2(\mu+\gamma+\alpha_1)}{2(\mu+\gamma+\alpha_1)(\mu+\alpha_2)} \pm \sqrt[2]{\left(E\frac{\kappa_1(\mu+\alpha_2)+\kappa_2(\mu+\gamma+\alpha_1)}{2(\mu+\gamma+\alpha_1)(\mu+\alpha_2)}\right)^2 - \kappa_1\kappa_2E^2}$$

$Our\ reproductive\ number\ is\ \Re_0 = \rho(G)$

$$= E\frac{\kappa_1(\mu+\alpha_2)+\kappa_2(\mu+\gamma+\alpha_1)}{2(\mu+\gamma+\alpha_1)(\mu+\alpha_2)} + \sqrt[2]{\left(E\frac{\kappa_1(\mu+\alpha_2)+\kappa_2(\mu+\gamma+\alpha_1)}{2(\mu+\gamma+\alpha_1)(\mu+\alpha_2)}\right)^2 - \kappa_1\kappa_2E^2}.$$

4.

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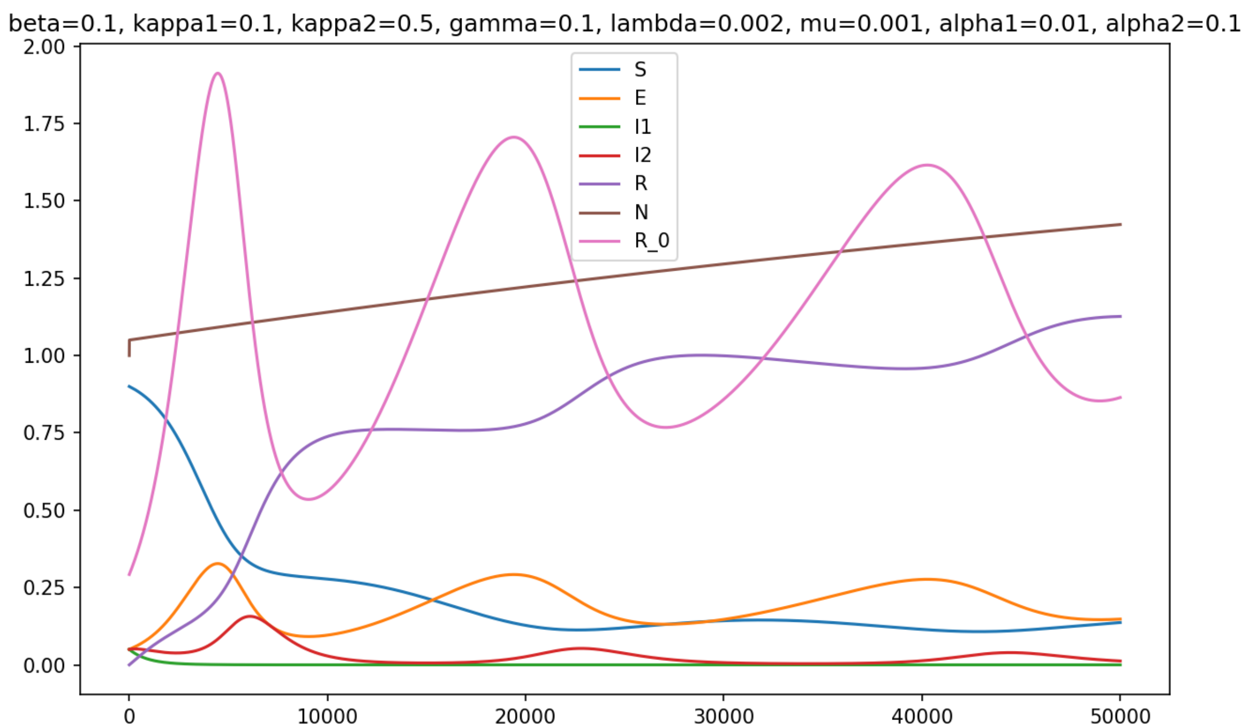
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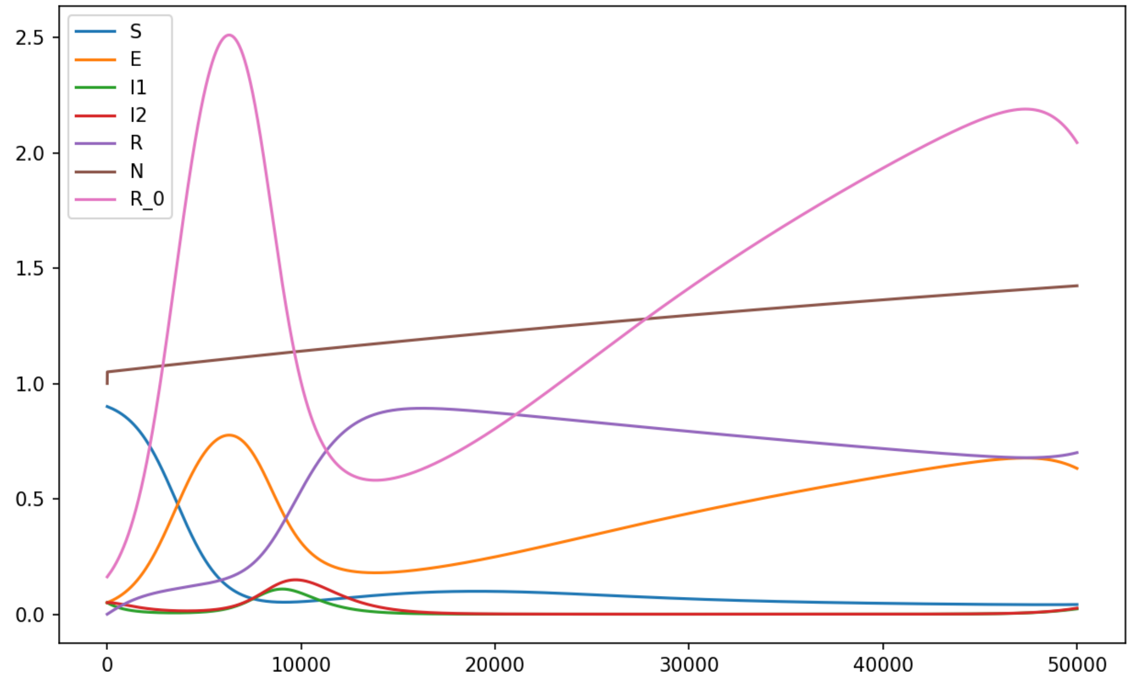

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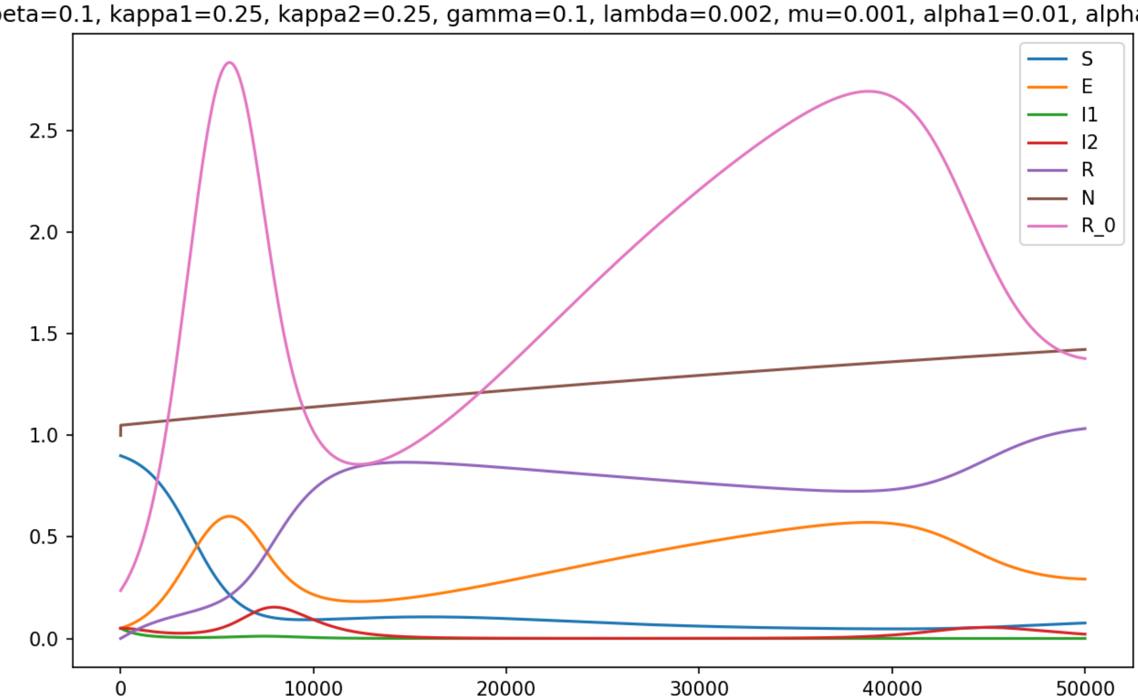

