Metric Space

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Chapter 1

Lecture 01: Definition and Examples

1.1 Definition of Metric space

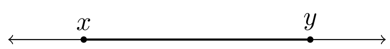
Metric Space

Definition 1.1. *A Metric Space* (X, d) *is a set* x *together with a map* $d : X \times X \to [0, \infty)$. *Such that,*

1. $\forall x, y \in X, d(x, y) = 0 \iff x = y$ *(Reflective)*
2. $d(x, y) = d(y, x)$ $, \forall x, y \in X$ *(Symmetric)*
3. $d(x, z) \leq d(x, y) + d(y, z)$ $, \forall x, y \in X$ *(Triangle Inequality)*

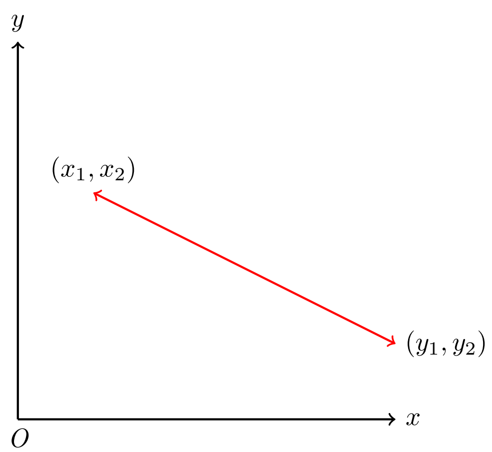
1.2 Example of Metric space

Example 1.1.1. $X = \mathbb{R}$, *together with* $\forall x, y \in X, (x, y) \mapsto d(x, y) = |x - y|, (X, d)$ *is Metric Space*



$$d(x, y) = |x - y| \, \forall x, y \in X = \mathbb{R}^n$$

Example 1.1.2. $X = \mathbb{R}^2 = \{(x_1, x_2) \, | x_i \in \mathbb{R}, i = 1, 2\}$ *together with* $\forall x, y \in X, (x, y) \mapsto d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. (X, d) *is Metric Space*



$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \, , \forall x, y \in X = \mathbb{R}^n$$

Example 1.1.3. $X = \mathbb{R}^n = \{(x_1, x_2, ..., x_n)\,|x_i \in \mathbb{R}, i = 1, 2, ...n\}$ *together with* $\forall x, y \in X, (x, y) \mapsto d(x, y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$. (X, d) *is Metric Space*

Proof: Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$

1. (Reflexive)

$$d(x, y) = 0$$
$$\iff \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^2 = 0$$

Since,

$$(x_i - y_i)^2 > 0$$

So ,

$$\left(\sum_{i=1}^{n}(x_i - y_i)^{1/2}\right)^2 = 0$$
$$\iff \forall i = \{1, ..n\} : \left((x_i - y_i)^2 = 0\right)$$
$$\iff x_i = y_i \ , \forall i \iff x = y$$

2. (Symmetry)

$$d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = d(y, x)$$

3. **Triangle inequality**

Cauchy-Schwar Inequality

Let $(x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{R}^n$then

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}$$
$$\iff \langle x, y\rangle \leq \|x\| \cdot \|y\|$$

W.T.S. $d(x, z) \leq d(x, y) + d(y, z)$

$$\iff \sqrt{\sum(x_i - z_i)^2} \leq \sqrt{\sum(x_i - y_i)^2} + \sqrt{\sum(y_i - z_i)^2}$$

Proof.

$$\sum (x_i - z_i)^2 = \sum (x_i - y_i + y_i - z_i)^2 = \sum (x_i - y_i)^2 + (y_i - z_i)^2 + 2(x_i - y_i)(y_i - z_i)$$

$$\sum (x_i - y_i)^2 + \sum (y_i - z_i)^2 + 2\sum (x_i - y_i)(y_i - z_i)$$
$$\leq \sum (x_i - y_i)^2 + \sum (y_i - z_i)^2 + 2\sqrt{\sum(x_i - y_i)^2}\sqrt{\sum (y_i - z_i)^2}$$
$$\leq \left(\sqrt{\sum x_i - y_i)^2} + \sqrt{\sum x_i - y_i)^2}\right)^2$$

thus

$$\sqrt{\sum(x_i - z_i)^2} \leq \sqrt{\sum(x_i - y_i)^2} + \sqrt{\sum(y_i - z_i)^2}$$

