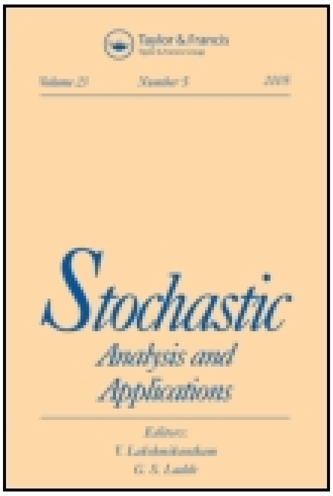

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A Two-Commodity Stochastic Inventory System with Lost Sales

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A Two-Commodity Stochastic Inventory System with Lost Sales

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ABSTRACT

This paper considers a two commodity continuous review inventory system with fixed individual and joint reorder level. The maximum storage capacity for the ith commodity is fixed as S_i $(i = 1, 2)$. It is assumed that demand for ith commodity is of unit size and has Poisson distribution with parameter $\lambda_i\,(i = 1, 2)$. The independent reorder level for ith commodity is fixed as r_i and whenever the inventory level of ith commodity falls on r_i an order for $P_i\,(=S_i - r_i)$ $i = 1, 2$ items is placed for that commodity irrespective of the inventory level of the other commodity. A joint

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inventory level s $(<\min(r_1, r_2)(x + y = s,\ x = 0, 1, \ldots, s,\ y = 0, 1, \ldots, s)$ is fixed for both commodities whenever the inventory level drops to a prefixed level s, an order for Q_x^1 $(=S_1 - x)$ and Q_y^2 $(=S_2 - y)$ items is placed for both commodities. The limiting probability distribution for the joint inventory levels is computed. Various operational characteristics, expression for the long run total expected cost rate is derived. The results are illustrated with numerical examples.

Key Word: Two commodity continuous review system.

1. INTRODUCTION

In many practical multi-item inventory systems concentrated the coordination of replenishment orders for group of items. Now a days it is very much applicable to run a successful Business and Industries. These systems unlike those dealing with single commodity involve more complexities in the reordering procedures. The modelling of multi-item inventory system under joint replenishment has been receiving considerable attention for the past three decades.

In continuous review inventory systems, Ballintify[1] and Silver[2] have considered a coordinated reordering policy which is represented by the triplet $(\mathbf{S}, \mathbf{c}, \mathbf{s})$, where the three parameters S_i, c_i and s_i are specified for each item i with $s_i \leq c_i \leq S_i$, under the unit sized Poisson demand and constant lead time. In this policy, if the level of ith commodity at any time is below s_i, an order is placed for $S_i - s_i$ items and at the same time, any other item $j\,(\neq i)$ with available inventory at or below its can-order level c_j, an order is placed so as to bring its level back to its maximum capacity S_j. Subsequently many articles have appeared with models involving the above policy and another article of interest is due to Federgruen et al.[3] which deals with the general case of compound Poisson demands and non-zero lead times. A review of inventory models under joint replenishment is provided by Goyal and Satir.[4]

Kalpakam and Arivarignan[5] have introduced $(\mathbf{s}, \mathbf{S})$ policy with a single reorder level s defined in terms of the total number of items in the stock. This policy avoids separate ordering for each commodity and hence a single processing of orders for both commodities has some advantages in situation where in procurement is made from the same supplies, items are produced on the same machine, or items have to be supplied by the same transport facility.

Krishnamoorthy et al.[6] have considered a two commodity continuous review inventory system without lead time. In their model, each demand is for one unit of first commodity or one unit of second commodity or one unit of each commodity 1 and 2, with prefixed probabilities. Krishnamoorthy and Varghese[7] have considered a two commodity inventory problem without lead time and with Markov shift in demand for the type of commodity namely "commodity-1", "commodity-2" or "both commodity", using the direct Markov renewal theoretical results. And also for the same problem, Sivasamy and Pandiyan[8] had derived various results by the application of filtering technique.

Anbazhagan and Arivarignan[9] have considered a two commodity inventory system with Poisson demands and a joint reorder policy which placed fixed ordering quantities for both commodities whenever both inventory levels are less than or equal to their respective reorder levels.

Anbazhagan and Arivarignan[10] have analysed models with a joint ordering policy which places orders for both commodities whenever the total net inventory level drops to a prefixed level s. More explicitly, if the maximum stock levels for the two commodities are denoted by S_1 and S_2, the reorder level is s, then an order for both commodities is placed whenever inventory levels L_1 and L_2 drops to $L_1(t) + L_2(t) = s$. In situations where the level of one commodity is considerably low, but the other commodity has high amount of stock, a joint order is not possible and may result in a perpetual shortage for that commodity. In order to handle this situation a separate order for the commodity whose stock is very low may be initiated. This aspect is introduced in this paper. As such we considered an individual reorder policy for each commodity and a joint reorder policy for both commodities.

For the individual reorder policy, the reorder level for ith commodity is fixed as r_i and whenever the inventory level of ith commodity falls on r_i an order for P_i $(=S_i - r_i)$ items is placed for that commodity irrespective of the inventory level of the other commodity. A joint reorder policy is used with prefixed reorder levels s and order for $Q_x^1(S_1 - x)$ and $Q_y^2(S_2 - y)$ items is placed for both commodities by cancelling the previous orders, whenever both commodities have their inventory level drops to a reorder level s, $(x + y = s)$ (see Fig. 1).