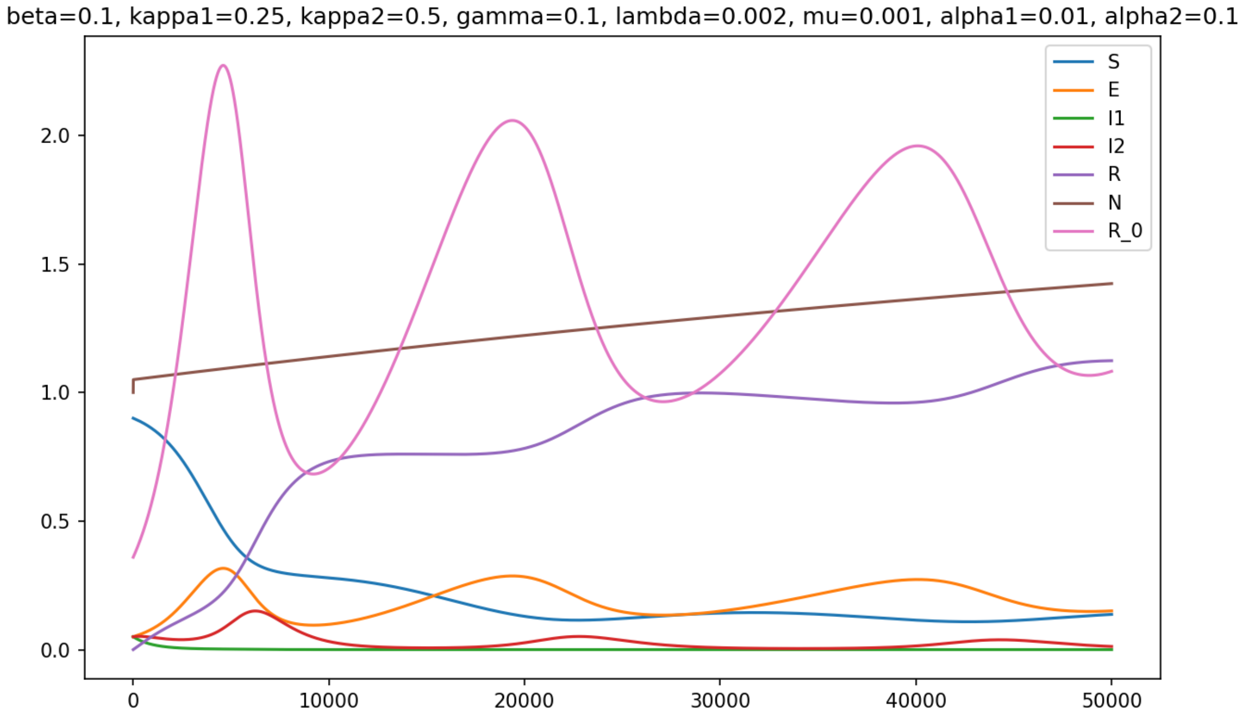
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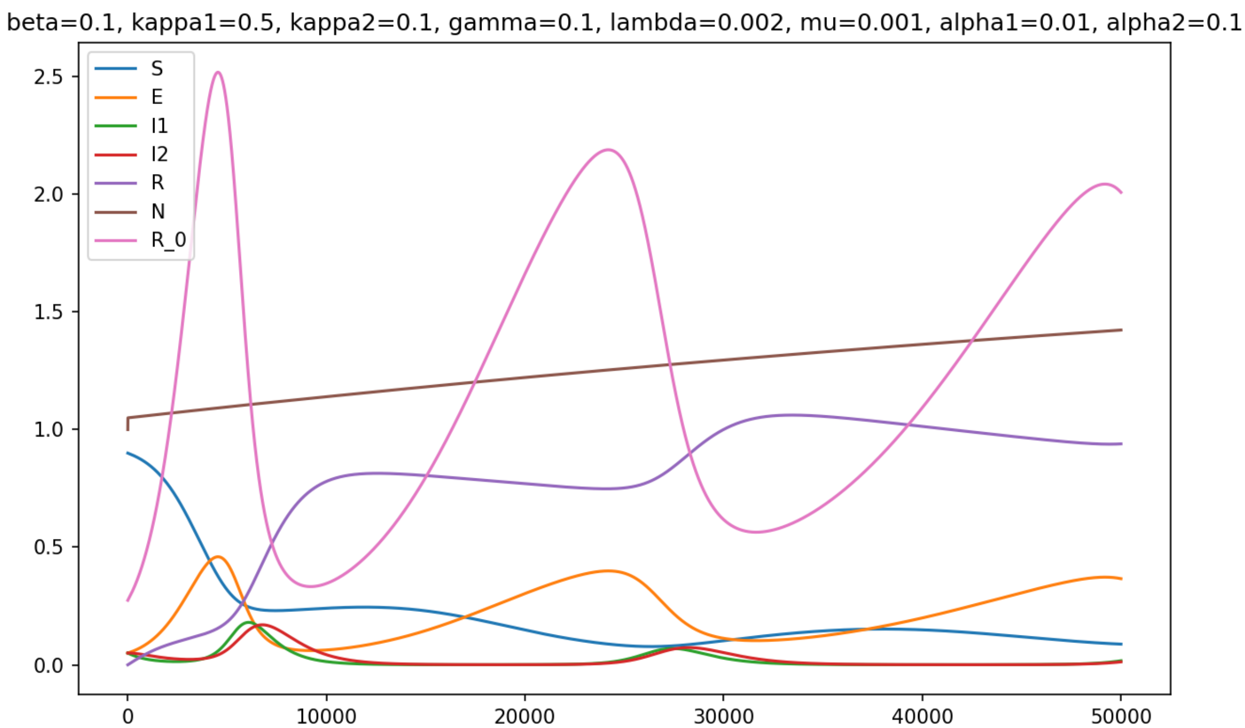
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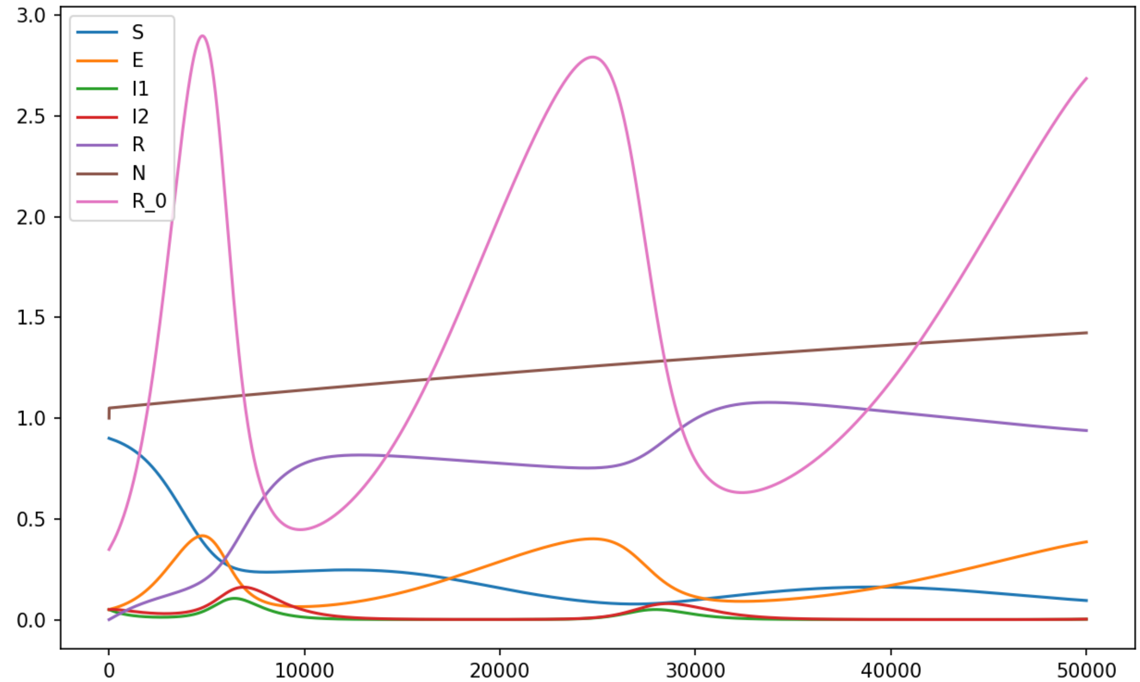

