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Invited Review

Lost-sales inventory theory: A review

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ABSTRACT

In classic inventory models it is common to assume that excess demand is backordered. However, studies analyzing customer behavior in practice show that most unfulfilled demand is lost or an alternative item/location is looked for in many retail environments. Inventory systems that include this lost-sales characteristic appear to be more difficult to analyze and to solve. Furthermore, lost-sales inventory systems require different replenishment policies to minimize costs compared to backorder systems. In this paper, we classify the models in the literature based on the characteristics of the inventory system and review the proposed replenishment policies. For each classification and type of replenishment policy we discuss the available models and their performance. Furthermore, directions for future research are proposed.

1. Introduction

The worldwide out-of-stock rate is rather high with 7–8% (Gruen et al., 2002; Verhoef and Sloot, 2006). The customer behavior to such stock-out occurrences appears to be rather complex. An extensive study by Gruen et al. (2002) reveals that only 15% of the customers who observe a stock out will wait for the item to be on the shelves again, whereas the remaining 85% will either buy a different product (45%), visit another store (31%) or do not buy any product at all (9%). Similar percentages are found by Verhoef and Sloot (2006), who conclude that 23% of the customers will delay the purchase in case of excess demand. These results show that most of the original demand can be considered to be lost in many practical settings. However, most inventory models in literature assume that excess demand is backordered (i.e., customers wait for a new delivery to arrive). As the abovementioned studies show, such models are not representative for a lot of real-world applications in a retail environment. Backorder models are more commonly seen in an industrial environment.

Even though people started studying lost-sales inventory systems around 1960, there are not many applications that consider lost sales. One of the major reasons is that inventory models that include lost sales are more difficult to analyze and to perform computations on compared to backorder models. Namely, the inventory level (inventory on hand minus backorders) cannot be negative in lost-sales models. Therefore, lost-sales systems require a different type of research approach. The number of scientific papers and case studies on this topic has increased the last couple of years. Based on both these observations and trends examined in practice we conclude that there is a growing interest and need for research in lost-sales inventory models. Currently, none of the available overview papers on inventory theory discuss lost-sales settings (see, e.g., Nahmias (1982), Guide and Srivastava (1997), Kennedy et al. (2002), Silver (2008), Williams and Tokar (2008)). The goal of this paper is threefold. First, we present a general classification scheme for the lost-sales inventory systems that have been studied in the literature. Second, we discuss the available literature on lost-sales inventory control models based on the classification scheme. Third, we identify the gap between literature and practice, and we propose directions for future research.

The outline of this paper is as follows. We start with more background information on lost-sales inventory models in Section 2 and compare them with backorder models. We also include a more detailed discussion on why optimal policies are difficult to find for lost-sales models, and show how this can be dealt with based on observations derived from the literature. In Section 3, we present a classification scheme that can be used to classify the models and solution techniques for lost-sales inventory control problems according to the characteristics of the inventory system. Sections 4–7 discuss literature from each of the classes. Section 4 provides an overview on the models with continuous reviews, whereas periodic review models are discussed in Section 5. We first provide a general description of the developments for both types of replenishment review processes, after which we discuss the available literature in more detail. We address models which assume a mixture of lost sales and backordering in Section 6. More related research on lost-sales inventory systems with specific characteristics is provided in Section 7 (e.g., emergency replenishments, multiple demand classes). Our conclusions regarding the literature

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overview and the gap between theory and practice are addressed in Section 8.

2. Lost-sales systems as research area

Inventory models with a backorder assumption have received by far the greatest attention in inventory literature. This is mainly because order-up-to policies are proven to be optimal for backorder models with periodic reviews by Karlin and Scarf (1958) and Scarf (1960). This optimality result has been studied extensively, and resulted in several modifications and extensions (see, e.g., Zabel (1962), Veinott Jr. (1966), Johnson (1968)). Different exact algorithms and approximation techniques have been developed to find optimal or near-optimal values for the reorder level and order-up-to level. Some examples are Federgruen and Zipkin (1984) and Zheng and Federgruen (1991). A comparison of the different procedures is performed by Porteus (1985). Similar developments are found for replenishment policies with fixed order sizes (e.g., Federgruen and Zheng, 1992). Only a few papers include a service level restriction (e.g., Tijms and Groenevelt, 1984). In practice, the latter type of backorder models is commonly used. However, the backorder assumption for excess demand is not realistic in many retail environments. When the lost-sales system is approximated by a backorder model, the cost deviations can run up to 30% (see, e.g., Zipkin, 2008a). Therefore, in general, the customer's behavior has to be modeled with a lost-sales model instead of a backorder model. Huh et al. (2009) show that only under strictly defined circumstances a backorder model can be used to approximate a lost-sales model. More detailed remarks on the comparison between backorder models and lost-sales models are provided in Sections 4 and 5 for continuous and periodic review models, respectively. The objective of this section is to explain the differences between the backorder models and lost-sales models, and to show why lost-sales models require a different approach to analyze inventory systems compared to backorder models.

In a backorder model, the inventory position (inventory on hand plus inventory on order minus backorders) is used as main indicator of the inventory status. It increases when an order is placed and decreases when a demand occurs. Notice that backorders are included in the definition for the inventory position. When the demand is lost instead of backordered, the inventory position does not decrease if the system is out of stock. It is no longer true that the amount of inventory after the lead time equals the inventory position after the order placement minus the demand during the lead time. This is illustrated in Fig. 1, in which an order is placed to increase the inventory position to level S since the inventory position is less than reorder level s at review instant R. This type of replenishment policy is denoted by (R, s, S), and will be explained in more detail in Section 3. Contrary to the backorder model, it is not possible to track the changes in the inventory position independently of the on-hand inventory level when excess demand is lost. Consequently, a lost-sales model has to keep track of the available inventory on hand and the quantities of the individual outstanding orders that were placed in the past and have not yet arrived. As a result, the information vector for a lost-sales model has a length equal to the lead time, and the state space to describe the inventory system grows exponentially fast with the length of the lead time. Therefore, inventory models with a lost-sales assumption on excess demand are more difficult to analyze compared to models where excess demand is assumed to be backordered. In order to keep the analysis tractable, almost all exact approaches assume that at most one (or two) order(s) can be outstanding at the same time. Such assumptions cannot be guaranteed in real-life systems.

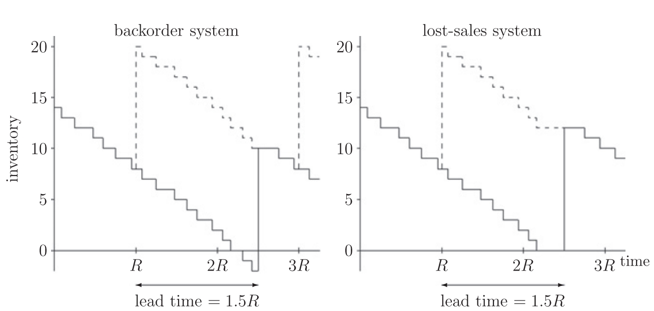

Fig. 1. In the backorder model the inventory level (solid line) is based solely on the inventory position (dashed line) and the demand during the lead time, whereas in the lost-sales model it depends on the individual orders outstanding. In this example an (R, s, S) replenishment policy is used with $s = 9$ and $S = 20$.

3. Classification of literature

In this section, we present a general approach that is commonly used in the literature to tackle lost-sales inventory control problems. It consists of five phases, which we will describe in this section to come to a classification of literature that we use in the remainder of this paper.