We impose certain restrictions on the values of r_i, $(i = 1, 2)$ and s. Since the individual reorder policy is invoked when the inventory level for that commodity is considerably low, r_i is taken to be less than half of the maximum capacity for that commodity. As a joint reordering policy will cancel the previous reorders, the reorder level s is fixed less than the previous reorder levels r_i. The other types of restrictions on r_i and s will be considered in our subsequent works.

The demand points for each commodity form independent Poisson processes and the lead times initiated by individual reorder policy and

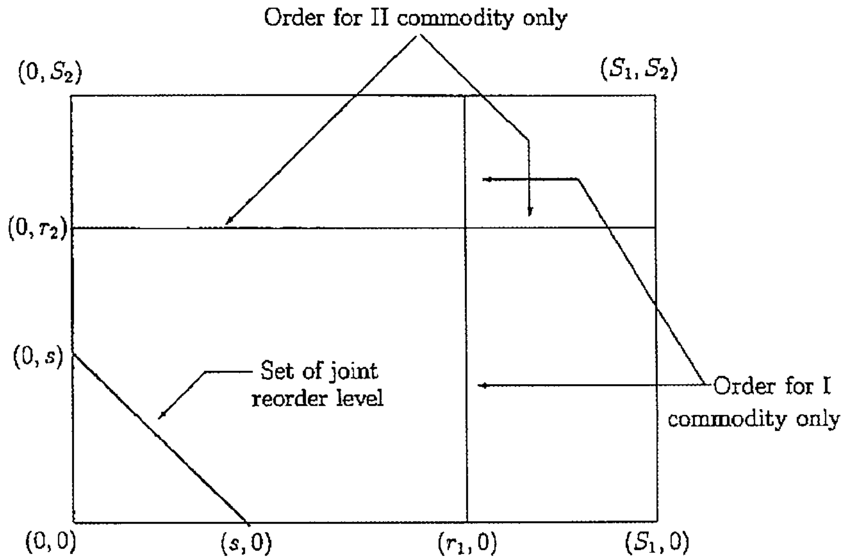


Figure 1. Space of inventory levels.

by joint reorder policy are assumed to be independent and distributed as negative exponential. The demands that occured during stockout periods are lost. The limiting probability distribution of the joint inventory level is derived. Various measures of system performance in the steady state are also obtained.

2. PROBLEM FORMULATION

Consider a two commodity inventory system with the maximum capacity S_i units for i-th commodity $(i = 1, 2)$. It is asssumed that demand for ith commodity is of unit size and the time points of demand occurrances form a Poisson process with parameter λ_i $(i = 1, 2)$. The demand process of the two commodities are further assumed to be independent. The reorder level for the ith commodity is fixed as r_i $(r_i < S_i/2)$ with ordering quantity $P_i\,(=S_i - r_i)$ $i = 1, 2$. The joint reorder level for both commodities is fixed as s $(s < \min(r_1, r_2))$ and a joint reorder for both commodities is placed with ordering quantity $Q_x^1\,(=S_1 - x)$ and $Q_y^2\,(=S_2 - y)$, when both inventory levels drops to $s(x + y = s)$ by cancelling the orders made previously. Let $L_i(t)$ represent inventory level of ith commodity at time t.

Then the ordering policy can be described as follows,

Reorder level	Ordering quantity	Remark
r_1	P_1	$L_1(t) = r_1,\ \forall L_2(t)$
r_2	P_2	$L_2(t) = r_2,\ \forall L_1(t)$
s	$(Q^1_{L_1(t)}, Q^2_{L_2(t)})$	$L_1(t) + L_2(t) = s$ (The previous pending orders are cancelled)

The lead time for the order placed at r_i is assumed to be distributed as negative exponential with parameter $\mu_i\,(>0)(i = 1, 2)$. The lead time for joint order is also assumed to be distributed as negative exponential with parameter $\mu\,(>0)$. The demands that occur during stockout periods are lost.

The process $\{(L_1(t), L_2(t)), t \geq 0\}$ has the state space,

$$E = \{(i, j) \mid i = 0, 1, 2, \ldots, S_1 \text{ and } j = 0, 1, \ldots, S_2\}.$$

Notations.

$\mathbf{0}$: zero matrix
$1'_N$: $(1, 1, \ldots, 1)_{1\times N}$
I_N: an identity matrix of order N
δ_{ij}: Kronecker delta.
$\sum_{j=0}^{i} a^j = \begin{cases} a^0 + a^1 + \cdots + a^i & \text{if } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$
$[A]_{ij}$: (i, j)th element of the matrix A.
$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

From the assumptions made on demand and on replenishment processes, it follows that $\{(L_1(t), L_2(t)), t \geq 0\}$ is a Markov process. To determine the infinitesimal generator $\tilde{A} = (a(i, j, k, l)),\ (i, j), (k, l) \in E$, of this process we use the following arguments:

(i) First we note that a demand takes the inventory level (i, j) to

(a) $(i - 1, j)$, if the demand is for the first commodity, or
(b) $(i, j - 1)$, if the demand is for the second commodity.

The intensity of transition for the case (a) is λ_1 and for that of case (b) is λ_2.