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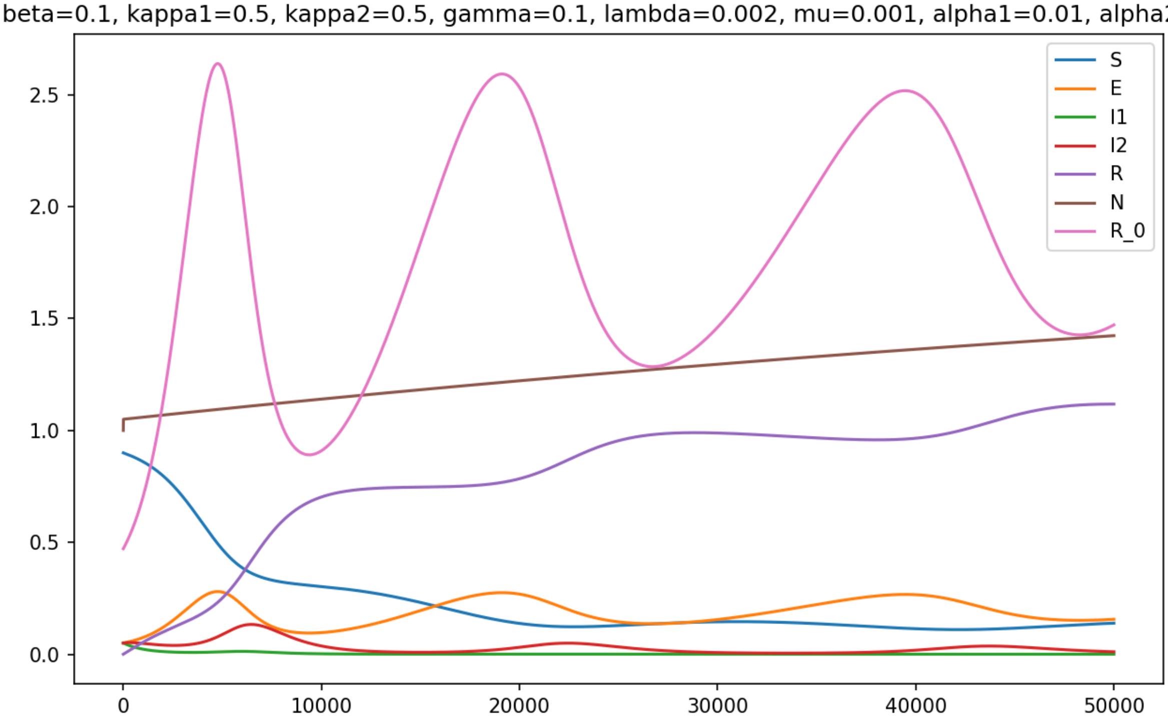

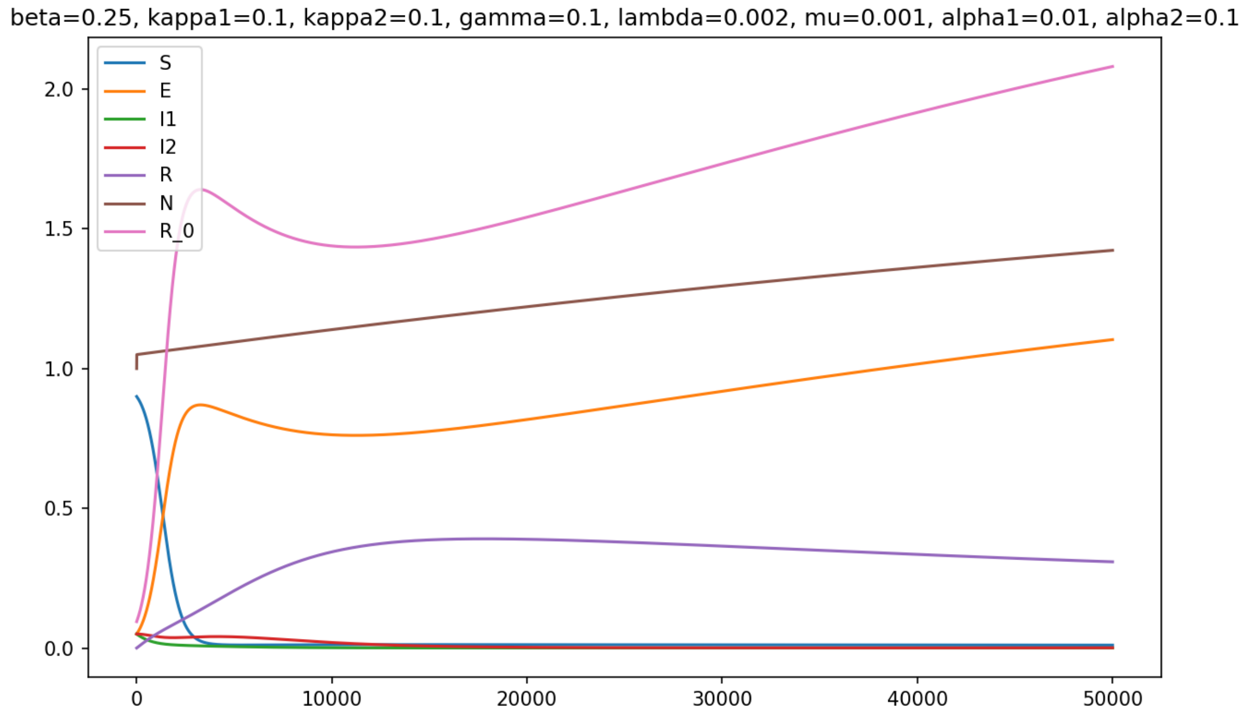
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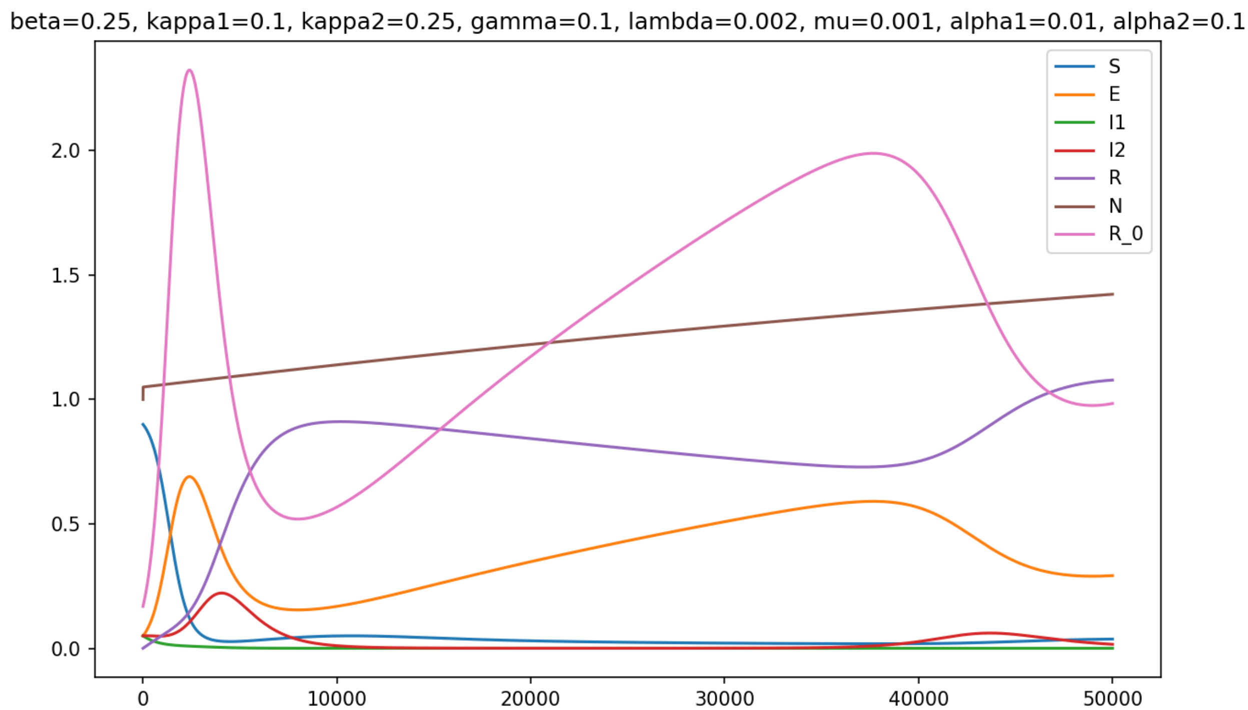
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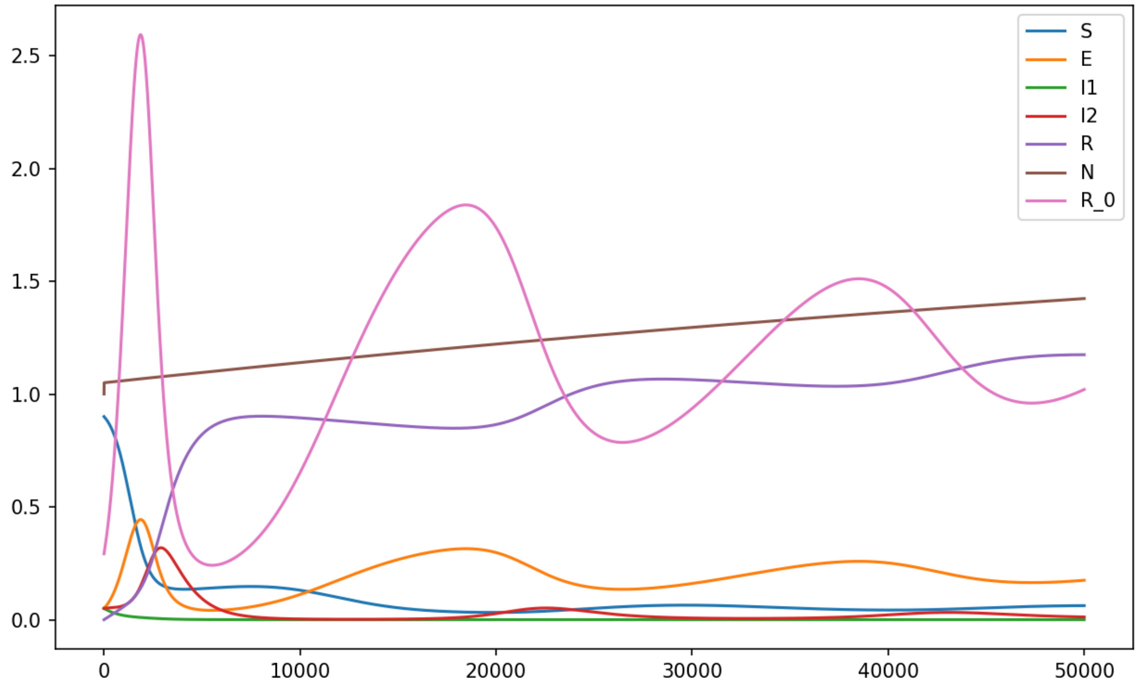