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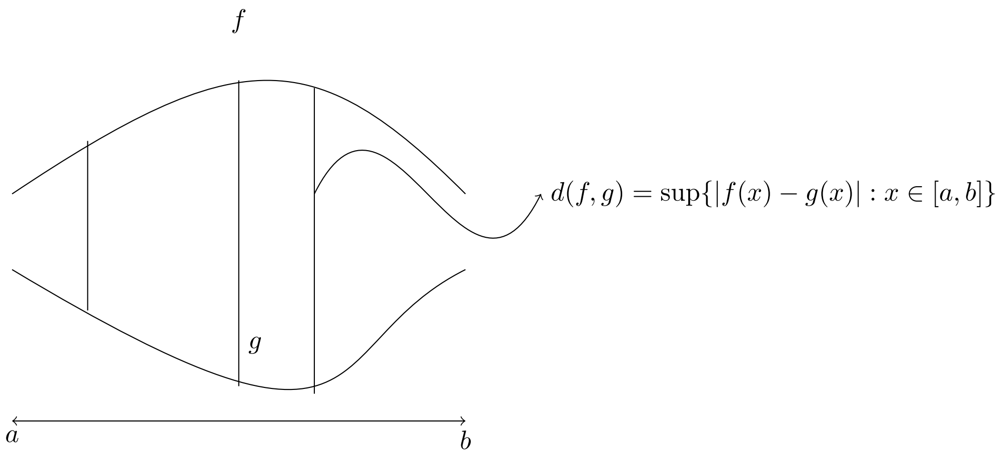
Example 1.1.4. ***(Non-Ecludiean Metric Space).*** *Let* $X = \{f : [a,b] \to \mathbb{R} \mid f$ *is continuous on* $[a,b]\}$ *together with*

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$



Proof. Let $f, g \in X$. Then since $|f - g|$ is continuous on a closed bounded interval $[a,b]$ by maximum minimum theorem ,$\{|f(x) - g(x)| : x \in [a,b]\}$ has Least upper bound (**sup**) which means

$$0 \le d(f,g) = sup\{|f(x) - g(x)| : x \in [a,b]\} < \infty$$

So $d(f,g)$ is well defined.

1. **Reflexive.**
$$d(f,g) = 0 \iff sup\{|f(x) - g(x)| : x \in [a,b]\} = 0$$
$$\iff |f(x) - g(x)| = 0, \forall x \in [a,b]$$
Since $|f - g|$ is no-zero and Sup is zero.
$$\iff f(x) = g(x), \forall x \in [a,b]$$
Thus, $d(x,y) = 0 \iff f = g$

2. **Symmetric.**
$$d(f,g) = sup\{|f(x) - g(x)| : x \in [a,b]\} = sup\{|g(x) - f(x)| : x \in [a,b]\}$$
Thus $d(f,g) = d(g,f)$

3. **Triangle Inequality.**
$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]\} = \sup\{|f(x) - h(x) + h(x) + g(x)| : x \in [a,b]\}$$
Since
$$|f(x) - g(x)| = |f(x) - h(x) + h(x) + g(x)| \le |f(x) - h(x)| + |h(x) - g(x)|$$

And
$$|f(x) - g(x)| \le \sup\{|f(x) - g(x)| : x \in [a,b]\}$$
Similarly,
$$|f(x) - h(x)| + |h(x) - g(x)| \le \; sup\{|f(x) - h(x)| : x \in [a,b]\} + \sup\{|h(x) - g(x)| : x \in [a,b]\}$$
$$|f(x) - h(x)| + |h(x) - g(x)| \le d(f,h) + d(h,g)$$
$$|f(x) - g(x) \le d(f,h) + d(h,g), \;\; \forall x \in [a,b]$$
$$\implies d(f,g) \le d(f,h) + d(h,g)$$

Chapter 2

Lecture 02: Notion Of Open and Closed Set

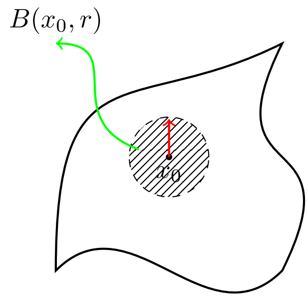
2.1 Definition of Open Ball.