(ii) In order to consider the transitions due to replenishment, we partition the set of states E into

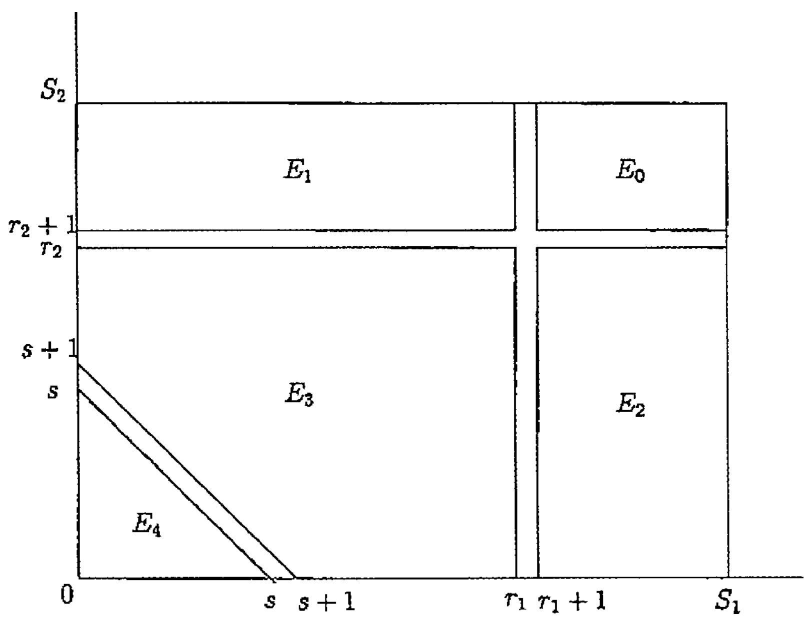
$$E_0 = \{(i, j) \mid r_1 + 1 \le i \le S_1, r_2 + 1 \le j \le S_2\}$$

$$E_1 = \{(i, j) \mid 0 \le i \le r_1, r_2 + 1 \le j \le S_2\}$$

$$E_2 = \{(i, j) \mid r_1 + 1 \le i \le S_1, 0 \le j \le r_2\}$$

$$E_3 = \{(i, j) \mid 0 \le i \le r_1, \ \max\{0, s + 1 - i\} \le j \le r_2\}$$

$$E_4 = \{(i, j) \mid 0 \le i \le s, 0 \le j \le s - i\}$$



With these notations we have,

(a) When $(i, j) \in E_1$ or E_3 a replenishment by the order for the first commodity takes the inventory level to $(i + P_1, j)$ with the intensity of transition μ_1.

(b) When $(i, j) \in E_2$ or E_3 a replenishment by the order for the second commodity takes the inventory level to $(i, j + P_2)$ with the intensity of transition μ_2.

(c) When $(i, j) \in E_4$ a replenishment by the delivery of orders for both commodities takes the inventory level to $(i + Q_i^1, j + Q_j^2)$ with the intensity of transition μ.
(d) We observe that no other transitions other than the above are possible.
(e) Finally the value of $a(i, j, i, j)$ is obtained by

$$a(i, j, i, j) = -\sum_{k \neq i} \sum_{l \neq j} a(i, j, k, l)$$

Hence we have,

$$a(i, j, k, l)$$
$$= \begin{cases} \lambda_1, & k = i - 1, & l = j, \\ & i = 1, 2, \ldots, S_1, & j = 0, 1, \ldots, S_2 \\ \lambda_2, & k = i, & l = j - 1, \\ & i = 0, 1, \ldots, S_1, & j = 1, 2, \ldots, S_2 \\ \mu_1, & k = i + P_1, & l = j \\ & (i, j) \in E_1 \text{ or } E_3 & \\ \mu_2, & k = i, & l = j + P_2 \\ & (i, j) \in E_2 \text{ or } E_3 & \\ \mu, & k = i + Q_i^1, & l = j + Q_j^2 \\ & (i, j) \in E_4 & \\ -(\lambda_1 + \lambda_2), & k = i, & l = j, \\ & (i, j) \in E_0 & \\ -(\lambda_1(1 - \delta_{i0}) + \lambda_2 + \mu_1) & k = i, & l = j \\ & (i, j) \in E_1 & \\ -(\lambda_1 + \lambda_2(1 - \delta_{0j}) + \mu_2) & k = i, & l = j \\ & (i, j) \in E_2 & \\ -(\lambda_1(1 - \delta_{i0}) & & \\ \quad +\lambda_2(1 - \delta_{0j}) + \mu_1 + \mu_2) & k = i, & l = j \\ & (i, j) \in E_3 & \\ -(\lambda_1(1 - \delta_{i0}) & & \\ \quad +\lambda_2(1 - \delta_{0j}) + \mu) & k = i, & l = j \\ & (i, j) \in E_4 & \\ 0, & \text{otherwise} & \end{cases}$$

Denoting $\underset{\sim}{q} = ((q, S_2), (q, S_2 - 1), \ldots, (q, 1), (q, 0))$ for $q = 0, 1, 2, \ldots, S_1$, the infinitesimal generator $\tilde{A}$ can be conveniently expressed as a partitioned matrix:

$$\widetilde{A} = (A_{ij}),$$

where A_{ij} is a $(S_2 + 1) \times (S_2 + 1)$ submatrix, and is given by,

$$A_{ij} = \begin{cases} L & \text{if} \quad j = i - 1, & i = S_1, S_1 - 1, \ldots, 1 \\ A & \text{if} \quad j = i, & i = S_1, S_1 - 1, \ldots, r_1 + 1 \\ A_0 & \text{if} \quad j = i, & i = r_1, r_1 - 1, \ldots, s + 1 \\ A_{s+1-i} & \text{if} \quad j = i, & i = s, s - 1, \ldots, 1, 0 \\ M_0 & \text{if} \quad j = i + P_1, & i = r_1, r_1 - 1, \ldots, s + 1 \\ M_{s+1-i} & \text{if} \quad j = i + P_1, & i = s, s - 1, \ldots, 1, 0 \\ N_{j-i-Q_s^1} & \text{if} \quad j = S_1, S_1 - 1, \ldots, & i = s, s - 1, \ldots, 0 \\ & \qquad\quad S_1 - (s - i), & \\ \mathbf{0} & \text{otherwise.} & \end{cases} \tag{1}$$