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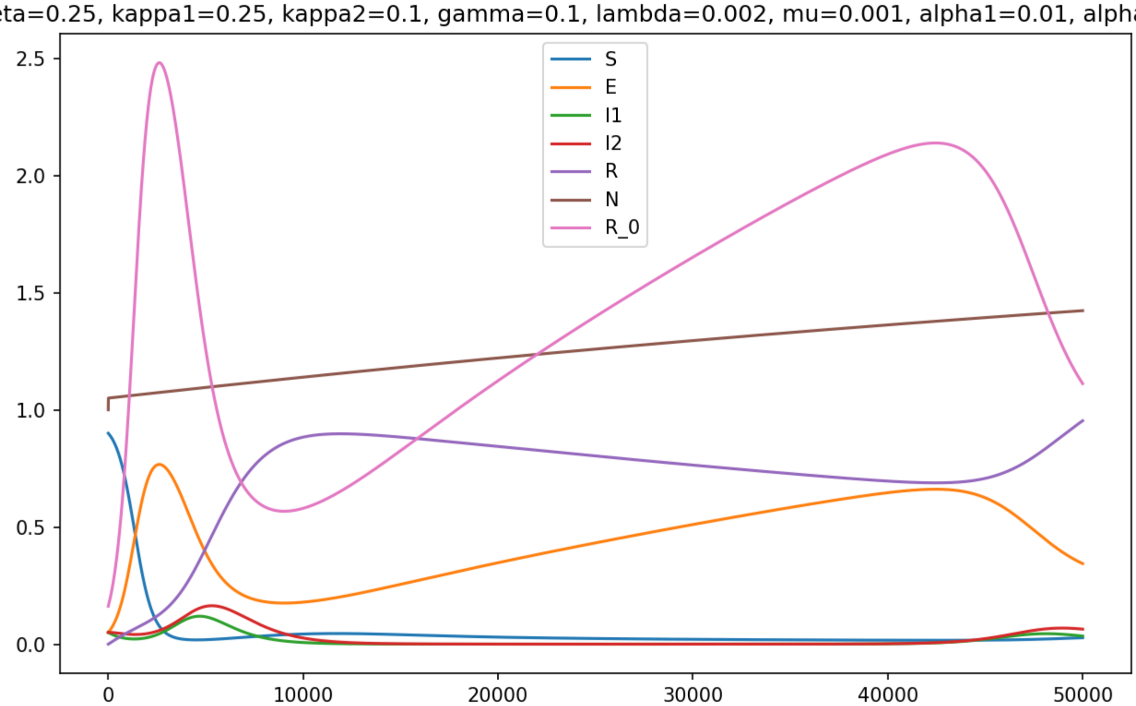