3.1. Characterize the inventory replenishment process

The reordering process is characterized by the review interval, the determination of the order size, the order costs and the objective function. Two types of review systems are widely used in business and industry. Either inventory is continuously monitored (continuous review) or inventory is reviewed at regular periodic intervals of length R (periodic review). Whether or not to order at a review instant is usually determined by a reorder level denoted by s. This is the inventory position at which a vendor is triggered to place a replenishment order in order to maintain an adequate supply of items to accommodate current and new customers. An order size can either be fixed (Q units) or variable. The type of replenishment policies with variable order quantities is called order-up-to policies in which the order size is such that the inventory position is increased to an order-up-to level S (see e.g. Fig. 1). There can also be a fixed order cost incurred with each order. When no fixed order cost is charged, there is no incentive not to place an order at a reorder instant in case of variable order sizes. Consequently, the reorder level does not play a role in order-up-to policies (i.e., $s = S - 1$). Based on these characteristics on the replenishment process we present the classification scheme as presented in Table 1 to classify the literature. Similar classifications can be found in Silver et al. (1998). Notice that this classification scheme does not include optimal policies, since none of the aforementioned policies are optimal in a lost-sales setting. See Sections 4 and 5 for a more detailed discussion on optimal policies. Furthermore, we distinguish between models with a total cost minimization objective function (*cost model*) or with a service level constraint (*service model*) for the replenishment process.

Table 1
The notation for the six types of replenishment policies most often applied in literature and practice as used in this paper to classify lost-sales inventory models.

		Order moment			
		Continuous review		Periodic review	
Order size	Fixed	(s, Q)		(R, s, Q)	
	Variable	No fixed cost: Fixed cost:	$(S-1, S)$ (s, S)	No fixed cost: Fixed cost:	(R, S) (R, s, S)

3.2. Identify assumptions

Common assumptions of the inventory model to represent the system concern the demand distribution, the lead time (deterministic or stochastic), and the maximum number of outstanding orders. When more than a single order can be outstanding at the same time and lead times are stochastic, difficulties are encountered in properly representing the lead time as a random variable. This is because in practice, orders are almost always received in the same sequence in which they were placed (i.e., orders cannot cross). When the lead time is, however, assumed to be an independent random variable, orders are allowed to cross in time. There is no easy solution to model this problem of dependency between the lead times of the orders outstanding. Therefore, it is common to assume independent lead times when they are stochastic. This assumption makes sense in most real-world applications, where the time interval between placement of two or more orders is usually large enough that there is no interaction between orders. Consequently, it is a good approximation to treat the lead time as an independent random variable while simultaneously assuming that orders do not cross.

3.3. Develop a Markov model

Usually a Markov model is developed to represent the on-hand inventory level and the individual orders outstanding based on the characteristics and assumptions.

3.4. Analyze the long-run behavior

Based on the transition probabilities and steady-state behavior of the system the long-run behavior of the inventory model is analyzed. The stationary distribution function of the on-hand inventory level is used to express the expected average cost and fill rate. Another approach is the use of dynamic programming to compute the performance measures of interest.

3.5. Determine the inventory control variables

In the final phase, the inventory control variables, such as the reorder level and order quantities, are set. Either an exact procedure or an approximation procedure can be used to find these values. Two types of exact procedures are commonly used in literature, namely a policy iteration algorithm or an extensive numerical search procedure. Such procedures can be performed more efficiently if convexity results are derived for the objective function. However, the amount of computational effort remains large. Therefore, approximation procedures are commonly derived. There are two types of heuristic approaches that are commonly found in literature. Either the EOQ model of Harris (1913) is used to determine the order quantities or the backorder model is used to approximate the steady-state behavior of inventory systems with lost sales. The performance of both heuristic approaches is discussed in Sections 4 and 5.

Different models have been developed in the literature based on the methodology as presented in this section. Following the classification scheme of Table 1, we discuss the literature on continuous and periodic review systems in Sections 4 and 5, respectively. More specific characteristics of the replenishment process are considered in Section 7.

4. Continuous review lost-sales models

The objective of this section is to discuss the available literature on lost-sales inventory systems with continuous reviews. First, we explain the use of the research methodology in more detail for such inventory systems. Next, characteristics of optimal order quantities are discussed. In the remainder of this section we discuss the available literature on all different replenishment policies according to our classification scheme of Table 1. We end this section with a short summary of the developments in the literature and propose directions for future research on continuous review systems.

As mentioned in Section 2, it is common to assume that at most one (or two) order(s) can be outstanding at any time when a lost-sales inventory system with continuous reviews is analyzed. Consequently, the Markov model representing the inventory system will be one (or two) dimensional and will consist of the on-hand inventory level (and inventory on order). The decision (or ordering) points in such a model are the time instants at which either a demand occurs and no order is outstanding, or an order is delivered. Next, the transition probabilities and stationary distribution function for the long-run behavior of the on-hand inventory level are derived conditionally on the number of outstanding orders. This distribution function is used to analyze the inventory system in terms of expected average cost and service level. The analysis is, however, different for different demand and lead time distributions and it also depends on the replenishment policy. In this section, we discuss different solution techniques available in the literature to find optimal or near-optimal order quantities. In practice, it depends on the specific situation which model to choose. For instance, sales data have to be analyzed to determine the most appropriate demand distribution.

Not much is known about an *optimal policy* for continuous review models with lost sales. Johansen and Thorstenson (1993, 1996) are two of the few authors who derive characteristics for optimal order quantities. The authors assume at most one order to be outstanding at the same time in their model. They prove with a semi-Markov decision model the following property for the optimal order quantities $a^*(x)$ where x equals the on-hand inventory level at a decision point:

$$\begin{aligned} &a^*(x+1) \leqslant a^*(x) \leqslant a^*(x+1)+1, && \text{for } x = 0,1,\ldots,s-1, \\ &a^*(x) = 0, && \text{for } x > s. \end{aligned} \tag{1}$$

This means that an optimal order quantity $a^*(x)$ is decreasing in the on-hand inventory level x, and the related rate of decrease is less than one. Furthermore, they show with a numerical example that an optimal policy is prescribed neither by an (s, Q) policy nor by an (s, S) policy. However, we will show in this section that most research is performed on such policies. Unfortunately, there is no comparison of the performances for such policies and an optimal policy in the literature.

We classify the available lost-sales inventory models with continuous reviews in *fixed order size policies* and *order-up-to policies* according to the general classification scheme as presented in Table 1. The majority of the research is performed on inventory systems with fixed order size policies and will be discussed first. Next, order-up-to policies are considered. Notice that fixed order size policies and order-up-to policies are equivalent to each other in a continuous review setting when the demand size of each individual customer is one (i.e., unit sized). For each replenishment policy we identify the restrictions on the inventory control variables which impose that at most one or two orders can be outstanding at the same time to make the analysis tractable (see Section 2). Besides exact analyses under these assumptions, we also discuss common heuristic procedures to determine the order quantities based on the EOQ formula and the backorder model (see Section 3). The performance of such approximation procedures is compared with exact procedures.

4.1. Fixed order size policies

Fixed order size policies are denoted as (s, Q) policies where a new order of size Q is immediately placed when the inventory position drops down to or falls below a reorder level s. Let n denote the maximum number of orders that can be outstanding at any time for this policy. The value of n is specified by $(n-1)Q \leqslant s < nQ$ since the inventory position cannot drop below kQ when k orders are outstanding. Consequently, the maximum number of outstanding orders equals the smallest integer strictly larger than s/Q.