More explicitly,

$$\widetilde{A} = \begin{array}{c} S_1 \\ S_1 - 1 \\ \\ \vdots \\ r_1 + 1 \\ r_1 \\ r_1 - 1 \\ \vdots \\ \\ \vdots \\ s + 1 \\ s \\ s - 1 \\ \\ \vdots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} A & L & & & & & & & & & & & & \\ & A & L & & & & & & & & & & & \\ & & \cdots & & & & & & & & & & & \\ & & \cdots & & & & & & & & & & & \\ & & \cdots & A & L & & & & & & & & & \\ M_0 & & & & A_0 & L & & & & & & & & \\ & M_0 & & & & A_0 & & & & & & & & \\ & & \cdots & & & & & & & & & & & \\ & & & & \cdots & & & & & & & & & \\ & & & & & & & & & & & & & \\ & & & & M_0 & & & \cdots & A_0 & L & & & & \\ N_0 & & & & & M_1 & & & & A_1 & L & & & \\ N_1 & N_0 & & & & & M_2 & & & & A_2 & & & \\ \ddots & \ddots & \ddots & & & & \cdots & & & & \cdots & & & \\ & & & & & & & & & & & & & \\ N_{s-2} & \ddots & N_1 & N_0 & & & & & & & & & & \\ N_{s-1} & N_{s-2} & \ddots & \ddots & \ddots & & & \cdots & & & & \cdots & A_s & L \\ N_s & N_{s-1} & N_{s-2} & \ddots & N_1 & N_0 & & & & & M_{s+1} & & & A_{s+1} \end{pmatrix}$$

where,

$$L = \lambda_1 I_{(S_2+1)\times(S_2+1)},$$

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$$
N_i = \begin{array}{c} S_2 \\ S_2-1 \\ \vdots \\ \\ i \\ i-1 \\ \vdots \\ 1 \\ 0 \end{array}
\begin{pmatrix}
\mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mu & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mu & \cdots & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\
\cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mu & \cdots & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \cdots & \mu & \cdots & \mathbf{0}
\end{pmatrix}
$$

$$
S_2 \quad S_2-1 \quad \cdots \quad S_2-i \quad \cdots \quad 0
$$

$i = 0, 1, 2, \ldots, s$

$$
M_i = \begin{array}{c} S_2 \\ S_2-1 \\ \vdots \\ i+1 \\ i \\ i-1 \\ \vdots \\ 1 \\ 0 \end{array}
\begin{pmatrix}
\mu_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mu_1 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mu_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mu_1 & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0}
\end{pmatrix}
$$

$$
S_2 \quad S_2-1 \quad \cdots \quad i+1 \quad i \quad i-1 \quad \cdots \quad 0
$$

$i = 0, 1, 2, \ldots, s+1$

$$
A = \begin{array}{c} S_2 \\ S_2-1 \\ \vdots \\ \\ r_2+1 \\ r_2 \\ r_2-1 \\ \vdots \\ \\ 1 \\ 0 \end{array}
\begin{pmatrix}
d & \lambda_2 & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & d & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & d & \lambda_2 & \cdots & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mu_2 & \mathbf{0} & \cdots & \cdots & d_1 & \lambda_2 & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mu_2 & \cdots & \cdots & \mathbf{0} & d_1 & \lambda_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \cdots & \mu_2 & \mathbf{0} & \cdots & \mathbf{0} & \cdots & d_1 & \lambda_2 \\
\mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mu_2 & \cdots & \mathbf{0} & \cdots & \mathbf{0} & -(\lambda_1+\mu_2)
\end{pmatrix}
$$

with $d = -(\lambda_1 + \lambda_2)$ and $d_1 = -(\lambda_1 + \lambda_2 + \mu_2)$

$$
A_0 = \begin{array}{c} S_2 \\ S_2 - 1 \\ \vdots \\ \\ r_2 + 1 \\ r_2 \\ r_2 - 1 \\ \vdots \\ \\ 1 \\ 0 \end{array}
\begin{pmatrix}
d_2 & \lambda_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & d_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & d_2 & \lambda_2 & \cdots & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mu_2 & \mathbf{0} & \cdots & \cdots & d_3 & \lambda_2 & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mu_2 & \cdots & \cdots & \mathbf{0} & d_3 & \lambda_2 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mu_2 & \cdots & \mathbf{0} & \cdots & d_3 & \lambda_2 \\
\mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mu_2 & \cdots & \mathbf{0} & d_4
\end{pmatrix}
$$

with $d_2 = -(\lambda_1 + \lambda_2 + \mu_1)$, $d_3 = -(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)$ and $d_4 = -(\lambda_1 + \mu_1 + \mu_2)$

