European Journal of Operational Research 220 (2012) 610–618

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Production, Manufacturing and Logistics

# Lost-sales inventory systems with a service level criterion

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ARTICLE INFO

*Article history:*
Received 5 July 2011
Accepted 8 February 2012
Available online 17 February 2012

*Keywords:*
Inventory
Lost sales
Service level
Base-stock policy
Optimal policy

ABSTRACT

Competitive retail environments are characterized by service levels and lost sales in case of excess demand. We contribute to research on lost-sales models with a service level criterion in multiple ways. First, we study the optimal replenishment policy for this type of inventory system as well as base-stock policies and $(R,s,S)$ policies. Furthermore, we derive lower and upper bounds on the order-up-to level, and we propose efficient approximation procedures to determine the order-up-to level. The procedures find values of the inventory control variables that are close to the best $(R,s,S)$ policy and comply to the service level restriction for most of the instances, with an average cost increase of 2.3% and 1.2% for the case without and with fixed order costs, respectively.



## 1. Introduction

In this paper, we consider a single-item inventory system with periodic reviews at a single retailer location. In classic inventory models it is common to assume that excess demand is backordered (Silver et al., 1998; Zipkin, 2000; Axsäter, 2000). An extensive study by Gruen et al. (2002) reveals that only 15% of the customers who observe a stock out will wait for the item to be on the shelves again, whereas the remaining 85% will either buy a different product (45%), visit another store (31%) or do not buy any product at all (9%). Similar percentages are found by Verhoef and Sloot (2006), who conclude that 23% of the customers will delay the purchase in case of excess demand. These results show that most of the original demand can be considered to be lost in many retail settings. However, the assumption that excess demand is backordered is more common in an industrial environment.

Another aspect that should be considered in inventory models for retail establishments is that the retail environment is very competitive. As a result, customer satisfaction is commonly used as differentiation strategy among competitors and a service level restriction is imposed on the replenishment process. This also has a more practical advantage, since it is easier to define a suitable service performance measure than to define all costs. For example, out-of-stock costs include loss of goodwill which is difficult to quantify. We refer to models with a penalty cost for lost sales as *cost models*. Models with a service level restriction instead of the penalty cost are referred to as *service models*. Throughout this paper, we use the average fill rate as service level, which is defined as the fraction of demand directly satisfied by stock on hand.

The main goal of this paper is to study lost-sales inventory systems with a service level criterion. We consider *optimal replenishment policies* and $(R,s,S)$ policies. In the latter type of policies, the inventory position (inventory on hand plus inventory on order minus backorders) is raised to a pre-defined order-up-to level $S$ when the inventory position is at or below a reorder level $s$ at a review instant. Such policies are very popular in practice, since they are easy to understand and to implement. We also consider *base-stock policies*, which are a special case of the $(R,s,S)$ policy where $s = S - 1$. Consequently, the inventory position is raised to order-up-to level $S$ at each review instant. This policy is denoted as the $(R,S)$ policy, and it is usually applied when there is no fixed order cost. The $(R,s,S)$ policy is more suitable when a fixed order cost is incurred with each order. In the remainder of this section, we will discuss related literature on lost-sales inventory systems to illustrate our contribution to the existing literature. For a complete overview of the literature on such systems, we refer to Bijvank and Vis (2011).

Inventory systems with a backorder assumption have received by far the greatest attention in inventory literature. This is mainly because $(R,s,S)$ policies are proven to be optimal replenishment policies for such systems (Karlin and Scarf, 1958; Scarf, 1960). When excess demand is lost instead of backordered, much less is known about the *optimal replenishment policy*. Recently, Zipkin (2008) reformulated the original lost-sales cost model of Karlin and Scarf (1958) and Morton (1969) when no fixed costs are

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doi:10.1016/j.ejor.2012.02.013

incurred with each order. Zipkin (2008) proves by induction that the cost function is $L$-natural convex under the optimal replenishment policy (i.e., convex, submodular and it contains a property related to diagonal-dominance). This result implies that the optimal order quantities are monotone decreasing in the inventory position. Our first goal is to develop inventory models with an optimal replenishment policy for lost-sales inventory models with a service level constraint. To our knowledge we are the first authors to study the performance of an optimal replenishment policy for this type of inventory systems.

*Cost models* have mostly been studied for lost-sales inventory systems, especially the base-stock policy. Huh et al. (2009) show that this type of policy is asymptotically optimal for lost-sales systems when the lost-sales penalty cost grows large. Downs et al. (2001) derive convexity results on the cost function, such that the best value of order-up-to level $S$ can be found with a bisection method. Janakiraman and Roundy (2004) and Chiang (2006) find similar convexity properties for the cost function with respect to the order-up-to level in case of stochastic lead times and fractional lead times (i.e., smaller than the review period length), respectively. Upper and lower bounds on the best value of the order-up-to level are presented by Huh et al. (2009). However, numerical results indicate that these bounds are not tight enough to perform well as approximation for the order-up-to level. Our second goal is to extend these results for a service model.

van Donselaar et al. (1996) are the first authors to study a lost-sales inventory model with a *service constraint* for the base-stock policy. However, they propose a myopic approach to determine the order-up-to level dynamically for each review period. Consequently, the order-up-to level varies over time. This is not preferred in practice. The authors also assume Erlang distributed demand. We do not restrict ourselves to any demand distribution. Sezen (2006) uses simulation to study the impact of the review period length on the average fill rate for the $(R,S)$ policy when lead times are fractional. No analytical procedure is proposed to determine the order-up-to level. Our third goal is to propose several approximation procedures to determine the order-up-to level for a service model with an $(R,S)$ replenishment policy.

The $(R,s,S)$ *replenishment policy* is hardly studied in a context with lost sales. Hill and Johansen (2006) show with the means of numerical results that this policy is not optimal for the cost model, but the expected total costs are close to optimal. Chiang (2007) and Bijvank et al. (2012) show similar results for the cost model. Tijms and Groenevelt (1984) briefly mention this policy for a service model with lost sales. However, the numerical results are restricted to a backorder system only. Kapalka et al. (1999) also consider a service model for the $(R,s,S)$ policy, but they restrict their model to fractional lead times and no procedure is proposed to determine the values of $s$ and $S$. Our final goal is to propose an efficient approximation procedure to set the inventory control variables for the $(R,s,S)$ policy.

The rest of this paper is organized as follows. In Section 2 we develop inventory models for the optimal replenishment policy and the $(R,s,S)$ policy. Numerical search procedures are required to determine the best values of reorder level $s$ and order-up-to level $S$. Therefore, upper and lower bounds on the order-up-to level are presented in Section 3 to narrow the search space. An alternative approach is to develop computationally efficient approximation procedures. Such procedures are derived for the base-stock policy and the $(R,s,S)$ replenishment policy in Sections 4 and 5, respectively. Numerical results on the bounds and the approximation procedures are presented in Section 6. We end the paper with our concluding remarks in Section 7.

## 2. Model

Before we specify the replenishment policy, we first introduce the notations to model the inventory system as a Markov chain. Consider a single-item inventory system in discrete time. The demands and state variables can be either integers or continuous, where $D_\tau$ represents the demand during $\tau$ time units. The probability distribution function of $D_\tau$ equals $g_\tau(d)$ with mean $\mu_\tau$ and standard deviation $\sigma_\tau$. Furthermore, $\mathcal{G}^0_\tau(d) = P(D_\tau < d)$ and $\mathcal{G}^1_\tau(d) = \int_0^d \mathcal{G}^0_\tau(\xi)d\xi = \mathbb{E}[(d - D_\tau)^+]$, where $(A)^+ = \max\{A,0\}$. Let $L$ denote the order lead time. For ease of notation, we assume the lead time to be an integral multiple of the review period length $R$. At the end of this section we demonstrate how to relax this assumption to obtain a general model.

Contrary to backorder models, the inventory position cannot be used as main indicator of the inventory status when excess demand is lost. This is due to the fact that the inventory position does not decrease if the system is out of stock. As a result, the on-hand inventory at order delivery does not equal the inventory position minus the demand during the lead time. Therefore, a lost-sales model has to keep track of the available inventory on hand and the individual outstanding orders that were placed in the past and have not yet arrived. We define the state space before ordering at a review instant $t$ as $\mathbf{x}_t = (x_{0t}, x_{1t}, \ldots, x_{L-1,t})$ where $x_{0t}$ is the on-hand inventory level after order arrival and $x_{it}$ for $i \in \{1,\ldots,L-1\}$ is the order placed $L-i$ review periods before. Consequently,

$$\mathbf{x}_{t+1} = \left((x_{0t} - D_R)^+ + x_{1t}, x_{2t}, \ldots, x_{L-1,t}, y_t\right), \tag{1}$$

where $y_t$ is the order size placed at review instant $t$. The actual order size depends on the replenishment policy. We consider the optimal replenishment policy and the $(R,s,S)$ policy in Sections 2.1 and 2.2, respectively.