The earliest work on lost-sales inventory models with a fixed order size replenishment policy dates back to Hadley and Whitin (1963). They derive an exact expression for the expected total cost under the assumption that there is never more than a single order outstanding (i.e., $s < Q$). This model is extended to deal with stochastic lead times by Ravichandran (1984), Buchanan and Love (1985), Beckmann and Srinivasan (1987) and Johansen and Thorstenson (1993) for phase type, Erlang and exponential lead time distributions, respectively. A phase type distribution represents a large class of distributions including the Erlang distribution and a mixture of exponentials. The Erlang distribution corresponds to the exponential distribution when the shape parameter is set to one and to a constant lead time when it is set to infinity. A discounted model for the same inventory system with Erlang distributed lead times is considered by Johansen and Thorstenson (1996). Numerical results show that discounting is only of importance when the lead time is uncertain (e.g., exponentially distributed), the interest rate is high and the penalty cost for a lost sale is low. A policy iteration algorithm (PIA) is developed by Johansen and Thorstenson (1993), Johansen and Thorstenson (1996) to find optimal values of s and Q.

The previous models discussed so far in this section assume a Poisson demand process. More general demand distributions are considered by Kalpakam and Arivarignan (1988, 1989a,b) and Mohebbi and Posner (1998a). In the former models demand is generated from a renewal process, whereas in the latter models demand is not assumed to be unit sized but the demand size of each customer follows an exponential distribution. Optimal values of s and Q can be found with an extensive numerical search procedure. The most general model under the assumption $s < Q$ is developed by Rosling (1998). In this model, any lead time distribution can be used and demand is assumed to be continuous or Poisson distributed. It is only required that the distribution function of the demand during the lead time is logconcave. This is the case in most demand distributions suggested for inventory control. An iterative cost minimization procedure is developed to find optimal values of s and Q similar to Hadley and Whitin (1963).

The restriction that at most two orders may be outstanding is specified by $Q \leqslant s < 2Q$. Inventory models with such restrictions are analyzed by Hill (1992, 1994) for deterministic and Erlang distributed lead times, respectively. Other studies which allow for more than one replenishment order to be outstanding at the same time are Morse (1958) for Poisson demand, Kalpakam and Arivarignan (1991) for a unit-sized renewal demand process and Mohebbi and Posner (2002) for a compound Poisson demand process. These models assume exponentially distributed lead times. Local search techniques are used to find optimal parameter values. Johansen and Thorstenson (2004) propose a PIA to find optimal values of s and Q for more general lead time distributions where orders do not cross in time.

Adding a minimal service level restriction to an inventory model with lost sales makes the model more realistic to represent a retail environment, but the analysis and computations become more difficult. Hardly any literature is available that studies this problem. Aardal et al. (1989) examine a continuous review (s, Q) model with a fill rate constraint. They show by using Lagrange multipliers that any service level restriction implies a penalty cost for lost sales, and relate the lost-sales model to the backorder model.

Table 2
An overview on lost-sales inventory models with an (s, Q) replenishment policy.

	Demand	Lead time	Assumption	Objective
Hadley and Whitin (1963)	P	D	$s < Q$	C/S
Ravichandran (1984)	P	PH	$s < Q$	–
Beckmann and Srinivasan (1987)	P	Ex	$s < Q$	C
Buchanan and Love (1985)	P	Er	$s < Q$	C
Johansen and Thorstenson (1993, 1996)	P	Er	$s < Q$	C
Kalpakam and Arivarignan (1988, 1989b)	uR	Ex	$s < Q$	C
Kalpakam and Arivarignan (1989a)	M	M	$s < Q$	C
Mohebbi and Posner (1998a)	CP	Er/HEx	$s < Q$	C/S
Rosling (1998)	P/Cont	G	$s < Q$	C
Hill (1992)	P	D	$Q \leqslant s < 2Q$	S
Hill (1994)	P	Er	$Q \leqslant s < 2Q$	S
Morse (1958)	P	Ex	–	C
Kalpakam and Arivarignan (1991)	uR	Ex	–	C
Mohebbi and Posner (2002)	CP	Ex	–	C
Johansen and Thorstenson (2004)	P	Er	–	C
Aardal et al. (1989)	G	G	–	S

An overview of all models as discussed so far with different demand and lead time distributions is provided in Table 2. The demand distribution is assumed to be general (G), Poisson (P), compound Poisson (CP), continuous (Cont), Markovian (M) or a unit-sized renewal process (uR). The lead time follows a deterministic (D), exponential (Ex), hyperexponential (HEx), Erlang (Er), phase type (PH), Markovian (M) or general (G) distribution. The fourth column indicates whether at most one or two orders can be outstanding at the same time ($s < Q$ or $Q \leqslant s < 2Q$, respectively) or no restriction is imposed on the maximum number of outstanding orders. The objective in the models is cost minimization (C) or a service level constraint is included next to a cost objective (S).

Based on this overview, we conclude that a lot of research has been performed on (s, Q) replenishment policies for lost-sales inventory systems. However, it remains difficult to analyze the model exactly and determine optimal values of s and Q that minimize the expected total cost C (denoted as s^* and Q^*, respectively). In the literature, either a PIA or an extensive search procedure is proposed to find the values of the inventory control variables (see also the methodology of Section 3). The computation time can increase rapidly for large inventory systems. Therefore, simple approximation procedures are also developed. Hadley and Whitin (1963) derive an approximate expression for the expected total cost in which the expected time period during which the system has no on-hand inventory is negligible. The authors propose an iterative procedure to determine s^* and Q^*, such that the derivative of C in s and Q equals zero (i.e., $\partial C/\partial s = \partial C/\partial Q = 0$). First they set $Q_1 = Q_w$, where Q_w is the order quantity based on the EOQ formula. The value Q_1 is used to compute s_1 based on $\partial C/\partial s = 0$ and the so obtained value s_1 to compute Q_2 based on $\partial C/\partial Q = 0$, etc. This procedure stops when s and Q are determined with sufficient accuracy. This is called the H–W procedure. Other approximation procedures in the literature are based on the EOQ model where $Q = Q_w$ and s minimizes the expected total cost, or the backorder model is used to determine values of s and Q (see Federgruen and Zheng, 1992). In general, the following relationships between these models and the optimal inventory control variables are identified by Hadley and Whitin (1963):

- $Q^* \geqslant Q_w$, because of the variability in the demand and the holding cost is less than the penalty cost for a lost sale.
- $Q^* < Q_{BO}$ and $s^* > s_{BO}$, where s_{BO} and Q_{BO} are the optimal reorder level and order quantity in the backorder model. This is because the expected holding cost is lower in the backorder model than in the lost-sales model, since the backorders reduce the average on-hand inventory level (provided that the values of s and Q are the same in both models). But also, the expected penalty cost is higher in the backorder model for the same reason. Consequently, a higher value of s and a lower value of Q are more advantageous in the lost-sales model.

The authors do not give any technical proofs for these results, but they use intuitive arguments. Numerical results show that the cost increases are very minor when one of these heuristic approaches is applied compared to the optimal cost, although the values of the policy variables show larger deviations from using the heuristics (see, e.g., Johansen and Thorstenson, 1993). This is due to the flatness of the cost function. The result it has on the service level is not investigated. The EOQ model results in values of s and Q that are too high and too low, respectively, compared to the exact solutions. The H–W procedure finds too high values of s whereas Q is about the same as in the exact solution. Furthermore, lead time variability shows a significant impact on the values of s^* and Q^* in comparison to the results of the H–W procedure. This is because the H–W procedure ignores the time during which the system is out of stock. With highly variable lead times this is not justified.

