1 Introduction

1.1 What is Celestial Mechanics?

Subject

- Motion of celestial bodies

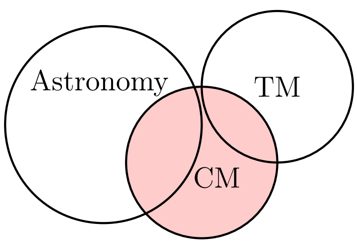
Basis: two laws by Isaac Newton (1642 - 1727)

- Newton's second law of dynamics
- Newton's universal law of gravitation

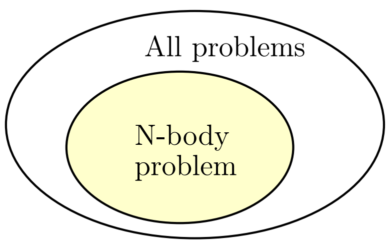
Extensions

- Relativistic CM:
  → replace Euclidean space and Newtonian mechanics by non-Eucledean space within the framework of General Relativity
- Non-gravitational CM:
  → add e.g. radiation pressure, Lorentz force, friction in Earth's atmosphere, ...
- Space flight / Astrodynamics:
  → study the motion of spacecraft. There is a possibility to actively affect the motion by turning thrusters on/off. What is the best strategy, e.g., to fly from Earth to Mars? Involves control theory, optimisation theory etc.
- Statistical CM:
  → study a large ensemble of bodies. E.g. insert a planet into a disk of test particles; how will disk properties change, what structures will be induced?

Relation to other sciences



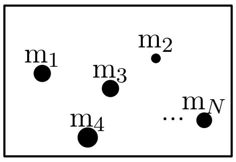
Problems



1.2 The N-body Problem

1.2.1 Formulation

Given:



N point masses,
exerting gravity force on each other

To find:

$$\bar{r}_i\,(t)\,,\dot{\bar{r}}_i\,(t) \quad i = 1, 2, \ldots, N$$

Why is the N-body problem so important in Celestial Mechanics? Because it is applicable to many astrophysical problems. But why that? For two reasons: (1) If the distances between the bodies are much larger than their sizes (which is true in most of the astronomical systems), they can well be approximated by point masses. (2) Furthermore, if a body is close to a sphere in shape (which is also usually true), one can replace a sphere with a point mass of the same mass in its center. The latter was first shown by Newton in his *"Principia Mathematica Philosophiae Naturalis"*.

1.2.2 Solution

1. Choose a reference system, to be able to define $\bar{r}_i$, $\dot{\bar{r}}_i$, $i = 1, 2, \ldots, N$
2. Write down the force on the particles as a function of the positions $\bar{r}_j$. This argument is sufficient for a classical, purely gravitational N-body problem. However, for the sake of generality, we can also include $\dot{\bar{r}}_j$ (to add frictional forces, if any), and t (to add forces that may depend on time explicitly):

$$\bar{F}_i = \bar{F}_i\,(m_j, \bar{r}_j, \dot{\bar{r}}_j, t)\,; \quad i, j = 1, 2, \ldots, N \tag{1.1}$$

3. Write down the equations of motion

$$\boxed{m_i \ddot{\bar{r}}_i = \bar{F}_i\,(m_j, \bar{r}_j, \dot{\bar{r}}_j, t)} \tag{1.2}$$

   Mathematically, this is a system of ODE of order $6N$
4. Specify initial conditions (positions and velocities; $6N$ conditions in total)

$$\boxed{\bar{r}_i\,(t = 0) \equiv \bar{r}_i^0; \quad \dot{\bar{r}}_i\,(t = 0) \equiv \dot{\bar{r}}_i^0} \tag{1.3}$$

5. Solve equations (1.2) with initial conditions (1.3). There is always one and only one solution!

1.2.3 Particular cases

N=0:
no particles, thus nothing to solve

N=1:
solution first described by Galileo Galilei (1564-1642), but got known as the Newton's 1st law: *"Each body that does not experience any forces moves along a straight line at a constant speed."*

N=2, restricted: $m \ll M$. Also known as problem of one attracting center.

- large mass M moves as if $N = 1$; "test particle" m feels gravitational force of M
- described by Johannes Kepler (1571-1630)
- together with the general $N = 2$ case thus often referred to as "Kepler problem"
- trajectories are conic sections (ellipse, parabola, hyperbola) and straight lines
- dependence on time $\bar{r}\,(t)$ very complex → time series needed to express

N=2, general case: M_1, M_2.
Mathematically identical to the restricted 2BP! Promise to show this in a future lecture.

N=3, restricted: $m \ll M_1, M_2$
"Two's a company, three's a crowd" (Proverb). Very complex trajectories, e.g. loops, lots of properties... tens of books, thousands of papers in scientific journals... Many applications: e.g. triple stellar systems such as α Cen, small bodies in the Solar system.

N=3:
formally solved by Karl Sundman, Finland (1912); but in the form of series that are converging extremely slowly: $\sim 10^{10000}$ (!) terms needed to reach reasonable accuracy

∀N:
formally solved by Quidong Wang, U. Arizona (CelMech 1991, **50**, 73–88)

1.3 Methods of Celestial Mechanics

Analytic(al)

Idea: find a general solution to equation (1.2) as analytic formulas

E.g. perturbation method. One seeks an approximate solution to a problem, by starting from the exact solution of a related simpler problem. Example is a system consisting of a star and N planets. Neglecting mutual grav interactions between the planets, we have "N times 2-body problem". Solution to it is known: N ellipses. We then include interaction between the planets and try to find corrections to those elliptic motions.

The groundwork was done by Joseph Louis Lagrange (1736-1813) and Karl Friedrich Gauss (1777-1855)

A great success of perturbation method: discovery of Neptune. Predicted from analysis of Uranus motion by John Couch Adams (1819-1892) and Urbain Le Verrier (1811-1877)
Discovered in 1846 by Johann Galle in Berlin

Qualitative

Idea: find essential properties of the solution without finding the solution itself

E.g. stability problem for the Solar system. Kolmogorov-Arnold-Moser (KAM) theory (1960s) found that the Solar system will be stable if (i) planet masses are sufficiently small and (ii) for most of the initial conditions. For (i), the original estimate gave... $M_{planet} < M_{electron}$. Later on, the estimates were dramatically improved, showing that actual planetary masses are fine. However, were the planets $\sim$ 30 times more massive than they are, the system would no longer be stable. For (ii), the difficulty is the "swiss cheese" structure of the initial condition space (every point of which corresponds to one possible initial state of the system). In an arbitrarily close vicinity of any "good" (stable) point, there is always at least one "bad" (unstable) point...

Numerical

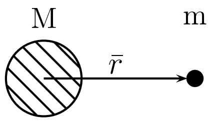
Idea: find a solution to equation (1.2) as numbers for one set of (1.3)

This way works always, e.g. to compute the motion of spacecraft with required accuracy. However, solutions for different initial conditions remain unknown.

2 The two-body problem

2.1 Equation of motion

Restricted two-body problem ≡ one-center problem ($m \ll M$)



According to the Newton's law of gravitation,

$$\bar F = -G\,\frac{M\,m}{r^3}\,\bar r$$

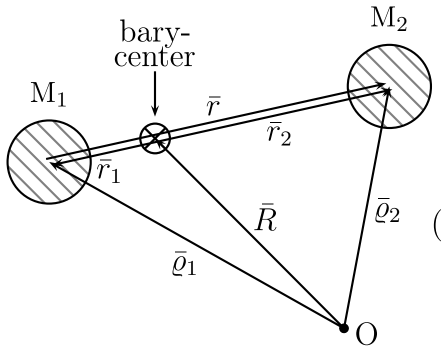
According to the Newton's 2nd law,

$$\bar F = m\,\ddot{\bar r}$$

Thus

$$\boxed{\ddot{\bar r} + G\,\frac{M}{r^3}\,\bar r = 0} \tag{2.1}$$

Two-body problem – relative motion



$$M_1\,\ddot{\bar\varrho}_1 = \quad G\,\frac{M_1\,M_2}{r^3}\,\bar r \tag{2.2}$$

$$M_2\,\ddot{\bar\varrho}_2 = -G\,\frac{M_1\,M_2}{r^3}\,\bar r \tag{2.3}$$

$$(2.2) + (2.3) \Rightarrow M_1\,\ddot{\bar\varrho}_1 + M_2\,\ddot{\bar\varrho}_2 = 0$$
$$\text{Integrate} \Rightarrow M_1\,\dot{\bar\varrho}_1 + M_2\,\dot{\bar\varrho}_2 = \bar A, \qquad \bar A = \text{const}$$
$$\text{Integrate} \Rightarrow M_1\,\bar\varrho_1 + M_2\,\bar\varrho_2 = \bar A\,t + \bar B, \qquad \bar B = \text{const}$$

Consider the center of mass (a.k.a. barycenter) of the system:

$$\bar R \equiv \frac{M_1\,\bar\varrho_1 + M_2\,\bar\varrho_2}{M_1 + M_2} \quad \Rightarrow \quad \bar R = \frac{\bar A}{M_1 + M_2}\,t + \frac{\bar B}{M_1 + M_2}; \quad \dot{\bar R} = \frac{\dot{\bar A}}{M_1 + M_2}, \tag{2.4}$$

which means that the center of mass moves along a straight line $\bar R$ at a constant speed.

Application: Sirius, the brightest star on the sky. In 1844, Friedrich Wilhelm Bessel analyzed the astrometric data and found the trajectory on Sirius on the sky to be "sine"-like. He interpreted this by a presence of an invisible companion of comparable mass. If such a companion ("Sirius B") were present, the center of mass of Sirius A + B could be moving along a straight line. Later on, this companion was indeed discovered (by Clark, 1862). It turned out to be the first white dwarf ever discovered. White dwarfs have normal stellar masses, but much lower luminosities, which makes them faint. Sirius B has mag = 8^m*. This explains why the companion was not observed earlier.*

We can now write down the equation of motion:

$$\frac{(2.3)}{M_2} - \frac{(2.2)}{M_1} \quad \Rightarrow \quad \boxed{\ddot{\bar{r}} + G\,\frac{M_1 + M_2}{r^3}\,\bar{r} = 0} \tag{2.5}$$

<u>*Two-body problem – barycentric motion*</u>

Using the equation for the center of mass

$$\bar{r}_2 = \frac{M_1}{M_1 + M_2}\,\bar{r}$$

converted to express $\bar{r}$

$$\bar{r} = \frac{M_1 + M_2}{M_1}\,\bar{r}_2$$

one can rewrite equation (2.5) as follows:

$$\frac{M_1 + M_2}{M_1}\,\ddot{\bar{r}}_2 + G\,\frac{M_1^2}{(M_1 + M_2)\;r_2^3}\,\bar{r}_2 = 0$$

$$\boxed{\ddot{\bar{r}}_2 + G\,\frac{M_1^3/\left(M_1 + M_2\right)^2}{r_2^3}\,\bar{r}_2 = 0} \tag{2.6}$$

<u>*Comparison*</u>

Comparing equations (2.1), (2.5) and (2.6), we see that they all have the same form

$$\boxed{\ddot{\bar{r}} + \kappa^2\,\frac{\bar{r}}{r^3} = 0}, \qquad \kappa = \text{const}\,. \tag{2.7}$$

Just the meaning of $\bar{r}$ and κ differs from one problem to another, but mathematically all these versions of the two-body problem are the same.

The equations of motion are too complicated to be solved directly. Instead of solving them, a usual approach is to seek the so-called integrals of the motion.

An integral of the motion is any combination of time, positions, and velocities of the bodies that remains constant at all times:

$$f\,(t, \bar{r}, \dot{\bar{r}}) = \mathit{const.}$$

Any integral of the motion usually tells us something, i.e., implies a certain property of the motion. Thus finding every integral is useful.

Furthermore, if we are able to find N independent integrals of the motion, where N is the order of the system, the problem will be solved. The problem is said to be "integrable in quadratures." Looking back at the (general) two-body problem, it had an order 12 originally. We were able to find 6 integrals, Eqs. (2.4) (with six arbitrary constants $\bar{A}, \bar{B}$). This allowed us to reduce the order of the systems from 12 to $12 - 6 = 6$, see eqs. (2.7). Thus it only remains to find another six integrals – and the two-body problem will be solved completely! Let's start looking for those integrals...

2.2 The angular momentum integral

Calculating the cross-product of $\bar{r}$ and equation (2.7) yields:

$$\bar{r} \times \ddot{\bar{r}} + \kappa^2 \frac{\overbrace{\bar{r} \times \bar{r}}^{0}}{r^3} = 0$$

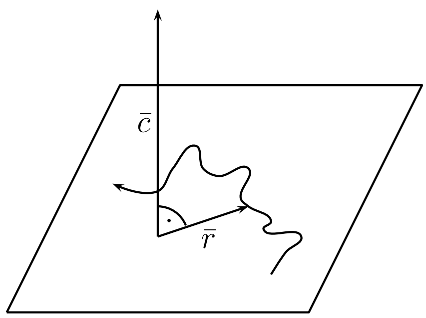
Integrating this gives:

$$\boxed{\bar{r} \times \dot{\bar{r}} = \bar{c}}, \qquad \bar{c} = \text{const}, \tag{2.8}$$

which is known as the angular momentum integral. In fact, these are three (scalar) integrals.

The most important implication is that motion is two-dimensional (planar). This is because the angular momentum vector $\bar{c}$ is perpendicular to the plane containing $\bar{r}$ and $\dot{\bar{r}}$ and that $\bar{c} = \text{const}$ (see figure).

By the way, this does not necessarily require a newtonian gravity force. The angular momentum integral obviously exists for every force that is radial (e.g. radiation pressure).



2.3 The energy integral

Calculating now the scalar product of $\bar{v} \equiv \dot{\bar{r}}$ and equation (2.7) yields:

$$\bar{v} \cdot \frac{\mathrm{d}\bar{v}}{\mathrm{d}t} + \kappa^2 \, \frac{\bar{r} \cdot \bar{v}}{r^3} = 0$$

The left-hand side can be transformed to:

$$\bar{v} \cdot \frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}\right)$$

and

$$\begin{aligned}
\kappa^2 \, \frac{\bar{r} \cdot \bar{v}}{r^3} &= \kappa^2 \, \frac{\bar{r} \cdot \dot{\bar{r}}}{r^3} = \kappa^2 \, \frac{r \, \dot{r}}{r^3} \\
&= \kappa^2 \, \frac{\dot{r}}{r^2} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\kappa^2}{r}\right),
\end{aligned}$$

where we used the fact that $\bar{r} \cdot \dot{\bar{r}} = r\dot{r}$, since in the polar coordinates $\bar{r} = \{r, 0, 0\}$ and $\dot{\bar{r}} = \{\dot{r}, 0, 0\}$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\kappa^2}{r}\right) = 0,$$

which can be integrated to

$$\boxed{\frac{v^2}{2} - \frac{\kappa^2}{r} = \frac{h}{2}}, \qquad h = \mathrm{const.} \tag{2.9}$$

This is known as energy integral (this time only one — not three, unfortunately). The name is easy to understand: multiplying with the mass, we get the sum of the kinetic energy and potential energy, i.e., the total mechanical energy. The integral tells us that the total energy is conserved.

We now consider a limiting case $r \to \infty$. Since h is an arbitrary constant, it can be negative, zero, or positive:

- let $h < 0$:
  $$\underbrace{\frac{v^2}{2}}_{\geq 0} = \underbrace{\frac{\kappa^2}{r}}_{0} + \underbrace{\frac{h}{2}}_{<0}$$
  $\to$ a problem! $\to$ we can not let r go to infinity $\to$ motion must be finite
- let $h = 0$: $\lim\limits_{r\to\infty} v^2 = 0 \to$ infinite motion possible
- let $h > 0$: $\lim\limits_{r\to\infty} v^2 > 0 \to$ infinite motion possible

Strictly speaking, we don't know yet what the shape of the orbit is. But we will learn later that the three cases considered above correspond to ellipses, parabolas, and hyperbolas...

2.4 The Laplace integral

In a similar style to derivation of previous integrals, one can compute the cross-product of $\bar{c}$ with Eq. (2.7) to derive the Laplace integral

$$\frac{\dot{\bar{r}} \times \bar{c}}{\kappa^2} - \frac{\bar{r}}{r} = \bar{e}, \qquad \bar{e} = \text{const.} \tag{2.10}$$

Unfortunately, this does not provide us with three new scalar integrals — rather, with one only. This is because there exist two relations between the angular momentum, energy, and Laplace integrals:

$$\bar{c} \cdot \bar{e} = 0 \qquad \text{(obvious)}$$

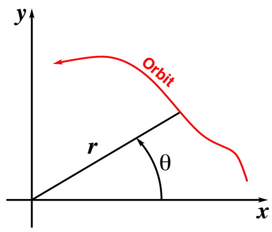
and

$$\kappa^4(e^2 - 1) = 2hc^2 \qquad \text{(see, e.g., Karttunen et al. book for derivation)}$$

By using the Laplace integral, one can finally determine the shape of the orbits. In what follows, we do it with a different method, through polar coordinates.

