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COMPUTING OPTIMAL (*s*, *S*) INVENTORY POLICIES*

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A complete computational approach for finding optimal (s, S) inventory policies is developed. The method is an efficient and unified approach for all values of the model parameters, including a non-negative set-up cost, a discount factor $0 \leqq \alpha \leqq 1$, and a lead time. The method is derived from renewal theory and stationary analysis, generalized to permit the unit interval range of values for α. Careful attention is given to the problem associated with specifying a starting condition (when $\alpha < 1$); a resolution is found that guarantees an (s, S) policy optimal for all starting conditions is produced by the computations. New upper and lower bounds on the optimal values of both s and S are established. The special case of linear holding and penalty costs is treated in detail. In the final section, a model in which there is a minimum guaranteed demand in each period is studied, and a simplified method of solution is developed.

1. Introduction

We consider the familiar dynamic inventory model in which demands for a single product in each of an unbounded sequence of periods are independent, identically distributed discrete random variables. There is a constant lead time, a discount factor $0 \leqq \alpha \leqq 1$, a fixed set-up cost, a linear purchase cost, a convex expected holding and penalty cost function, and (usually) total backlogging of unfilled demand. The objective is to choose from among the class of all ordering policies one which minimizes the long run "equivalent" average cost per period; such a policy is called optimal. Under these assumptions, the results of [9, 10, 17, 22] together with a short additional argument imply that there is an optimal policy of the (s, S) type.

Our principal objective in this paper is to develop an efficient computing procedure for finding an optimal (s, S) policy. We formulate the problem in detail and discuss several possible computational approaches in Section 2. In Section 3 we develop formulas for computing the equivalent average cost per period, $a_\alpha(x \mid s, S)$, associated with a given (s, S) policy and a fixed initial amount x of stock on hand and on order. The formulas are developed in two closely related ways—by the aid of renewal theory [2, 3, 7, 13] and by means of a generalized (to the case $\alpha < 1$) form of stationary analysis [2, 7, 12, 21]. The formulas are new for the case $\alpha < 1, x \geqq s$. We construct an algorithm for searching the (s, S) policies to find one that simultaneously minimizes $a_\alpha(x \mid s, S)$ for all x in Section 4. The procedure involves two steps. The first is to find the collection $\mathcal{S}$ of all

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(s, S) policies that minimize $a_\alpha(x \mid s, S)$ for some suitably small and *fixed* value of x. To simplify this task we develop new and easily computed upper and lower bounds on the optimal values of both s and S. The second step is to search the policies in $\mathcal{S}$ to find *one* that minimizes $a_\alpha(x \mid s, S)$ for *every* x. A typical result useful in this step is that if the probability of every non-negative integral demand is positive, then one optimal policy is an (s, S) policy in $\mathcal{S}$ with maximal value of s. This second step is necessary when $\alpha < 1$, because some of the policies in $\mathcal{S}$ may not minimize $a_\alpha(x \mid s, S)$ for every x, as was first recognized in [2]. An example demonstrating the need for searching the policies in $\mathcal{S}$ is given in Section 5, where a special case in which there is a minimal guaranteed demand is studied. This case is of independent interest, because the natural optimal (s, S) policy for the single period model is also optimal for the infinite and all finite horizon models.

Several types of ordering rules that are characterized by a small number of parameters have been proposed in the inventory theory literature as alternatives to the (s, S) policy. These papers typically suggest selecting from among a specified class of policies one that performs best. Although this is a meaningful approach if $\alpha = 1$, it is ordinarily impossible when $\alpha < 1$. The reason is that when $\alpha < 1$ the policy that is best will in general depend upon the particular initial amount of stock on hand and on order. Consequently, in sharp contrast to the (s, S) case, no one of these alternative policies will in general be uniformly best for all initial conditions. We give an example of this phenomenon in Section 4.

2. Model Formulation

Basic Definitions

We assume that the demands $\xi_1, \xi_2, \cdots$ for a single item in periods $1, 2, \cdots$ are independent, non-negative, discrete random variables with common probability distribution φ, $\varphi(k) = \Pr(\xi_t = k)$, $(k = 0, 1, \cdots ; t = 1, 2, \cdots)$. At the beginning of each period the stock on hand and on order is reviewed. An order may then be placed for any positive integral amount of stock. An order placed in period $t(=1, 2, \cdots)$ is delivered at the beginning of the period $t + \lambda$, where λ is a known non-negative integer. When the demand during a period exceeds the inventory on hand after receipt of incoming orders in the period, the excess demand is backlogged until it is subsequently filled by a delivery.

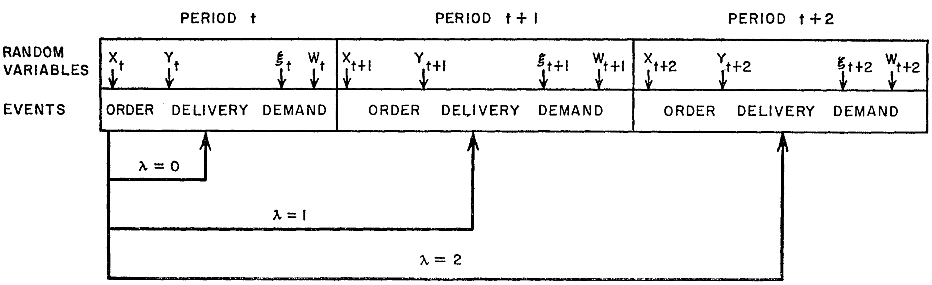

FIGURE 1

In Figure 1 we exhibit three successive periods, and indicate the precise sequence of events. We utilize the following notation for the random variables of interest:

X_t = inventory on hand plus on order prior to placing any order in period t

Y_t = inventory on hand plus on order subsequent to placing any order but before the demand occurs in period t. Y_t is permitted to be any integer valued function of the information accumulated up to the beginning of period t, that is, of $\xi_1, \cdots, \xi_{t-1}, X_1, \cdots, X_t, Y_1, \cdots, Y_{t-1}$. Of course Y_t must also satisfy $Y_t \geqq X_t$. Observe that $Y_t - \xi_t = X_{t+1}$. Finally, we remark that Y_t itself is not a random variable. However, when its argument includes random variables, for example $\xi_1, \cdots, \xi_{t-1}$, then the composite function is a random variable. Although we do not express this difference in our notation, we hope that the correct interpretation will be clear from the context in each case.

W_t = inventory on hand after period t demand occurs.

Since full backlogging of unfilled demand is assumed, each of the above random variables may take on negative values indicating the existence of a backlog.

The economic parameters employed are described as

α = single period discount factor, $0 \leqq \alpha \leqq 1$.

$K\delta(z) + c \cdot z$ = ordering cost incurred at the time of delivery, that is, in period $t + \lambda$, of an order for $z(\geqq 0)$ units placed in period t, where $K \geqq 0$, $c \geqq 0$, and $\delta(0) = 0$, $\delta(z) = 1$ if $z > 0$.

$L_{t+\lambda}$ = holding and penalty cost incurred in period $t + \lambda$. $L_{t+\lambda}$ is a specified non-negative[1] function of $Y_t, \xi_t, \xi_{t+1}, \cdots, \xi_{t+\lambda}$.

We postulate that

$$L(y) \equiv E(L_{t+\lambda} \mid Y_t = y) \quad \text{exists for each } y,$$

and denote

$$G_\alpha(y) \equiv (1 - \alpha)cy + L(y).$$

We assume that $L(y)$ is convex and $G_\alpha(y) \to \infty$ as $|y| \to \infty$.

A few words of explanation about these economic parameters are in order. As defined above, all the economic consequences immediately attributable to an ordering decision in period t are to be measured in period $t + \lambda$. If, instead, the entire ordering cost is actually incurred in period t, and if $\lambda > 0$ and $\alpha > 0$, then $K\delta(z) + c \cdot z$ should be replaced by $\alpha^{-\lambda}[K\delta(z) + c \cdot z]$. Of course, other timing assumptions are also possible, and should be handled by discounting the cost of placing an order in period t to period $t + \lambda$. Generally $\alpha = 1/(1 + i)$ where i is the (decimal) interest rate per period; when $\alpha = 0$, the model is equivalent to a single period inventory problem.

Frequently the holding and penalty costs for period $t + \lambda$ are postulated as

[1] The non-negativity requirement may be replaced by the assumption of the existence of a lower bound for $L_{t+\lambda}$, as will be seen later.

$$hW_{t+\lambda} \quad \text{for} \quad W_{t+\lambda} \geqq 0$$
$$-pW_{t+\lambda} \quad \text{for} \quad W_{t+\lambda} < 0,$$

where $h > 0$ and $p > 0$. In this instance, since $W_{t+\lambda} = Y_t - \xi_t - \cdots - \xi_{t+\lambda}$,

$$(1) \qquad L(y) = \begin{cases} h\sum_{k=0}^{y} (y-k)\varphi^{\lambda+1}(k) + p\sum_{k=y+1}^{\infty} (k-y)\varphi^{\lambda+1}(k) & y \geqq 1 \\ p\sum_{k=0}^{\infty} (k-y)\varphi^{\lambda+1}(k) & y \leqq 0, \end{cases}$$

where $\varphi^n(k)$ is the n-fold convolution of $\varphi(k)$. Also let $\Phi^n(k) = \sum_{t=0}^{k} \varphi^n(t)$. When $n = 1$, we often drop the superscript.

When $\lambda = 0$, our model is easily modified to encompass the case where unsatisfied demand is lost instead of backlogged. Following the reasoning in [20, Section 5; 21, p. 113], we replace $L(y)$ by

$$L(y) - \alpha c\sum_{j=y+1}^{\infty} (j-y)\varphi(j) = L(y) - \alpha c[\mu - y + \sum_{j=0}^{y} (y-j)\varphi(j)]$$
$$= L(y) - \alpha c[\mu - y + \sum_{j=0}^{y-1} \Phi(j)]$$

where here and elsewhere we assume

$$\mu = E(\xi) < \infty.$$

The rationale for the substitution can be put roughly as follows: When demand in a period exceeds inventory on hand, a potential backlog exists at the start of the next period. If the "total backlogging of unsatisfied demand" assumption applies, and $\lambda = 0$, then in this next period an order will be placed including an amount sufficient to remove the backlog. In contrast, if unsatisfied demand is lost, then this order will not include the amount of the potential backlog. Thus, the latter situation can be viewed as a backlog model in which a credit of αc is given to each unit of demand actually backlogged.