Besides fixed order size policies, we also consider order-up-to policies in the remainder of this section as indicated in Table 1.

4.2. Base-stock $(S-1, S)$ policies

When $Q = 1$ and the demand is unit sized, the (s, Q) policy corresponds to a base-stock $(S-1,S)$ policy with order-up-to level $S = s + 1$. In this policy every customer's demand is immediately reordered. This policy is optimal in case excess demand is backordered, the lead time is deterministic and no fixed order cost is charged. Karush (1957) models a lost-sales inventory system with this policy as a queueing system. Since the demand process is assumed to be Poisson and the lead times are mutually independent, the author uses the well-known steady-state probability formulas from queueing theory to calculate the average cost per unit time. The author also proves that the out-of-stock probability is strictly convex in the order-up-to level. A similar convexity property is found in an empirical study by Weinstock and Young (1957) when random replenishment lead times of successive orders are correlated. Convexity has been proven differently by Jagers and van Doorn (1986). An explicit approximation for the order-up-to level S is presented by Smith (1977). The use of indifference curves to find the optimal value of S graphically for this inventory system is mentioned by Silver and Smith (1977). These results are applicable for any type of lead time distribution when mutual independency is assumed between lead times. For a compound Poisson demand process, Feeney and Sherbrooke (1966) derive the steady-state probabilities for the inventory on order when the $(S-1, S)$ policy is applied. They show that the fill rate in a lost-sales model is higher compared to a backorder model and the inventory on order is always smaller. Chen et al. (2011) correct some of the expressions derived by Feeney and Sherbrook (1996). When the replenishment orders are not able to cross in time, the lead times of simultaneously outstanding orders are dependent. Johansen (2005) solves this problem with Erlang distributed lead times. A simple and effective procedure is developed to compute the optimal order-up-to level S with respect to the average cost.

Hill (1999) shows with intuitive justification and an illustrative example that the $(S-1, S)$ policy is never optimal if $S \geqslant 2$. The author proposes a new type of base-stock policy in which a second replenishment order is only placed when the lead time remaining on the outstanding order is less than T time units. Otherwise, the order waits until this occurs. If T equals the lead time, then a new order is immediately placed. If T equals zero, at most one order is outstanding at any time. The author calls this the base-stock policy with delay. As a result, the order process is smoothed over time. A similar remark is made by Karlin and Scarf (1958) and it corresponds with the findings of Johansen and Thorstenson (1993) on optimal order quantities (i.e., Inequality (1)). A more detailed policy with delay is discussed by Hill (2007), where the value of T depends on the remaining lead time of all outstanding orders. Hence, the value of T has to be determined every time a decision is required (i.e., when a demand is satisfied). A simple to compute lower bound on T is derived. These policies with delay can only be evaluated by simulation. The cost benefits to be gained from delay policies compared to base-stock policies are not large (1–2%) but non-trivial.

Reiman (2004) compares the base-stock policy to a policy in which an order of a fixed size is placed every T time units. The author performs an asymptotic analysis for both policies as the penalty cost and lead time grow large, and derives expressions for the best order-up-to policy S and constant order quantity Q.

4.3. (s, S) policies

When a fixed order cost is incurred with each order, a reorder level should be incorporated in the replenishment policy. In case the demand is unit sized, the models for an (s, Q) policy are the same as for an (s, S) policy with $Q = S - s$. To ensure that at most one order can be outstanding in an (s, S) policy, the inventory position should remain above the reorder level s when an order is outstanding, i.e., $S - s > s$. Archibald (1981) studies an (s, S) policy with this assumption for a continuous review inventory system with a compound Poisson demand process. An extensive search procedure is required to find the optimal values of s and S. An approximate solution is also presented as alternative, where the average satisfied demand during the lead time is added to the reorder level to set the order-up-to level. The exact and approximation procedure are compared with a backorder model and an adjusted backorder model, in which the optimal S in the backorder model is decreased by the expected number of shortages during the lead time. The approximation procedure and adjusted backorder model find near-optimal solutions with cost increases less than 0.1%.

In case a service level constraint has to be satisfied, Tijms and Groenevelt (1984) provide a simple approximation to set the reorder level for (s, S) replenishment systems with backlogging. The quantity $S - s$ equals the EOQ value. They make a remark on how to modify the model in case excess demand is lost, but never perform any numerical results to validate the model.

An overview of the different demand and lead time distributions for base-stock policies and (s, S) policies is provided in Table 3. The abbreviations are the same as in Table 2. Column four indicates whether it concerns a base-stock policy ($s = S - 1$) or an (s, S) policy where at most one order can be outstanding at any time ($S - s > s$).

4.4. Concluding remarks on systems with continuous reviews

Based on the overview on continuous review models with lost sales, we conclude that most of the work has been performed on the (s, Q) replenishment policy. PIAs and extensive numerical search procedures are proposed to find optimal values of s and Q. The H–W procedure is a popular approximation procedure to find near-optimal reorder levels and order sizes for the cost model. This procedure has been extended and improved for different demand and lead time distributions. A service model is, however, hardly discussed in the literature. Another topic for future research is

Table 3
An overview on the lost-sales inventory models with an order-up-to replenishment policy.

	Demand	Lead time	Assumption	Objective
Karush (1957)	P	G	$s = S - 1$	C
Smith (1977)	P	G	$s = S - 1$	C
Silver and Smith (1977)	P	G	$s = S - 1$	C
Feeney and Sherbrooke (1966)	CP	G	$s = S - 1$	C
Chen et al. (2011)	CP	G	$s = S - 1$	C
Johansen (2005)	P	Er	$s = S - 1$	C
Hill (1999, 2007)	P	D	$s = S - 1$	C
Reiman (2004)	P	D	$s = S - 1$	C
Archibald (1981)	CP	D	$S - s > s$	C
Tijms and Groenevelt (1984)	G	G	$S - s > s$	S

the investigation of optimal replenishment policies. There are hardly any comparisons with an optimal policy, but only the performance of approximation procedures is illustrated within a specific class of replenishment policies. How well such policies perform compared to an optimal policy is unknown.

5. Periodic review lost-sales models

Optimality results for periodic review inventory models with backorders are well known. As discussed in Section 2, this is not the case when excess demand is lost. The majority of the models that include the lost-sales characteristic in a periodic review setting assume no or negligible fixed order cost. Consequently, an order is placed each review period to minimize holding and penalty costs. This results in a regular replenishment process, whereas orders are placed less regular when fixed order costs are charged. Both situations are observed in practical settings. The classification of models with and without fixed order costs is used in this section as indicated in Table 1. As we will show in this section, the main focus of most lost-sales models for periodic reviews is on identifying near-optimal replenishment policies and deriving bounds on optimal order quantities. These results are first derived for the case where the lead time equals the length of a review period ($L = R$), and next for the case where the lead time is an integral multiple of the review period length ($L = nR$). The bounds on the optimal order quantities can be used as a myopic replenishment policy, but they do not always perform well. More common policies to be found in practice and literature are also investigated, as well as some modified policies where a delay in the order process is included. Similar alterations to replenishment policies have already been discussed for continuous review models in Section 4. An overview on the different types of policies for periodic review lost-sales systems is presented in Fig. 2, in which we use a similar classification as in Table 1. A similar line of research developments is found in the literature when the lead time is stochastic or when it is smaller than or equal to the review period length ($L \leqslant R$). All literature will be discussed in this section.