To summarize, we have got 5 independent integrals. The last, 6th integral is related to time (or to the position of a particle in its orbit, rather than orbit itself). Indeed, from theoretical mechanics, it is known that an autonomous system of the 6th order has no more than 5 autonomous integrals. ["Autonomous" = "not depending on time explicitly."]

2.5 Integrals in polar coordinates



We choose the reference frame is such a way that the planar motion occurs within the x-y-plane. Then,

$$\bar{r} = \{x, y, 0\}\,, \qquad \dot{\bar{r}} = \{\dot{x}, \dot{y}, 0\}\,, \qquad \bar{c} = \{0, 0, c\}\,,$$

and the angular momentum integral $\bar{r} \times \dot{\bar{r}} = \bar{c}$ takes the form:

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$

or simply

$$x\,\dot{y} - \dot{x}\,y = c \tag{2.11}$$

The energy integral is

$$\dot{x}^2 + \dot{y}^2 = \frac{2\kappa^2}{r} + h \tag{2.12}$$

We now express both integrals in polar coordinates by using

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases}$$

Differentiation gives

$$\begin{cases} \dot{x} &= \dot{r}\cos\theta - r\sin\theta\dot{\theta} \\ \dot{y} &= \dot{r}\sin\theta + r\cos\theta\dot{\theta} \end{cases}$$

Substituting these exprerssions for x, y, $\dot{x}$, $\dot{y}$ into Eqs. (2.11) and (2.12), after some algebra we obtain

$$\boxed{r^2\,\dot{\theta} = c} \tag{2.13}$$

$$\boxed{\dot{r}^2 + r^2\,\dot{\theta}^2 = \frac{2\kappa^2}{r} + h} \tag{2.14}$$

2.6 Geometry of the orbits, Kepler's 1st law

We now rework Eqs. (2.13)–(2.14) a bit. First of all, we use Eq. (2.13) to exclude time from (2.14):

$$\mathrm{d}t = \frac{r^2}{c}\ \mathrm{d}\theta,$$

assuming $c \neq 0$. This can be inserted into Eq. (2.14):

$$\frac{c^2}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + \frac{c^2}{r^2} = \frac{2\kappa^2}{r} + h$$

Let us use the Binet variable $u \equiv 1/r$ instead of r:

$$c^2u^4\left(\frac{\mathrm{d}\,(1/u)}{\mathrm{d}\theta}\right)^2 + c^2u^2 = 2\kappa^2 u + h$$

or

$$c^2u'^2 + c^2u^2 = 2\kappa^2 u + h,$$

where prime denotes the derivative with respect to θ, i.e., $' \equiv \mathrm{d}/\,\mathrm{d}\theta$.
We now differentiate once again:

$$2c^2u'u'' + 2c^2uu' = 2\kappa^2 u'.$$

Assuming that $u' \neq 0$ we finally get:

$$u'' + u = \frac{\kappa^2}{c^2}.$$

A general solution to this equation is known to be:

$$u = \frac{\kappa^2}{c^2} + A\ \cos\left(\theta + B\right), \qquad A, B = \text{const},$$

where can can set $B \equiv 0$ (because it only affects the orientation of the orbit). We now define some new constants e and p instead of A and c by means of

$$\frac{\kappa^2}{c^2} \equiv \frac{1}{p}\ \ , \quad A \equiv \frac{e}{p}$$

to get

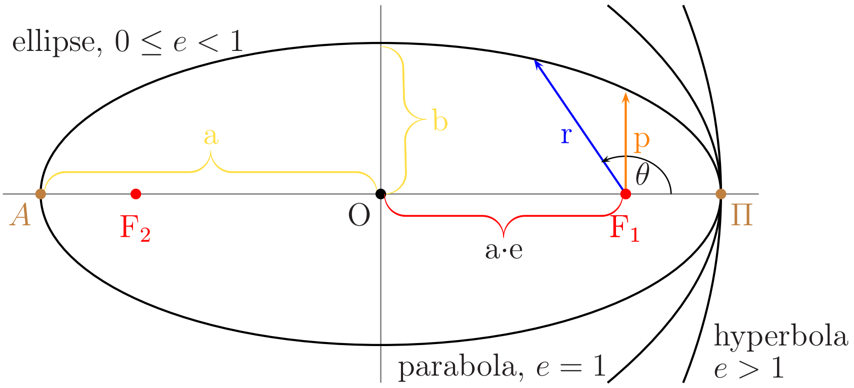
$$u = \frac{1 + e\ \cos\theta}{p}$$

or

$$\boxed{r = \frac{p}{1 + e\ \cos\theta}}, \tag{2.15}$$

which is a well-known equation of a conic section.

In Eq. (2.15),
θ is the *true anomaly*
e is the eccentricity,
p is called *semilatus rectum* ("senkrechte Halbseite").



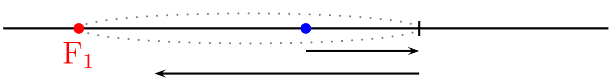
F_1, F_2	-	foci (engl. [’fəʊsaɪ])
Π, A	-	pericenter and apocenter, the apsides [’æpsidiːs]
a	-	semimajor axis
b	-	semiminor axis; $b = a\ \sqrt{1-e^2}$
p	-	semilatus rectum, $p = a\ (1-e^2)$
r_Π	-	pericenter distance, $r_\Pi = a\ (1-e)$
r_A	-	apocenter distance, $r_A = a\ (1+e)$

Applied to the motion of planets around the Sun, **Kepler's first law** is proven:

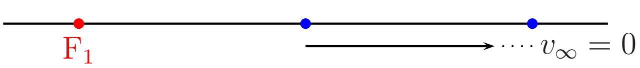
“Each planet moves in an ellipse with the Sun at one focus.“

We now go back to two assumptions we made (highlighted in light green) and check what changes if these are not satisfied:

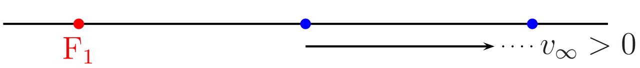
- let $u' = 0$: $u = \text{const} \Leftrightarrow r = \text{const} \Leftrightarrow$ circular case($e = 0$) $\Leftrightarrow$ nothing new
- let $\bar{c} = 0 : \bar{r} \times \bar{v} = 0 \Leftrightarrow \bar{r} \parallel \bar{v} \Leftrightarrow$ rectilinear motion with three new cases:



$h < 0$: degenerate ellipse;
particle falls to F_1



$h = 0$: degenerate parabola;
particle stops at infinity



$h > 0$: degenerate hyperbola;
reaches infinity with $v_\infty > 0$

One can show that for all three degenerate types the eccentricity $e = 1$!

Summary: There are 6 different types of orbits in the two-body problem: ellipse, parabola, hyperbola, rectilinear motion of elliptic type, parabolic type, and hyperbolic type.

2.7 The energy constant

Consider the elliptical case, so that $p = a(1-e^2)$.
Let the body of interest be at the pericenter Π. Then, $\theta = 0$, and:

$$r = \frac{p}{1+e\ \cos\theta} = \frac{a\,(1-e^2)}{1+e} = a\,(1-e) \tag{2.16}$$

At Π, the angular momentum integral takes the form:

$$\begin{aligned} |\bar{r} \times \bar{v}| &= c \\ r\ v &= \kappa\sqrt{p} \\ a\,(1-e)\ v &= \kappa\sqrt{a\ (1-e^2)} \\ v &= \frac{\kappa}{\sqrt{a}}\ \sqrt{\frac{1+e}{1-e}} \end{aligned}$$

(By the way, the velocity v at Π is the maximum velocity. The minimum velocity,

$$v = \frac{\kappa}{\sqrt{a}}\sqrt{\frac{1-e}{1+e}},$$

is reached at the apocenter A.)
At Π, the energy integral transforms to:

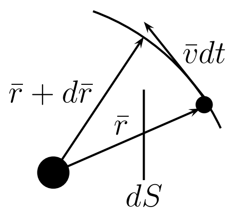
$$\begin{aligned} v^2 &= \frac{2\ \kappa^2}{r} + h \\ \frac{\kappa^2}{a}\ \left(\frac{1+e}{1-e}\right) &= \frac{2\ \kappa^2}{a\ (1-e)} + h \\ \Rightarrow & \boxed{h = -\frac{\kappa^2}{a}} \qquad (2.17) \\ \Rightarrow & \boxed{v^2 = \kappa^2\ \left(\frac{2}{r} - \frac{1}{a}\right)} \qquad (2.18) \end{aligned}$$

Eq. (2.17) shows that the total energy only depends on the semimajor axis and not, for instance, on the eccentricity.

Further, Eq. (2.18) shows that the semimajor axis only depends on the absolute value of the velocity at a given point and does not depend on the *directon* of the velocity vector at that point: $a = a\,(r, v)$.

Imagine a stone thrown from the Earth surface with the first cosmic velocity of 8 km/s. Eq. (2.18) *suggests that $a = R$, regardless of the direction:*
- *Thrown horizontically, it will fly round the Earth in a circle of radius $r = a = R$.*
- *Thrown at an angle, it will move in an ellipse with the semimajor axis $a = R$.*
- *Thrown vertically, it will reach a height R above the Earth surface and fall back, moving in a degenerated ellipse with $a = R$ as well.*

2.8 Kepler's 2nd and 3rd laws



Consider an infinitesimal sector described by the radius vector in time dt. Its area is approximately equal to the area of the triangle formed by the vectors $\bar{r}$ and $\bar{v}dt$ (see figure):

$$\mathrm{d}S = \frac{1}{2}\,|\bar{r}\times\bar{v}dt| = \frac{1}{2}\kappa\sqrt{p}dt$$

This imples that the *sectorial velocity* $\mathrm{d}S/\,\mathrm{d}t$ of a particle is constant:

$$\boxed{\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{2}\,\kappa\,\sqrt{p} = \mathrm{const}}. \tag{2.19}$$

We have proven **Kepler's second law** of planetary motion:

"The radius vector of a planet sweeps out equal area in equal times."

We can now integrate Eq. (2.19) over one orbital period P to find the area of the orbital ellipse S:

$$\begin{aligned} S &= \int \mathrm{d}S \\ &= \frac{1}{2}\,\kappa\,\sqrt{p}\,\int\limits_0^P \mathrm{d}t \\ &= \frac{1}{2}\,\kappa\,\sqrt{p}\,P. \end{aligned}$$

On the other hand,

$$S = \pi\,a\,b = \pi\,a^2\,\sqrt{1-e^2}.$$

Equating these two results, we obtain:

$$\begin{aligned} \pi\,a^2\,\sqrt{1-e^2} &= \frac{1}{2}\,\kappa\,\sqrt{p}\,P \\ 2\pi\,a^{3/2} &= \kappa\,P \\ \boxed{\frac{P^2}{a^3} = \frac{4\,\pi^2}{\kappa^2}} \end{aligned} \tag{2.20}$$

Let us apply this to the motion of a planet (of mass M) around the Sun (of mass $M_\odot$). For the relative ("heliocentric") motion, $\kappa^2 = G(M_\odot + M)$, and

$$\boxed{\frac{P^2}{a^3} = \frac{4\,\pi^2}{G(M_\odot + M)}}, \tag{2.21}$$

which is known as the Kepler's *generalized* third law. A standard (non-generalized) law can be obtained by noting that planetary masses are much smaller than the solar mass, so that M can be neglected:

$$\boxed{\frac{P^2}{a^3} \approx \frac{4\,\pi^2}{GM_\odot} = \text{const}}. \tag{2.22}$$

In words, **Kepler's third law** is:

"The square of the orbital period of a planet is proportional to the cube of the semimajor axis of its orbit."

Example 1: The orbital period of Pluto

$$P_♇ = P_\oplus \left(\frac{a_♇}{a_\oplus}\right)^{3/2} = 1 \text{ yr} \left(\frac{40 \text{ au}}{1 \text{ au}}\right)^{3/2} \approx 250 \text{ yr}$$

Example 2: The mass of the Sun

$$M_\odot = \frac{4\pi^2}{G}\frac{a_\oplus^3}{P_\oplus^2} = \frac{4 \cdot 10}{7 \cdot 10^{-8}} \frac{(1.5 \cdot 10^{13})^3}{(3 \cdot 10^7)^2} = 5 \cdot 10^8 \cdot \frac{1}{3} \cdot 10^{39-14} = 2 \cdot 10^{33} \text{ g}.$$

Example 3: The mass of Pluto
The mass of Pluto (discovered in 1930 by Clyde Tombaugh) was unknown until the discovery of its companion Charon (Christy 1978). Kepler's 3rd law gave for the binary system: $1/400 M_\oplus$.

Example 4: Ida
Galileo spacecraft flew by the asteroid Ida (1993) and found a companion asteroid Dactyl. (This was the first binary asteroid dscovered in the history of astronomy, many others since then.) Measuring the oribital period of the binary and their separation, and then applying Kepler's 3rd law, the mass of Ida was determined. The mass, together with the measured size, give the mean density. This turned out to be surprizingly low, just above that of water, leading to the conclusion that asteroids must have a "rubble pile" structure. That structure must have resulted from the formation and/or collisional evolution history of asteroids. In general, the logical chain is as follows:

$$\boxed{\text{sizes and masses}} \Rightarrow \boxed{\text{densities}} \Rightarrow \boxed{\text{interior structure}} \Rightarrow \boxed{\text{origin and evolution}}$$

2.9 Kepler's equation

By now, we have found x, y, and r as functions of θ, but we want to have them also as functions of time t. Hence we need to find $\theta = \theta\,(t)$. which will obviously be a nonlinear relation. We only know that at the pericenter, $\theta = 0$, and half a period later, at the apocenter, $\theta = \pi$. But we have no idea about θ in between...

In what follows, we consider elliptic motion only. From Kepler's third law,

$$\frac{P^2}{a^3} = \frac{4\pi}{\kappa^2} \qquad \Rightarrow \qquad P = \frac{2\pi}{\kappa\; a^{-3/2}}$$

Thus the orbital frequency is

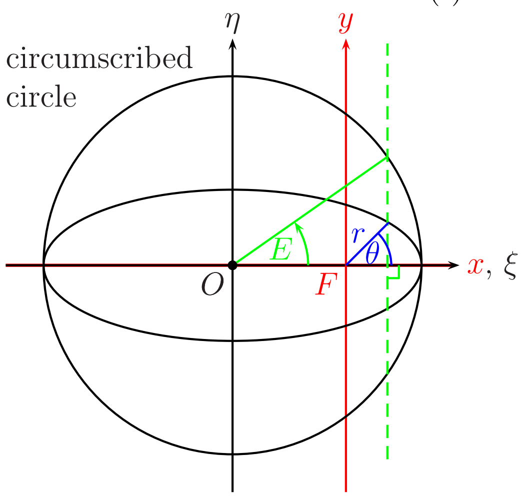
$$\boxed{n \equiv \kappa\; a^{-3/2}}.$$

In celestial mechanics, n is called *mean motion*. We now define the *mean anomaly* M:

$$\boxed{M \equiv n\;(t - T)}, \tag{2.23}$$

where T is the *time of pericenter passage*. Like θ, $M = 0$ at Π and $M = \pi$ at A. Unlike θ, M is a linear function of time. Unfortunately, M does not have any simple geometric interpretation.

However, we can introduce yet another anomaly that does. It is the *eccentric anomaly* E (see figure). Obviously, $E = 0$ at Π and $E = \pi$ at A. $E(t)$ is non-linear.



One can use two different coordinate systems (see figure). One system, $\{x, y\}$, is centered at the focus F. Another one, $\{\xi, \eta\}$, has the origin at the center of the ellipse O. We write:

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases} \tag{2.24}$$

and use the parametric equation of an ellipse

$$\begin{cases} \xi &= a\cos E \\ \eta &= b\sin E \end{cases} \tag{2.25}$$

(which is consistent with the well-known relation $\xi^2/a^2 + \eta^2/b^2 = 1$). Further,

$$\begin{aligned} x &= \xi - |\mathrm{OF}| = a\,(\cos E - e) \\ y &= \eta = a\sqrt{1-e^2}\,\sin E \end{aligned} \tag{2.26}$$

and

$$r = \sqrt{x^2+y^2} \overset{(2.26)}{=} \ldots = a\,(1 - e\cos E) \tag{2.27}$$

Equating (2.24) and (2.26) gives

$$\begin{aligned} r\cos\theta &= a\,(\cos E - e) \\ r\sin\theta &= a\sqrt{1-e^2}\,\sin E \end{aligned} \tag{2.28}$$

The second of equations (2.28) can be re-written as

$$\sin\theta = \frac{a\sqrt{1-e^2}\ \sin E}{r} \overset{(2.27)}{=} \frac{a\sqrt{1-e^2}\ \sin E}{a\,(1-e\cos E)} = \frac{\sqrt{1-e^2}\ \sin E}{1-e\cos E}$$

Differentiating with respect to time yields:

$$\begin{aligned} \cos\theta\cdot\dot\theta &= \frac{\sqrt{1-e^2}\cos E\cdot\dot E\,(1-e\cos E) - \sqrt{1-e^2}\sin E\cdot e\sin E\cdot\dot E}{(1-e\cos E)^2} \\ &= \frac{\sqrt{1-e^2}}{(1-e\cos E)^2}\dot E\left[\cos E - e\cos^2 E - e\sin^2 E\right] \\ &= \frac{\sqrt{1-e^2}}{(1-e\cos E)^2}\dot E\,(\cos E - e) \end{aligned} \tag{2.29}$$

We now work out both factors on the left-hand side, $\cos\theta$ and $\dot\theta$. From the first of equations (2.28), we find

$$\cos\theta = \frac{a\,(\cos E - e)}{r} \overset{(2.27)}{=} \frac{a\,(\cos E - e)}{a\,(1-e\cos E)} = \frac{\cos E - e}{1-e\cos E}$$

and from the angular momentum integral (2.13), we obtain

$$\dot\theta = \frac{\kappa\sqrt{p}}{r^2} = \frac{\kappa\sqrt{a}}{a^2}\frac{\sqrt{1-e^2}}{(1-e\cos E)^2} = n\frac{\sqrt{1-e^2}}{(1-e\cos E)^2}$$

Substituting these expressions for $\cos\theta$ and $\dot\theta$ into Eq. (2.29) results in

$$\frac{(\cos E - e)}{(1-e\cos E)}\ n\ \frac{\sqrt{1-e^2}}{(1-e\cos E)^2} = \frac{\sqrt{1-e^2}}{(1-e\cos E)^2}\ \dot E\ (\cos E - e)$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{n}{1-e\cos E}$$

Since $\mathrm{d}M = n\,\mathrm{d}t$, we get:

$$(1-e\cos E)\,\mathrm{d}E = \mathrm{d}M$$

and

$$\boxed{E - e\sin E = M} \tag{2.30}$$

Equation (2.30) is called *Kepler's equation* (KE).