Inclusion of the Ordering Cost in the Holding and Penalty Cost Function

Let $X_1 = x$ and let the sequence of functions $Y = (Y_1, Y_2, \cdots)$ be called an ordering policy. We define the expected discounted cost over the n periods $\lambda + 1, \lambda + 2, \cdots, \lambda + n$ when following the policy Y, all discounted to the beginning of period $\lambda + 1$, by

$$f_n(x \mid Y) = E\{\sum_{t=1}^{n} \alpha^{t-1}[K\delta(Y_t - X_t) + c\cdot(Y_t - X_t) + L_{t+\lambda}]\}$$
$$- E\alpha^n c[X_{n+1} - \sum_{t=n+1}^{n+\lambda} \xi_t]$$
$$= \sum_{t=1}^{n} \alpha^{t-1}[KE\delta(Y_t - X_t) + cE(Y_t - X_t) + EL(Y_t)]$$
$$- \alpha^n c[EX_{n+1} - \mu\lambda]$$
$$= \sum_{t=1}^{n} \alpha^{t-1}[KE\delta(Y_t - X_t) + EG_\alpha(Y_t)]$$
$$- cx + \alpha\mu c\sum_{t=0}^{n-1} \alpha^t + \alpha^n c\mu\lambda.$$

The last equality follows by using the fact that $X_t = Y_{t-1} - \xi_{t-1}$ and regrouping terms.

Note that we have assumed that stock left over at the end of period $\lambda + n$ can be salvaged with a return of the initial purchase cost. Similarly, any backlogged demand remaining at the end of period $\lambda + n$ can be satisfied by a purchase at this same cost.[2]

Since the term $-cx + \alpha\mu c \sum_{t=0}^{n-1} \alpha^t + \alpha^n c\mu\lambda$ is not affected by the choice of Y, we find it convenient to redefine $f_n(x \mid Y)$ by setting

$$f_n(x \mid Y) = \sum_{t=1}^{n} \alpha^{t-1}[KE\delta(Y_t - X_t) + EG_\alpha(Y_t)]. \tag{2}$$

Observe[3] that this revised formula for $f_n(x \mid Y)$ is that which would be obtained if the original model were changed by setting the unit purchase cost c equal to zero, and replacing $L(y)$ by $G_\alpha(y)$, which is a composite of $L(y)$ and c. For this reason we refer to $G_\alpha(\cdot)$ hereafter as the conditional expected holding and penalty cost function.

Optimality of a Stationary (s, S) Policy

We may now formulate the n-period model as one of finding a policy Y that minimizes $f_n(x \mid Y)$. Let $Y^n = (Y_1^n, Y_2^n, \cdots)$ denote a minimizing policy, termed an optimal policy, and let $f_n(x) = f_n(x \mid Y^n)$.

By making slight modifications (to account for discrete demands and order quantities) of the proofs in [17, 22], one can establish the existence of an optimal policy and show further that one optimal policy is of the (s, S) type. That is to say, for each period t, there are integers s_t, S_t with $s_t \leqq S_t$ that characterize the ordering rule in period t as follows: If upon review in period t it is discovered that the stock on hand plus that on order has fallen to the level $X_t < s_t$, then the amount $S_t - X_t$ is ordered; otherwise, no order is placed. This rule implies that

$$Y_t = \begin{cases} S, & X_t < s_t \\ X_t, & X_t \geqq s_t. \end{cases}$$

When $s_t = s$ and $S_t = S$ for all t, the (s, S) policy is called *stationary.*

Our principal interest in this paper is the infinite period model. When $0 \leqq \alpha < 1$, we let

$$f(x \mid Y) = \lim_{n\to\infty} f_n(x \mid Y) \tag{3}$$

which may be infinite. We seek Y that minimizes $f(x \mid Y)$. Let Y^* denote a minimizing policy, termed an optimal policy, and let $f(x) = f(x \mid Y^*)$.

When $\alpha = 1$, $f(x \mid Y)$ typically equals $+\infty$ for all policies, so that an alternative criterion is required. A reasonable objective function based on the notion of

[2] In most of the literature the stock on hand at the end of period $\lambda + n$ is assumed to have no value and backlogged demand remaining at the end of period $\lambda + n$ is never satisfied. This difference in treatment of the stock on hand at the end of period $\lambda + n$ affects the choice of an "optimal" policy for the n-period model. But it can easily be shown that the effect of the difference in treatment vanishes as $n \to \infty$.

[3] This reduction of a model with a unit purchase cost to an equivalent model with no unit purchase cost is based on a suggestion of Martin Beckmann. The possibility of such a simplification was noted in his paper [3].

average cost per period is

$$\lim_{n\to\infty} (1/n)f_n(x \mid Y).$$

But this limit does not exist for all Y. Thus, we consider instead

$$a_1(x \mid Y) = \underline{\lim}_{n\to\infty} (1/n)f_n(x \mid Y). \tag{4}$$

We seek Y^* that minimizes $a_1(x \mid Y)$ and call such a policy optimal.

We justify our choice of $\underline{\lim}$ rather than some other limit point of $\{(1/n)f_n(x \mid Y)\}$, say $\overline{\lim}$, as follows. As we shall see later, $a_1(x \mid Y^*) = \lim_{n\to\infty} (1/n)f_n(x \mid Y^*)$. Thus,

$$a_1(x \mid Y^*) \leqq \underline{\lim}\ (1/n)f_n(x \mid Y) \leqq \overline{\lim}\ (1/n)f_n(x \mid Y)$$

for all Y. Hence Y^* would also be optimal if $\underline{\lim}$ in (4) were replaced by $\overline{\lim}$ (or any other limit point). Thus, Y^* is optimal in a very strong sense.

On the other hand, suppose we had used $\overline{\lim}$ in (4) and defined Y^* as a policy that minimizes (4). Then even if we could be sure that $a_1(x \mid Y^*) = \lim_{n\to\infty}(1/n)f_n(x \mid Y^*)$, it might still be possible to find a policy Y such that

$$\underline{\lim}\ (1/n)f_n(x \mid Y) < a_1(x \mid Y^*) \leqq \overline{\lim}\ (1/n)f_n(x \mid Y).$$

This would mean that for infinitely many values of n, Y would give a lower expected cost for n periods than would Y^*.

We now seek to demonstrate for the infinite period model, with either $0 \leqq \alpha < 1$ or $\alpha = 1$, that at least one optimal policy exists which is of the stationary (s, S) type. We consider first the case $\alpha = 1$.

In Sections 4 and 5 of [10], it is shown that there is a stationary (s, S) policy such that

$$\lim_{n\to\infty} (1/n)f_n(x) = a_1(x \mid s, S)$$

with $a_1(x \mid s, S)$ being independent of x. (Actually we here require a discrete demand version of the results in [10].) Now let Y be any policy. Then

$$f_n(x) \leqq f_n(x \mid Y), \qquad n = 1, 2, \cdots .$$

Thus,

$$a_1(x \mid s, S) = \underline{\lim}\ (1/n)f_n(x) \leqq \underline{\lim}\ (1/n)f_n(x \mid Y) = a_1(x \mid Y),$$

which establishes the optimality of (s, S). We are indebted to C. Derman for suggesting this argument.

Next, suppose $0 < \alpha < 1$ (the case $\alpha = 0$ is trivial since it amounts to a single period model). Denote by (s_n, S_n) an optimal (s, S) ordering rule for the first period in an n-period model. Thus, Y^n consists of using (s_n, S_n) in period 1, (s_{n-1}, S_{n-1}) in period 2, $\cdots$, (s_1, S_1) in period n. The numbers s_n, S_n do not depend on the initial inventory on hand and on order for any n. Notice that (s_t, S_t) is here not the policy followed in period t as was the case earlier in this subsection. It is proved in Sections 4 and 5 of [9] that

$$\lim_{n\to\infty} f_n(x) = g(x) \quad \text{(say)}$$

where $g(\cdot)$ is the unique [since $g(\cdot)$ is non-negative] finite solution of

$$(5) \qquad g(x) = \min_{y \geqq x} \{K\delta(y - x) + G_\alpha(y) + \alpha \textstyle\sum_{k=0}^{\infty} g(y - k)\varphi(k)\}.$$

(As usual, we are giving a discrete demand version of the results of [9].) It is also shown in [9] that the sequences $\{s_n\}$ and $\{S_n\}$ are bounded and that any limit point, (s, S) say, of (s_n, S_n) satisfies

$$g(x) = \begin{cases} K + G_\alpha(S) + \alpha \sum_{k=0}^{\infty} g(S - k)\varphi(k), & x < s \\ G_\alpha(x) + \alpha \sum_{k=0}^{\infty} g(x - k)\varphi(k), & x \geqq s. \end{cases}$$

By solving this equation one sees that

$$g(x) = f(x \mid s, S).$$

Now let Y be any policy. Then

$$f_n(x) \leqq f_n(x \mid Y), \qquad n = 1, 2, \cdots.$$

Letting $n \to \infty$ gives

$$(6) \qquad f(x \mid s, S) \leqq f(x \mid Y) \quad \text{for all } x,$$

establishing[4] the optimality of (s, S).

Computational Methods for Finding an Optimal Stationary (s, S) Policy

The principal numerical methods for finding an optimal (stationary) (s, S) policy are related intimately to the functional equation (5), where $\alpha < 1$. (If $\alpha = 1$, a somewhat different equation is appropriate [10].) Three principal methods have been proposed which lead to a solution of (5): successive approximations [5], policy iteration [8] (or the related linear programming methods [6], [16]), and stationary or renewal analysis [2], [3], [10], [12], [13], [21]. A comprehensive survey of the relative merits of these approaches for finding optimal (s, S) policies (as well as an excellent survey of inventory theory) is given in [19]. We only repeat here the remark that policy iteration does not appear computationally attractive since (in the form given in the literature) it does not exploit the property that there is an optimal policy of the (s, S) type, whereas the other two methods do take advantage of this information. The method of stationary or renewal analysis is developed in this paper.

3. Equivalent Average Cost Per Period

Renewal Approach

The key relation to be derived in this section is a formula yielding the equivalent average cost per period $a_\alpha(x \mid s, S)$, $(0 \leqq \alpha \leqq 1)$ resulting from a given

[4] In the event that $L_{t+\lambda}$ is bounded below by B, a negative number, the following changes in the above arguments are required: Let $L'_{t+\lambda} = L_{t+\lambda} - B(\geqq 0)$. Replace $L_{t+\lambda}$ by $L'_{t+\lambda}$ everywhere. Then add $B(1 - \alpha)^{-1}$ and B, respectively, to the formulas (3) and (4).

(s, S) [in this section and hereafter, (s, S) policies will be understood to be stationary] policy [in terms of S and $D \equiv S - s$] and starting condition $X_1 = x$. Two closely related mathematical approaches for this purpose are renewal theory [2, 3, 13] and generalized stationary analysis [2, 12, 21]. We have found that the renewal approach seems to provide the required results with less expository effort, and so we shall use that method to derive the relationships of interest. On the other hand, a generalized stationary analysis seemingly has merit in providing more readily comprehensible interpretations of the results. Therefore, in the next subsection we shall provide an explanatory bridge between the two views. The reader who is mainly interested in how to use the formulas developed in this section can skip immediately to Section 4.

Let $T(d)$ be the first period in which the cumulative demand exceeds the nonnegative integer d; that is, $T(d)$ is the smallest integer for which $\xi_1 + \cdots + \xi_{T(d)} > d$. If $X_1 = S$ and $d = D$, then period $t = T(D) + 1$ will be the first time $X_t < s$, and consequently the first order will be placed at the start of that period.