Open Ball

Definition 2.1. *Let (X, d) be ca metric space. Let $x_0 \in X$ and $r > 0$. Then the* ***open ball*** *with radius r centered at x_0 is defined as:*

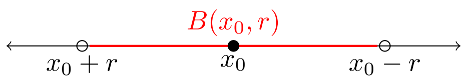
$$B(x_0, r) = \{y \in X \mid d(x_0, y) < r)\}$$

2.1.1 Example of Open ball.



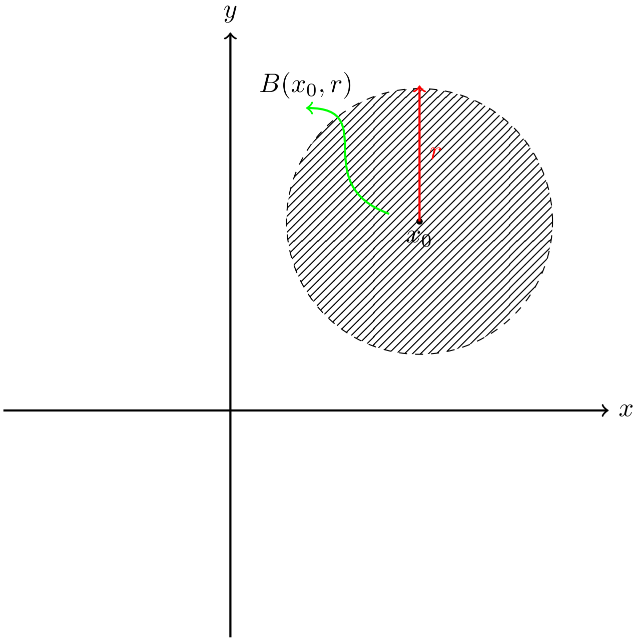
Example 2.1.1. *In $\mathbb{R}$-line.*

$$X = \mathbb{R},\ d(x, y) = |x - y|\ x, y \in \mathbb{R}$$



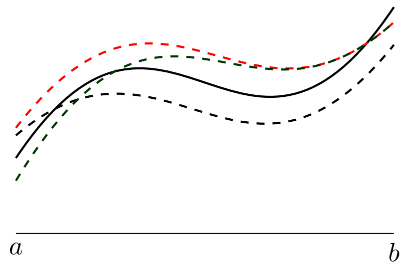
Example 2.1.2. *In 1-D $\mathbb{R}$-line.*

$X = \mathbb{R}^2 = \{(x_1, x_2)\,|x_i \in \mathbb{R}, i = 1, 2\}$ together with $\forall x, y \in X, (x, y) \mapsto d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.



Example 2.1.3.

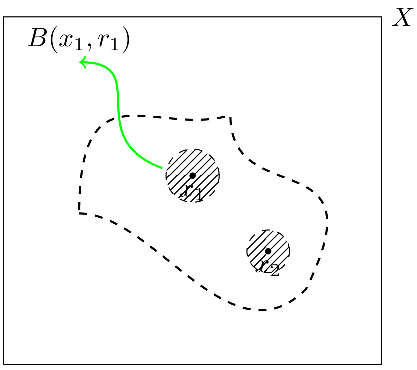
$$X = C[a, b], \text{ together with } d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$



Note: These dashed line around our Lines is the open ball $B(x_0, r)$.

Open Set

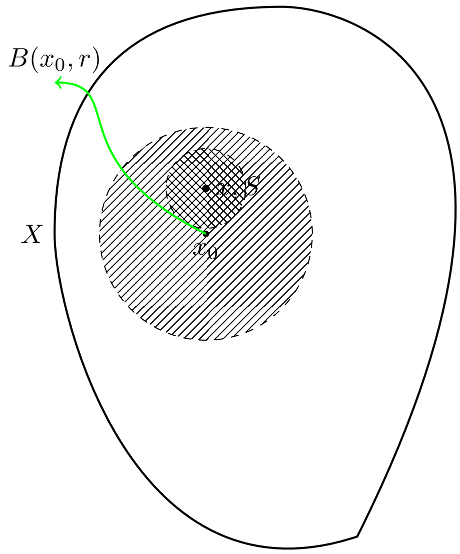
Definition 2.2. *Let $O \subseteq X$, O is* open *if $\forall x \in O$, $\exists r > 0$ such that $B(x_0, r) \subseteq O$.*