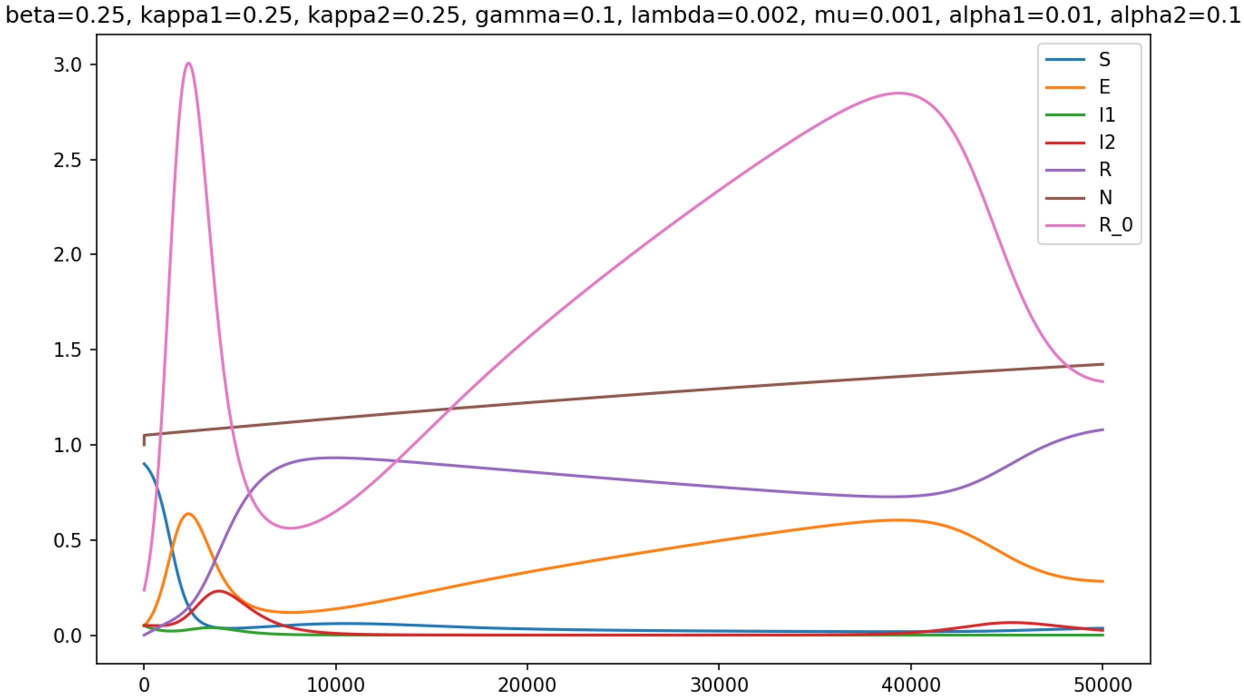
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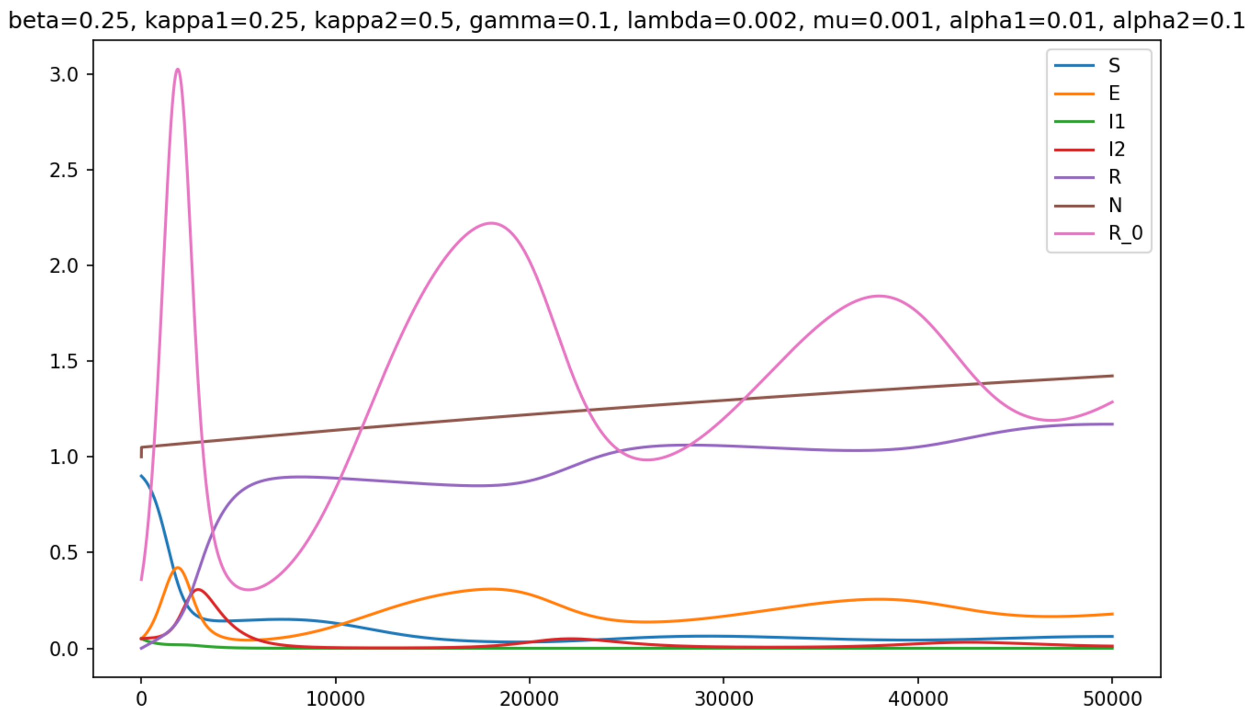
I1
I2
R
N
R_0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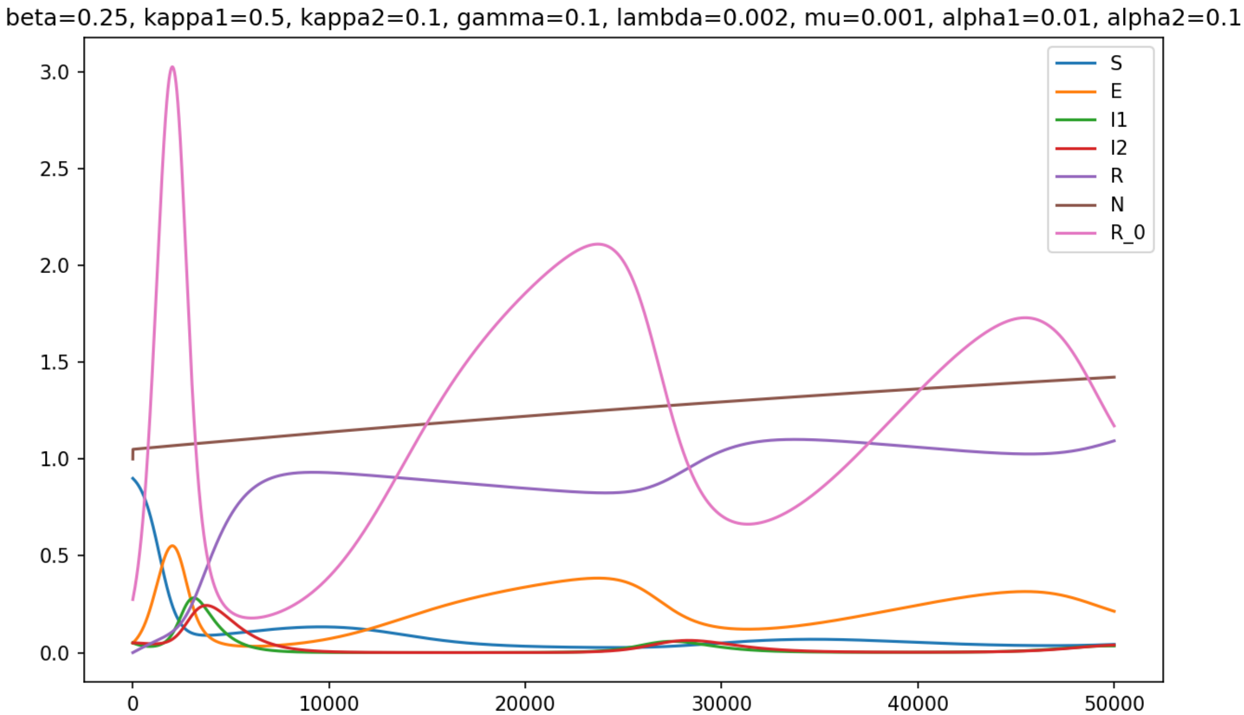
S
E
I1
I2
R
N
R_0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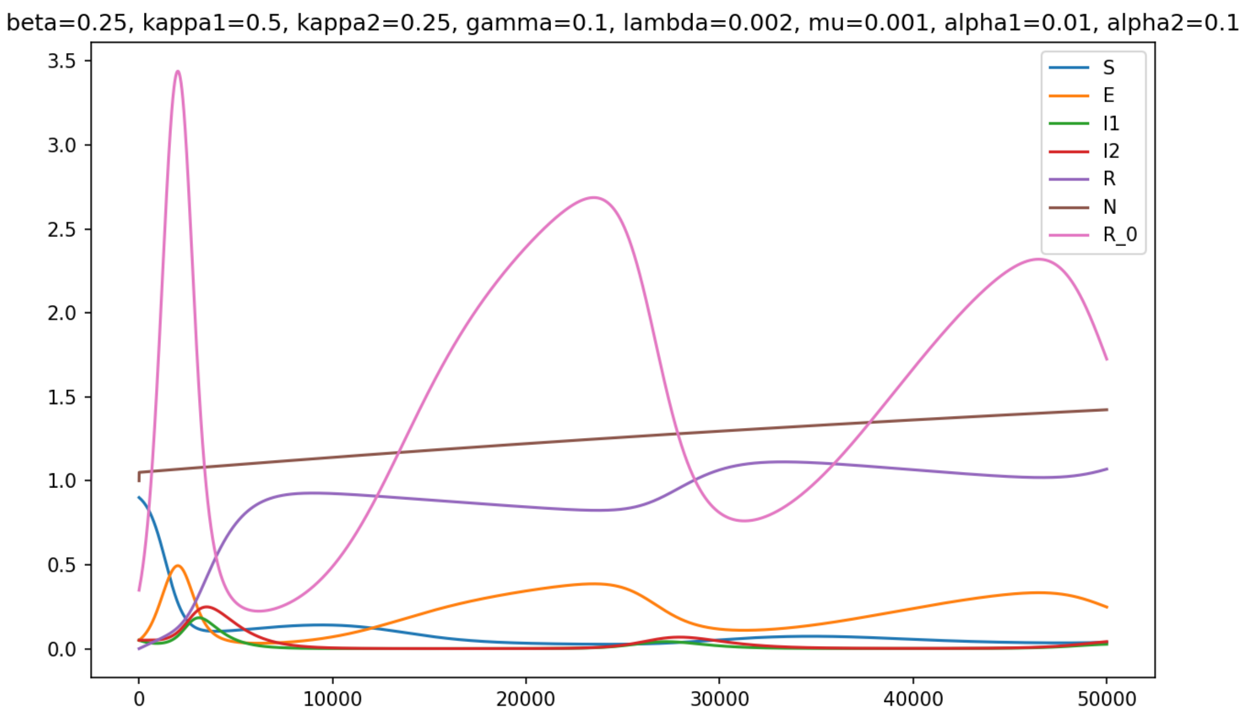
S
E
I1
I2
R
N
R_0
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
10000
20000
30000
40000
50000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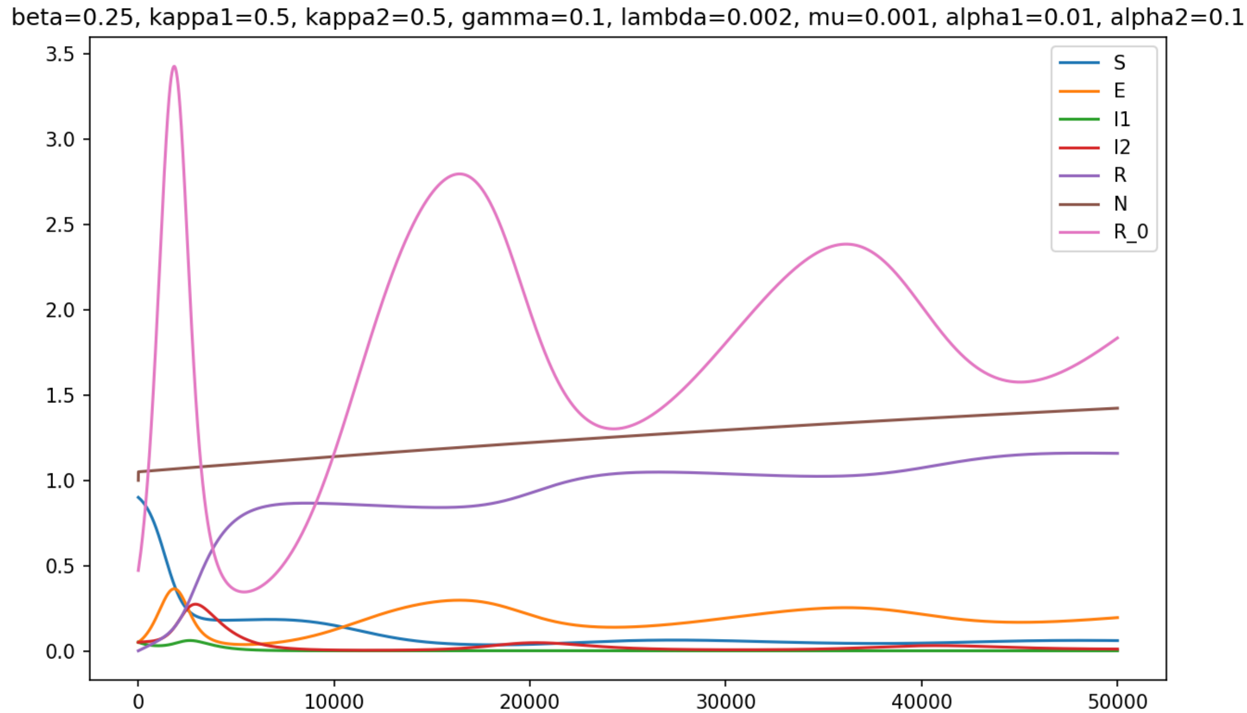
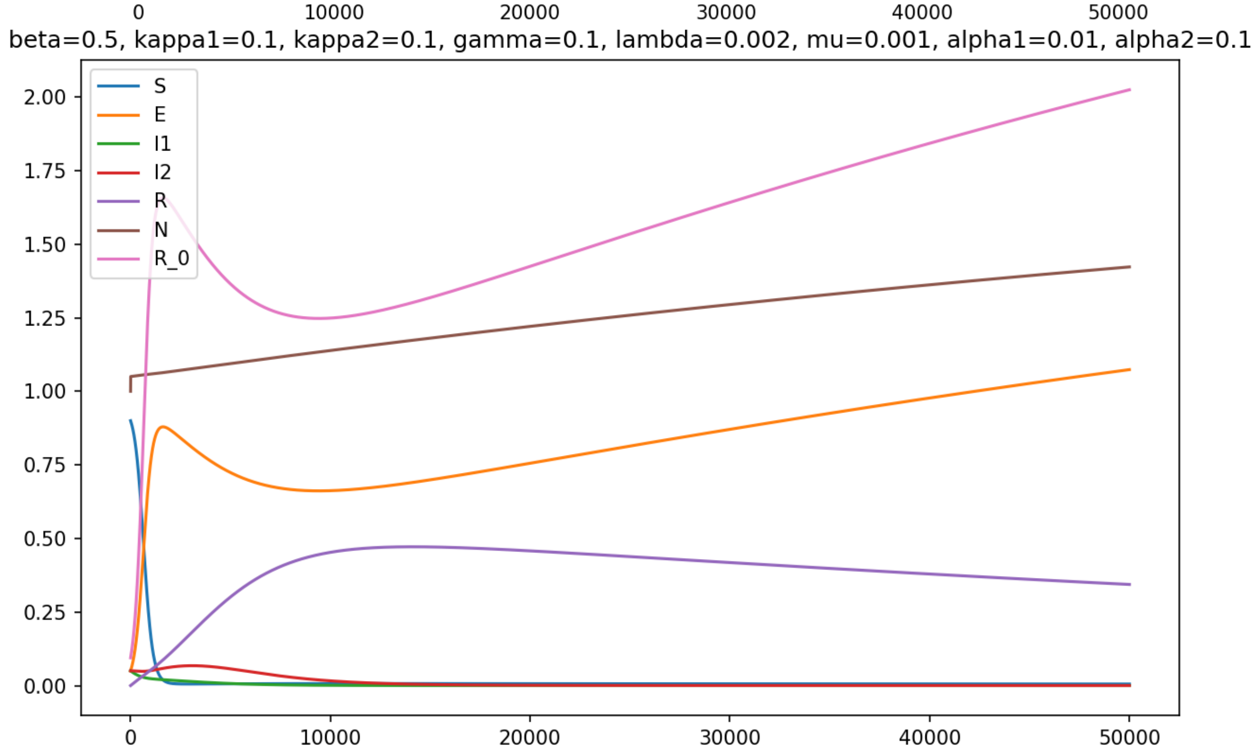
S
E
I1
I2
R
N
R_0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
10000
20000
30000
40000
50000