Assume $X_1 = x$ and no new order is placed until the period after the cumulative demand exceeds d, that is, no new order is placed in periods $1, 2, \cdots, T(d)$. We want to obtain the conditional *expected* holding and penalty cost incurred during the interval $\lambda + 1, \cdots, \lambda + T(d)$ given that $X_1 = x$ and $Y_i = X_i$, $i = 1, \cdots, T(d)$. These costs are discounted to the beginning of period $\lambda + 1$. In taking this expectation, we must account for $T(d)$ being a random variable. Since no order is placed in period 1, $G_\alpha(x)$ is the contribution to this expectation in period $\lambda + 1$. If $\xi_1 \leqq d$, no order is placed in period 2. Then $X_2 = Y_2 = x - \xi_1$, and the contribution to this expectation in period $\lambda + 2$ is the discounted average of $G_\alpha(x - \xi_1)$, namely, $\alpha \sum_{k=0}^{d} G_\alpha(x - k)\varphi(k)$. In general, if $\xi_1 + \cdots + \xi_i \leqq d$, no order is placed in period $i + 1$. Then $X_{i+1} = Y_{i+1} = x - \xi_1 - \cdots - \xi_i$, and the contribution to this expectation in period $\lambda + i + 1$ is the discounted average of $G_\alpha(x - \xi_1 - \cdots - \xi_i)$, namely, $\alpha^i \sum_{k=0}^{d} G_\alpha(x - k)\varphi^i(k)$. Therefore, letting $L_\alpha(x, d)$ be the desired conditional expected holding and penalty cost during the interval $\lambda + 1, \cdots, \lambda + T(d)$, we have

$$L_\alpha(x, d) = G_\alpha(x) + \sum_{i=1}^{\infty} \sum_{k=0}^{d} \alpha^i G_\alpha(x - k)\varphi^i(k) \qquad (x = \cdots, -1, 0, 1, \cdots; d = 0, 1, \cdots). \tag{7}$$

We define

$$m_\alpha(k) \equiv \sum_{i=1}^{\infty} \alpha^i \varphi^i(k) \qquad (k = 0, 1, \cdots)$$

$$M_\alpha(k) \equiv \sum_{i=1}^{\infty} \alpha^i \Phi^i(k) = \sum_{j=0}^{k} m_\alpha(j) \qquad (k = 0, 1, \cdots).$$

We call M_α the discount renewal function. We can interpret[5] $M_\alpha(k)$ as the discounted number of periods before the cumulative demand exceeds k. It is known

[5] This interpretation may be justified as follows: Let $I(t) = \alpha^t$ if $\xi_1 + \cdots + \xi_t \leqq k$, and let $I(t) = 0$ otherwise. Then

$$E[\textstyle\sum_{t=1}^{\infty} I(t)] = \sum_{t=1}^{\infty} E[I(t)] = \sum_{t=1}^{\infty} \alpha^t \Phi^t(k) = M_\alpha(k).$$

[2] that $M_\alpha(k)$ [and hence $m_\alpha(k)$] is finite if $\alpha\varphi(0) < 1$, an assumption that we impose to avoid trivialities. (For completeness we prove this finiteness property in Section 1 of the Appendix.) It follows that the series (7) converges absolutely. If we interchange the order of summation in (7) we find

$$L_\alpha(x, d) = G_\alpha(x) + \textstyle\sum_{j=0}^{d} G_\alpha(x-j)m_\alpha(j) \qquad (x = \cdots, -1, 0, 1, \cdots; d = 0, 1, \cdots). \tag{8}$$

This formula is useful because, as we shall see in Section 4, there is a simple recursive procedure for calculating $m_\alpha(k)$.

Continuing the analysis under the assumption that an order occurs in the period $T(d) + 1$, we proceed to find the expected set-up cost of this order. The order is delivered in period $T(d) + 1 + \lambda$, and the appropriate set-up cost discounted to the beginning of period $\lambda + 1$ is $K\alpha^{T(d)}$. Therefore, we seek the expected value of $\alpha^{T(d)}$. Let $\tau_\alpha(d) \equiv E[\alpha^{T(d)}]$. Now

$$\Pr[T(d) = i] = \Phi^{i-1}(d) - \Phi^i(d), \quad (i = 1, 2, \cdots; d = 0, 1, \cdots)$$

where $\Phi^0(d) \equiv 1$.

Thus by using the definition of $M_\alpha(\cdot)$ we obtain the result implicit in [2]

$$\begin{aligned} \tau_\alpha(d) &= \textstyle\sum_{i=1}^{\infty} \alpha^i[\Phi^{i-1}(d) - \Phi^i(d)] \\ &= [\alpha - (1-\alpha)M_\alpha(d)] \qquad (d = 0, 1, \cdots). \end{aligned} \tag{9}$$

Assume $\alpha < 1$ and that a fixed (s, S) policy is followed. As in Section 2, let $f(x \mid s, S)$ denote the total expected cost incurred in periods $\lambda + 1, \lambda + 2, \cdots$ discounted to the beginning of period $\lambda + 1$ when $X_1 = x$. Where no ambiguity will result, we write $f(x)$ for brevity instead of $f(x \mid s, S)$. Here there is no implication in the abbreviated notation that (s, S) is optimal. If $X_1 = S$, then no order will be placed in periods $1, 2, \cdots, T(D)$, where $D \equiv S - s$. In the following period the (s, S) policy ensures that $Y_{T(D)+1} = S$, so that a renewal of the process takes place. Making use of this observation and the results (8) and (9) above, we have[6]

$$f(S) = L_\alpha(S, D) + K\tau_\alpha(D) + f(S)\tau_\alpha(D),$$

so that

$$f(S) = (L_\alpha(S, D) + K\tau_\alpha(D))/(1 - \tau_\alpha(D)).$$

It follows immediately that if $X_1 = x < s$, then

$$f(x) = K + f(S) \qquad x < s.$$

Now if $X_1 = x \geqq s$, no order will be placed in periods $1, 2, \cdots, T(x - s)$ and $Y_{T(x-s)+1} = S$, at which time a renewal occurs. Therefore

$$f(x) = L_\alpha(x, x - s) + K\tau_\alpha(x - s) + f(S)\tau_\alpha(x - s) \qquad x \geqq s.$$

[6] This approach was suggested by similar arguments on related problems in [13].

Substituting the formula for $f(S)$ into the above gives

$$(10)\quad f(x) = \begin{cases} (L_\alpha(S, D) + K)/(1 - \tau_\alpha(D)) & x < s \\ L_\alpha(x, x - s) \\ \qquad + [(L_\alpha(S, D) + K)/(1 - \tau_\alpha(D))]\tau_\alpha(x - s) & x \geqq s. \end{cases}$$

This formula was first given in essence in [2] for $x < s$, although the derivation there is different.

It is convenient to convert the total expected discounted cost $f(x \mid s, S)$ to an equivalent cost per period $a_\alpha(x \mid s, S)$, that is, $a_\alpha(x \mid s, S)$ is chosen so that $f(x \mid s, S) \equiv \sum_{i=1}^{\infty} \alpha^{i-1} a_\alpha(x \mid s, S)$. Thus $a_\alpha(x \mid s, S) = (1 - \alpha)\ f(x \mid s, S)$. Again we write $a_\alpha(x)$ for brevity instead of $a_\alpha(x \mid s, S)$. From (9) we have

$$(1 - \alpha)[1 - \tau_\alpha(D)]^{-1} = [1 + M_\alpha(D)]^{-1}.$$

Thus we obtain from (10)

$$(11)\quad a_\alpha(x) = \begin{cases} (L_\alpha(S, D) + K)/(1 + M_\alpha(D)) & x < s \\ (1 - \alpha)\, L_\alpha(x, x - s) \\ \qquad + [(L_\alpha(S, D) + K)/(1 + M_\alpha(D))]\tau_\alpha(x - s) & x \geqq s. \end{cases}$$

Observe that if we define $a_1(x) = \lim_{\alpha \to 1} a_\alpha(x)$, then since $\tau_1(x - s) = 1$, we conclude from (11) that

$$(12)\qquad a_1(x) = (L_1(S, D) + K)/(1 + M_1(D)) \equiv a$$

which is completely independent of x. As was noted in [2], (12) gives the long run average cost per period.

Generalized Stationary Analysis Approach[7]

The above derivations can also be accomplished by means of stationary probability analysis [2, 10, 12, 21] appropriately generalized to encompass $0 \leqq \alpha \leqq 1$. Without going into all the details of the approach, we outline the fundamental concepts and their relation to the renewal quantities previously established.

For brevity we restrict our discussion to the case where the value of X_1 is less than s, and consequently $X_t \leqq S$ for all t. Define[8]

$$p_{X_t}(x) \equiv \Pr(X_t = x).$$

For $0 \leqq \alpha < 1$, let

$$(13)\qquad \begin{aligned} \pi_X(x) &\equiv \textstyle\sum_{t=1}^{\infty} \alpha^{t-1} p_{X_t}(x) \\ &= p_{X_1}(x) + \alpha \textstyle\sum_{t=1}^{\infty} \alpha^{t-1} p_{X_{t+1}}(x). \end{aligned}$$

We can interpret $\pi_X(x)$ as the discounted expected number of periods t in which $X_t = x$ over the infinite horizon. It is easily verified that $\sum_{x=-\infty}^{S} \pi_X(x) = (1 - \alpha)^{-1}$, which can be interpreted as the total discounted number of periods. We

[7] The development of the algorithm for finding an optimal (s, S) policy in Section 4 is not dependent on this subsection.

[8] Note that throughout this subsection, we use the symbol x to represent a value of X_t and not just the specific value of X_1, unless we state otherwise.

normalize by letting

$$P_X(x) = (1 - \alpha)\, \pi_X(x). \tag{14}$$

$P_X(x)$ is the discounted fraction of the time that $X_t = x$.

It follows from the transition law for X_t that

$$p_{X_{t+1}}(x) = \begin{cases} \varphi(S - x) \sum_{j=-\infty}^{s-1} p_{X_t}(j) + \sum_{j=s}^{S} \varphi(j - x) p_{X_t}(j) & x < s, \\ \varphi(S - x) \sum_{j=-\infty}^{s-1} p_{X_t}(j) + \sum_{j=x}^{S} \varphi(j - x) p_{X_t}(j) & s \leqq x \leqq S. \end{cases} \tag{15}$$

Substituting (15) into (13), multiplying (13) by $(1 - \alpha)$, and noting that $p_{X_1}(x) = 0$ for $x \geqq s$, we obtain

$$P_X(x) = \begin{cases} (1 - \alpha) p_{X_1}(x) + \alpha[\varphi(S - x) \sum_{j=-\infty}^{s-1} P_X(j) \\ \qquad + \sum_{j=s}^{S} \varphi(j - x) P_X(j)] & x < s, \\ \alpha[\varphi(S - x) \sum_{j=-\infty}^{s-1} P_X(j) + \sum_{j=x}^{S} \varphi(j - x) P_X(j)] & s \leqq x \leqq S. \end{cases} \tag{16}$$

This equation is also valid for $\alpha = 1$; in that event the $P_X(x)$ are ordinary stationary probabilities.