Discussion

For a given moment of time t, we can calculate M by means of (2.23).
Then we have to solve KE for E.
Once E is known, we can use (2.25), (2.26), and (2.27) to compute the coordinates.

Unluckily, KE is not easy to solve. It is transcendental in E ("transcendental" means containing non-algebraic functions, such as exponential or trigonometric).

One can show that that $\forall 0 \leq e < 1$ and $\forall M$ $\exists!$ solution to equation (2.30).

.

Methods of solution

1) Iterations

$$\begin{cases} E_0 &= M \\ E_{i+1} &= M + e \sin E_i \end{cases}$$

converge for $0 \leq e < 1$

2) Expansion to Fourier series

$$E = M + \sum_{k=1}^{\infty} a_k\left(e\right) \sin kM = M + \left(e + \ldots\right) \sin M + \left(\frac{e^2}{2} + \ldots\right) \sin 2M + \ldots$$

converges for $0 \leq e < 1$

3) Taylor series in powers of e

$$E = M + \sum_{k=1}^{\infty} b_k\left(M\right)\ e^k = M + \left(\sin M + \ldots\right) \cdot e + \left(\frac{1}{2} \sin 2M + \ldots\right)\ e^2 + \ldots$$

converges for $0 \leq e < 0.6627...$ (Laplace limit)

4) Taylor series in time

$$E = M + \sum_{k=1}^{\infty} c_k\left(e\right) \cdot \left(t - T\right)^k$$

converges on relatively short timescales

Application

Find the time-average value of distance, $\langle r \rangle$, and of inverse distance, $\langle 1/r \rangle$, in the elliptic motion.

An educated guess would be something like $\langle r \rangle = a\,(1 + \ldots \cdot e)$ (can you explain why?). But let us solve this problem rigorously. This can be done easily, using eccentric anomaly, KE, and some other two-body problem formulas. The definition of the time-average distance is

$$\langle r \rangle = \frac{1}{P} \int\limits_0^P r \, \mathrm{d}t$$

From KE,

$$\begin{aligned} E - e \sin E &= M \\ \mathrm{d}E(1 - e\cos E) &= n \, \mathrm{d}t \\ \mathrm{d}E \, \frac{r}{a} &= n \, \mathrm{d}t \\ \mathrm{d}t &= \frac{r}{na} \, \mathrm{d}E \end{aligned}$$

Thus

$$\begin{aligned} \langle r \rangle &= \frac{n}{2\pi} \int\limits_0^{2\pi} \frac{r^2}{na} \, \mathrm{d}E \\ &= \frac{a}{2\pi} \int\limits_0^{2\pi} (1 - e\cos E)^2 \, \mathrm{d}E \\ &= \frac{a}{2\pi} \int\limits_0^{2\pi} \left(1 - 2e\cos E + \frac{e^2}{2} + \frac{e^2}{2} \cdot \cos 2E\right) \mathrm{d}E \end{aligned}$$

so that

$$\boxed{\langle r \rangle = a\left(1 + \frac{e^2}{2}\right)}$$

In a similar way, we can find

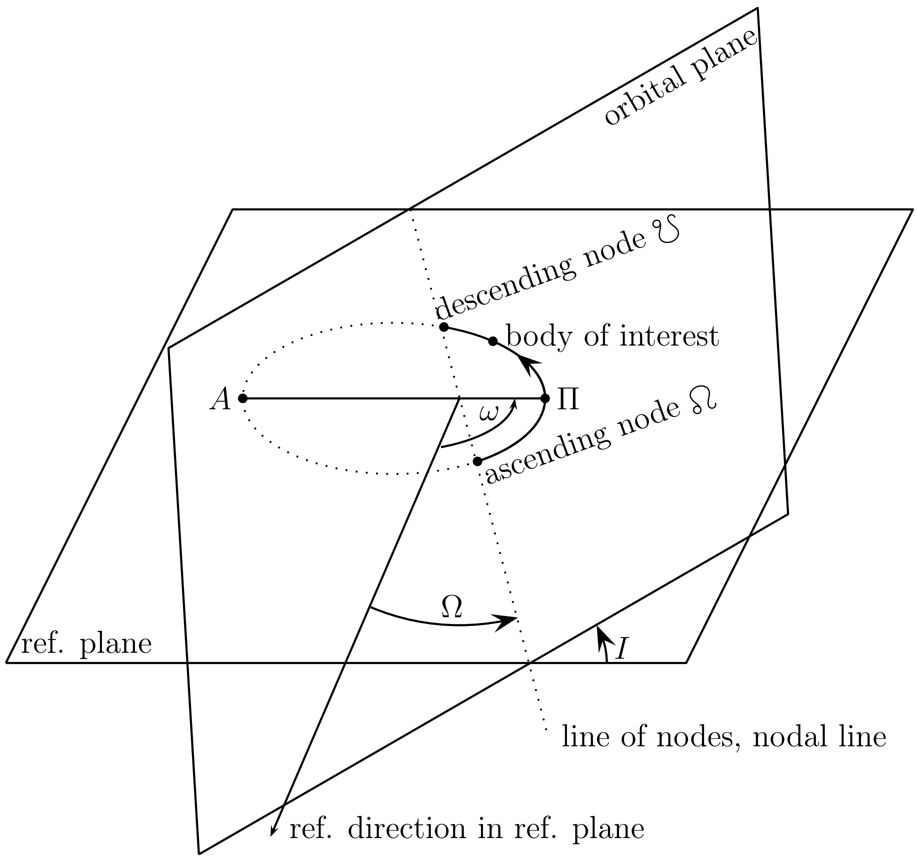
$$\boxed{\left\langle \frac{1}{r} \right\rangle = \ldots = \frac{1}{a}}$$

This reminds us that the average of the inverse is not the inverse of the average:

$$1/\langle r \rangle \neq \langle 1/r \rangle$$

2.10 Orbital elements

Six quantities $\{\bar{r}_0, \dot{\bar{r}}_0\}$ uniquely determine the orbital motion. However, positions and velocities are not very illustrative – if you know them, you will not even be able to say whether the orbit is elliptic, parabolic, or hyperbolic... Thus it is a common practice to use another set of six quantities that have a clear geometrical meaning. These are called *orbital elements* or just *elements* and are introduced as follows.



description	element
shape & size of orbit	a - semimajor axis e - eccentricity
orientation of orbit in its plane	ω - argument of pericenter
orientation of orbital plane in space	I - inclination Ω - longitude of ascending node
position of the body in orbit	T - time of pericenter passage

Hence we have a set of six orbital elements: $\{a, e, I, \Omega, \omega, T\}$. In this set, instead of the time of pericenter passage T one often uses the mean anomaly $M = n(t - T)$, although it is not a constant.

2.11 Calculation of coordinates

For a given set of orbital elements $\{a, e, \omega, \Omega, I, T\}$ we want to determine the coordinates and velocities, $\{x, y, z, \dot{x}, \dot{y}, \dot{z}\}$.

This can be done by performing the following steps:

1. Compute $n = \kappa a^{-3/2}$; $M = n\,(t - T)$
2. Solve the KE to find E
3. Calculate the position and velocity:

$$
\begin{aligned}
x &= a\,(\cos E - e) & \dot{x} &= \ldots \\
y &= a\sqrt{1 - e^2}\,\sin E & \dot{y} &= \ldots \\
r &= a\,(1 - e\cos E) & \dot{r} &= \ldots
\end{aligned}
$$

   Alternatively, one can calculate θ from E with (2.28) and use the formulas for coordinates as functions of θ.

   The coordinates x,y are in the orbital plane ($z = 0$), not in the reference plane (X, Y, Z).

4. Conversion of $(x, y, z = 0) \to (X, Y, Z)$

   Perform three rotations:

   - rotate by ω around $\bar{c}$-axis
   - rotate by I around the line of nodes |☊☋|
   - rotate by Ω around OZ

   Omitting the algebra, the result is

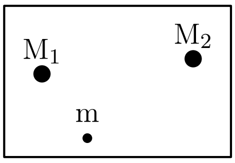
$$
\boxed{
\begin{aligned}
X &= r\,(\cos\Omega\cos u - \sin\Omega\sin u\cos I) \\
Y &= r\,(\sin\Omega\cos u + \cos\Omega\sin u\cos I) \\
Z &= r\,(\sin u\sin I)
\end{aligned}
}
\tag{2.31}
$$

   where $u \equiv \omega + \theta$ is the *argument of latitude.*

There also exists the opposite problem: we may wish to determine the orbital elements $\{a, e, \omega, \Omega, I, T\}$ if the coordinates and velocities $\{x, y, z, \dot{x}, \dot{y}, \dot{z}\}$ are known. In practice, what is usually known is several positions of the body, i.e. its radius vector at several moments of time. This *orbit determination* problem is quite complicated. Even famous scientists, such as Lagrange and Gauss, studied it. We will not consider this problem.

3 The restricted circular three-body problem

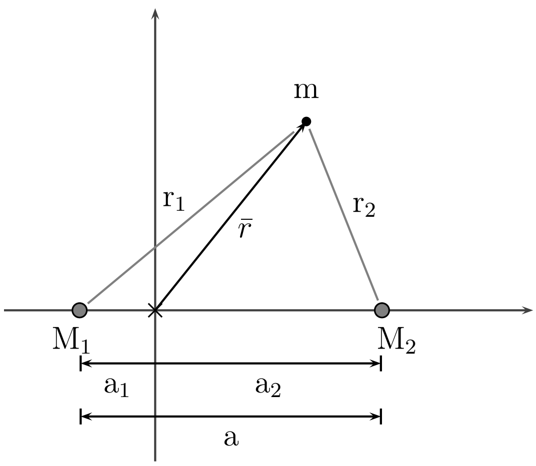
As mentioned in the Introduction, the 3-body problem is already so complicated that there is no (practically useful) solution to it in the general case. We therefore make simplifying assumptions and formulate the so-called *restricted circular 3-body problem*:



- Assume that $m \ll M_1, M_2 \rightarrow$ "restricted"
- Assume that the orbit of M_1 relative to M_2, or that M_2 relative to M_1, or either body relative to their barycenter is circular $\rightarrow$ "circular"

Examples for $M_1 - M_2 - m$:
- Sun – Earth – Moon
- Sun – Jupiter – any of its moons / any other planet / any comet / any asteroid
- binary star – exoplanet

3.1 Equation of motion



Introduce a rotating coordinate system (figure). Its origin is in the barycenter, the X-axis goes through M_1 and M_2, the system rotates around Z-axis as the two finite masses orbit each other, and the Y-axis completes the right triad. In this system, M_1 and M_2 are sitting at fixed positions.

The angular velocity of rotation is given by:

$$n = \kappa\, a^{-3/2} = \sqrt{G\,(M_1 + M_2)}\; a^{-3/2} \tag{3.1}$$

Obviously, $a_1 M_1 = a_2 M_2$, and

$$\begin{aligned} r_1 &= \sqrt{(x + a_1)^2 + y^2 + z^2} \\ r_2 &= \sqrt{(x - a_2)^2 + y^2 + z^2} \end{aligned} \tag{3.2}$$

The gravitational potential ($\equiv -$ potential energy) per unit mass (i.e., for $m \equiv 1$) is:

$$U = \frac{GM_1}{r_1} + \frac{GM_2}{r_2} \tag{3.3}$$

We first calculate the kinetic energy. The *absolute* velocity of m in the *rotating* frame is

$$\overline{v_{\text{abs}}} = \overline{v_{\text{rel}}} + \overline{v_{\text{trans}}}$$

with

$$\overline{v_{\text{rel}}} = \{\dot{x}, \dot{y}, \dot{z}\}$$

$$\overline{v_{\text{trans}}} = \overline{\omega} \times \overline{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & n \\ x & y & z \end{vmatrix} = \{-ny, nx, 0\}$$

so that

$$\overline{v_{\text{abs}}} = \{\dot{x} - ny, \dot{y} + nx, \dot{z}\}$$

Thus the kinetic energy (per unit mass, i.e., for $m \equiv 1$) is:

$$T = \frac{v_{\text{abs}}^2}{2} = \frac{1}{2}\left[(\dot{x} - ny)^2 + (\dot{y} + nx)^2 + \dot{z}^2\right] \tag{3.4}$$

The Lagrangian (cf. Hamiltonian $H = T - U$) is then:

$$L = T + U$$

We now write the Lagrange equations of second kind:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\overline{r}}}\right) - \frac{\partial L}{\partial \overline{r}} = 0$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\overline{r}}}\right) - \frac{\partial T}{\partial \overline{r}} = \frac{\partial U}{\partial \overline{r}} \tag{3.5}$$

Substituting (3.4) into (3.5) we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\dot{x} - ny) - &\quad (\dot{y} + nx)\, n &= U_x \\ \frac{\mathrm{d}}{\mathrm{d}t}(\dot{y} + nx) - &\quad (\dot{x} - ny)\,(-n) &= U_y \\ \frac{\mathrm{d}}{\mathrm{d}t}(\dot{z} \qquad) & &= U_z \end{aligned}$$

where $U_x = \partial U / \partial x$, ..., or

$$\boxed{\begin{aligned} \ddot{x} - 2n\dot{y} - n^2 x &= U_x \\ \ddot{y} + 2n\dot{x} - n^2 y &= U_y \\ \ddot{z} \qquad\qquad &= U_z \end{aligned}} \tag{3.6}$$

These are the equations of motion in the restricted, circular three-body problem. Remember that U is given by (3.3), and r_1, r_2 by (3.2).

The first column of Eq. (3.6) contains accelerations,
hence the other columns have to be (specific) forces:

- $\dot{x}/\dot{y}$-terms $=$ – Coriolis force
- x/y-terms $=$ – centrifugal force
- $U_{\dots}$-terms $=$ gradient of the grav potential $=$ grav force

3.2 The Jacobi Integral

Let us calculate a scalar product of Eq. (3.6) and the vector $2\dot{\bar{r}} = \{2\dot{x}, 2\dot{y}, 2\dot{z}\}$:

$$\begin{aligned}(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{z}\ddot{z}) + \left(-\cancel{2n\dot{x}\dot{y}} + \cancel{2n\dot{x}\dot{y}}\right) \\ - \left(2n^2 x\dot{x} + 2n^2 y\dot{y}\right) = 2\left(U_x\dot{x} + U_y\dot{y} + U_z\dot{z}\right)\end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - n^2\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2 + y^2\right) = 2\frac{\mathrm{d}}{\mathrm{d}t}\left(U\right)$$

This can be integrated to give

$$\boxed{v^2 = n^2\left(x^2 + y^2\right) + 2U - C}, \qquad C = \mathrm{const} \tag{3.7}$$

where $\bar{v}$ is actually v_{rel}, the relative velocity in the rotating frame – we have omitted the subscript "rel" for brevity.

Equation (3.7) is known as the Jacobi integral, and C is called Jacobi constant.

These are named after a German mathematician Jacob Jacobi (1804–1851).

One can define the so-called generalized potential

$$\Omega \equiv U + \frac{1}{2}n^2\left(x^2 + y^2\right) \tag{3.8}$$

which is the gravitational potential U plus the centrifugal potential, or just the total potential of the forces that act on the particle in our non-inertial reference frame. Note that the Coriolis force does not show up in Eqs. (3.7)–(3.8): this force does no work, and therefore does not change energy.