### 2.1. Optimal replenishment policy

Let us first consider the optimal replenishment policy for the *cost model*. By $h$ and $p$ we denote the unit holding cost per unit time and the unit penalty cost for each lost demand, respectively. The fixed order cost is denoted by $K$. Define $V_t(\mathbf{x}_t)$ as the expected total costs incurred from time $t$ to time $N$ when the system incurs no costs at or after time $N$. Consequently, $V_N(\mathbf{x}_N) = 0$ for all states $\mathbf{x}_N$ and, recursively for $t < N$

$$V_t(\mathbf{x}_t) = c(x_{0t}) + \min_{y_t \geqslant 0}\left\{K\delta(y_t) + \mathbb{E}\left[V_{t+1}((x_{0t} - D_R)^+ + x_{1t}, x_{2t}, \ldots, x_{L-1,t}, y_t)\right]\right\}, \tag{2}$$

where

$$c(x_{0t}) = h\mathbb{E}[(x_{0t} - D_t)^+] + p\mathbb{E}[(D_t - x_{0t})^+], \tag{3}$$

and indicator function $\delta(i)$ is zero when $i = 0$ and one otherwise. For an infinite horizon $N$, we can find the optimal policy and the corresponding long-run expected total costs per review period with value iteration (Puterman, 2005). Such a procedure with accuracy number $\epsilon$ repeats to decrease $t$ by one and compute the value function in Eq. (2) until $M_t - m_t < \epsilon$, where

$$m_t = \min_{\mathbf{x}}\{V_t(\mathbf{x}) - V_{t+1}(\mathbf{x})\},$$

$$M_t = \max_{\mathbf{x}}\{V_t(\mathbf{x}) - V_{t+1}(\mathbf{x})\}.$$

When this value-iteration algorithm is stopped, then $(m_t + M_t)/2$ cannot deviate more than $100\epsilon\%$ from the long-run average costs per review period. Let us denote this amount by $g$.

To compute the optimal replenishment policy for the *service model*, we use the dynamic programming formulation of Eq. (2). However, in order to derive this policy we have to solve a constrained dynamic programming (CDP) problem. The solution approach for this CDP problem is to transform the problem into the unconstrained dynamic programming formulation of Eq. (2). Therefore, we formulate the CDP problem as a Lagrange relaxation, where $\gamma \in [0,\infty)$ denotes the Lagrange multiplier (see Altman (1999)). The idea behind this approach is to include the service constraint in the objective function that minimizes holding and order costs, such that the constrained problem is transformed in an unconstrained problem:

$$\begin{array}{ll} \text{minimize} & \text{costs} \\ \text{subject to} & \text{service level} \geqslant \bar{\beta} \Rightarrow \text{minimize costs} + \gamma(\bar{\beta} - \text{service level}) \end{array}$$

For more information on the connection between the cost model and the service model, we refer to van Houtum and Zijm (2000).

In the Lagrange relaxation of the original CDP problem, we have to find the value of $\gamma$ such that the service level constraint $\bar{\beta}$ is satisfied. To do so, we can use the cost model as presented already, where $\gamma$ represent the penalty cost $p$ for lost demand. Moreover, for a certain value of $\gamma$ let $y^*_\gamma(\mathbf{x})$ denote the optimal policy for state $\mathbf{x}$ in the cost model where $p = \gamma$. Next, the corresponding fill rate for this policy can be computed. Therefore, we set $h = 0$ and $p = \gamma = 1$ in the cost function $c(x_{0t})$ and use the policy $y^*_\gamma$ in Eq. (2) to set the value of $y_t$. The result of the value-iteration algorithm (denoted by $g$) represents the average demand lost during a review period. Consequently, the fill rate equals $\beta_{LS} = 1 - g/\mu_R$. If $\beta_{LS}$ is sufficiently close to $\bar{\beta}$, the value of $\gamma$ is found. Otherwise, we adjust $\gamma$ according to a bisection method. This is possible because lost demand is penalized more when the Lagrange multiplier $\gamma$ increases, which results in a higher service level (since higher inventory levels should balance the increase in the penalty cost $p = \gamma$). Therefore, if for the value of $\gamma$ the service level constraint is satisfied ($\beta_{LS} \geqslant \bar{\beta}$), the value of $\gamma$ should decrease and, otherwise, it should increase. This is repeated until the fill rate is sufficiently close to $\bar{\beta}$. This procedure to solve the service model is summarized in Algorithm 1. Line 5 to line 9 prescribes to exclude half the interval on $\gamma$ by checking the middle value of this interval. When the right-hand side of the interval is not defined (i.e., $\gamma_{\text{RHS}} = \infty$), the service level has never been satisfied for $\gamma$ and the value of $\gamma$ is increased by 10. This value is chosen based on numerical experiments.

**Algorithm 1:** Optimal replenishment policy for service model.

1 set $\gamma = 1$, $\gamma_{\text{LHS}} = 0$, $\gamma_{\text{RHS}} = \infty$
2 **repeat**
3     solve the cost model with $p = \gamma$ to find $y^*_\gamma$
4     compute the expected fill rate $\beta_{\text{LS}}$ where $y_t = y^*_\gamma(\mathbf{x}_t)$
5     **if** $\beta_{LS} \leqslant \bar{\beta}$ **then**
6         $\gamma_{\text{LHS}} = \gamma$ and $\gamma = \frac{1}{2}(\gamma_{\text{LHS}} + \min\{\gamma_{\text{RHS}}, \gamma + 20\})$
7     **else**
8         $\gamma_{\text{RHS}} = \gamma$ and $\gamma = \frac{1}{2}(\gamma_{\text{LHS}} + \gamma_{\text{RHS}})$
9     **end**
10 **until**$(|\beta_{LS} - \bar{\beta}| \leqslant \varepsilon)$

### 2.2. (R,s,S) replenishment policy

The computation of the performance measures for the inventory system with an $(R,s,S)$ replenishment policy can also be performed with the value-iteration algorithm as presented in Section 2.1. For this type of policy, the order quantities $y_t$ are defined as

$$y_t = \begin{cases} S - \sum_{i=0}^{L-1} x_{it} & \text{if } \sum_{i=0}^{L-1} x_{it} \leqslant s, \\ 0 & \text{otherwise}. \end{cases}$$

Consequently, Eq. (2) is simplified to

$$V_t(\mathbf{x}_t) = c(x_{0t}) + K\delta(y_t) + \mathbb{E}\left[V_{t+1}\left((x_{0t} - D_R)^+ + x_{1t}, x_{2t}, \ldots, x_{L-1,t}, y_t\right)\right]. \tag{4}$$

Next, we can find the values of $s$ and $S$ that minimize the inventory costs (sum of holding and order costs) subject to a service constraint $\bar{\beta}$. Let us denote these values by $s_{\text{LS}}$ and $S_{\text{LS}}$, respectively. It requires a numerical search procedure to determine these values. For each set $(s,S)$, first the service level is computed with the value-iteration algorithm when $K = h = 0$ and $p = 1$, and next if the service constraint is satisfied the corresponding average inventory costs have to be computed. If base-stock policies are considered when $K = 0$, the total inventory costs consist of holding costs only. Consequently, $S_{\text{SL}} = \min\{S | \beta_{\text{SL}}(S) \geqslant \bar{\beta}\}$, where $\beta_{\text{LS}}(S)$ is the fill rate for a lost-sales inventory system with order-up-to level $S$.

### 2.3. Lead time extension

When the lead time can be any number of time units instead of an integral multiple of the review period length, we have to split the review period in two periods: from review instant until order delivery, and from order delivery until the next review instant. The first time period has length $l = L \bmod R$, whereas the second time period contains $R - l$ time units. Consequently, we have to specify two recursive expressions like Eq. (2). When the inventory system is at a review instant

$$V_t(\mathbf{x}_t) = c_l(x_{0t}) + \min_{y_t \geqslant 0}\left\{K\delta(y_t) + \mathbb{E}\left[V'_t\left((x_{0t} - D_l)^+ + x_{1t}, x_{2t}, \ldots, x_{L-1,t}, y_t\right)\right]\right\}, \tag{5}$$

and when an order is delivered

$$V'_t(\mathbf{x}_t) = c_{R-l}(x_{0t}) + \mathbb{E}\left[V_{t+1}\left((x_{0t} - D_{R-l})^+, x_{1t}, \ldots, x_{L-1,t}\right)\right]. \tag{6}$$

Note that the expected costs $c_\tau(x_{0t})$ depend on the time period length $\tau$. The transformation from cost model to service model remains the same, as well as the procedure for an $(R,s,S)$ replenishment policy. More details on the value functions expressed in Eqs. (5) and (6) can be found in Bijvank and Johansen (2012) for the cost model.

## 3. Boundaries on order-up-to level *S*

It requires a two-dimensional search space to find the best values of the inventory control parameters for the $(R,s,S)$ policy which we modeled in Section 2. The goal of this section is to derive lower and upper bounds on the order-up-to level $S$ to decrease the size of the search space and to perform any search procedure more efficiently. Kapalka et al. (1999) show by numerical experiments that $2\,EOQ + 3\sigma_L$ can be used as an upper bound on $S$ in the $(R,s,S)$ policy, where $EOQ$ is the economic order quantity. We use the same upper bound in the $(R,s,S)$ policy. Any lower bound on $S$ in the base-stock policy can be used in the $(R,s,S)$ policy as well. As a result, we only need to derive upper and lower bounds on $S$ in the base-stock policy in this section.

### 3.1. Upper bound

The on-hand inventory levels in backorder models are by definition never higher than the inventory levels in lost-sales models when the replenishment policy and the inventory control variables are the same. Therefore, the order-up-to level in a backorder system has to exceed the order-up-to level in a lost-sales system to reach the same average on-hand inventory level (or service level for that matter). For a more detailed comparison between backorder systems and lost-sales systems we refer to Janakiraman et al.