Let

$$\sigma = \sum_{j=-\infty}^{s-1} P_X(j) = \text{discounted fraction of the periods in which an order is placed.}$$

Let $k = S - x$ for $s \leqq x \leqq S$, that is, $k = 0, 1, \cdots, D$. Then (16) for $s \leqq x \leqq S$ can be written as

$$P_X(S - k)/\sigma = \alpha\{\varphi(k) + \sum_{j=0}^{k} \varphi(k - j)\, [P_X(S - j)/\sigma]\} \qquad (k = 0, 1, \cdots, D). \tag{17}$$

It is shown in Section 1 of the Appendix that (17) is a renewal equation, and

$$P_X(S - k)/\sigma = m_\alpha(k),$$

where $m_\alpha(k)$ is defined in the preceding section, so that

$$P_X(S - k) = m_\alpha(k)\sigma \qquad (k = 0, 1, \cdots, D). \tag{18}$$

Now by the definition of σ and the fact that the $P_X(x)$ sum to 1

$$\sum_{j=0}^{D} P_X(S - j) = 1 - \sigma.$$

Therefore from (18)

$$\sigma = [1 + \sum_{j=0}^{D} m_\alpha(j)]^{-1} = [1 + M_\alpha(D)]^{-1}.$$

Substituting this formula into (18) and then (18) into (16) yields, for X_1 less than s,

$$P_X(x) = \begin{cases} (1 - \alpha) p_{X_1}(x) \\ \quad + \alpha[\varphi(S - x) + \sum_{j=0}^{D} \varphi(S - j - x) m_\alpha(j)]/(1 + M_\alpha(D)), & x < s \\ m_\alpha(S - x)/(1 + M_\alpha(D)), & s \leqq x \leqq S. \end{cases} \tag{19}$$

This formula was established in [12] for $\alpha = 1$.

Letting $X_1 = x < s$ and using (19), we can write $a_\alpha(x)$ in (11) as

$$a_\alpha(x) = G_\alpha(S)\sigma + \sum_{j=s}^{S} G_\alpha(j)P_X(j) + K\sigma. \tag{20}$$

When the cost function $L(y)$ has the form (1), we may write (20) in a more intuitive (but computationally less convenient) alternative form. To this end, let

$$p_{W_t}(w) = \Pr(W_t = w) \quad \text{and} \quad P_W(w) = (1 - \alpha) \sum_{t=1}^{\infty} \alpha^{t-1} p_{W_{t+\lambda}}(w).$$

Clearly $P_W(w)$ is the discounted fraction of the periods in which the end of period inventory on hand is w. By elementary algebraic manipulation it can be demonstrated that for $X_1 = x < s$

$$a_\alpha(x) + \alpha c\mu - (1 - \alpha)cx = K\sigma + c \sum_{k=-\infty}^{s-1} (S - k)P_X(k) + h \sum_{w=0}^{S} wP_W(w) - p \sum_{w=-\infty}^{-1} wP_W(w), \qquad x < s.$$

4. Computation of an Optimal (s, S) Policy

We now develop an algorithm for finding an optimal (s, S) policy, say (s^*, S^*). We emphasize at the outset that (s^*, S^*) simultaneously minimizes the (equivalent) average cost per period $a_\alpha(x \mid s, S)$ [see (11) above] over the class of all (s, S) policies for *every* fixed (integer) value of x. We established the existence of such a policy in Section 2. We shall carefully distinguish between (s^*, S^*) and an (s, S) policy that is optimal for only certain values of x. Specifically if $\underline{X}$ is a set of integers, we say that (s', S') is optimal for $\underline{X}$ if for each fixed $x \in \underline{X}$, (s', S') minimizes $a_\alpha(x \mid s, S)$ over the class of all (s, S) policies.

Unfortunately $a_\alpha(x \mid s, S)$ is not a convex function of (s, S), nor, in general, are all of its local minima [with respect to (s, S)] global minima. However, from (11) it is clear that for $x < s = S - D$, $a_\alpha(x \mid S - D, S)$ equals a function of S and D only, say $\mathcal{L}_\alpha(S, D)$. In addition, since $G_\alpha(\cdot)$ is convex, $a_\alpha(x \mid S - D, S)$ is convex in S for fixed x and D satisfying $x < S - D$. We exploit these facts in a three step algorithm for finding (s^*, S^*); we first outline the algorithm briefly below and then develop it in detail.

Step i *Determine bounds on s^* and S^*.*

There are easily computed integers $\underline{s} \leqq \bar{s} \leqq \underline{S} \leqq \bar{S}$ such that $\underline{s} \leqq s^* \leqq \bar{s}$ and $\underline{S} \leqq S^* \leqq \bar{S}$. The definitions of these constants are given in the first subsection below. One immediate consequence is that $\underline{S} - \bar{s} \leqq D^* \equiv S^* - s^* \leqq \bar{S} - \underline{s}$.

Step ii *Find (s, S) policies optimal for $x < \underline{s}$.*

Since (s^*, S^*) is optimal, it is certainly optimal for $x < \underline{s}$. Thus, (s^*, S^*) minimizes $\mathcal{L}_\alpha(S, D)$ over the class of all (s, S) policies. In this step we find the collection $\mathcal{S}$ of *all* (s, S) policies that minimize $\mathcal{L}_\alpha(S, D)$ over the class of (s, S) policies falling within the bounds defined in Step i. From what we have said above, $\mathcal{S}$ contains (s^*, S^*).

Thus, each policy in $\mathcal{S}$ is optimal for $x < \underline{s}$. Detailed procedures for minimizing $\mathcal{L}_\alpha(S, D)$ are given in the second subsection below.

Step iii *Choose an optimal* (s, S) *policy from* $\mathcal{S}$.

When $\alpha = 1$, every policy in $\mathcal{S}$ is optimal since $a_1(x \mid s, S) = \mathcal{L}_1(S, D)$ for all x. However, if $\alpha < 1$ and if $\mathcal{S}$ contains more than one policy, some of the policies in $\mathcal{S}$ may not be optimal. (We give a significant example of this phenomenon in Section 5.) Therefore, in this step we give methods of identifying which policies in $\mathcal{S}$ are optimal. These techniques are described in the third subsection below.

Bounds on s^* *and* S^*

Let $\underline{S}$ be the smallest integer that minimizes the function $G_\alpha(y)$. Since $\lim_{|y|\to\infty} G_\alpha(y) = \infty$, $\underline{S}$ exists. Specifically $\underline{S}$ is the unique integer satisfying

$$\Delta G_\alpha(\underline{S} - 1) < 0 \leqq \Delta G_\alpha(\underline{S}),$$

where $\Delta G_\alpha(y) = G_\alpha(y + 1) - G_\alpha(y)$. When the cost function $L(y)$ has the form (1), $\underline{S}$ is determined from

$$\Phi^{\lambda+1}(\underline{S} - 1) < (p - (1 - \alpha)c)/(p + h) \leqq \Phi^{\lambda+1}(\underline{S}).$$

Let $\bar{S}$ be the smallest integer not less than $\underline{S}$ for which

$$G_\alpha(\bar{S} + 1) \geqq G_\alpha(\underline{S}) + \alpha K. \tag{21}$$

Let $\underline{s}$ be the smallest integer for which

$$G_\alpha(\underline{s}) \leqq G_\alpha(\underline{S}) + K. \tag{22}$$

Let $\bar{s}$ be the smallest integer for which

$$G_\alpha(\bar{s}) \leqq G_\alpha(\underline{S}) + (1 - \alpha)K. \tag{23}$$

The existence of the above parameters is ensured by the fact that $\lim_{|y|\to\infty} G_\alpha(y)$

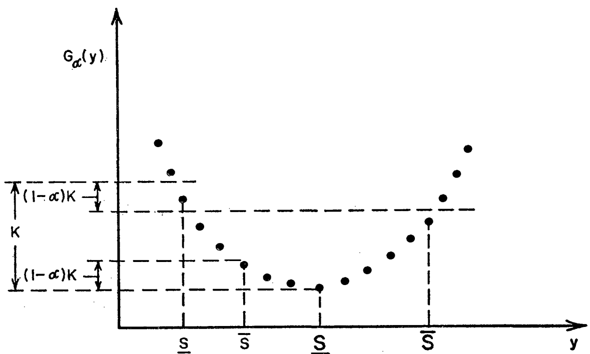

FIGURE 2

$= \infty$. Figure 2 illustrates how the preceding relations determine the bounds used in Step i.

The bounds on s^* and S^* given in Step i are all new[9] and are established in Section 2 of the Appendix. It is also shown there that the bounds are valid for the n-period model.

The bounds have meaningful interpretations. Observe $(\underline{s}, \underline{S})$ is optimal for the one-period model in which $G_\alpha(\cdot)$ is the expected holding and penalty cost function. If $K = 0$, $\underline{s} = \bar{s} = \underline{S} = \bar{S}$ so that $(\underline{S}, \underline{S})$ is an optimal policy for the n-period and infinite-period models as is known from [5, 11, 14, 20]. When $\alpha = 0$, $\underline{s} = \bar{s}$ and $\underline{S} = \bar{S}$ so that $(\underline{s}, \underline{S})$ is optimal. Finally, as we show in Section 5, if $\varphi(k) = 0$ for $k = 0, 1, \cdots, \bar{S} - \underline{s}$, then $(\underline{s}, \underline{S})$ is optimal for the n-period and infinite-period models.

We digress here to justify an assertion which we made in the introduction, namely, that if $0 \leqq \alpha < 1$ and we restrict ourselves to a class of policies not containing the (s, S) policies, then the optimal policy from the given class will in general depend upon X_1. As an illustration, suppose we consider the single-period model ($\alpha = 0$) and seek an optimal policy from among those characterized by the parameters s and Q as follows:

$$\text{if } X_1 < s, \text{ order } Q; \text{ otherwise do not order.}$$

Observe from Figure 2 that if $X_1 < \underline{s}$, then $s = \underline{s}$ and $Q = \underline{S} - X_1$ are optimal. Thus, the optimal choice of Q depends on X_1.

(s, S) *Policies Optimal for* $x < \underline{s}$

In this section we describe in detail how the minimization of

$$\mathfrak{L}_\alpha(S, D) = (L_\alpha(S, D) + K)/(1 + M_\alpha(D)) \tag{24}$$

in Step ii of our algorithm may be carried out. Our approach is first to find the values of S that minimize $\mathfrak{L}_\alpha(S, D)$ for each fixed $D \,\varepsilon\, \{\underline{S} - \bar{s}, \cdots, \bar{S} - \underline{s}\}$. This enables us to tabulate the function $\mathfrak{L}_\alpha(D) \equiv \min_S \mathfrak{L}_\alpha(S, D)$. We then minimize $\mathfrak{L}_\alpha(D)$ over $D \,\varepsilon\, \{\underline{S} - \bar{s}, \cdots, \bar{S} - \underline{s}\}$.