beta=0.25, kappa1=0.5, kappa2=0.5, gamma=0.1, lambda=0.002, mu=0.001, alpha1=0.01, alpha2=0.1
S
E
I1
I2
R
N
R_0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
10000
20000
30000
40000
50000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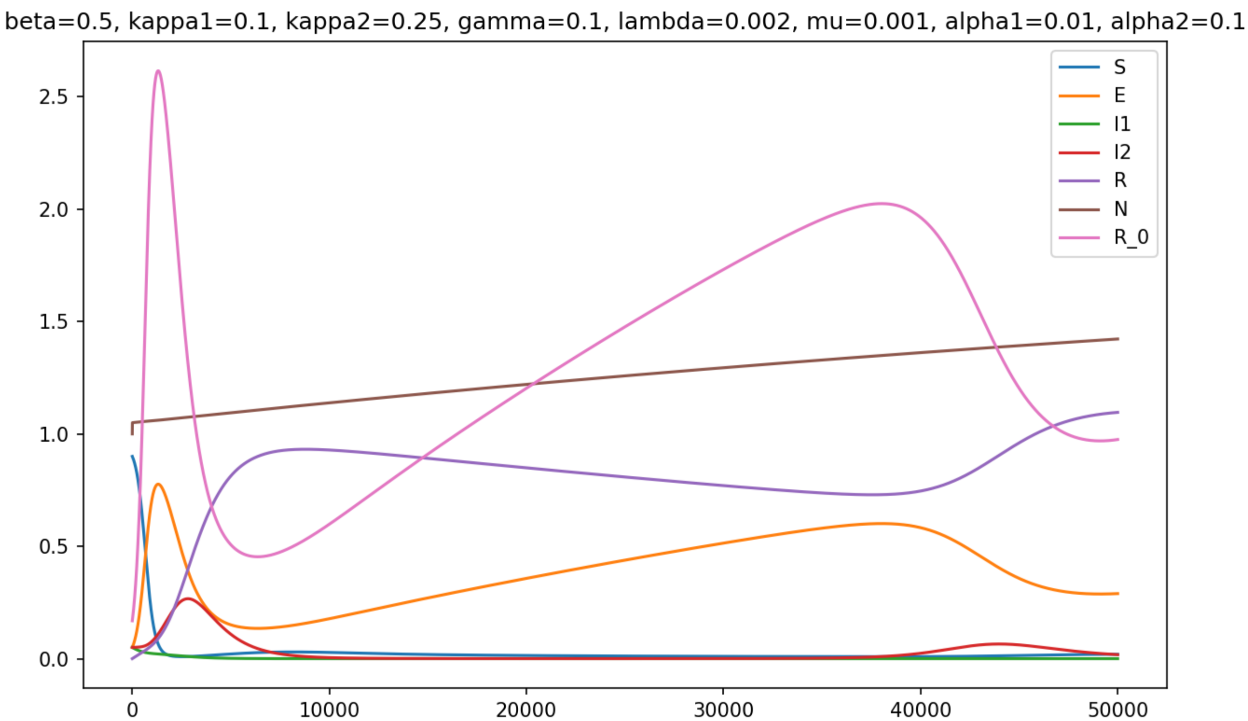
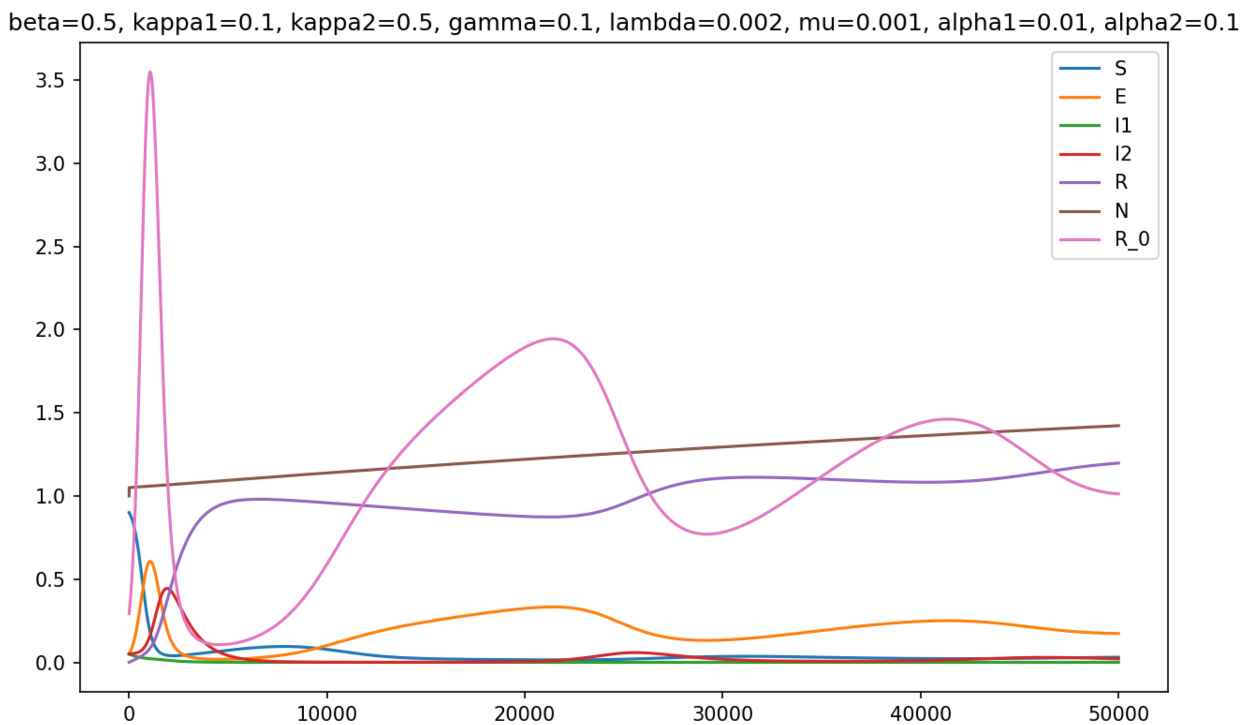
S
E
I1
I2
R
N
R_0
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
10000
20000
30000
40000
50000