(2007) and Huh et al. (2009). Based on the previous statements, $\beta_{\rm BO}(S) \leqslant \beta_{\rm LS}(S)$, where $\beta_{\rm BO}(S)$ denotes the service level for the backorder model for a given order-up-to level $S$, and is computed by

$$\beta_{\rm BO}(S) = 1 - \frac{1}{\mu_R}(\mathbb{E}[(D_{R+L} - S)^+] - \mathbb{E}[(D_L - S)^+]). \tag{7}$$

See Zipkin (2000) for more details. Note that in general $\mathbb{E}[(D_\tau - S)^+] = \mu_\tau - S + \mathcal{G}^1_\tau(S)$, and more specifically for a normally distributed demand process, it is

$$\mathbb{E}[(D_\tau - S)^+] = \sigma_\tau\left[\frac{\mu_\tau - S}{\sigma_\tau}\left[1 - \Phi\left(\frac{S-\mu_\tau}{\sigma_\tau}\right)\right] + \phi\left(\frac{S-\mu_\tau}{\sigma_\tau}\right)\right], \tag{8}$$

where $\phi(\cdot)$ is the probability density function of the standard normal distribution, and $\Phi(\cdot)$ is the cumulative distribution function. Define $S_{\rm BO} = \min\{S|\beta_{\rm BO}(S) \geqslant \bar{\beta}\}$ as the upper bound on $S_{\rm LS}$ in the base-stock policy.

### 3.2. Lower bound

We derive two lower bounds on the order-up-to level $S_{\rm LS}$. The first lower bound is applicable for Poisson distributed demand, whereas the second lower bound can be used for any demand distribution.

Inventory levels are higher in continuous review models compared to periodic review models with the same replenishment policy. Based on a similar reasoning as in Section 3.1, the smallest order-up-to level which satisfies $\beta_{\rm CR}(S) \geqslant \bar{\beta}$ is a lower bound on $S_{\rm LS}$, where $\beta_{\rm CR}(S)$ is the fill rate for a continuous review inventory system with lost sales. Let us denote this value by $S_{\rm CR}$. When the demand follows a Poisson process

$$\beta_{\rm CR}(S) = 1 - \frac{(\mu_L)^S/S!}{\sum_{j=0}^{S}(\mu_L)^j/j!}. \tag{9}$$

See Smith (1977) for more details.

When the demand does not follow a Poisson process, there is no closed-form expression for $\beta_{\rm CR}(S)$. In that case, we use a lower bound based on a lost-sales inventory model with zero lead times. Such models also result in higher inventory levels compared to models with positive lead times when the same replenishment policy is applied. Denote the fill rate for models with zero lead time by $\beta_{\rm ZL}(S)$, where

$$\beta_{\rm ZL}(S) = 1 - \frac{\mathbb{E}[(D_R - S)^+]}{\mu_R}. \tag{10}$$

Consequently, $S_{\rm ZL}(S) = \min\{S|\beta_{\rm ZL} \geqslant \bar{\beta}\}$ is a lower bound on order-up-to level $S_{\rm LS}$.

Although the derived bounds narrow the search space, it still requires a considerable amount of processing time to find the best order-up-to levels with a numerical search procedure. Therefore, we will develop several approximation procedures to efficiently determine the order-up-to levels in the next sections. These methods are based on the expressions discussed in this section. First, we develop approximation procedures for the base-stock policy in Section 4, whereas a procedure for the $(R,s,S)$ policy is derived in Section 5.

## 4. Heuristics for the base-stock policy

In this section, we use the expressions of Section 3 to derive approximations for the order-up-to level in inventory systems with a service level restriction when the $(R,S)$ replenishment policy is applied. The performance of each of the approximation procedures is studied in Section 6.

### 4.1. Based on backorder model

As mentioned in Section 3.1, the average inventory levels in backorder models are lower in comparison to the inventory levels in lost-sales models. The difference equals, on average, the expected number of units short in a replenishment cycle. This amount is given by the term between brackets in Eq. (7), and should not be included in the order-up-to level for lost-sales models. Therefore, we subtract this amount from the upper bound $S_{\rm BO}$ and express our first approximation $\widehat{S}_1$ for the order-up-to level as $S_{\rm BO} - (1-\bar{\beta})\mu_R$.

### 4.2. Based on continuous review model

In comparison to continuous review models, an order has to wait until the next review instant in a periodic review model. We propose three approaches to add this time period to the lead time in Eq. (9) such that it approximates the fill rate for a period review inventory system. The additional lead time follows a uniform distribution on $[0,R]$ since demand is as likely to occur over time. Consequently, we replace $L$ by $L + \frac{1}{2}R$ in Eq. (9). This results in an approximation for $S_{\rm LS}$ denoted by $\widehat{S}_{2a}$. Johansen (2001) proposed a similar approximation by adding an additional delay of $R/(1-e^{-\mu_R}) - 1/\mu_1$ instead of $\frac{1}{2}R$. We denote this approximation by $\widehat{S}_{2b}$, and the performance will be tested in Section 6.

Both approximations assume orders to be able to cross in time, whereas in real-world applications orders are required to arrive in the same sequence as they are placed. Johansen (2005) assumes Erlang distributed lead times with shape parameter $r$ and mean $r/\lambda$ to prevent orders to cross in time. To keep the first two moments of the lead time equal to the first two moments of $L$ plus the extra delay which is uniformly distributed on $[0,R]$, we set

$$r = \mathrm{round}\left(\frac{(L+\frac{1}{2}R)^2}{R^2/12}\right) \quad \text{and} \quad \lambda = \frac{r}{L+\frac{1}{2}R},$$

where $r$ is rounded to the nearest integer value. We use the results of Johansen (2005) to calculate the service level given by $1 - n_{r,S}q^{S-1}\mu/(C\lambda)$, where $n_{r,k} = \binom{r-1+k}{k}$, $q = \mu_1/(\mu_1+\lambda)$ and $C$ equals the normalizing constant $C = \sum_{k=0}^{S-1} n_{r,k}q^k + n_{r,S}q^{S-1}(\mu_1/\lambda)$. The smallest value of $S$ satisfying the service level criterion is denoted as approximation $\widehat{S}_{2c}$.

### 4.3. Based on zero lead time model

When inventory models with zero lead times are used to approximate inventory models with positive lead times, they have to compensate for the lack of safety stock during the actual lead time. Therefore, we increase the review period $R$ with the lead time $L$ to increase the average inventory level. So, $R$ is replaced by $R + L$ in Eq. (10). The smallest value of $S$ where $\beta_{\rm ZL}(S) \geqslant \bar{\beta}$ is denoted by $\widehat{S}_3$.

### 4.4. Based on mean-value analysis

This approximation procedure is based on the analysis of mean values. Let us denote the order size by random variable $Q$, where the average order size equals the average demand satisfied during a review period, i.e., $\mathbb{E}[Q] = \bar{\beta}\mu_R$. Since $Q$ is directly related to the demand process, we assume the same distribution for $Q$ as the demand distribution. At a review instant, there are $M = \lceil L/R \rceil$ orders outstanding after ordering at this review instant. The inventory on order is a random variable $Q^M = \sum_{i=1}^{M} Q_i$ with mean $M\bar{\beta}\mu_R$. Consequently, the on-hand inventory level at a review instant equals

$S - Q^M$. The expected satisfied demand until the next order delivery equals

$$\mathbb{E}[\min\{(S-Q^M)^+, D_l\}] = \mathbb{E}[D_l] - \mathbb{E}[(D_l - (S-Q^M)^+)^+], \tag{11}$$

where $l = L \bmod R$. Next, we specify the average on-hand inventory level just before order delivery as

$$IL^- = \mathbb{E}[((S-Q^M)^+ - D_l)^+]. \tag{12}$$

The expected demand satisfied between order delivery and the next review equals

$$\mathbb{E}[\min\{IL^- + \delta(l)Q, D_{R-l}\}] = \mathbb{E}[D_{R-l}] - \mathbb{E}[(D_{R-l} - (IL^- + \delta(l)Q))^+], \tag{13}$$

where $\delta(l)$ is the indicator function.

The sum of Eqs. (11) and (13) approximates the expected demand satisfied in a review period. Since $\bar{\beta}\mu_R$ represents the same, we let $\widehat{S}_4$ be the order-up-to level which sets both amounts equal to each other.

In this section, we have developed several heuristic procedures to set the value of $S$ for the base-stock policy when a minimal service level is imposed. In the next section we discuss such a procedure for the $(R,s,S)$ policy.

## 5. Heuristic for the (R,s,S) policy

As mentioned in Section 1, Tijms and Groenevelt (1984) propose a solution method based on renewal theory to set the value of reorder level $s$ for a service model with the $(R,s,S)$ policy. They set $S - s$ equal to the economic order quantity ($EOQ$) to determine the order-up-to level, and show by means of numerical results that their approximation procedure performs well for models with a backorder assumption. The goal of this section is to propose a new approximation procedure to determine the order-up-to level for the $(R,s,S)$ policy where excess demand is lost. We follow partly the procedure of Tijms and Groenevelt (1984) to set the value of the reorder level. However, we use the order-up-to level resulting from our approximation procedure instead of the economic order quantity. In Section 6, we compare the performance of both solution techniques for lost-sales models.

The new procedure to determine the order-up-to level is closely related to the mean-value analysis of Section 4.4. However, in the $(R,s,S)$ policy it is not required to place an order at each review instant. Therefore, we introduce the notion of a replenishment cycle which represents the time between two subsequent orders. Let $N$ denote the average number of review periods in a replenishment cycle. Similar to the procedure for the base-stock policy, the order size is a random variable denoted by $Q$. For the $(R,s,S)$ policy, $\mathbb{E}[Q]$ equals the average demand satisfied in a replenishment cycle, i.e., $\mathbb{E}[Q] = \bar{\beta}\mu_R N$. We set $\mathbb{E}[Q] = EOQ$ to determine the value of $N$. Next, we express the service level similar to the procedure of Section 4.4 to approximate the order-up-to level $S_{LS}$. When we assume at most one replenishment order to be outstanding at any time, the average on-hand inventory level before ordering at a review instant equals $(S - Q)^+$. Consequently, the expected demand satisfied between ordering and order delivery equals

$$\mathbb{E}[\min\{(S-Q)^+, D_L\}] = \mathbb{E}[D_L] - \mathbb{E}[(D_L - (S-M)^+)^+], \tag{14}$$

and between order delivery and the next order it equals

$$\mathbb{E}[\min\{IL^- + Q, D_{RN-L}\}] = \mathbb{E}[D_{RN-L}] - \mathbb{E}[(D_{RN-L} - (IL^- + Q))^+], \tag{15}$$

where

$$IL^- = \mathbb{E}[((S-Q)^+ - D_L)^+]. \tag{16}$$

Hence, the sum of Eqs. (14) and (15) expresses the expected demand satisfied in a replenishment cycle. This should be equal to $\bar{\beta}\mu_R N$. Let $\widehat{S}_5$ be the value of the order-up-to level such that both expressions result in the same amount of satisfied demand. The value of the corresponding reorder level is denoted by $\hat{s}_5$.