To carry out these computations it is convenient to recall from Section 3 that

$$L_\alpha(S, D) = G_\alpha(S) + \textstyle\sum_{j=0}^{D} G_\alpha(S - j) m_\alpha(j)$$

and

$$M_\alpha(k) = \textstyle\sum_{j=0}^{k} m_\alpha(j).$$

In Section 1 of the Appendix we show that $m_\alpha(k)$ may be calculated recursively by means of the known formula [7]

[9] We should qualify this claim by mentioning that our lower bound on S^* can be deduced easily from Lemma 1 and Theorem 3 of [9] for the case $0 \leqq \alpha < 1$. We also remark that a lower bound on s^* and an upper bound on S^* are given in [9] and [10]. However, our bounds are sharper.

$$(25)\qquad m_\alpha(k) = \begin{cases} \alpha\varphi(0)/(1-\alpha\varphi(0)) & (k=0) \\ \alpha[\varphi(k) + \sum_{j=0}^{k-1}\varphi(k-j)m_\alpha(j)]/(1-\alpha\varphi(0)) & (k=1,2,\cdots). \end{cases}$$

Since $\mathcal{L}_\alpha(S, D)$ is convex in S, an integer S minimizes $\mathcal{L}_\alpha(S, D)$ for fixed D if and only if

$$\Delta_1\mathcal{L}_\alpha(S-1, D) \leqq 0 \leqq \Delta_1\mathcal{L}_\alpha(S, D)$$

where Δ_1 signifies the first forward difference of $\mathcal{L}_\alpha$ with respect to S. The above inequality is equivalent to

$$(26)\qquad \Delta G_\alpha(S-1) + \sum_{j=0}^{D}\Delta G_\alpha(S-1-j)m_\alpha(j) \leqq 0 \leqq \Delta G_\alpha(S) + \sum_{j=0}^{D}\Delta G_\alpha(S-j)m_\alpha(j).$$

Notice that (26) does not involve K.

When the cost function $L(y)$ has the form (1), (26) is equivalent to[10]

$$(27)\qquad \frac{\Phi^{\lambda+1}(S-1) + \sum_{j=0}^{D}\Phi^{\lambda+1}(S-1-j)m_\alpha(j)}{1+M_\alpha(D)} \leqq \frac{p-(1-\alpha)c}{p+h} \leqq \frac{\Phi^{\lambda+1}(S) + \sum_{j=0}^{D}\Phi^{\lambda+1}(S-j)m_\alpha(j)}{1+M_\alpha(D)}.$$

The inequality (27) has the following interpretation suggested by an equivalent result for the case $\alpha = 1$ in [21]. Choose S just large enough so that the expected discounted number of periods between two successive orders during which there is no shortage of stock [the numerator of the right side of (27)] does not fall below a fraction $(p-(1-\alpha)c)/(p+h)$ of the expected discounted number of periods between two successive orders $[1+M_\alpha(D)]$.

The inequality (27) can be interpreted more directly by proceeding as follows. Using the definition of $P_W(w)$, (13), (14), (19), the fact that $\sum_{x=-\infty}^{s-1}P_X(x) = [1+M_\alpha(D)]^{-1}$, and assuming that $X_1 < s$, we have

$$P_W(w) = (1-\alpha)\sum_{t=1}^{\infty}\alpha^{t-1}\left[\sum_{k=-\infty}^{s-1}\varphi^{\lambda+1}(S-w)p_{X_t}(k) + \sum_{k=s}^{S}\varphi^{\lambda+1}(k-w)p_{X_t}(k)\right]$$
$$= \frac{\varphi^{\lambda+1}(S-w) + \sum_{j=0}^{D}\varphi^{\lambda+1}(S-j-w)m_\alpha(j)}{1+M_\alpha(D)} \qquad w \leqq S.$$

Thus (27) takes the alternative simple form that was first given in [21] for the case $\alpha = 1$

$$(27)'\qquad \sum_{w=1}^{S}P_W(w) \leqq (p-(1-\alpha)c)/(p+h) \leqq \sum_{w=0}^{S}P_W(w).$$

[10] This inequality is a generalization of a result given in [3]. Note that $\varphi^{\lambda+1}(k) = \Phi^{\lambda+1}(k) = 0$ for $k < 0$, so that in fact the indicated upper limit of summation of the index j on the right of (27) can be expressed as min (S, D).

As we remarked above, when $K = 0$, there exists an optimal policy of the form (S, S). In this event (27) reduces to

$$\Phi^{\lambda+1}(S-1) \leqq (p-(1-\alpha)c)/(p+h) \leqq \Phi^{\lambda+1}(S).$$

If $\lambda = 0$ and unsatisfied demand is lost, then the ratio $(p-(1-\alpha)c)/(p+h)$ in (27) should be replaced by $(p-c)/(p+h-\alpha c)$, which is known from [5].

We recall from the outline of Step i that we need consider only those (s, S) policies for which

$$\underline{S}(D) \equiv \max(\underline{s}+D, \underline{S}) \leqq S \leqq \min(\bar{s}+D, \bar{S}) \equiv \bar{S}(D).$$

These limits can be utilized in solving (26) [or (27)] for S. Let $R(S)$ be the right side of (26). First compute $R[\underline{S}(D)-1]$ and $R[\bar{S}(D)]$. If it is not true that

$$R[\underline{S}(D)-1] \leqq 0 \leqq R[\bar{S}(D)] \tag{28}$$

then the value of S, say S', that minimizes $\mathcal{L}_\alpha(S, D)$ does not lie between $\underline{S}(D)$ and $\bar{S}(D)$. Thus we can immediately reject D as a possible value of D^*. If (28) does hold, then the familiar technique of interval bisection may be employed with the limits $\underline{S}(D)$ and $\bar{S}(D)$ as the starting interval. Specifically, at iteration i suppose we have $\underline{r}^i \leqq S \leqq \bar{r}^i$. Let $S^i =$ integer part $[.5(\underline{r}^i + \bar{r}^i)]$. Then

$$\underline{r}^{i+1} = \underline{r}^i \quad \text{and} \quad \bar{r}^{i+1} = S^i \quad \text{if} \quad R(S^i) > 0$$
$$\underline{r}^{i+1} = S^i \quad \text{and} \quad \bar{r}^{i+1} = \bar{r}^i \quad \text{if} \quad R(S^i) \leqq 0$$

until, for $i = k$, $\bar{r}^k = \underline{r}^k + 1$, when

$$\text{Optimal } S = \begin{cases} \underline{r}^k & \text{if} \quad R(\underline{r}^k) > 0 \\ \bar{r}^k & \text{if} \quad R(\underline{r}^k) \leqq 0. \end{cases} \tag{29}$$

An obvious modification of the above procedure enables us to solve (27) in the same way as we solve (26).

The bisection method is a more efficient search procedure than directly minimizing $\mathcal{L}_\alpha(S, D)$ with respect to S using, say, a Fibonacci approach [15]. With the bisection method, no more than n evaluations need be made if S is known to lie in an interval of $2^n - 1$ consecutive integers. Note that in general the bisection procedure above does not yield every optimal value for S when more than one exists, but it does locate the largest.[11] In general it is necessary to find all optimal values of S to guarantee that in Step iii an optimal (s, S) policy will be obtained. As we indicate in the next subsection, there are important special cases where it will suffice only to locate the largest value of S for each D.

The above computations provide us with one or more values of S satisfying (28) that minimize $\mathcal{L}_\alpha(S, D)$ for each fixed $D \varepsilon \{\underline{S} - \bar{s}, \cdots, \bar{S} - \underline{s}\}$. Thus we may compute $\mathcal{L}_\alpha(D)$ for each such D, compare the results, and select the values

[11] To locate the smallest value, replace the conditions "$R(S^i) >$" and "$R(S^i) \leqq$" by "$R(S^i) \geqq$" and "$R(S^i) <$", respectively.

TABLE 1

Example of Fibonacci Approximation

$\varphi(k)$ = Poisson Holding Cost = 1/unit $K = 64$
$E(\xi_t) = \mu$ Penalty Cost = 9/unit $\alpha = 1$
$\lambda = 0$

μ	Optimal Policy				Approximate Policy				$\frac{a_1(\cdot\mid s, S)100}{a_1(\cdot\mid s^*, S^*)}$
	s^*-1	S^*	D^*+1	$a_1(\cdot\mid s^*, S^*)$	$s-1$	S	$D+1$	$a_1(\cdot\mid s, S)$	
21	15	65	50	50.40590	15	65	50	50.40590	100
22	16	68	52	51.63222	16	68	52	51.63222	100
23	17	52	35	52.75658	17	52	35	52.75658	100
24	18	54	36	53.51777	18	54	36	53.51777	100
51	43	110	67	71.61085	43	110	67	71.61085	100
52	44	112	68	72.24602	44	61	17	77.01544	106.6
55	47	118	71	74.14860	45	65	20	77.38106	104.4
59	51	126	75	76.67902	49	69	20	77.82948	101.5
61	52	131	79	77.92867	50	71	21	78.05713	100.2
63	54	73	19	78.28676	50	73	23	78.28676	100
64	55	74	19	78.40221	50	74	24	78.40221	100

of D that minimize $\mathfrak{L}_\alpha(D)$. This procedure provides us with the desired set $\mathfrak{S}$ of (s, S) policies that are optimal for $X_1 = x < \underline{s}$.

In the remainder of this subsection we outline a procedure that substantially reduces the computations needed in Step ii, but does *not* guarantee that an optimal policy will be found. However, the second author has conducted extensive empirical tests for the case $\alpha = 1$ (which will be reported elsewhere) that have shown that the method often does yield an optimal policy and so far has always produced a policy that is nearly optimal as measured in terms of average cost per period.

The method proposed is to apply the Fibonacci search procedure to minimize $\mathfrak{L}_\alpha(D)$ even though $\mathfrak{L}_\alpha(D)$ is not unimodal as it should be for theoretical validity. To illustrate the idea, this approach was applied to the model in Table 1 and the results are summarized in the final columns. Notice that the approximation prematurely reduced D at $\mu = 52$ resulting in an expected cost 6.6% above optimal. It probably wound be possible to detect that such an error has occurred by the substantial jump evident in the expected cost function.[12]

The attractive feature of the Fibonacci search procedure, which is explained in detail in [4, 15], is its speed of convergence. If we can assert that an optimal D lies in an interval of width R, say $\bar{D} + 1 \leqq D \leqq \bar{D} + R$, then the Fibonacci search method requires testing only N_R values of D, as given in Table 2. Thus if $R = 54$, we need test only 8 values of D.

We digress to remark here that Table 1 shows that the optimal D is not a

[12] There are a variety of other safeguard features that can be employed. For example, split the interval for D into two (or more) parts and perform the Fibonacci search on each part.

TABLE 2

Fibonacci Search

Interval Length	R	1	2	4	7	12	20	33	54	88	143
Number of Tests	N_R	1	2	3	4	5	6	7	8	9	10

monotone function of μ. A complete table for $1 \leqq \mu \leqq 64$ would demonstrate that the graph of optimal values of D dips twice (at the points indicated in Table 1) in the interval $1 \leqq \mu \leqq 64$.