5.1. No fixed order costs

Bellman et al. (1955) are one of the first authors to address the lost-sales inventory control problem in a periodic review system with non-zero lead times. However, they restrict their attention to the special case that the *lead time equals one review period*. The objective in their model is to minimize the total expected order and penalty costs. Since holding costs are not included, this model is of less relevance in practice. Karlin and Scarf (1958) extended this model to the case with holding costs and no restriction imposed on the lead time. They develop optimal dynamic programming equations for the inventory system when excess demand is lost. In their model, the lead time is assumed to be a fixed number of review periods. For the special case that the lead time equals one review period, they show that the cost function and the optimal order quantities are well-defined and bounded. Yaspan (1961) derived similar properties for this special case.

Morton (1969) extends the results of Karlin and Scarf (1958) with the use of induction to the general case where the *lead time is any integral multiple of the review period length*. The author derives bounds on the optimal order quantities when linear and proportional holding, penalty and ordering costs are assumed. Notice that this assumption prohibits fixed order costs. More recently, Zipkin (2008b) reformulated the original problem of Karlin and Scarf (1958) and Morton (1969) with a new state definition in which partial sums of the outstanding order quantities are considered. Based on the new state variables, the author proves by induction that the optimal cost function is L-natural convex (i.e., convex, submodular and it contains a property related to diagonal-dominance). This implies that the optimal order quantities are monotone decreasing in the inventory position and the optimal order quantities are more sensitive to recent orders. These properties are useful to determine optimal order quantities. L-natural convexity is also used by Huh and Janakiraman (2010b) to derive properties of the optimal replenishment policy in multi-echelon inventory systems with lost sales (see also Section 7). Furthermore, Zipkin (2008b) derived additional bounds on the optimal policy, and he extended the model to include capacity constraints, correlated demands, stochastic lead times and multiple demand classes. Johansen (2001) proposes a policy iteration algorithm (PIA) to determine the optimal replenishment policy.

A comparison between optimal replenishment policies in backorder models and lost-sales models are first presented by Yaspan (1961) for the case when $L = R$, and later generalized by Janakiraman et al. (2007). The studies show that a lost-sales system has lower optimal cost compared to a backorder system when the cost function is the same in both systems. Furthermore, Yaspan (1961) shows that the optimal order quantity is smaller for the lost-sales system when the same inventory status is observed at a review instant.

As indicated by Fig. 2, several replenishment policies have been proposed in the literature to set the order quantities. One of the first policies is a *myopic policy* proposed by Morton (1971), in which the upper bound on the optimal order quantity derived by Morton (1969) is used as the actual order quantity. In this policy the order quantity has to be sufficient to fulfill the demand until the delivery of the next order. Denote the probability of not stocking out during this time period by $P_{NS}(x, \mathbf{y}, z)$ where x is the on-hand inventory level, $\mathbf{y}$ the order sizes of the individual orders outstanding and z the order quantity at the current review instant. The

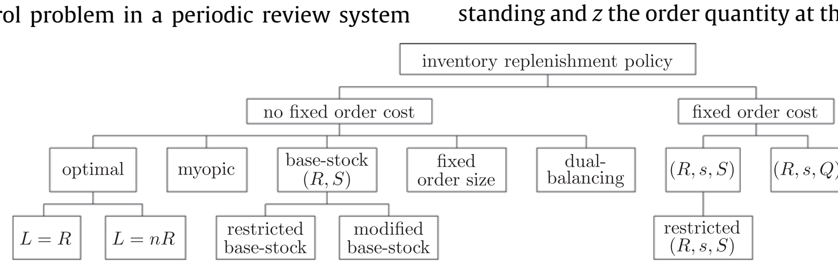

Fig. 2. The developments in replenishment policies for periodic review lost-sales models.

value of z is set such that $P_{NS}(x, \mathbf{y}, z)$ equals the fractional benefit of an increase in the order size by one extra unit. If a unit is ordered but not required, the total cost increases with the unit holding cost h. If, on the other hand, the unit is needed, a penalty cost p is saved minus the extra ordering cost c. The *myopic policy* of Morton (1971) prescribes to order z units that satisfies

$$P_{NS}(x,\mathbf{y},z) = \frac{p-c}{p-c+h}. \tag{2}$$

This policy, however, requires to compute the P_{NS} function at each review. A recursive procedure for these calculations is discussed by Morton (1969). If the demand follows a normal distribution, Yaspan (1972) shows that multi-dimensional normal tables are required to perform the computations.

The first research on *base-stock policies* in a lost-sales setting with periodic reviews dates back to Gaver (1959) and Morse (1959). Both authors restrict to cases where the lead time equals one review period length. This model is extended by Pressman (1977) to the case where the lead time can be any integral multiple of the review period length. To find the optimal order-up-to level, a bisection method can be used due to convexity results on the cost function derived by Downs et al. (2001). They prove by induction that the expected excess demand and on-hand inventory are convex functions of the order-up-to level S when fixed lead times are involved, i.e., the cost function is a convex function.

Huh et al. (2009) compare the performance of base-stock policies in lost-sales and backorder models similar to the analysis of Janakiraman et al. (2007) for optimal replenishment policies. The authors present upper and lower bounds on the expected total cost and the optimal order-up-to levels in the lost-sales model based on the backorder model with different holding and penalty costs for each unit. Numerical results indicate that these bounds are not tight enough to perform well as approximation for the order-up-to level. Their main result states that the optimal order-up-to level for the backorder model is asymptotically optimal for the lost-sales model. This means that under certain circumstances the backorder model can be used as approximation model for a lost-sales inventory system, especially when the penalty cost for a lost demand is high enough. We remark that this asymptotic analysis shows that base-stock policies perform well for large values of the penalty cost and a fixed lead time, whereas Reiman (2004) shows that base-stock policies perform worse than a constant order size policy for large values of the lead time and a fixed penalty cost.

Johansen (2001) relates the performance of periodic review inventory systems with a base-stock policy to the continuous review model of Smith (1977). The difference with a continuous review model is that orders have to wait in a periodic review system for the next review moment. Therefore, a delay is included in the continuous review model by increasing the demand rate during the lead time. Numerical results illustrate that the fill rate and average on-hand inventory level are approximated very well with this approach compared to the exact values. Besides this base-stock policy, Johansen (2001) also proposes a *modified base-stock policy* in which a minimum number of review periods between two subsequent orders is specified to smooth the ordering process over time (similar to the delay policy of Hill (1999, 2007) for continuous reviews). Even though the numerical results find near-optimal cost for this policy, the test instances are restricted to unrealistic values of S for many practical settings ($S \leqslant 4$).