With (3.8), the Jacobi integral transforms to

$$\boxed{v^2 = 2\Omega - C} \tag{3.9}$$

Obviously, the Jacobi integral is nothing else but the conservation law of the mechanical energy. Unfortunately, this is the only known integral in the restricted circular 3-body problem. Therefore, the problem cannot be solved analytically in closed form.

3.3 Zero-Velocity Surfaces (ZVS)

The usefulness of the Jacobi integral can be appreciated by considering the locations in space where the velocity $v = 0$. The surfaces on which $v = 0$ are called *zero-velocity surfaces* (ZVSs). They separate the regions where the motion of the particle is possible ($v^2 \geq 0$) from those that are excluded dynamically ($v^2 < 0$).

From (3.9), the equation of ZVS is

$$\boxed{2\Omega = C}$$

or, in expanded form,

$$\boxed{2\frac{GM_1}{r_1} + 2\frac{GM_2}{r_2} + n^2(x^2 + y^2) = C} \tag{3.10}$$

For the subsequent calculations we choose special units, such that

$$\begin{aligned} a &= 1 \\ G(M_1 + M_2) &= 1 \end{aligned}$$

so that $n = \sqrt{G\,(M_1 + M_2)a^{-3/2}} = 1$. This means that:

$$\begin{aligned} &\text{The unit of length} &&= a \\ &\text{The unit of time} &&= P/(2\pi) \end{aligned}$$

Furthermore, we denote (assuming $M_1 \geq M_2$)

$$\begin{aligned} GM_1 &\equiv \mu_1 &&= 1 - \mu \\ GM_2 &\equiv \mu_2 &&\equiv \mu \end{aligned}$$

Then, Eq. (3.10) takes the form

$$\boxed{2\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) + x^2 + y^2 = C} \tag{3.11}$$

where

$$\begin{aligned} r_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2} \\ r_2 &= \sqrt{(x+\mu-1)^2 + y^2 + z^2} \end{aligned}$$

Let us look at the geometry of ZVSs which, of course, depends on the value of C. (See figure on the next page.) We start by assuming that C is large and look what happens when it decreases:

(a) C is large ($\gg 1$)

How to achieve this, i.e., which terms in the left-hand side can be large?

- $|x|, |y|$ are large $\rightarrow r_1, r_2$ are large $\rightarrow x^2 + y^2 \approx C \rightarrow$ "quasi-cylinder"
- r_1 is small $\rightarrow \frac{2(1-\mu)}{r_1} \approx C \rightarrow$ "quasi-sphere" around M_1
- r_2 is small $\rightarrow \frac{2\mu}{r_2} \approx C \rightarrow$ "quasi-sphere" around M_2

Motion is possible outside the cylinder and inside the spheres

(b) C gets smaller $\rightarrow$ cylinder shrinks, spheres grow until they get in touch.

The touching point of the spheres is denoted $\mathcal{L}_1$

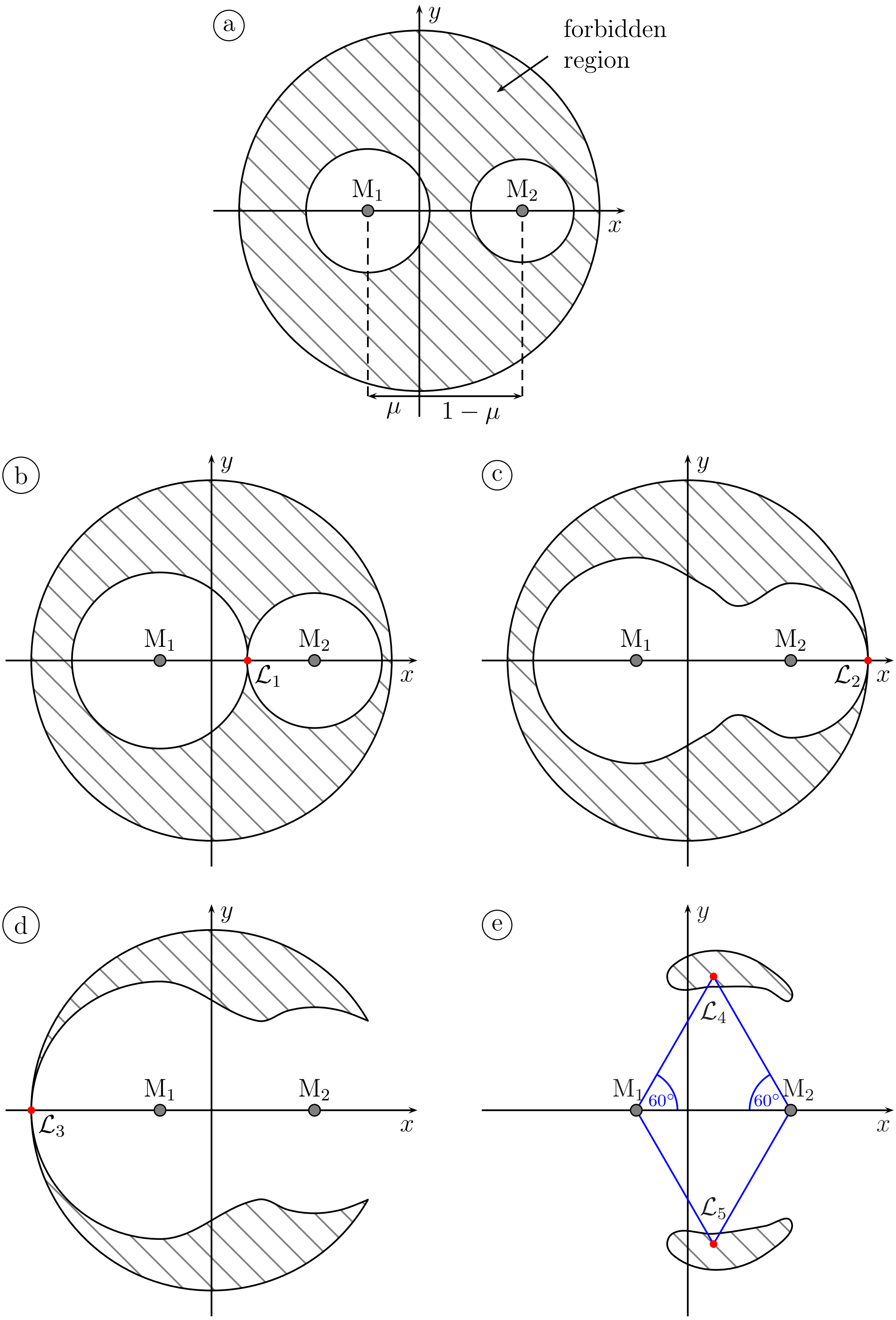
(c) C gets even smaller, until the smaller sphere hits the cylinder at $\mathcal{L}_2$

(d) C gets even smaller, until the larger sphere hits the cylinder at $\mathcal{L}_3$

(e) for very small C, only kidney-shaped regions around $\mathcal{L}_4$ and $\mathcal{L}_5$ are prohibited

In the case (a), the test particle that is close to μ cannot escape from the vicinity of μ. This is the case for the Moon ($1 - \mu =$ Sun, $\mu =$ Earth). But: the Moon can strike to Earth! This is the concept of *Hill stability* (a weak stability!). Named after American astronomer George William Hill (1838–1914) .

In the case (b) and between (b)–(c), the particle can "walk" between the two masses, yet it can never escape from the system. This is often the case for material in close binaries.


a
y
forbidden
region
M1
M2
x
μ
1 − μ
b
y
M1
M2
L1
x
c
y
M1
M2
L2
x
d
y
M1
M2
L3
x
e
y
L4
M1
60°
60°
M2
x
L5

3.4 Lagrangian equilibrium points $\mathcal{L}_i$

The special points $\mathcal{L}_i$ that we identified in the analysis of the ZVSs are named *Lagrangian points* after Joseph Louis Lagrange (1736-1813). [Actually, Lagrange discovered $\mathcal{L}_4$ and $\mathcal{L}_5$. The first three points, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, were discovered earlier by Leonhard Euler (1707–1783).] Sometimes, $\mathcal{L}_i$ are also called *libration points*.

Definition of $\mathcal{L}_i$

Consider equation (3.11) of ZVS again:

$$\underbrace{\frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} + x^2 + y^2}_{2\Omega} - C = 0 \tag{3.12}$$

with

$$\begin{aligned} r_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2} \\ r_2 &= \sqrt{(x+\mu-1)^2 + y^2 + z^2} \end{aligned}$$

and find *fixed*, or *equilibrium*, points, defined as

$$\frac{\partial \Omega}{\partial \overline{r}} = \{\Omega_x, \Omega_y, \Omega_z\} \equiv 0.$$

Location of $\mathcal{L}_i$

We have:

$$-\frac{1-\mu}{r_1^3}(x+\mu) \qquad -\frac{\mu}{r_2^3}(x+\mu-1) \qquad +x = 0 \tag{3.13}$$

$$-\frac{1-\mu}{r_1^3}y \qquad -\frac{\mu}{r_2^3}y \qquad +y = 0 \tag{3.14}$$

$$-\frac{1-\mu}{r_1^3}z \qquad -\frac{\mu}{r_2^3}z \qquad = 0 \tag{3.15}$$

From equation (3.15),

$$\underbrace{z\left[\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right]}_{>0} = 0 \qquad \Leftrightarrow \qquad z = 0,$$

so that all equilibrium points (if any exist) must lie in the x-y-plane:

$$\boxed{\mathcal{L}_i = (x_i, y_i, 0)}$$

From equation (3.14), we now obtain:

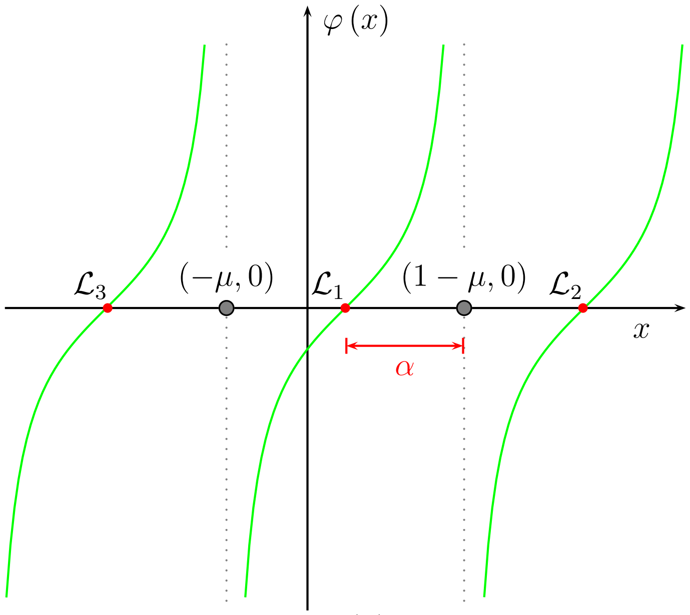
$$y\left[1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right] = 0$$

so that either $y = 0$ or the expression in brackets $= 0$.

Let $y = 0$ first (i.e., let us seek the points on the x-axis. Equation (3.13) gives

$$\underbrace{x - \frac{1-\mu}{|x+\mu|^3}(x+\mu) - \frac{\mu}{|x+\mu-1|^3}(x+\mu-1)}_{\equiv\varphi(x)} = 0$$

A qualitative plot looks as follows:



x	$\varphi(x)$
$-\infty$	$-\infty$
$-\mu-\varepsilon$	$+\infty$
$-\mu+\varepsilon$	$-\infty$
$1-\mu-\varepsilon$	$+\infty$
$1-\mu+\varepsilon$	$-\infty$
$+\infty$	$+\infty$

showing that the function $\phi(x)$ has 3 zeroes, i.e., there must be 3 points ($\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$).

Let us find, for instance, an approximate location of $\mathcal{L}_1 = \{x_1, 0, 0\}$. We seek x such that $-\mu < x < 1-\mu$. Equation (3.13) is then

$$x - \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(x+\mu-1)^2} = 0$$

Obviously, for small μ the solution should be close to $1-\mu$. Accordingly, let us substitute

$$x \equiv 1 - \mu - \alpha$$

and search for α instead of x. We get

$$1 - \mu - \alpha - \frac{1-\mu}{(1-\alpha)^2} + \frac{\mu}{\alpha^2} = 0$$

Now, we can use the relations $1-\mu = \mu_1$ and $\mu = \mu_2$. With them,

$$\mu_1 - \alpha - \frac{\mu_1}{(1-\alpha)^2} + \frac{\mu_2}{\alpha^2} = 0$$

But $\alpha = \mu_1\alpha + \mu_2\alpha$, so that

$$\mu_1\left[(1-\alpha) - \frac{1}{(1-\alpha)^2}\right] = \mu_2\left[\alpha - \frac{1}{\alpha^2}\right]$$

and

$$\frac{\mu_2}{\mu_1} = \frac{(1-\alpha) - \frac{1}{(1-\alpha)^2}}{\alpha - \frac{1}{\alpha^2}}$$

The right-hand side simplifies to

$$= \frac{\alpha^2}{(1-\alpha)^2} \frac{(1-\alpha)^3 - 1}{\alpha^3 - 1} = \alpha^2 \frac{(-3\alpha)}{-1} + o\left(\alpha^3\right) = 3\alpha^3 + o\left(\alpha^3\right).$$

Thus $\mu_2/\mu_1 \approx 3\alpha^3$, and

$$\alpha \approx \sqrt[3]{\frac{\mu_2}{3\mu_1}} \equiv R_H \tag{3.16}$$

This quantity R_H, the distance from the smaller body M_2 to $\mathcal{L}_1$, is called the *Hill radius of M_2 with respect to M_1*. The sphere of radius R_H around M_2 is called the *Hill sphere*. It is usually considered that everything inside the Hill sphere of M_2 "belongs" to that body, i.e., its gravity dominates there over M_1.

As an example we take the Sun–Earth–Moon system. Sun is M_1, Earth is M_2. The distance between Sun and Earth is unity in our set of units. The fraction μ_2/μ_1 corresponds to the mass ratio between Sun-Earth with $\mu_\oplus/\mu_\odot \approx 1/\left(3 \cdot 10^5\right)$. We find that

$$\alpha_{\oplus\odot} = \sqrt[3]{\frac{\mu_\oplus}{3\mu_\odot}} \approx 10^{-2} \approx 4 r_{\oplus☾} ,$$

i.e. the Moon is orbiting at one-quarter of the Hill radius and so deeply inside the Hill sphere of the Earth.

<u>*Let $y \neq 0$:*</u>

If we set $y \neq 0$ in equation (3.14), then the expression in brackets has to be zero:

$$1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} = 0 \tag{3.17}$$

This expression appears in equation (3.13). Indeed, grouping all terms in that equation containing x we get:

$$\underbrace{\left[1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right]}_{\stackrel{!}{=}0} x - \frac{\mu\,(1-\mu)}{r_1^3} - \frac{\mu\,(\mu-1)}{r_2^3} = 0$$

$$\Rightarrow r_1 = r_2$$

and from (3.17)

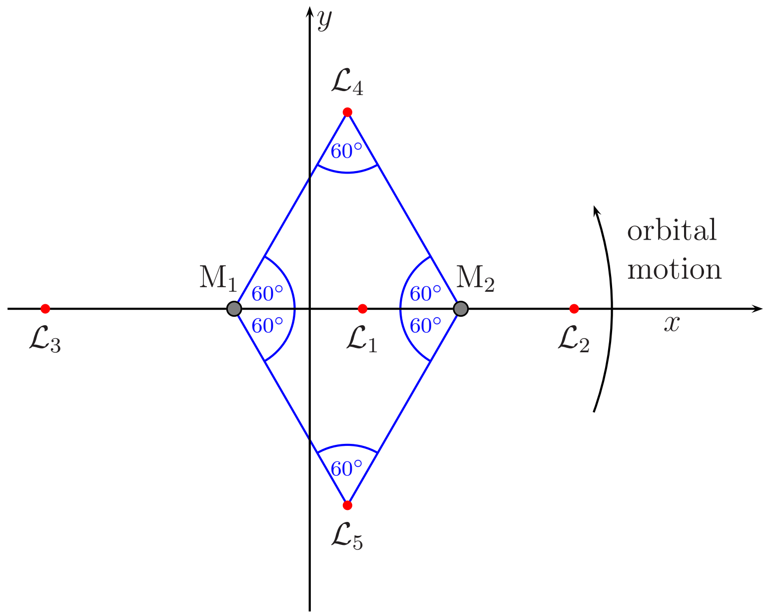
$$1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_1^3} = 0 \Rightarrow r_1 = 1.$$

Thus $r_1 = r_2 = 1$, which can be achieved at two different positions $\mathcal{L}_4$ and $\mathcal{L}_5$ (see figure).

$\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are called collinear points

$\mathcal{L}_4$ and $\mathcal{L}_5$ are named triangular points.

With respect to the orbital motion, $\mathcal{L}_4$ is the leading, $\mathcal{L}_5$ the trailing point.