## 6. Numerical results

The goal of this section is threefold. First, we compare the performance of the $(R,S)$ policy and the $(R,s,S)$ policy to the optimal replenishment policy. Second, we show the state space reduction based on the upper and lower bounds introduced in Section 3. Third, the performance of the approximation procedures is investigated to determine reorder level $s$ and order-up-to level $S$ (see Sections 4 and 5). The applicability of our models and the performance of the policies are demonstrated for a test bed of three different demand distributions, where the review period length $R = 1$, the lead time $L = 2R$, the holding cost $h$ equals 1, the service level restriction $\bar{\beta}$ ranges from 75% to 99%, and the fixed order cost $K$ ranges from 0 to 100. We use a Poisson demand process, a negative binomial distribution and a normal distribution to represent the demand. This illustrates that our models and procedures can be used for any type of demand distribution. As average demand, we consider $\mu_R = 2.5$, 5 and 10. The variance-to-mean (VTM) ratio is always 1 for a Poisson demand distribution, whereas we use VTM = 2,4 for a negative binomial distribution. For a normal distribution, we consider the pairs $(\mu_R, \text{VTM}) = (2.5, 0.5), (5, 0.5), (5, 1), (10, 0.5), (10, 1), (10, 2)$ such that $P(D_R \leqslant 0) \leqslant 0.01267$ (i.e., the probability of negative demand should be small). This illustrates the performance for a wide range of demand settings. Scenarios without and with fixed order costs are considered in Sections 6.1 and 6.2, respectively.

### 6.1. The (R,S) policy

When there are no fixed order costs, it is common to use an $(R,S)$ replenishment policy. For this policy, we compute the reorder level $S_{LS}$ that minimizes the expected total costs. We also compute upper bound $S_{BO}$ and lower bounds $S_{CR}$ and $S_{ZL}$ as described in Section 3, and the values of $\widehat{S}_i$ based on the approximation procedure $i$ as discussed in Section 4. Furthermore, we report the average costs $C$ for the optimal replenishment policy, which is computed according to the models presented in Section 2.

The results for the different demand distributions are presented in Tables 1–3, respectively. When the approximation procedure results in an underestimation of the order-up-to level, the corresponding service level is presented, whereas the average cost increases ($CI$) are presented when the approximation procedure results in an overestimation of the order-up-to level (since the service level requirement is satisfied). When the procedure results in under- and overestimations, the service level is presented with a gray background if the order-up-to level is underestimated and the average cost increase is presented with a white background otherwise. For the Poisson demand distribution (Table 1) we also included the results when we vary the lead time $L$ for the base-case where $\mu_R = 5$.

When we compare the $(R,S)$ policy to the optimal replenishment policy (column 4 versus column 6), it is clear that the $(R,S)$ policy is not close to optimality (the average gap is 31.0%, 15.9% and 33.4% for each demand distribution, respectively). However, this policy is used in many applications in practice. The bounds on order-up-to level $S$ (column 7 to column 9) are not tight enough to be used as approximation for $S_{LS}$. However, the bounds narrow the search space to find the value of $S_{LS}$. With a binary search procedure, such as a bisection method, only small parts of the search space have to be examined to find $S_{LS}$. To illustrate the computation times, we performed our experiments on a 2.0 Gigahertz AMD Opteron 246 processor running Linux and C++. The CPU times

**Table 1**
The results for the service model when $K = 0$, $h = 1$ and the demand follows a Poisson distribution.

| L | $\mu_R$ | $\bar{\beta}(\%)$ | $C^*$ | $S_{LS}$ | $C(S_{LS})$ | $S_{BO}$ | $S_{CR}$ | $S_{ZL}$ | $\hat{S}_1$ | $CI(\hat{S}_1)(\%)$ | $\hat{S}_{2a}$ | $\beta(\hat{S}_{2a})(\%)$ | $\hat{S}_{2b}$ | $CI(\hat{S}_{2b})(\%)$ | $\hat{S}_{2c}$ | $\beta(\hat{S}_{2c})(\%)$ | $\hat{S}_3$ | $\beta(\hat{S}_3)(\%)$ | $\hat{S}_4$ | $CI(\hat{S}_4)(\%)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2.5 | 75 | 0.90 | 7 | 1.19 | 9 | 6 | 3 | 8 | 41.2 | 7 | 77.4 | 7 | 0.0 | 7 | 77.4 | 7 | 77.4 | 7 | 0.0 |
| 2 | 2.5 | 80 | 1.21 | 8 | 1.69 | 9 | 6 | 3 | 9 | 35.7 | 8 | 84.2 | 8 | 0.0 | 8 | 84.2 | 7 | 77.4 | 8 | 0.0 |
| 2 | 2.5 | 85 | 1.62 | 9 | 2.29 | 10 | 7 | 4 | 10 | 30.8 | 8 | 84.2 | 9 | 0.0 | 8 | 84.2 | 8 | 84.2 | 9 | 0.0 |
| 2 | 2.5 | 90 | 2.23 | 10 | 2.99 | 11 | 8 | 4 | 11 | 26.6 | 9 | 89.5 | 10 | 0.0 | 9 | 89.5 | 9 | 89.5 | 10 | 0.0 |
| 2 | 2.5 | 95 | 3.30 | 11 | 3.79 | 12 | 9 | 5 | 12 | 22.9 | 11 | 96.1 | 11 | 0.0 | 11 | 96.1 | 10 | 93.4 | 11 | 0.0 |
| 2 | 2.5 | 99 | 5.70 | 14 | 6.54 | 14 | 11 | 6 | 14 | 0.0 | 13 | 98.9 | 14 | 0.0 | 13 | 98.9 | 13 | 98.9 | 14 | 0.0 |
| 2 | 5 | 75 | 0.78 | 13 | 1.23 | 16 | 10 | 5 | 15 | 66.4 | 12 | 73.9 | 13 | 0.0 | 12 | 73.9 | 12 | 73.9 | 13 | 0.0 |
| 2 | 5 | 80 | 1.13 | 14 | 1.60 | 17 | 11 | 5 | 16 | 60.4 | 13 | 78.5 | 14 | 0.0 | 13 | 78.5 | 13 | 78.5 | 14 | 0.0 |
| 2 | 5 | 85 | 1.62 | 15 | 2.04 | 17 | 12 | 6 | 16 | 25.5 | 14 | 82.7 | 15 | 0.0 | 14 | 82.7 | 14 | 82.7 | 15 | 0.0 |
| 2 | 5 | 90 | 2.40 | 17 | 3.16 | 19 | 13 | 6 | 19 | 45.0 | 16 | 89.6 | 17 | 0.0 | 16 | 89.6 | 16 | 89.6 | 17 | 0.0 |
| 2 | 5 | 95 | 3.85 | 19 | 4.58 | 20 | 15 | 8 | 20 | 17.7 | 18 | 94.4 | 19 | 0.0 | 18 | 94.4 | 18 | 94.4 | 19 | 0.0 |
| 2 | 5 | 99 | 7.17 | 23 | 8.09 | 23 | 18 | 10 | 23 | 0.0 | 21 | 98.3 | 23 | 0.0 | 22 | 99.0 | 21 | 98.3 | 23 | 0.0 |
| 2 | 10 | 75 | 0.59 | 24 | 0.94 | 30 | 18 | 8 | 28 | 119.8 | 21 | 68.4 | 25 | 23.4 | 22 | 71.3 | 23 | 74.2 | 24 | 0.0 |
| 2 | 10 | 80 | 0.93 | 26 | 1.41 | 31 | 19 | 9 | 29 | 73.6 | 23 | 74.2 | 27 | 21.2 | 24 | 76.9 | 25 | 79.5 | 26 | 0.0 |
| 2 | 10 | 85 | 1.49 | 28 | 2.06 | 32 | 21 | 10 | 31 | 65.3 | 25 | 79.5 | 29 | 19.2 | 26 | 82.0 | 27 | 84.3 | 28 | 0.0 |
| 2 | 10 | 90 | 2.46 | 30 | 2.90 | 34 | 23 | 11 | 33 | 57.6 | 28 | 86.5 | 31 | 17.3 | 28 | 86.5 | 29 | 88.5 | 31 | 17.3 |
| 2 | 10 | 95 | 4.38 | 34 | 5.24 | 36 | 26 | 13 | 36 | 28.4 | 31 | 92.0 | 35 | 13.7 | 32 | 93.5 | 32 | 93.5 | 34 | 0.0 |
| 2 | 10 | 99 | 8.95 | 39 | 9.28 | 40 | 30 | 16 | 40 | 9.9 | 36 | 97.6 | 40 | 9.9 | 38 | 98.7 | 37 | 98.2 | 40 | 9.9 |
| 1 | 5 | 75 | 0.72 | 9 | 1.07 | 10 | 6 | 5 | 9 | 0.0 | 8 | 72.8 | 9 | 0.0 | 8 | 72.8 | 8 | 72.8 | 9 | 0.0 |
| 1 | 5 | 80 | 1.02 | 10 | 1.52 | 11 | 6 | 5 | 10 | 0.0 | 9 | 79.3 | 10 | 0.0 | 9 | 79.3 | 9 | 79.3 | 10 | 0.0 |
| 1 | 5 | 85 | 1.43 | 11 | 2.07 | 12 | 7 | 6 | 11 | 0.0 | 9 | 79.3 | 11 | 0.0 | 10 | 84.8 | 10 | 84.8 | 11 | 0.0 |
| 1 | 5 | 90 | 2.08 | 12 | 2.72 | 13 | 8 | 6 | 13 | 27.3 | 10 | 84.8 | 12 | 0.0 | 11 | 89.3 | 11 | 89.3 | 12 | 0.0 |
| 1 | 5 | 95 | 3.25 | 13 | 3.46 | 14 | 9 | 8 | 14 | 23.7 | 12 | 92.8 | 14 | 23.7 | 13 | 95.4 | 13 | 95.4 | 13 | 0.0 |
| 1 | 5 | 99 | 5.91 | 16 | 6.09 | 17 | 11 | 10 | 17 | 15.7 | 15 | 98.4 | 17 | 15.7 | 16 | 99.1 | 16 | 99.1 | 16 | 0.0 |
| 3 | 5 | 75 | 0.82 | 17 | 1.35 | 21 | 14 | 5 | 20 | 84.3 | 16 | 74.6 | 17 | 0.0 | 16 | 74.6 | 16 | 74.6 | 17 | 0.0 |
| 3 | 5 | 80 | 1.18 | 18 | 1.67 | 22 | 15 | 5 | 21 | 78.8 | 17 | 78.3 | 18 | 0.0 | 17 | 78.3 | 17 | 78.3 | 18 | 0.0 |
| 3 | 5 | 85 | 1.73 | 20 | 2.49 | 23 | 16 | 6 | 22 | 43.0 | 19 | 84.8 | 20 | 0.0 | 19 | 84.8 | 18 | 81.6 | 20 | 0.0 |
| 3 | 5 | 90 | 2.61 | 21 | 2.99 | 24 | 18 | 6 | 24 | 63.5 | 20 | 87.6 | 22 | 19.0 | 21 | 90.1 | 20 | 87.6 | 22 | 19.0 |
| 3 | 5 | 95 | 4.29 | 24 | 4.89 | 26 | 20 | 8 | 26 | 32.1 | 23 | 94.1 | 25 | 15.5 | 23 | 94.1 | 22 | 92.2 | 25 | 15.5 |
| 3 | 5 | 99 | 8.11 | 29 | 9.13 | 30 | 24 | 10 | 30 | 10.4 | 27 | 98.4 | 29 | 0.0 | 28 | 99.0 | 27 | 98.4 | 29 | 0.0 |