Choosing an Optimal (s, S) *Policy from* $\mathcal{S}$

The computations in Step ii determine a set $\mathcal{S} \equiv \{(s^i, S^i) \mid i = 1, \cdots, n; s^1 \leqq \cdots \leqq s^n\}$ of (s, S) policies, each of which is optimal for $x < \underline{s}$, and at least one of which is an optimal (s, S) policy (for all x). In this subsection we develop methods for determining which of the policies in $\mathcal{S}$ are optimal. We assume that $\alpha < 1$ and $n > 1$, for otherwise, as we have already noted, our task would be trivial. We start by giving a universally applicable procedure, and then discuss simplifications for important special cases.

Theorem 1

If $0 \leqq \alpha < 1$, if the two policies (s, S), (s', S') are such that $s \leqq s'$, and if $a_\alpha(x \mid s, S) = a_\alpha(x \mid s', S')$ for $x < s'$, then $a_\alpha(x \mid s, S) = a_\alpha(x \mid s', S')$ for all x.

This theorem, which is proved in Section 3 of the Appendix, may be applied as follows to determine the policies in $\mathcal{S}$ that are optimal. First observe that by hypothesis $a_\alpha(x \mid s^{n-1}, S^{n-1}) = a_\alpha(x \mid s^n, S^n)$ for $x < s^{n-1}$. Next for each x, $s^{n-1} \leqq x < s^n$, compute $a_\alpha(x \mid s^{n-1}, S^{n-1})$ and compare with $a_\alpha(x \mid s^n, S^n) = \mathcal{L}_\alpha(S^n, D^n)$, $(D^n \equiv S^n - s^n)$. As soon as one policy proves less favorable than the other, eliminate it. If both policies are equally favorable for all x in this interval, then by the theorem they are equally favorable for all x. In this event, select one arbitrarily, say (s^{n-1}, S^{n-1}). Repeat the procedure with (s^{n-2}, S^{n-2}) and with the policy remaining. In general, policies (s^i, S^i) and (s^j, S^j), $i < j$, are to be compared by testing the equality of $a_\alpha(x \mid s^i, S^i)$ and $a_\alpha(x \mid s^j, S^j)$ for $s^i \leqq x < s^j$. Continue the test procedure until one policy remains, which is then optimal.

One special application of Theorem 1 is:

Corollary 1.1

If $s^1 = \cdots = s^n$, then each (s^i, S^i), $(i = 1, \cdots, n)$ is optimal.

The general procedure given above for choosing an optimal policy from $\mathcal{S}$ can frequently be substantially simplified. For this purpose we require a definition. Given a specific (s, S) policy, we say that the value x' is accessible from $X_1 = x$ if there exists a $t > 1$ such that

$$\Pr(X_t = x' \mid X_1 = x) > 0.$$

We have the following result which is proved in Section 3 of the Appendix.

Lemma 1

If (s, S) is optimal for $X_1 = x$ and if $0 < \alpha < 1$, then (s, S) is optimal for every x' accessible from x.

This lemma together with Theorem 1 enables us to establish the following useful theorem.

Theorem 2

If $0 < \alpha < 1$, if (s^i, S^i) is optimal, and if each x' for which $\min(s^i, s^j) \leqq x' < \max(s^i, s^j)$ is accessible from S^j under (s^j, S^j), then (s^j, S^j) is optimal.

Proof:

By construction of $\mathcal{S}$, $a_\alpha(x \mid s^i, S^i) = a_\alpha(x \mid s^j, S^j)$ for $x < \min(s^i, s^j)$. Thus, since (s^i, S^i) is optimal, (s^j, S^j) is optimal for $x < \min(s^i, s^j)$. Now every x' accessible from S^j under (s^j, S^j) is also accessible from $x < \min(s^i, s^j)$ under (s^j, S^j). Therefore, by an hypothesis of the theorem and Lemma 1, (s^j, S^j) is optimal for $x < \max(s^i, s^j)$. Then by Theorem 1, (s^j, S^j) is optimal.

Since $s^1 \leqq \min(s^i, s^j)$ and $\max(s^i, s^j) \leqq s^n$, we have the following immediate corollaries of practical significance.

Corollary 2.1

If $0 < \alpha < 1$ and if each x' for which $s^1 \leqq x' < s^n$ is accessible from S^j under (s^j, S^j), then (s^j, S^j) is optimal.

Corollary 2.2

If $0 < \alpha < 1$ and if $\varphi(k) > 0$ for $k = 1, \cdots, s^n - s^1$, then (s^n, S^n) is optimal.

Obviously the hypothesis on $\varphi(k)$ in Corollary 2.2 is satisfied for a Poisson or Negative Binomial distribution. Observe that if $\varphi(k) > 0$ for $k = 1, \cdots, \bar{S} - \underline{s}$, then it follows from Corollary 2.2 that (s^n, S^n) is optimal, since $s^n - s^1 \leqq \bar{S} - \underline{s}$. In this situation, it is only necessary to compute $\bar{S}$ and $\underline{s}$ to verify the hypothesis, and there is no need to develop the entire set $\mathcal{S}$; consequently the procedure in Steps ii and iii of our algorithm is as simple for $\alpha < 1$ as for $\alpha = 1$. In Step ii, for each D considered, it suffices to select the largest S that minimizes $\mathcal{L}_\alpha(S, D)$ by the bisection procedure as outlined above.

In the event that the value $X_1 = x$ is actually specified and that an optimal policy for this particular value is being sought, the above general procedure can be shortened. Every policy (s^i, S^i) where $x < s^i$ is equally favorable; evaluate $a_\alpha(x \mid s^i, S^i)$ by (11) for any one of these policies. Then also evaluate $a_\alpha(x \mid s^j, S^j)$ for each j for which $s^j \leqq x$. Finally, select a policy having the minimum value for $a_\alpha(x \mid \cdot, \cdot)$.

Function Limits and Tabled Quantities

Steps ii and iii of the algorithm as we have expressed it require only finite summations. We summarize here the limits of the functions which must be evaluated and stored for the general case in Steps ii and iii. It is necessary to have available $m_\alpha(k)$ and $M_\alpha(k)$ for $k = 0, 1, \cdots, \bar{S} - \underline{s}$, and $G_\alpha(y)$ and $\Delta G_\alpha(y)$ for $\underline{s} \leqq y \leqq \bar{S}$.

When the cost function $L(y)$ has the form (1), it is possible to exploit the special formulas for this case as given above. Specifically, the following sequence of computations can be performed which will provide all the tabled information required for Steps i, ii, and iii.

Calculation 1: Starting at $k = 0$, generate and store $\Phi^{\lambda+1}(k)$ and $G_\alpha(k)$, obtaining in the process $\underline{S}$, until the condition for $k = \bar{S}$ is satisfied. Record $\underline{S}$ and $\bar{S}$.

Calculation 2: Determine $\underline{s}$ and $\bar{s}$. If $\underline{s} < 0$, generate and store

$$G_\alpha(y) \quad \text{for} \quad y = -1, -2, \cdots, \underline{s}.$$

Calculation 3: Generate and store $m_\alpha(k)$ and $M_\alpha(k)$ for

$$k = 0, 1, \cdots, \bar{S} - \underline{s}.$$

Notice in this case it is not necessary to compute $\Delta G_\alpha(y)$, $\varphi(k)$, $\Phi(k)$, and there is no need to store $\varphi^{\lambda+1}(k)$.

The computation of $L(y)$ needed for Calculations 1 and 2 may be simplified with the aid of the following formulas.

$$\begin{aligned} L(y) &= p[(\lambda + 1)\mu - y] + (h + p) \textstyle\sum_{k=0}^{y} (y - k)\varphi^{\lambda+1}(k) \\ &= p[(\lambda + 1)\mu - y] + (h + p) \textstyle\sum_{k=0}^{y-1} \Phi^{\lambda+1}(k), \qquad & y \geqq 1. \\ L(y) &= p[(\lambda + 1)\mu - y] & y \leqq 0. \end{aligned}$$

This representation can also be viewed recursively

$$L(y) = L(y - 1) + \Delta L(y - 1) \qquad y \geqq 1$$

where

$$\begin{aligned} \Delta L(y - 1) &= (h + p)\Phi^{\lambda+1}(y - 1) - p \\ &= \Delta L(y - 2) + (h + p)\varphi^{\lambda+1}(y - 1) \qquad & y \geqq 2, \\ \Delta L(0) &= (h + p)\varphi^{\lambda+1}(0) - p & y = 0. \end{aligned}$$

Further reductions in computations are possible if $\varphi(k)$ is either a Poisson or Negative Binomial distribution:

$$\varphi^n(k) = e^{-n\mu}(n\mu)^k/k!, \qquad \textstyle\sum_{k=0}^{\infty} k\varphi^n(k) = n\mu \qquad \text{Poisson,}$$

$$\varphi^n(k) = \binom{nr - 1 + k}{nr - 1} q^{nr}(1 - q)^k, \qquad \textstyle\sum_{k=0}^{\infty} k\varphi^n(k) = n\mu = nr(1 - q)/q \qquad \text{Negative Binomial.}$$

Then since $\varphi^n(k) = (n\mu/k)\varphi^n(k - 1)$ for the Poisson distribution

$$L(y) = p[(\lambda + 1)\mu - y] + (h + p)\,[y\Phi^{\lambda+1}(y - 1) - (\lambda + 1)\mu\Phi^{\lambda+1}(y - 2)], \qquad y \geqq 1.$$

Similarly since $\varphi^n(k) = [(1 - q)\,(nr - 1 + k)/k]\varphi^n(k - 1)$ for the Negative Binomial distribution,

$$L(y) = p[(\lambda + 1)\mu - y] + (h + p)[y\Phi^{\lambda+1}(y - 1) - (\lambda + 1)\mu\Phi^{\lambda+1}(y - 2) + (y - 1)(1 - q)\varphi^{\lambda+1}(y - 1)/q], \qquad y \geqq 1.$$

5. A Special Case of Guaranteed Demand

In this section we examine a case having twofold interest: the demand assumption is an important special case leading to an extreme simplification of the computational requirements for determining an optimal policy, and the example illustrates the need for Step iii of our algorithm. The following theorem is proved in Section 4 of the Appendix.

Theorem 3

If $\varphi(k) = 0, k = 0, 1, \cdots, \bar{S} - \underline{s}$, then $(\underline{s}, \underline{S})$ is optimal (for all values of X_1) for the finite and infinite period models.

When the hypothesis of the theorem applies, the demand in each period is guaranteed to exceed $\bar{S} - \underline{s}$. The computation of an optimal policy is then of the same order of magnitude as finding an optimal policy in a one period model. Once a period occurs in which an order is placed, an order will be placed in every succeeding period.