$$
A_i = \begin{array}{c} S_2 \\ S_2 - 1 \\ \vdots \\ \\ r_2 \\ r_2 - 1 \\ \vdots \\ \\ i + 1 \\ i \\ i - 1 \\ \vdots \\ \\ 1 \\ 0 \end{array}
\begin{pmatrix}
d_2 & \lambda_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & d_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mu_2 & \mathbf{0} & \cdots & d_3 & \lambda_2 & \cdots & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mu_2 & \cdots & \mathbf{0} & d_3 & \cdots & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mu_2 & \cdots & d_3 & \lambda_2 & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & d_3 & \lambda_2 & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & D_{i-1} & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & D_1 & \lambda_2 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -(\lambda_1 + i * \mu)
\end{pmatrix}
$$

with $d_2 = -(\lambda_1 + \lambda_2 + \mu_1)$, $d_3 = -(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)$ and $D_j = -(\lambda_1 + \lambda_2 + (i - j) * \mu)$, $(j = 1, 2, \ldots, i - 1)$

$$
A_{s+1} = \begin{array}{c} S_2 \\ S_2-1 \\ \vdots \\ \\ r_2 \\ r_2-1 \\ \vdots \\ \\ i+1 \\ i \\ i-1 \\ \vdots \\ \\ 1 \\ 0 \end{array}
\begin{pmatrix}
d_5 & \lambda_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & d_5 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mu_2 & \mathbf{0} & \cdots & d_6 & \lambda_2 & \cdots & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mu_2 & \cdots & \mathbf{0} & d_6 & \cdots & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mu_2 & \cdots & d_6 & \lambda_2 & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & d_6 & \lambda_2 & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & D_{i-1} & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & D_1 & \lambda_2 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -i*\mu
\end{pmatrix}
$$

with $d_5 = -(\lambda_2 + \mu_1)$, $d_6 = -(\lambda_2 + \mu_1 + \mu_2)$ and $D_j = -(\lambda_2 + (i-j)*\mu)$, $(j = 1, 2, \ldots, i-1)$

3. STEADY STATE RESULTS

It can be seen from the structure of A that the homogeneous Markov process $\{(L_1(t), L_2(t)), t \geq 0\}$ on the state space E is irreducible. Hence the limiting distribution

$$\Phi = (\phi^{S_1}, \phi^{S_1-1}, \ldots, \phi^1, \phi^0) \tag{2}$$

with $\phi^q = (\phi^{(q,S_2)}, \phi^{(q,S_2-1)}, \ldots, \phi^{(q,0)})$, where $\phi^{(i,j)}$ denotes the steady state probability for the state (i, j) of the inventory level process, exists and is given by

$$\Phi\widetilde{A} = \mathbf{0} \quad \text{and} \quad \sum_{(i,j)\in E}\sum \phi^{(i,j)} = 1 \tag{3}$$

The first equation of the above yields the following set of equations:

$$\phi^i\lambda_1 + \phi^{i-1}A_{s-i+2} = \mathbf{0}, \quad i = 1, 2, \ldots, s+1$$

$$\phi^i\lambda_1 + \phi^{i-1}A_0 = \mathbf{0}, \quad i = s+2, s+3, \ldots, r_1+1$$

$$\phi^i\lambda_1 + \phi^{i-1}A = \mathbf{0}, \quad i = r_1+2, r_1+2, \ldots, P_1$$

$$\phi^i\lambda_1 + \phi^{i-1}A + H(P_1+s+1-i)\phi^{i-1-P_1}M_{P_1+s+2-i}$$
$$+ H(i-P_1-s-2)\phi^{i-1-P_1}M_0 = \mathbf{0}, \quad i = P_1+1, \ldots, Q_s^1$$

$$
\begin{aligned}
&\phi^i\lambda_1 + \phi^{i-1}A + H(P_1+s+1-i)\phi^{i-1-P_1}M_{P_1+s+2-i} \\
&\quad + H(i-P_1-s-2)\phi^{i-1-P_1}M_0 + \sum_{j=0}^{i-Q_s^1-1}\phi^j N_{i-Q_s^1-1-j} = \mathbf{0}, \\
&\qquad i = Q_s^1+1,\ldots,S_1 \\
&\phi^{S_1}A + \phi^{r_1}M_0 + \sum_{j=0}^{s}\phi^j N_{s-j} = \mathbf{0}.
\end{aligned}
$$