**Table 2**
The results for the service model when $K = 0$, $h = 1$, $L = 2R$ and the demand follows a negative binomial distribution.

| $\mu_R$ | VTM | $\bar{\beta}(\%)$ | $C^*$ | $S_{LS}$ | $C(S_{LS})$ | $S_{BO}$ | $S_{CR}$ | $S_{ZL}$ | $\hat{S}_1$ | $CI(\hat{S}_1)(\%)$ | $\hat{S}_{2a}$ | $\beta(\hat{S}_{2a})(\%)$ | $\hat{S}_{2b}$ | $\beta(\hat{S}_{2b})(\%)$ | $\hat{S}_{2c}$ | $\beta(\hat{S}_{2c})(\%)$ | $\hat{S}_3$ | $\beta(\hat{S}_3)(\%)$ | $\hat{S}_4$ | $CI(\hat{S}_4)(\%)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.5 | 2 | 75 | 1.84 | 8 | 2.23 | 10 | 6 | 4 | 9 | 27.2 | 7 | 70.7% | 7 | 70.7 | 7 | 70.7 | 7 | 70.7 | 8 | 0.0 |
| 2.5 | 2 | 80 | 2.35 | 9 | 2.83 | 11 | 6 | 4 | 11 | 50.0 | 8 | 77.0% | 8 | 77.0 | 8 | 77.0 | 8 | 77.0 | 9 | 0.0 |
| 2.5 | 2 | 85 | 3.04 | 10 | 3.51 | 12 | 7 | 5 | 12 | 43.8 | 8 | 77.0 | 9 | 82.2 | 8 | 77.0 | 9 | 82.2 | 10 | 0.0 |
| 2.5 | 2 | 90 | 4.08 | 12 | 5.05 | 13 | 8 | 5 | 13 | 16.7 | 9 | 82.2 | 10 | 86.5 | 9 | 82.2 | 10 | 86.5 | 12 | 0.0 |
| 2.5 | 2 | 95 | 5.89 | 14 | 6.78 | 15 | 9 | 7 | 15 | 13.5 | 11 | 90.0 | 11 | 90.0 | 11 | 90.0 | 12 | 92.7 | 14 | 0.0 |
| 2.5 | 2 | 99 | 9.95 | 18 | 10.56 | 19 | 11 | 10 | 19 | 9.3 | 13 | 94.7 | 14 | 96.3 | 13 | 94.7 | 16 | 98.2 | 18 | 0.0 |
| 2.5 | 4 | 75 | 3.57 | 10 | 4.20 | 12 | 6 | 5 | 11 | 17.2 | 7 | 62.4 | 7 | 62.4 | 7 | 62.4 | 9 | 73.0 | 10 | 0.0 |
| 2.5 | 4 | 80 | 4.45 | 11 | 4.92 | 13 | 6 | 6 | 13 | 31.7 | 8 | 68.1 | 8 | 68.1 | 8 | 68.1 | 10 | 77.3 | 11 | 0.0 |
| 2.5 | 4 | 85 | 5.65 | 13 | 6.48 | 14 | 7 | 7 | 14 | 12.8 | 8 | 68.1 | 9 | 73.0 | 8 | 68.1 | 11 | 81.0 | 13 | 0.0 |
| 2.5 | 4 | 90 | 7.41 | 15 | 8.16 | 16 | 8 | 8 | 16 | 10.8 | 9 | 73.0 | 10 | 77.3 | 9 | 73.0 | 13 | 86.9 | 15 | 0.0 |
| 2.5 | 4 | 95 | 10.49 | 18 | 10.86 | 19 | 9 | 10 | 19 | 8.6 | 11 | 81.0 | 11 | 81.0 | 11 | 81.0 | 16 | 92.8 | 19 | 8.6 |
| 2.5 | 4 | 99 | 17.54 | 26 | 18.56 | 26 | 11 | 16 | 26 | 0.0 | 13 | 86.9 | 14 | 89.2 | 13 | 86.9 | 23 | 98.4 | 26 | 0.0 |
| 5 | 2 | 75 | 1.73 | 14 | 2.35 | 17 | 10 | 5 | 16 | 43.2 | 12 | 69.7 | 13 | 73.8 | 12 | 69.7 | 13 | 73.8 | 14 | 0.0 |
| 5 | 2 | 80 | 2.34 | 15 | 2.83 | 18 | 11 | 6 | 17 | 39.7 | 13 | 73.8 | 14 | 77.7 | 13 | 73.8 | 14 | 77.7 | 15 | 0.0 |
| 5 | 2 | 85 | 3.18 | 17 | 3.96 | 19 | 12 | 7 | 18 | 16.2 | 14 | 77.7 | 15 | 81.1 | 14 | 77.7 | 15 | 81.1 | 17 | 0.0 |
| 5 | 2 | 90 | 4.49 | 19 | 5.29 | 21 | 13 | 8 | 21 | 28.8 | 16 | 84.2 | 17 | 87.0 | 16 | 84.2 | 17 | 87.0 | 19 | 0.0 |
| 5 | 2 | 95 | 6.82 | 22 | 7.63 | 24 | 15 | 10 | 24 | 22.8 | 18 | 89.3 | 19 | 91.4 | 18 | 89.3 | 20 | 93.1 | 22 | 0.0 |
| 5 | 2 | 99 | 12.09 | 28 | 13.11 | 29 | 18 | 13 | 29 | 7.4 | 21 | 94.6 | 23 | 96.7 | 22 | 95.8 | 25 | 98.1 | 28 | 0.0 |
| 5 | 4 | 75 | 3.58 | 16 | 4.43 | 19 | 10 | 7 | 18 | 27.9 | 12 | 63.9 | 13 | 67.6 | 12 | 63.9 | 14 | 71.0 | 16 | 0.0 |
| 5 | 4 | 80 | 4.63 | 18 | 5.66 | 21 | 11 | 8 | 20 | 24.2 | 13 | 67.6 | 14 | 71.0 | 13 | 67.6 | 16 | 77.2 | 18 | 0.0 |
| 5 | 4 | 85 | 6.07 | 20 | 7.03 | 23 | 12 | 9 | 22 | 21.3 | 14 | 71.0 | 15 | 74.2 | 14 | 71.0 | 18 | 82.3 | 20 | 0.0 |
| 5 | 4 | 90 | 8.22 | 23 | 9.32 | 25 | 13 | 11 | 25 | 17.7 | 16 | 77.2 | 17 | 79.8 | 16 | 77.2 | 20 | 86.5 | 23 | 0.0 |
| 5 | 4 | 95 | 12.02 | 27 | 12.70 | 29 | 15 | 13 | 29 | 14.2 | 18 | 82.3 | 19 | 84.5 | 18 | 82.3 | 24 | 92.5 | 28 | 7.0 |
| 5 | 4 | 99 | 20.66 | 36 | 21.14 | 38 | 18 | 19 | 38 | 9.2 | 21 | 88.2 | 23 | 91.2 | 22 | 89.8 | 33 | 98.3 | 37 | 4.6 |
| 10 | 2 | 75 | 1.48 | 25 | 2.06 | 31 | 18 | 9 | 29 | 73.8 | 21 | 66.4 | 25 | 76.5 | 22 | 69.1 | 24 | 74.1 | 25 | 0.0 |
| 10 | 2 | 80 | 2.16 | 27 | 2.75 | 33 | 19 | 10 | 31 | 66.1 | 23 | 71.6 | 27 | 80.8 | 24 | 74.1 | 26 | 78.7 | 28 | 14.5 |
| 10 | 2 | 85 | 3.17 | 30 | 4.06 | 34 | 21 | 11 | 33 | 40.7 | 25 | 76.5 | 29 | 84.7 | 26 | 78.7 | 28 | 82.8 | 31 | 12.6 |
| 10 | 2 | 90 | 4.79 | 33 | 5.71 | 37 | 23 | 13 | 36 | 34.9 | 28 | 82.8 | 31 | 88.1 | 28 | 82.8 | 31 | 88.1 | 34 | 11.0 |
| 10 | 2 | 95 | 7.84 | 37 | 8.44 | 40 | 26 | 15 | 40 | 28.3 | 31 | 88.1 | 35 | 93.3 | 32 | 89.6 | 35 | 93.3 | 38 | 9.1 |
| 10 | 2 | 99 | 14.91 | 45 | 15.29 | 47 | 30 | 19 | 47 | 12.4 | 36 | 94.3 | 40 | 97.2 | 38 | 96.0 | 42 | 98.1 | 46 | 6.1 |
| 10 | 4 | 75 | 3.37 | 27 | 4.20 | 34 | 18 | 10 | 32 | 59.8 | 21 | 63.1 | 25 | 72.0 | 22 | 65.4 | 25 | 72.0 | 28 | 10.6 |
| 10 | 4 | 80 | 4.59 | 30 | 5.63 | 36 | 19 | 12 | 34 | 40.6 | 23 | 67.7 | 27 | 76.0 | 24 | 69.9 | 27 | 76.0 | 31 | 9.4 |
| 10 | 4 | 85 | 6.33 | 33 | 7.30 | 38 | 21 | 13 | 37 | 35.6 | 25 | 72.0 | 29 | 79.6 | 26 | 74.1 | 30 | 81.2 | 34 | 8.4 |
| 10 | 4 | 90 | 9.00 | 37 | 9.89 | 42 | 23 | 15 | 41 | 30.1 | 28 | 77.8 | 31 | 82.8 | 28 | 77.8 | 34 | 87.0 | 39 | 14.6 |
| 10 | 4 | 95 | 13.83 | 43 | 14.48 | 47 | 26 | 19 | 47 | 23.6 | 31 | 82.8 | 35 | 88.2 | 32 | 84.3 | 40 | 93.0 | 45 | 11.6 |
| 10 | 4 | 99 | 24.90 | 55 | 25.29 | 57 | 30 | 26 | 57 | 7.6 | 36 | 89.3 | 40 | 93.0 | 38 | 91.3 | 51 | 98.3 | 57 | 7.6 |