As an example, suppose that the demand occurring during a period is uniformly distributed between β and $\beta + \gamma (\beta > 0, \gamma > 0)$. For expositional simplicity we discuss the case where there is a density function $\varphi(k)$ of demand and where the order quantities need not be integral. Thus

$$\varphi(k) = \begin{cases} 1/\gamma & \beta \leqq k \leqq \beta + \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Assume the cost function $L(y)$ has the form (1) and $\lambda = 0$. Let

$$\theta = [p - (1 - \alpha)c]/(p + h).$$

Elementary calculations show that

$$\bar{S} = (K + .5\gamma(p + h)(1 - \theta^2))/(h + (1 - \alpha)c) + \beta,$$

$$\underline{s} = \theta\gamma - (2K\gamma/(p + h))^{1/2} + \beta,$$

$$\bar{S} - \underline{s} = (K + .5\gamma(p + h)(1 - \theta^2))/(h + (1 - \alpha)c) - \theta\gamma + (2K\gamma/(p + h))^{1/2},$$

$$\underline{S} = \beta + \theta\gamma,$$

$$D^* = \underline{S} - \underline{s} = (2K\gamma/(p + h))^{1/2},$$

if $\geqq 0$, $\bar{S} \geqq \beta + \gamma$, and $s \geqq \beta$. (Under our assumptions these conditions usually hold. If they do not hold, the formulas are easily modified.) Observe that

$$\bar{S} - \underline{s} \geqq K/[h + (1 - \alpha)c] - \gamma + (2K\gamma/(p + h))^{1/2}$$

which gives some indication of how large the guaranteed demand β must be in order to apply the theorem.

Theorem 3 provides us with a simple example of a situation in which several (s, S) policies are optimal for sufficiently small values of X_1 while not all of them are optimal (for all values of X_1). Specifically, suppose that $\underline{s} < s \leqq \bar{s}$. Then under the hypothesis of Theorem 3, $(s, \underline{S})$ is optimal for $X_1 < \underline{s}$ since $(s, \underline{S})$ and $(\underline{s}, \underline{S})$ incur the same costs in each period. However, if $\underline{s} < X_1 < s$, then $(\underline{s}, \underline{S})$ would involve no ordering in period 1, while $(s, \underline{S})$ would call for ordering in period 1. Both policies incur the same costs after period 1. Thus,

$$\begin{aligned} a_\alpha(X_1 \mid s, \underline{S}) - a_\alpha(X_1 \mid \underline{s}, \underline{S}) &= (1 - \alpha)[K + G_\alpha(\underline{S}) - G_\alpha(X_1)] \\ &> (1 - \alpha)[K + G_\alpha(\underline{S}) - G_\alpha(\underline{s})] \geqq 0 \end{aligned}$$

by (22). Hence, $(s, \underline{S})$ is not optimal for $\underline{s} < X_1 < s$.

Appendix

Here we provide proofs of several propositions appearing in the text.

1. *Some Renewal Formulas*

We recall from Section 3 the definition

$$M_\alpha(k) = \sum_{i=1}^{\infty} \alpha^i \Phi^i(k).$$

We show now that this series converges absolutely for $\alpha\varphi(0) < 1$. It is convenient to consider two cases, $\alpha < 1$ and $\alpha = 1$. In the former event

$$M_\alpha(k) \leqq \sum_{i=1}^{\infty} \alpha^i \leqq 1/(1 - \alpha) < \infty.$$

In the latter event it follows that $\varphi(0) < 1$. But this means that $\Phi^{k+1}(k) < 1$ for each non-negative integer k. Now[13]

$$\Phi^{n(k+1)}(k) \leqq [\Phi^{k+1}(k)]^n, \qquad (n = 1, 2, \cdots).$$

Also $\Phi^t(k)$ is non-increasing in t. Hence, letting $\Phi^0(k) \equiv 1$, we have

$$\begin{aligned} M_1(k) &= \sum_{i=0}^{\infty} \Phi^i(k) - 1 = \sum_{n=0}^{\infty} \sum_{t=0}^{k} \Phi^{n(k+1)+t}(k) - 1 \\ &\leqq (k + 1) \sum_{n=0}^{\infty} \Phi^{n(k+1)}(k) - 1 \leqq (k + 1) \sum_{n=0}^{\infty} [\Phi^{k+1}(k)]^n - 1 \\ &\leqq \frac{k + 1}{1 - \Phi^{k+1}(k)} - 1 < \infty \end{aligned}$$

which completes the proof.

[13] We use the fact that if U and V are independent non-negative random variables, then

$$\Pr(U + V \leqq t) \leqq \Pr(U \leqq t, V \leqq t) = \Pr(U \leqq t)\Pr(V \leqq t).$$

Next we establish that $m_\alpha(k)$ satisfies the renewal equation

$$m_\alpha(k) = \alpha[\varphi(k) + \textstyle\sum_{j=0}^{k}\varphi(k-j)m_\alpha(j)]. \tag{A1}$$

Now from the definition of $m_\alpha(k)$,

$$\begin{aligned} m_\alpha(k) &= \alpha\varphi(k) + \alpha \textstyle\sum_{i=1}^{\infty}\alpha^i\varphi^{i+1}(k) \\ &= \alpha[\varphi(k) + \textstyle\sum_{i=1}^{\infty}\alpha^i \sum_{j=0}^{k}\varphi(k-j)\varphi^i(j)] \\ &= \alpha[\varphi(k) + \textstyle\sum_{j=0}^{k}\varphi(k-j)m_\alpha(j)]. \end{aligned}$$

The equation (A1) can be solved recursively as follows:

$$m_\alpha(0) = \alpha[\varphi(0) + \varphi(0)m_\alpha(0)] \quad \text{so that}$$

$$m_\alpha(0) = \alpha\varphi(0)/(1 - \alpha\varphi(0)).$$

For $k > 0$,

$$\begin{aligned} m_\alpha(k) &= \alpha[\varphi(k) + \textstyle\sum_{j=0}^{k}\varphi(k-j)m_\alpha(j)] \\ &= \alpha[\varphi(k) + \textstyle\sum_{j=0}^{k-1}\varphi(k-j)m_\alpha(j)] + \alpha\varphi(0)m_\alpha(k) \quad \text{so that} \\ m_\alpha(k) &= \alpha[\varphi(k) + \textstyle\sum_{j=0}^{k-1}\varphi(k-j)m_\alpha(j)]/[1 - \alpha\varphi(0)]. \end{aligned}$$

This establishes the recursion (25).

2. Justification of the Bounds on s^* and S^*

Denote by (s_n, S_n) an optimal (s, S) ordering rule for the first period of an n-period model. Thus, an optimal policy for the n-period model, denoted Y^n, consists of using (s_n, S_n) in period 1, (s_{n-1}, S_{n-1}) in period 2, $\cdots$, (s_1, S_1) in period n.

Lemma 2

$$\underline{S} \leqq S_n, \qquad n = 2, 3, \cdots.$$

Proof:

Suppose $S_n < \underline{S}$. Let Y' be an alternative policy defined as follows:

$$Y'_1 = \begin{cases} \underline{S}, & X_1 < s_n \\ X_1, & X_1 \geqq s_n \end{cases}$$

$$Y'_t = \max(X'_t, Y_t^n) \quad \text{with} \quad X'_t = Y'_{t-1} - \xi_{t-1}, \qquad t = 2, 3, \cdots$$

Clearly

$$K[\delta(Y_t^n - X_t) - \delta(Y'_t - X'_t)] \geqq 0 \quad \text{and} \quad G_\alpha(Y_t^n) - G_\alpha(Y'_t) \geqq 0$$

for $t > 1$. Thus, for $X_1 < s_n$,

$$f_n(X_1 \mid Y^n) - f_n(X_1 \mid Y') \geqq G_\alpha(S_n) - G_\alpha(\underline{S}) > 0$$

by the definition of $\underline{S}$. Hence, Y^n is not optimal, which is a contradiction and completes the proof.

Lemma 3

If for some n ($n = 2, 3, \cdots$) $\bar{s} < s_n$, then there is a policy Y' (say) such that $(\bar{s}, S_n)$ is used in period 1 and $f_n(x \mid Y') \leqq f_n(x \mid Y^n)$ for all x.

Proof:

Suppose $\bar{s} < s_n$. Let Y' be an alternative policy defined as follows:

$$Y'_1 = \begin{cases} S_n, & X_1 < \bar{s} \\ X_1, & X_1 \geqq \bar{s} \end{cases}$$

$$Y'_t = Y_t^n \quad \text{with} \quad X'_t = Y'_{t-1} - \xi_{t-1}, \qquad t = 2, 3, \cdots.$$

Then

$$\begin{aligned} f_n(X_1 \mid Y^n) &- f_n(X_1 \mid Y') \\ &= K[\delta(Y_1^n - X_1) - \delta(Y'_1 - X_1)] + [G_\alpha(Y_1^n) - G_\alpha(Y'_1)] \\ &\qquad + \alpha K E[\delta(Y_2^n - X_2) - \delta(Y'_2 - X'_2)]. \end{aligned}$$

Now if $X_1 < \bar{s}$ or $s_n \leqq X_1$, the bracketed terms vanish. If $\bar{s} \leqq X_1 < s_n$, then

$$\begin{aligned} K[\delta(Y_1^n - X_1) &- \delta(Y'_1 - X_1)] + \alpha K[\delta(Y_2^n - X_2) - \delta(Y'_2 - X'_2)] \\ &+ G_\alpha(Y_1^n) - G_\alpha(Y'_1) \geqq (1 - \alpha)K + G_\alpha(S_n) - G_\alpha(X_1) \\ &\geqq (1 - \alpha)K + G_\alpha(\underline{S}) - G_\alpha(\bar{s}) \geqq 0. \end{aligned}$$

The final inequality follows from (23). The second inequality may be justified as follows: If $\underline{S} \leqq X_1$, then

$$G_\alpha(S_n) - G_\alpha(X_1) \geqq 0 \geqq G_\alpha(\underline{S}) - G_\alpha(\bar{s}).$$

If $X_1 < \underline{S}$, then

$$G_\alpha(S_n) \geqq G_\alpha(\underline{S}) \quad \text{and} \quad G_\alpha(X_1) \leqq G_\alpha(\bar{s}).$$

In both cases, $G_\alpha(S_n) - G_\alpha(X_1) \geqq G_\alpha(\underline{S}) - G_\alpha(\bar{s})$. Thus, $f_n(x \mid Y^n) - f_n(x \mid Y') \geqq 0$, which proves that using $(\bar{s}, S_n)$ in period 1 is also optimal. This proves the lemma.

Lemma 4

If for some $n(=2, 3, \cdots)\bar{S} < S_n$, then there is a policy Y' (say) such that $(s_n, \underline{S})$ is used in period 1 and such that $f_n(x \mid Y') \leqq f_n(x \mid Y^n)$ for all x.