Mechanical meaning of $\mathcal{L}_i$

Equation of motion (3.6) can be written as

$$
\begin{aligned}
\ddot{x} \quad & -2n\dot{y} && = \Omega_x \\
\ddot{y} \quad & +2n\dot{x} && = \Omega_y \\
\ddot{z} \quad & && = \Omega_z
\end{aligned}
\tag{3.18}
$$

By definition of Lagrangian points, at $\mathcal{L}_i$ the rhs vanishes: $\Omega_x = \Omega_y = \Omega_z = 0$.
Further, since $\mathcal{L}_i$ are on ZVS, there must be $\dot{x} = \dot{y} = \dot{z} = 0$.
Thus in any of the Lagrangian point, the acceleration $\ddot{x} = \ddot{y} = \ddot{z} = 0$.

This means, if we place a test particle in any $\mathcal{L}_i$ with zero initial valocity, it will reside there forever! This also means, 5 fixed points represent 5 particular solutions to the restricted, circular 3-body problem.

Stability of $\mathcal{L}_i$

Questions to answer:

- What happens to a body put *near* the position of an $\mathcal{L}_i$?
- What happens to a body put at an $\mathcal{L}_i$, but with a *small non-zero initial velocity*?

Should such a body stay close to that $\mathcal{L}_i$, this Larganian point is said to be *stable*.
Otherwise it is called *unstable*.

A standard method is the so-called *linear stability analysis*.
We start again with equations of motion (3.18).
Consider one of the $\mathcal{L}_i$'s, with coordinates $\{x_i, y_i, 0\}$.
We can make a transformation of variables

$$x = x_i + X, \qquad y = y_i + Y, \qquad z = Z$$

and expand (3.18) in Taylor series in X, Y, Z around $\{x_i, y_i, 0\}$.

Keeping only linear terms in X, Y, Z, we will get a system of *linear* ODE.
For instance, at $\mathcal{L}_4$ the result will be:

$$\begin{aligned}
\ddot{X} - 2\dot{Y} &= \frac{3}{4}X + \frac{3\sqrt{3}}{4}(1-2\mu)\,Y \\
\ddot{Y} + 2\dot{X} &= \frac{3\sqrt{3}}{4}(1-2\mu) + \frac{9}{4}Y \\
\ddot{Z} &= -\,Z.
\end{aligned}$$

With $u = \{X, Y, \dot{X}, \dot{Y}\}$, we can write this system in the form

$$\dot{u} = A\,u,$$

where A is the matrix of coefficients.
Then we write down the so-called characteristic equation:

$$\det(A - \lambda I) = 0,$$

where I is the unit matrix. In the case of $\mathcal{L}_4$, this equation is a bi-quadratic:

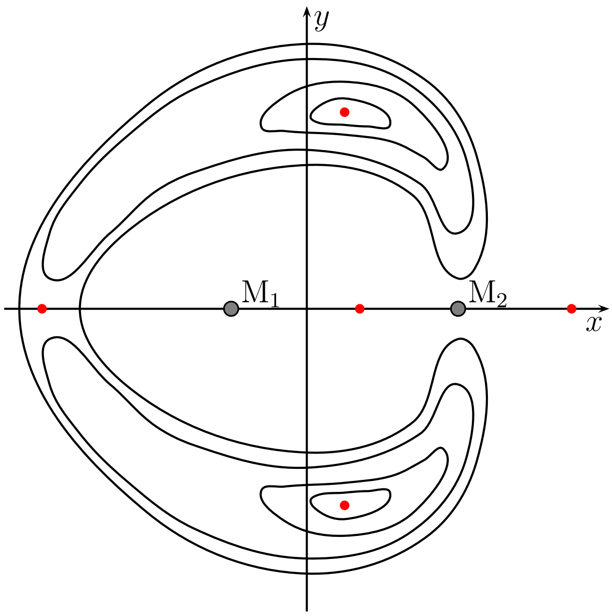
$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0.$$

This has four (complex) roots, the so-called eigenvalues λ_1, λ_2, λ_3, and λ_4. The Lagrangian point is stable, if the eigenvalues are purely imaginary ($\mathrm{Re}\,(\lambda_i) = 0$).
The results (for all five points):

- $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are always unstable
- $\mathcal{L}_4$ and $\mathcal{L}_5$ are only stable if:

$$\mu < \frac{27 - \sqrt{621}}{54} \approx \frac{1}{26} \qquad \text{or, in terms of the mass ratio,} \qquad \frac{\mu_2}{\mu_1} = \frac{\mu}{1-\mu} < \frac{1}{25}$$

Motion around $\mathcal{L}_i$



- "banana"-shaped small closed curves around $\mathcal{L}_4$ or $\mathcal{L}_5$
  → 'tadpole orbits"
- large motion embracing $\mathcal{L}_3$, $\mathcal{L}_4$ and $\mathcal{L}_5$
  → "horseshoe orbits"

Astronomical examples

M_1	M_2	libration point	shape	m
Sun	Earth	$\mathcal{L}_1$	–	SOHO[1]
		$\mathcal{L}_2$	–	JWST[2]
Sun	Jupiter	$\mathcal{L}_4$	tadpole	Greeks[3]
		$\mathcal{L}_5$	tadpole	Trojans[3]
Saturn	Tethys[4]	$\mathcal{L}_4$	tadpole	Telesto
		$\mathcal{L}_5$	tadpole	Calypso
Saturn	Janus	$\mathcal{L}_3$	horseshoe	Epimetheus[5]
	Epimetheus	$\mathcal{L}_3$	horseshoe	Janus[5]

[1] SOlar and Heliospheric Observatory. Launched in 1995, originally as a 2-year mission — but still in operation after more than 25 years in space!
The mission will likely end in 2025.
[2] James Webb Space Telescope (launched in 2021) — space-based infrared telescope. The largest telescope in space so far.
[3] Following Homer's "Iliad" that described the war between Greeks and Trojans. Both groups are often called "Trojans" or "Trojan asteroids". More than 7000 of Jupiter trojans have been discovered so far! Other planets have Trojans, too, but much fewer. As of 2021: Venus (1), Earth (1), Mars (9), Uranus (2), Neptune (28).
[4] Another satellite of Saturn, Dione, also has co-orbitals.
[5] Epimetheus moves in a horseshoe orbit of the Saturn-Janus system, and Janus moves in a horseshoe orbit of the Saturn-Epimetheus system.

3.5 The Tisserand criterion

In this section, we will return to the Jacobi integral and find an approximate expression for the Jacobi constant, which is useful for many applications — e.g. in studies of comets. As before, we use the "Jacobian" system of units:

$$n = 1, \qquad G(M_1 + M_2) = 1$$

and the notation

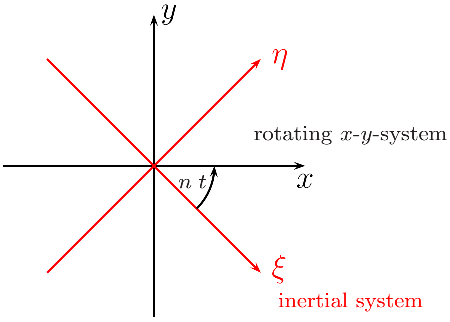
$$GM_1 \equiv \mu_1 \equiv 1 - \mu, \qquad GM_2 \equiv \mu_2 \equiv \mu.$$

Then, in the *rotating* reference frame, the Jacobi integral is

$$\boxed{C = \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + (x^2 + y^2) - \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)} \tag{3.19}$$

Step 1.

First, we wish to write the Jacobi integral in the *inertial* reference frame (ξ, η, ζ).



Since $n \equiv 1$, the angle between the ξ-axis and the x-axis is $nt \equiv t$. Denoting $c \equiv \cos(t)$ and $s \equiv \sin(t)$, the radius-vector in the rotating frame is expressed through that in the inertial frame, $(\dot{\xi}, \dot{\eta})$, as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$
$$z = \zeta$$

Similarly, the absolute velocity vector in the rotating frame is expressed through the absolute velocity in the inertial frame, as:

$$\begin{pmatrix} \dot{x} - \cancelto{1}{n} y \\ \dot{y} + \cancelto{1}{n} x \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix}$$
$$\dot{z} = \dot{\zeta}.$$

Then,

$$\boxed{x^2 + y^2 = \xi^2 + \eta^2} \tag{3.20}$$

Further,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} + \begin{pmatrix} -s & c \\ -c & -s \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

so that

$$\begin{aligned} \dot{x} &= c\dot{\xi} + s\dot{\eta} - s\xi + c\eta \\ \dot{y} &= -s\dot{\xi} + c\dot{\eta} - c\xi - s\eta \end{aligned}$$

giving

$$\begin{aligned}
\dot{x}^2 + \dot{y}^2 &= \left[c\dot{\xi} + s\dot{\eta} - s\xi + c\eta\right]^2 + \left[-s\dot{\xi} + c\dot{\eta} - c\xi - s\eta\right]^2 \\
&= \dots \\
&= \left(\dot{\xi}^2 + \dot{\eta}^2\right) + \left(\xi^2 + \eta^2\right) + 2\left(\dot{\xi}\eta - \xi\dot{\eta}\right)
\end{aligned}$$

and

$$\boxed{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) + \left(\xi^2 + \eta^2\right) + 2\left(\dot{\xi}\eta - \xi\dot{\eta}\right)} \tag{3.21}$$

Inserting (3.20) and(3.21) into (3.19) gives

$$\boxed{C \;=\; \left(\frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2}\right) - \left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) + 2\left(\xi\dot{\eta} - \dot{\xi}\eta\right)} \tag{3.22}$$

<u>Step 2</u>

Second, we wish to replace the coordinates and velocities by the orbital elements of the test particle. Conceptually, the very approach does not seem to be valid, as we introduced the orbital elements for a particle in the two-body problem, moving in an ellipse. But now we are considering the three-body problem, and the orbit is not an ellipse. However, we can argue that, if the particle spends most of the time far from the body M_2, the motion of the test particle will be dominated by the gravity of M_1 and so closely follow the solution of the 2-body problem "M_1–test particle."

Specifically, we:
- assume that $\mu_2 \ll \mu_1$,
- assume that r_2 is not small,

and express C through orbital elements of the test particle with respect to μ_1.

As explained above, under these conditions we are in the framework of the 2-body problem. Thus we are allowed to apply the integrals of the motion that exist in the 2-body problem. These are the energy integral

$$\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = \kappa^2\left(\frac{2}{r_1} - \frac{1}{a}\right)$$

and the angular momentum integral (see Section 2.5)

$$\xi\dot{\eta} - \dot{\xi}\eta = c_z = c\cos I = \kappa\sqrt{a\left(1 - e^2\right)}\cos I$$

where $c_z = \cos I$ is the Z-component of the angular momentum vector $\bar{c}$.

Since $\mu_2 \ll \mu_1$, we can set $\mu_1 \approx 1$ and $r_1 \approx r$. Thus, equation (3.22) becomes:

$$C \approx \frac{2}{r} + \frac{2\mu_2}{r_2} - \kappa^2\left(\frac{2}{r} - \frac{1}{a}\right) + 2\kappa\sqrt{a\left(1 - e^2\right)}\cos I$$

Since in our system of units with $\kappa = 1$, this simplifies to

$$C \approx \frac{2\mu_2}{r_2} + \frac{1}{a} + 2\sqrt{a\left(1 - e^2\right)}\cos I.$$

Further, we can neglect the first term — but this is only valid far from the small mass, i.e. if r_2 is not small. Then,

$$\boxed{C \approx T,}$$

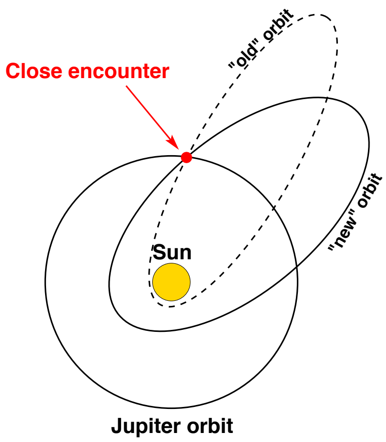
where

$$\boxed{T \equiv \frac{1}{a} + 2\sqrt{a\left(1 - e^2\right)} \cos I} \tag{3.23}$$

is called *Tisserand constant* (after François Félix Tisserand who obtained it in 1896). The Jacobi constant is approximately equal to the Tisserand constant, expect during close encounters between the test particle and M_2.

Application to comets:

Many comets in the Solar system are in orbits that cross the Jupiter orbit. Sooner or later in its life, such a comet will have a close encounter with Jupiter. Before the encounter, the cometary orbit is close to an ellipse. After the encounter, it is close to an ellipse — but possibly a completely different one. However, even though the changes in the orbital elements can be drastic, the Tisserand constant must be the same before and after the encounter. This can be used, and we now explain how.



It is possible that a comet was discovered and observed for some time, but then it experienced a close encounter with Jupiter — and got lost! We may not know what the new orbit looks like. Many years later, the comet can be re-discovered, and the astronomers may think this is a comet that has not been known previously. However, it is easy to check whether a comet that has been lost and the one newly discovered are one and the same object. If

$$\boxed{T' = T} \tag{3.24}$$

$$\tag{3.25}$$

or

$$\frac{1}{a'} + 2\sqrt{a'\left(1 - e'^2\right)} \cos I' = \frac{1}{a} + 2\sqrt{a\left(1 - e^2\right)} \cos I,$$

then it is most probably the same comet.

An example: the story of **comet Lexell**.

- Discovery (Messier, 1770): $a = 3$ au, $e = 0.8$, $r_\Pi = 0.6$ au
- Traceback (Lexell, 1776): showed that there was an encounter with Jupiter in 1767; computed elements before the encounter: $a = 4$ au, $e = 0.3$, $r_\Pi = 3$ au; thus explained why it was not known before — large r_Π
- Another encounter happened in 1779; the comet was then lost. Could it have been ejected from the Solar system? (as proposed by LeVerrier in the 19^{th} century)
- Many attempts to find the comets, many false identifications...
- New elements were finally computed (in the 1960s): $a = 43$ au, $e = 0.87$, $r_\Pi = 5.6$ au (thus $P \sim 300$ yr!)
- Recent work by Ye et al. (2018): Express doubts that the above orbit is correct. Suggest that Lexell could be in the inner Solar system as an unidentified asteroid. One candidate is asteroid 2010JL$_{33}$.

In general, close encounters of comets with giant planets change their orbits. There are three extreme possibilities:

- crash into M$_2$ (like Shoemaker-Levy 9 that smashed into Jupiter in 1994)
- ejection from the Solar system
- crash into M$_1$ ("sun-grazing" comets, or "sun-grazers", such as those frequently discovered by SOHO, about 1 comet/day!)

Further remarks:

One can derive an analog of the Tisserand constant for the case when the test particle is bound to the smaller mass M_2, rather than being far from it (Hamilton & Krivov 1997). To this end, one shall express C through orbital elements relative to M_2 rather than M_1.

4 Theory of perturbed motion

4.1 The concept of perturbed motion

In Chapter 2, we studied the two-body problem, with the equation of motion:

$$\ddot{\overline{r}} + \kappa^2 \frac{\overline{r}}{r^3} = 0. \tag{4.1}$$

We now slighly generalize the problem: let the body of interest, moving around another body, experience an additional small force or forces, $\overline{F}$. Then, the equation of motion is

$$\ddot{\overline{r}} + \kappa^2 \frac{\overline{r}}{r^3} = \overline{F} \tag{4.2}$$

(strictly speaking, $\overline{F}$ as written is the acceleration, but we assume $m = 1$).
The additional force $\overline{F}$ is called *pertubuing force*, or *perturbation*.
Motion that obeys Eq. (4.1) is called *unperturbed motion*.
Motion that obeys Eq. (4.2) is called *perturbed motion*.

It is usually assumed (and in what follows it will be assumed) that $\overline{F}$ is small, i.e., $|\overline{F}| \ll \kappa^2/r^2$. In that case, one can use the Keplerian motion as a reference motion, and only seek deviations from it. For example, smallness of perturbations facilitate making expansions etc.

However, the assumption that $\overline{F}$ is small is not obligatory. Formally, one can consider the motion of the body under an arbitrary force $\overline{F}^*$, so that

$$\ddot{\overline{r}} = \overline{F}^*,$$

and put this equation in the form (4.2) by setting

$$\overline{F} \equiv \overline{F}^* + \kappa^2 \frac{\overline{r}}{r^3}.$$

Examples of perturbed motion

Body of interest	Primary	Perturbing force(s)
Moon	Earth	Gravity of the Sun
Planet (or comet, or asteroid)	Sun	Gravity of other planets, notably Jupiter
Earth's artificial satellite	Earth	- High-altitude satellites: gravity of the Moon and Sun - Low-altitude satellites: non-sphericity of the Earth, atmosphere, ...

4.2 Osculating elements

Solution of the homogeneous equation (4.1) has the form:

$$\begin{aligned} \overline{r} &= \overline{r}\,(t;\; a, e, I, \Omega, \omega, T) \\ \dot{\overline{r}} &= \dot{\overline{r}}\,(t;\; a, e, I, \Omega, \omega, T) \end{aligned} \tag{4.3}$$

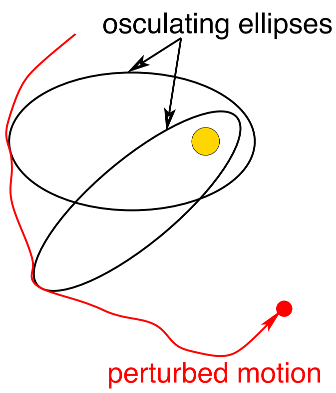
with a, e, I, Ω, ω, T being 6 integration constants.