**Table 3**
The results for the service model when $K = 0$, $h = 1$, $L = 2R$ and the demand follows a normal distribution.

| $\mu_R$ | VTM | $\bar{\beta}(\%)$ | $C^*$ | $S_{LS}$ | $C(S_{LS})$ | $S_{BO}$ | $S_{CR}$ | $S_{ZL}$ | $\widehat{S}_1$ | $CI(\widehat{S}_1)$ | $\widehat{S}_{2a}$ | $CI(\widehat{S}_{2a})$ / $\beta(\widehat{S}_{2a})(\%)$ | $\widehat{S}_{2b}$ | $CI(\widehat{S}_{2b})$/ $\beta(\widehat{S}_{2b})(\%)$ | $\widehat{S}_{2c}$ | $CI(\widehat{S}_{2c})$ / $\beta(\widehat{S}_{2c})(\%)$ | $\widehat{S}_3$ | $\beta(\widehat{S}_3)(\%)$ | $\widehat{S}_4$ | $CI(\widehat{S}_4)$ /$\beta(S_4)(\%)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.5 | 0.5 | 75 | 0.41 | 7 | 0.80 | 8 | 6 | 3 | 7 | 0.0 | 7 | 0.0 | 7 | 0.0 | 7 | 0.0 | 6 | 73.6 | 7 | 0.0 |
| 2.5 | 0.5 | 80 | 0.59 | 7 | 0.80 | 9 | 6 | 3 | 9 | 135.1 | 8 | 57.4 | 8 | 57.4 | 8 | 57.4 | 7 | 82.7 | 7 | 0.0 |
| 2.5 | 0.5 | 85 | 0.83 | 8 | 1.26 | 9 | 7 | 3 | 9 | 49.3 | 8 | 0.0% | 9 | 49.3 | 8 | 0.0 | 7 | 82.7 | 8 | 0.0 |
| 2.5 | 0.5 | 90 | 1.20 | 9 | 1.88 | 9 | 8 | 3 | 9 | 0.0 | 9 | 0.0 | 10 | 41.3 | 9 | 0.0 | 8 | 89.9 | 9 | 0.0 |
| 2.5 | 0.5 | 95 | 1.83 | 10 | 2.66 | 10 | 9 | 4 | 10 | 0.0 | 11 | 33.6 | 11 | 33.6 | 11 | 33.6 | 9 | 94.9 | 10 | 0.0 |
| 2.5 | 0.5 | 99 | 3.28 | 11 | 3.55 | 11 | 11 | 5 | 11 | 0.0 | 13 | 54.9 | 14 | 82.9 | 13 | 54.9 | 11 | 99.3 | 11 | 0.0 |
| 5 | 0.5 | 75 | 0.33 | 12 | 0.50 | 15 | 10 | 4 | 14 | 107.0 | 12 | 0.0 | 13 | 45.6 | 12 | 0.0 | 12 | 76.6 | 12 | 0.0 |
| 5 | 0.5 | 80 | 0.50 | 13 | 0.73 | 16 | 11 | 5 | 15 | 97.5 | 13 | 0.0 | 14 | 42.2 | 13 | 0.0 | 13 | 81.8 | 13 | 0.0 |
| 5 | 0.5 | 85 | 0.79 | 14 | 1.04 | 16 | 12 | 5 | 15 | 38.9 | 14 | 0.0 | 15 | 38.9 | 14 | 0.0 | 14 | 86.4 | 14 | 0.0 |
| 5 | 0.5 | 90 | 1.24 | 15 | 1.44 | 17 | 13 | 6 | 17 | 79.2 | 16 | 35.6 | 17 | 79.2 | 16 | 35.6 | 15 | 90.4 | 15 | 0.0 |
| 5 | 0.5 | 95 | 2.13 | 17 | 2.59 | 18 | 15 | 6 | 18 | 28.6 | 18 | 28.6 | 19 | 61.0 | 18 | 28.6 | 16 | 93.6 | 17 | 0.0 |
| 5 | 0.5 | 99 | 4.25 | 20 | 5.07 | 20 | 18 | 8 | 20 | 0.0 | 21 | 18.8 | 23 | 57.7 | 22 | 38.1 | 19 | 98.9 | 20 | 0.0 |
| 5 | 1 | 75 | 0.86 | 13 | 1.29 | 16 | 10 | 5 | 15 | 60.1 | 12 | 73.3 | 13 | 0.0 | 12 | 73.3 | 12 | 73.3 | 13 | 0.0 |
| 5 | 1 | 80 | 1.20 | 14 | 1.64 | 17 | 11 | 5 | 16 | 56.1 | 13 | 78.1 | 14 | 0.0 | 13 | 78.1 | 13 | 78.1 | 14 | 0.0 |
| 5 | 1 | 85 | 1.67 | 15 | 2.06 | 17 | 12 | 6 | 16 | 24.2 | 14 | 82.4 | 15 | 0.0 | 14 | 82.4 | 14 | 82.4 | 15 | 0.0 |
| 5 | 1 | 90 | 2.39 | 17 | 3.14 | 18 | 13 | 6 | 18 | 21.0 | 16 | 89.6 | 17 | 0.0 | 16 | 89.6 | 16 | 89.6 | 17 | 0.0 |
| 5 | 1 | 95 | 3.73 | 19 | 4.54 | 20 | 15 | 7 | 20 | 17.8 | 18 | 94.7 | 19 | 0.0 | 18 | 94.7 | 17 | 92.4 | 19 | 0.0 |
| 5 | 1 | 99 | 6.64 | 22 | 7.12 | 23 | 18 | 9 | 23 | 13.3 | 21 | 98.6 | 23 | 13.3 | 22 | 0.0 | 21 | 98.6 | 22 | 0.0 |
| 10 | 0.5 | 75 | 0.21 | 23 | 0.33 | 29 | 18 | 8 | 27 | 207.5 | 21 | 69.4 | 25 | 79.6 | 22 | 72.5 | 23 | 75.6 | 23 | 0.0 |
| 10 | 0.5 | 80 | 0.35 | 25 | 0.58 | 30 | 19 | 9 | 28 | 120.1 | 23 | 75.6 | 27 | 71.2 | 24 | 78.5 | 25 | 81.4 | 25 | 0.0 |
| 10 | 0.5 | 85 | 0.65 | 27 | 1.00 | 31 | 21 | 9 | 30 | 105.4 | 25 | 81.4 | 29 | 63.5 | 26 | 84.1 | 26 | 84.1 | 27 | 0.0 |
| 10 | 0.5 | 90 | 1.20 | 29 | 1.64 | 32 | 23 | 10 | 31 | 56.0 | 28 | 89.0 | 31 | 56.0 | 28 | 89.0 | 28 | 89.0 | 29 | 0.0 |
| 10 | 0.5 | 95 | 2.37 | 32 | 3.13 | 33 | 26 | 11 | 33 | 21.1 | 31 | 94.8 | 35 | 70.4 | 32 | 0.0 | 31 | 94.8 | 32 | 0.0 |
| 10 | 0.5 | 99 | 5.41 | 36 | 6.20 | 36 | 30 | 13 | 36 | 0.0 | 36 | 0.0 | 40 | 61.5 | 38 | 30.0 | 34 | 98.2 | 36 | 0.0 |
| 10 | 1 | 75 | 0.67 | 24 | 1.03 | 30 | 18 | 8 | 28 | 105.3 | 21 | 68.1 | 25 | 20.7 | 22 | 71.0 | 23 | 73.9 | 24 | 0.0 |
| 10 | 1 | 80 | 1.02 | 26 | 1.49 | 31 | 19 | 9 | 29 | 67.2 | 23 | 73.9 | 27 | 19.4 | 24 | 76.6 | 25 | 79.2 | 26 | 0.0 |
| 10 | 1 | 85 | 1.56 | 28 | 2.11 | 32 | 21 | 10 | 31 | 61.5 | 25 | 79.2 | 29 | 18.0 | 26 | 81.7 | 27 | 84.1 | 28 | 0.0 |
| 10 | 1 | 90 | 2.49 | 30 | 2.92 | 33 | 23 | 11 | 32 | 35.2 | 28 | 86.3 | 31 | 16.6 | 28 | 86.3 | 29 | 88.4 | 31 | 16.6 |
| 10 | 1 | 95 | 4.30 | 34 | 5.20 | 36 | 26 | 12 | 36 | 28.4 | 31 | 92.0 | 35 | 13.7 | 32 | 93.5 | 32 | 93.5 | 34 | 0.0 |
| 10 | 1 | 99 | 8.52 | 39 | 9.24 | 40 | 30 | 15 | 40 | 10.0 | 36 | 97.7 | 40 | 10.0 | 38 | 98.9 | 37 | 98.4 | 39 | 0.0 |
| 10 | 2 | 75 | 1.75 | 25 | 2.29 | 32 | 18 | 9 | 30 | 81.0 | 21 | 65.5 | 25 | 0.0 | 22 | 68.2 | 24 | 73.3 | 26 | 13.3 |
| 10 | 2 | 80 | 2.40 | 27 | 2.93 | 33 | 19 | 10 | 31 | 57.8 | 23 | 70.8 | 27 | 0.0 | 24 | 73.3 | 26 | 78.0 | 28 | 12.6 |
| 10 | 2 | 85 | 3.34 | 30 | 4.15 | 34 | 21 | 11 | 33 | 37.5 | 25 | 75.7 | 29 | 84.3 | 26 | 78.0 | 28 | 82.3 | 30 | 0.0 |
| 10 | 2 | 90 | 4.81 | 33 | 5.70 | 36 | 23 | 12 | 35 | 21.7 | 28 | 82.3 | 31 | 87.9 | 28 | 82.3 | 31 | 87.9 | 33 | 0.0 |
| 10 | 2 | 95 | 7.48 | 37 | 8.33 | 39 | 26 | 14 | 39 | 18.5 | 31 | 87.9 | 35 | 93.5 | 32 | 89.5 | 34 | 92.3 | 37 | 0.0 |
| 10 | 2 | 99 | 13.32 | 44 | 14.24 | 45 | 30 | 18 | 45 | 6.6 | 36 | 94.6 | 40 | 97.7 | 38 | 96.4 | 41 | 98.2 | 43 | 98.9 |