After long simplifications, the above equations, except the last one, yield

$$
\begin{aligned}
\phi^i &= \phi^0(A_{s+1}A_s\cdots A_{s+2-i})(-1/\lambda_1)^i, \quad i=1,2,\ldots,s+1 \\
&= \phi^0(A_{s+1}A_s\cdots A_1)A_0^{i-s-1}(-1/\lambda_1)^i, \quad i=s+2,\ldots,r_1+1 \\
&= \phi^0(A_{s+1}A_s\cdots A_1)A_0^{r_1-s}A^{i-r_1-1}(-1/\lambda_1)^i, \quad i=r_1+2,\ldots,P_1 \\
&= \phi^0(A_{s+1}A_s\cdots A_1)A_0^{r_1-s}A^{i-r_1-1}(-1/\lambda_1)^i + \phi^0A^{i-P_1-1}M_{s+1}(-1/\lambda_1)^{i-P_1} \\
&\quad + \left\{\sum_{k=1}^{i-P_1-1}H(P_1+s-i+k)M_{P_1+s+1-i+k}\right. \\
&\qquad \left. + \sum_{k=1}^{i-P_1-1}H(i-P_1-s-1-k)M_0\right\} \\
&\qquad \times\{H(s+1-i+P_1+k)\phi^0(A_{s+1}A_s\cdots A_{s+2-i+P_1+k}) \\
&\qquad\quad + H(i-P_1-k-s-1)\phi^0(A_{s+1}A_s\cdots A_1)A_0^{i-P_1-k-s-1}\} \\
&\qquad \times A^{k-1}(-1/\lambda_1)^{i-P_1}, \quad i=P_1+1,P_1+2,\ldots,Q_s^1 \\
&= \phi^0(A_{s+1}A_s\cdots A_1)A_0^{r_1-s}A^{i-r_1-1}(-1/\lambda_1)^i + \phi^0A^{i-P_1-1}M_{s+1}(-1/\lambda_1)^{i-P_1} \\
&\quad + \left\{\sum_{k=1}^{Q_s^1-P_1-1}H(P_1+s-Q_s^1+k)M_{P_1+s+1-Q_s^1+k}\right. \\
&\qquad \left. + \sum_{k=1}^{Q_s^1-P_1-1}H(Q_s^1-P_1-s-1-k)M_0\right\} \\
&\qquad \times\left\{H(s+1-Q_s^1+P_1+k)\phi^0(A_{s+1}A_s\cdots A_{s+2-Q_s^1+P_1+k})\right. \\
&\qquad\quad \left. + H(Q_s^1-P_1-k-s-1)\phi^0(A_{s+1}A_s\cdots A_1)A_0^{Q_s^1-P_1-k-s-1}\right\} \\
&\qquad \times A^{i-Q_s^1+k-1}(-1/\lambda_1)^{i-P_1} + \sum_{k=0}^{i-Q_s^1-1}\phi^0N_kA^{i-Q_s^1-1-k}(-1/\lambda_1)^{i-Q_s^1-k}
\end{aligned}
$$

$$+\sum_{k=1}^{i-Q_s^1-1}\sum_{j=1}^{k}\{H(s+1-j)\phi^0(A_{s+1}A_s\cdots A_{s+2-j})$$
$$+H(j-s-1)\phi^0(A_{s+1}A_s\cdots A_1)A_0^{j-s-1}\}$$
$$\times N_{k-j}A^{i-Q_s^1-1-k}(-1/\lambda_1)^{i+j-Q_s^1-k}\quad i=Q_s^1+1,Q_s^1+2,\ldots,S_1$$

where ϕ^0 can be obtained by solving,

$$\phi^{S_1}A+\phi^{r_1}M_0+\sum_{j=0}^{s}\phi^jN_{s-j}=\mathbf{0}\quad\text{and}\quad\sum_{i=0}^{S_1}\phi^i1_{(S_2+1)\times 1}=1,$$

that is

$$\phi^0\Big[A^{S_1-P_1}M_{s+1}(-1/\lambda_1)^{S_1-P_1}$$
$$+\Big\{\sum_{k=1}^{Q_s^1-P_1-1}H(P_1+s-Q_s^1+k)M_{P_1+s+1-Q_s^1+k}$$
$$+\sum_{k=1}^{Q_s^1-P_1-1}H(Q_s^1-P_1-s-1-k)M_0\Big\}$$
$$\times\{H(s+1-Q_s^1+P_1+k)(A_{s+1}A_s\cdots A_{s+2-Q_s^1+P_1+k})$$
$$+H(Q_s^1-P_1-k-s-1)(A_{s+1}A_s\cdots A_1)A_0^{Q_s^1-P_1-k-s-1}\}$$
$$\times A^{S_1-Q_s^1+k}(-1/\lambda_1)^{S_1-P_1}$$
$$+\sum_{k=0}^{S_1-Q_s^1-1}N_kA^{S_1-Q_s^1-k}(-1/\lambda_1)^{S_1-Q_s^1-k}$$
$$+\sum_{k=1}^{S_1-Q_s^1-1}\sum_{j=1}^{k}\{H(s+1-j)(A_{s+1}A_s\cdots A_{s+2-j})$$
$$+H(j-s-1)(A_{s+1}A_s\cdots A_1)A_0^{j-s-1}\}$$
$$\times N_{k-j}A^{S_1-Q_s^1-k}(-1/\lambda_1)^{S_1+j-Q_s^1-k}$$
$$+(A_{s+1}A_s\cdots A_1)A_0^{r_1-s-1}(-1/\lambda_1)^{r_1}M_0$$
$$+N_s+\sum_{j=1}^{s}(A_{s+1}A_s\cdots A_{s+2-j})(-1/\lambda_1)^jN_{s-j}\Big]=\mathbf{0}$$