Another policy related to base-stock policies is the *simple approximation policy* proposed by Morton (1971), in which the order size cannot exceed the percentile of the demand to be expected in a review period given by the right-hand side of Eq. (2). Let L denote the lead time and D_n the demand during n review periods with cumulative distribution function $F_n(\cdot)$. This approximation policy prescribes to order

$$z = \min\left\{S - x - \sum_{i=1}^{L-1} y_i, \bar{z}\right\}, \tag{3}$$

where

$$S = F_{L+1}^{-1}\left(\frac{p-c}{p-c+h}\right) \Longleftrightarrow P(D_{L+1} \leqslant S) = \frac{p-c}{p-c+h},$$
$$\bar{z} = F_1^{-1}\left(\frac{p-c}{p-c+h}\right) \Longleftrightarrow P(D_1 \leqslant \bar{z}) = \frac{p-c}{p-c+h}.$$

A similar approach is used by Johansen and Thorstenson (2008). The authors set S equal to the optimal order-up-to level (which can be found with bisection method) and $\bar{z}$ equals $S/(L+1)$ rounded to the nearest integer. They call this the *restricted base-stock policy*. Numerical results show that the increase in average cost for this policy is less than 1% compared to the optimal cost. This policy is extended by Bijvank and Johansen (2010) in which the lead time and review period can be of any length instead of an integral multiple, and the cost and demand occur in continuous time instead of after the demand has occurred at the beginning of a review period.

In all policies mentioned so far, the sum of the expected holding and penalty costs is minimized. Lost-sales penalty costs incur due to the risk of ordering too few units, whereas holding costs incur due to the risk of ordering too many units. Levi et al. (2008) propose a *dual-balancing policy*, in which the two risks are balanced. The authors prove that the expected total cost of this policy is at most twice the expected cost of the optimal policy.

To complete the overview on all replenishment policies in this class, we mention the model developed by Reiman (2004) in which a *fixed amount* is ordered at each review. Such policies do not perform well in practice, since it is a static policy in which the order quantities do not depend on demand occurrences. The author compares this policy to a base-stock policy in a continuous review setting (see also Section 4).

Zipkin (2008a) provides a numerical comparison of several of the aforementioned policies for the cost model. The author concludes that base-stock policies do not perform well, whereas myopic policies (like Eq. (2)) are fairly good. Such policies are however not straightforward to implement in real life. However, the restricted base-stock policy of Johansen and Thorstenson (2008) is not included in this comparison, whereas it does perform well and it is fairly easy to implement.

Besides the cost model, van Donselaar et al. (1996) consider base-stock policies when a prespecified target fill rate β has to be satisfied (i.e., the *service model*). Demand is assumed to be Erlang distributed and the lead time is fixed. The authors develop a procedure to set the order-up-to level. They also propose a dynamic replenishment policy, in which the order size is determined such that the service level constraint is satisfied in the review period after the lead time (i.e., $P_{NS}(x, \mathbf{y}, z) \geqslant \beta$). Consequently, the order quantities depend on the individual orders outstanding. The amount to order has to be determined at each review instant. This policy can be seen as a myopic policy similar to Morton (1971) for the cost model (see Eq. (2)). Such a dynamic strategy results in a relatively smooth ordering pattern and lower on-hand inventory levels compared to the base-stock policy to obtain a given service level.

Stochastic lead times are studied by Nahmias (1979) as an extension to the lost-sales model of Morton (1971). To ensure that the lead time is an integral multiple of the review period length, the lead time distribution is reformulated such that orders do not cross as well. Besides the dynamic programming equations to find the optimal policy, an approximation policy is proposed. Janakiraman and Roundy (2004) establish some sample-path properties for the lost-sales inventory model with random lead times and a base-stock policy. Similar to Nahmias (1979), orders are not allowed to cross in the lead time process. Their main contribution is the convexity of the cost function with respect to order-up-to level S. The authors prove by induction that the expected sum of the on-hand inventory is convex at the start of each period, similar to Downs et al. (2001) for deterministic lead times. This result justifies the use of common search techniques to determine optimal order-up-to levels.

In most of the papers mentioned so far, the lead time is assumed to be a fixed or a random integral multiple of the review period. Only a few papers address an inventory system with *fractional lead times*, i.e., the lead time is smaller than the length of a review period. Such a system can be modeled as a one-dimensional Markov chain. Chiang (2006) analyzes such a model with dynamic programming. The author demonstrates that the optimal cost function is convex (similar to Downs et al. (2001)). Furthermore, it is shown that the optimal policy contains the same property as Inequality (1) for continuous reviews (see also Johansen and Thorstenson (1993, 1996)). The impact of the review period length on the average on-hand inventory levels and the fill rate is studied through a simulation approach by Sezen (2006) in case of fractional lead times. The numerical results show that the variability in the demand process is the most important factor to set the duration of a review period. No analytical procedure is proposed to determine the length of a review period or on how to set the order-up-to level.

An overview on the different replenishment policies is provided in Table 4 for different demand and lead time distributions. The demand is assumed to follow a Poisson (P), Erlang (Er), normal (N) or general (G) distribution. The lead time is either deterministic (D) or stochastic (G). Moreover, the lead time is assumed to be an integral multiple of the review period ($L = nR$) or fractional ($L \leqslant R$). In the objective function a cost function is minimized (C) or a target service level has to be satisfied (S).

Table 4
An overview on the lost-sales inventory models with no fixed order cost in a periodic review setting.

	Demand	Lead time	Assumption	Obj	Policy
Bellman et al. (1955)	G	D	$L = R$	C	Optimal
Karlin and Scarf (1958)	G	D	$L = R$	C	Optimal
Yaspan (1961)	G	D	$L = R$	C	Optimal
Morton (1969)	G	D	$L = nR$	C	Optimal
Morton (1971)	G	D	$L = nR$	C	Myopic, restricted base-stock
Yaspan (1972)	N	D	$L = nR$	C	Myopic
Zipkin (2008b)	G	D	$L = nR$	C	Optimal
Janakiraman et al. (2007)	G	D	$L = nR$	C	Optimal
Gaver (1959)	G	D	$L = R$	C	Base-stock
Morse (1959)	G	D	$L = R$	C	Base-stock
Pressman (1977)	G	D	$L = nR$	C	Base-stock
Downs et al. (2001)	G	D	$L = nR$	C	Base-stock
Huh et al. (2009)	G	D	$L = nR$	C	Base-stock
Johansen (2001)	P	D	$L = nR$	C	Optimal, (modified) base-stock
Johansen and Thorstenson (2008)	G	D	$L = nR$	C	Restricted base-stock
Bijvank and Johansen (2010)	G	D	–	C	Restricted base-stock
Levi et al. (2008)	G	D	$L = nR$	C	Dual-balancing
Reiman (2004)	G	D	$L = nR$	C	Fixed order size
Zipkin (2008a)	G	D	$L = nR$	C	–
Nahmias (1979)	G	G	$L = nR$	C	Optimal
Janakiraman and Roundy (2004)	G	G	$L = nR$	C	Base-stock
van Donselaar et al. (1996)	Er	D	$L = nR$	S	Base-stock, myopic
Chiang (2006)	G	D	$L \leqslant R$	C	Optimal, base-stock
Sezen (2006)	N	D	$L \leqslant R$	–	Base-stock

5.2. Fixed order costs

When a fixed order cost is incurred with each order in a lost-sales system, the (R, s, S) policy is proven to be optimal when the lead time is zero (see, e.g., Veinott Jr. and Wagner (1965), Veinott Jr. (1966), Shreve (1976), Bensoussan et al. (1983), Cheng and Sethi (1999) and Xu et al. (2010)). In case of positive lead times, there is no simple optimal replenishment policy. Nahmias (1979) is the first author to study such an inventory system and extends the model of Morton (1971) to include fixed order cost. In this model the lead time can be deterministic or a random variable. Under the assumption that no more than one order may be outstanding, Hill and Johansen (2006) propose a policy iteration algorithm to find optimal order quantities. The authors show with the means of a numerical example that the optimal policy is neither an (R, s, S) policy nor an (R, s, Q) policy. Both policies are, however, close to optimal and easy to implement in real-world applications. Therefore, some research has been performed on such policies. Notice that the two replenishment policies are not equivalent when the demand is unit sized, contrary to the continuous review systems (as discussed in Section 4).