to find $S_{LS}$ are on average 20–70 milliseconds, whereas finding the optimal policy can take as long as 5–7 minutes. The average CPU times are summarized in Table 7.

When we consider the approximation procedures to determine the values of order-up-to level $S$, most of the procedures either underestimate or overestimate $S_{LS}$. Based on these results, we conclude that the approximation procedure based on our mean-value analysis ($\widehat{S}_4$) finds very good results for the order-up-to level. It satisfies the service level restriction in all instances except for one, and it results in an average cost increase of 2.3% compared to the best $(R,S)$ policy. In almost 80% of the instances the best value of the order-up-to level is found, whereas in the other instances the order-up-to level is overestimated by one or two units. It seems that the performance of $\widehat{S}_4$ decreases when the average demand is higher, even though the deviation from $S_{LS}$ remains small (only 1 unit in most instances).

**Table 4**
The average cost increase of the best $(R,s,S)$ policy compared to the optimal policy for the service model when $K \in \{25,50,100\}$, $h = 1$, $L = 2R$.

| $\mu_R$ | VTM | $K$ | $\bar{\beta}$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 75% | 80% | 85% | 90% | 95% | 99% |
| *Poisson* | | | | | | | | |
| {2.5,5,10} | 1.0 | 25 | 1.93 | 1.96 | 2.02 | 1.51 | 2.16 | 0.54 |
| | | 50 | 1.37 | 1.08 | 1.18 | 0.83 | 1.01 | 1.25 |
| | | 100 | 1.05 | 0.62 | 0.78 | 0.53 | 0.39 | 0.47 |
| *Negative Binomial* | | | | | | | | |
| {2.5,5,10} | 2.0 | 25 | 2.31 | 2.19 | 2.06 | 1.40 | 1.06 | 0.66 |
| | | 50 | 1.07 | 1.24 | 1.13 | 0.65 | 0.30 | 0.38 |
| | | 100 | 0.74 | 0.65 | 0.49 | 0.61 | 0.55 | 0.24 |
| {2.5,5,10} | 4.0 | 25 | 2.23 | 2.15 | 1.76 | 1.61 | 1.20 | 0.58 |
| | | 50 | 1.25 | 1.30 | 0.84 | 0.56 | 0.70 | 0.25 |
| | | 100 | 0.72 | 0.62 | 0.66 | 0.54 | 0.45 | 0.29 |
| *Normal* | | | | | | | | |
| {2.5,5,10} | 0.5 | 25 | 1.89 | 2.24 | 1.50 | 1.10 | 1.66 | 1.07 |
| | | 50 | 0.97 | 0.94 | 1.31 | 1.38 | 0.85 | 0.71 |
| | | 100 | 1.17 | 0.64 | 0.37 | 0.77 | 0.68 | 0.95 |
| {5,10} | 1.0 | 25 | 2.10 | 1.68 | 1.79 | 1.23 | 1.64 | 1.49 |
| | | 50 | 1.28 | 0.59 | 0.94 | 0.95 | 0.33 | 0.09 |
| | | 100 | 0.72 | 0.58 | 0.57 | 0.72 | 0.27 | 0.52 |
| {10} | 2.0 | 25 | 2.72 | 3.07 | 2.42 | 2.47 | 2.99 | 1.26 |
| | | 50 | 0.99 | 0.97 | 0.80 | 0.63 | 0.71 | 0.34 |
| | | 100 | 0.65 | 0.79 | 0.61 | 0.58 | 0.44 | 0.75 |

### 6.2. The $(R,s,S)$ policy

The $(R,s,S)$ policy is applied in many practical applications when fixed costs are incurred with each order. For the numerical experiments we consider the same settings as in the previous section and include fixed order cost $K = 25$,50 and 100. Similarly, we compute the optimal policy with the corresponding expected total costs $C^*$ and the $(R,s,S)$ policy with the best values of reorder level $s$ and order-up-to level $S$ ($s_{LS}$ and $S_{LS}$, respectively). The lower bounds on $S_{LS}$ are the same as in the previous section.

First, we consider the cost increase for the best $(R,s,S)$ policy compared to the optimal policy in Table 4. Since we have 54, 108, 108 instances when demand follows a Poisson, negative binomial and normal distribution, respectively, we only report on the average cost increase over the different average demand scenarios. It becomes clear that the $(R,s,S)$ policy performs close to optimal, with an average cost increase of 1.1% (1.15%, 0.98% and 1.10% for each demand distribution, respectively). When the fixed order cost $K$ increases, the $(R,s,S)$ policy performs in general better. Similarly, when the minimal fill rate $\bar{\beta}$ increases. Note that these observations only hold on average and not for individual instances, due to the discrete demand distribution and the integer-valued policy parameters of the problem. Therefore, no monotonic relation exists.