and

$$\phi_0\Bigg[\Bigg(\sum_{i=1}^{s+1}(A_{s+1}A_s\cdots A_{s+2-i})+\sum_{i=s+2}^{r_1+1}(A_{s+1}A_s\cdots A_1)A_0^{i-s-1}$$
$$+\sum_{i=r_1+2}^{P_1}(A_{s+1}A_s\cdots A_1)A_0^{r_1-s}A^{i-r_1-1}\Bigg)(-1/\lambda_1)^i$$
$$+\sum_{i=P_1+1}^{Q_s^1}\Bigg(A^{i-P_1-1}M_{s+1}+\Bigg\{\sum_{k=1}^{i-P_1-1}H(P_1+s-i+k)M_{P_1+s+1-i+k}$$
$$+\sum_{k=1}^{i-P_1-1}H(i-P_1-s-1-k)M_0\Bigg\}$$
$$\times\{H(s+1-i+P_1+k)(A_{s+1}A_s\cdots A_{s+2-i+P_1+k})$$
$$+H(i-P_1-k-s-1)(A_{s+1}A_s\cdots A_1)A_0^{i-P_1-k-s-1}\}A^{k-1}\Bigg)(-1/\lambda_1)^{i-P_1}$$
$$+\sum_{i=Q_s^1+1}^{S_1}\Bigg((A_{s+1}A_s\cdots A_1)A_0^{r_1-s}A^{i-r_1-1}(-1/\lambda_1)^i$$
$$+A^{i-P_1-1}M_{s+1}(-1/\lambda_1)^{i-P_1}$$
$$+\Bigg\{\sum_{k=1}^{Q_s^1-P_1-1}H(P_1+s-Q_s^1+k)M_{P_1+s+1-Q_s^1+k}$$
$$+\sum_{k=1}^{Q_s^1-P_1-1}H(Q_s^1-P_1-s-1-k)M_0\Bigg\}$$
$$\times\big\{H(s+1-Q_s^1+P_1+k)(A_{s+1}A_s\cdots A_{s+2-Q_s^1+P_1+k})$$
$$+H(Q_s^1-P_1-k-s-1)(A_{s+1}A_s\cdots A_1)$$
$$\times A_0^{Q_s^1-P_1-k-s-1}\big\}A^{i-Q_s^1+k-1}(-1/\lambda_1)^{i-P_1}$$
$$+\sum_{k=0}^{i-Q_s^1-1}N_kA^{i-Q_s^1-1-k}(-1/\lambda_1)^{i-Q_s^1-k}$$
$$+\sum_{k=1}^{i-Q_s^1-1}\sum_{j=1}^{k}\big\{H(s+1-j)(A_{s+1}A_s\cdots A_{s+2-j})$$
$$+H(j-s-1)(A_{s+1}A_s\cdots A_1)A_0^{j-s-1}\big\}$$
$$\times N_{k-j}A^{i-Q_s^1-1-k}(-1/\lambda_1)^{i+j-Q_s^1-k}\Bigg)\Bigg]1_{(S_2+1)\times 1}=1.$$

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The marginal probability distribution $\{\phi_{1i}, i = 0, 1, 2, \ldots, S_1\}$ of the first commodity is given by

$$\phi_{1i} = \sum_{j=0}^{S_2} \phi^{(i,j)}, \quad i = 0, 1, 2, \ldots, S_1$$

and the marginal probability distribution $\{\phi_{2j}, j = 0, 1, 2, \ldots, S_2\}$ of the second commodity is given by

$$\phi_{2j} = \sum_{i=0}^{S_1} \phi^{(i,j)}, \quad j = 0, 1, 2, \ldots, S_2.$$

The expected inventory level in the steady state, for the ith commodity is given by

$$E[L_i] = \sum_{k=0}^{S_i} k\phi^{(i,k)}, \quad i = 1, 2. \tag{4}$$

4. REORDERS AND SHORTAGES

In this section we shall study reorder and shortages. This requires the study of time points at which a transition occurs in the inventory level process.

Let $0 = T_0 < T_1 < T_2 < \cdots$ be the instances of transitions of the process. Let $(L_n^{(1)}, L_n^{(2)}) = (L_1(T_n+), L_2(T_n+)), n = 0, 1, 2, \ldots$.

From the well known theory of Markov processes, $\{(L_n^{(1)}, L_n^{(2)}), n=0, 1, 2, \ldots\}$ is a Markov chain and with the transition probability matrix (*tpm*)

$$P = (p(i, j, k, l))_{(i,j)\in E,(k,l)\in E},$$

where,

$$p(i, j, k, l) = \begin{cases} 0, & (i, j) = (k, l) \\ a(i, j, k, l)/\theta_{ij}, & (i, j) \neq (k, l). \end{cases}$$

Here $\theta_{ij} = -a(i, j, i, j)$. Moreover for all n, we also have,

$$\begin{aligned} &Pr[(L_1(T_{n+1}+), L_2(T_{n+1}+)) \\ &\quad = (k, l), T_{n+1} - T_n > t \mid (L_1(T_n+), L_2(T_n+)) = (i, j)] \\ &\quad = p(i, j, k, l)e^{\theta_{ij}t}. \end{aligned}$$

4.1. Reorders

A reorder for ith commodity is made when the inventory level at any time t, drops to its individual reorder level r_i $(i = 1, 2)$ and a joint order is made for both commodities when the joint inventory level at any time t, drops to either (s_1, s_2) or (s_1, j), $j < s_2$ or (i, s_2), $i < s_1$.