An (R, s, Q) policy is applied in a cost model by Johansen and Hill (2000). The authors assume at most one order to be outstanding at any time. The expected total cost is approximated and the best values of s and Q are determined with a policy iteration algorithm. In the approximation procedure, they introduce a period of risk consisting of the lead time and the undershoot period (time when the inventory position is below the reorder level). The first and second moment of the demand distribution during this risk period are derived, where the demand is assumed to be normally distributed. Consequently, the expected on-hand inventory level and expected demand lost is approximated based on this distribution.

No assumptions on the number of outstanding orders are made by Bijvank et al. (2010). The authors develop mathematical models for different replenishment policies (fixed order size policies and order-up-to policies) and they perform a numerical comparison. They also propose a modified order-up-to policy, in which the maximum order size is restricted to an upper bound (similar to Bijvank and Johansen (2010)). This policy results in near-optimal order quantities with an average cost increase of less than 1% compared to the cost of the optimal policy.

Tijms and Groenevelt (1984) briefly discuss a service model in which a target fill rate has to be satisfied when order-up-to policies are considered in a lost-sales system. The same analysis can be used for periodic review systems as for continuous review systems (see also Section 4). However, the numerical results are restricted to a backorder system only. Bijvank and Vis (2010) provide bounds and an approximation procedure to determine the reorder level and order-up-to level when a service level constraint is imposed on a lost-sales inventory system with an (R, s, S) replenishment policy.

Fractional lead times are considered by Kapalka et al. (1999) where the objective is to minimize the long-run average cost subject to a fill rate constraint. The system is modeled as a one-dimensional Markov chain and the steady-state behavior of the on-hand inventory is studied. A cost objective function is studied by Chiang (2007) in case of fractional lead times. No properties of the optimal policy are developed, only an example is provided to illustrate the optimal policy.

Table 5 provides an overview on the replenishment policies for periodic review inventory models with lost sales where a fixed cost is incurred with each order. The abbreviations are the same as in Table 4.

Table 5
An overview on the lost-sales inventory models with fixed order cost in a periodic review setting.

	Demand	Lead time	Assumption	Objective	Policy
Nahmias (1979)	G	G	$L = nR$	C	Optimal
Hill and Johansen (2006)	G	D	–	C	Optimal
Johansen and Hill (2000)	N	D	$L = nR$	C	(R, s, Q)
Bijvank et al. (2010)	G	D	–	C/S	Optimal, (R, s, Q), (modified) (R, s, S)
Tijms and Groenevelt (1984)	G	G	$L = nR$	S	(R, s, S)
Bijvank and Vis (2010)	G	G	–	S	Optimal, (R, s, S)
Kapalka et al. (1999)	P	Ex	$L \leqslant R$	S	(R, s, S)
Chiang (2007)	G	G	$L \leqslant R$	C	Optimal

5.3. Concluding remarks on systems with periodic reviews

Based on the overview of periodic review models with lost sales, we conclude that most of the developed models focus on inventory systems in which no fixed order cost is charged. In comparison to continuous review models, much more research has been performed on deriving properties and bounds on the optimal order quantities for periodic review inventory systems. However, it still requires quite some computational effort to find the optimal order quantities especially for large inventory systems. Many alternative replenishment policies have been proposed. Myopic policies seem to perform well, but they are less insightful for practical use. Base-stock policies are much more popular for many practical applications. Due to the many convexity results for this type of policies, the order-up-to level can be determined with common search techniques. The restricted base-stock policies of Johansen and Thorstenson (2008) and Bijvank et al. (2010) with restrictions on the maximum order size are a natural extension to deal with lost sales. Such policies perform close to optimal, and they can easily be implemented without any drastic changes to current base-stock policies.

6. Mixture of lost sales and backorders

Besides inventory systems in which excess demand is either backordered or lost, there are also systems in which a fraction of the excess demand is backordered and the remaining fraction is lost. Such inventory systems encounter similar difficulties to analyze the performance as for lost-sales systems (see Section 2). Montgomery et al. (1973) are the first authors to analyze inventory systems with this mixture of backorders and lost sales. Similarly, Rosenberg (1979) and Leung (2008, 2009) reformulate this model to simplify the analysis. There is a lot of literature to be found on such partial backorder models. In this class, most continuous review models focus on (s, Q) replenishment policies whereas most periodic review models focus on (R, S) replenishment policies. In this section we address some recent papers which can be used as point of reference.

Partial backordering models with deterministic demand are studied by Pentico et al. (2009), Pentico and Drake (2009) and San-José et al. (2009). Models in which the lead time is considered to be a decision variable are developed by, e.g., Ouyang et al. (2005, 2007), Liang et al. (2008), Chang and Lo (2009) and the references therein. In these models, the fraction of backorders is based on a constant backorder probability. This probability can, however, also depend linearly on the number of outstanding backorders (see the overview by Lodree (2007) and Hu et al. (2009)). The review period could also be a decision variable (see Chuang and Lee (2008)).

Besides customers that are either willing or not willing to wait for a backorder with some probability, customers can also be willing to wait for some maximum period of time. Posner (1981) proposes a model when this time period is stochastic, whereas Das (1977) considers a constant patience time. More recent research on the trade-off between time and costs can be found in Chu et al. (2004). Another possibility to incorporate a mixture of backorders and lost sales in an inventory model is to limit the number of outstanding backorders to a maximum (Chu et al., 2001; Krishnamoorthy and Islam, 2005). To complete this overview on models with mixtures of backorders and lost-sales, we mention the models in which an incentive is included to backorder a demand during a stock-out period. Such models to prevent lost sales are considered by Netessine et al. (2006), Lee et al. (2006) and Bhargava et al. (2006) and the references therein.

7. Related research

From a practical point of view, there are many different inventory systems where excess demand is lost, each with its own characteristics. In the previous sections, we have discussed the inventory literature for single-item systems where demand depletes the inventory levels and orders are delivered after a lead time according to a general replenishment policy. However, there is also literature available on lost-sales inventory systems which can be classified based on the classification scheme described in Section 3, but the models include more specific characteristics such as emergency replenishments and multiple demand classes. These different types of inventory systems are discussed in this section.

7.1. Supply interruptions

As mentioned in Section 4, the variability of the lead time greatly influences the optimal values of the inventory control variables, like the reorder level and order quantity. When there are random supply interruptions, the supply process is unreliable and the variability increases. In such inventory systems the source of supply can change from available to unavailable. During an unavailable period, the supplier is not able to deliver any orders. One can think of a machine with regular breakdowns. Parlar and Berkin (1991) introduce this problem in a continuous review inventory system with lost sales. They derive the optimal order quantity when the reorder level equals zero. Their model is further investigated by Bar-Lev et al. (1993) and Berk and Arreola-Risa (1994). The aforementioned models assume deterministic demand and zero lead times. Kalpakam and Sapna (1997) model a lost-sales (s, Q) inventory system with unit-sized renewal demands in the presence of supply interruptions. Gupta (1996) and Mohebbi (2003, 2004) study a similar inventory system with (compound) Poisson demand. They assume deterministic and Erlang distributed lead times, respectively.