When we compare the CPU times (Table 7) it becomes clear that finding the optimal policy takes on average 2–6 minutes, whereas the best values of reorder level $s$ and order-up-to level $S$ in the $(R,s,S)$ policy are found within 10–25 seconds (this requires a search procedure in a two dimensional search space). However,

**Table 5**
The average cost increase of the approximation procedures compared to the best $(R,s,S)$ policy for the service model when $K \in \{25,50,100\}$, $h = 1$, $L = 2R$.

| $\mu_R$ | VTM | $K$ | $CI(s_{TG},S_{TG})$ $\bar{\beta}$ | | | | | | $CI(\hat{s}_5,\widehat{S}_5)$ $\bar{\beta}$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 75% | 80% | 85% | 90% | 95% | 99% | 75% | 80% | 85% | 90% | 95% | 99% |
| *Poisson* | | | | | | | | | | | | | | |
| {2.5,5,10} | 1.0 | 25 | 1.26 | 2.44 | 2.05 | 1.24 | 1.94 | 2.85 | 0.38 | 2.09 | 1.10 | 2.18 | 0.34 | 2.23 |
| | | 50 | 0.77 | 1.62 | 1.54 | 0.81 | 0.82 | 1.80 | 0.23 | 0.31 | 1.21 | 1.29 | 1.28 | 2.07 |
| | | 100 | 3.66 | 0.59 | 1.57 | 0.00 | 0.48 | 2.19 | 2.17 | 0.23 | 0.57 | 0.40 | 0.57 | 1.55 |
| *Negative Binomial* | | | | | | | | | | | | | | |
| {2.5,5,10} | 2.0 | 25 | 0.17 | 0.00 | 1.90 | 0.39 | 3.60 | 1.41 | 1.41 | 1.18 | 2.35 | 0.83 | 3.02 | 1.47 |
| | | 50 | 1.26 | 1.56 | 0.82 | 1.19 | 2.14 | 0.58 | 0.00 | 0.74 | 0.79 | 1.73 | 0.71 | 2.16 |
| | | 100 | 1.85 | 0.77 | 1.09 | 0.75 | 1.37 | 0.55 | 1.74 | 0.36 | 1.16 | 0.87 | 0.88 | 2.20 |
| {2.5,5,10} | 4.0 | 25 | 0.00 | 2.71 | 0.43 | 1.31 | 1.65 | 2.52 | 2.59 | 0.39 | 1.26 | 0.36 | 1.62 | 2.33 |
| | | 50 | 0.07 | 0.17 | 0.65 | 0.73 | 0.39 | 0.29 | 0.18 | 0.00 | 0.92 | 1.27 | 1.24 | 2.38 |
| | | 100 | 1.13 | 0.81 | 1.19 | 0.69 | 1.11 | 0.59 | 0.07 | 0.79 | 0.12 | 0.92 | 2.32 | 2.46 |
| *Normal* | | | | | | | | | | | | | | |
| {2.5,5,10} | 0.5 | 25 | 3.05 | 0.36 | 1.37 | 0.79 | 3.40 | 3.56 | 0.68 | 3.10 | 1.46 | 1.12 | 2.51 | 2.33 |
| | | 50 | 1.66 | 0.98 | 1.77 | 1.18 | 0.42 | 1.55 | 0.90 | 0.55 | 1.42 | 0.46 | 1.36 | 1.77 |
| | | 100 | 3.88 | 1.13 | 0.79 | 1.00 | 2.05 | 0.30 | 1.25 | 0.61 | 0.68 | 0.47 | 1.64 | 0.35 |
| {5,10} | 1.0 | 25 | 1.28 | 2.93 | 1.41 | 1.32 | 1.23 | 1.52 | 0.61 | 1.63 | 0.34 | 2.70 | 1.32 | 1.22 |
| | | 50 | 1.18 | 1.63 | 1.09 | 1.57 | 0.54 | 0.86 | 0.34 | 0.47 | 0.64 | 0.78 | 1.58 | 1.01 |
| | | 100 | 0.90 | 0.48 | 1.31 | 2.02 | 0.75 | 0.92 | 0.04 | 0.24 | 0.86 | 1.11 | 0.37 | 0.96 |
| {10} | 2.0 | 25 | 0.61 | 0.00 | 1.03 | 0.94 | 0.93 | 1.90 | | 1.74 | 0.00 | 0.31 | 3.56 | 2.90 |
| | | 50 | 2.56 | 1.55 | 0.61 | 1.78 | 0.00 | 1.32 | 0.81 | 0.62 | 0.53 | 0.00 | 1.06 | 0.90 |
| | | 100 | 0.32 | 0.06 | 1.83 | 1.43 | 1.23 | 0.77 | 0.00 | 0.00 | 0.41 | 0.31 | 0.92 | 1.02 |

**Table 6**
The number of instances that the service level constraint is not satisfied including the average deviation from the minimal service level (over all three demand distributions).

| Policy | K | | $\bar{\beta}$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 75% | 80% | 85% | 90% | 95% | 99% |
| $(s_{TG},S_{TG})$ | 25 | nr. instances | 4 | 4 | 3 | 1 | 2 | 0 |
| | | average deviation | 0.49% | 0.63% | 0.49% | 0.01% | 0.35% | |
| | 50 | nr. instances | 4 | 4 | 4 | 2 | 1 | 0 |
| | | average deviation | 0.45% | 0.30% | 0.18% | 0.15% | 0.00% | |
| | 100 | nr. instances | 4 | 2 | 1 | 0 | 0 | 0 |
| | | average deviation | 0.96% | 0.19% | 0.07% | | | |
| $(\hat{s}_5,\widehat{S}_5)$ | 25 | nr. instances | 3 | 1 | 0 | 0 | 0 | 0 |
| | | average deviation | 0.43% | 0.14% | | | | |
| | 50 | nr. instances | 0 | 2 | 2 | 0 | 0 | 0 |
| | | average deviation | | 0.16% | 0.09% | | | |
| | 100 | nr. instances | 2 | 3 | 1 | 1 | 0 | 0 |
| | | average deviation | 0.68% | 0.51% | 0.34% | 0.05% | | |

the number of stock keeping units (SKUs) at retailers can be quite large. In the grocery industry, supermarkets often carry more than 30,000 SKUs (Kök and Fisher, 2007). As a result, it would take many hours/days to set the replenishment policy for each SKU. Therefore, we also study the performance of our heuristic procedure as proposed in Section 5 as well as the procedure proposed by Tijms and Groenevelt (1984). The average cost increase for each of the $(R,s,S)$ policies based on either one of the two procedures is denoted by $CI(\hat{s}_5,\widehat{S}_5)$ and $CI(s_{TG},S_{TG})$, respectively. The results are presented in Table 5, where we average over the demand similar to Table 4. We note that we only take the average over the instances where the service level constraint is satisfied, otherwise the cost increase can be negative. As a result, the cell for the instance where the demand follows a normal distribution with $\mu_R = 10$ and VTM = 2 is empty since the service level is too low. The results illustrate that both approximation procedures find results that set the inventory control variables close to their best values; the average cost increase of our procedure based on a mean-value analysis compared to the best $(R,s,S)$ policy is 1.2%, whereas the one of Tijms and Groenevelt (1984) results in an average cost increase of 1.3%. We also considered the number of instances that the service level constraint is violated as well as the average service deviation from this minimal level. The results are presented in Table 6. It becomes clear that the service restriction is less satisfied when $\bar{\beta}$ is smaller, but the average service deviation remains

**Table 7**
The CPU times (seconds) to find the optimal policy as well as the best $(R,s,S)$ policy when either $K = 0$ or $K > 0$ under different demand distributions. We excluded the setting to find the best base-stock policy (when $K = 0$) since the average CPU times are less than 0.1 seconds.

| Setting | $\bar{\beta}$ | | | | | |
|---|---|---|---|---|---|---|
| | 75% | 80% | 85% | 90% | 95% | 99% |
| *$K = 0$, optimal policy* | | | | | | |
| Poisson | 66.9 | 70.6 | 68.1 | 55.9 | 53.4 | 37.6 |
| Negative Binomial | 216.8 | 284.9 | 278.4 | 133.3 | 173.7 | 130.9 |
| Normal | 167.2 | 145.8 | 175.5 | 158.3 | 131.4 | 49.2 |
| *$K > 0$, optimal policy* | | | | | | |
| Poisson | 107.5 | 137.0 | 134.9 | 133.0 | 105.7 | 129.9 |
| Negative Binomial | 340.5 | 319.6 | 293.7 | 330.4 | 285.6 | 386.4 |
| Normal | 207.9 | 249.3 | 230.1 | 228.3 | 216.5 | 186.5 |
| *$K > 0$, $(R,s,S)$ policy* | | | | | | |
| Poisson | 9.7 | 10.6 | 12.6 | 15.8 | 16.8 | 23.3 |
| Negative Binomial | 6.8 | 8.1 | 9.3 | 11.9 | 14.5 | 24.2 |
| Normal | 13.5 | 14.9 | 17.0 | 18.5 | 22.6 | 26.8 |

within 1%. Furthermore, it can be seen that our approximation procedure satisfies the service level restriction in more instances compared to the procedure of Tijms and Groenevelt (1984).

## 7. Conclusion

In this paper we have developed a service model for a periodic review inventory system with lost sales. We are the first authors to evaluate the optimal replenishment policy for this type of inventory systems and to compare this policy to the $(R,s,S)$ policy. In our numerical results, we showed that even though the $(R,s,S)$ policy is not optimal, it results in an average cost increase of only 1.1% when fixed order costs are incurred. This justifies the popularity of the $(R,s,S)$ policy in many practical applications.

It requires a two-dimensional search procedure to find the best values of the inventory control variables for the $(R,s,S)$ policy. We introduced lower and upper bounds on these variables to reduce the search space. The numerical results showed that these bounds are not tight enough to perform well as approximations for the variables. Therefore, we also developed efficient approximation procedures to determine the order-up-to level based on a mean-value analysis. In the case of no fixed order costs, this procedure satisfies the service level restriction in almost all instances, and it finds the best order-up-to level in 80% of the instances and otherwise its value is overestimated by only one or two units. In the case that fixed order costs are involved, our approximation procedure satisfies the service level restriction in almost all test instances against an average cost increase of only 1.2%, compared to the best $(R,s,S)$ policy. Furthermore, we performed a comparative analysis to the procedure proposed by Tijms and Groenevelt (1984), which has only been demonstrated to perform well in case excess demand is backordered. We extended their numerical results to the lost-sales case and demonstrated that their procedure results in a similar cost increase of 1.3%. However, the procedure of Tijms and Groenevelt (1984) does not satisfy the service level restriction in as many instances compared to our mean-value analysis procedure. Based on these results, we conclude that our approximation procedure outperforms the procedures from the literature. The models, solution approaches and results can be extended to more complicated settings, such as a multi-echelon setting.

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