According to the method of variation of arbitrary constants, solution of the inhomogeneous equation (4.2) can be sought in the form:

$$\begin{aligned} \overline{r} &= \overline{r}\,(t;\; a\,(t)\,, e\,(t)\,, I\,(t)\,, \Omega\,(t)\,, \omega\,(t)\,, T\,(t)) \\ \dot{\overline{r}} &= \dot{\overline{r}}\,(t;\; a\,(t)\,, e\,(t)\,, I\,(t)\,, \Omega\,(t)\,, \omega\,(t)\,, T\,(t))\,, \end{aligned} \tag{4.4}$$

in which we now allow the orbital elements to be functions of time.

Equation (4.4) shows that the coordinates and velocities in the perturbed motion will be given by same formulas as in the two-body problem: equation (4.4) has the same functional form as equation (4.3)! The elements appearing in (4.4), which are no longer constant, are called *osculating elements*.



Geometrical/physical meaning of osculating elements:

- Unperturbed motion is an ellipse
- Perturbed motion is close to an ellipse, but is not an exact ellipse
- By introducing osculating elements, we treat the perturbed motion as an exact ellipse, but the one changing in time!

The naming "osculating elements" was introduced by Lagrange.
"Osculation"(lat.) = "a kiss."
This reflects closeness of the perturbed and unperturbed orbits.

How to use osculating elements in practice? We wish to calculate $\overline{r}$, $\dot{\overline{r}}$ in the perturbed motion. To this end, we just use (4.4), i.e. the formulas of the two-body problem, with osculating elements instead of usual Keplerian ones. Thus what remains to be found is those osculating elements $a\,(t)\,, e\,(t)\,, \ldots T\,(t)$. They can be found by solving differential equations for osculating elements that will be derived in the next sections. Obviously, these equations will contain the perturbing force $\overline{F}$.

4.3 Gauss' perturbation equations

As an example, let us derive the equation for the osculating semi-major axis. Consider the energy integral

$$v^2 = \kappa^2 \left(\frac{2}{r} - \frac{1}{a} \right)$$

and differentiate it with respect to time ($\dot{a} \neq 0$ now!):

$$\begin{aligned}
2\overline{v}\,\dot{\overline{v}} &= \kappa^2 \left(-\frac{2\dot{r}}{r^2} + \frac{\dot{a}}{a^2} \right) \\
2\dot{\overline{r}}\,\ddot{\overline{r}} &= \kappa^2 \left(-\frac{2r\dot{r}}{r^3} + \frac{\dot{a}}{a^2} \right) \\
2\dot{\overline{r}}\,\ddot{\overline{r}} &= \kappa^2 \left(-\frac{2\overline{r}\,\dot{\overline{r}}}{r^3} + \frac{\dot{a}}{a^2} \right) \\
\frac{2\dot{\overline{r}}\,\ddot{\overline{r}}}{\kappa^2} &= -\frac{2\overline{r}\,\dot{\overline{r}}}{r^3} + \frac{\dot{a}}{a^2} \\
\frac{\dot{a}}{a^2} &= \frac{2\dot{\overline{r}}\,\ddot{\overline{r}}}{\kappa^2} + \frac{2\overline{r}\,\dot{\overline{r}}}{r^3} \\
\dot{a} &= \frac{a^2}{\kappa^2} \left[2\dot{\overline{r}}\,\ddot{\overline{r}} + 2\kappa^2 \frac{\overline{r}\,\dot{\overline{r}}}{r^3} \right]
\end{aligned}$$

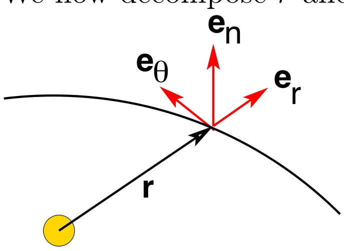
Substituting $\ddot{\overline{r}}$ from (4.2), we obtain

$$\dot{a} = \frac{a^2}{\kappa^2} \left[2\dot{\overline{r}} \left(-\frac{\kappa^2 \overline{r}}{r^3} + \overline{F} \right) + 2\kappa^2 \frac{\overline{r}\,\dot{\overline{r}}}{r^3} \right]$$

or

$$\dot{a} = \frac{2a^2}{\kappa^2}\, \dot{\overline{r}}\, \overline{F}.$$

We now decompose $\dot{\overline{r}}$ and $\overline{F}$ into radial, tangential, and normal components:



$$\begin{aligned}
\overline{r} &= r\ \overline{e_r} + 0\ \overline{e_\theta} + 0\ \overline{e_n} \\
\dot{\overline{r}} &= \dot{r}\ \overline{e_r} + r\dot{\theta}\ \overline{e_\theta} + 0\ \overline{e_n} \\
\overline{F} &= S\ \overline{e_r} + T\ \overline{e_\theta} + W\ \overline{e_n}
\end{aligned}$$

We get:

$$\dot{a} = \frac{2a^2}{\kappa^2} \left(\dot{r}\ S\ +\ r\dot{\theta}\ T \right).$$

From Chapter 2,

$$r = \frac{p}{1 + e\cos\theta} \qquad \text{and} \qquad r\dot{\theta} = \kappa \frac{\sqrt{p}}{r}$$

so that

$$\dot{a} = \frac{2a^2}{\kappa^2} \left(\frac{\kappa e \sin\theta}{\sqrt{p}} \, S \; + \; \frac{\kappa\sqrt{p}}{r} \, T \right)$$

and

$$\dot{a} = \frac{2a^2}{\kappa\sqrt{p}} \, \left[e \sin\theta \, S \; + \; (1 + e\cos\theta) \, T \right].$$

Denoting

$$(S', T', W') = \frac{1}{\kappa\sqrt{p}} \, (S, T, W)$$

we get the desired equation for the osculating a:

$$\dot{a} = 2a^2 \left[e \sin\theta \, S' + (1 + e\cos\theta) \, T' \right] \tag{4.4a}$$

The equations for the other 5 elements can be derived in a similar way.
The full set of these **Gauss perturbation equations** is as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 2a^2 \left[e \sin\theta \, S' + (1 + e\cos\theta) \, T' \right] \tag{4.5a}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = p \sin\theta \, S' + p \, (\cos\theta + \cos E) \, T' \tag{4.5e}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = r \cos u \, W' \tag{4.5I}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = r \sin u \,\mathrm{cosec}\, I \, W' \tag{4.5Ω}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{e} \left[-p \cos\theta \, S' + (r + p) \sin\theta T' \right] - \cos I \, \frac{\mathrm{d}\Omega}{\mathrm{d}t} \tag{4.5ω}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n + \frac{\sqrt{1 - e^2}}{e} \left[(p \cos\theta - 2er) \, S' - (r + p) \sin\theta \, T' \right] \tag{4.5M}$$

Comments:

- Equations (4.5) are equations for osculating elements. The right-hand sides (rhs) contain the components of the perturbing force (S, T, W), the elements themselves, as well as r, θ, E. Distance r can be expressed through θ or E. The anomalies θ and E can, in turn, be expressed through M by means of Kepler's equation. Thus equations (4.5) are a closed system of ODE, although in fact more complicated than it appears.
- In the unperturbed case we have $S' = T' = W' = 0$. Here we distinguish between:
  - "slow" variables a, e, I, Ω, ω (which are constant in the unperturbed case)
  - "fast" variable M (for which $\dot{M} = n \neq 0$ in the unperturbed case)
- Which of the force components may change which elements?
  - a, e, M are affected by S, T (these lie in the orbital plane)
  - I, Ω are only affected by the normal component W
  - ω is affected by all three: S, T, W

4.4 Lagrange perturbation equations

Often $\overline{F}$ has a potential, i.e. there exists some R such that $F = \partial R/\partial \overline{r}$. The potential R of the perturbing force is called *disturbing function*.

We wish to derive another set of perturbation equations, alternative to (4.5), the rhs of which would contain $\partial R/\partial a$, $\partial R/\partial e$, $\ldots \partial R/\partial M$ instead of S, T, W. Obviously,

$$\begin{aligned}\frac{\partial R}{\partial a} &= \frac{\partial R}{\partial \overline{r}} \cdot \frac{\partial \overline{r}}{\partial a} \\ &= \overline{F} \cdot \frac{\overline{r}}{a} = S\ \frac{r}{a} \\ &= \kappa\sqrt{p}\ S'\ \frac{r}{a} \\ &= \frac{r}{a}\ \kappa\sqrt{a(1-e^2)}\ S' \\ &= rna\sqrt{1-e^2}S'\end{aligned}$$

or

$$S' = \frac{1}{rna\sqrt{1-e^2}}\frac{\partial R}{\partial a}$$

Similarly, we could derive

$$T' = \text{linear combination of } \left(\frac{\partial R}{\partial a}, \frac{\partial R}{\partial e}, \ldots \frac{\partial R}{\partial M}\right)$$

and

$$W' = \text{linear combination of } \left(\frac{\partial R}{\partial a}, \frac{\partial R}{\partial e}, \ldots \frac{\partial R}{\partial M}\right)$$

Substituting these expressions for S', T', W' into Gauss' equations (4.5), we obtain the **Langrange perturbation equations:**

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial R}{\partial M} \tag{4.6a}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1-e^2}{ena^2}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{ena^2}\frac{\partial R}{\partial \omega} \tag{4.6e}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\operatorname{cotan} I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} - \frac{\operatorname{cosec} I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \Omega} \tag{4.6I}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\operatorname{cosec} I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I} \tag{4.6Ω}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{ena^2}\frac{\partial R}{\partial e} - \frac{\operatorname{cotan} I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I} \tag{4.6ω}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{ena^2}\frac{\partial R}{\partial e} \tag{4.6M}$$

Comments:

- The full derivation is cumbersome and can only be found in a few textbooks – e.g., Brower & Clemence (1961).
- General structure of the Lagrange equations:

	R_a	R_e	R_I	R_Ω	R_ω	R_M
$\dot{a}$						x
$\dot{e}$					x	x
$\dot{I}$				x	x	
$\dot{\Omega}$			x			
$\dot{\omega}$		x	x			
$\dot{M}$	x	x				

In the table, the non-zero terms in the linear Lagrange perturbation equations are marked by crosses. All the others are absent.

We see that all six elements split into two groups, one consisting of a, e, I and another one of Ω, ω, M. The time derivatives of the elements of the first group only depend on the R-derivatives with respect to the elements of the second group, and vice versa. Furthermore, the matrix of coefficients is almost diagonal (except for the elements marked with circles). With all this, Lagrange equations look almost like a canonical/Hamiltonian system.

4.5 Perturbation equations in Hamiltonian form

We have seen that Lagrange equations (4.6) are almost a Hamiltonian (or canonical) system, but not exactly. We now want to put them into the Hamiltonian form. What we would like to have is:

$$\frac{\mathrm{d}L_k}{\mathrm{d}t} = \frac{\partial R}{\partial l_k} \tag{4.7}$$

$$\frac{\mathrm{d}l_k}{\mathrm{d}t} = -\frac{\partial R}{\partial L_k} \tag{4.8}$$

where R is the Hamiltonian function of Hamiltonian, L_k and l_k represent the "coordinates" and "momenta", respectively, while $k = 1, 2, 3$.

To bring the system in this form, we have to modify the variables. As momenta l_k, let us try to simply keep our angles as they are:

$$l_1 \equiv M, \qquad l_2 \equiv \omega, \qquad l_3 = \Omega,$$

and only modify the coordinates $L_k = L_k\,(a, e, I)$.

We have:

$$\frac{\mathrm{d}L_1}{\mathrm{d}t} = \frac{\partial L_1}{\partial a}\dot{a} + \frac{\partial L_1}{\partial e}\dot{e} + \frac{\partial L_1}{\partial I}\dot{I} \overset{\text{must be}}{=} \frac{\partial R}{\partial M}$$

This is an equation in partial derivatives with respect to L_1. Let us find *any* solution to it. For instance, assuming that L_1 only depends on the semimajor axis, i.e. $L_1 = L_1(a)$, we get

$$\frac{\partial L_1}{\partial a}\dot{a} = \frac{\partial R}{\partial M}$$

and, using the Lagrange eq. for $\dot{a}$, we obtain

$$\frac{\partial L_1}{\partial a}\left(\frac{2}{na}\frac{\partial R}{\partial M}\right) = \frac{\partial R}{\partial M}$$

$$\frac{\partial L_1}{\partial a} = \frac{na}{2}$$

$$\Rightarrow L_1 = \kappa\sqrt{a} + \text{const}$$

where, since we wish to obtain *any* solution, we simply set the constant to zero:

$$\boxed{L_1 = \kappa\sqrt{a}} \tag{4.9}$$

We now seek L_2 in a similar way:

$$\frac{\mathrm{d}L_2}{\mathrm{d}t} = \frac{\partial L_2}{\partial a}\dot{a} + \frac{\partial L_2}{\partial e}\dot{e} + \frac{\partial L_2}{\partial I}\dot{I} \overset{\text{must be}}{=} \frac{\partial R}{\partial \omega}$$

For simplicity, let us assume that $L_2 = L_2\,(a, e)$. Then,

$$\frac{\partial L_2}{\partial a}\dot{a} + \frac{\partial L_2}{\partial e}\dot{e} = \frac{\partial R}{\partial \omega}$$

and, using the Lagrange equations for $\dot{a}$ and $\dot{e}$,

$$\frac{\partial L_2}{\partial a}\left(\frac{2}{na}\frac{\partial R}{\partial M}\right)+\frac{\partial L_2}{\partial e}\left(\frac{1-e^2}{ena^2}\frac{\partial R}{\partial M}-\frac{\sqrt{1-e^2}}{ena^2}\frac{\partial R}{\partial \omega}\right)=\frac{\partial R}{\partial \omega}$$

We see that in the left-hand sides (lhs) the coefficient of the $\partial R/\partial M$-terms should be zero, while the coefficient of $\partial R/\partial \omega$ should be unity. This gives us two equations:

$$\begin{aligned}\frac{2}{na}\frac{\partial L_2}{\partial a}+\frac{1-e^2}{ena^2}\frac{\partial L_2}{\partial e}&=0\\ -\frac{\sqrt{1-e^2}}{ena^2}\frac{\partial L_2}{\partial e}&=1\end{aligned}$$

The second equation can be solved directly:

$$\begin{aligned}\frac{\partial L_2}{\partial e}&=-\frac{e}{\sqrt{1-e^2}}na^2=-\frac{e}{\sqrt{1-e^2}}\kappa\sqrt{a}\\ \Rightarrow L_2&=\kappa\sqrt{a\left(1-e^2\right)}+f\left(a\right),\end{aligned}$$

where $f(a)$ is an arbitrary function of a. Using this result in the first equation, we find:

$$\begin{aligned}\frac{2}{na}\left(\cancel{\frac{\kappa\sqrt{1-e^2}}{2\sqrt{a}}}+f'\left(a\right)\right)+\frac{1-e^2}{\cancel{ena^2}}\left(\cancel{-\frac{(\kappa\sqrt{a}e}{\sqrt{1-e^2}}}\right)&=0\\ \Rightarrow f'\left(a\right)&=0\\ \Rightarrow f\left(a\right)&=\text{const}\end{aligned}$$

Why don't we simply take $f(a)=0$, so that

$$\boxed{L_2=\kappa\sqrt{a(1-e^2)}} \tag{4.10}$$

In a very similar way (we will omit these calculations), one can obtain the third "coordinate":

$$\boxed{L_3=\kappa\sqrt{a(1-e^2)}\cos I} \tag{4.11}$$

Altogether, we have a set of new variables:

$$\boxed{\begin{array}{ll}L_1=\kappa\sqrt{a} & l_1=M\\ L_2=\kappa\sqrt{a(1-e^2)} & l_2=\omega\\ L_3=\kappa\sqrt{a(1-e^2)}\cos I & l_3=\Omega\end{array}} \tag{4.12}$$

From our derivation, we guarantee that equations (4.7) are correct Lagrange equations (4.6). But what about equations (4.8)? All degrees of freedom are fixed, since all six elements (L_k, l_k) are set. Let us check whether equations (4.8) are correct Lagrange equations, too, and hope they are...