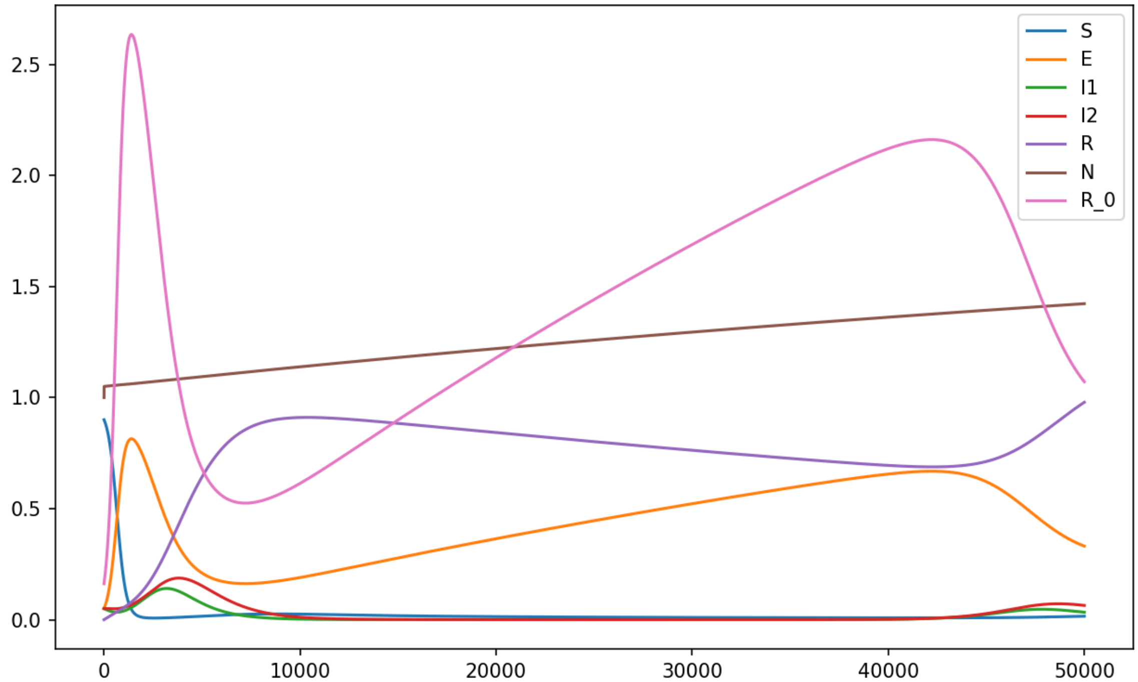


beta=0.5, kappa1=0.1, kappa2=0.25, gamma=0.1, lambda=0.002, mu=0.001, alpha1=0.01, alpha2=0.1
S
E
I1
I2
R
N
R_0
0.0
0.5
1.0
1.5
2.0
2.5
0
10000
20000
30000
40000
50000



beta=0.5, kappa1=0.1, kappa2=0.5, gamma=0.1, lambda=0.002, mu=0.001, alpha1=0.01, alpha2=0.1
S
E
I1
I2
R
N
R_0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
0
10000
20000
30000
40000
50000

beta=0.5, kappa1=0.25, kappa2=0.1, gamma=0.1, lambda=0.002, mu=0.001, alpha1=0.01, alpha2=0.1