Proof:

Suppose $\bar{S} < S_n$. By lemma 3 we may assume that $s_n \leqq \bar{s}$. Let Y' be an alternative policy defined as follows:

$$Y'_1 = \begin{cases} \underline{S}, & X_1 < s_n \\ X_1, & X_1 \geqq s_n \end{cases}$$

$$Y'_t = Y_t^n \quad \text{with} \quad X'_t = Y'_{t-1} - \xi_{t-1}, \qquad t = 2, 3, \cdots$$

Then

$$f_n(X_1 \mid Y^n) - f_n(X_1 \mid Y') = G_\alpha(Y_1^n) - G_\alpha(Y'_1) + \alpha K[\delta(Y_2^n - X_2) - \delta(Y'_2 - X'_2)].$$

Both bracketed terms vanish if $X_1 \geqq s_n$, while if $X_1 < s_n$,

$$G_\alpha(Y_1^n) - G_\alpha(Y'_1) + \alpha K[\delta(Y_2^n - X_2) - \delta(Y'_2 - X'_2)] \geqq G_\alpha(\bar{S} + 1) - G_\alpha(\underline{S}) - \alpha K \geqq 0$$

by (21). Thus, it is also optimal to use $(s_n, \underline{S})$ in period 1, which completes the proof.

Lemma 5

$$\underline{s} \leqq s_n, \qquad n = 2, 3, \cdots.$$

Proof:

Suppose $s_n < \underline{s}$. By lemma 2 we must have that $\underline{S} \leqq S_n$. Let Y' be an alternative policy defined as follows:

$$Y'_1 = \begin{cases} S_n, & X_1 < s_n \\ \underline{S}, & s_n \leqq X_1 < \underline{s} \\ X_1, & \underline{s} \leqq X_1 \end{cases}$$

$$Y'_t = Y_t, \qquad t = 2, 3, \cdots \quad \text{for} \quad X_1 < s_n \quad \text{and} \quad \underline{s} \leqq X_1$$

$$Y'_t = \max (X'_t, Y_t) \quad \text{for} \quad s_n \leqq X_1 < \underline{s}$$

with $X'_t = Y'_{t-1} - \xi_{t-1}$ for $t = 2, 3, \cdots$.

Now if $X_1 < s_n$ or $\underline{s} \leqq X_1$, Y^n and Y' incur the same costs over the n-periods. If $s_n \leqq X_1 < \underline{s}$, observe that for $t > 1$,

$$K[\delta(Y_t^n - X_t) - \delta(Y'_t - X'_t)] \geqq 0$$

and

$$G_\alpha(Y_t^n) - G_\alpha(Y'_t) \geqq 0.$$

In addition,

$$K[\delta(Y_1^n - X_1) - \delta(Y'_1 - X_1)] + G_\alpha(Y_1^n) - G_\alpha(Y'_1) \geqq G_\alpha(\underline{s} + 1) - G_\alpha(\underline{S}) - K > 0$$

by (22). Hence, $f_n(X_1 \mid Y^n) > f_n(X_1 \mid Y')$, contradicting the optimality of Y^n. Thus, our assumption that $s_n < \underline{s}$ is false, and the lemma is proved.

Theorem 4

(a) For the n-period model $(n = 2, 3, \cdots)$ there exists an optimal (s, S) policy with $\underline{s} \leqq s_n \leqq \bar{s} \leqq \underline{S} \leqq S_n \leqq \bar{S}$.

(b) For the infinite period model there exists an optimal policy (s^*, S^*) such that $\underline{s} \leqq s^* \leqq \bar{s} \leqq \underline{S} \leqq S^* \leqq \bar{S}$.

Proof:

Part (a) follows by applying Lemmas 2, 3, 4, 5. We prove part (b) by considering separately the cases $0 \leqq \alpha < 1$ and $\alpha = 1$.

For the case $0 \leqq \alpha < 1$, we noted in Section 2 that any limit point, (s^*, S^*) say, of the sequence $\{(s_n, S_n)\}$ is optimal for the infinite period model. In view of part (a) of the theorem, (s^*, S^*) evidently satisfies the inequalities in part (b), which completes the proof in this case.

Now consider the case $\alpha = 1$. Note that $\underline{s}$, $\bar{s}$, $\underline{S}$, $\bar{S}$ are each functions of α. It is convenient to express this dependence explicitly by writing $\underline{s}(\alpha)$, $\bar{s}(\alpha)$, $\underline{S}(\alpha)$, $\bar{S}(\alpha)$. Observe that $\underline{s}(1)$, $\bar{s}(1)$, $\underline{S}(1)$, $\bar{S}(1)$ are defined by

$$\Delta G_\alpha[\underline{S}(1) - 1] < 0 \leqq \Delta G_\alpha[\underline{S}(1)]$$

$$G_\alpha[\bar{S}(1)] < G_\alpha[\underline{S}(1)] + \alpha K \leqq G_\alpha[\bar{S}(1) + 1]$$

$$G_\alpha[\underline{s}(1)] \leqq G_\alpha[\underline{S}(1)] + K < G_\alpha[\underline{s}(1) - 1]$$

$$G_\alpha[\bar{s}(1)] \leqq G_\alpha[\underline{S}(1)] + (1 - \alpha)K < G_\alpha[\bar{s}(1) - 1],$$

where $\alpha = 1$ and $K > 0$. Examination of these inequalities shows that they also hold for all sufficiently large $\alpha \leqq 1$. Thus,

$$\text{(A2)} \quad \underline{s}(\alpha) = \underline{s}(1), \qquad \bar{s}(\alpha) = \bar{s}(1), \qquad \underline{S}(\alpha) = \underline{S}(1), \qquad \bar{S}(\alpha) = \bar{S}(1),$$

for all sufficiently large $\alpha \leqq 1$.

If $K = 0$, the same argument applies except that only the first of the group of four inequalities above is used.

Now let $\{\alpha_i : i = 1, 2, \cdots; 0 < \alpha_i < 1\}$ be an infinite sequence such that (A2) holds for all α_i and such that $\lim \alpha_i = 1$. Denote by $[s(i), S(i)]$ an (s, S) policy that is optimal for α_i and satisfies

$$\underline{s}(1) \leqq s(i) \leqq \bar{s}(1) \leqq \underline{S}(1) \leqq S(i) \leqq \bar{S}(1).$$

Since there are only finitely many (s, S) policies satisfying these inequalities, at least one of the policies in the sequence $\{[s(i), S(i)]\}$ occurs infinitely often. Denote it by (s^*, S^*) and let $\beta_1, \beta_2, \cdots$ denote the subsequence of $\{\alpha_i\}$ for which (s^*, S^*) is optimal. Now consider any arbitrary (s, S) policy. We have

$$\text{(A3)} \qquad a_{\beta_i}(x \mid s^*, S^*) \leqq a_{\beta_i}(x \mid s, S), \qquad i = 1, 2, \cdots$$

for all x. Since $a_\alpha(\cdot \mid \cdot, \cdot)$ is continuous in α from the left at $\alpha = 1$, we obtain by letting $i \to \infty$ in (A3),

$$a_1(x \mid s^*, S^*) \leqq a_1(x \mid s, S) \quad \text{for all } x.$$

Thus, (s^*, S^*) is optimal for $\alpha = 1$ and satisfies the hypothesis of part (b) of the theorem, completing the proof.

3. Proofs of Theorem 1 and Lemma 1

Proof of Theorem 1:

Let

$$b(d, t, k) \equiv \Pr[T(d) = t, \textstyle\sum_{r=1}^{t} \xi_r = k].$$

Then by hypothesis, for $x \geqq s'$

$$
\begin{aligned}
a_\alpha(x \mid s, S)/(1 - \alpha) \\
&= f(x \mid s, S) \\
&= L_\alpha(x, x - s') + \textstyle\sum_{k>x-s'} \sum_{t=1}^{\infty} \alpha^t b(x - s', t, k) f(x - k \mid s, S) \\
&= L_\alpha(x, x - s') + \textstyle\sum_{k>x-s'} \sum_{t=1}^{\infty} \alpha^t b(x - s', t, k) f(x - k \mid s', S') \\
&= f(x \mid s', S') = a_\alpha(x \mid s', S')/(1 - \alpha),
\end{aligned}
$$

which proves Theorem 1.

Proof of Lemma 1:

Let (s^*, S^*) be an optimal policy. Let the random variable T be the smallest positive integer t such that $X_{t+1} = x'$. If no such integer exists, let $T = +\infty$.

Let q denote the total discounted cost incurred during periods $\lambda + 1, \lambda + 2, \cdots, \lambda + T$ when using (s, S) and let $Q = Eq$. Since x' is accessible from x under (s, S) and $0 < \alpha < 1$, we have $0 < E(\alpha^T) \equiv A < 1$. Also

$$Q + Af(x' \mid s^*, S^*) \geqq f(x \mid s^*, S^*) = f(x \mid s, S) = Q + Af(x' \mid s, S),$$

where the inequality follows from the fact that using (s, S) in periods $1, 2, \cdots, T$ and (s^*, S^*) in periods $T + 1, T + 2, \cdots$, cannot be better than using (s^*, S^*) in every period. Since $0 < A < 1$, and $|Q| < \infty$, the inequality implies that

$$f(x' \mid s, S) \leqq f(x' \mid s^*, S^*).$$

But the reverse inequality is also true by the optimality of (s^*, S^*). Consequently equality holds, proving the lemma.

4. Proof of Theorem 3:

In view of Theorem 4 we may here restrict our analysis to (s, S) policies falling within the bounds given there. We consider first the infinite period model.

Suppose that $\alpha = 1$ and $X_1 < \underline{s}$. Then if we use (s, S), by an hypothesis of Theorem 3 we order in every period up to S. Thus,

$$a_1(X_1 \mid s, S) = K + G_1(S).$$

Clearly S should be chosen to minimize $G_1(S)$; for example, $S = \underline{S}$ will do. And s may assume any value in the interval $[\underline{s}, \bar{s}]$; in particular, $s = \underline{s}$ will do. Thus, $(\underline{s}, \underline{S})$ is optimal for $\alpha = 1$.

Now suppose $0 \leqq \alpha < 1$. Then if $X_1 \leqq \bar{s}$ and if (s, S) is followed, there are two possibilities. If $X_1 < s$, an order is placed in every period. If $s \leqq X_1 \leqq \bar{s}$, an order is not placed in period 1, but an order is placed in every period thereafter. Thus,

$$a_\alpha(X_1 \mid s, S) = \begin{cases} (1 - \alpha)[K + G_\alpha(S)] + \alpha[K + G_\alpha(S)], & X_1 < s \\ (1 - \alpha)G_\alpha(X_1) + \alpha[K + G_\alpha(S)], & s \leqq X_1 \leqq \bar{s}. \end{cases}$$

Now for $X_1 < \underline{s}$, we clearly have

$$a_\alpha(X_1 \mid s, S) \geqq a_\alpha(X_1 \mid s, \underline{S}) = a_\alpha(X_1 \mid \underline{s}, \underline{S}).$$

Also for $\underline{s} \leqq X_1 < s$, we have

$$a_\alpha(X_1 \mid s, S) - a_\alpha(X_1 \mid \underline{s}, \underline{S}) = (1 - \alpha)[K + G_\alpha(S) - G_\alpha(X_1)]$$
$$\geqq (1 - \alpha)[K + G_\alpha(\underline{S}) - G_\alpha(\underline{s})] \geqq 0$$

by (22). Thus, by Theorem 1, ($\underline{s}$, $\underline{S}$) is optimal, which completes the proof.

Obvious modifications of the above arguments show that ($\underline{s}$, $\underline{S}$) is also optimal for the n-period model.

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