To check whether these equations (4.8)

$$\frac{\mathrm{d}l_k}{\mathrm{d}t} = -\frac{\partial R}{\partial L_k}$$

are fullfilled, we have to calculate $\partial R/\partial L_k$, but how to do this? The problem is that R is a function of a, e, I, and not of L_k... To do this, one uses obvious relations

$$\frac{\partial R}{\partial a} = \sum_{k=1}^{3} \frac{\partial R}{\partial L_k}\frac{\partial L_k}{\partial a}$$
$$\frac{\partial R}{\partial e} = \sum_{k=1}^{3} \frac{\partial R}{\partial L_k}\frac{\partial L_k}{\partial e}$$
$$\frac{\partial R}{\partial I} = \sum_{k=1}^{3} \frac{\partial R}{\partial L_k}\frac{\partial L_k}{\partial I}$$

which are a system of three *algebraic* equations to solve for three unknowns $\partial R/\partial L_k$. Having solved them, we compare the results with Lagrange equations (4.6). The result is:

$$\begin{aligned}\frac{\mathrm{d}l_1}{\mathrm{d}t} &= n - \frac{\partial R}{\partial L_1}\\ \frac{\mathrm{d}l_2}{\mathrm{d}t} &= \quad - \frac{\partial R}{\partial L_2}\\ \frac{\mathrm{d}l_3}{\mathrm{d}t} &= \quad - \frac{\partial R}{\partial L_3}\end{aligned} \tag{4.13}$$

We see that that everything is fine for $k = 2$ and $k = 3$, but — unfortunately — the first equation is "slightly wrong", as there is an unwanted term n in it. Hence, we do not have a Hamiltonian system yet. Fortunately, we still have a possibility to fix this. Although we cannot change the variables anymore, we can alter the Hamiltonian. Since

$$n = \kappa a^{-3/2} = \frac{\kappa}{(\sqrt{a})^3}\frac{\kappa^3}{\kappa^3} = \frac{\kappa^4}{L_1^3} = \frac{\partial}{\partial L_1}\left[-\frac{\kappa^4}{2L_1^2}\right]$$

we can take

$$\mathcal{H} \equiv \frac{\kappa^4}{2L_1^2} + R$$

instead of R as the Hamiltonian function. This will bring the first equation to the form

$$\frac{\mathrm{d}l_1}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial L_1},$$

while not changing the other two equations (4.13) (since they do not contain $\partial/\partial L_1$).

Thus we have obtained the **perturbation equations in the Hamiltonian form** (or order 6, or equivalently, with 3 degrees of freedom):

$$\boxed{\begin{aligned} \frac{\mathrm{d}L_k}{\mathrm{d}t} &= \frac{\partial\mathcal{H}}{\partial l_k} \\ \frac{\mathrm{d}l_k}{\mathrm{d}t} &= -\frac{\partial\mathcal{H}}{\partial L_k} \end{aligned}}$$

with the Hamiltonian

$$\boxed{\mathcal{H} = \frac{\kappa^4}{2L_1^2} \ + \ R},$$

where R should be expressed through L_k, l_k given by (4.12).

Comments:

- In the particular case of the unperturbed two-body motion ($R = 0$), the Hamiltonian is just
$$\mathcal{H} = \frac{\kappa^4}{2L_1^2} = \frac{\kappa^2}{2a}.$$
This is nothing else than the (minus) total mechanical energy of the system. This result is not surprising: for conservative systems, the Hamiltonian is the full mechanical energy.
- Variables (4.12) are called the **Delaunay variables**. This choice is not unique. There exist other combinations of osculating elements that make the perturbation equations Hamiltonian. For instance, the so-called Poincaré variables I and II.

A general look at Hamiltonian systems

A Hamiltonian system has the form

$$\begin{aligned} \frac{\mathrm{d}\bar{q}}{\mathrm{d}t} &= \frac{\partial \mathcal{H}}{\partial \bar{p}} \\ \frac{\mathrm{d}\bar{p}}{\mathrm{d}t} &= -\frac{\partial \mathcal{H}}{\partial \bar{q}}. \end{aligned} \tag{4.14}$$

Having a Hamiltonian system is a dream for theorists, because they have a number of useful properties. For example:

1 *Canonical transformations:*
A transformation $\{\bar{q}, \bar{p}\} \longrightarrow \{\bar{Q}, \bar{P}\}$ is canonical, if it preserves the Hamiltonian form of the equations. There are known criteria of "canonicity" of transformations. It is also known how the Hamiltonian $\mathcal{H}$ changes into a new one $\mathcal{H}^*$ under such transformations.

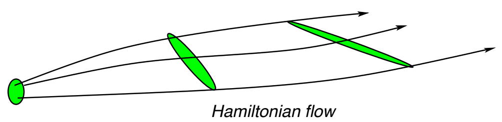
For example, you can easily find other variables that may be the most appropriate to your specific problem, and directly write the Hamiltonian in the these new variables. For instance, starting with Delaunay variables, you define Poincare I or II variables and then write the Hamiltonian for those.

2 *Conservation of energy:*
Consider the full time derivative of $\mathcal{H} = \mathcal{H}(t, \bar{q}, \bar{p})$:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial \bar{q}} \cdot \underbrace{\frac{\partial \bar{q}}{\partial t}}_{\frac{\partial \mathcal{H}}{\partial \bar{p}}} + \frac{\partial \mathcal{H}}{\partial \bar{p}} \cdot \underbrace{\frac{\partial \bar{p}}{\partial t}}_{-\frac{\partial \mathcal{H}}{\partial \bar{q}}} = \frac{\partial \mathcal{H}}{\partial t} + \cancel{\frac{\partial \mathcal{H}}{\partial \bar{q}} \cdot \frac{\partial \mathcal{H}}{\partial \bar{p}}} - \cancel{\frac{\partial \mathcal{H}}{\partial \bar{p}} \cdot \frac{\partial \mathcal{H}}{\partial \bar{q}}} = \frac{\partial \mathcal{H}}{\partial t}$$

Thus the full derivative with respect to time is the same as the partial derivative with respect to time. If the Hamiltonian is *autonomous* (i.e., does not depend on time explicitly, that is, $\mathcal{H} = \mathcal{H}(\bar{q}, \bar{p})$), then $\mathrm{d}\mathcal{H}/\,\mathrm{d}t = \partial \mathcal{H}/\partial t = 0$, and thus $\mathcal{H}(\bar{q}, \bar{p}) = \mathrm{const}$ is an integral of motion!

3 *Conservation of volume:*
Time evolution of $(\bar{q}, \bar{p})$ defined by Eqs. (4.14) is called *Hamiltonian flow*.



Imagine a set of possible initial conditions in the $(\bar{q}, \bar{p})$-space. No matter how it evolves along the flow, its volume remains constant ("the Liouville theorem").

This means, in particular, that the so-called *attractors* (manifolds of dimension smaller than the number of degrees of freedom, to which the flow collapses) do not exist. A particular consequence of this is that *asymptotic stability* is not possible.

Gauss perturbation equations

$$\frac{da}{dt} = 2a^2 \left[e \sin\theta \; S' + (1 + e\cos\theta) \; T' \right]$$

$$\frac{de}{dt} = p \sin\theta \; S' + p(\cos\theta + \cos E) \; T'$$

$$\frac{dI}{dt} = r \cos u \, W'$$

$$\frac{d\Omega}{dt} = r \sin u \; \mathrm{cosec} \; I \; W'$$

$$\frac{d\omega}{dt} = \frac{1}{e} \left[-p \cos\theta \; S' + (r + p) \sin\theta \; T' \right] \; - \; \cos I \; \frac{d\Omega}{dt}$$

$$\frac{dM}{dt} = n \; + \; \frac{\sqrt{1 - e^2}}{e} \left[(p \cos\theta - 2er) \; S' - (r + p) \sin\theta \; T' \right]$$

Lagrange perturbation equations

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M}$$

$$\frac{de}{dt} = \frac{1-e^2}{ena^2}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{ena^2}\frac{\partial R}{\partial \omega}$$

$$\frac{dI}{dt} = \frac{\mathrm{cotan}\ I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} - \frac{\mathrm{cosec}\ I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \Omega}$$

$$\frac{d\Omega}{dt} = \frac{\mathrm{cosec}\ I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{ena^2}\frac{\partial R}{\partial e} - \frac{\mathrm{cotan}\ I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I}$$

$$\frac{dM}{dt} = n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{ena^2}\frac{\partial R}{\partial e}$$

Perturbation equations in Hamiltonian form

$$\begin{aligned} \frac{dL_k}{dt} &= \frac{\partial \mathcal{H}}{\partial l_k}, \\ \frac{dl_k}{dt} &= -\frac{\partial \mathcal{H}}{\partial L_k}, \end{aligned} \qquad \text{(k = 1,2,3)}$$

with the Hamiltonian

$$\mathcal{H} = \frac{æ^4}{2L_1^2} + R$$

and the *Delaunay variables*

$$\begin{aligned} L_1 &= æ\sqrt{a}, & l_1 &= M, \\ L_2 &= æ\sqrt{a(1-e^2)}, & l_2 &= \omega, \\ L_3 &= æ\sqrt{a(1-e^2)}\cos I, & l_3 &= \Omega \end{aligned}$$

In the previous sections (§§ 4.3–4.5), we derived three equivalent systems of perturbation equations. In the subsequent two sections (§§ 4.6–4.7), we will consider two possible methods on how to solve them.

4.6 The small parameter method

Equations of perturbed motion (Gauss, Lagrange, or Hamiltonian) all have the following form:

$$\begin{cases} \dot{x} &= \mu\, f\,(x, y) \\ \dot{y} &= n\,(x) + \mu\, g\,(x, y) \end{cases} \tag{4.15}$$

Here,

$x \equiv \{a, e, I, \Omega, \omega\}$ (a vector of *slow* variables)
$y \equiv \{M\}$ (a vector of *fast* variables – actually, we only have one such variable)
μ is a small parameter that characterizes the strength of a perturbations.
For instance, if we consider the motion of an asteroid around the Sun perturbed by Jupiter, μ can be the mass ratio of Jupiter and the Sun: $\mu = M_{jup}/M_{\odot}$.

Note that M is a linear function of time (in the unperturbed motion), so that time t "sits" in M, and t usually does not appear in the equations explicitly.

Lyapunov and Poincaré suggested a method of solving (4.15).
The idea to to make successive iterations, starting from the unpertured case.

In the zero-order approximation the perturbation is set to zero (i.e., $\mu \equiv 0$), and we get:

$$\begin{cases} \dot{x} &= 0 \\ \dot{y} &= n\,(x) \end{cases} \quad \Rightarrow \quad \begin{cases} x = x_0 \\ y = y_0 + n\,(x_0)\, t \end{cases} \qquad \text{where } x_0,\ y_0 \text{ are arbitrary constants}$$

The first-order solution is

$$\begin{cases} \dot{x} &= \mu\, f\,(x_0, y_0 + n\,(x_0)\, t) \\ \dot{y} &= n\,(x_0) + \mu\, g\,(x_0, y_0 + n\,(x_0)\, t) \end{cases} \quad \Rightarrow \quad \begin{aligned} x =& x_0 + \mu \overbrace{\int f\,(x_0, y_0 + n\,(x_0)\, t)\, \mathrm{d}t}^{\equiv x_1} \\ y =& y_0 + n\,(x_0)\, t + \mu \underbrace{\int g\,(x_0, y_0 + n\,(x_0)\, t)\, \mathrm{d}t}_{\equiv y_1} \end{aligned}$$

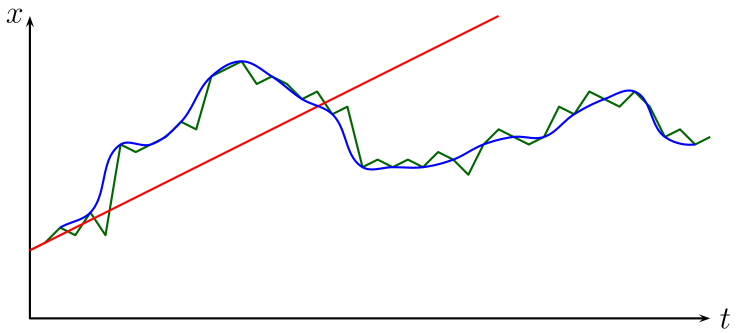
In most of the problems, this first-order solution is sufficient. However, if necessary, we can continue indefinitely to find the exact solution to Eqs. (4.15) in the form of series:

$$\begin{cases} x &= x_0 + \sum\limits_{k=1}^{\infty} \mu^k x_k \\ y &= y_0 + n\,(x_0)\, t + \sum\limits_{k=1}^{\infty} \mu^k y_k \end{cases}$$

4.7 The orbit-averaging method

Consider Eqs. (4.15) again. The most important variables are x: they tell us everything about the orbit, whereas y only characterizes the instantaneous position of the body in that orbit. Hence we are most interested in the first of the two Eqs. (4.15). But even this equation contains y in the right-hand sides (r.h.s.). How to get rid of y there?

We may also wish to exclude y from the r.h.s. for another reason: y is a *fast* variable, thus rapidly changing from 0 to 2π in one orbital period. Its presence in the r.h.s. makes the r.h.s. changing rapidly as well. This necessitates taking a very small integration step when solving Eqs. (4.15) numerically, slowing down the calculations.



A schematic plot, with green curve respesenting the exact behavior of $x(t)$

(a) The simplest way of eliminating y from the r.h.s. of Eqs. (4.15) is to replace the r.h.s. by their initial value:

$$\dot{x} = \mu\, f\,(x, y)\,\big|_{t=0} \qquad \Rightarrow \qquad x = \mu\, f\,(x, y)\,\big|_{t=0} \cdot t$$

This gives a tangent to the solution at $t = 0$ (red line in the figure) — too crude!

(b) A better method is to replace the r.h.s. of Eqs. (4.15) by their time averages — or, which is the same, by their averages over the fast variable y:

$$\begin{cases} \dot{x} = \mu\, \bar{f}\,(x) \\ \dot{y} = n\,(x) + \mu \bar{g}\,(x) \end{cases} \tag{4.16}$$

with

$$\bar{f}\,(x) \equiv \frac{1}{2\pi} \int\limits_0^{2\pi} f\,(x, y)\, \mathrm{d}y, \qquad \bar{g}\,(x) \equiv \frac{1}{2\pi} \int\limits_0^{2\pi} g\,(x, y)\, \mathrm{d}y$$

The result is a relatively smooth curve (yellow in the figure) that follows the (unknown) exact solution more closely than in the previous method.

(c) Finally, it is possible to treat Eq. (4.16) not as an approximate system, but as an *exact* one – but written in some different variables $\{X, Y\}$:

$$\begin{cases} \dot{X} = \mu\, \bar{f}\,(Y) \\ \dot{Y} = n\,(X) + \mu \bar{g}\,(X) \end{cases} \tag{4.17}$$

The art is to find a transformation $\{x, y\} \leftrightarrow \{X, Y\}$ that will bring Eqs. (4.16) to Eqs. (4.17)... not easy. But if we managed to find it, and to solve Eqs. (4.17), we can then go back from $\{X, Y\}$ to $\{x, y\}$ and thus find the *exact* solution to the original system!

The version (b) of the orbit-averaging method is most commonly used in practice.

4.8 The Hansen coefficients

So far, we have derived perturbation equations in three different forms and found two methods to solve them. In practice, there are two major difficulties:

(i) How to express the right-hand sides of the equations through orbital elements?

(ii) If we want to apply the orbit-averaging method (see section 4.7), how to calculate the time averages of the r.h.s., i.e., how to exclude the fast variable y from them?

Both problems are easy to solve by using the so-called Hansen coefficients, which we will now introduce. It turns out that all of the (unwanted) dependence of the r.h.s. on the coordinates is through the terms

$$\left(\frac{r}{a}\right)^n \exp\left(ik\theta\right) \qquad \text{with} \qquad n, k \in \mathbb{Z}.$$

This combination of coordinates can be expanded in Fourier series:

$$\left(\frac{r}{a}\right)^n \exp\left(ik\theta\right) = \sum_{q=-\infty}^{\infty} X_q^{n,k}\left(e\right) \exp\left(iqM\right) \tag{4.18}$$

The symbols $X_q^{n,k}$, which are functions of eccentricity only, are called *Hansen coefficients*. With Eq. (4.18), the problem (i) is solved: the r.h.s. only depends on elements (e, M) and no longer on r and θ.

How to find the time-average of Eq. (4.18)? By definition of the time average of any quantity (which we will denote by a bar over it),

$$\overline{\left(\frac{r}{a}\right)^n \exp\left(ik\theta\right)} = \frac{1}{2\pi} \int\limits_0^{2\pi} \left(\frac{r}{a}\right)^n \exp\left(ik\theta\right) \mathrm{d}M,$$

or, using Eq. (4.18),

$$\overline{\left(\frac{r}{a}\right)^n \exp\left(ik\theta\right)} = \frac{1}{2\pi} \int\limits_0^{2\pi} \sum_{q=-\infty}^{\infty} X_q^{n,k}\left(e\right) \exp\left(iqM\right) \mathrm{d}M.$$

Imaginary part:
$\overline{\left(\frac{r}{a}\right)^n \sin\left(k\theta\right)} = 0$ (because $\int_0^{2\pi} \sin(qM)\,\mathrm{d}M = 0$ for any q).
Real part:
$\overline{\left(\frac{r}{a}\right)^n \cos\left(k\theta\right)} = \frac{1}{2\pi}\int_0^{2\pi} X_0^{n,k}(e)\,\mathrm{d}M = X_0^{n,k}\left(e\right)$ (since $\int_0^{2\pi} \cos(qM)\,\mathrm{d}M = 0$ for any $q \neq 0$).