Lemma

Lemma 2.2.1. *$B(x_0, r)$ is an open set.*

Proof.
Let $x \in B(x_0, r)$



From fig S is radius at x inside $B(x_0, r)$

So, $S = r - d(x, x_0)$

Claim: $B(x, S) \subseteq B(x_0, r)$

$$\iff y \in B(x, S) \implies y \in B(x_0, r)$$

Proof.
Let $y \in B(x, S)$ By definition, $d(x, y) < S$

$$d(x, y) < r - d(x, x_0)$$

Since By Triangle Inequality,

$$d(x_0, y) \leq d(x_0, x) + d(x, y)$$

$$\implies d(x, y) + d(x, x_0) < r$$
$$\implies d(x_0 y) \leq d(x, y) + d(x, x_0) < r$$
$$\iff d(x_0, y) < r$$
$$\implies y \in B(x_0, r)$$

Since y is arbitrary so,

$$y \in B(x, S) \implies y \in B(x_0, r), \; \forall y \in B(x, S)$$

Thus,

$$B(x, S) \subseteq B(x_o, r)$$

In nutshell, We have shown that,

$B(x_0, r) \subseteq X$, $\forall x \in B(x_0, r)$ there is S such that $B(x, S) \subseteq B(x_0, r)$

Hooray!!! We have shown that $B(x_0, r)$ is an open set.

Chapter 3

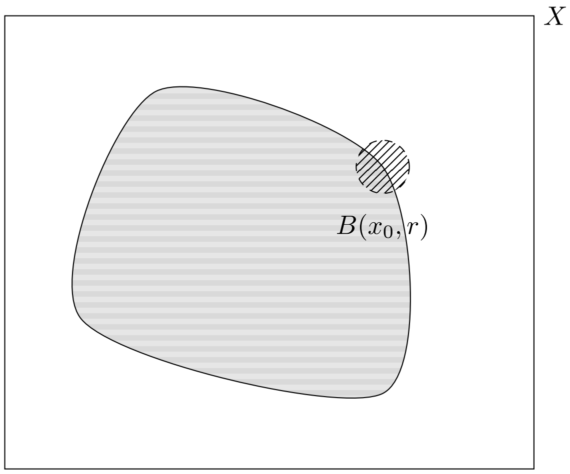
Lecture 03: Properties of set.

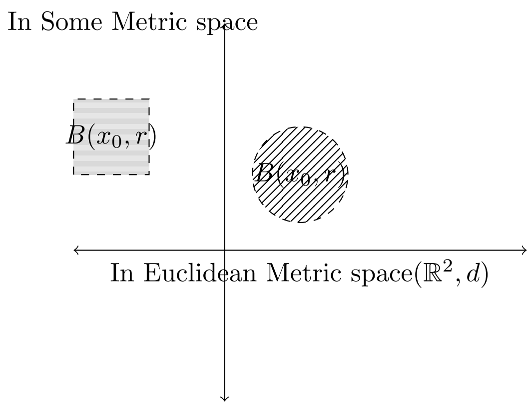
3.1 Properties of open set

Contact point

Definition 3.1. *Let (X, d) be a Metric Space, Let $A \subseteq X$. A point $x_0 \in X$ is contact point of the $A \subseteq X$ if $\forall r > 0, \exists x \in A : x \in B(x_0, r)$*

Example 3.1.1.





3.2 Convergence of Sequence Using Metric Space.

Limit of Sequence in Metric Space

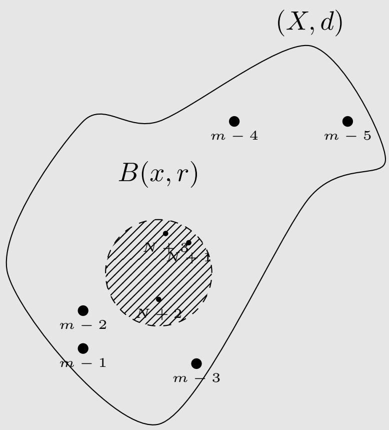
Let $\{x_n\}_{n=1}^{\infty}$ be the Sequence in Metric space(X, d). Let $x \in X$.
We say that

$$\lim_{n\to\infty} x_n = x$$

If $\forall r > 0$, $\exists N(r) \in \mathbb{N}$ s.t if $\forall n > N(r)$ then

$$d(x_n, x) < r \iff x_n \in B(x, r)$$

Illustration



Lemma 3.1.1. *Let* (X, d) *be a metric space.*

1. $$\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} d(x_n, x) = 0$$

2. x is contact point of $A \subseteq X \iff \lim_{n\to\infty} x_n = x$; $\{x_n\}_{n=1}^{\infty} \subseteq A$