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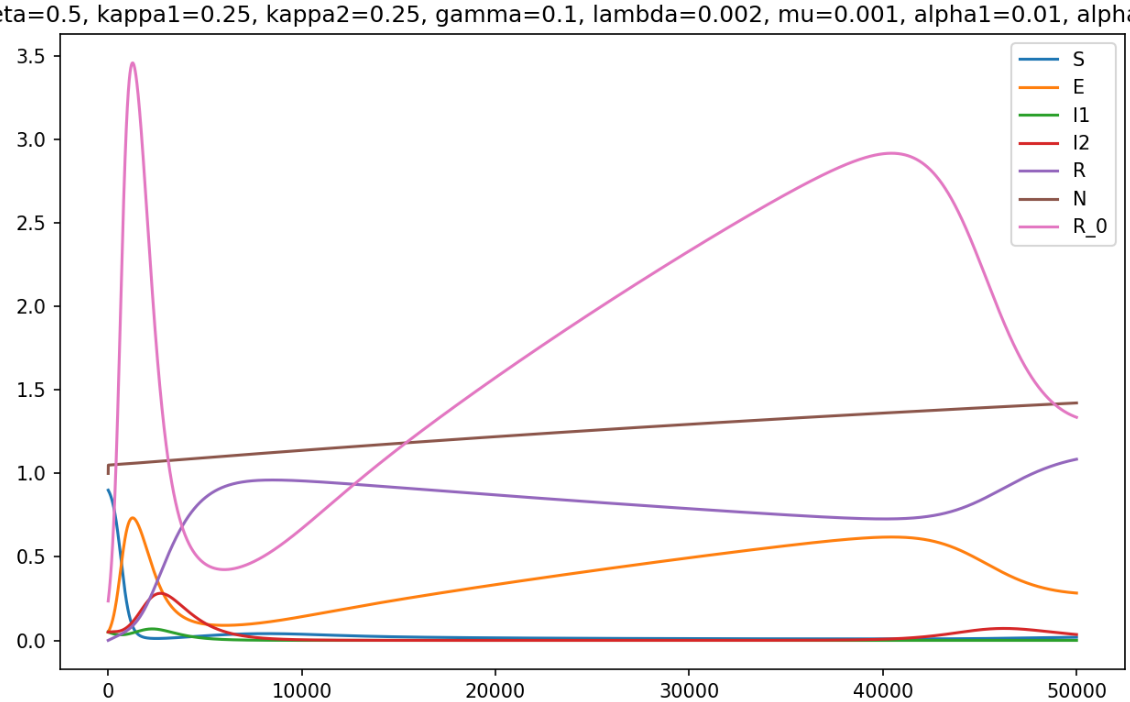
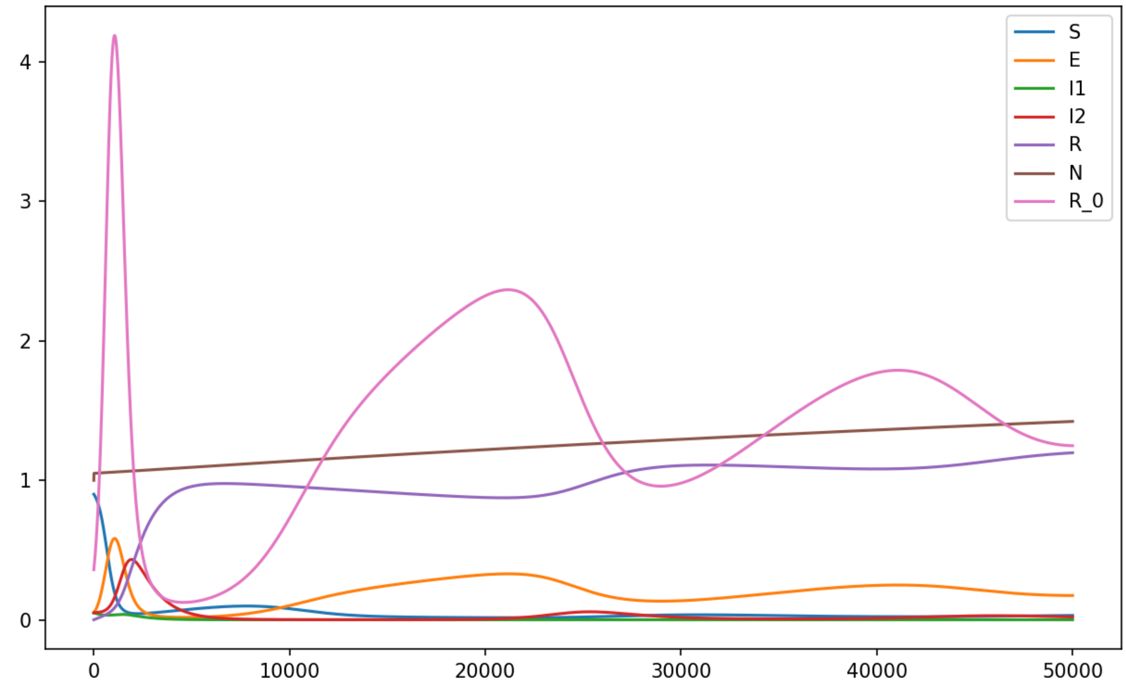
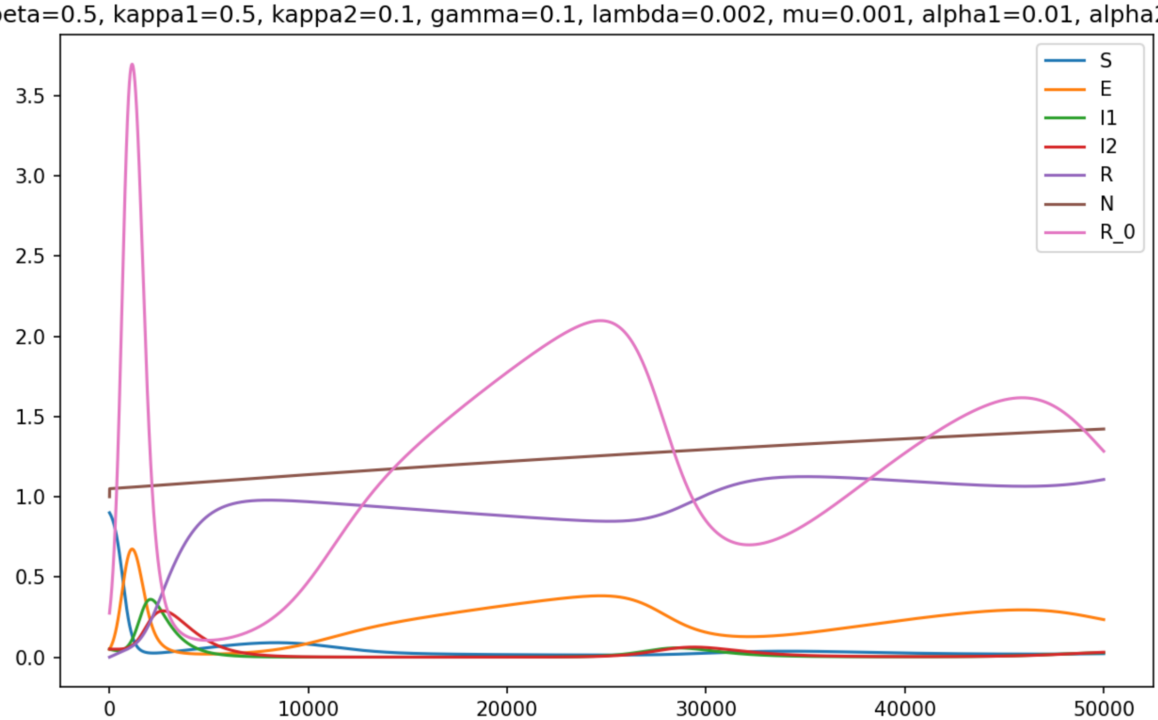
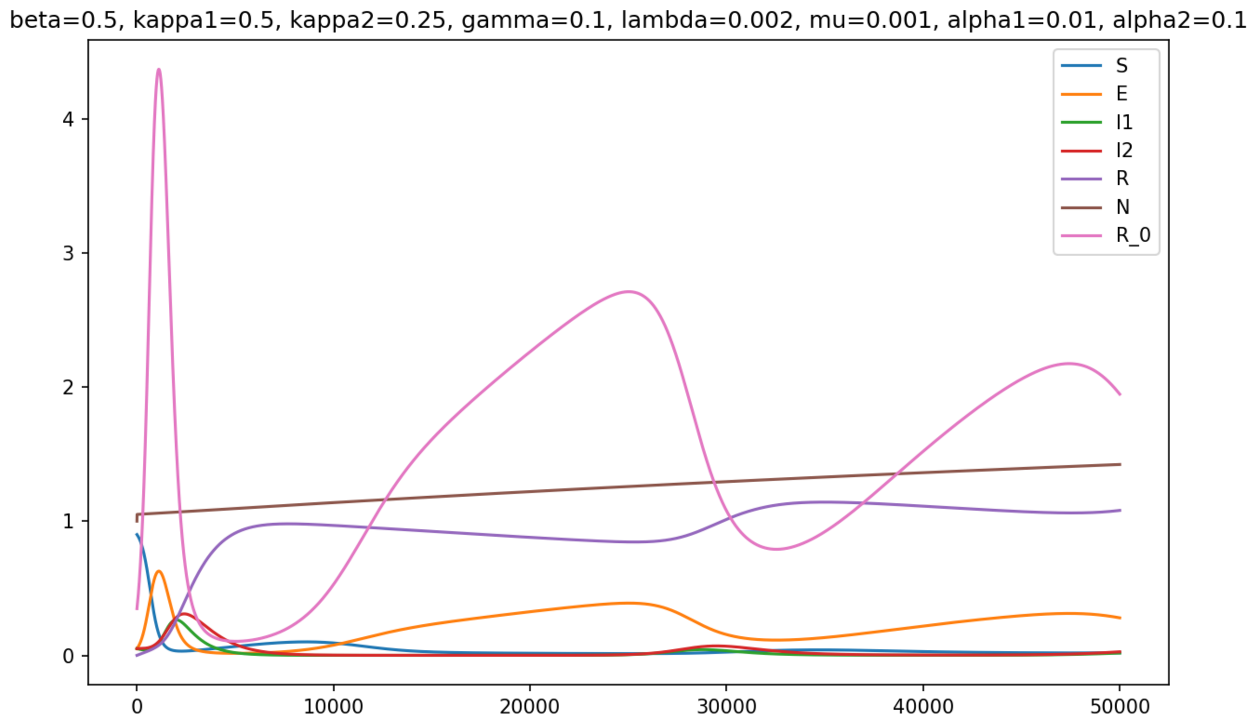


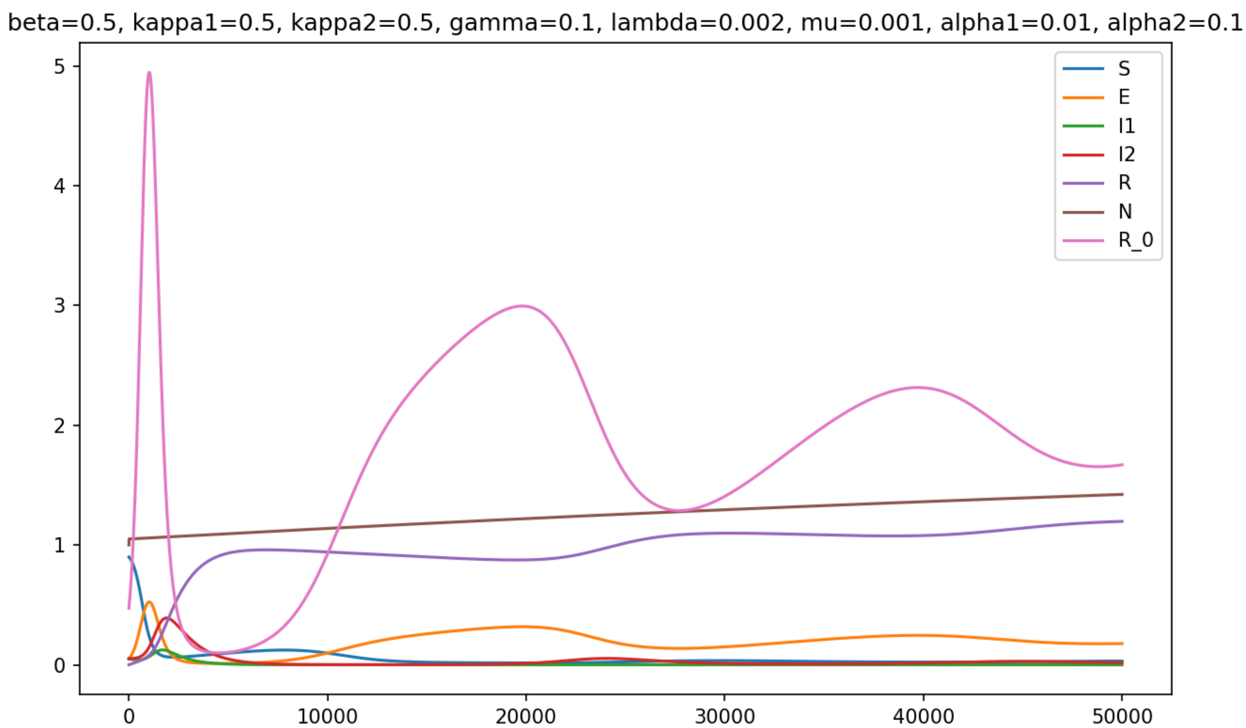
beta=0.5, kappa1=0.25, kappa2=0.5, gamma=0.1, lambda=0.002, mu=0.001, alpha1=0.01, alpha2=0.1



beta=0.5, kappa1=0.5, kappa2=0.1, gamma=0.1, lambda=0.002, mu=0.001, alpha1=0.01, alpha2=0.1







There is a lot going on. First, R_0 changes over time because E is not really an infected class and most figures shows R_0 that skyrocket above 1 and then below 1 and repeat with less intensity. This shows how intense ("height" of the peak) and fast ("width" of the peak) the epidemic is. We can see that the disease is endemic because obviously the disease "waits" until the E class becomes low to break out again. This is due to deaths, births, all the recovers and change from E to I_1 and I_2.