7.2. Emergency replenishments

Besides the regular inventory replenishment process, a supplier can also offer a second means to supply items with a faster mode of resupply at higher cost. This is referred to as emergency replenishments. The first continuous review models with lost sales and emergency orders, invoke a replenishment policy where orders of size Q and s are placed when the on-hand inventory level falls down to s and zero, respectively. See for example Morse (1958), Bhat (1984), Kalpakam and Sapna (1993b, 1995). A more general type of policies is considered by Mohebbi and Posner (1999) where two (s, Q) policies are used for regular and emergency orders where $s < Q$. The numerical results indicate that the emergency mode of resupply is most beneficial when the penalty cost for a stock out is high or the target service level is high.

Moinzadeh and Schmidt (1991) study a lost-sales system with emergency replenishments where an $(S - 1, S)$ replenishment policy is used. Whether a normal or an emergency order is placed depends on the age of the outstanding orders and the amount of remaining on-hand inventory at the time an order is placed. Kalpakam and Sapna (1993a) consider a more general order-up-to policy where the inventory position is raised to the order-up-to level S when the on-hand inventory level reaches s or zero. The authors assume no more than one order outstanding of each type at the same time.

An interesting reference related to this topic is Sheopuri et al. (2010), who show the connection between lost-sales inventory systems and dual sourcing inventory systems with backorders; orders placed as an emergency replenishment can be interpreted as lost in the regular replenishment process.

7.3. Multiple demand classes

The demand for an item may be categorized into classes of different importance. The first lost-sales model with priority demand classes is developed by Cohen et al. (1988) for a periodic review system. In each review period, inventory is used to meet the high-priority demand first, and the low-priority demand is satisfied with the remaining inventory. They consider an (R, s, S) replenishment policy. Veinott (1965) introduced critical level policies for multiple demand models when excess demand is backordered. In such policies each demand class has a threshold which prescribes the lower level on the inventory level for which the demand of each class is satisfied such that low-priority demand is rejected in anticipation of future high-priority demand. Such thresholds are considered in an (s, Q) replenishment policy for a lost-sales model with two demand classes by Melchiors et al. (2000), Isotupa (2006) and Fadiloglu and Bulut (2010) for deterministic and exponentially distributed lead times, respectively. Melchiors (2003) extends this work to more than two demand classes. Sivakumar and Arivarignan (2008) use a similar model for a mixture of lost sales and backorders. A continuous review (s, S) policy with a critical level is studied by Lee and Hong (2003). Multiple demand classes in an $(S - 1, S)$ replenishment policy are considered by Ha (1997, 2000) and Dekker et al. (2002). Consequently, the critical stock level has to be determined for each demand class. Kranenburg and van Houtum (2007) present three very effective heuristic algorithms to solve this problem.

7.4. Order splitting

We also mention the work of Hill (1996) where a continuous review (s, Q) policy is studied in a lost-sales setting, and an order for Q units is split equally between identical suppliers. The author assumes that at most one order can be outstanding at any time. Order splitting is also considered by Mohebbi and Posner (1998b) for two non-identical suppliers.

7.5. Perishable items

For items with a fixed life time (i.e., perishable items), Schmidt and Nahmias (1985) adopt a continuous review $(S - 1, S)$ policy for lost-sales models with constant lead times. An (s, Q) policy is considered by Berk and Gürler (2008). Perishable inventory systems with exponential life and lead times are analyzed by Kalpakam and Sapna (1996) and Kalpakam and Shanthi (2001). A constant life time in a periodic review system is studied by Broekmeulen and van Donselaar (2009) and Minner and Transchel (2010). When excess demand is partially backlogged (see also Section 6), we refer to San José et al. (2005, 2006), Yang et al. (2008), Abad (2008) and the references therein for a recent overview on inventory control models with different deterioration circumstances.

7.6. Pricing and inventory control

When the demand depends on the price of an item, the inventory control model should incorporate the selling price as a decision variable. Zabel (1990) performs an analysis for such systems in case of lost sales. The author concludes that the lost-sales and backorder assumption share common features. For a recent summary of the developments in pricing and inventory control models with lost sales, we refer to Chen et al. (2006), Huh and Janakiraman (2008), Song et al. (2009) and the references therein.

7.7. Joint replenishments

When fixed order costs are shared among all items that are ordered simultaneously, there is a cost benefit to place orders jointly. Such multi-item inventory systems are studied by Goyal (1989) and Yadavalli et al. (2004) in case excess demand is lost.

7.8. Multi-echelon

A two-echelon inventory system with one central warehouse and an arbitrary number of retailers is studied by Nahmias and Smith (1994), Anupindi and Bassok (1999) and Paul and Rajendran (2011)for an (R, S) replenishment policy with zero lead times. Positive lead times are considered for an optimal policy by Huh and Janakiraman (2010b), for a base-stock policy by Andersson and Melchiors (2001) and Huh and Janakiraman (2010a), and for a fixed order size policy by Jokar and Seifbarghy (2006), Seifbarghy and Jokar (2006) and Hill et al. (2007).

7.9. Production

In this overview we do not consider production-inventory control models. The decision issues for such systems are beyond the scope of this paper. Such decisions concern for instance production rates (Doshi et al., 1978; de Kok, 1985; Nobel and van der Heeden, 2000), setup times and batching in a multi-product system (Kim and van Oyen, 1998; Krieg and Kuhn, 2002, 2004; Gurgur and Altiok, 2007; Grasman et al., 2008), scheduling of maintenance activities (Kenné et al., 2007).

8. Conclusion and future research

In this paper, we have classified the literature addressing lost-sales inventory systems according to the classification scheme presented in Section 3. In Sections 4 and 5, we have discussed the literature for continuous review and period review inventory systems with different types of replenishment policies according to this classification scheme. To complete the literature overview, we have also discussed models in which a mixture of lost sales and backorders can occur (Section 6) and in which the replenishment process contains specific characteristics (Section 7).

From this literature overview on lost-sales inventory systems, we conclude that not much is known about an optimal replenishment policy when excess demand is lost. The properties and numerical results that have been derived for the optimal order quantities show that there is no structure for an easy-to-understand optimal replenishment policy which can be implemented in real-life applications. However, the most effective approximation policies that have been proposed in the literature include some kind of delay in the ordered quantities. For continuous review policies this is explicitly included, whereas for periodic review systems it is implicitly included with a maximum order size. This delay prevents too many orders to be outstanding after a period with many customer demands. In a backorder setting these items are already allocated to excess demand, whereas in a lost-sales setting this demand is lost. Consequently, in a lost-sales setting the order sizes do not have to raise the inventory position as much as in a backorder setting in case the inventory position is low. Numerical experiments show that such policies with delay result in near to optimal cost for periodic review models. There is no such comparison for continuous review systems. This is an interesting aspect to investigate in the future.

Another promising direction for future research is the influence of the aforementioned policies with delay on the entire supply chain. When the ordering process is smoothed over time, the variance in the demand for production and transportation also reduces. There is hardly any analysis on the impact of lost sales in a multi-echelon setting.

Modeling the customer behavior in case of excess demand is also an interesting new aspect in inventory theory. Besides backlogging, lost sales and mixtures of the two, a customer can also buy a substitute product or go to a different store. Incorporating the different customer reactions towards stock outs would also influence the replenishment policies.

The final suggestion regarding directions for future research is to focus more research on non-stationary situations. Even though steady-state conditions are more analytically tractable, they become less relevant in practical situations due to non-stationary demand and a shorter lifespan of products. Consequently, the methodology to analyze lost-sales inventory systems should change accordingly.

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