We associate with a reorder a counting process $N(t)$. Define

$$h(i, j, t) = \lim_{\Delta \to 0} P_{ij}[N(t+\Delta) - N(t) = 1]\frac{1}{\Delta}$$

where $P_{ij}[\cdots]$ represents $Pr[\cdots \mid (L_0^{(1)}, L_0^{(2)}) = (i, j)]$

The fact that the reorder at time t is either due to the first transition or due to a subsequent one, gives the following equations:

$$\begin{aligned} h(i, j, t) &= \lim_{\Delta \to 0} \frac{1}{\Delta}\Bigg\{P_{ij}[N(t+\Delta) - N(t) = 1, t < T_1 < t + \Delta] \\ &\quad + \sum_{(k,l)\in E} \int_0^t P_{ij}[(L_1(T_1+), L_2(T_1+)) \\ &\qquad = (k, l), u < T_1 < u + \Delta] \\ &\quad \times Pr[N(t-u+\Delta) - N(t-u) = 1 \mid \\ &\quad \times (L_1(T_1+), L_2(T_1+)) = (k, l)]\Bigg\} \\ &= \tilde{h}(i, j, t) + \sum_{(k,l)\in E} \int_0^t p(i, j, k, l)\theta_{ij} e^{\theta_{ij} u} h(k, l, t-u)\,du \end{aligned}$$

where $\tilde{h}(i, j, t)$ is given by

$$\tilde{h}(i, j, t) = \begin{cases} \lambda_2 e^{-\theta_{ij} t} & j = r_2 + 1, \quad i = 0, 1, \ldots, S_1 \\ \lambda_1 e^{-\theta_{ij} t} & j = r_1 + 1, \quad i = 0, 1, \ldots, S_2 \\ e^{-\theta_{ij} t}[\delta_{i0}\delta_{0(s+1)}\lambda_2 & \\ + \sum_{k=1}^{s} \delta_{ik}\delta_{j(s-k)}(\lambda_1 + \lambda_2) & \\ + \delta_{i(s+1)}\delta_{j0}\lambda_1] & j = s + 1 - i, \quad i = 0, 1, \ldots, s \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

In the above expression we have used the fact that, when $i = 0, 1, 2, \ldots, S_1$ and $j = r_2 + 1$, then the next demand for commodity 2 alone will trigger a reorder of commodity 2 only. Similarly when

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$j = 0, 1, 2, \ldots, S_2$ and $i = r_1 + 1$, then a demand for commodity 1 will trigger a reorder of commodity 1 only. When $i + j = s + 1$ the next demand for commodity 1 or 2 triggers a joint reorder for both commodities.

As the Markov process $\{(L_1(t), L_2(t)), t \geq 0\}$ is irreducible and recurrent (due to finite state space),

$$h = \lim_{t \to \infty} h(i, j, t)$$

exists and will be equal to the steady state mean reorder rate. Moreover, we have from Cinlar[11],

$$h = \sum_{(i,j) \in E} \pi^{(i,j)} \int_0^\infty \frac{\tilde{h}(i, j, t)dt}{\sum_{(i,j) \in E} \pi^{(i,j)} m_{ij}}, \tag{6}$$

where m_{ij} is the mean sojourn time in the inventory level (i, j) and is given by $1/\theta_{ij}$, and $\pi^{(i,j)}$ is the stationary distribution of the Markov chain

$$\{(L_n^{(1)}, L_n^{(2)}), n = 0, 1, 2, \ldots\}.$$

Since for a Markov process,

$$\phi^{(i,j)} = \frac{\pi^{(i,j)} m_{ij}}{\sum_{(k,l) \in E} \pi^{(k,l)} m_{kl}}, \tag{7}$$

we have from (6)

$$\begin{aligned} h &= \sum_{(i,j) \in E} \left(\frac{\phi^{(i,j)}}{m_{ij}} \right) \int_0^\infty \tilde{h}(i, j, t)dt \\ &= \sum_{(i,j) \in E} \phi^{(i,j)} \theta_{ij} \int_0^\infty \tilde{h}(i, j, t)dt \\ &= \lambda_2 \sum_{k=0}^{S_1} \phi^{(k, r_2+1)} + \lambda_1 \sum_{k=0}^{S_2} \phi^{(r_1+1, k)} + \lambda_2 \sum_{k=0}^{s} \phi^{(k, s+1-k)} + \lambda_1 \sum_{k=0}^{s} \phi^{(s+1-k, k)} \end{aligned}$$

4.2. Shortages

A shortage for a commodity occurs when a demand occurs during a stockout period. We associate with a shortage a counting process $M(t)$. Define

$$g(i, j, t) = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr[M(t+\Delta) - M(t) = 1 \mid (I_0^{(1)}, I_0^{(2)}) = (i, j)] \tag{8}$$

which satisfies the equation,

$$g(i, j, t) = \tilde{g}(i, j, t) + \sum_{(k,l)\in E} \int_0^t p(i, j, k, l)\theta_{ij} e^{\theta_{ij}u} h(k, l, t-u) du.$$

We have used the fact that the shortage at time t is due to the first demand or a subsequent one. Hence

$$\tilde{g}(i, j, t) = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr[M(t+\Delta) - M(t) = 1, \\ \times t < T_1 < t + \Delta \mid (L_0^{(1)}, L_0^{(2)}) = (i, j)]$$

and is given by,

$$\tilde{g}(i, j, t) = \begin{cases} \lambda_1 e^{-\theta_{ij}t}, & j = 1, 2, \ldots, S_2, \quad i = 0 \\ \lambda_2 e^{-\theta_{ij}t}, & i = 1, 2, \ldots, S_1, \quad j = 0 \\ (\lambda_1 + \lambda_2) e^{\theta_{ij}t}, & i = 0, \qquad\qquad\quad j = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Derivations similar to the one used to derive h (refer subsection Reorder) yields,

$$\begin{aligned} g &= \lim_{t \to \infty} g(i, j, t) \\ &= \sum_{i=0}^{S_1} \lambda_2 \phi^{(i,0)} + \sum_{j=0}^{S_2} \lambda_1 \phi^{(0,j)}. \end{aligned}$$

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