This means, we have also solved the problem (ii): the averages of $(r/a)^n \sin\left(k\theta\right)$ are simply zeroes, whereas the averages of $(r/a)^n \cos\left(k\theta\right)$ are Hansen coefficients with zero lower indices, $X_0^{n,k}$. The latter thus play a special role. They can be calculated as follows:

$$X_0^{n,k} \equiv \overline{\left(\frac{r}{a}\right)^n \cos\left(k\theta\right)} = \frac{1}{2\pi} \int\limits_0^{2\pi} \left(\frac{r}{a}\right)^n \cos\left(k\theta\right) \mathrm{d}M \tag{4.19}$$

Examples

Let us compute several particular Hansen coefficients $X_0^{n,k}$
(actually those that we will need for the next lecture):

$n = 1$, $k = 0$ (see section 2.9 where we calculated the average distance):

$$X_0^{1,0} = \overline{\left(\frac{r}{a}\right)} = \frac{1}{a} \cdot \overline{r} = \frac{1}{a} \cdot a \left(1 + \frac{e^2}{2}\right) = 1 + \frac{e^2}{2}$$

$n = -1$, $k = 0$ (see section 2.9 for the average inverse distance as well):

$$X_0^{-1,0} = \overline{\left(\frac{r}{a}\right)^{-1}} = a \cdot \overline{\frac{1}{r}} = a \cdot \frac{1}{a} = 1$$

$n = -2$, $k = 0$:

$$X_0^{-2,0} = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{a^2}{r^2} \,\mathrm{d}M \overset{(*)}{=} \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{1-e^2}} = \frac{1}{\sqrt{1-e^2}}$$

(*) We use the angular momentum integral:

$$\begin{aligned}
r^2\dot{\theta} &= \kappa\sqrt{p} \\
r^2 \,\mathrm{d}\theta &= \kappa\sqrt{p}\,\mathrm{d}t \\
r^2 \,\mathrm{d}\theta &= \frac{\kappa\sqrt{p}}{n}\,\mathrm{d}M \\
r^2 \,\mathrm{d}\theta &= \frac{\kappa\sqrt{a\left(1-e^2\right)}}{\kappa a^{-3/2}}\,\mathrm{d}M \\
r^2 \,\mathrm{d}\theta &= a^2\sqrt{1-e^2}\,\mathrm{d}M \\
\frac{a^2}{r^2}\,\mathrm{d}M &= \frac{\mathrm{d}\theta}{\sqrt{1-e^2}}
\end{aligned}$$

$n = -3$, $k = 0$:

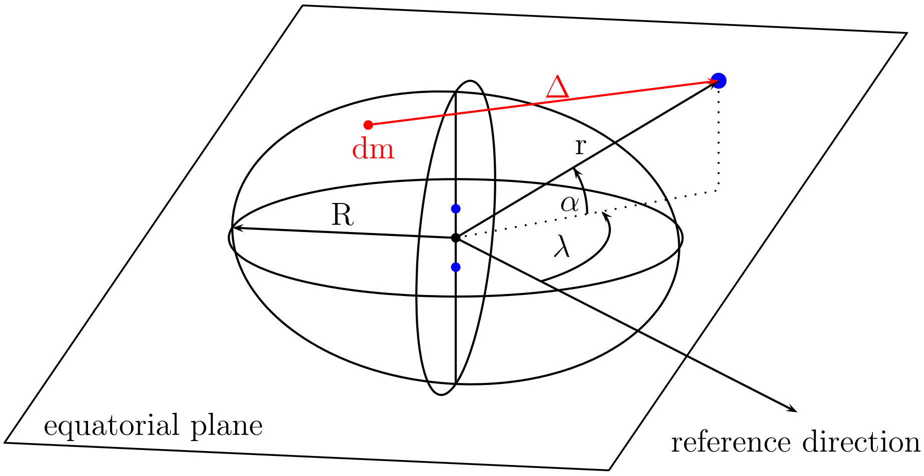
$$\begin{aligned}
X_0^{-3,0} &= \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{a^3}{r^3} \,\mathrm{d}M = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{a}{r}\frac{a^2}{r^2} \,\mathrm{d}M = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{a}{r}\frac{\mathrm{d}\theta}{\sqrt{1-e^2}} \\
&= \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{a\left(1+e\cos\theta\right)}{a\left(1-e^2\right)}\frac{\mathrm{d}\theta}{\sqrt{1-e^2}} \\
&= \frac{1}{2\pi\left(1-e^2\right)^{3/2}} \int\limits_0^{2\pi} \left(1+e\cos\theta\right)\mathrm{d}\theta = \frac{1}{\left(1-e^2\right)^{3/2}}
\end{aligned}$$

$n = -3$, $k = 2$:

$$X_0^{-3,2} = \ldots = \frac{1}{2\pi\left(1-e^2\right)^{3/2}} \int\limits_0^{2\pi} \left(1+e\cos\theta\right)\cos 2\theta \,\mathrm{d}\theta = 0$$

4.9 Example: motion around an oblate body

Now, as both perturbation equations and methods of their solution are at our disposal, I will show how to use them practically. Let us solve one particular problem, namely the motion of a test particle around an oblate body (i.e., spheroid, or two-axial ellipsoid).



This problem was originally motivated by the need to accurately describe the motion of satellites orbiting the Earth in times when space flight became reality (in 1960s). Indeed, for low-altitude satellites the non-sphericity of the Earth is everything else than negligible. There are several possibilities to address the problem. For instance, instead of one point mass in the center, one can also imagine two point masses set symmetrically on the polar axis ("problem of two fixed centers", yellow points in the figure). Unfortunately, it was shown that such two point masses would well approximate the gravitational field of a *prolate* rather than oblate body, hence the method does not seem to be applicable to our problem. However, M.D. Kislik and collaborators in Moscow found a non-trivial possibility to approximate the gravity of an oblate body with two point masses: it is sufficient to allow them to be... complex and located at a purely imaginary distance from the center of the Earth!

We will focus here on another, more general method applicable to an extended body of an (almost) *arbitrary* shape (not only to oblate or prolate spheroids). Omitting the proof, the potential of such a body can be written as an expansion in spherical functions:

$$U \equiv G \int\limits_{\text{body}} \frac{\mathrm{d}m}{\Delta} = \dots$$

$$= \frac{GM}{r}\left[1-\sum_{k=2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k\left(\sin\alpha\right)+\sum_{k=2}^{\infty}\sum_{j=1}^{k}\left(\frac{R}{r}\right)^k P_{k,j}\left(\sin\alpha\right)\left(C_{k,j}\cos j\lambda+S_{k,j}\sin j\lambda\right)\right]$$

where M is the mass, R is the (equatorial) radius, P_k are Legendre polynomials and $P_{k,j}$ are associated Legendre polynomials. The leading "1" in brackets describes the point-like potential. The first (single) sum contains the *zonal* terms that describe latitudinal asymmetries of the body. Finally, the double sum is composed of the *sectorial* ($j = k$) harmonics that describe longitudinal asymmetries, and the *tesseral* ($j \neq k$) terms that describe variation both the latitude α and longitude λ.

If the particular case of a spheroidal body, we can neglect the sectorial and tesseral parts. Furthermore, only $k = 2$ zonal term is needed. Hence the potential can we written as:

$$\boxed{U = \frac{GM}{r}\left[1 - J_2\left(\frac{R}{r}\right)^2 P_2\left(\sin\alpha\right)\right]}$$

Here, the first term represents the point-like gravity. The second one is the correction to potential for oblateness, i.e., the disturbing function $\mathcal{R}$ for our problem:

$$\mathcal{R} = -J_2\frac{GMR^2}{r^3}P_2\left(\sin\alpha\right)$$

The coefficient J_2 is called *oblateness coefficient*. It is $J_2 \sim 0.001$ for the Earth (but much larger, $J_2 \sim 0.015$, for rapidly-rotating Jupiter and Saturn!).
Next, the first Legendre polynomials are

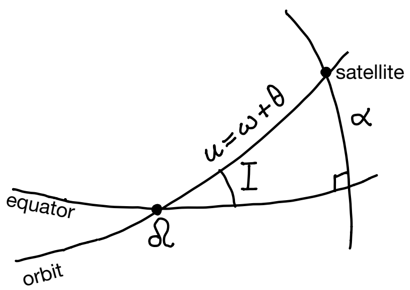
$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \qquad \ldots$$

Thus, the disturbing function $\mathcal{R}$ is given by:

$$\mathcal{R} = -\frac{\nu}{r^3}\left(3\sin^2\alpha - 1\right)$$

with

$$\nu \equiv \frac{1}{2}J_2GMR^2.$$



Applying the sine formula to the spherical triangle depicted on the left, we can find:

$$\frac{\sin u}{\sin 90^\circ} = \frac{\sin\alpha}{\sin I}$$

or

$$\sin\alpha = \sin\left(\omega + \theta\right)\sin I.$$

Therefore,

$$\begin{aligned}\mathcal{R} &= -\frac{\nu}{r^3}\left[3\sin^2 I\sin^2\left(\omega+\theta\right) - 1\right]\\ &= -\frac{\nu}{r^3}\left[3\sin^2 I\left(\frac{1}{2} - \frac{1}{2}\cos\left(2\omega + 2\theta\right)\right) - 1\right]\\ &= -\frac{\nu}{r^3}\left[\left(\frac{3}{2}\sin^2 I - 1\right) - \frac{3}{2}\sin^2 I\cos 2\omega\cos 2\theta + \frac{3}{2}\sin^2 I\sin 2\omega\sin 2\theta\right]\\ &= -\frac{\nu}{a^3}\left[\left(\frac{3}{2}\sin^2 I - 1\right)\underbrace{\left(\frac{r}{a}\right)^{-3}} - \frac{3}{2}\sin^2 I\cos 2\omega\underbrace{\left(\frac{r}{a}\right)^{-3}\cos 2\theta} + \frac{3}{2}\sin^2 I\sin 2\omega\underbrace{\left(\frac{r}{a}\right)^{-3}\sin 2\theta}\right]\end{aligned}$$

The terms marked in the last line are not expressed in terms of orbital elements and therefore are unwanted. We shall express them through orbital elements and — in the same step – orbit-average the disturbing function, both using the Hansen coefficients introduced in the previous section. The averaged disturbing finction is:

$$\langle\mathcal{R}\rangle = -\frac{\nu}{a^3}\left[\left(\frac{3}{2}\sin^2 I - 1\right)\underbrace{X^{-3,0}}_{(1-e^2)^{-\frac{3}{2}}} - \frac{3}{2}\sin^2 I\cos 2\omega\,\underbrace{X_0^{-3,2}}_{0}\right] + \frac{3}{2}\sin^2 I\sin 2\omega\cdot 0$$

$$\boxed{\langle\mathcal{R}\rangle = -\frac{\nu}{a^3}\frac{1}{(1-e^2)^{\frac{3}{2}}}\left(\frac{3}{2}\sin^2 I - 1\right)}$$

where angular brackets denote averaging (same meaning as bar in the previous section). As we see, $\langle\mathcal{R}\rangle = \langle\mathcal{R}\rangle(a, e, I)$, and is independent of the other three elements (Ω, ω and M). The Lagrange equations (4.6a)–(4.6M) immediately tell us that

$$\langle\dot{a}\rangle = \langle\dot{e}\rangle = \langle\dot{I}\rangle = 0,$$

i.e., the oblateness of the central body does not change the size and shape of the orbit and the inclination of the orbital plane. Important: strictly speaking, this conclusion is approximate. Exact, non-averaged disturbing function would depend on ω, Ω and M as well. Thus in reality, the semi-major axis, eccentricity, and inclination will exhibit small "oscillations" around a constant value.

We now calculate changes in Ω and ω (M is simply less interesting) from the Lagrange equations. For $\langle\dot{\Omega}\rangle$, we use Eq. (4.6$\Omega$) to get:

$$\langle\dot{\Omega}\rangle = \frac{\operatorname{cosec} I}{na^2\sqrt{1-e^2}}\frac{\partial\langle\mathcal{R}\rangle}{\partial I} \tag{4.19O}$$

$$= \frac{1}{\kappa\sqrt{a(1-e^2)}\sin I}\left(-\frac{\nu}{a^3}\right)\cdot 3\sin I\cos I\left(1-e^2\right)^{-3/2} \tag{4.20}$$

or

$$\boxed{\langle\dot{\Omega}\rangle = -\frac{3\nu\cos I}{\kappa a^{7/2}\left(1-e^2\right)^2}} \tag{4.21}$$

Similarly, for $\langle\dot{\omega}\rangle$ we use Eq. (4.6ω) find:

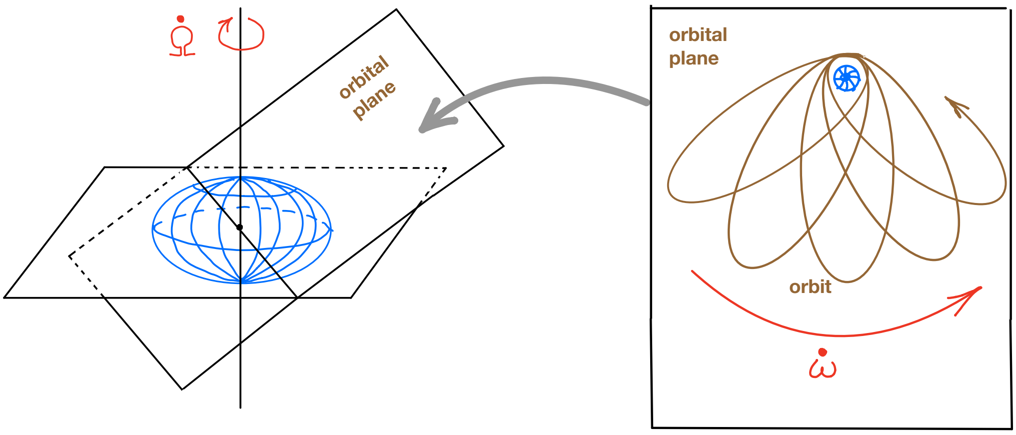
$$\boxed{\langle\dot{\omega}\rangle = +\frac{6\nu\left(1-\frac{5}{4}\sin^2 I\right)}{\kappa a^{7/2}\left(1-e^2\right)^2}} \tag{4.22}$$

Geometry of the orbit

Using the small parameter method (the first approximation is sufficient), we find

$$\langle\Omega\rangle = \langle\Omega_0\rangle + \langle\dot{\Omega}\rangle\cdot t$$
$$\langle\omega\rangle = \langle\omega_0\rangle + \langle\dot{\omega}\rangle\cdot t,$$

where $\langle\dot{\Omega}\rangle$ and $\langle\dot{\omega}\rangle$ are given by Eqs. (4.21) and (4.22). This shows that the orbital plane rotates clockwise around the Z-axis at a rate $\langle\dot{\Omega}\rangle$, and that the orbital ellipse itself rotates counterclockwise in the orbital plane at a rate $\langle\dot{\omega}\rangle$:



Dependence on a

$\langle\dot{\Omega}\rangle, \langle\dot{\omega}\rangle \propto a^{-7/2} \to$ very important for low-altitude orbits

Dependence on e

$\langle\dot{\Omega}\rangle, \langle\dot{\omega}\rangle \propto (1-e^2)^{-2} \to$ important for very elliptic orbits

Dependence on I

- If $I = 90°$ (a polar satellite),
  then $\langle\dot{\Omega}\rangle = 0$, $\langle\dot{\omega}\rangle \neq 0$.
  The orbital plane stays constant, and only the orbital ellipse precesses.
- If $I = 63.4°$ (so that $1 = \frac{5}{4}\sin^2 I$),
  then $\langle\dot{\omega}\rangle = 0$, $\langle\dot{\Omega}\rangle \neq 0$.
  The orbital planet precesses, while the orbital ellipse does not.

  By choosing a high eccentricity ($e \lesssim 1$), we can force the satellite with $I = 63.4°$ to spend most of the time near the apogee. Next, by properly choosing the semimajor axis, we can also make the orbital period equal to one day[1]. Such a satellite will "hang" over some region on the Earth surface for most of the time. This can be useful – e.g, for monitoring (civil or military). There also exist the so-called "Molniya" (from Russian: "Lightning") orbits with a period of 12 hours, allowing one to monitor *two* regions on the Earth with one satellite.

Numerical estimate.

Eqs. (4.21) and (4.22) both contain the same factor, which can be re-written more elegantly:

$$\frac{\nu}{\kappa a^{7/2}} = \frac{GMJ_2R^2}{2\kappa a^{7/2}} = \frac{\kappa a^{-3/2}J_2R^2}{2a^2} = \frac{1}{2}nJ_2\left(\frac{R}{a}\right)^2.$$

Imagine a low altitude satellite ($a \sim R$) in a nearly-circular orbit ($e \approx 0$) in the equatorial plane ($I \approx 0$). Since for the Earth $J_2 \sim 10^{-3}$, we obtain roughly (neglecting pre-factors "3", "3/2" or such like): $\langle\dot{\Omega}\rangle \sim 10^{-3}n$. This means that the precession period of Ω and ω is about 1000 times the orbital period ("Gagarin time" of 90 min), or just a couple of months!

[1]Strictly speaking, "nodal day", which is slightly less that 24 hours, to compensate for $\langle\dot{\Omega}\rangle$