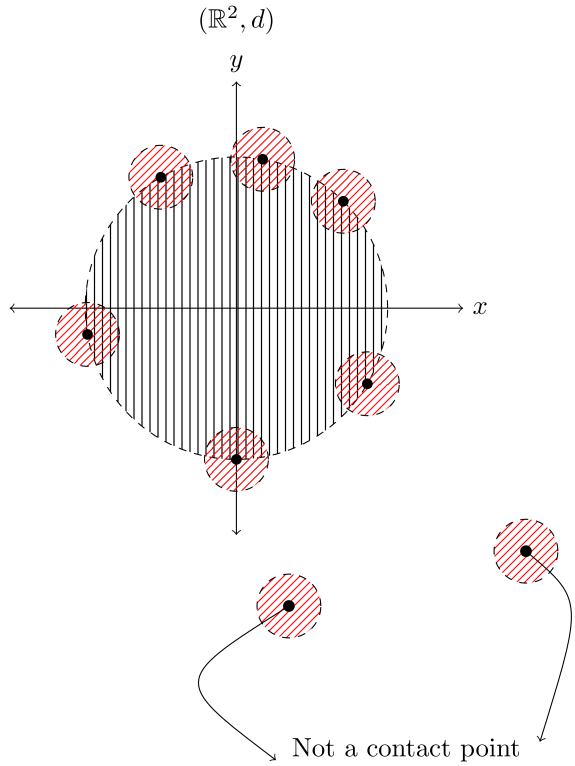
Closed set

A set $E \subseteq X$ is closed if E contains all it's contact point (By the Lemma 3.1.1)

$$E \text{ is closed} \iff \text{whenever}\{x_n\} \subseteq E : x_n \to x \text{ then } x \in E.$$

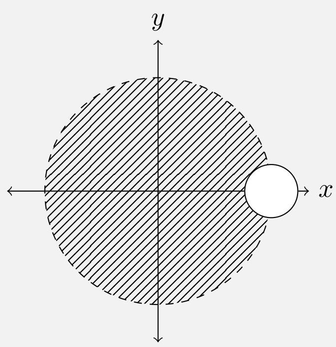
3.2.1 Example of Closed Set

Example 3.1.2. *Example of Closed set in* $(\mathbb{R}^2, d)$, $E := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$



Remark

But $F := E - \{(0,1)\}$ is not closed set.



Since contact point of F should lies in both $B((1,0), \forall r > 0)$ and $F = E - \{(1,0)\}$.
But contact is not in $F = E - \{(1,0)\}$. Thus F is not closed set.

Lemma 3.1.2. ***(Another Definition of Closed set.)***

Let (X, d) be a metric space. Let $A \subseteq X$. Then

$$A \text{ is open } \iff A^c = X/A \text{ is close}$$

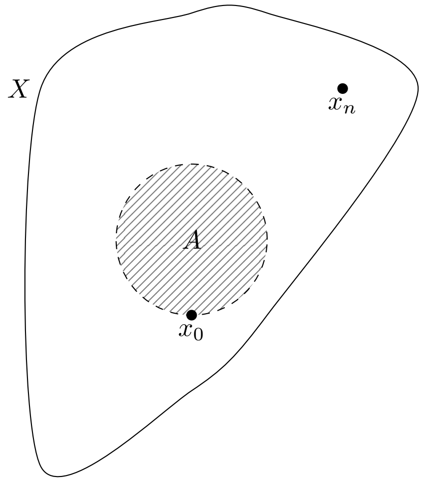
Since $(A^c)^c = A$

$$A^c = X/A \text{ is open } \iff A \text{ is close.}$$

$$A \text{ is close } \iff A^c = X/A \text{ is open.}$$

Pf.

Suppose A is open. Let x_0 be a contact pooint of A^c



Then $\exists\{x_n\}_{n=1}^{\infty}$ in A^c such that $x_n \to x$. We need to show that A^c is closed $\iff$ all contact point of A^c is contained in A^c. So $x_0 \in A^c$ If not , $x_0 \in A$

But since A is open $\exists r > 0$ such that $B(x_0, r) \subseteq A$.

But since $x_n \to x_o$, so $x_n \in B(x_0, r)$ for sufficiently large n. so $x_n \in A$